Waves Answer Set

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Answer 1 (3)

Answer 1: For

$$x(t) = A\cos(\omega t + \phi) + B,$$

$$\dot{x}(t) = -A\omega\sin(\omega t + \phi),$$
$$\ddot{x}(t) = -A\omega^2\cos(\omega t + \phi).$$

But

$$x - B = A\cos(\omega t + \phi) \;\Rightarrow\; \ddot{x} = -\omega^2(x - B).$$

$$\boxed{\ddot{x} = -\omega^2\,(x - B)}$$

For simple harmonic motion about the origin we require

$$\ddot{x} = -\omega^2 x,$$

so comparing with the above gives

$$B = 0.$$

$$\boxed{B = 0 \text{ for SHM about the origin}}$$

The speed is

$$v(t) = \dot{x}(t) = -A\omega\sin(\omega t + \phi),$$

so the maximum speed occurs when $|\sin(\omega t + \phi)| = 1$:

$$v_{\text{max}} = A\omega = 0.040\,\text{m} \times 12.0\,\text{s}^{-1} = 0.48\,\text{m s}^{-1}.$$

$$\boxed{v_{\text{max}} = 4.8 \times 10^{-1}\ \text{m s}^{-1}}$$

Answer 2 (3)

Answer 2: In $y(x,t) = A\cos(kx - \omega t)$, increasing x by one wavelength λ must change the phase by 2π:

$$k(x+\lambda) - \omega t - kx + \omega t = k\lambda = 2\pi$$

so

$$\boxed{\lambda = \frac{2\pi}{k}}.$$

Similarly, increasing t by one period T at fixed x gives

$$kx - \omega(t+T) - kx + \omega t = -\omega T = -2\pi$$

so

$$\boxed{T = \frac{2\pi}{\omega}}.$$

The wave speed is $v = \lambda/T$:

$$v = \frac{2\pi/k}{2\pi/\omega} = \frac{\omega}{k}.$$

$$\boxed{v = \frac{\omega}{k}}$$

For $k = 12.5\ \mathrm{m}^{-1}$ and $\omega = 220\ \mathrm{rad\ s}^{-1}$:

$$\lambda = \frac{2\pi}{12.5} \approx 5.03 \times 10^{-1}\ \mathrm{m},$$

$$v = \frac{220}{12.5} = 17.6\ \mathrm{m\ s}^{-1},$$

$$f = \frac{\omega}{2\pi} \approx \frac{220}{2\pi} \approx 3.50 \times 10^{1}\ \mathrm{Hz}.$$

$$\boxed{\lambda \approx 0.503\ \mathrm{m}, \quad v = 17.6\ \mathrm{m\ s}^{-1}, \quad f \approx 35.0\ \mathrm{Hz}}$$

Answer 3 (4)

Answer 3: For

$$y(x,t) = A\sin(kx)\sin(\omega t),$$

the fixed end at $x = 0$ is automatically satisfied since $\sin(0) = 0$.

At $x = L$ we require

$$y(L,t) = A\sin(kL)\sin(\omega t) = 0 \quad \forall t.$$

Since $\sin(\omega t)$ is not identically zero, we need

$$\sin(kL) = 0 \;\Rightarrow\; kL = n\pi, \quad n = 1, 2, 3, \ldots$$

thus

$$\boxed{k_n = \frac{n\pi}{L}}.$$

For a wave speed v, the dispersion relation is

$$\omega_n = vk_n = v\frac{n\pi}{L}.$$

$$\boxed{\omega_n = \frac{n\pi v}{L}, \qquad n = 1, 2, 3, \ldots}$$

The corresponding frequencies are

$$f_n = \frac{\omega_n}{2\pi} = \frac{nv}{2L}.$$

With $L = 0.80$ m and $v = 120$ m s^{-1}:

$$f_1 = \frac{1 \times 120}{2 \times 0.80} = \frac{120}{1.6} = 75 \text{ Hz},$$

$$f_2 = 2f_1 = 150 \text{ Hz}, \qquad f_3 = 3f_1 = 225 \text{ Hz}.$$

$$\boxed{f_1 = 75 \text{ Hz}, \quad f_2 = 150 \text{ Hz}, \quad f_3 = 225 \text{ Hz}}$$

Answer 4 (4)

Answer 4: For a stretched string with tension T and linear density μ, the wave speed is

$$v = \sqrt{\frac{T}{\mu}}.$$

$$\boxed{v = \sqrt{\frac{T}{\mu}}}$$

With $T = 90$ N and $\mu = 2.0 \times 10^{-3}$ kg m^{-1}:

$$v = \sqrt{\frac{90}{2.0 \times 10^{-3}}} = \sqrt{4.5 \times 10^4} \approx 2.12 \times 10^2 \text{ m s}^{-1}.$$

$$\boxed{v \approx 2.1 \times 10^2 \text{ m s}^{-1}}$$

For a string of length L fixed at both ends, the fundamental (first harmonic) has wavelength $\lambda_1 = 2L$ and frequency

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}.$$

For $L = 0.75$ m and $v \approx 2.12 \times 10^2$ m s^{-1}:

$$f_1 = \frac{2.12 \times 10^2}{2 \times 0.75} = \frac{2.12 \times 10^2}{1.50} \approx 1.41 \times 10^2 \text{ Hz}.$$

$$\boxed{f_1 \approx 1.4 \times 10^2 \text{ Hz}}$$

If the tension is increased by a factor of 4, then

$$v' = \sqrt{\frac{4T}{\mu}} = 2\sqrt{\frac{T}{\mu}} = 2v,$$

so

$$f_1' = \frac{v'}{2L} = \frac{2v}{2L} = 2f_1.$$

Thus the fundamental frequency doubles.

$$\boxed{\text{New fundamental} = 2f_1 \text{ (frequency increases by a factor of 2)}}$$

Answer 5 (5)

Answer 5:

(a) The transverse displacement $y(x,t)$ of the string obeys the one-dimensional wave equation with fixed-end boundary conditions. The general solution can be written as a superposition of normal modes

$$y(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right] \sin\left(\frac{n\pi x}{L}\right),$$

where the $\sin(n\pi x/L)$ are standing-wave solutions that individually satisfy the boundary conditions. Any initial shape (including the triangular one) can be expanded in this basis, so the subsequent motion is a superposition of standing waves.

$$\boxed{\text{Motion is a superposition of normal modes } \sin(n\pi x/L)}$$

(b) Take the initial displacement at $t = 0$ as

$$y(x,0) = \begin{cases} \dfrac{2A}{L}x, & 0 \le x \le \dfrac{L}{2}, \\ \dfrac{2A}{L}(L-x), & \dfrac{L}{2} \le x \le L. \end{cases}$$

Expanding in normal modes,

$$y(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right),$$

with

$$a_n = \frac{2}{L}\int_0^L y(x,0)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Evaluating the integral piecewise and simplifying gives

$$a_n = \frac{8A}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right).$$

For even n, $\sin(n\pi/2) = 0$, so $a_n = 0$. For odd n, $\sin(n\pi/2) = \pm 1$, so $a_n \neq 0$. Thus only odd-n modes appear.

$$\boxed{\text{Only odd harmonics } n = 1, 3, 5, \ldots \text{ have nonzero amplitude}}$$

(c) The normal-mode frequencies are $f_n = n f_1$ but only odd n occur. With $f_1 = 110$ Hz, the next three present modes are

$$f_3 = 3f_1 = 330 \text{ Hz}, \quad f_5 = 5f_1 = 550 \text{ Hz}, \quad f_7 = 7f_1 = 770 \text{ Hz}.$$

$$\boxed{f_3 = 330 \text{ Hz}, \quad f_5 = 550 \text{ Hz}, \quad f_7 = 770 \text{ Hz}}$$

(d) The vibrating string drives the guitar body and surrounding air at the same oscillation frequency. The air then supports a sound wave with the same frequency as the string, but with a different wave speed v_{air}, so its wavelength is different:

$$f_{\text{sound}} = f_{\text{string}}, \qquad \lambda_{\text{sound}} = \frac{v_{\text{air}}}{f_{\text{sound}}} \neq \lambda_{\text{string}}.$$

$$\boxed{\text{Frequency is the same; wavelength in air is different from that on the string}}$$