Probability & Statistics

Problem set №6. Week starting April 1st

1. Suppose that random variable X has a geometric distribution ($X \sim \mathrm{Geom}(p)$). Prove that $M_X(t)$ is of the form $M_X(t) = \dfrac{pe^t}{1-qe^t}$.
2. Let $X \sim \mathrm{Geom}(p)$. Using $M_X(t)$ find $\mathrm{E}(X)$ and $\mathrm{V}(X)$.
3. Let $X \sim \mathrm{N}(\mu, \sigma^2)$. That means $f(x) = \dfrac{1}{\sqrt{2\pi}\sigma}\exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right)$, $x \in \mathbb{R}$. Prove (using definition $M_X(t) = \mathrm{E}(\mathrm{e}^{tX})$) that $M_X(t) = \exp\left(\mu t + \dfrac{1}{2}\sigma^2 t^2\right)$.
4. Random variables $X_1, \dots, X_n$ are independent and $X_k \sim \mathrm{N}(\mu, \sigma^2)$. Find MGF $M_{\bar{X}}(t)$ of random variable $\bar{X}$ ($\bar{X}$ is the mean of $X_1, \dots, X_n$). Try to identify the distribution of $\bar{X}$.

   [**Ex. 5–6**] Assuming that if $X \sim \mathrm{Gamma}(b, p)$ then MGF is the form $M_X(t) = (1 - \frac{t}{b})^{-p}$ give answers in form Gamma with (correct) parameters. If necessary you may use, without proof, that $\Gamma(1/2) = \sqrt{\pi}$.
5. Let $X \sim \mathrm{N}(\mu, \sigma^2)$. Find distribution of $Y = \left(\dfrac{X-\mu}{\sigma}\right)^2$.
6. Random variables $X_1, \dots, X_n$ are independent and $X_k \sim \mathrm{N}(\mu, \sigma^2)$. Find distribution of random variable $Z_n = \displaystyle\sum_{k=1}^{n}\left(\frac{X_k-\mu}{\sigma}\right)^2$.

   [**Ex. 7–8**] What is the distribution of random variable $Z = \displaystyle\sum_{k=1}^{n} X_k$? We assume that random variables X_k are independent. Try find the solution using "*MGFs*".
7. $X_k \sim \mathrm{Gamma}(b, p_k)$, $k = 1, \dots, n$.
8. $X_k \sim \mathrm{B}(m_k, p)$, $k = 1, \dots, n$.
9. 2-dimensional random variable (X, Y) has density of the form $f(x, y) = \dfrac{15}{2}x^2 y$ (on triangle with vertices $(0,0)$, $(2,0)$, $(0,1)$). Find density function of $T = X/Y$.
10. Assume that profit of the firm is the random variable U. MGF of the profit is given by $M_U(t) = \dfrac{2}{2-3t}$. Find:
    (a) expected value of the profit,
    (b) variance of the profit,
    (c) find MGF of *netto profit* when the flat tax ratio equals 0.9 (90%).
11. MGF of random variable X equals $M_X(t)$. Random variable Y is some function of r.vs with distributions like X. What we can say about Y (assumptions, distribution of initial random variables) if:
    (a) $M_Y(t) = M_X(2t) \cdot M_X(4t)$,
    (b) $M_Y(t) = \mathrm{e}^{2t} M_X(t)$,
    (c) $M_Y(t) = 4M_X(t)$.

Witold Karczewski