Stochastic SIR model

Karolina Schmidt

Introduction to complex networks
Wrocław University of Science and Technology

30 juin 2021

Politechnika
Wrocławska

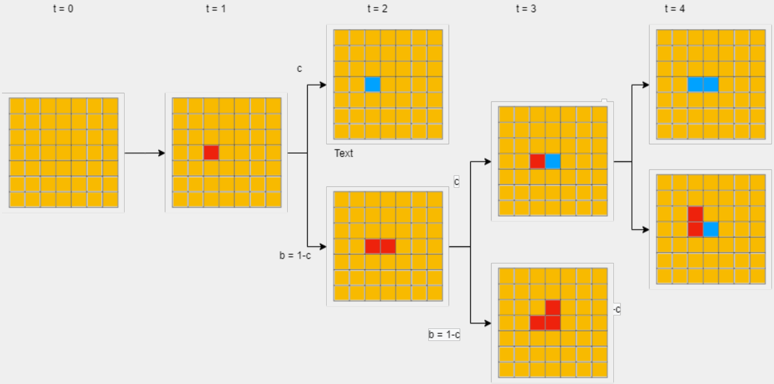
A new scale-invariant ratio and finite-size scaling for the stochastic susceptible–infected–recovered model

David R de Souza[1], Tânia Tomé[1] and Robert M Ziff[2]

Abstract. The critical behavior of the stochastic susceptible–infected–recovered model on a square lattice is obtained by numerical simulations and finite-size scaling. The order parameter as well as the distribution in the number of recovered individuals is determined as a function of the infection rate for several values of the system size. The analysis around criticality is obtained by exploring the close relationship between the present model and standard percolation theory. The quantity UP, equal to the ratio U between the second moment and the squared first moment of the size distribution multiplied by the order parameter P, is shown to have, for a square system, a universal value 1.0167(1) that is the same for site and bond percolation, confirming further that the SIR model is also in the percolation class.

Politechnika Wrocławska

The SIR model can be described by just one parameter, either the reduced infection rate b or the reduced immunization rate $c = 1 - b$.


t = 0
t = 1
t = 2
t = 3
t = 4
c
Text
c
b = 1-c
b = 1-c
-c

Stochastic SIR model

Stochastic SIR model - percolation

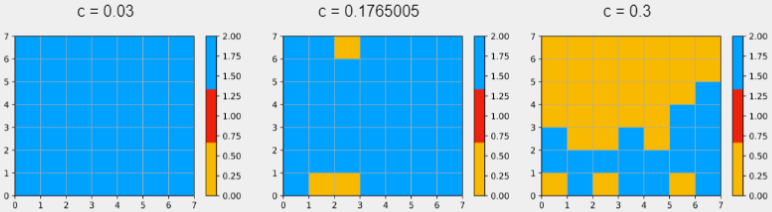
- Simulations were performed on a square lattice of $N = L^2$ sites and periodic boundary conditions. .
- We begin with an infected individual placed at the center of the lattice full of susceptible individuals.
- To speed up the simulations we keep a list of the I sites.
- At each step of the simulation we choose randomly an I site among the list of the N_I I sites with probability $1/N_I$.
  - With probability c the chosen I site becomes an R site.
  - Otherwise (with the complementary probability b = 1 c), we choose one of its four nearest-neighbor sites
  - If the nearest-neighbor site is an S site, it becomes an I site

Politechnika Wrocławska

Stationary states

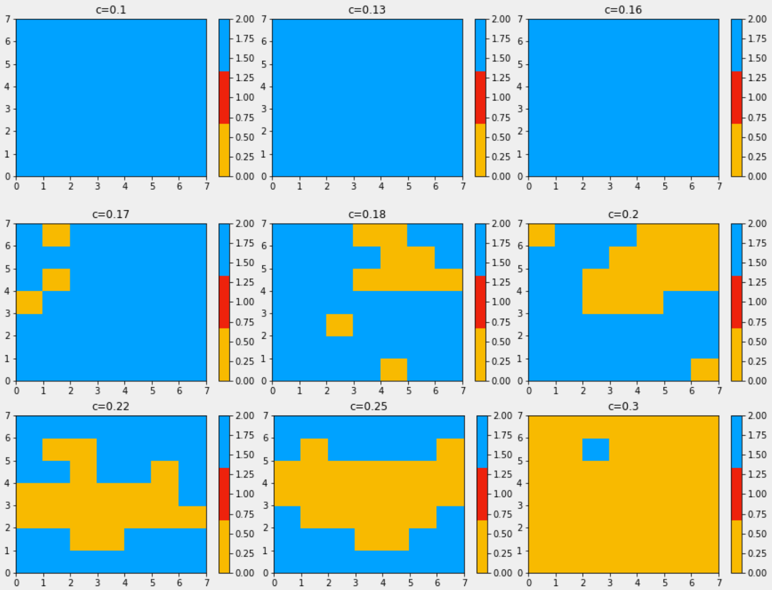
Stationary states are characterized by the absence of infected sites.The phase transition occurs at a critical value $c = c_c$, which has been estimated as $c_c = 0.1765005(10)$.

- if $c < c_c$ then all population becomes infectious and recovered
- if $c > c_c$ then not all population becomes infectious and recovered



Politechnika Wrocławska

Critical value



Quantities

- N - total number of sites item N_S - number of susceptible
- N_I - number of infectious
- N_R - number of recovered

$$N = N_S + N_I + N_R \tag{1}$$

The mean number of recovered individuals :

$$S = \langle N_R \rangle \tag{2}$$

The mean value of the square of the number of recovered individuals :

$$M = \langle N_R^2 \rangle \tag{3}$$

Ratio U :

$$U = \frac{M}{S^2}$$

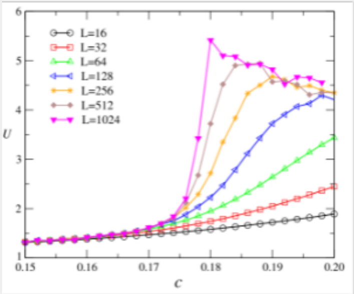
Order parameter P :

$$P = \frac{S}{N} \tag{4}$$

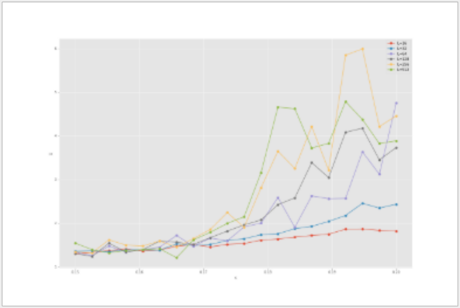
Politechnika Wrocławska

ratio U

mc_steps ~ 1e8



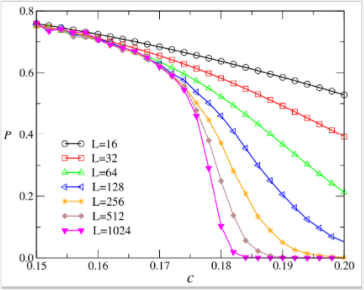
mc_steps ~ 100



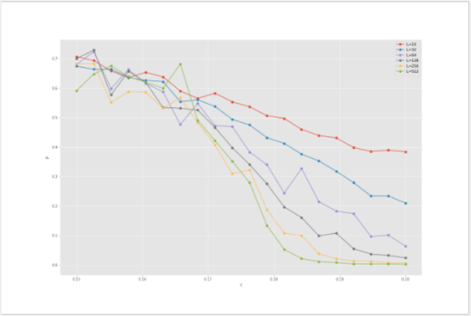
Politechnika Wrocławska

parametr P

mc_steps ~ 1e8



mc_steps ~ 100



In standard percolation theory the probability that an occupied site belongs to a cluster of size s is

$$P_s = sn_s \tag{5}$$

where n_s is the mean number of clusters of size s per lattice site.
The mean of epidemic size S :

$$S = \sum_s sP_s = \sum_s s^2 n_s \tag{6}$$

The mean-square epidemic cluster size M :

$$S = \sum_s s^2 P_s = \sum_s s^3 n_s \tag{7}$$

Politechnika Wrocławska

Relation to percolation

For critical point $p = p_c$ quantities behave as

$$P \sim \epsilon^{\beta}, \epsilon \geq 0 \tag{8}$$

$$S \sim |\epsilon|^{-\gamma} \tag{9}$$

$$M \sim |\epsilon|^{-\beta-2\gamma} \tag{10}$$

where $\epsilon = p - p_c$, and β and γ are critical exponents associated with the order parameter and with the mean cluster size, respectively.The quantity p is the parameter associated with the percolation problem and p_c is its critical value. In site (bond) percolation, p is the probability that a site (bond) is occupied.
Critical behavior is given by equations (8)-(10) with

$$\epsilon = c - c_c \tag{11}$$

Then ratio $U = M/S^2$:

$$U \sim |\epsilon|^{-\beta} \tag{12}$$

Politechnika Wrocławska

Finite-size scaling analysis

We assume that the phase transition in the SIR model is characterized by a correlation length ξ which diverges in the limit where the system is infinite as

$$\xi = |\epsilon|^{-\nu_\perp} \tag{13}$$

Using the finite-size scaling, we may write the following relations for the quantities P, S, M and U

$$P = L^{-\beta/\nu_\perp} \hat{P}(L^{1/\nu_\perp} \epsilon) \tag{14}$$

$$S = L^{\gamma/\nu_\perp} \hat{S}(L^{1/\nu_\perp} \epsilon) \tag{15}$$

$$M = L^{\beta+2\gamma/\nu_\perp} \hat{M}(L^{1/\nu_\perp} \epsilon) \tag{16}$$

$$U = L^{\beta//\nu_\perp} \hat{U}(L^{1/\nu_\perp} \epsilon) \tag{17}$$

Politechnika Wrocławska

Finite-size scaling analysis

At the critical point $\epsilon = 0$ we have

$$P \sim L^{-\beta/\nu_\perp} \tag{18}$$

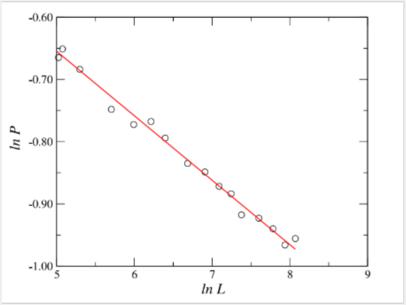
$$S \sim L^{\gamma/\nu_\perp} \tag{19}$$

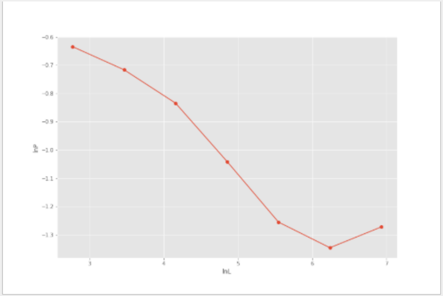
$$U \sim L^{\beta//\nu_\perp} \tag{20}$$

Politechnika Wrocławska

Fitting parameters

parameter P



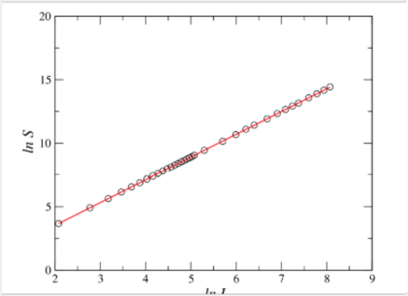


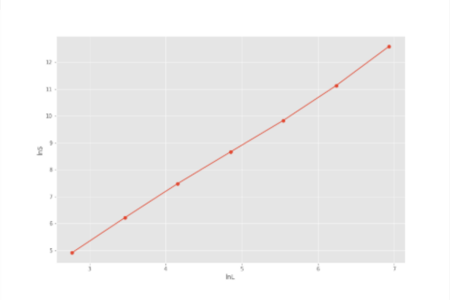
The slope of data from article gives value $\beta/v_{\perp} = 0.1048$. The slope of data from simulation gives value $\beta/v_{\perp} = 0.1150449$.

Politechnika Wrocławska

Fitting parameters

mean value S



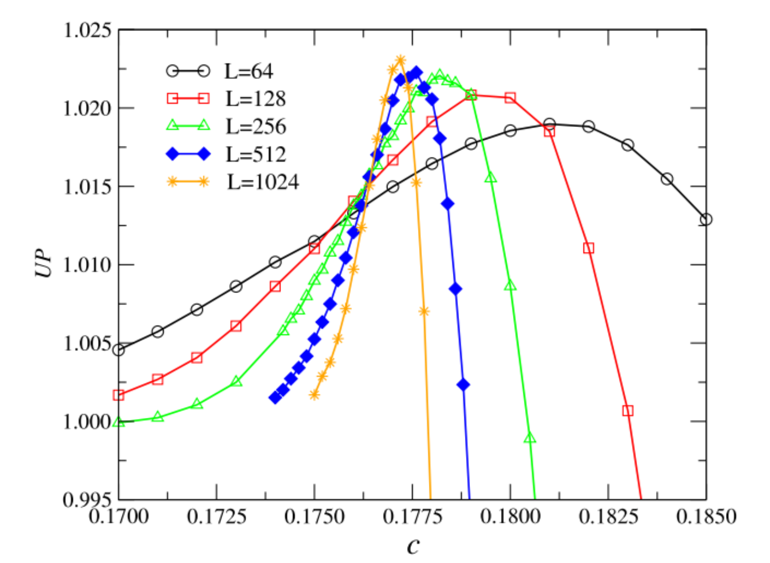


The slope of data from article gives value $\gamma/v_{\perp} = 1.7923$ The slope of data from simulation gives value $\gamma/v_{\perp} = 1.814691$

Politechnika Wrocławska

UP product to estimate critical value

mean value S



Politechnika Wrocławska

Conclusions

- Stochastic SIR mode on finite square lattice implemented
- Determined by Monte Carlo simulation the order parameter P, the mean number of recovered individuals S and the mean squared number of recovered individuals M.
- Determined parameters to finite-size scaling analysis
- Determined UP product to define critical value $c_c = 1.0167$
- Defined critical behavior of the stochastic susceptible–infected– recovered model on a square lattice.

Politechnika Wrocławska