

Probability & Statistics

Problem set №9. Week starting April 23rd

1. Let r.v.s. X_1, X_2, \dots, X_n be independent and have the same distribution $\text{Exp}(\lambda)$. Let $Y_i = X_1 + \dots + X_i$, dla $i = 1, \dots, n$. Prove that density of variable (Y_1, \dots, Y_n) equals $f_{Y_1, \dots, Y_n}(y_1, \dots, y_n) = \lambda^n \exp(-\lambda y_n)$, with $0 < y_1 < y_2 < \dots < y_n$.
2. Given density from the previous exercise $f_{Y_1, \dots, Y_n}(y_1, \dots, y_n)$, prove that marginal density with respect to Y_n equals $f_{Y_n}(y_n) = \lambda^n \frac{y_n^{n-1}}{(n-1)!} \exp(-\lambda y_n)$, with $0 < y_n$.
3. Given is value of parameter a . Find MLE estimator of parameter θ of uniform distribution on interval $[\theta - a; \theta + a]$.
4. Find MLE estimator of parameter θ of uniform distribution on interval $[\theta - a; \theta + a]$ in case when value of parameter a is unknown.
5. Let X_1, \dots, X_5 be independent and identically distributed continuous random variables. Symbol p denotes probability $P(X_1 < X_2 > X_3 < X_4 > X_5)$. Prove that p does not depend on density $f(x)$ of variables X_k . Find value of p .
6. X, Y, Z are independent variables of $U[0, 1]$ distribution. Find $P(X \geq YZ)$.
7. X_1, X_2, X_3 are independent random variables of $\text{Exp}(\lambda)$ distribution. Find (3-dimensional) density of variable $(Y_1, Y_2, Y_3) = (X_1 + X_2, X_1 + X_3, X_2 + X_3)$.
8. (2p.) X_1, \dots, X_n are independent and of identical continuous distribution random variables. At time j record occurs ($j \leq n$), if $X_j \geq X_i$ for $1 \leq i \leq j$. Let r.v. Z equals number of record values in sequence $\{X_k\}$. Prove that $E(Z) = \sum_{i=1}^n \frac{1}{i}$.

[Ex. 9–10] Density of r.v. (X, Y) is given by:

$$f(x, y) = 1, \quad 0 < x, y < 1.$$

9. Find density of r.v. $Z = X/Y$.
10. Find probability that first significant digit of Z equals 1.

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