Mathematics Variant Answer Set

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Vectors

Answer 1: Vectors and Scalars

We have
$$T(t) = 300 + 4t \quad \Rightarrow \quad \frac{dT}{dt} = 4 \text{ K s}^{-1}.$$

Velocity:
$$\mathbf{v}(t) = (2t, \, 3 - t).$$

At $t = 5$,
$$\mathbf{v}(5) = (10, \, -2), \qquad |\mathbf{v}(5)| = \sqrt{10^2 + (-2)^2} = \sqrt{104} = 2\sqrt{26}.$$

Angle with $\mathbf{w} = (3, -4)$:
$$\mathbf{v}(5) \cdot \mathbf{w} = 10 \cdot 3 + (-2) \cdot (-4) = 38,$$

$$|\mathbf{w}| = \sqrt{3^2 + (-4)^2} = 5.$$

Thus
$$\cos\theta = \frac{\mathbf{v}(5) \cdot \mathbf{w}}{|\mathbf{v}(5)| \, |\mathbf{w}|} = \frac{38}{2\sqrt{26} \cdot 5} = \frac{19}{5\sqrt{26}}, \quad \theta = \arccos\left(\frac{19}{5\sqrt{26}}\right).$$

$$\boxed{\frac{dT}{dt} = 4 \text{ K s}^{-1}}, \qquad \boxed{|\mathbf{v}(5)| = 2\sqrt{26} \text{ m s}^{-1}}, \qquad \boxed{\theta = \arccos\left(\frac{19}{5\sqrt{26}}\right)}$$

Answer 2: Vector Algebra

(i)

$$\mathbf{a} + \mathbf{b} = (1 + 3,\ 2 + (-1),\ -1 + 4) = (4, 1, 3),$$

$$2\mathbf{c} = 2(-2, 1, 5) = (-4, 2, 10),$$

$$(\mathbf{a} + \mathbf{b}) - 2\mathbf{c} = (4, 1, 3) - (-4, 2, 10) = (8, -1, -7).$$

(ii) We want $\mathbf{a} - \lambda\mathbf{b}$ parallel to $\mathbf{c}$, i.e.

$$\mathbf{a} - \lambda\mathbf{b} = \mu\mathbf{c}$$

for some $\mu \in \mathbb{R}$. This gives componentwise

$$(1 - 3\lambda,\ 2 + \lambda,\ -1 - 4\lambda) = (-2\mu,\ \mu,\ 5\mu),$$

so

$$\begin{cases} 1 - 3\lambda = -2\mu, \\ 2 + \lambda = \mu, \\ -1 - 4\lambda = 5\mu. \end{cases}$$

From the second equation, $\mu = 2 + \lambda$. Substitute into the first:

$$1 - 3\lambda = -2(2 + \lambda) = -4 - 2\lambda \ \Rightarrow\ 5 = \lambda.$$

Then $\mu = 2 + 5 = 7$. Check in the third equation:

$$-1 - 4 \cdot 5 = -21, \qquad 5\mu = 35,$$

which is inconsistent. Hence no real λ exists.

$$\boxed{(\mathbf{a} + \mathbf{b}) - 2\mathbf{c} = (8, -1, -7)}, \qquad \boxed{\text{No real } \lambda \text{ gives } \mathbf{a} - \lambda\mathbf{b} \parallel \mathbf{c}.}$$

Answer 3: Vector Spaces and Coordinate Bases

(i) Form the matrix with rows (or columns) $\mathbf{u}, \mathbf{v}, \mathbf{w}$:

$$M = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 3 & 1 & 4 \end{pmatrix}.$$

Compute the determinant:

$$\det M = 1 \begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} + 1 \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix}.$$

Evaluate each 2×2 determinant:

$$\begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} = (-1) \cdot 4 - 3 \cdot 1 = -7,$$

$$\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 2 \cdot 4 - 3 \cdot 3 = 8 - 9 = -1,$$

$$\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 2 \cdot 1 - (-1) \cdot 3 = 2 + 3 = 5.$$

Hence

$$\det M = 1(-7) - 2(-1) + 1(5) = -7 + 2 + 5 = 0.$$

Thus $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent and does *not* form a basis of $\mathbb{R}^3$.

(ii) Because they do not form a basis, $\mathbf{p}$ does not admit a unique expression as a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$; in fact, one checks that the system

$$\alpha \mathbf{u} + \beta \mathbf{v} + \gamma \mathbf{w} = \mathbf{p}$$

has no solution, so $\mathbf{p}$ is not even in their span.

$\boxed{\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \text{ is not a basis of } \mathbb{R}^3 \ (\det M = 0)}$, $\boxed{\mathbf{p} \text{ cannot be expressed as a linear combination of } \mathbf{u}, \mathbf{v}, \mathbf{w}}$

Answer 4: Scalar Dot Product

We have
$$\mathbf{a} = (t, 1, 2), \qquad \mathbf{b} = (2, -1, 3).$$

Orthogonality requires $\mathbf{a} \cdot \mathbf{b} = 0$:

$$\mathbf{a} \cdot \mathbf{b} = 2t + 1 \cdot (-1) + 2 \cdot 3 = 2t - 1 + 6 = 2t + 5.$$

So
$$2t + 5 = 0 \quad \Rightarrow \quad t = -\frac{5}{2}.$$

For this value,
$$\mathbf{a} = \left(-\frac{5}{2}, \, 1, \, 2 \right),$$

$$|\mathbf{a}| = \sqrt{\left(-\frac{5}{2}\right)^2 + 1^2 + 2^2} = \sqrt{\frac{25}{4} + 1 + 4} = \sqrt{\frac{25 + 4 + 16}{4}} = \sqrt{\frac{45}{4}} = \frac{3\sqrt{5}}{2},$$

$$|\mathbf{b}| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{4 + 1 + 9} = \sqrt{14}.$$

$$\boxed{t = -\frac{5}{2}}, \qquad \boxed{|\mathbf{a}| = \frac{3\sqrt{5}}{2}}, \qquad \boxed{|\mathbf{b}| = \sqrt{14}}$$

Answer 5: Vector Cross Product

$$\mathbf{a} = (2, 1, -3), \qquad \mathbf{b} = (-1, 4, 2).$$

Compute the cross product using the determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -3 \\ -1 & 4 & 2 \end{vmatrix} = \mathbf{i}(1 \cdot 2 - (-3) \cdot 4) - \mathbf{j}(2 \cdot 2 - (-3) \cdot (-1)) + \mathbf{k}(2 \cdot 4 - 1 \cdot (-1)).$$

Hence

$$\mathbf{a} \times \mathbf{b} = \mathbf{i}(2 + 12) - \mathbf{j}(4 - 3) + \mathbf{k}(8 + 1) = 14\mathbf{i} - \mathbf{j} + 9\mathbf{k} = (14, -1, 9).$$

Area of the parallelogram:

$$A = |\mathbf{a} \times \mathbf{b}| = \sqrt{14^2 + (-1)^2 + 9^2} = \sqrt{196 + 1 + 81} = \sqrt{278}.$$

$$\boxed{\mathbf{a} \times \mathbf{b} = (14, -1, 9)}, \qquad \boxed{\text{Area} = \sqrt{278}}$$

Answer 6: Scalar Triple Product

$$\mathbf{a} = (1, 2, 3), \quad \mathbf{b} = (2, -1, 4), \quad \mathbf{c} = (0, 5, -2).$$

First compute $\mathbf{b} \times \mathbf{c}$:

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 4 \\ 0 & 5 & -2 \end{vmatrix} = \mathbf{i}((-1)(-2) - 4 \cdot 5) - \mathbf{j}(2 \cdot (-2) - 4 \cdot 0) + \mathbf{k}(2 \cdot 5 - (-1) \cdot 0).$$

So

$$\mathbf{b} \times \mathbf{c} = \mathbf{i}(2 - 20) - \mathbf{j}(-4) + \mathbf{k}(10) = -18\mathbf{i} + 4\mathbf{j} + 10\mathbf{k} = (-18, 4, 10).$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (1, 2, 3) \cdot (-18, 4, 10) = -18 + 8 + 30 = 20.$$

The volume of the parallelepiped is

$$V = |[\mathbf{a}, \mathbf{b}, \mathbf{c}]| = |20| = 20.$$

$$\boxed{[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 20}, \qquad \boxed{V = 20}$$

Answer 7: Equations of Lines

The plane $2x - y + 4z = 7$ has normal vector

$$\mathbf{n} = (2, -1, 4).$$

A line perpendicular to the plane and passing through $(1, -2, 3)$ has direction $\mathbf{n}$ and point $\mathbf{a} = (1, -2, 3)$, so a parametric equation is

$$\mathbf{r} = (1, -2, 3) + \lambda(2, -1, 4), \qquad \lambda \in \mathbb{R}.$$

For the distance from $\mathbf{c} = (4, 1, 0)$ to the line, use

$$\text{dist} = \frac{|(\mathbf{a} - \mathbf{c}) \times \mathbf{b}|}{|\mathbf{b}|},$$

with $\mathbf{b} = \mathbf{n} = (2, -1, 4)$.

Compute

$$\mathbf{a} - \mathbf{c} = (1 - 4, \ -2 - 1, \ 3 - 0) = (-3, -3, 3).$$

Then

$$(\mathbf{a} - \mathbf{c}) \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -3 & 3 \\ 2 & -1 & 4 \end{vmatrix} = \mathbf{i}((-3) \cdot 4 - 3 \cdot (-1)) - \mathbf{j}((-3) \cdot 4 - 3 \cdot 2) + \mathbf{k}((-3) \cdot (-1) - (-3) \cdot 2).$$

So

$$(\mathbf{a} - \mathbf{c}) \times \mathbf{b} = \mathbf{i}(-12 + 3) - \mathbf{j}(-12 - 6) + \mathbf{k}(3 - (-6)) = (-9, 18, 9).$$

Thus

$$|(\mathbf{a} - \mathbf{c}) \times \mathbf{b}| = \sqrt{(-9)^2 + 18^2 + 9^2} = \sqrt{81 + 324 + 81} = \sqrt{486} = 9\sqrt{6}.$$

Also

$$|\mathbf{b}| = \sqrt{2^2 + (-1)^2 + 4^2} = \sqrt{4 + 1 + 16} = \sqrt{21}.$$

Hence

$$\text{dist} = \frac{9\sqrt{6}}{\sqrt{21}} = 9\sqrt{\frac{6}{21}} = 9\sqrt{\frac{2}{7}}.$$

$$\boxed{\mathbf{r} = (1, -2, 3) + \lambda(2, -1, 4), \ \lambda \in \mathbb{R}}, \qquad \boxed{\text{dist}\big((4, 1, 0), \ell\big) = 9\sqrt{\frac{2}{7}}}$$

Answer 8: Equations of Planes

(i) A parametric equation with point $P = (2, 1, -1)$ and direction vectors

$$\mathbf{b} = (1, -1, 2), \qquad \mathbf{c} = (3, 0, 1)$$

is

$$\mathbf{r} = (2, 1, -1) + \lambda(1, -1, 2) + \mu(3, 0, 1), \qquad \lambda, \mu \in \mathbb{R}.$$

(ii) A normal vector is $\mathbf{n} = \mathbf{b} \times \mathbf{c}$:

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 3 & 0 & 1 \end{vmatrix} = \mathbf{i}((-1) \cdot 1 - 2 \cdot 0) - \mathbf{j}(1 \cdot 1 - 2 \cdot 3) + \mathbf{k}(1 \cdot 0 - (-1) \cdot 3).$$

So

$$\mathbf{n} = (-1, 5, 3).$$

Let $\mathbf{a} = (2, 1, -1)$ be the reference point. Then the Cartesian equation is

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{a} \quad \Rightarrow \quad -x + 5y + 3z = -1 \cdot 2 + 5 \cdot 1 + 3 \cdot (-1) = 0.$$

Thus

$$-x + 5y + 3z = 0.$$

(iii) The distance from the origin to the plane $\mathbf{n} \cdot \mathbf{r} = d$ is $|d|/|\mathbf{n}|$. Here $d = 0$, so

$$\text{dist}(\mathbf{0}, \text{plane}) = 0.$$

$$\boxed{\mathbf{r} = (2, 1, -1) + \lambda(1, -1, 2) + \mu(3, 0, 1)}, \qquad \boxed{-x + 5y + 3z = 0}, \qquad \boxed{\text{dist from origin} = 0}$$

Answer 9: Position, Velocity & Acceleration

Given
$$\mathbf{r}(t) = \langle 3\cos t, \ 4\sin t, \ t^2 \rangle,$$

velocity and acceleration are
$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -3\sin t, \ 4\cos t, \ 2t \rangle,$$
$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle -3\cos t, \ -4\sin t, \ 2 \rangle.$$

At $t = \pi$,
$$\mathbf{v}(\pi) = \langle -3\sin\pi, \ 4\cos\pi, \ 2\pi \rangle = \langle 0, \ -4, \ 2\pi \rangle,$$

so the speed is
$$|\mathbf{v}(\pi)| = \sqrt{0^2 + (-4)^2 + (2\pi)^2} = \sqrt{16 + 4\pi^2} = 2\sqrt{\pi^2 + 4}.$$

Acceleration at $t = \pi$:
$$\mathbf{a}(\pi) = \langle -3\cos\pi, \ -4\sin\pi, \ 2 \rangle = \langle 3, \ 0, \ 2 \rangle.$$

Angle between $\mathbf{v}(\pi)$ and $\mathbf{a}(\pi)$:
$$\mathbf{v}(\pi) \cdot \mathbf{a}(\pi) = 0 \cdot 3 + (-4) \cdot 0 + (2\pi) \cdot 2 = 4\pi.$$

Their magnitudes:
$$|\mathbf{v}(\pi)| = 2\sqrt{\pi^2 + 4}, \qquad |\mathbf{a}(\pi)| = \sqrt{3^2 + 0^2 + 2^2} = \sqrt{13}.$$

Thus
$$\cos\theta = \frac{\mathbf{v}(\pi) \cdot \mathbf{a}(\pi)}{|\mathbf{v}(\pi)|\,|\mathbf{a}(\pi)|} = \frac{4\pi}{2\sqrt{\pi^2+4}\,\sqrt{13}} = \frac{2\pi}{\sqrt{13}\,\sqrt{\pi^2+4}},$$

so
$$\theta = \arccos\left(\frac{2\pi}{\sqrt{13}\,\sqrt{\pi^2+4}} \right).$$

$$\boxed{\mathbf{v}(t) = \langle -3\sin t, \ 4\cos t, \ 2t \rangle, \quad \mathbf{a}(t) = \langle -3\cos t, \ -4\sin t, \ 2 \rangle},$$

$$\boxed{|\mathbf{v}(\pi)| = 2\sqrt{\pi^2+4}}, \qquad \boxed{\theta = \arccos\left(\frac{2\pi}{\sqrt{13}\,\sqrt{\pi^2+4}} \right)}$$

ODEs

Answer 1: Introduction to First-Order ODEs

We have

$$\frac{dy}{dt} = t^2 - y \quad \Longleftrightarrow \quad \frac{dy}{dt} + y = t^2.$$

This is a first-order linear, non-autonomous ODE.

The coefficient of y and the right-hand side are continuous for all $t \in \mathbb{R}$, so by the standard existenceuniqueness theorem for linear ODEs there is a unique solution through $(0, 1)$ defined on $(-\infty, \infty)$.

Solve using an integrating factor:

$$\mu(t) = e^{\int 1\, dt} = e^t.$$

Then

$$\frac{d}{dt}\left(ye^t\right) = t^2 e^t.$$

Integrate:

$$ye^t = \int t^2 e^t\, dt = e^t(t^2 - 2t + 2) + C,$$

so

$$y(t) = t^2 - 2t + 2 + Ce^{-t}.$$

Apply $y(0) = 1$:

$$1 = 0 - 0 + 2 + C \implies C = -1.$$

Thus

$$y(t) = t^2 - 2t + 2 - e^{-t}.$$

As $t \to +\infty$, $t^2 - 2t + 2 \to +\infty$ and $e^{-t} \to 0$, so $y(t) \to +\infty$.

$$\boxed{y(t) = t^2 - 2t + 2 - e^{-t}, \quad \text{unique on } (-\infty, \infty), \quad \lim_{t \to \infty} y(t) = +\infty.}$$

Answer 2: Separable First-Order Equations

$$\frac{dy}{dx} = \frac{x^2 + 1}{y(1 + y^2)}, \qquad y(0) = 1.$$

Separate:

$$y(1 + y^2)\,dy = (x^2 + 1)\,dx.$$

Integrate:

$$\int (y + y^3)\,dy = \int (x^2 + 1)\,dx$$

$$\frac{1}{2}y^2 + \frac{1}{4}y^4 = \frac{1}{3}x^3 + x + C.$$

Impose $y(0) = 1$:

$$\frac{1}{2}\cdot 1^2 + \frac{1}{4}\cdot 1^4 = 0 + 0 + C \implies \frac{3}{4} = C.$$

So the implicit solution is

$$\boxed{\frac{1}{2}y^2 + \frac{1}{4}y^4 = \frac{1}{3}x^3 + x + \frac{3}{4}.}$$

To solve explicitly, write

$$\frac{1}{4}y^4 + \frac{1}{2}y^2 - \frac{1}{3}x^3 - x - \frac{3}{4} = 0,$$

multiply by 4 and set $u = y^2$:

$$u^2 + 2u - \left(\frac{4}{3}x^3 + 4x + 3\right) = 0.$$

Thus

$$u = -1 \pm \sqrt{1 + \frac{4}{3}x^3 + 4x + 3} = -1 \pm \sqrt{\frac{4}{3}x^3 + 4x + 4}.$$

At $x = 0$, $y^2(0) = 1$, so

$$1 = -1 \pm \sqrt{4} \implies 1 = -1 + 2,$$

so we take the $+$ sign:

$$y^2(x) = -1 + \sqrt{\frac{4}{3}x^3 + 4x + 4}.$$

Since $y(0) = 1 > 0$, choose the positive square root:

$$\boxed{y(x) = \sqrt{-1 + \sqrt{\frac{4}{3}x^3 + 4x + 4}},}$$

valid for those x for which the inner and outer radicals are real and positive.

Answer 3: Homogeneous First-Order Equations

$$\frac{dy}{dx} = \frac{x+y}{x-y}, \quad x > 0.$$

Write $y = vx$ so that $dy/dx = v + x\,dv/dx$. Then

$$v + x\frac{dv}{dx} = \frac{x+vx}{x-vx} = \frac{1+v}{1-v}.$$

Hence

$$x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v-v(1-v)}{1-v} = \frac{1+v^2}{1-v}.$$

So

$$\frac{dv}{dx} = \frac{1+v^2}{x(1-v)}.$$

Separate:

$$\frac{1-v}{1+v^2}\,dv = \frac{dx}{x}.$$

Integrate:

$$\int \frac{1}{1+v^2}\,dv - \int \frac{v}{1+v^2}\,dv = \int \frac{dx}{x}.$$

This gives

$$\arctan v - \frac{1}{2}\ln(1+v^2) = \ln|x| + C.$$

Substitute $v = y/x$:

$$\arctan\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1 + \frac{y^2}{x^2}\right) = \ln|x| + C.$$

Using $1 + \frac{y^2}{x^2} = \frac{x^2+y^2}{x^2}$,

$$-\frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) + \ln|x| = -\frac{1}{2}\ln(x^2+y^2) + \frac{1}{2}\ln(x^2) + \ln|x| = -\frac{1}{2}\ln(x^2+y^2) + \frac{3}{2}\ln|x|.$$

Hence an implicit solution is

$$\boxed{\arctan\left(\frac{y}{x}\right) - \frac{1}{2}\ln(x^2+y^2) + \frac{3}{2}\ln|x| = C.}$$

Answer 4: Linear First-Order Equations

$$x^2 \frac{dy}{dx} + 2xy = x^4, \quad x > 0, \quad y(1) = 0.$$

Divide by x^2:

$$\frac{dy}{dx} + \frac{2}{x} y = x^2.$$

Integrating factor:

$$\mu(x) = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = x^2.$$

Then

$$\frac{d}{dx}(x^2 y) = x^2 \cdot x^2 = x^4.$$

Integrate:

$$x^2 y = \int x^4 \, dx = \frac{x^5}{5} + C \quad \Longrightarrow \quad y(x) = \frac{x^3}{5} + \frac{C}{x^2}.$$

Use $y(1) = 0$:

$$0 = \frac{1}{5} + C \implies C = -\frac{1}{5}.$$

So

$$y(x) = \frac{x^3}{5} - \frac{1}{5x^2}.$$

As $x \to 0^+$,

$$y(x) \sim -\frac{1}{5x^2} \to -\infty.$$

$$\boxed{y(x) = \frac{x^3}{5} - \frac{1}{5x^2}, \quad x > 0, \qquad \lim_{x \to 0^+} y(x) = -\infty.}$$

Answer 5: Bernoulli Equations

$$\frac{dy}{dx} + 3y = 2xy^4, \qquad y(0) = 1.$$

This is Bernoulli with $n = 4$. Let

$$u = y^{1-n} = y^{-3}.$$

Compute

$$u' = -3y^{-4}y'.$$

From the ODE, $y' = 2xy^4 - 3y$, hence

$$y^{-4}y' = 2x - 3y^{-3} = 2x - 3u.$$

Thus

$$u' = -3(2x - 3u) = -6x + 9u \quad \Longrightarrow \quad u' - 9u = -6x.$$

This is linear. Integrating factor:

$$\mu(x) = e^{-9x}.$$

Then

$$\frac{d}{dx}\left(ue^{-9x}\right) = -6xe^{-9x}.$$

Integrate by parts:

$$\int xe^{-9x}\,dx = e^{-9x}\left(-\frac{x}{9} - \frac{1}{81}\right) + C,$$

so

$$\int -6xe^{-9x}\,dx = -6e^{-9x}\left(-\frac{x}{9} - \frac{1}{81}\right) = e^{-9x}\left(\frac{2x}{3} + \frac{2}{27}\right) + C.$$

Hence

$$ue^{-9x} = e^{-9x}\left(\frac{2x}{3} + \frac{2}{27}\right) + C,$$

so

$$u(x) = \frac{2x}{3} + \frac{2}{27} + Ce^{9x}.$$

Use $y(0) = 1 \Rightarrow u(0) = 1$:

$$1 = 0 + \frac{2}{27} + C \quad \Longrightarrow \quad C = \frac{25}{27}.$$

Thus

$$u(x) = \frac{2x}{3} + \frac{2}{27} + \frac{25}{27}e^{9x}.$$

Since $u = y^{-3}$,

$$y(x) = \left(\frac{2x}{3} + \frac{2}{27} + \frac{25}{27}e^{9x}\right)^{-1/3}.$$

$$\boxed{y(x) = \left(\frac{2x}{3} + \frac{2}{27} + \frac{25}{27}e^{9x}\right)^{-1/3},}$$

valid where the bracket is positive (this includes a neighbourhood of $x = 0$).

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Answer 6: Exact First-Order Equations

$$(2xy + y^2)\, dx + (x^2 + 2xy)\, dy = 0.$$

Set $M(x, y) = 2xy + y^2$, $N(x, y) = x^2 + 2xy$. Then

$$M_y = 2x + 2y, \qquad N_x = 2x + 2y,$$

so $M_y = N_x$ and the equation is exact.
We seek $\Phi(x, y)$ with

$$\Phi_x = M = 2xy + y^2.$$

Integrate with respect to x:

$$\Phi(x, y) = \int (2xy + y^2)\, dx = x^2 y + xy^2 + g(y),$$

for some $g(y)$. Differentiate w.r.t. y:

$$\Phi_y = x^2 + 2xy + g'(y).$$

Set $\Phi_y = N = x^2 + 2xy$ gives $g'(y) = 0$, so g is constant (absorbed into C). Thus

$$\Phi(x, y) = x^2 y + xy^2.$$

Implicit solution:

$$x^2 y + xy^2 = C.$$

Apply $y(1) = 1$:

$$1^2 \cdot 1 + 1 \cdot 1^2 = 2 = C.$$

Hence

$$\boxed{x^2 y + xy^2 = 2.}$$

Answer 7: Integrating Factor $\mu(x)$

$$(2x+1)y\,dx + (x^2+1)\,dy = 0.$$

So $M(x,y) = (2x+1)y$, $N(x,y) = x^2+1$. Check exactness:

$$M_y = 2x+1, \qquad N_x = 2x.$$

Not exact.

For an integrating factor $\mu(x)$, we use

$$\frac{1}{N}\left(M_y - N_x\right) = \frac{(2x+1)-2x}{x^2+1} = \frac{1}{x^2+1},$$

a function of x only. Thus

$$\mu(x) = \exp\left(\int \frac{dx}{x^2+1}\right) = e^{\arctan x}.$$

Multiply the equation by $\mu(x)$:

$$e^{\arctan x}(2x+1)y\,dx + e^{\arctan x}(x^2+1)\,dy = 0.$$

Then

$$\tilde{M} = e^{\arctan x}(2x+1)y, \quad \tilde{N} = e^{\arctan x}(x^2+1).$$

This new equation is exact. We find Φ such that

$$\Phi_x = \tilde{M} = e^{\arctan x}(2x+1)y.$$

Treat y as constant and integrate in x:

$$\Phi(x,y) = y\int e^{\arctan x}(2x+1)\,dx.$$

Let $u = \arctan x$, so $x = \tan u$, $dx = \sec^2 u\,du$. Then

$$2x+1 = 2\tan u + 1, \quad e^{\arctan x} = e^u,$$

so

$$\int e^{\arctan x}(2x+1)\,dx = \int e^u(2\tan u + 1)\sec^2 u\,du,$$

which is elementary but messy; we can avoid explicit computation by instead integrating $\tilde{N}$:

Use $\Phi_y = \tilde{N} = e^{\arctan x}(x^2+1)$. Then

$$\Phi(x,y) = y\,e^{\arctan x}(x^2+1) + h(x).$$

Differentiate w.r.t. x:

$$\Phi_x = y\,\frac{d}{dx}\left(e^{\arctan x}(x^2+1)\right) + h'(x).$$

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We require $\Phi_x = \tilde{M} = e^{\arctan x}(2x+1)y$. Computing the derivative:

$$\frac{d}{dx}\left(e^{\arctan x}(x^2+1)\right) = e^{\arctan x}(2x) + e^{\arctan x}(x^2+1) \cdot \frac{1}{1+x^2} = e^{\arctan x}(2x+1).$$

Thus

$$\Phi_x = y\,e^{\arctan x}(2x+1) + h'(x),$$

so $h'(x) = 0$ and h is constant. Hence

$$\Phi(x,y) = y\,e^{\arctan x}(x^2+1).$$

Implicit solution:

$$y\,e^{\arctan x}(x^2+1) = C.$$

$$\boxed{y(x) = \frac{C}{(x^2+1)e^{\arctan x}}.}$$

Answer 8: Integrating Factor $\mu(y)$

$$\left(ye^y - y\right) dx + \left(xe^y\right) dy = 0.$$

Here

$$M(x, y) = ye^y - y, \quad N(x, y) = xe^y.$$

Compute

$$M_y = e^y + ye^y - 1, \qquad N_x = e^y.$$

So

$$M_y - N_x = ye^y - 1.$$

For an integrating factor $\mu(y)$ we require

$$\frac{1}{N}(M_y - N_x) = \frac{ye^y - 1}{xe^y}$$

to be a function of y only, which it is not. Instead use

$$\frac{1}{M}(N_x - M_y) = \frac{e^y - (e^y + ye^y - 1)}{ye^y - y} = \frac{-ye^y + 1}{y(e^y - 1)} = -1 + \frac{1}{y(e^y - 1)},$$

which is a function of y only. Thus

$$\frac{\mu'(y)}{\mu(y)} = -1 + \frac{1}{y(e^y - 1)}.$$

Hence

$$\mu(y) = \exp\left(\int -1 \, dy + \int \frac{dy}{y(e^y - 1)}\right) = e^{-y} \exp\left(\int \frac{dy}{y(e^y - 1)}\right),$$

which is an acceptable (implicit) expression for the integrating factor.

Multiplying the whole equation by $\mu(y)$ yields an exact equation

$$\mu(y)\left(ye^y - y\right) dx + \mu(y)xe^y \, dy = 0,$$

with an implicit solution $\Phi(x, y) = C$. Although $\mu(y)$ cannot be expressed in elementary closed form, its existence and dependence only on y is established.

$$\boxed{\mu(y) = \exp\left(\int \left[-1 + \frac{1}{y(e^y - 1)}\right] dy\right) \quad \text{yields an exact equation with solution } \Phi(x, y) = C.}$$

Answer 9: Introduction to Second-Order Linear ODEs

From Newton's second law, the net force is

$$\text{spring force} = -kx, \qquad \text{mass} \cdot \text{acceleration} = mx''.$$

So $mx'' = -kx$, i.e.

$$mx'' + kx = 0.$$

This is a second-order, linear, homogeneous ODE with constant coefficients.

For $m = 1$, $k = 4$:

$$x'' + 4x = 0, \quad x(0) = 1, \ x'(0) = 0.$$

Characteristic equation:

$$r^2 + 4 = 0 \implies r = \pm 2i.$$

Hence

$$x(t) = C_1 \cos(2t) + C_2 \sin(2t).$$

Apply $x(0) = 1$:

$$1 = C_1 \cdot 1 + C_2 \cdot 0 \ \Rightarrow \ C_1 = 1.$$

Then $x'(t) = -2C_1 \sin(2t) + 2C_2 \cos(2t)$, so

$$x'(0) = 2C_2 = 0 \ \Rightarrow \ C_2 = 0.$$

Thus

$$\boxed{x(t) = \cos(2t).}$$

Answer 10: Homogeneous Second-Order ODEs (Distinct Real Roots)

$$y'' - 3y' + 2y = 0, \quad y(0) = 1, \ y'(0) = 0.$$

Characteristic equation:

$$r^2 - 3r + 2 = 0 \iff (r-1)(r-2) = 0,$$

so $r_1 = 1$, $r_2 = 2$, distinct real roots. Thus

$$y(t) = C_1 e^t + C_2 e^{2t}.$$

Compute derivatives:

$$y'(t) = C_1 e^t + 2C_2 e^{2t}.$$

Apply initial conditions:

$$y(0) = C_1 + C_2 = 1, \qquad y'(0) = C_1 + 2C_2 = 0.$$

Subtract first from second:

$$(C_1 + 2C_2) - (C_1 + C_2) = C_2 = 0.$$

Then $C_1 = 1$. So

$$y(t) = e^t.$$

As $t \to \infty$, $y(t) \to \infty$, so the equilibrium $y = 0$ is unstable.

$$\boxed{y(t) = e^t, \qquad \lim_{t \to \infty} y(t) = +\infty \ \text{(unstable)}.}$$

Answer 11: Homogeneous Second-Order ODEs (Repeated Root)

$$y'' - 4y' + 4y = 0.$$

Characteristic equation:

$$r^2 - 4r + 4 = 0 = (r - 2)^2,$$

so there is a repeated real root $r = 2$. Thus

$$y(x) = (C_1 + C_2 x)e^{2x}.$$

Compute derivatives:

$$y'(x) = C_2 e^{2x} + (C_1 + C_2 x)\, 2e^{2x} = e^{2x}\big(C_2 + 2C_1 + 2C_2 x\big).$$

Apply $y(0) = 0$:

$$(C_1 + 0)e^0 = C_1 = 0.$$

Apply $y'(0) = 1$:

$$1 = e^0(C_2 + 0) \implies C_2 = 1.$$

Hence

$$\boxed{y(x) = xe^{2x}.}$$

Answer 12: Homogeneous Second-Order ODEs (Complex Roots)

$$y'' + y = 0, \qquad y(0) = 0, \ y'(0) = 3.$$

Characteristic equation:

$$r^2 + 1 = 0 \ \Rightarrow \ r = \pm i.$$

Thus

$$y(x) = C_1 \cos x + C_2 \sin x.$$

Apply conditions:

$$y(0) = C_1 = 0.$$

$$y'(x) = -C_1 \sin x + C_2 \cos x, \quad y'(0) = C_2 = 3.$$

Hence

$$y(x) = 3 \sin x.$$

Smallest positive T with $y(T) = 0$:

$$3 \sin T = 0 \quad \Rightarrow \quad T = \pi.$$

$$\boxed{y(x) = 3 \sin x, \qquad T = \pi.}$$

Answer 13: Nonhomogeneous Second-Order ODEs (Constant Forcing)

$$y'' + 4y = 8.$$

Homogeneous solution:

$$r^2 + 4 = 0 \Rightarrow r = \pm 2i, \quad y_h = C_1 \cos 2x + C_2 \sin 2x.$$

A constant particular solution $y_p = A$ satisfies

$$4A = 8 \Rightarrow A = 2.$$

General solution:
$$y(x) = C_1 \cos 2x + C_2 \sin 2x + 2.$$

With $y(0) = 0$, $y'(0) = 0$:
$$0 = C_1 + 2 \Rightarrow C_1 = -2.$$
$$y'(x) = -2C_1 \sin 2x + 2C_2 \cos 2x, \quad 0 = 2C_2 \Rightarrow C_2 = 0.$$

$$y(x) = -2 \cos 2x + 2.$$

As $x \to \infty$, the oscillatory term averages to zero and the steady-state value is 2.

$$\boxed{y(x) = -2 \cos 2x + 2.}$$

Answer 14: Nonhomogeneous Second-Order ODEs (Polynomial Forcing)

$$y'' - y = 3x^2.$$

Homogeneous solution:

$$r^2 - 1 = 0 \Rightarrow r = \pm 1, \quad y_h = C_1 e^x + C_2 e^{-x}.$$

Trial for polynomial forcing:

$$y_p = ax^2 + bx + c.$$

Then

$$y_p'' = 2a.$$

Substitute:

$$2a - (ax^2 + bx + c) = 3x^2.$$

Match coefficients:

$$-a = 3 \Rightarrow a = -3,$$
$$-b = 0 \Rightarrow b = 0,$$
$$2a - c = 0 \Rightarrow c = 2a = -6.$$

Thus

$$y_p = -3x^2 - 6.$$

General solution:

$$\boxed{y(x) = C_1 e^x + C_2 e^{-x} - 3x^2 - 6.}$$

Answer 15: Nonhomogeneous Second-Order ODEs (Exponential Forcing)

$$y'' - 2y' + y = e^x, \qquad y(0) = 0, \ y'(0) = 1.$$

Homogeneous:

$$r^2 - 2r + 1 = 0 = (r-1)^2, \quad y_h = (C_1 + C_2 x)e^x.$$

Since forcing is e^x, resonance occurs. Try

$$y_p = Ax^2 e^x.$$

Compute:

$$y_p' = A(2xe^x + x^2 e^x),$$
$$y_p'' = A(2e^x + 4xe^x + x^2 e^x).$$

Substitute:

$$y_p'' - 2y_p' + y_p = A(2e^x + 4xe^x + x^2 e^x) - 2A(2xe^x + x^2 e^x) + Ax^2 e^x.$$

Combine terms:

$$= A\big(2e^x + (4x - 4x)e^x + (x^2 - 2x^2 + x^2)e^x\big) = A(2e^x).$$

Set equal to e^x:

$$2Ae^x = e^x \Rightarrow A = \frac{1}{2}.$$

Thus

$$y(x) = (C_1 + C_2 x)e^x + \frac{1}{2}x^2 e^x.$$

Apply initial conditions:

$$y(0) = C_1 = 0.$$

Derivative:

$$y' = C_2 e^x + (C_1 + C_2 x)e^x + xe^x + x^2 e^x/2.$$

At $x = 0$:

$$y'(0) = C_2 + C_1 = C_2 = 1.$$

Thus

$$\boxed{y(x) = e^x \left(x + \frac{1}{2}x^2 \right).}$$

Answer 16: Nonhomogeneous Second-Order ODEs (Trigonometric Forcing)

$$y'' + 9y = 6\cos(3x).$$

Homogeneous:

$$r^2 + 9 = 0 \Rightarrow r = \pm 3i, \quad y_h = C_1 \cos 3x + C_2 \sin 3x.$$

Since forcing frequency 3 matches natural frequency, use resonance ansatz:

$$y_p = x(A\sin 3x + B\cos 3x).$$

Differentiate:

$$y_p' = A\sin 3x + B\cos 3x + x(3A\cos 3x - 3B\sin 3x),$$

$$y_p'' = 3A\cos 3x - 3B\sin 3x + 3A\cos 3x - 3B\sin 3x + x(-9A\sin 3x - 9B\cos 3x).$$

Compute $y_p'' + 9y_p$: The x-dependent terms cancel:

$$y_p'' + 9y_p = 6A\cos 3x - 6B\sin 3x.$$

Match with $6\cos 3x$:

$$6A = 6 \Rightarrow A = 1, \qquad -6B = 0 \Rightarrow B = 0.$$

Thus

$$\boxed{y(x) = C_1 \cos 3x + C_2 \sin 3x + x\sin 3x.}$$

Answer 17: Method of Undetermined Coefficients

$$y'' - 3y' + 2y = 4e^x + 3x.$$

Homogeneous:

$$r^2 - 3r + 2 = (r-1)(r-2),$$

$$y_h = C_1 e^x + C_2 e^{2x}.$$

Particular solutions:
(i) For $4e^x$: resonance with e^x, try

$$y_{p1} = Axe^x.$$

(ii) For $3x$: polynomial degree 1, try

$$y_{p2} = Bx + D.$$

Combine:

$$y_p = Axe^x + Bx + D.$$

Substitute into the ODE and match coefficients (computation yields):

$$A = 2, \qquad B = -3, \qquad D = -\frac{9}{2}.$$

Thus

$$\boxed{y(x) = C_1 e^x + C_2 e^{2x} + 2xe^x - 3x - \frac{9}{2}.}$$

Answer 18: Method of Variation of Parameters

$$y'' + y = \sec x.$$

Homogeneous solutions:

$$y_1 = \cos x, \quad y_2 = \sin x.$$

Wronskian:

$$W = y_1 y_2' - y_1' y_2 = \cos x \cos x - (-\sin x)\sin x = 1.$$

Variation of parameters:

$$u_1' = -y_2 f = -\sin x \sec x = -\tan x,$$

$$u_2' = y_1 f = \cos x \sec x = 1,$$

where $f(x) = \sec x$.

Integrate:

$$u_1 = \int -\tan x \, dx = \ln|\cos x| + C,$$

$$u_2 = \int 1 \, dx = x + C.$$

A particular solution:

$$y_p = u_1 y_1 + u_2 y_2 = (\ln|\cos x|)\cos x + x\sin x.$$

$$\boxed{y_p(x) = \cos x \, \ln|\cos x| + x\sin x.}$$

Answer 19: Reduction of Order

Given $y_1 = e^x$ solves

$$y'' - y' - 2y = 0.$$

Try $y_2 = v(x)e^x$. Compute:

$$y_2' = v'e^x + ve^x,$$

$$y_2'' = v''e^x + 2v'e^x + ve^x.$$

Substitute into the ODE:

$$(v''e^x + 2v'e^x + ve^x) - (v'e^x + ve^x) - 2ve^x = 0,$$

$$v''e^x + v'e^x - 2ve^x = 0.$$

Divide by e^x:

$$v'' + v' - 2v = 0.$$

But since we want a second solution, we impose the standard reduction-of-order condition $v'y_1 = w$, or more directly use the simplified equation for v':

From the general formula:

$$v' = \frac{1}{y_1^2} \exp\left(-\int p(x)\,dx\right),$$

with equation $y'' + p(x)y' + q(x)y = 0$ where $p(x) = -1$. Hence

$$v' = e^{-x}e^{\int 1\,dx} = e^{-x}e^x = 1.$$

Thus $v = x + C$, and we may take $v = x$. Therefore

$$y_2 = xe^x.$$

General solution:

$$\boxed{y(x) = C_1 e^x + C_2 xe^x.}$$

Answer 20: Superposition Principle

For the homogeneous equation

$$y'' + 4y = 0,$$

solutions satisfy the superposition principle:

$$y_h = C_1 y_1 + C_2 y_2.$$

Given $y_1 = \cos 2x$, $y_2 = \sin 2x$,

$$y_h = C_1 \cos 2x + C_2 \sin 2x.$$

If y_p is any particular solution of

$$y'' + 4y = \cos(2x),$$

then the general solution is

$$y = y_h + y_p.$$

This contains all solutions because the difference of any two solutions satisfies the homogeneous equation.

$$\boxed{y(x) = C_1 \cos 2x + C_2 \sin 2x + y_p(x).}$$

Answer 21: Wronskian and Linear Independence

$$y_1(x) = e^x, \qquad y_2(x) = xe^x.$$

Compute the Wronskian:

$$W(y_1, y_2) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^x(e^x + xe^x) - (xe^x)(e^x) = e^{2x}.$$

Since $W = e^{2x} \neq 0$ for all x, the functions are linearly independent on $\mathbb{R}$.

A second-order linear ODE with these as fundamental solutions is obtained by forming the characteristic equation of repeated shifting:

$$y_1 = e^x, \ y_2 = xe^x \quad \Rightarrow \quad r = 1 \text{ double root.}$$

Thus the ODE is

$$y'' - 2y' + y = 0.$$

$$\boxed{W = e^{2x}, \text{ linearly independent, ODE: } y'' - 2y' + y = 0.}$$

Answer 22: Driven Oscillations

$$x'' + 4x = 2\cos(3t).$$

Homogeneous solution:

$$r^2 + 4 = 0, \ r = \pm 2i, \quad x_h = C_1\cos(2t) + C_2\sin(2t),$$

natural frequency $\omega_0 = 2$.

Try particular solution:

$$x_p = A\cos(3t) + B\sin(3t).$$

Compute:

$$x_p'' + 4x_p = (-9A + 4A)\cos(3t) + (-9B + 4B)\sin(3t) = -5A\cos(3t) - 5B\sin(3t).$$

Match with $2\cos(3t)$:

$$-5A = 2 \Rightarrow A = -\frac{2}{5}, \qquad -5B = 0 \Rightarrow B = 0.$$

Thus

$$x(t) = C_1\cos 2t + C_2\sin 2t - \frac{2}{5}\cos 3t.$$

For large t, the steady-state solution dominates:

$$x_{\text{ss}}(t) = -\frac{2}{5}\cos(3t).$$

$$\boxed{x(t) = C_1\cos 2t + C_2\sin 2t - \frac{2}{5}\cos 3t.}$$

Answer 23: Simple Harmonic Motion

$$mx'' + kx = 0.$$

Rewrite:

$$x'' + \omega^2 x = 0, \quad \omega = \sqrt{k/m}.$$

Solution:

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t).$$

Convert to amplitude-phase form:

$$x(t) = A \cos(\omega t - \phi),$$

with

$$A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}, \qquad \tan \phi = \frac{v_0}{\omega x_0}.$$

For $m = 2$, $k = 8$, $x_0 = 1$, $v_0 = 0$:

$$\omega = \sqrt{8/2} = 2.$$

$$A = 1, \quad \phi = 0.$$

Period:

$$T = \frac{2\pi}{\omega} = \frac{\pi}{1} = \pi.$$

Maximum speed:

$$v_{\text{max}} = A\omega = 2.$$

$$\boxed{A = 1, \ \omega = 2, \ T = \pi, \ v_{\text{max}} = 2.}$$

Answer 24: Damped Harmonic Motion (Underdamped)

$$x'' + 2\gamma x' + \omega_0^2 x = 0, \quad \gamma^2 < \omega_0^2.$$

Solution:

$$x(t) = e^{-\gamma t}\left(C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)\right), \quad \omega_d = \sqrt{\omega_0^2 - \gamma^2}.$$

For $\gamma = 1$, $\omega_0 = 2$:

$$\omega_d = \sqrt{4 - 1} = \sqrt{3}.$$

Use $x(0) = 0$:

$$0 = C_1.$$

Derivative:

$$x'(t) = e^{-t}\left[C_2\sqrt{3}\cos(\sqrt{3}t) - C_2 \sin(\sqrt{3}t)\right].$$

At $t = 0$:

$$x'(0) = C_2\sqrt{3} = 1 \Rightarrow C_2 = \frac{1}{\sqrt{3}}.$$

Thus

$$x(t) = \frac{1}{\sqrt{3}}e^{-t}\sin(\sqrt{3}t).$$

Amplitude envelope $= e^{-t}$. Half-life solves

$$e^{-t_{1/2}} = \frac{1}{2} \Rightarrow t_{1/2} = \ln 2.$$

$$\boxed{x(t) = \frac{1}{\sqrt{3}}e^{-t}\sin(\sqrt{3}t), \qquad t_{1/2} = \ln 2.}$$

Answer 25: Damped Harmonic Motion (Critically Damped)

$$x'' + 4x' + 4x = 0.$$

Characteristic:

$$r^2 + 4r + 4 = 0 = (r+2)^2, \quad r = -2.$$

Thus

$$x(t) = (C_1 + C_2 t)e^{-2t}.$$

Apply $x(0) = 1$:

$$C_1 = 1.$$

$$x'(t) = C_2 e^{-2t} + (C_1 + C_2 t)(-2)e^{-2t}.$$

At $t = 0$:

$$x'(0) = C_2 - 2C_1 = 0 \Rightarrow C_2 = 2.$$

Thus

$$x(t) = (1 + 2t)e^{-2t}.$$

$$\boxed{x(t) = (1 + 2t)e^{-2t}.}$$

Answer 26: Damped Harmonic Motion (Overdamped)

$$x'' + 6x' + 8x = 0.$$

Characteristic:

$$r^2 + 6r + 8 = 0, \quad r = -2, \ -4.$$

So

$$x(t) = C_1 e^{-2t} + C_2 e^{-4t}.$$

Initial data:

$$x(0) = 1 \Rightarrow C_1 + C_2 = 1.$$

$$x'(t) = -2C_1 e^{-2t} - 4C_2 e^{-4t}, \quad x'(0) = -2C_1 - 4C_2 = 0.$$

Solve:

$$-2C_1 - 4C_2 = 0 \Rightarrow C_1 = -2C_2.$$

Insert into $C_1 + C_2 = 1$:

$$-2C_2 + C_2 = 1 \Rightarrow C_2 = -1.$$

$$C_1 = 2.$$

Thus

$$x(t) = 2e^{-2t} - e^{-4t}.$$

Monotone decay follows since both exponentials decrease.

$$\boxed{x(t) = 2e^{-2t} - e^{-4t}.}$$

Answer 27: Resonance

$$x'' + cx' + 4x = 2\cos(2t).$$

Homogeneous:

$$r^2 + cr + 4 = 0, \quad x_h = e^{-ct/2}\big(C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)\big),$$

with

$$\omega_d = \sqrt{4 - \frac{c^2}{4}}.$$

Assume c small. Try

$$x_p = A\cos(2t) + B\sin(2t).$$

Substitute into full ODE, solve linear system (omitted algebra):

$$A = \frac{8 - 2c^2}{(4-4)^2 + (2c \cdot 2)^2} = \frac{8 - 2c^2}{16c^2}, \qquad B = \frac{4c}{16c^2} = \frac{1}{4c}.$$

The particular solution is finite for $c > 0$.
For $c = 0$:

$$x'' + 4x = 2\cos(2t).$$

Resonant forcing: try

$$x_p = t(A\sin 2t).$$

Compute and find

$$A = \frac{1}{2}.$$

Thus

$$x(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{t}{2}\sin 2t,$$

whose amplitude grows without bound.

$$\boxed{x_{c=0}(t) = C_1 \cos 2t + C_2 \sin 2t + \tfrac{t}{2}\sin 2t.}$$

Answer 28: RLC Circuits

Given

$$Lq'' + Rq' + \frac{1}{C}q = E_0 \cos(\omega t).$$

Kirchhoff voltage law gives the stated equation.
For $L = 1$, $R = 2$, $C = \frac{1}{5}$, $E_0 = 10$:

$$q'' + 2q' + 5q = 10\cos(\omega t).$$

Homogeneous:

$$r^2 + 2r + 5 = 0, \quad r = -1 \pm 2i.$$

Thus

$$q_h = e^{-t}(C_1 \cos 2t + C_2 \sin 2t).$$

Steady-state (trial):

$$q_p = A\cos(\omega t) + B\sin(\omega t).$$

(Algebra omitted) One obtains:

$$A = \frac{10(5 - \omega^2)}{(5 - \omega^2)^2 + (2\omega)^2}, \qquad B = \frac{20\omega}{(5 - \omega^2)^2 + (2\omega)^2}.$$

$$\boxed{q(t) = e^{-t}(C_1 \cos 2t + C_2 \sin 2t) + A\cos(\omega t) + B\sin(\omega t).}$$

Answer 29: Population Models

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right).$$

Separate:

$$\frac{dP}{P(1 - P/K)} = r\,dt.$$

Use partial fractions:

$$\frac{1}{P(1 - P/K)} = \frac{1}{P} + \frac{1}{K - P}.$$

Integrate:

$$\ln|P| - \ln|K - P| = rt + C.$$

$$\frac{P}{K - P} = Ce^{rt}.$$

Solve:

$$P = \frac{KCe^{rt}}{1 + Ce^{rt}}.$$

Use $P(0) = P_0$:

$$\frac{P_0}{K - P_0} = C.$$

Thus

$$P(t) = \frac{K}{1 + \left(\frac{K - P_0}{P_0}\right)e^{-rt}}.$$

For numerical values $r = 0.5, K = 1000, P_0 = 100$:

Solve $P(t) = 500$:

$$500 = \frac{1000}{1 + 9e^{-0.5t}} \Rightarrow 1 + 9e^{-0.5t} = 2 \Rightarrow 9e^{-0.5t} = 1$$

$$e^{-0.5t} = \frac{1}{9} \Rightarrow t = 2\ln 9.$$

$$\boxed{P(t) = \frac{1000}{1 + 9e^{-0.5t}}, \qquad t = 2\ln 9.}$$

Answer 30: Mixing Problems

Salt amount $S(t)$.

Rate in:

$$0.3 \cdot 5 = 1.5 \text{ kg/min.}$$

Rate out:

$$\frac{S(t)}{100} \cdot 5 = \frac{S(t)}{20}.$$

ODE:

$$\frac{dS}{dt} = 1.5 - \frac{S}{20}.$$

Solve:

$$S' + \frac{1}{20}S = 1.5.$$

Integrating factor:

$$\mu = e^{t/20}.$$

$$\frac{d}{dt}\left(Se^{t/20}\right) = 1.5e^{t/20}.$$

Integrate:

$$Se^{t/20} = 30e^{t/20} + C.$$

Thus

$$S = 30 + Ce^{-t/20}.$$

Use $S(0) = 0$:

$$0 = 30 + C \Rightarrow C = -30.$$

$$S(t) = 30(1 - e^{-t/20}).$$

Limit:

$$\lim_{t \to \infty} S(t) = 30.$$

$$\boxed{S(t) = 30(1 - e^{-t/20}), \qquad S(\infty) = 30 \text{ kg.}}$$

Answer 31: Newton's Cooling Law

$$\frac{dT}{dt} = -k(T - 20), \quad T(0) = 90.$$

Solve:

$$T(t) - 20 = Ce^{-kt}.$$

At $t = 0$: $90 - 20 = 70 = C$.

$$T(t) = 20 + 70e^{-kt}.$$

Given $T(10) = 60$:

$$60 = 20 + 70e^{-10k} \Rightarrow 40 = 70e^{-10k} \Rightarrow e^{-10k} = \frac{4}{7} \Rightarrow k = \frac{1}{10} \ln \frac{7}{4}.$$

Cooling to $T = 30$:

$$30 = 20 + 70e^{-kt} \Rightarrow 10 = 70e^{-kt} \Rightarrow e^{-kt} = \frac{1}{7}$$

$$t = \frac{1}{k} \ln 7 = \frac{10 \ln 7}{\ln(7/4)}.$$

$$\boxed{T(t) = 20 + 70e^{-kt}, \ k = \tfrac{1}{10} \ln(7/4), \ t = \tfrac{10 \ln 7}{\ln(7/4)}.}$$

Answer 32: Free Fall with Quadratic Drag

$$m\frac{dv}{dt} = mg - kv^2, \qquad v(0) = 0.$$

Separate:

$$\frac{dv}{g - (k/m)v^2} = dt.$$

Let $a = \sqrt{mg/k}$. Then

$$g - (k/m)v^2 = g\left(1 - \frac{v^2}{a^2}\right).$$

Integrate:

$$\int \frac{dv}{1 - v^2/a^2} = \int \frac{k}{m}dt.$$

This yields (standard integral):

$$\frac{a}{2}\ln\left|\frac{a+v}{a-v}\right| = gt.$$

Solve for v:

$$\frac{a+v}{a-v} = e^{2gt/a},$$

$$v = a\frac{e^{2gt/a} - 1}{e^{2gt/a} + 1} = a\tanh\left(\frac{gt}{a}\right).$$

Terminal velocity:

$$v_{\text{term}} = a = \sqrt{\frac{mg}{k}}.$$

Half-terminal velocity:

$$\frac{a}{2} = a\tanh\left(\frac{gt}{a}\right) \Rightarrow \tanh(\frac{gt}{a}) = \frac{1}{2}.$$

Thus

$$t_{1/2} = \frac{a}{g}\tanh^{-1}\left(\frac{1}{2}\right) = \frac{a}{2g}\ln\frac{3}{1}.]$$

$$\boxed{v(t) = \sqrt{\frac{mg}{k}}\tanh\left(t\sqrt{\frac{gk}{m}}\right), \qquad v_{\text{term}} = \sqrt{\frac{mg}{k}}, \qquad t_{1/2} = \frac{\sqrt{mg/k}}{2g}\ln 3.}$$

Fourier Series

Answer 1: Fourier Coefficients – Eulers Formulae

We have the 2π-periodic function

$$f(x) = \begin{cases} x, & -\pi < x < 0, \\ \pi - x, & 0 \le x \le \pi. \end{cases}$$

(a) Fourier coefficients.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi} \left(\int_{-\pi}^{0} x\, dx + \int_{0}^{\pi} (\pi - x)\, dx \right).$$

Compute each integral:

$$\int_{-\pi}^{0} x\, dx = \left[\frac{x^2}{2} \right]_{-\pi}^{0} = -\frac{\pi^2}{2}, \quad \int_{0}^{\pi} (\pi - x)\, dx = \left[\pi x - \frac{x^2}{2} \right]_{0}^{\pi} = \frac{\pi^2}{2},$$

so

$$a_0 = \frac{1}{2\pi} (0) = 0.$$

For $n \ge 1$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx = \frac{1}{\pi} \left(\int_{-\pi}^{0} x \cos(nx)\, dx + \int_{0}^{\pi} (\pi - x) \cos(nx)\, dx \right).$$

First integral (integration by parts):

$$\int x \cos(nx)\, dx = \frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2},$$

hence

$$\int_{-\pi}^{0} x \cos(nx)\, dx = \left[\frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right]_{-\pi}^{0} = \frac{1 - \cos(n\pi)}{n^2}.$$

Second integral:

$$\int_{0}^{\pi} (\pi - x) \cos(nx)\, dx = \pi \int_{0}^{\pi} \cos(nx)\, dx - \int_{0}^{\pi} x \cos(nx)\, dx.$$

Since $\int_{0}^{\pi} \cos(nx)\, dx = 0$ and

$$\int_{0}^{\pi} x \cos(nx)\, dx = \left[\frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right]_{0}^{\pi} = \frac{\cos(n\pi) - 1}{n^2},$$

we obtain

$$\int_{0}^{\pi} (\pi - x) \cos(nx)\, dx = -\frac{\cos(n\pi) - 1}{n^2} = \frac{1 - \cos(n\pi)}{n^2}.$$

Thus

$$a_n = \frac{1}{\pi} \left(\frac{1 - \cos(n\pi)}{n^2} + \frac{1 - \cos(n\pi)}{n^2} \right) = \frac{2}{\pi} \frac{1 - \cos(n\pi)}{n^2} = \frac{2}{\pi} \frac{1 - (-1)^n}{n^2}.$$

For b_n:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx = \frac{1}{\pi} \left(\int_{-\pi}^{0} x \sin(nx)\, dx + \int_{0}^{\pi} (\pi - x) \sin(nx)\, dx \right).$$

Integration by parts:

$$\int x \sin(nx)\, dx = -\frac{x \cos(nx)}{n} + \frac{\sin(nx)}{n^2},$$

so

$$\int_{-\pi}^{0} x \sin(nx)\, dx = \left[-\frac{x \cos(nx)}{n} + \frac{\sin(nx)}{n^2} \right]_{-\pi}^{0} = -\frac{\pi \cos(n\pi)}{n}.$$

Next,

$$\int_{0}^{\pi} (\pi - x) \sin(nx)\, dx = \pi \int_{0}^{\pi} \sin(nx)\, dx - \int_{0}^{\pi} x \sin(nx)\, dx.$$

Now

$$\int_{0}^{\pi} \sin(nx)\, dx = \left[-\frac{\cos(nx)}{n} \right]_{0}^{\pi} = \frac{1 - \cos(n\pi)}{n},$$

and

$$\int_{0}^{\pi} x \sin(nx)\, dx = \left[-\frac{x \cos(nx)}{n} + \frac{\sin(nx)}{n^2} \right]_{0}^{\pi} = -\frac{\pi \cos(n\pi)}{n}.$$

Thus

$$\int_{0}^{\pi} (\pi - x) \sin(nx)\, dx = \pi \frac{1 - \cos(n\pi)}{n} + \frac{\pi \cos(n\pi)}{n} = \frac{\pi}{n}.$$

Therefore

$$b_n = \frac{1}{\pi} \left(-\frac{\pi \cos(n\pi)}{n} + \frac{\pi}{n} \right) = \frac{1 - \cos(n\pi)}{n} = \frac{1 - (-1)^n}{n}.$$

So

$$\boxed{a_0 = 0, \quad a_n = \frac{2}{\pi} \frac{1 - (-1)^n}{n^2}, \quad b_n = \frac{1 - (-1)^n}{n}}$$

(b) Fourier series.
For even n, $1 - (-1)^n = 0$, so $a_n = b_n = 0$. For odd $n = 2k + 1$,

$$a_{2k+1} = \frac{4}{\pi(2k + 1)^2}, \qquad b_{2k+1} = \frac{2}{2k + 1}.$$

Hence

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{4}{\pi(2k + 1)^2} \cos\big((2k + 1)x\big) + \frac{2}{2k + 1} \sin\big((2k + 1)x\big) \right).$$

$$\boxed{f(x) = \sum_{k=0}^{\infty} \left(\frac{4}{\pi(2k + 1)^2} \cos\big((2k + 1)x\big) + \frac{2}{2k + 1} \sin\big((2k + 1)x\big) \right)}$$

(c) Convergence at $x = 0$.

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The one-sided limits are

$$\lim_{x \to 0^-} f(x) = 0, \qquad \lim_{x \to 0^+} f(x) = \pi.$$

Thus f has a jump at $x = 0$. The Fourier series converges at a jump to the midpoint:

$$\lim_{N \to \infty} S_N(0) = \frac{0 + \pi}{2} = \frac{\pi}{2}.$$

$$\boxed{\text{The series converges at } x = 0 \text{ to } \frac{\pi}{2}.}$$

Answer 2: Even and Odd Functions

We are given $f(x) = x(\pi - x)$ on $0 \le x \le \pi$, extended periodically.

(a) **Choice of extension.**

On $[0, \pi]$, $f(0) = f(\pi) = 0$ and $f(x) \ge 0$. Extending evenly via $f(-x) = f(x)$ produces a continuous, non-negative even function on $[-\pi, \pi]$, leading to a cosine-only Fourier series. This is simpler than the odd extension, which would change sign.

$$\boxed{\text{The even extension is the more natural choice.}}$$

(b) **Vanishing coefficients.**

For the even extension, $f(x)$ is even on $[-\pi, \pi]$, so

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx = 0, \quad \forall n \ge 1.$$

Thus

$$\boxed{b_n = 0 \text{ for all } n, \text{ only } a_0, a_n \ne 0.}$$

(c) **Non-zero coefficients and series.**

For an even function,

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx, \qquad a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx.$$

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First,

$$a_0 = \frac{1}{\pi} \int_0^\pi x(\pi - x)\, dx = \frac{1}{\pi} \int_0^\pi (\pi x - x^2)\, dx = \frac{1}{\pi} \left[\frac{\pi x^2}{2} - \frac{x^3}{3} \right]_0^\pi = \frac{1}{\pi} \left(\frac{\pi^3}{2} - \frac{\pi^3}{3} \right) = \frac{\pi^2}{3}.$$

Next, for $n \geq 1$,

$$a_n = \frac{2}{\pi} \int_0^\pi x(\pi - x) \cos(nx)\, dx = \frac{2}{\pi} \left(\pi \int_0^\pi x \cos(nx)\, dx - \int_0^\pi x^2 \cos(nx)\, dx \right).$$

Compute

$$\int_0^\pi x \cos(nx)\, dx = \left[\frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right]_0^\pi = \frac{\cos(n\pi) - 1}{n^2},$$

and, after two integrations by parts,

$$\int_0^\pi x^2 \cos(nx)\, dx = \left[\frac{x^2 \sin(nx)}{n} \right]_0^\pi - \frac{2}{n} \int_0^\pi x \sin(nx)\, dx = -\frac{2\pi \cos(n\pi)}{n^2}.$$

So

$$a_n = \frac{2}{\pi} \left(\pi \frac{\cos(n\pi) - 1}{n^2} + \frac{2\pi \cos(n\pi)}{n^2} \right) = \frac{2}{\pi} \cdot \frac{\pi}{n^2} \left(3 \cos(n\pi) - 1 \right).$$

Using $\cos(n\pi) = (-1)^n$ and simplifying,

$$a_n = \frac{2}{n^2} \left(3(-1)^n - 1 \right) = 2 \frac{(-1)^{n+1} - 1}{n^2}.$$

Equivalently,

$$a_n = \begin{cases} 0, & n \text{ even}, \\ -\dfrac{4}{n^2}, & n \text{ odd}. \end{cases}$$

Thus the Fourier series on $[-\pi, \pi]$ and hence on $[0, \pi]$ is

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^\infty a_n \cos(nx) = \frac{\pi^2}{3} - \sum_{k=0}^\infty \frac{4}{(2k+1)^2} \cos\left((2k+1)x \right).$$

$$\boxed{f(x) = \frac{\pi^2}{3} - \sum_{k=0}^\infty \frac{4}{(2k+1)^2} \cos\left((2k+1)x \right), \quad b_n = 0.}$$

Answer 3: Odd and Even Extensions

We have $g(x) = x$ on $0 \le x \le \pi$.

(a) Odd extension and Fourier series.

For the odd periodic extension g_{odd}, we define

$$g_{\text{odd}}(x) = \begin{cases} x, & 0 < x \le \pi, \\ -g_{\text{odd}}(-x), & -\pi \le x < 0, \end{cases}$$

extended 2π-periodically. This is an odd function, so its Fourier series is

$$g_{\text{odd}}(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \quad b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx)\, dx.$$

Integration by parts:

$$\int_0^{\pi} x \sin(nx)\, dx = \left[-\frac{x\cos(nx)}{n} \right]_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos(nx)\, dx = -\frac{\pi \cos(n\pi)}{n}.$$

Thus

$$b_n = \frac{2}{\pi} \left(-\frac{\pi \cos(n\pi)}{n} \right) = -\frac{2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}.$$

So

$$\boxed{g_{\text{odd}}(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx).}$$

(b) Even extension and Fourier series.

For the even extension,

$$g_{\text{even}}(x) = |x|, \quad -\pi \le x \le \pi,$$

extended 2π-periodically. This is even, so

$$g_{\text{even}}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx), \quad a_0 = \frac{2}{\pi} \int_0^{\pi} x\, dx, \quad a_n = \frac{2}{\pi} \int_0^{\pi} x\cos(nx)\, dx.$$

We already have

$$\int_0^{\pi} x\, dx = \frac{\pi^2}{2}, \quad \int_0^{\pi} x\cos(nx)\, dx = \frac{\cos(n\pi) - 1}{n^2}.$$

Hence

$$a_0 = \frac{2}{\pi} \cdot \frac{\pi^2}{2} = \pi, \quad a_n = \frac{2}{\pi} \cdot \frac{\cos(n\pi) - 1}{n^2} = \frac{2}{\pi} \cdot \frac{(-1)^n - 1}{n^2}.$$

Thus

$$g_{\text{even}}(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{(-1)^n - 1}{n^2} \cos(nx).$$

Since $(-1)^n - 1 = 0$ for even n and -2 for odd $n = 2k + 1$,

$$g_{\text{even}}(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos\big((2k+1)x\big)}{(2k+1)^2}.$$

$$\boxed{g_{\text{even}}(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos\big((2k+1)x\big)}{(2k+1)^2}.}$$

(c) Infinite series representations of π.
For the odd extension, evaluate at $x = \pi/2$:

$$g_{\text{odd}}\left(\frac{\pi}{2}\right) = \frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{2}\right).$$

Only odd $n = 2k + 1$ contribute (even terms have $\sin(n\pi/2) = 0$). Using $\sin\left(\frac{(2k+1)\pi}{2}\right) = (-1)^k$, we get

$$\frac{\pi}{2} = 2\sum_{k=0}^{\infty} \frac{(-1)^{k+1}(-1)^k}{2k+1} = 2\sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{2k+1} = 2\sum_{k=0}^{\infty} \frac{-1}{2k+1},$$

so

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$

Thus

$$\boxed{\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right).}$$

For the even extension, evaluate at $x = 0$:

$$g_{\text{even}}(0) = 0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2},$$

hence

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

Therefore,

$$\pi^2 = 8\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \quad \Rightarrow \quad \pi = \sqrt{8\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}}.$$

$$\boxed{\pi = \sqrt{8\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right)}.}$$

Answer 4: Parsevals Theorem

We have $f(x) = x$ on $(-\pi, \pi]$, extended 2π-periodically.

(a) Fourier series.

f is odd, so only sine terms:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \quad b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx)\, dx.$$

As before,

$$\int_0^{\pi} x \sin(nx)\, dx = -\frac{\pi \cos(n\pi)}{n},$$

hence

$$b_n = \frac{2}{\pi} \left(-\frac{\pi \cos(n\pi)}{n} \right) = \frac{2(-1)^{n+1}}{n}.$$

Thus

$$\boxed{f(x) = x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx).}$$

(b) Apply Parsevals theorem.

Parseval for 2π-periodic f:

$$\int_{-\pi}^{\pi} |f(x)|^2\, dx = 2\pi a_0^2 + \pi \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2 \right).$$

Here $a_0 = 0$, $a_n = 0$ and $b_n = \dfrac{2(-1)^{n+1}}{n}$, so

$$\int_{-\pi}^{\pi} x^2\, dx = \pi \sum_{n=1}^{\infty} \left(\frac{2(-1)^{n+1}}{n} \right)^2 = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

But

$$\int_{-\pi}^{\pi} x^2\, dx = 2 \int_0^{\pi} x^2\, dx = 2 \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2\pi^3}{3}.$$

So

$$\frac{2\pi^3}{3} = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(c) Sum of the series.

Divide by 4π:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^3/3}{4\pi} = \frac{\pi^2}{6}.$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.}$$

Answer 5: Periods Other Than 2π

We have period $2L = 4$ so $L = 2$, and $f(x) = x$ on $(-2, 2]$, extended periodically.

(a) General form for period $2L$.

For a $2L$-periodic function,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

with

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

> Use the above cosinesine expansion with frequency $\dfrac{n\pi}{L}$.

(b) Coefficients for $f(x) = x$, $L = 2$.

Here $f(x) = x$ is odd on $[-2, 2]$, so

$$a_0 = \frac{1}{4} \int_{-2}^{2} x\, dx = 0, \quad a_n = \frac{1}{2} \int_{-2}^{2} x \cos\left(\frac{n\pi x}{2}\right) dx = 0$$

(since integrand is odd). For b_n:

$$b_n = \frac{1}{2} \int_{-2}^{2} x \sin\left(\frac{n\pi x}{2}\right) dx.$$

The integrand is even (odd $\times$ odd), so

$$b_n = \frac{1}{2} \cdot 2 \int_{0}^{2} x \sin\left(\frac{n\pi x}{2}\right) dx = \int_{0}^{2} x \sin\left(\frac{n\pi x}{2}\right) dx.$$

Let $k = \dfrac{n\pi}{2}$. Then

$$\int_{0}^{2} x \sin(kx)\, dx = \left[-\frac{x \cos(kx)}{k}\right]_{0}^{2} + \frac{1}{k} \int_{0}^{2} \cos(kx)\, dx = -\frac{2\cos(2k)}{k} + \frac{1}{k}\left[\frac{\sin(kx)}{k}\right]_{0}^{2}.$$

Since $k = \dfrac{n\pi}{2}$, we have $2k = n\pi$ and $\sin(2k) = \sin(n\pi) = 0$, so

$$\int_{0}^{2} x \sin\left(\frac{n\pi x}{2}\right) dx = -\frac{2\cos(n\pi)}{k} = -\frac{2(-1)^n}{n\pi/2} = -\frac{4(-1)^n}{n\pi}.$$

Thus

$$b_n = -\frac{4(-1)^n}{n\pi} = \frac{4(-1)^{n+1}}{n\pi}.$$

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$$a_0 = 0, \quad a_n = 0, \quad b_n = \frac{4(-1)^{n+1}}{n\pi}, \quad n \geq 1.$$

(c) Explicit Fourier series.

Therefore,

$$f(x) = x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi x}{2}\right).$$

$$x = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi x}{2}\right), \quad -2 < x \leq 2.$$

Answer 6: Complex Exponential Form of Fourier Series

We consider

$$f(x) = x, \qquad -\pi < x \leq \pi,$$

extended periodically with period 2π.

(a) Complex Fourier coefficients. The complex Fourier coefficients are

$$d_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x\, e^{-inx}\, dx.$$

Integrating by parts with $u = x$, $dv = e^{-inx} dx$ gives $du = dx$, $v = \frac{e^{-inx}}{-in}$:

$$d_n = \frac{1}{2\pi} \left[\frac{x e^{-inx}}{-in} \right]_{-\pi}^{\pi} - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-inx}}{-in}\, dx$$

$$= \frac{1}{2\pi} \left(\frac{\pi e^{-in\pi} - (-\pi)e^{in\pi}}{-in} \right) + \frac{1}{2\pi in} \int_{-\pi}^{\pi} e^{-inx}\, dx.$$

But

$$\int_{-\pi}^{\pi} e^{-inx}\, dx = \left[\frac{e^{-inx}}{-in}\right]_{-\pi}^{\pi} = \frac{e^{-in\pi} - e^{in\pi}}{-in} = 0,$$

since $e^{-in\pi} = e^{in\pi} = (-1)^n$. Hence

$$d_n = \frac{1}{2\pi} \cdot \frac{\pi\left(e^{-in\pi} + e^{in\pi}\right)}{-in} = \frac{1}{2\pi} \cdot \frac{2\pi(-1)^n}{-in} = \frac{i(-1)^n}{n}, \qquad n \neq 0.$$

Also,

$$d_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x\, dx = 0.$$

$$\boxed{d_0 = 0, \qquad d_n = \frac{i(-1)^n}{n} \quad (n \neq 0).}$$

(b) Complex exponential Fourier series. Thus the complex Fourier expansion is

$$f(x) = \sum_{n \in \mathbb{Z}} d_n e^{inx} = \sum_{n \neq 0} \frac{i(-1)^n}{n} e^{inx}.$$

$$\boxed{f(x) = \sum_{n \neq 0} \frac{i(-1)^n}{n} e^{inx}.}$$

(c) Verification of relationship with real coefficients. For $n > 0$,

$$d_n = \frac{a_n - ib_n}{2}, \qquad d_{-n} = \frac{a_n + ib_n}{2}.$$

Since

$$d_n = \frac{i(-1)^n}{n}, \qquad d_{-n} = -\frac{i(-1)^n}{n},$$

we obtain

$$a_n = d_n + d_{-n} = 0,$$

$$b_n = i(d_{-n} - d_n) = i\left(-\frac{i(-1)^n}{n} - \frac{i(-1)^n}{n}\right) = \frac{2(-1)^{n+1}}{n},$$

which agrees with the known real Fourier series

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx).$$

$$\boxed{a_n = 0, \qquad b_n = \frac{2(-1)^{n+1}}{n}.}$$