3

3.1 $Kx = \lambda x$ (λ is Eigenvalues, X is Eigenvectors)

$x^T K x = x^T \lambda x = \lambda x^T x$

$x^T K x = \lambda x^T x$

for $x^T x = |x|^2$ and x is non-zero

we have $x^T x = |x|^2 > 0$

So, If $x^T K x > 0$

we have $\lambda > 0$

So, we have

① K is a symmetric matrix

② λ of $K > 0$

we get K is positive semidefinite

3.2

1. $K_{ij} = k(x^{(i)}, x^{(j)}) = \alpha$

since ① $K_{ij} > 0$ for all i, j

eigenvalues of $K_{ij} > 0$

② K_{ij} is symmetric

We get K_{ij} is positive semi-definite

Thus, $k(x, y) = \alpha$ is a kernel

3.2

2. Because $f: R^d \to R$

we get $f(x) \cdot f(y) = \langle f(x), f(y) \rangle$

Thus, $k(x,y) = \langle \phi(x), \phi(y) \rangle$ where $\phi(x) = f(x)$

we got $k(x,y) = f(x) \cdot f(y)$ is a kernel

3. $K_{ij} = a \cdot K_{1ij} + b \cdot K_{2ij}$

$$X^T K_{ij} X = a \cdot \underline{X^T K_{1ij} X} + b \cdot \underline{X^T K_{2ij} X} \qquad (a, b > 0)$$

we have, $X^T K_{1ij} X > 0$ and $X^T K_{2ij} X > 0$

Thus $X^T K_{ij} X > 0$

$\longrightarrow$ K_{ij} is positive semidefine

$\longrightarrow$ $k(x,y)$ is a kernel

4. $$\frac{k_1(x,y)}{\sqrt{k_1(x,x)}\sqrt{k_1(y,y)}} = \frac{\langle \phi(x), \phi(y) \rangle}{\sqrt{\langle \phi(x), \phi(x) \rangle}\sqrt{\langle \phi(y), \phi(y) \rangle}}$$

$$= \frac{\langle \phi(x), \phi(y) \rangle}{\|\phi(x)\| \, \|\phi(y)\|}$$

$$= \left\langle \frac{\phi(x)}{\|\phi(x)\|}, \frac{\phi(y)}{\|\phi(y)\|} \right\rangle$$

We get $k(x,y) = \langle \phi'(x), \phi'(y) \rangle$

where $\phi'(x) = \dfrac{\phi(x)}{\|\phi(x)\|}$ $\quad \phi'(y) = \dfrac{\phi(y)}{\|\phi(y)\|}$