UNDERSTANDING PROGRAM EFFICIENCY

WANT TO UNDERSTAND EFFICIENCY OF PROGRAMS

- computers are fast and getting faster so maybe efficient

  but data sets can be very large

  thus, simple solutions may simply not scale with size in acceptable manner

- so how could we decide which option for program is most efficient?

- separate **time and space efficiency** of a program
- tradeoff between them will focus on time efficiency

WANT TO UNDERSTAND EFFICIENCY OF PROGRAMS

Challenges in understanding efficiency of solution to a computational problem:

- a program can be **implemented in many different ways**
- you can solve a problem using only a handful of different **algorithms**
- would like to separate choices of implementation from choices of more abstract algorithm

HOW TO EVALUATE EFFICIENCY OF PROGRAMS

- measure with a **timer**
- **count** the operations
- abstract notion of **order of growth**

will argue that this is the most appropriate way of assessing the impact of choices of algorithm in solving a problem; and in measuring the inherent difficulty in solving a problem

TIMING A PROGRAM

- use time module
- recall that importing means to bring in that class into your own file
- start clock
- call function
- stop clock

```
import time

def c_to_f(c):
    return c*9/5 + 32
```

Celsius to Fahrenheit

```
t0 = time.clock()
c_to_f(100000)
t1 = time.clock() - t0
Print("t =", t, ":", t1, "s,")
```

TIMING PROGRAMS IS INCONSISTENT

- GOAL: to evaluate different algorithms
- running time **varies between algorithms** ✓
- running time **varies between implementations** ✗
- running time **varies between computers** ✗
- running time is **not predictable** based on small inputs ✗

- time varies for different inputs but cannot really express a relationship between inputs and time ✗

COUNTING OPERATIONS

- assume these steps take **constant time**:
  - mathematical operations
  - comparisons
  - assignments
  - accessing objects in memory
- then count the number of operations executed as function of size of input

```
def c_to_f(c):
    return c*9.0/5 + 32
```

3 ops

```
def mysum(x):
    total = 0
    for i in range(x+1):
        total += i
    return total
```

1 op

loop x times

1 op

2 ops

mysum → 1+3x ops

as I vary the size of x, it tells me how this is going to scale

COUNTING OPERATIONS IS BETTER, BUT STILL…

- GOAL: to evaluate different algorithms
- count **depends on algorithm** ✓
- count **depends on implementations** ✗
- count **independent of computers** ✓
- no real definition of **which operations** to count ✗

- count varies for different inputs and can come up with a relationship between inputs and the count ✓

STILL NEED A BETTER WAY

- timing and counting **evaluate implementations**
- timing **evaluates machines**

- want to **evaluate algorithm**
- want to **evaluate scalability**
- want to **evaluate in terms of input size**

NEED TO CHOOSE WHICH INPUT TO USE TO EVALUATE A FUNCTION

- want to express **efficiency in terms of input**, so need to decide what your input is

what is the input that matters?
And how am I going to measure the size of that as I talk about the efficiency of the algorithm?

- could be an **integer**
  -- `mysum(x)`
- could be **length of list**
  -- `list_sum(L)`
- **you decide** when multiple parameters to a function
  -- `search_for_elmt(L, e)`

For example, if I'm searching to see if a particular element's in a list, probably I want to use the length of the listas the size of the problem.
And not the size of the element, since I'm simply looking to see if it's present

DIFFERENT INPUTS CHANGE HOW THE PROGRAM RUNS

- a function that searches for an element in a list

```
def search_for_elmt(L, e):
    for i in L:
        if i == e:
            return True
    return False
```

- when `e` is **first element** in the list → BEST CASE
- when `e` is **not in list** → WORST CASE
- when **look through about half** of the elements in list → AVERAGE CASE
- want to measure this behavior in a general way

BEST, AVERAGE, WORST CASES

- suppose you are given a list `L` of some length `len(L)`
- **best case**: minimum running time over all possible inputs of a given size, `len(L)`
  - constant for `search_for_elmt`
  - first element in any list

No matter how long the list is, I find it in the first element

- **average case**: average running time over all possible inputs of a given size, `len(L)`
  - practical measure

generally will focus on this case

- **worst case**: maximum running time over all possible inputs of a given size, `len(L)`
  - linear in length of list for `search_for_elmt`
  - must search entire list and not find it

As that list grows, the amount of time is going to grow equally, or at the same ratio

(Complexity of algorithms)

ORDERS OF GROWTH

Goals:

- want to evaluate programs efficiency when **input is very big**

How does it grow as we scale the size of the input?

- want to express the **growth of program's run time** as input size grows
- want to put an **upper bound** on growth

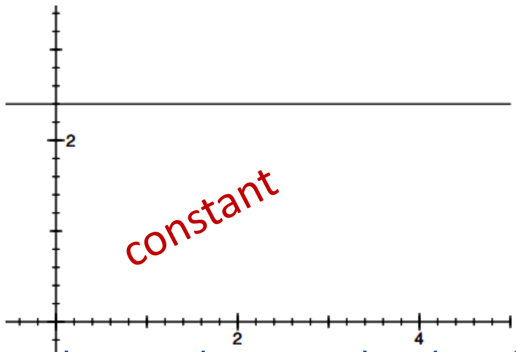
it will grow no more than this quickly as I deal with that

- do not need to be precise: **"order of" not "exact"** growth
- we will look at **largest factors** in run time (which section of the program will take the longest to run?)

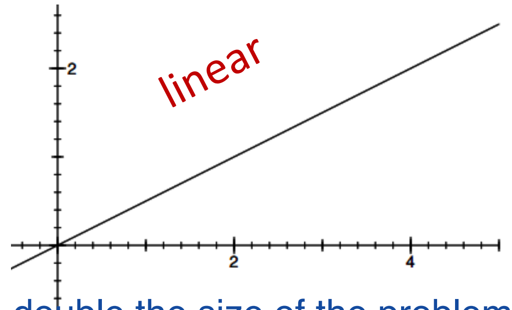
And what's the worst case behavior for that?
And how do I put a bound on that as I describe the complexity of that particular program?

TYPES OF ORDERS OF GROWTH

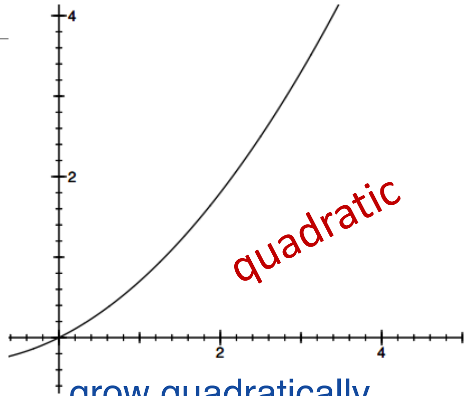
y = size of problem
x = running time



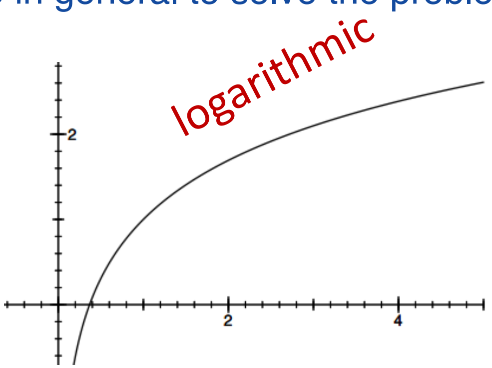
no matter how we increase the size of the input, it takes the same amount of time in general to solve the problem



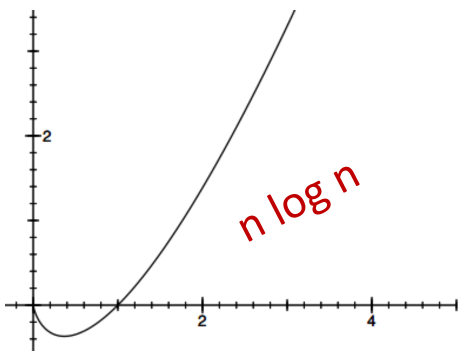
I double the size of the problem, I roughly double the amount of time it takes.



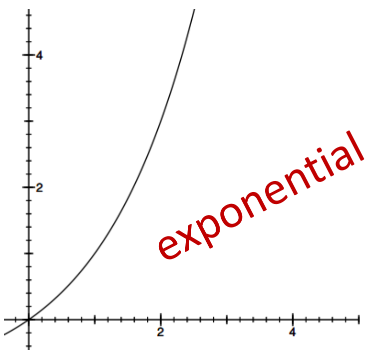
grow quadratically, or with the square, of the time



grow logarithmically, that is, with the log of the size of the problem



not as bad as quadratic, but a little bit more than linear



MEASURING ORDER OF GROWTH: BIG OH NOTATION

- Big Oh notation measures an **upper bound on the asymptotic growth**, often called order of growth

symptotic means as the problem input size gets really big what is the behavior?

- **Big Oh or O()** is used to describe worst case
  - worst case occurs often and is the bottleneck when a program runs
  - express rate of growth of program relative to the input size
  - evaluate algorithm not machine or implementation

EXACT STEPS vs O()

```
def fact_iter(n):
    """assumes n an int >= 0"""
    answer = 1
    while n > 1:
        answer *= n
        n -= 1
    return answer
```

temp = n-1
n = temp

- computes factorial
- number of steps: $1 + 5n + 1$
  * 5*(n-1) times, since n > 1
- worst case asymptotic complexity: $O(n)$
  linear / order n algorithm
  • ignore additive constants
  • ignore multiplicative constants
  as n gets very large they do not matter

SIMPLIFICATION EXAMPLES

- drop constants and multiplicative factors
- focus on **dominant terms**

asymptotically as n gets really big, what's the dominant term ?

$O(n^2)$: $n^2 + 2n + 2$ n*2 grows more rapidly than 2n

$O(n^2)$: $n^2 + 100000n + 3^{1000}$

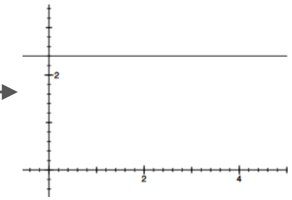
$O(n)$: $\log(n) + n + 4$ because n grows more rapidly than log n

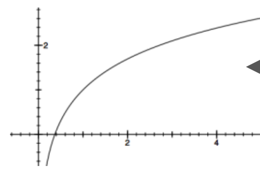
$O(n \log n)$: $0.0001*n*\log(n) + 300n$

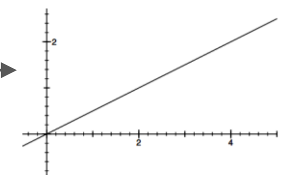
$O(3^n)$: $2n^{30} + 3^n$ as n gets really big, 3*n grows much more rapidly than n*30

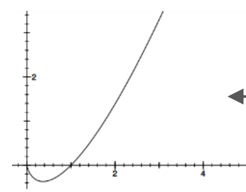
COMPLEXITY CLASSES ORDERED LOW TO HIGH

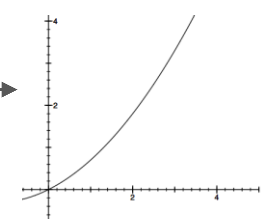
Ideally, I'd like an algorithm that is as close to this point in the hierarchy as possible, because the higher up I am in this hierarchy, the more efficient the algorithm is

O(1) : constant → 

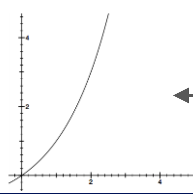
O(log n) :  ← logarithmic

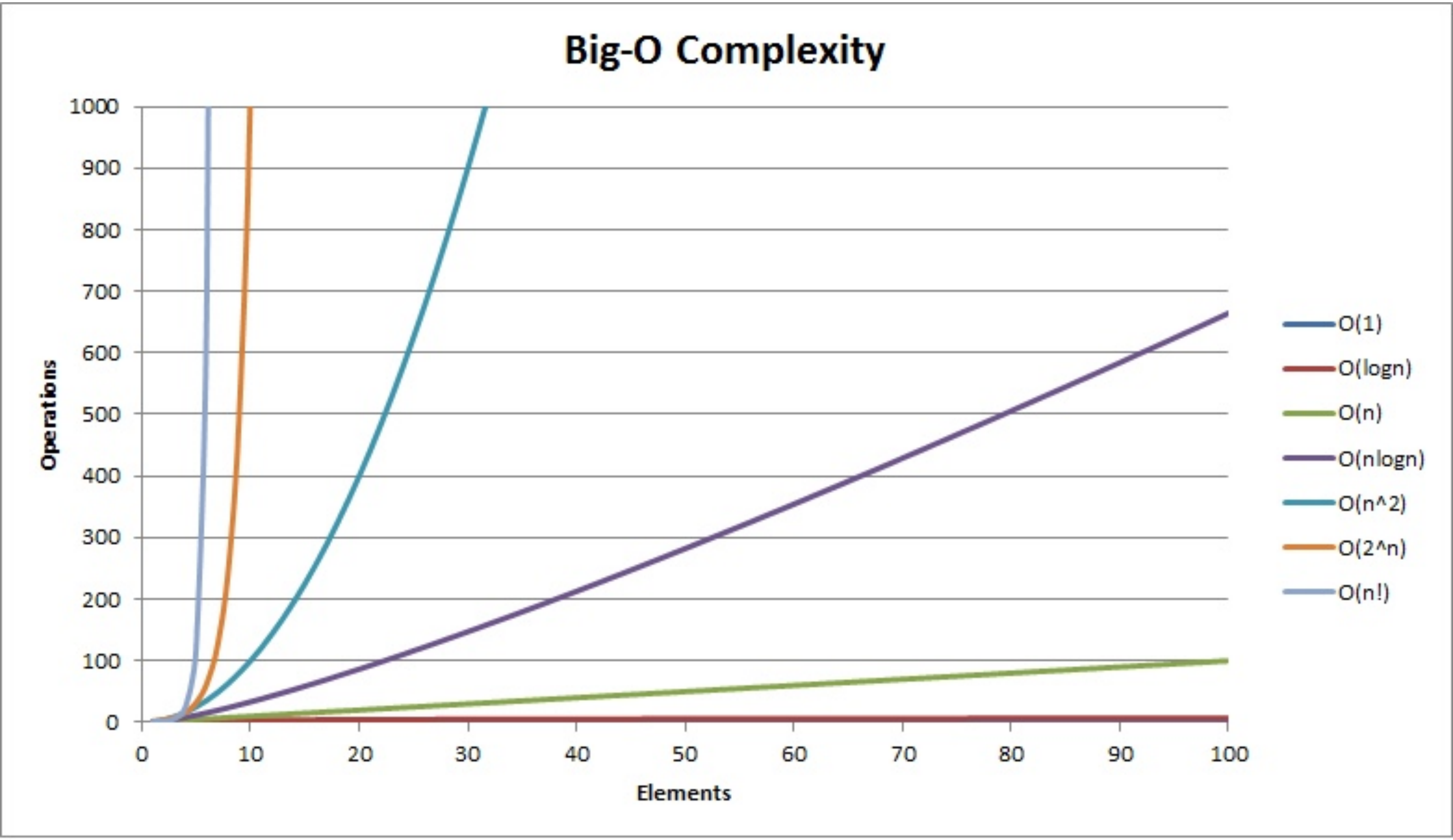
O(n) : linear → 

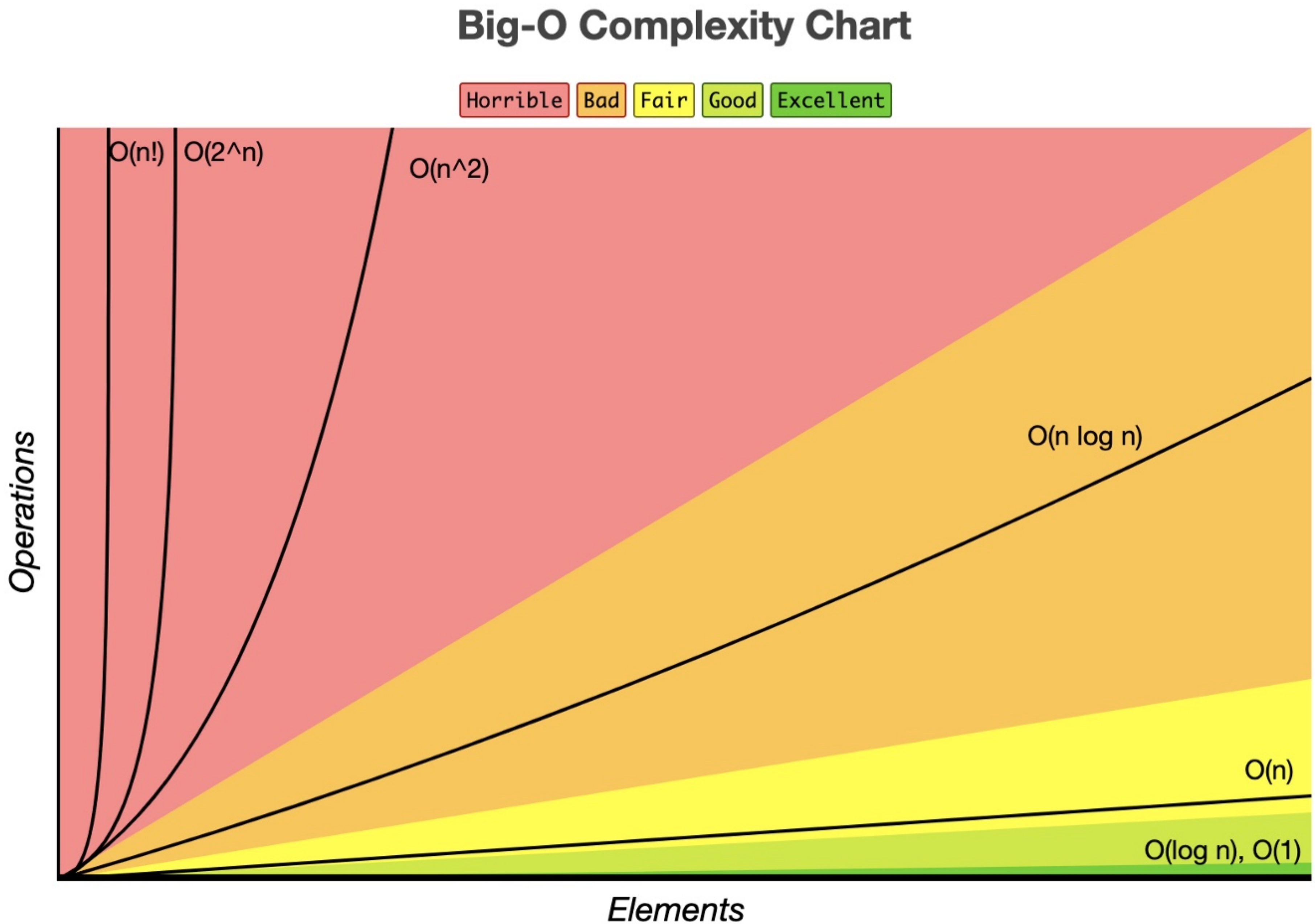
O(n log n):  ← loglinear

O(n^c) : polynomial → 

c is a constant

O(c^n) :  ← exponential


Big-O Complexity
Operations
1000
900
800
700
600
500
400
300
200
100
0
0
10
20
30
40
50
60
70
80
90
100
Elements
O(1)
O(logn)
O(n)
O(nlogn)
O(n^2)
O(2^n)
O(n!)


Big-O Complexity Chart
Horrible
Bad
Fair
Good
Excellent
O(n!)
O(2^n)
O(n^2)
O(n log n)
O(n)
O(log n), O(1)
Operations
Elements

ANALYZING PROGRAMS AND THEIR COMPLEXITY

- **combine** complexity classes
  - analyze statements inside functions
  - apply some rules, focus on dominant term

Law of Addition for O():

- used with **sequential** statements
- O(f(n)) + O(g(n)) is O(f(n) + g(n))
- for example,

```
for i in range(n):
    print('a')
for j in range(n*n):
    print('b')
```

This is going to take much longer as n gets really big

is $O(n) + O(n*n) = O(n+n^2) = O(n^2)$ because of dominant term

ANALYZING PROGRAMS AND THEIR COMPLEXITY

- **combine** complexity classes
  - analyze statements inside functions
  - apply some rules, focus on dominant term

Law of Multiplication for O():

- used with **nested** statements/loops
- O(f(n)) * O(g(n)) is O(f(n) * g(n))
- for example,

```
for i in range(n):
    for j in range(n):
        print('a')
```

is $O(n)*O(n) = O(n*n) = O(n^2)$ because the outer loop goes n times and the inner loop goes n times for every outer loop iter.

COMPLEXITY CLASSES

- *O(1)* denotes constant running time
- *O(log n)* denotes logarithmic running time
- *O(n)* denotes linear running time
- *O(n log n)* denotes log-linear running time
- $O(n^c)$ denotes polynomial running time (c is a constant)
- $O(c^n)$ denotes exponential running time (c is a constant being raised to a power based on size of input)

CONSTANT COMPLEXITY

- complexity independent of inputs
- very few interesting algorithms in this class, but can often have pieces that fit this class
- can have loops or recursive calls, but number of iterations or calls independent of size of input

LOGARITHMIC COMPLEXITY

- complexity grows as log of size of one of its inputs

I'm reducing the size of the problem by a constant factor at each step

- example:
  - bisection search
  - binary search of a list

divides the space of the search in half at each step

LOGARITHMIC COMPLEXITY

```
def intToStr(i):
    digits = '0123456789'
    if i == 0:
        return '0'
    result = ''
    while i > 0:
        result = digits[i%10] + result
        i = i//10
    return result
```

LOGARITHMIC COMPLEXITY

```
def intToStr(i):
    digits = '0123456789'
    if i == 0:
        return '0'
    res = ''
    while i > 0:
        res = digits[i%10] + res
        i = i//10
    return result
```

*res

constant, only do it once

constant, only do it once

only have to look at loop as no function calls

within while loop, constant number of steps

how many times through loop?

- how many times can one divide i by 10?
- *O(log(i))*

each time through the loop I reduce i by a factor of 10

It is linear in the length or the size of n.
But it is a log in the number of digits in n (decided to measure this in terms of the size of the input)

LINEAR COMPLEXITY

- searching a list in sequence to see if an element is present
- add characters of a string, assumed to be composed of decimal digits

```
def addDigits(s):
    val = 0
    for c in s:
        val += int(c)
    return val
```

take constant time

conversion to an int,
and add, and an assignment.
So it's 3 steps inside loop

take constant time

take constant time

All I need to worry about is
how many times
do I go through this loop.
And the answer is however
many characters
there are in s, the string

- *O(len(s))*

LINEAR COMPLEXITY

- complexity can depend on number of recursive calls

```
def fact_iter(n):
    prod = 1
    for i in range(1, n+1):
        prod *= i
    return prod
```

take constant time

question is how many times do I go around the loop?
n, that's the size of my problem

- number of times around loop is n
- number of operations inside loop is a constant
- overall just *O(n)*

loop in it with a constant number of operations or work inside the loop

O() FOR RECURSIVE FACTORIAL

```
def fact_recur(n):
    """ assume n >= 0 """
    if n <= 1:
        return 1
    else:
        return n*fact_recur(n - 1)
```

doing a recursive call once for each increment in the size of the problem. And inside of that operation I'm just doing a constant number of things

- computes factorial recursively
- if you time it, may notice that it runs a bit slower than iterative version due to function calls
- still **O(n)** because the number of function calls is linear in n
- **iterative and recursive factorial** implementations are the **same order of growth**

LOG-LINEAR COMPLEITY

- many practical algorithms are log-linear
- very commonly used log-linear algorithm is merge sort
- will return to this

POLYNOMIAL COMPLEXITY

- most common polynomial algorithms are quadratic, i.e., complexity grows with square of size of input

loop inside of a loop

- commonly occurs when we have nested loops or recursive function calls

in a way where the recursive function call has some costs other than constant (Because as we saw with factorial, you can have recursive function calls and have it still be linear) given two lists is going to try to decide: is the first list a subset of the second list, meaning every element of the first list does it also occur in the second list even though there may be other elements inside the second list

QUADRATIC COMPLEXITY

nested loops.
I've got an outer loop on L1.
I've got an inner loop on L2.

```
def isSubset(L1, L2):
    for e1 in L1:
        matched = False
        for e2 in L2:
            if e1 == e2:
                matched = True
                break
        if not matched:
            return False
    return True
```

even though I could break out of either of those lists, either here or here, we know that it's the worst case behavior that we're interested in. And that's going to happen when in fact L1 is in fact a subset of L2

QUADRATIC COMPLEXITY

```
def isSubset(L1, L2):
    for e1 in L1:
        matched = False
        for e2 in L2:
            if e1 == e2:
                matched = True
                break
        if not matched:
            return False
    return True
```

fact that we might break out of the loop earlier doesn't change the order of growth of the algorithm because we always look at worst case

outer loop executed len(L1) times

each iteration will execute inner loop up to len(L2) times

*O(len(L1)*len(L2))*

worst case when L1 and L2 same length, none of elements of L1 in L2

O(len(L1)²)

QUADRATIC COMPLEXITY

find intersection of two lists, return a list with each element appearing only once

```
def intersect(L1, L2):
    tmp = []
    for e1 in L1:
        for e2 in L2:
            if e1 == e2:
                tmp.append(e1)
    res = []
    for e in tmp:
        if not(e in res):
            res.append(e)
    return res
```

A loop is typically linear. A loop over a loop is typically quadratic

nested pair of loops
-> quadratic

single loop
-> linear

QUADRATIC COMPLEXITY

```
def intersect(L1, L2):
    tmp = []
    for e1 in L1:
        for e2 in L2:
            if e1 == e2:
                tmp.append(e1)
    res = []
    for e in tmp:
        if not(e in res):
            res.append(e)
    return res
```

first nested loop takes *len(L1)*len(L2)* steps

second loop takes at most *len(L1)* steps

ignore that second term.

latter term overwhelmed by former term

*O(len(L1)*len(L2))*

O() FOR NESTED LOOPS

```
def g(n):
    """ assume n >= 0 """
    x = 0
    for i in range(n):
        for j in range(n):
            x += 1
    return x
```

- computes n^2 very inefficiently
- when dealing with nested loops, **look at the ranges**

n gets generated incrementally as I need it

- nested loops, **each iterating n times**
- **O(n^2)**

EXPONENTIAL COMPLEXITY

- recursive functions where more than one recursive call for each size of problem
  - Towers of Hanoi
- many important problems are inherently exponential
  - unfortunate, as cost can be high
  - will lead us to consider approximate solutions more quickly

most expensive algorithms

generate a list of all the subsets of that list [1, 2, 3, 4]. So what does that mean?
The empty list [] will be a subset. Every list of [1], of [2], or [3], of [4] would be a subset.
Lists of [1, 2], [1, 3], [1, 4], [2, 3], [2, 4], ... would be a subset.

EXPONENTIAL COMPLEXITY

```
def genSubsets(L):
    res = []
    if len(L) == 0:
        return [[]] #list of empty list
    smaller = genSubsets(L[:-1]) # all subsets without
 last element
    extra = L[-1:] # create a list of just last element
    new = []
    for small in smaller:
        new.append(small+extra)  # for all smaller
 solutions, add one with last element
    return smaller+new  # combine those with last
 element and those without
```

EXPONENTIAL COMPLEXITY

I need to know where the end of the list is without having to walk down the list

```
def genSubsets(L):
    res = []
    if len(L) == 0:
        return [[]]
    smaller = genSubsets(L[:-1])
    extra = L[-1:]
    new = []
    for small in smaller:
        new.append(small+extra)
    return smaller+new
```

assuming append is constant time

time includes time to solve smaller problem, plus time needed to make a copy of all elements in smaller problem

I've got K element-- say they're integers-- for each integer I can decide is this one in or not? So for every integer I got a choice of two. I've got k of them. So there are 2 to the k possible cases that could be generated as I solve the smaller problem.

EXPONENTIAL COMPLEXITY

```
def genSubsets(L):
    res = []
    if len(L) == 0:
        return [[]]
    smaller = genSubsets(L[:-1])
    extra = L[-1:]
    new = []
    for small in smaller:
        new.append(small+extra)
    return smaller+new
```

solve a problem of size 2 to the n minus 1 (all the subsets with n minus 1 elements)
I need to solve the problem with 2 to the n minus 2, a problem with 2 to the n minus c, all the way down to 2 to the 0 or 1 case

- but important to think about size of smaller
- know that for a set of size k there are 2^k cases

breaking this down into sub problems that I have to call multiple times

- so to solve need $2^{n-1} + 2^{n-2}$ +....+2^^{0} steps
- math tells us this is $O(2^n)$

The base is 2.
But it grows as 2 to the n.

COMPLEXITY CLASSES

- *O(1)* denotes constant running time no matter what the size of the problem is
- *O(log n)* denotes logarithmic running time I'm breaking the problem in half or in portions
- *O(n)* denotes linear running time loop / sth that recursively calls itself some number of times, that number based on the size of the problem
- *O(n log n)* denotes log-linear running time
- $O(n^c)$ denotes polynomial running time (c is a constant) nested loops, recursive calls
- $O(c^n)$ denotes exponential running time (c is a constant being raised to a power based on size of input) recursive calls, but we're breaking the problem down into multiple calls each time around

EXAMPLES OF ANALYZING COMPLEXITY

TRICKY COMPLEXITY

```
def h(n):
    """ assume n an int >= 0 """
    answer = 0
    s = str(n)
    for c in s:
        answer += int(c)
    return answer
```

linear O(len(s)) but what in terms of input n?

important to think about what am I using to measure the size of the problem and then how do I characterize the complexity of the algorithm in terms of that size.

- adds digits of a number together
- tricky part
  - convert integer to string
  - iterate over **length of string**, not magnitude of input n
  - think of it like dividing n by 10 each iteration
- **O(log n)** base here doesn't matter

COMPLEXITY OF ITERATIVE FIBONACCI

```
def fib_iter(n):
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        fib_i = 0
        fib_ii = 1
        for i in range(n-1):
            tmp = fib_i
            fib_i = fib_ii
            fib_ii = tmp + fib_ii
        return fib_ii
```

constant O(1)

constant O(1)

linear O(n)

constant O(1)

- Best case:

  O(1)

- Worst case:

  O(1) + O(n) + O(1) ➔ **O(n)**

want to look at is inside of the loop: an assignment, an assignment, an addition, an assignment, so four operations.
And how many times do I do that? However many times go I go through the loop, which is order n because of range

COMPLEXITY OF RECURSIVE FIBONACCI

part of what you want to recognize is, can I find a solution that's lower complexity and still gives me the answer I want? recursively this is exponential. Iteratively it was linear. So same problem, different algorithms, different complexity.

```
def fib_recur(n):
    """ assumes n an int >= 0 """
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return fib_recur(n-1) + fib_recur(n-2)
```

two base cases

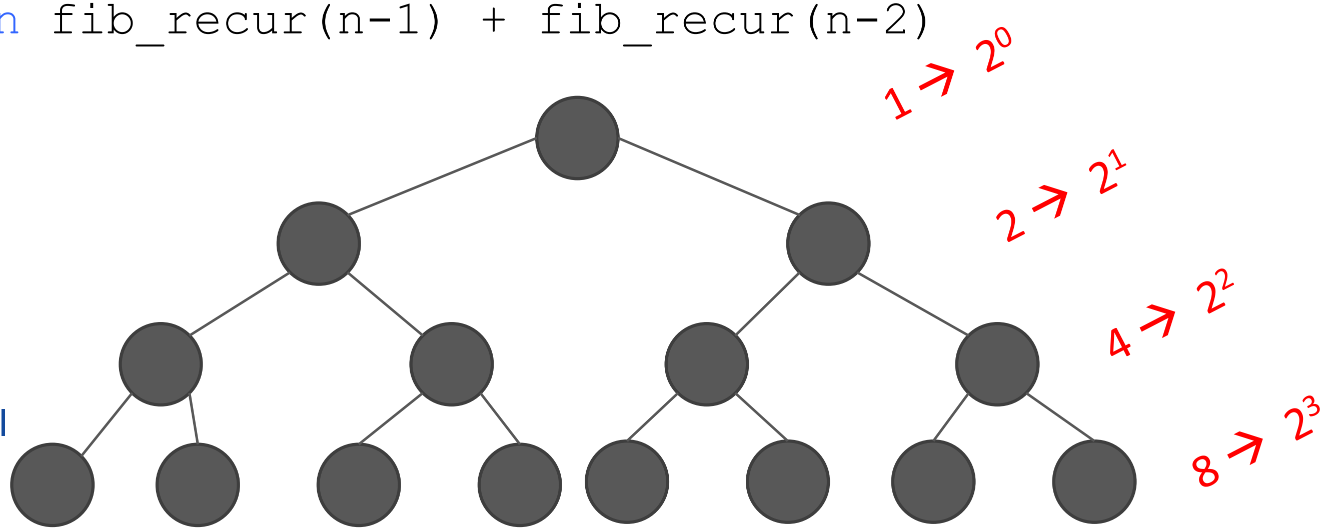
recursive call is actually two calls

if I want to solve Fib of n, I've got to do that by solving two versions of Fib of n minus 1, which is going to have to be solved by four versions of a Fib of n minus 2, which has got to be solved by 8 versions of Fib of n minus 3

- Worst case:

  $O(2^n)$

sequence of 2 to the 0 plus 2 to the 1 plus 2 squared all the way up to 2 to the n. And that overall sums up to being an exponential or 2 to the n problem



WHEN THE INPUT IS A LIST...

```
def sum_list(L):
    total = 0
    for e in L:
        total = total + e
    return total
```

- O(n) where n is the length of the list
- O(len(L))
- must **define what size of input means**
  - previously it was the magnitude of a number
  - here, it is the length of list

BIG OH SUMMARY

- compare **efficiency of algorithms**
  - notation that describes growth asymptotically as the algorithm takes on bigger and bigger-sized problems.
  - **lower order of growth** is better
  - independent of machine or specific implementation

- use Big Oh
  - describe order of growth
  - **asymptotic notation**
  - **upper bound**
  - **worst case** analysis

COMPLEXITY OF COMMON PYTHON FUNCTIONS

I get more power with a dictionary. But it comes with a cost in terms of the complexity of the algorithm

- **Lists**: `n` is `len(L)`
  - index O(1)
  - store O(1)
  - length O(1)
  - append O(1)
  - == O(n)
  - remove O(n)
  - copy O(n)
  - reverse O(n)
  - iteration O(n)
  - in list O(n)

implementation in Python knows exactly how to get to that spot and do something

have to walk down the list/ look at every element

- **Dictionaries**: `n` is `len(d)`
- **worst case**
  - index O(n)
  - store O(n)
  - length O(n)
  - delete O(n)
  - iteration O(n)

could be stored in any order. It gives me more power, but has a cost, which is that it is linear compared to a list, which was constant

- **average case**
  - index O(1)
  - store O(1)
  - delete O(1)
  - iteration O(n)