# Probability & Statistics

## Problem set №2. Week starting on March 4th

1. Let $\Sigma$ be a $\sigma$-field on a set $\Omega$.
   (a) Check that $\Omega \in \Sigma$.
   (b) Suppose $A_k \in \Sigma$, with $k = 1, 2, 3, \ldots$. Prove that $\bigcap_{k \in \mathbb{N}} A_k \in \Sigma$.
2. Let $\Omega = \{a, b, c\}$.
   (a) Describe $\sigma$-fields on a set $\Omega$.
   (b) Give examples of functions $X, Y$ such that $X$ is random variable, and $Y$ is not.
3. Let $\Omega = \{1, 2, 3, 4, 5\}$ and $S = \{1, 4\}$. Find the smallest $\sigma$-field containing $S$.
4. Find distribution function (CDF) and expected value $\mathrm{E}(X)$ of the random variable $X$:

| | | | | |
|---|---|---|---|---|
| $x_i$ | 2 | 3 | 4 | 5 |
| $p_i$ | 0.2 | 0.4 | 0.1 | 0.3 |

5. CDF $F$ of random variable $X$ is given by:

| | | | | |
|---|---|---|---|---|
| $x$ | $(-\infty; -2)$ | $[-2; 3)$ | $[3; 5)$ | $[5; \infty)$ |
| $F(x)$ | 0 | 0.2 | 0.7 | 1 |

6. Let $X$ be a discrete random variable. Check that $\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$.
7. Let $X$ be a continuous random variable. Prove that $\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$.
8. **2p.** Check that
   (a) $B(p, q+1) = B(p, q)\,\dfrac{q}{p+q}$,
   (b) $B(p, q) = B(p, q+1) + B(p+1, q)$.
9. **2p.** Prove that $\Gamma(p)\,\Gamma(q) = \Gamma(p+q)\,B(p, q)$, where $p, q \in \mathbb{R}$ and $> 0$ (so all the involved integrals exist).

Def. 1. Non-empty set $\Omega$ is called a **sample space**.

Def. 2. Set of subsets $\Sigma \subset 2^{\Omega}$ is called **$\sigma$-field** on sample space iff
1. $\overline{\overline{\Sigma}} > 0$.
2. $A \in \Sigma \;\Rightarrow A^C \in \Sigma$.
3. $A_1, A_2, \ldots \in \Sigma \;\Rightarrow \bigcup_{k \in \mathbb{N}} A_k \in \Sigma$.

Def. 3. Function (from events to interval $[0, 1]$) $P : \Sigma \to [0, 1]$ is called **probability** iff
1. $P(\Omega) = 1$.
2. $A_i \cap A_j = \emptyset,\ i \neq j \;\Rightarrow P\left(\bigcup_{k \in \mathbb{N}} A_k\right) = \sum_{k \in \mathbb{N}} P(A_k)$.

Def. 4. A triple $(\Omega, \Sigma, P)$ is called a **probability space**.

Def. 5. Let $(\Omega, \Sigma, P)$ be a probability space. Function $X : \Omega \to \mathbb{R}$ is called **random variable** iff

$$\forall\, a \in \mathbb{R} \quad X^{-1}((-\infty, a]) \in \Sigma.$$

Def. 6. **Beta function** is the value of the integral

$$B(p, q) = \int_0^1 t^{p-1}\,(1-t)^{q-1}\,dt, \quad p > 0,\ q > 0.$$

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