RMS® CCRA® Training Program Study Topic Supplement
Incorporating Secondary Uncertainty into XSAAL Calculations

Excess average annual loss (XSAAL) is the contribution to the total average annual loss of events that exceed a certain threshold. The threshold is usually chosen as a point on the EP curve, such as the 100 or 250-year return period loss.

The calculation of the XSAAL is often done on an expected mode basis, i.e. without consideration of secondary uncertainty. Following is an example of such a calculation.

Suppose we have an event loss table (ELT) with only ten events, as shown below in Table 1.

Table 1

Event	Rate	Loss
1	0.006	97,743
2	0.012	62,767
3	0.023	57,861
4	0.024	49,976
5	0.034	48,167
6	0.048	33,251
7	0.222	18,826
8	0.255	4,357
9	0.301	2,070
10	0.395	1,545

Note that Table 1 is sorted by loss in descending order. Suppose that we want to calculate the XSAAL with a loss threshold of 50,000. Events 1, 2, and 3 exceed this threshold, so the XSAAL is calculated as the sumproduct of the rate and the loss for these three events: (0.006 * 97,743) + (0.012 * 62,767) + (0.023 * 57,861) = 2,670.

This can be expressed symbolically as

equation [1]

$$\sum_{x_i \geq T} x_i r_i$$

where

T = chosen threshold
x_i = mean loss for event i
r_i = rate for event i

Another way to look at this calculation is to add another column to the ELT, as shown below in Table 2.

Table 2

Event	Rate	Loss	Over Threshold
1	0.006	97,743	1
2	0.012	62,767	1
3	0.023	57,861	1
4	0.024	49,976	0
5	0.034	48,167	0
6	0.048	33,251	0
7	0.222	18,826	0
8	0.255	4,357	0
9	0.301	2,070	0
10	0.395	1,545	0

Now the calculation is to take the sumproduct of the three columns (Rate, Loss, and Over Threshold) for all of the events in the ELT shown in Table 2:
(0.006 * 97,743 * 1) + (0.012 * 62,767 * 1) + (0.023 * 57,861 * 1) + (0.024 * 49,976 * 0) + (0.034 * 48,167 * 0) + … + (0.395 * 1,545 * 0) = 2,670.

A limitation of this calculation is that it ignores events with losses near the threshold, even though there is a high probability that the losses would exceed the threshold if the event was to occur. For example, event 4 above has a mean loss of 49,976, which is very close to our threshold of 50,000. If event 4 was to occur, then the chance that the losses exceed 50,000 could be significant.

To incorporate secondary uncertainty into our calculation, we first need to expand the ELT to include the standard deviations and exposure values for the events. This is shown in Table 3.

Table 3

Event	Rate	Loss	Std. Dev.	Exp. Value	μ	CV	α	β
1	0.006	97,743	45,980	828,931	0.118	0.47	3.87	28.9
2	0.012	62,767	23,891	883,720	0.071	0.381	6.34	82.9
3	0.023	57,861	23,405	611,870	0.095	0.405	5.44	52.1
4	0.024	49,976	24,036	949,073	0.053	0.481	4.04	72.7
5	0.034	48,167	4,860	492,570	0.098	0.101	88.5	817
6	0.048	33,251	5,743	407,444	0.082	0.173	30.7	346

In RiskLink, the event standard deviations are split into two pieces: the correlated piece and the independent piece. To get the total event standard deviation, we add the two pieces together. This sum is what is shown in the Std.Dev. column in Table 3.

The columns μ, CV, α, and β are calculated from the Loss, Std.Dev., and Exp.Value columns as follows:

μ = Loss / Exp.Value
CV = Std.Dev / Loss
α = (1-μ) / CV^2 - μ
β = α (1-μ) / μ

The approach we take to include secondary uncertainty into the XSAAL calculation is to replace the Over Threshold column of 1's and 0's from Table 2 by a measure of the propensity for the event losses to exceed the threshold.

A generalization to equation [1] for a continuous case (including secondary uncertainty) is

equation [2]

$$\int_T^\infty x f(x) dx$$

where

x = size of loss
$f(x)$ = probability density function (pdf) of x.

Splitting this up by event, we get

equation [3]

$$\sum_{i=1}^{N} \int_T^\infty x_i f_i(x_i) dx_i$$

where

N = the number of events in the ELT
x_i = loss amount for event i
$f_i(x_i)$ = pdf of the losses for event i.

If we look at a particular event, drop the indexes, and then divide everything by the exposure value for the event we get

equation [4]

$$\int_U^1 yf(y)dy$$

where

U = T / exposure value
y = x / exposure value = damage ratio
$f(y)$ = pdf of the damage ratio

Equation [4] is what we will use for each event to determine its contribution to the XSAAL.

The appendix shows how equation [4] can be expressed as

equation [5]

$$\mu[1 - \mathrm{B}(U \mid \alpha + 1, \beta)]$$

where

μ = mean damage ratio for the event
α = alpha parameter of the distribution for the event
β = beta parameter of the distribution for the event
B(x | α, β) = cumulative distribution function for the beta distribution with parameters α and β

We use [1 - B(U | α + 1, β)] for the Over Threshold column to calculate the XSAAL because when we multiply it by the mean loss for an event it gives us the contribution of that event to the XSAAL. This is shown in Table 4.

Table 4

Event	Rate	Loss	Over Threshold
1	0.006	97,743	0.947263
2	0.012	62,767	0.800927
3	0.023	57,861	0.738927
4	0.024	49,976	0.627855
5	0.034	48,167	0.381152
6	0.048	33,251	0.006977
7	0.222	18,826	0
8	0.255	4,357	0
9	0.301	2,070	0
10	0.395	1,545	0

We now take the sumproduct of the three columns (Rate, Loss, and Over Threshold) for all of the events in the ELT to calculate the XSAAL:
(0.006 * 97,743 * 0.947263) + (0.012 * 62,767 * 0.800927) + … + (0.395 * 1,545 * 0) = 3,531.

Using XSAAL for allocation

A common use for XSAAL is to allocate capital or reinsurance costs to business units or regions. The traditional (expected mode) approach is to fix the set of events that exceed the threshold based on the portfolio, and then calculate the average annual loss by region for only those events. The corresponding distributed mode approach is to fix the Over Threshold column based on the portfolio, then take the sumproduct of this, the event rates, and the mean event losses by region.

Suppose our event losses by region are as shown below in Table 5.

Table 5

Event	A	B	C	D	Total
1	78,118	-	5,140	14,485	97,743
2	-	8,039	29,363	25,365	62,767
3	35,879	6,220	-	15,762	57,861
4	49,976	-	-	-	49,976
5	30,320	15,515	2,332	-	48,167
6	-	-	-	33,251	33,251
7	18,826	-	-	-	18,826
8	-	4,357	-	-	4,357
9	-	-	2,070	-	2,070
10	-	-	-	1,545	1,545

Note that the sum of the losses across regions for each event equals the event loss for the portfolio, as shown in Tables 1 to 4.

To calculate the XSAAL for each region, we insert two columns from Table 4: the Rate and Over Threshold. This is shown in Table 6.

Table 6

Event	Rate	Over Threshold	A	B	C	D	Total
1	0.006	0.947263	78,118	-	5,140	14,485	97,743
2	0.012	0.800927	-	8,039	29,363	25,365	62,767
3	0.023	0.738927	35,879	6,220	-	15,762	57,861
4	0.024	0.627855	49,976	-	-	-	49,976
5	0.034	0.381152	30,320	15,515	2,332	-	48,167
6	0.048	0.006977	-	-	-	33,251	33,251
7	0.222	0	18,826	-	-	-	18,826
8	0.255	0	-	4,357	-	-	4,357
9	0.301	0	-	-	2,070	-	2,070
10	0.395	0	-	-	-	1,545	1,545
XSAAL			**2,200**	**384**	**342**	**605**	**3,531**
Percent of total			**62.30%**	**10.90%**	**9.70%**	**17.10%**	**100.00%**

Note that the sum of the XSAALs across the regions equals the portfolio XSAAL of 3,531.

Appendix - Derivation of equation [5]

Because the mean of the loss distribution for an event is

equation [A-1]

$$\int_0^1 yf(y)dy,$$

equation [4] in the main section of the paper can be expressed as

equation [A-2]

$$\mu - \int_0^U yf(y)dy$$

where μ = mean damage ratio for the event

RiskLink assumes that the damage ratios follow a beta distribution for each event. The cumulative distribution function (cdf) of the beta distribution with parameters α and β can be expressed as

equation [A-3]

$$\mathrm{B}(x\,|\,\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt$$

Because the *f* (*y*) in equation [A-2] is a beta distribution, equation [A-2] can be expressed as

equation [A-4]

$$\mu - \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^U y \cdot y^{\alpha-1}(1-y)^{\beta-1}dy$$

The beta distribution with parameters *α*+1 and *β* can be expressed as

equation [A-5]

$$\mathrm{B}(x \mid \alpha+1, \beta) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta)}\int_0^x t^{\alpha}(1-t)^{\beta-1}dt$$

Because Γ(n+1) = n Γ(n), equation [A-5] can be expressed as

equation [A-6]

$$\mathrm{B}(x \mid \alpha+1, \beta) = \frac{(\alpha+\beta)}{\alpha}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x t \cdot t^{\alpha-1}(1-t)^{\beta-1}dt$$

Because *α* / (*α* + *β*) = *μ* for the beta distribution, it can be shown that equation [A-4] can be expressed as

equation [A-7]

$$\mu - \mu \cdot B(U \mid \alpha+1, \beta)$$

which is equal to

equation [A-8]

$$\mu\,[1 - B(U \mid \alpha + 1, \beta)]$$