L1 Mathematics Interim Assessment 1 Solutions

Prepared by Karsilo

Instructions

1. Time allow: 40 minutes.
2. **Use of electronic calculators is forbidden**.

Question 1: Find all the cube roots of unity ($z^3 = 1$) and represent them in the form $a + bi$ where $a, b \in \mathbb{R}$

Answer

Step 1. General formula for nth roots of unity

$$z_k = e^{\frac{2\pi ik}{n}}, \quad k = 0, 1, 2, ..., n-1$$

Step 2. For $n = 3$:

$$z_0 = e^{2\pi i \cdot 0/3} = 1$$
$$z_1 = e^{2\pi i \cdot 1/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$
$$z_2 = e^{2\pi i \cdot 2/3} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\boxed{z = 1,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i,\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i}$$

Question 2: Find the general solution $y = y(x)$ to the differential equation

$$\frac{dy}{dx} + 3y = 2y^4 e^{5x}$$

Answer

Step 1. Divide by y^4

$$y^{-4}\frac{dy}{dx} + 3y^{-3} = 2e^{5x}$$

Step 2. Let

$$f(x) = 2, \quad g(x) = 2e^{5x}, \quad n = 4, \quad z = y^{-3}$$

Step 3. Calculate $\dfrac{dz}{dx}$

$$\frac{dz}{dx} = -3y^{-4}\frac{dy}{dx}, \qquad y^{-4}\frac{dy}{dx} = -\frac{1}{3}\frac{dz}{dx}$$

Step 4. Substitute

$$-\frac{1}{3}\frac{dz}{dx} + 3z = 2e^{5x}$$

$$\frac{dz}{dx} - 9z = -6e^{5x}$$

Step 5. Integrating factor

$$\mu(x) = e^{\int -9\,dx} = e^{-9x}$$

$$e^{-9x}\frac{dz}{dx} - 9e^{-9x}z = -6e^{-4x}$$

Step 6. Integrate both sides

$$ze^{-9x} = \frac{3}{2}e^{-4x} + C$$

$$z = \frac{3}{2}e^{5x} + Ce^{9x}$$

Step 6. Back-substitute $z = y^{-3}$

$$\boxed{y = (\frac{3}{2}e^{5x} + Ce^{9x})^{-\frac{1}{3}}}$$

Question 3: Find the general solution $y = y(x)$ to the differential equation

$$y'' + 4y = \sin(2x)$$

Answer

Step 1. Solve the homogeneous equation $y'' + 4y = 0$
Characteristic equation:

$$r^2 + 4 = 0 \quad \Rightarrow \quad r = \pm 2i$$

Homogeneous solution:

$$y_h(x) = C_1 \cos(2x) + C_2 \sin(2x)$$

Step 2. Find a particular solution $y_p(x)$
Since $\sin(2x)$ and $\cos(2x)$ are solutions to the homogeneous equation, try

$$y_p(x) = x(A\cos(2x) + B\sin(2x))$$

Step 3. Compute derivatives

$$y_p'(x) = A\cos(2x) + B\sin(2x) + x(-2A\sin(2x) + 2B\cos(2x))$$

$$y_p''(x) = -2A\sin(2x) + 2B\cos(2x) + (-2A\sin(2x) + 2B\cos(2x)) + x(-4A\cos(2x) - 4B\sin(2x))$$

$$= -4A\sin(2x) + 4B\cos(2x) + x(-4A\cos(2x) - 4B\sin(2x))$$

Step 4. Substitute into $y_p'' + 4y_p$

$$y_p'' + 4y_p = -4A\sin(2x) + 4B\cos(2x) + x(-4A\cos(2x) - 4B\sin(2x)) + 4x(A\cos(2x) + B\sin(2x))$$

$$= -4A\sin(2x) + 4B\cos(2x)$$

Set equal to $\sin(2x)$:

$$4B\cos(2x) - 4A\sin(2x) = \sin(2x)$$

Coefficients:

$$4B = 0 \quad \Rightarrow \quad B = 0, \qquad -4A = 1 \quad \Rightarrow \quad A = -\frac{1}{4}$$

Step 5. Particular solution

$$y_p(x) = x\left(-\frac{1}{4}\cos(2x)\right) = -\frac{1}{4}x\cos(2x)$$

Step 6. General solution

$$y(x) = y_h(x) + y_p(x) = C_1\cos(2x) + C_2\sin(2x) - \frac{1}{4}x\cos(2x)$$

$$\boxed{y = C_1\cos(2x) + C_2\sin(2x) - \frac{1}{4}x\cos(2x)}$$

Question 4: Evaluate the following limit

$$\lim_{x \to 0} \frac{\sin 7x}{4x}$$

Answer

Step 1. Rewrite the expression to use the standard limit $\lim_{x \to 0} \frac{\sin x}{x} = 1$

$$\frac{\sin 7x}{4x} = \frac{\sin 7x}{7x} \cdot \frac{7x}{4x} = \frac{\sin 7x}{7x} \cdot \frac{7}{4}$$

Step 2. Take the limit as $x \to 0$

$$\lim_{x \to 0} \frac{\sin 7x}{4x} = \lim_{x \to 0} \left(\frac{\sin 7x}{7x} \cdot \frac{7}{4} \right) = \left(\lim_{x \to 0} \frac{\sin 7x}{7x} \right) \cdot \frac{7}{4}$$

Step 3. Apply the standard limit (let $u = 7x$, as $x \to 0$ then $u \to 0$)

$$\lim_{x \to 0} \frac{\sin 7x}{7x} = \lim_{u \to 0} \frac{\sin u}{u} = 1$$

Step 4. Conclude

$$\lim_{x \to 0} \frac{\sin 7x}{4x} = 1 \cdot \frac{7}{4} = \frac{7}{4}$$

$$\boxed{\frac{7}{4}}$$

Question 5: For $A \in \mathbb{R}$ let $f : \mathbb{C} \to \mathbb{C}$ be the function given by

$$f(z) = -Ax^2 - y^2 + x + 2ixy + iy$$

Find a specific value of A for which f satisfies the Cauchy-Riemann equations for any $(x, y) \in \mathbb{R}^2$

Answer

Step 1. Calculate u and v

$$u = -Ax^2 - y^2 + x, \qquad v = 2xy + y$$

Step 2. Calculate derivatives

$$\frac{\delta u}{\delta x} = -2Ax + 1, \qquad \frac{\delta v}{\delta y} = 2x + 1$$

$$\frac{\delta u}{\delta y} = -2y, \qquad \frac{\delta v}{\delta x} = 2y$$

Step 3. Evaluate Cauchy-Riemann equations

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} = -2Ax + 1 = 2x + 1$$

$$\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x} = -2y = -2y$$

hence

$$\boxed{A = -1}$$