

Probability & Statistics

Problem set №7. Week starting April 8th

1. Independent random variables X, Y have distribution $U[0, 1]$. Let x, y are values of r.vs. X, Y . The interval $[0, 1]$ is therefore divided into three parts (it is possible that the length of one part equals 0). What is the probability that a triangle can be constructed from these three parts?

[Problems 7–9] Let (X_1, X_2) be two-dimensional random variable with density function $f(x_1, x_2) = \frac{1}{\pi}$, when $0 < x_1^2 + x_2^2 < 1$.

2. Find marginal densities of variable (1-dimensional) X_1, X_2 .
3. Prove that correlation coefficient of variables X_1, X_2 equals 0. Check that variables are dependent (not independent).
4. Let $X_1 = Y_1 \cos Y_2$, $X_2 = Y_1 \sin Y_2$ ($0 < Y_1 < 1$, $0 \leq Y_2 \leq 2\pi$). Find density $g(y_1, y_2)$ of variable (Y_1, Y_2) . Check if variables Y_1, Y_2 are independent.
5. Let n -dimensional random variable $\mathbf{X} = (X_1, \dots, X_n)^T$ be given. Define $\mathbf{Y} = (Y_1, \dots, Y_n)^T$ as follow:

$$Y_1 = \bar{\mathbf{X}}, \quad Y_k = X_k - \bar{\mathbf{X}}, \quad \text{when } k = 2, \dots, n.$$

Find form (entries) of Jacobian determinant

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}.$$

6. Given are random variables X_1, \dots, X_n . Prove that

$$\sum_{k=1}^n (X_k - \mu)^2 = \sum_{k=1}^n (X_k - \bar{\mathbf{X}})^2 + n (\bar{\mathbf{X}} - \mu)^2. \quad (1)$$

[Problems 7–8] Assume that independent random variables X_k have the distribution $N(\mu, \sigma^2)$.

7. Find (with proof) distribution of $M = \frac{n}{\sigma^2} \cdot (\bar{\mathbf{X}} - \mu)^2$
8. Let r.vs. $\bar{\mathbf{X}} = \frac{1}{n} \sum_{k=1}^n X_k$ and $S^2 = \frac{1}{n} \sum_{k=1}^n (X_k - \bar{\mathbf{X}})^2$ be independent. Using equation (1) prove that $\frac{nS^2}{\sigma^2} \equiv \text{Gamma}\left(\frac{1}{2}, \frac{n-1}{2}\right) \sim \chi^2(n-1)$

[Problems 8–9] Sides' length of rectangular are independent r.vs. X_1 i X_2 with distribution $U[1, 2]$. $Y_1 = 2X_1 + 2X_2$ is a perimeter of the rectangular and $Y_2 = X_1X_2$ denotes its area.

9. Find expected values and variances of Y_1, Y_2 . (ANSWER: $Y_1 - 6, \frac{2}{3}, Y_2 - \frac{9}{4}, \frac{55}{144}$).
10. Find the correlation coefficient ρ of variables Y_1, Y_2 . (ANSWER: $3\sqrt{330}/55$).

[Ex. 11–12] Random variable X is of normal distribution with parameters given below:

$$N \sim \left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 38 & -5 \\ -5 & 4 \end{bmatrix} \right).$$

11. Let $Y_1 = 3X_1 + X_2, Y_2 = -4X_1 + 2X_2$. Find distribution of Y .
12. Let $Y_1 = 2X_1 - 3X_2, Y_2 = 4X_1 + 2X_2$. Find value of the correlation coefficient ρ_{y_1, y_2} .

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