Probability & Statistics

Problem set №11. Week starting on May 12th.

1. Ex. 1–4. Given independent random variable X_1, X_2 with density $f(x) = \dfrac{x}{2}$, $x \in (0, 2)$ each. Let additionally $Y_1 = X_1 \cdot X_2$, $\;Y_2 = \dfrac{X_1}{X_2}$.
   (a) Find expected values $\mathrm{E}\,(X_1)$ and $\mathrm{E}\left(\dfrac{1}{X_1}\right)$.
   (b) Find expected values $\mathrm{E}\,(Y_1)$, $\mathrm{E}\,(Y_2)$.
2. Find 2-dimensional density $g(y_1, y_2)$ of variable (Y_1, Y_2).
3. Find marginal distribution of variables Y_1, Y_2.
4. Check if random variables Y_1, Y_2 are independent.
5. Ex. 5–6. Random variable X has multivariate normal distribution
$$N\left(\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 9 \end{bmatrix}\right).$$
   Find ρ_{x_1,x_3} (correlation of variables X_1, X_3).
6. Let $Y_1 = X_1 + 2X_3$, $\;Y_2 = X_2 - X_3$ and $Y = [Y_1, Y_2]^T$. Find distribution of variable Y.
7. Exp. 7–8. Let $X \sim \mathrm{Exp}(\lambda)$, i.e. $f(x) = \lambda\,\exp(-\lambda x)$, when $x \in (0, \infty)$. Find – from definition – expected value $\mathrm{E}\,(X)$.
8. Prove that $M_X(t) = \dfrac{\lambda}{\lambda - t}$.
9. (X, Y) is a discrete random variable with probabilities p_{ij}. Prove that $\mathrm{E}\,(X + Y) = \mathrm{E}\,(X) + \mathrm{E}\,(Y)$.
10. Complete equation below (with proof)
$$\mathrm{V}\,(X - Y) = V(X) \pm \ldots$$
    Symbol $\pm$ means that there is at least one term, with “plus” or “minus” sign.

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11. **(E1)** Data (in columns) presents the measurement of weight before and after the period of application of a specific diet for 16 people. Perform test of the hypothesis: **diet affects weight**.
12. **(E2)** 10 experimental plots were divided into two parts, in one of them additional agrotechnical activities were performed. The row contains the yield of the part subjected to additional treatments and the part of the field traditionally grown. Test the hypothesis: **additional factor affects crop yields**, i.e. give the form of the null hypothesis and specify p-value.
13. **(E2)** Independent observations $x_1, \ldots, x_n,\; y_1, \ldots, y_k$ are from the $N(\mu, \sigma^2)$ distribution. Find distribution of random variable Z:
$$Z = \frac{\bar{X} - \bar{Y}}{\sigma\sqrt{\frac{n+k}{nk}}}.$$

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