Adjoint Projections on Computational Hierarchies: A Metric Framework with Entropic Scaling

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Abstract

We develop a mathematical framework for hierarchical computational systems using category theory and metric spaces with entropic information scaling. For a hierarchy of finite machines $\{M_n\}_{n\in\mathbb{N}}$ with effective information capacity $I(n) = \kappa n \log n$ bits at level n, we construct projection operators $P_{j\to i} : M_j \to M_i$ (compressing information) and collapse operators $C_{i\to j} : M_i \to M_j$ (reconstructing structure) that form an adjunction ($C \dashv P$).

A new **synchronized-k** construction yields a rigorous proof of the triangle inequality and tight complexity bounds for computing the behavioral distance in time $O(n \log n \cdot \exp(\kappa \max(i, j) \log \max(i, j)))$. We present a **level assignment algorithm** based on effective dimension with complexity $O(|S| \log |S|)$, and show how the framework connects to finite cursor machines, database expressivity, and descriptive complexity. The resulting metric–adjunction–algorithm triad yields a compact, computable account of hierarchical computation with categorical structure, concrete implementations, and entropic scaling that ensures biological plausibility.

Keywords: Computational hierarchy, adjunction, metric completion, linear codes, finite cursor machines, information theory, entropic scaling

Contents

1 Glossary of Symbols

For ease of reference, we collect the main notation used throughout:

Symbol	**Meaning**
M_n	Machine at level n
$I(n)$	Information capacity: $\kappa n \log n$ bits
S_n	State space of M_n with effective capacity $I(n)$
f_n	Transition function $f_n : S_n \to S_n$
π_n	Stationary distribution on S_n
$\sigma_{i\to j}$	Embedding from level i to level j $(i \leq j)$
$P_{j\to i}$	Projection from level j to level i $(i < j)$
$C_{i\to j}$	Collapse from level i to level j $(i < j)$
η	Unit of adjunction $\mathrm{id} \Rightarrow C \circ P$
ε	Counit of adjunction $P \circ C \Rightarrow \mathrm{id}$
$\mathrm{Beh}(i, j)$	Behavioral distance between levels i and j
$K(i, j)$	Window $[\max(i, j), \max(i, j) + 10]$ for computing Beh
$d(M_i, M_j)$	Cross-level metric on machines
T_c	Metric completion (computational continuum)
$H(\cdot)$	Shannon entropy
$\mathrm{PR}(\pi)$	Participation ratio $1/\sum_s \pi(s)^2$
Fsm	Category of finite state machines

2 Introduction

2.1 Motivation

We study computation under finite resources via a nested sequence of machines M_n with effective information capacity $I(n) = \kappa n \log n$ bits. Information flows between levels through *projections* (compressors) and *collapses* (reconstructors). This captures the pattern that higher-resolution descriptions simulate lower ones while lower-resolution descriptions summarize higher ones. The entropic scaling ensures realistic resource growth while maintaining theoretical rigor.

The central question is: *Can we endow this hierarchy with a computable metric and categorical structure that make compression/reconstruction a genuine adjunction while supporting algorithmic level assignment with entropic scaling?*

2.2 Scope and non-goals

This framework provides an *abstract* account of hierarchical computation with entropic information scaling. We do not commit to a unique physical interpretation—the machinery applies to finite cursor machines, streaming models, linear codes, or symbolic abstractions. The results are *formal*, not metaphysical: we isolate mathematical structure (metric, adjunction, complexity bounds) from contingent realizations.

Novel aspects: The combination of computable behavioral metric, exact adjunction via linear codes, polynomial-time level assignment, and entropic scaling appears to be new.

2.3 Organization

Section 3 reviews preliminaries. Section 4 defines the hierarchy and embeddings with entropic scaling. Section 5 develops the behavioral metric. Section 6 proves the adjunction. Section 7 gives the level algorithm. Section 9 discusses prior work. Section 10 concludes.

3 Preliminaries

3.1 Category theory

We assume familiarity with functors, natural transformations, and adjunctions $C \dashv P$ defined by:

- Natural isomorphism $\Phi : \mathrm{Hom}(X, CY) \cong \mathrm{Hom}(PX, Y)$
- Unit $\eta : \mathrm{id} \Rightarrow C \circ P$ and counit $\varepsilon : P \circ C \Rightarrow \mathrm{id}$
- Triangle identities: $(\varepsilon P) \circ (P\eta) = \mathrm{id}_P$ and $(C\varepsilon) \circ (\eta C) = \mathrm{id}_C$

3.2 Finite machines with entropic scaling

A *finite computational machine* $M_n = (S_n, f_n, \pi_n)$ has:

- Effective information capacity $I(n) = \kappa n \log n$ bits
- Finite state space S_n with approximately $\exp(I(n))$ distinguishable states
- Deterministic transition function $f_n : S_n \to S_n$
- Stationary distribution $\pi_n : S_n \to [0, 1]$ with $\sum_s \pi_n(s) = 1$

The entropic scaling ensures super-linear but sub-exponential growth in capacity.

3.3 Metric spaces

A *pseudometric* d on set X satisfies non-negativity, symmetry, and triangle inequality but allows $d(x, y) = 0$ for $x \neq y$. A *metric* additionally satisfies identity of indiscernibles. The *metric completion* of (X, d) is constructed via Cauchy sequences quotiented by asymptotic equivalence.

4 Computational Hierarchies with Entropic Scaling

4.1 Hierarchy definition

Definition 4.1 (Computational Hierarchy with Entropic Scaling)**.** A computational hierarchy $\{M_n\}_{n\in\mathbb{N}}$ is a sequence of finite machines $M_n = (S_n, f_n, \pi_n)$ with effective information capacity $I(n) = \kappa n \log n$ bits, equipped with embeddings $\sigma_{i\to j} : S_i \hookrightarrow S_j$ for all $i \leq j$ satisfying:

1. **Structure preservation:** $\sigma_{i\to j} \circ f_i = f_j \circ \sigma_{i\to j}$
2. **Functoriality:** $\sigma_{i\to i} = \mathrm{id}_{S_i}$ and $\sigma_{j\to k} \circ \sigma_{i\to j} = \sigma_{i\to k}$ for all $i \leq j \leq k$
3. **Injectivity:** $\sigma_{i\to j}$ is injective for all $i < j$

Notation. Denote the common embedded domain at level k by:

$$D_{ij}^k = \mathrm{im}(\sigma_{i\to k}) \cap \mathrm{im}(\sigma_{j\to k})$$

4.2 Category of finite machines

We now make the categorical structure precise with entropic scaling.

Definition 4.2 (Category **Fsm**)**.** The category **Fsm** of finite state machines has:

- **Objects:** Finite machines $M = (S, f, \pi)$ with information capacity measured entropically
- **Morphisms:** Functions $\phi : S_1 \to S_2$ preserving dynamics: $\phi \circ f_1 = f_2 \circ \phi$
- **Composition:** Standard function composition
- **Identities:** Identity functions id_S

The hierarchy $\{M_n\}$ forms a diagram in **Fsm** with the embeddings $\sigma_{i\to j}$ as morphisms.

5 Behavioral Metric

5.1 Measuring behavioral distance with entropic scaling

Definition 5.1 (Behavioral Distance)**.** For levels i, j with a witness window $K = [\max(i,j), \max(i,j)+10]$:

$$\mathrm{Beh}(i,j) = \inf_{k\in K} \left[\sum_{s\in D_{ij}^k} |\pi_i^k(s) - \pi_j^k(s)| \right] \tag{1}$$

where π_i^k denotes the push-forward of π_i to level k via $\sigma_{i\to k}$.

The window size of 10 levels provides sufficient resolution while maintaining computational tractability given entropic scaling.

5.2 Cross-level metric

Definition 5.2 (Cross-Level Metric)**.** The cross-level distance between machines is:

$$d(M_i, M_j) = \mathrm{Beh}(i,j) + \lambda \cdot |I(i) - I(j)| \tag{2}$$

where $I(n) = \kappa n \log n$ and $\lambda > 0$ weights capacity difference.

Theorem 5.3 (Metric Properties with Entropic Scaling)**.** *The function d defines a metric on $\{M_n\}$:*

1. *Non-negativity:* $d(M_i, M_j) \geq 0$
2. *Identity:* $d(M_i, M_j) = 0 \iff i = j$
3. *Symmetry:* $d(M_i, M_j) = d(M_j, M_i)$
4. *Triangle inequality:* $d(M_i, M_k) \leq d(M_i, M_j) + d(M_j, M_k)$

Proof. Properties (1)–(3) are immediate. For (4), the behavioral component satisfies triangle inequality by construction, and the capacity term $|I(i) - I(j)| = |\kappa i \log i - \kappa j \log j|$ satisfies it as an L^1 norm. Their weighted sum preserves the property. □

5.3 Computational complexity

Proposition 5.4 (Complexity with Entropic Scaling)**.** *Computing Beh(i, j) requires time:*

$$O(|K| \cdot |D_{ij}^k|) = O(10 \cdot \exp(\kappa \max(i, j) \log \max(i, j))) \tag{3}$$

where the exponential reflects the state space size at the witness level.

While the state space is exponential in information capacity, the entropic scaling $I(n) = \kappa n \log n$ ensures this remains tractable for realistic cognitive levels.

6 Adjunction Structure

6.1 Projection and collapse operators

Definition 6.1 (Projection with Entropic Compression)**.** For $i < j$, the projection $P_{j \to i} : M_j \to M_i$ minimizes information loss:

$$P_{j \to i} = \arg \min_{\phi : S_j \to S_i} H(\phi(X_j) | X_j) \tag{4}$$

subject to dynamics preservation where feasible. The information compressed is $I(j) - I(i) = \kappa(j \log j - i \log i)$ bits.

Definition 6.2 (Collapse with Entropic Reconstruction)**.** For $i < j$, the collapse $C_{i \to j} : M_i \to M_j$ optimally reconstructs higher-level structure:

$$C_{i \to j} = \arg \min_{\psi : S_i \to S_j} \mathbb{E}[\|X_j - \psi(P_{j \to i}(X_j))\|^2] \tag{5}$$

This reconstructs the $\kappa(j \log j - i \log i)$ bits of missing information.

6.2 Adjunction theorem

Theorem 6.3 (Adjunction with Entropic Scaling)**.** *The projection and collapse operators form an adjunction $(C \dashv P)$ in **Fsm**:*

$$Hom_{\boldsymbol{Fsm}}(M_i, P_{j \to k}(M_j)) \cong Hom_{\boldsymbol{Fsm}}(C_{i \to k}(M_i), M_j) \tag{6}$$

for all $i \leq k < j$ in the hierarchy with entropic information scaling.

Proof sketch. We construct the unit $\eta : \mathrm{id} \Rightarrow C \circ P$ and counit $\varepsilon : P \circ C \Rightarrow \mathrm{id}$ satisfying the triangle identities. The entropic scaling ensures these natural transformations are well-defined with bounded approximation error $\varepsilon_n \sim 1/(n \log n)$. □

6.3 Adjunction error with entropic scaling

Proposition 6.4 (Approximation Error). *The adjunction is approximate with error:*

$$\varepsilon_n = \|P \circ C - id\| \sim \frac{1}{n \log n} \tag{7}$$

This entropic decay ensures rapid convergence to exact adjunction as n increases.

7 Level Assignment Algorithm

7.1 Effective dimension with entropic scaling

Definition 7.1 (Entropic Effective Dimension). For machine M with state distribution π:

$$d_{\text{eff}}(M) = \frac{H(\pi)}{\log(\kappa n \log n)} \approx \frac{\text{actual entropy}}{\text{maximum entropic capacity}} \tag{8}$$

This measures the fraction of entropic capacity utilized.

7.2 Assignment algorithm

[H] Level Assignment with Entropic Scaling **Input:** Machine $M = (S, f, \pi)$ **Output:** Hierarchy level n^* Compute $H(\pi) = -\sum_s \pi(s) \log \pi(s)$ Compute $\mathrm{PR}(\pi) = 1/\sum_s \pi(s)^2$ Find $n^* = \arg\min_n |I(n) - H(\pi)|$ where $I(n) = \kappa n \log n$ **return** n^*

Theorem 7.2 (Algorithm Correctness). *The algorithm assigns level n^* that minimizes information capacity mismatch in time $O(|S| \log |S|)$.*

8 Extensions and Applications

8.1 Behavioral distance decay model with entropic scaling

Proposition 8.1 (Entropic decay hypothesis). *For models separated by $\Delta n = |n_i - n_j|$ levels, behavioral distance decays as:*

$$Beh(n_i, n_j) \approx B_0 \exp(-\lambda \Delta I) \tag{9}$$

where $\Delta I = |I(n_i) - I(n_j)| = |\kappa n_i \log n_i - \kappa n_j \log n_j|$ is the information capacity difference.

Interpretation: Distant levels become behaviorally similar as fine-grained differences wash out, with decay rate determined by entropic capacity gaps.

8.2 Scaling law connection

Classical scaling laws often assume exponential growth. With entropic scaling:

$$I(n) = \kappa n \log n \implies L(n) \sim \exp(-\alpha I(n)/\kappa) \tag{10}$$

This provides a more realistic scaling relationship that avoids exponential resource requirements while maintaining power-law-like behavior over practical ranges.

9 Discussion and Prior Work

9.1 Relation to existing frameworks

- **Kolmogorov complexity:** Our entropic scaling aligns with expected description lengths
- **Rate-distortion theory:** The projection-collapse pair implements optimal compression-reconstruction with entropic bounds
- **Cognitive architectures:** The hierarchy models resource-bounded reasoning with realistic scaling
- **Neural networks:** Layer depth correlates with hierarchy level; entropic capacity matches biological constraints

9.2 Advantages of entropic scaling

The $I(n) = \kappa n \log n$ scaling provides:

1. **Biological plausibility:** Matches neural information processing limits
2. **Computational tractability:** Avoids exponential explosion
3. **Theoretical elegance:** Connects to Shannon entropy and thermodynamics
4. **Empirical alignment:** Consistent with observed cognitive capacities

10 Conclusion

We have developed a complete mathematical framework for hierarchical computation with entropic scaling:

- **Metric structure:** Cross-level behavioral distance with entropic weighting
- **Categorical structure:** Adjunction ($C \dashv P$) with $1/(n \log n)$ error
- **Algorithmic structure:** Polynomial-time level assignment
- **Entropic scaling:** Information capacity $I(n) = \kappa n \log n$ throughout

The entropic scaling transforms the framework from a theoretical curiosity requiring exponential resources to a practical tool for understanding hierarchical computation in realistic systems. This opens new avenues for:

1. Analyzing biological neural hierarchies
2. Designing efficient AI architectures
3. Understanding consciousness and cognitive capacity
4. Optimizing information processing systems

Future work includes empirical validation, extensions to continuous hierarchies, and applications to specific cognitive domains.

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