

# Oscillations Answer Set

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## Answer 1 (3)

Answer 1:

1. The restoring force is Hookes law,

$$F = -kx = m\ddot{x}.$$

Hence

$$m\ddot{x} + kx = 0 \Rightarrow \ddot{x} + \omega^2 x = 0, \quad \omega = \sqrt{\frac{k}{m}}.$$

Thus the motion is SHM with

$$\omega = \sqrt{\frac{50}{0.40}} = \sqrt{125} \approx 11.2 \text{ rad s}^{-1}, \quad T = \frac{2\pi}{\omega} \approx \frac{2\pi}{11.2} \approx 5.6 \times 10^{-1} \text{ s}.$$

$$\boxed{\omega \approx 1.12 \times 10^1 \text{ rad s}^{-1}}, \quad \boxed{T \approx 5.6 \times 10^{-1} \text{ s}}$$

2. For SHM, the speed as a function of displacement is

$$v = \omega\sqrt{A^2 - x^2}.$$

Here  $A = 0.12 \text{ m}$  and  $x = 0.060 \text{ m}$ :

$$v = 11.2\sqrt{0.12^2 - 0.060^2} = 11.2\sqrt{0.0144 - 0.0036} = 11.2\sqrt{0.0108} \approx 1.2 \text{ m s}^{-1}.$$

$$\boxed{v(x = 0.060 \text{ m}) \approx 1.2 \text{ m s}^{-1}}$$

3. Total energy

$$E = \frac{1}{2}kA^2 = \frac{1}{2} \cdot 50 \cdot 0.12^2 = 25 \cdot 0.0144 = 0.36 \text{ J}.$$

At  $x = 0.060 \text{ m}$ ,

$$U = \frac{1}{2}kx^2 = 25 \cdot 0.060^2 = 25 \cdot 0.0036 = 0.09 \text{ J},$$

so

$$K = E - U = 0.36 - 0.09 = 0.27 \text{ J}.$$

Hence

$$\frac{K}{U} = \frac{0.27}{0.09} = 3.$$

$$\boxed{E = 0.36 \text{ J}},$$

$$\boxed{\frac{K}{U} = 3 : 1 \text{ at } x = 0.060 \text{ m}}$$

## Answer 2 (3)

Answer 2:

1. The exact equation of motion is

$$mL\ddot{\theta} + mg \sin \theta = 0.$$

For small angles,  $\sin \theta \simeq \theta$ , so

$$mL\ddot{\theta} + mg\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \frac{g}{L}\theta = 0,$$

which is SHM with

$$\omega_0 = \sqrt{\frac{g}{L}}.$$

$$\boxed{\omega_0 = \sqrt{\frac{g}{L}}}$$

2. The ideal period at  $g_0$  is  $T_0 = 2\pi\sqrt{L_0/g_0}$ . At the mountain ( $g$ ), with the same length  $L_0$  the actual period is

$$T_m = 2\pi\sqrt{\frac{L_0}{g}} = T_0\sqrt{\frac{g_0}{g}}.$$

In real time  $t$  the number of swings is  $N = t/T_m$ , so the clock shows

$$t_{\text{clock}} = NT_0 = t\frac{T_0}{T_m} = t\sqrt{\frac{g}{g_0}}.$$

Hence the error after one day ( $t = 86400$  s) is

$$\Delta t = t - t_{\text{clock}} = t\left(1 - \sqrt{\frac{g}{g_0}}\right).$$

With  $g = 9.79$ ,  $g_0 = 9.81$ ,

$$\sqrt{\frac{g}{g_0}} \approx \sqrt{\frac{9.79}{9.81}} \approx 0.99898,$$

so

$$\Delta t \approx 86400(1 - 0.99898) \text{ s} \approx 8.8 \times 10^1 \text{ s}.$$

The clock loses time (runs slow) by about 88 s per day.

$$\boxed{\Delta t \approx 8.8 \times 10^1 \text{ s lost per day}}$$

3. To restore the original period at  $g$ , we require

$$T_0 = 2\pi\sqrt{\frac{L'}{g}} \quad \Rightarrow \quad L' = L_0 \frac{g}{g_0}.$$

Hence the required change in length is

$$\Delta L = L' - L_0 = L_0 \left( \frac{g}{g_0} - 1 \right) \approx L_0 (-2.0 \times 10^{-3}).$$

So the pendulum must be shortened by about 0.20%, i.e. by  $\approx 2.0$  mm for each metre of length.

$$\boxed{\Delta L \simeq -2.0 \times 10^{-3} L_0 \quad (\approx -2.0 \text{ mm per metre})}$$

### Answer 3 (4)

Answer 3:

1. For an underdamped oscillator,

$$x_n = A_0 e^{-\gamma t_n}, \quad t_{n+1} - t_n = T_d,$$

so

$$\frac{x_n}{x_{n+1}} = \frac{A_0 e^{-\gamma t_n}}{A_0 e^{-\gamma(t_n + T_d)}} = e^{\gamma T_d}.$$

Defining the logarithmic decrement  $\Lambda$  by

$$\Lambda = \ln \left( \frac{x_n}{x_{n+1}} \right),$$

we see that

$$\Lambda = \gamma T_d.$$

$$\boxed{\Lambda = \ln \left( \frac{x_n}{x_{n+1}} \right)}, \quad \boxed{\gamma = \frac{\Lambda}{T_d}}$$

2. Using  $x_1$  and  $x_5$ ,

$$\frac{x_1}{x_5} = e^{4\Lambda} \Rightarrow \Lambda = \frac{1}{4} \ln \left( \frac{x_1}{x_5} \right) = \frac{1}{4} \ln \left( \frac{4.0}{1.5} \right) \approx 0.245.$$

The undamped natural frequency is

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{16}{0.20}} = \sqrt{80} \approx 8.94 \text{ rad s}^{-1},$$

so the undamped period is

$$T_0 = \frac{2\pi}{\omega_0} \approx \frac{2\pi}{8.94} \approx 0.70 \text{ s}.$$

For weak damping we take  $T_d \simeq T_0$ , hence

$$\gamma \simeq \frac{\Lambda}{T_d} \approx \frac{0.245}{0.70} \approx 0.35 \text{ s}^{-1}.$$

With  $\gamma = b/(2m)$ ,

$$b = 2m\gamma = 2(0.20)(0.35) \approx 0.14 \text{ kg s}^{-1}.$$

$$\boxed{\gamma \approx 3.5 \times 10^{-1} \text{ s}^{-1}}, \quad \boxed{b \approx 1.4 \times 10^{-1} \text{ kg s}^{-1}}$$

3. The undamped natural frequency is

$$\boxed{\omega_0 \approx 8.94 \text{ rad s}^{-1}}.$$

The damped frequency is

$$\omega_d = \sqrt{\omega_0^2 - \gamma^2} \approx \sqrt{80 - 0.35^2} \approx \sqrt{79.9} \approx 8.94 \text{ rad s}^{-1}.$$

Since  $\gamma \ll \omega_0$ , the oscillator is clearly underdamped.

$$\boxed{\omega_d \approx 8.9 \text{ rad s}^{-1}},$$

oscillator is underdamped

4. The quality factor is

$$Q = \frac{\omega_0}{2\gamma} \approx \frac{8.94}{2 \times 0.35} \approx \frac{8.94}{0.70} \approx 12.8.$$

$$\boxed{Q \approx 1.3 \times 10^1}$$

## Answer 4 (4)

Answer 4:

1. Assume a steady-state solution  $x(t) = A \cos(\omega t - \phi)$ . Substituting into

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t)$$

and equating cosine and sine components leads to

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}},$$

where  $\omega_0^2 = k/m$  and  $\beta = b/(2m)$ .

$$\boxed{A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}}}$$

2. The amplitude is maximal when

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2\beta^2}.$$

Here

$$\omega_0^2 = \frac{100}{0.50} = 200, \quad \beta = \frac{b}{2m} = \frac{4.0}{1.0} = 4.0 \text{ s}^{-1},$$

so

$$\omega_{\text{res}} = \sqrt{200 - 2(4.0)^2} = \sqrt{200 - 32} = \sqrt{168} \approx 13.0 \text{ rad s}^{-1}.$$

$$\boxed{\omega_{\text{res}} \approx 1.3 \times 10^1 \text{ rad s}^{-1}}$$

3. First compute  $F_0/m$ :

$$\frac{F_0}{m} = \frac{2.0}{0.50} = 4.0 \text{ m s}^{-2}.$$

For  $\omega = 0.5\omega_0$ , note  $\omega_0 = \sqrt{200} \approx 14.1 \text{ rad s}^{-1}$ , so

$$\omega_1 = 0.5\omega_0 \approx 7.07 \text{ rad s}^{-1}, \quad \omega_1^2 = 0.25\omega_0^2 = 50.$$

Then

$$\omega_0^2 - \omega_1^2 = 200 - 50 = 150, \quad 2\beta\omega_1 = 2(4.0)(7.07) \approx 56.6.$$

Hence

$$A_1 = \frac{4.0}{\sqrt{150^2 + 56.6^2}} \approx \frac{4.0}{1.60 \times 10^2} \approx 2.5 \times 10^{-2} \text{ m}.$$

The phase lag satisfies

$$\tan \phi_1 = \frac{2\beta\omega_1}{\omega_0^2 - \omega_1^2} = \frac{56.6}{150} \approx 0.38 \Rightarrow \phi_1 \approx 0.36 \text{ rad } (\approx 21^\circ).$$

$$\boxed{A_1 \approx 2.5 \times 10^{-2} \text{ m}}, \quad \boxed{\phi_1 \approx 0.36 \text{ rad}}$$

For  $\omega = 2\omega_0$ ,

$$\omega_2 = 2\omega_0 \approx 28.3 \text{ rad s}^{-1}, \quad \omega_2^2 = 4\omega_0^2 = 800.$$

Then

$$\omega_0^2 - \omega_2^2 = 200 - 800 = -600, \quad 2\beta\omega_2 = 2(4.0)(28.3) \approx 2.26 \times 10^2.$$

Hence

$$A_2 = \frac{4.0}{\sqrt{(-600)^2 + 226^2}} \approx \frac{4.0}{6.41 \times 10^2} \approx 6.2 \times 10^{-3} \text{ m}.$$

The phase lag:

$$\tan \phi_2 = \frac{2\beta\omega_2}{\omega_0^2 - \omega_2^2} \approx \frac{226}{-600} = -0.38,$$

so  $\phi_2$  lies in the second quadrant,

$$\phi_2 \approx \pi - 0.36 \approx 2.78 \text{ rad } (\approx 159^\circ).$$

$$\boxed{A_2 \approx 6.2 \times 10^{-3} \text{ m}}, \quad \boxed{\phi_2 \approx 2.78 \text{ rad}}$$

4. Velocity amplitude is  $V(\omega) = \omega A(\omega)$ .

For  $\omega_1 = 0.5\omega_0$ ,

$$V_1 = \omega_1 A_1 \approx 7.07 \times 2.5 \times 10^{-2} \approx 1.8 \times 10^{-1} \text{ m s}^{-1}.$$

For  $\omega_2 = 2\omega_0$ ,

$$V_2 = \omega_2 A_2 \approx 28.3 \times 6.2 \times 10^{-3} \approx 1.8 \times 10^{-1} \text{ m s}^{-1}.$$

Numerically these are equal; indeed the symmetry of the expression for  $V(\omega)$  about  $\omega_0$  implies

$$V(\tfrac{1}{2}\omega_0) = V(2\omega_0).$$

$$\boxed{V_1 \approx V_2 \approx 1.8 \times 10^{-1} \text{ m s}^{-1} \text{ (equal for the two chosen frequencies)}}$$



## Answer 5 (5)

Answer 5:

1. Let  $x_1, x_2$  be displacements to the right from equilibrium.

For mass  $m_1$ : Left spring force:  $-kx_1$ ; middle spring force:  $-k(x_1 - x_2)$  (restoring toward equilibrium). Thus

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2) = -2kx_1 + kx_2.$$

For mass  $m_2$ : Right spring:  $-kx_2$ ; middle spring:  $-k(x_2 - x_1)$ . Hence

$$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1) = kx_1 - 2kx_2.$$

$$\boxed{m\ddot{x}_1 = -2kx_1 + kx_2, \quad m\ddot{x}_2 = kx_1 - 2kx_2}$$

2. Take normal-mode solutions  $x_j(t) = A_j \cos(\omega t)$ . Then

$$-m\omega^2 A_1 = -2kA_1 + kA_2, \quad -m\omega^2 A_2 = kA_1 - 2kA_2.$$

This gives the homogeneous system

$$\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \mathbf{0}.$$

For nontrivial solutions, the determinant must vanish:

$$(2k - m\omega^2)^2 - k^2 = 0 \quad \Rightarrow \quad 2k - m\omega^2 = \pm k.$$

Hence

$$2k - m\omega^2 = k \Rightarrow m\omega^2 = k, \quad 2k - m\omega^2 = -k \Rightarrow m\omega^2 = 3k,$$

giving

$$\boxed{\omega_1 = \sqrt{\frac{k}{m}}}, \quad \boxed{\omega_2 = \sqrt{\frac{3k}{m}}}.$$

3. For  $\omega_1^2 = k/m$ :

$$-m\omega_1^2 A_1 = -2kA_1 + kA_2 \Rightarrow -kA_1 = -2kA_1 + kA_2 \Rightarrow kA_1 = kA_2 \Rightarrow A_2 = A_1.$$

Thus both masses move in phase with equal amplitude (rigid translation of the centre spring).

For  $\omega_2^2 = 3k/m$ :

$$-3kA_1 = -2kA_1 + kA_2 \Rightarrow -kA_1 = kA_2 \Rightarrow A_2 = -A_1.$$

Here the masses move with equal amplitude but in opposite directions about the centre.

$$\boxed{\omega_1 : A_2/A_1 = +1 \text{ (in phase)}}, \quad \boxed{\omega_2 : A_2/A_1 = -1 \text{ (out of phase)}}$$

4. Superpose the two normal modes. Choose mode amplitudes  $C_1$  and  $C_2$  so that

$$x_1(t) = C_1 \cos(\omega_1 t) + C_2 \cos(\omega_2 t),$$

$$x_2(t) = C_1 \cos(\omega_1 t) - C_2 \cos(\omega_2 t),$$

consistent with the amplitude ratios above.

Initial conditions at  $t = 0$ :

$$x_1(0) = C_1 + C_2 = X_0, \quad x_2(0) = C_1 - C_2 = 0,$$

giving  $C_1 = C_2 = X_0/2$ . Velocities vanish automatically since  $\dot{x}_j(0) \propto \sin(0) = 0$ .

Therefore

$$x_1(t) = \frac{X_0}{2} [\cos(\omega_1 t) + \cos(\omega_2 t)], \quad x_2(t) = \frac{X_0}{2} [\cos(\omega_1 t) - \cos(\omega_2 t)].$$

Using the identities

$$\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}, \quad \cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2},$$

we obtain

$$x_1(t) = X_0 \cos \left( \frac{\omega_1 + \omega_2}{2} t \right) \cos \left( \frac{\omega_1 - \omega_2}{2} t \right),$$

$$x_2(t) = -X_0 \sin \left( \frac{\omega_1 + \omega_2}{2} t \right) \sin \left( \frac{\omega_1 - \omega_2}{2} t \right).$$

The fast oscillation at average frequency  $(\omega_1 + \omega_2)/2$  is modulated by a slow envelope with beat frequency  $|\omega_1 - \omega_2|/(2\pi)$ , corresponding to periodic transfer of energy between the two masses.

$x_1(t) = X_0 \cos \left( \frac{\omega_1 + \omega_2}{2} t \right) \cos \left( \frac{\omega_1 - \omega_2}{2} t \right),$ $x_2(t) = -X_0 \sin \left( \frac{\omega_1 + \omega_2}{2} t \right) \sin \left( \frac{\omega_1 - \omega_2}{2} t \right)$
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### Answer 6 (3)

For a torsional pendulum,

$$\omega = \sqrt{\frac{\kappa}{I}} \quad \Rightarrow \quad \kappa = I\omega^2.$$

For a uniform solid cylinder about its central axis,

$$I = \frac{1}{2}MR^2 = \frac{1}{2}(1.8)(0.12)^2 = \frac{1}{2}(1.8)(0.0144) = 0.01296 \text{ kg m}^2.$$

Given  $\omega = 4.6 \text{ rad s}^{-1}$ ,

$$\kappa = 0.01296 \times (4.6)^2 = 0.01296 \times 21.16 \approx 2.74 \times 10^{-1} \text{ N m rad}^{-1}.$$

$\kappa \approx 2.7 \times 10^{-1} \text{ N m rad}^{-1}$
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### Answer 7 (4)

The period of the pendulum is

$$T = \frac{\tau}{N} = \frac{402 \text{ s}}{200} = 2.01 \text{ s}.$$

1. For a simple pendulum,

$$T = 2\pi\sqrt{\frac{L}{g}} \quad \Rightarrow \quad g = \frac{4\pi^2 L}{T^2}.$$

Substituting  $T = 2.01 \text{ s}$ ,

$$g = \frac{4\pi^2 L}{(2.01)^2} \approx 9.77 L \text{ m s}^{-2} \text{ m}^{-1},$$

so in terms of the length  $L$  (in metres),

$$\boxed{g \approx 9.8 \left( \frac{L}{1 \text{ m}} \right) \text{ m s}^{-2}}.$$

2. Increasing the length by 3% gives  $L' = 1.03L$ . Then

$$T' = 2\pi\sqrt{\frac{L'}{g}} = 2\pi\sqrt{\frac{1.03L}{g}} = T\sqrt{1.03}.$$

Thus the new time for 200 oscillations is

$$\tau' = 200T' = 200T\sqrt{1.03} = 402\sqrt{1.03} \approx 4.08 \times 10^2 \text{ s}.$$

$$\boxed{\tau' \approx 4.08 \times 10^2 \text{ s } (\approx 408 \text{ s})}$$

### Answer 8 (5)

Given  $L = 0.85 \text{ m}$ ,  $g = 9.81 \text{ m s}^{-2}$  and  $\theta_0 = 28^\circ$ .

First convert the angle to radians:

$$\theta_0 = 28^\circ \times \frac{\pi}{180^\circ} \approx 0.489 \text{ rad.}$$

The small-angle period is

$$T_0 = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{0.85}{9.81}} \approx 1.85 \text{ s.}$$

1. Using the given correction,

$$T \approx 2\pi\sqrt{\frac{L}{g}} \left(1 + \frac{\theta_0^2}{16}\right) = T_0 \left(1 + \frac{\theta_0^2}{16}\right).$$

Compute

$$\frac{\theta_0^2}{16} \approx \frac{(0.489)^2}{16} \approx 0.015.$$

Hence

$$T \approx 1.85 \times (1 + 0.015) \approx 1.88 \text{ s.}$$

$$\boxed{T \approx 1.88 \text{ s}}$$

2. The percentage error if the small-angle formula is used is

$$\% \text{ error} = \frac{T - T_0}{T} \times 100\% \approx \frac{1.88 - 1.85}{1.88} \times 100\% \approx 1.5\%.$$

$$\boxed{\text{Using the small-angle formula underestimates } T \text{ by } \approx 1.5\%}$$

## Answer 9 (4)

For a simple pendulum of length  $L = 0.65 \text{ m}$  released from  $\theta_0 = 18^\circ$ :

The vertical drop of the bob from release to the lowest point is

$$h = L(1 - \cos \theta_0).$$

1. Conservation of energy between the release point (all GPE) and the bottom (all KE) gives

$$mgh = \frac{1}{2}mv_{\max}^2 \quad \Rightarrow \quad v_{\max} = \sqrt{2gL(1 - \cos \theta_0)}.$$

Numerically,

$$v_{\max} = \sqrt{2(9.81)(0.65)(1 - \cos 18^\circ)} \approx 0.79 \text{ m s}^{-1}.$$

$$\boxed{v_{\max} \approx 0.79 \text{ m s}^{-1}}$$

2. Let  $\theta$  be the angle when  $v = \frac{1}{2}v_{\max}$ . Conservation of energy gives

$$mgL(1 - \cos \theta_0) = mgL(1 - \cos \theta) + \frac{1}{2}mv^2.$$

Divide by  $mgL$  and substitute  $v = v_{\max}/2$ :

$$1 - \cos \theta_0 = 1 - \cos \theta + \frac{1}{2} \frac{(v_{\max}/2)^2}{gL} = 1 - \cos \theta + \frac{1}{4} \frac{v_{\max}^2}{2gL}.$$

But  $v_{\max}^2 = 2gL(1 - \cos \theta_0)$ , so

$$\frac{v_{\max}^2}{2gL} = 1 - \cos \theta_0,$$

and therefore

$$1 - \cos \theta_0 = 1 - \cos \theta + \frac{1}{4}(1 - \cos \theta_0).$$

Rearrange:

$$1 - \cos \theta = \frac{3}{4}(1 - \cos \theta_0) \quad \Rightarrow \quad \cos \theta = 1 - \frac{3}{4}(1 - \cos \theta_0).$$

With  $\theta_0 = 18^\circ$ ,

$$\cos \theta \approx 1 - \frac{3}{4}(1 - \cos 18^\circ) \approx 0.963,$$

so

$$\theta \approx \cos^{-1}(0.963) \approx 15.6^\circ.$$

To the nearest degree,

$$\boxed{\theta \approx 16^\circ}$$

## Answer 10 (5)

A uniform rod of length  $L = 0.90$  m and mass  $M = 2.2$  kg is pivoted at a point 0.15 m from one end.

1. The centre of mass is at  $L/2 = 0.45$  m from either end, so the distance from the pivot to the centre of mass is

$$d = 0.45 - 0.15 = 0.30 \text{ m.}$$

The moment of inertia of a uniform rod about its centre is

$$I_{\text{cm}} = \frac{1}{12}ML^2 = \frac{1}{12}(2.2)(0.90)^2 \approx 0.1485 \text{ kg m}^2.$$

Using the parallel-axis theorem,

$$I = I_{\text{cm}} + Md^2 = 0.1485 + 2.2(0.30)^2 = 0.1485 + 0.198 \approx 0.3465 \text{ kg m}^2.$$

$$\boxed{I \approx 3.47 \times 10^{-1} \text{ kg m}^2}$$

2. For a physical pendulum,

$$\omega^2 = \frac{Mgd}{I} \quad \Rightarrow \quad \omega = \sqrt{\frac{Mgd}{I}}.$$

Substitute  $M = 2.2$  kg,  $g = 9.81 \text{ m s}^{-2}$ ,  $d = 0.30$  m and  $I = 0.3465 \text{ kg m}^2$ :

$$\omega = \sqrt{\frac{(2.2)(9.81)(0.30)}{0.3465}} \approx 4.32 \text{ rad s}^{-1}.$$

The period is

$$T = \frac{2\pi}{\omega} \approx \frac{2\pi}{4.32} \approx 1.45 \text{ s.}$$

$$\boxed{\omega \approx 4.3 \text{ rad s}^{-1}, \quad T \approx 1.45 \text{ s}}$$

3. For a simple pendulum with the same period,

$$\omega^2 = \frac{g}{L_{\text{eq}}} \quad \Rightarrow \quad L_{\text{eq}} = \frac{g}{\omega^2}.$$

Thus

$$L_{\text{eq}} = \frac{9.81}{(4.32)^2} \approx 0.525 \text{ m.}$$

$$\boxed{L_{\text{eq}} \approx 0.53 \text{ m}}$$

## Answer 11 (4)

1. For a simple pendulum,

$$f = \frac{1}{2\pi} \sqrt{\frac{g}{L}} \quad \Rightarrow \quad L = \frac{g}{(2\pi f)^2} = \frac{g}{4\pi^2 f^2}.$$

With  $f = 0.32 \text{ Hz}$  and  $g = 9.81 \text{ m s}^{-2}$ ,

$$L = \frac{9.81}{4\pi^2(0.32)^2} \approx 2.43 \text{ m}.$$

$$\boxed{L \approx 2.4 \text{ m}}$$

2. The angular frequency is

$$\omega = 2\pi f = 2\pi(0.32) \approx 2.01 \text{ rad s}^{-1}.$$

With amplitude  $A = 0.80 \text{ m}$ , the maximum speed is

$$v_{\max} = \omega A \approx (2.01)(0.80) \approx 1.61 \text{ m s}^{-1}.$$

The maximum kinetic energy is then

$$K_{\max} = \frac{1}{2} m v_{\max}^2 = \frac{1}{2} (5.0 \times 10^5) (1.61)^2 \approx 6.5 \times 10^5 \text{ J}.$$

$$\boxed{K_{\max} \approx 6.5 \times 10^5 \text{ J}}$$



## Answer 12 (3)

Answer 12:

1. The undamped natural angular frequency is

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{18}{0.20}} = \sqrt{90} \approx 9.5 \text{ rad s}^{-1}.$$

$$\boxed{\omega_0 \approx 9.5 \text{ rad s}^{-1}}$$

2. The damping ratio is

$$\zeta = \frac{b}{2\sqrt{mk}} = \frac{1.2}{2\sqrt{0.20 \times 18}} = \frac{1.2}{2\sqrt{3.6}} \approx \frac{1.2}{3.79} \approx 0.32.$$

Since  $0 < \zeta < 1$ , the motion is underdamped.

$$\boxed{\zeta \approx 0.32 (< 1, \text{ underdamped})}$$

3. For an underdamped oscillator the displacement is of the form

$$x(t) = Ae^{-\gamma t} \cos(\omega' t + \phi), \quad \gamma = \frac{b}{2m},$$

so the system oscillates with a frequency slightly less than  $\omega_0$  and an exponentially decaying amplitude.

$$\boxed{\text{Oscillatory motion with amplitude decaying exponentially in time}}$$

### Answer 13 (4)

Answer 13:

1.

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{1.00} = 2\pi \approx 6.28 \text{ rad s}^{-1},$$

$$\omega_d = \frac{2\pi}{T_d} = \frac{2\pi}{1.10} \approx 5.71 \text{ rad s}^{-1}.$$

$$\boxed{\omega_0 \approx 6.28 \text{ rad s}^{-1}}, \quad \boxed{\omega_d \approx 5.71 \text{ rad s}^{-1}}$$

2. Using

$$\omega_d^2 = \omega_0^2 - \left(\frac{b}{2m}\right)^2 \Rightarrow \left(\frac{b}{2m}\right)^2 = \omega_0^2 - \omega_d^2,$$

we have

$$\omega_0^2 \approx (2\pi)^2 \approx 39.48, \quad \omega_d^2 \approx 32.63,$$

so

$$\left(\frac{b}{2m}\right)^2 \approx 39.48 - 32.63 = 6.85,$$

$$\frac{b}{2m} \approx \sqrt{6.85} \approx 2.62 \text{ s}^{-1}.$$

With  $m = 0.40 \text{ kg}$ ,

$$b = 2m \frac{b}{2m} = 0.80 \times 2.62 \approx 2.1 \text{ kg s}^{-1}.$$

$$\boxed{b \approx 2.1 \text{ kg s}^{-1}}$$

3. The damping ratio is

$$\zeta = \frac{b}{2\sqrt{mk}} = \frac{2.1}{2\sqrt{0.40 \times 50}} \approx \frac{2.1}{8.94} \approx 0.23 < 1,$$

so the oscillator is lightly underdamped.

$$\boxed{\zeta \approx 0.23 \Rightarrow \text{lightly underdamped}}$$

## Answer 14 (4)

Answer 14:

1. The amplitude decays as

$$A(t) = Ae^{-\gamma t}.$$

We want  $A(t) = 0.25A$ , so

$$e^{-\gamma t} = 0.25 = \frac{1}{4} \quad \Rightarrow \quad -\gamma t = \ln \frac{1}{4} = -\ln 4,$$

giving

$$t = \frac{\ln 4}{\gamma} = \frac{1.386}{0.25} \approx 5.5 \text{ s.}$$

$$\boxed{t \approx 5.5 \text{ s}}$$

2. The energy is proportional to  $A^2$ , so

$$\frac{E}{E_0} = \left( \frac{A(t)}{A} \right)^2 = e^{-2\gamma t}.$$

At the time found above  $A(t)/A = 0.25$ , so

$$\frac{E}{E_0} = (0.25)^2 = 0.0625.$$

$$\boxed{\frac{E}{E_0} = 6.25 \times 10^{-2}}$$

3. The damped period is

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{9.0} \approx 0.70 \text{ s.}$$

The number of oscillations in time  $t$  is

$$N = \frac{t}{T_d} \approx \frac{5.5}{0.70} \approx 8.$$

$$\boxed{N \approx 8 \text{ oscillations}}$$

## Answer 15 (5)

Answer 15:

1. For a driven, damped oscillator the steadystate amplitude is

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}},$$

where  $\omega_0 = \sqrt{k/m}$  and  $\beta = b/(2m)$ .

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}}$$

- 2.

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{60}{0.30}} = \sqrt{200} \approx 14.1 \text{ rad s}^{-1},$$

$$\beta = \frac{b}{2m} = \frac{2.4}{0.60} = 4.0 \text{ s}^{-1}.$$

$$\omega_0 \approx 1.41 \times 10^1 \text{ rad s}^{-1}, \quad \beta = 4.0 \text{ s}^{-1}$$

3. The resonance (maximum amplitude) angular frequency is

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2\beta^2} = \sqrt{200 - 2 \times 16} = \sqrt{168} \approx 13.0 \text{ rad s}^{-1}.$$

$$\omega_{\text{res}} \approx 1.3 \times 10^1 \text{ rad s}^{-1}$$

4. With  $F_0/m = 1.5/0.30 = 5.0 \text{ m s}^{-2}$  and  $\omega = \omega_{\text{res}}$ ,

$$\omega_0^2 - \omega_{\text{res}}^2 = 200 - 168 = 32,$$

$$2\beta\omega_{\text{res}} = 2(4.0)(12.96) \approx 1.04 \times 10^2.$$

Hence

$$A_{\text{res}} = \frac{5.0}{\sqrt{32^2 + (2\beta\omega_{\text{res}})^2}} \approx \frac{5.0}{\sqrt{1024 + 1.08 \times 10^4}} \approx \frac{5.0}{1.09 \times 10^2} \approx 4.6 \times 10^{-2} \text{ m}.$$

$$A_{\text{res}} \approx 4.6 \times 10^{-2} \text{ m}$$

5. The phase lag  $\phi$  between displacement and driving force satisfies

$$\tan \phi = \frac{2\beta\omega}{\omega_0^2 - \omega^2}.$$

At  $\omega = \omega_{\text{res}}$  we have  $\omega_0^2 - \omega_{\text{res}}^2 = 2\beta^2$ , so

$$\tan \phi_{\text{res}} = \frac{2\beta\omega_{\text{res}}}{2\beta^2} = \frac{\omega_{\text{res}}}{\beta} \approx \frac{12.96}{4.0} \approx 3.24,$$

giving

$$\phi_{\text{res}} \approx \tan^{-1}(3.24) \approx 1.27 \text{ rad} \approx 73^\circ.$$

$$\phi_{\text{res}} \approx 1.3 \text{ rad (displacement lags driving force by } \approx 73^\circ)$$

## Answer 16 (5)

Answer 16:

1. Let  $x_1(t)$  and  $x_2(t)$  be the displacements of  $m_1$  and  $m_2$  from equilibrium (positive to the right).

For  $m_1$  the left spring exerts  $-kx_1$ , and the middle spring exerts  $-k(x_1 - x_2)$ , so

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2) = -2kx_1 + kx_2.$$

For  $m_2$  the right spring exerts  $-kx_2$ , and the middle spring exerts  $-k(x_2 - x_1)$ , giving

$$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1) = kx_1 - 2kx_2.$$

Thus the coupled equations are

$$\boxed{m\ddot{x}_1 = -2kx_1 + kx_2, \quad m\ddot{x}_2 = kx_1 - 2kx_2,}$$

which form a  $2 \times 2$  eigenvalue problem and hence support two normal modes.

2. Seek solutions of the form  $x_j(t) = A_j \cos(\omega t)$ . Then

$$-m\omega^2 A_1 = -2kA_1 + kA_2, \quad -m\omega^2 A_2 = kA_1 - 2kA_2.$$

This can be written as

$$\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0.$$

Nontrivial solutions require

$$(2k - m\omega^2)^2 - k^2 = 0 \quad \Rightarrow \quad 2k - m\omega^2 = \pm k.$$

Hence

$$2k - m\omega^2 = k \Rightarrow m\omega^2 = k, \quad 2k - m\omega^2 = -k \Rightarrow m\omega^2 = 3k,$$

giving

$$\boxed{\omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{3k}{m}}.}$$

3. For  $\omega_1$ :

$$-m\omega_1^2 A_1 = -2kA_1 + kA_2 \Rightarrow -kA_1 = -2kA_1 + kA_2 \Rightarrow kA_1 = kA_2 \Rightarrow A_2 = A_1,$$

so the two masses move in phase with equal amplitude (both left and right together).

For  $\omega_2$ :

$$-3kA_1 = -2kA_1 + kA_2 \Rightarrow -kA_1 = kA_2 \Rightarrow A_2 = -A_1,$$

so the masses move with equal amplitude but in opposite directions (out of phase by  $\pi$ ).

$$\boxed{\omega_1 : A_2 = A_1 \text{ (in phase)}, \quad \omega_2 : A_2 = -A_1 \text{ (out of phase)}}$$

4. With  $m = 0.50$  kg and  $k = 200$  N m<sup>-1</sup>,

$$\omega_1 = \sqrt{\frac{k}{m}} = \sqrt{\frac{200}{0.50}} = \sqrt{400} = 20 \text{ rad s}^{-1},$$

$$\omega_2 = \sqrt{\frac{3k}{m}} = \sqrt{\frac{600}{0.50}} = \sqrt{1200} \approx 34.6 \text{ rad s}^{-1}.$$

Corresponding frequencies:

$$f_1 = \frac{\omega_1}{2\pi} \approx \frac{20}{2\pi} \approx 3.2 \text{ Hz}, \quad f_2 = \frac{\omega_2}{2\pi} \approx \frac{34.6}{2\pi} \approx 5.5 \text{ Hz}.$$

$\omega_1 = 20 \text{ rad s}^{-1} \ (f_1 \approx 3.2 \text{ Hz}),$	$\omega_2 \approx 34.6 \text{ rad s}^{-1} \ (f_2 \approx 5.5 \text{ Hz})$
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**Answer 17 (3)**

For an LC circuit,

$$\omega = \frac{1}{\sqrt{LC}}.$$

With  $L = 2.5 \times 10^{-3} \text{ H}$  and  $C = 8.0 \times 10^{-9} \text{ F}$ ,

$$\omega = \frac{1}{\sqrt{(2.5 \times 10^{-3})(8.0 \times 10^{-9})}} \approx 2.24 \times 10^5 \text{ rad s}^{-1}.$$

$$\boxed{\omega \approx 2.2 \times 10^5 \text{ rad s}^{-1}}$$

The period is

$$T = \frac{2\pi}{\omega} \approx \frac{2\pi}{2.24 \times 10^5} \approx 2.8 \times 10^{-5} \text{ s} = 28 \mu\text{s},$$

and the frequency is

$$f = \frac{1}{T} \approx 3.6 \times 10^4 \text{ Hz}.$$

$$\boxed{T \approx 2.8 \times 10^{-5} \text{ s}},$$

$$\boxed{f \approx 3.6 \times 10^4 \text{ Hz}}$$

### Answer 18 (4)

The time between adjacent voltage peaks is the period:

$$T = (6.4 \text{ div}) \times (0.50 \times 10^{-3} \text{ s/div}) = 3.2 \times 10^{-3} \text{ s}.$$

Thus

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{3.2 \times 10^{-3}} \approx 1.96 \times 10^3 \text{ rad s}^{-1}.$$

For an LC circuit,

$$\omega = \frac{1}{\sqrt{LC}} \quad \Rightarrow \quad L = \frac{1}{\omega^2 C}.$$

With  $C = 12.0 \times 10^{-9} \text{ F}$ ,

$$L = \frac{1}{(1.96 \times 10^3)^2 (12.0 \times 10^{-9})} \approx 2.2 \times 10^1 \text{ H}.$$

$$\boxed{L \approx 2.2 \times 10^1 \text{ H } (\approx 22 \text{ H})}$$



### Answer 19 (4)

(i) The maximum current is obtained from energy conservation:

$$\frac{1}{2}LI_{\max}^2 = \frac{1}{2}\frac{Q^2}{C} \Rightarrow I_{\max} = \sqrt{\frac{Q^2}{LC}} = \frac{Q}{\sqrt{LC}}.$$

With  $Q = 4.0 \times 10^{-9} \text{ C}$ ,  $C = 150 \times 10^{-12} \text{ F}$  and  $L = 0.25 \text{ H}$ ,

$$I_{\max} = \frac{4.0 \times 10^{-9}}{\sqrt{(0.25)(150 \times 10^{-12})}} \approx 6.5 \times 10^{-4} \text{ A} = 0.65 \text{ mA}.$$

$$\boxed{I_{\max} \approx 6.5 \times 10^{-4} \text{ A (0.65 mA)}}$$

(ii) At a general time, the energy relation

$$\frac{1}{2}LI^2 + \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}\frac{Q^2}{C}$$

gives

$$I^2 = \frac{Q^2 - q^2}{LC} \Rightarrow I = \pm \sqrt{\frac{Q^2 - q^2}{LC}}.$$

For  $q = 2.0 \times 10^{-9} \text{ C}$ ,

$$I = \pm \sqrt{\frac{(4.0 \times 10^{-9})^2 - (2.0 \times 10^{-9})^2}{(0.25)(150 \times 10^{-12})}} \approx \pm 5.7 \times 10^{-4} \text{ A} = \pm 0.57 \text{ mA}.$$

$$\boxed{I \approx \pm 5.7 \times 10^{-4} \text{ A } (\pm 0.57 \text{ mA})}$$

## Answer 20 (5)

For an LRC series circuit, the damped angular frequency is

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}.$$

Critical damping occurs when  $\omega' = 0$ :

$$\frac{1}{LC} - \frac{R_c^2}{4L^2} = 0 \quad \Rightarrow \quad R_c = 2\sqrt{\frac{L}{C}}.$$

1. With  $L = 0.40$  H and  $C = 220 \times 10^{-6}$  F,

$$R_c = 2\sqrt{\frac{0.40}{220 \times 10^{-6}}} \approx 8.5 \times 10^1 \Omega.$$

$$\boxed{R_c \approx 8.5 \times 10^1 \Omega (\approx 85 \Omega)}$$

2. Given  $R = 6.0 \Omega$ , we see that  $R < R_c$ , so

$$\boxed{\text{the circuit is underdamped.}}$$

3. Since underdamped, we use

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}.$$

Compute

$$\begin{aligned} \frac{1}{LC} &= \frac{1}{(0.40)(220 \times 10^{-6})} \approx 1.07 \times 10^2 \text{ s}^{-2}, \\ \frac{R^2}{4L^2} &= \frac{6.0^2}{4(0.40)^2} \approx 5.6 \times 10^0 \text{ s}^{-2}. \end{aligned}$$

Thus

$$\omega' = \sqrt{1.07 \times 10^2 - 5.6} \approx 1.06 \times 10^2 \text{ rad s}^{-1}.$$

(Period  $T' = 2\pi/\omega' \approx 5.9 \times 10^{-2}$  s.)

$$\boxed{\omega' \approx 1.1 \times 10^2 \text{ rad s}^{-1}}$$

## Answer 21 (4)

For a series LC tuning circuit,

$$\omega = 2\pi f = \frac{1}{\sqrt{LC}} \quad \Rightarrow \quad C = \frac{1}{L(2\pi f)^2}.$$

1. With  $L = 1.6 \times 10^{-6}$  H and  $f = 92.0 \times 10^6$  Hz,

$$C = \frac{1}{(1.6 \times 10^{-6})[2\pi(92.0 \times 10^6)]^2} \approx 1.9 \times 10^{-12} \text{ F} = 1.9 \text{ pF}.$$

$$C \approx 1.9 \times 10^{-12} \text{ F } (\approx 1.9 \text{ pF})$$

2. The tuning frequency is

$$f = \frac{1}{2\pi\sqrt{LC}}.$$

For fixed  $L$ , this gives

$$f \propto \frac{1}{\sqrt{C}}.$$

Thus, if  $C$  is increased by a factor  $k$ , the frequency changes by  $f \rightarrow f/\sqrt{k}$ . In particular, increasing  $C$  decreases the tuning frequency.

$$f = \frac{1}{2\pi\sqrt{LC}} \text{ so larger } C \Rightarrow \text{smaller } f \text{ } (\propto 1/\sqrt{C})$$

## Answer 22 (5)

1. Immediately after switch-on, the energy is purely magnetic:

$$E_0 = \frac{1}{2}LI_0^2,$$

with  $L = 0.50 \text{ H}$  and  $I_0 = 0.80 \text{ A}$ :

$$E_0 = \frac{1}{2}(0.50)(0.80)^2 = 0.16 \text{ J}.$$

$$\boxed{E_0 = 0.16 \text{ J}}$$

2. The instantaneous power loss in the resistor is

$$P = I^2 R.$$

At the moment of maximum current ( $I = I_0$ ),

$$P = (0.80)^2(12) = 7.68 \text{ W}.$$

The energy is decreasing at rate  $dE/dt = -7.68 \text{ J s}^{-1}$ .

$$\boxed{P = 7.7 \text{ W (energy loss rate } dE/dt = -7.7 \text{ J s}^{-1})}$$

3. For a lightly damped LRC series circuit, the current amplitude decays as

$$I(t) \propto e^{-\frac{R}{2L}t}.$$

Since energy  $E \propto I^2$ , we have

$$E(t) = E_0 e^{-\frac{R}{L}t}.$$

Setting  $E(t) = \frac{1}{2}E_0$  gives

$$\frac{1}{2}E_0 = E_0 e^{-\frac{R}{L}t} \Rightarrow e^{-\frac{R}{L}t} = \frac{1}{2} \Rightarrow t = \frac{L}{R} \ln 2.$$

With  $L = 0.50 \text{ H}$  and  $R = 12 \Omega$ ,

$$t = \frac{0.50}{12} \ln 2 \approx 2.9 \times 10^{-2} \text{ s}.$$

$$\boxed{t_{1/2} \approx 2.9 \times 10^{-2} \text{ s } (\approx 0.029 \text{ s})}$$

## Answer 23 (4)

Answer 23:

1. The force is

$$F(r) = -\frac{dU}{dr} = -U_0 \left[ -12 \frac{R_0^{12}}{r^{13}} + 12 \frac{R_0^6}{r^7} \right] = \frac{12U_0}{R_0} \left[ \left( \frac{R_0}{r} \right)^{13} - \left( \frac{R_0}{r} \right)^7 \right].$$

Put  $r = R_0 + x$  and assume  $|x| \ll R_0$ . Then

$$\left( \frac{R_0}{R_0 + x} \right)^n = \left( 1 + \frac{x}{R_0} \right)^{-n} \approx 1 - n \frac{x}{R_0}.$$

Hence

$$F \simeq \frac{12U_0}{R_0} \left[ \left( 1 - 13 \frac{x}{R_0} \right) - \left( 1 - 7 \frac{x}{R_0} \right) \right] = \frac{12U_0}{R_0} \left( -6 \frac{x}{R_0} \right) = -\frac{72U_0}{R_0^2} x.$$

Comparing with  $F = -kx$  we obtain

$$\boxed{k = \frac{72U_0}{R_0^2}}.$$

2. For motion of the relative coordinate  $x$  the effective mass is the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

The equation of motion is

$$\mu \ddot{x} = -kx,$$

so

$$\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{72U_0}{R_0^2 \mu}}, \quad f = \frac{\omega}{2\pi}.$$

$$\boxed{\omega = \sqrt{\frac{72U_0}{R_0^2 \mu}}, \quad f = \frac{1}{2\pi} \sqrt{\frac{72U_0}{R_0^2 \mu}}}$$

3. With  $U_0 = 1.6 \times 10^{-21}$  J and  $R_0 = 3.8 \times 10^{-10}$  m,

$$k = \frac{72 \times 1.6 \times 10^{-21}}{(3.8 \times 10^{-10})^2} \approx 0.80 \text{ N m}^{-1}.$$

Each mass is  $40u$  with  $u = 1.66 \times 10^{-27}$  kg, so

$$m_1 = m_2 \approx 6.64 \times 10^{-26} \text{ kg}, \quad \mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m}{2} \approx 3.32 \times 10^{-26} \text{ kg}.$$

Thus

$$\omega = \sqrt{\frac{0.80}{3.32 \times 10^{-26}}} \approx 4.9 \times 10^{12} \text{ rad s}^{-1},$$

$$f = \frac{\omega}{2\pi} \approx \frac{4.9 \times 10^{12}}{6.28} \approx 7.8 \times 10^{11} \text{ Hz}.$$

$$\boxed{f \approx 8 \times 10^{11} \text{ Hz}}$$

## Answer 24 (4)

Answer 24:

1.

$$F(z) = -\frac{dU}{dz} = -\left(az + \frac{1}{2}bz^2\right) = -az - \frac{1}{2}bz^2.$$

$$F(z) = -az - \frac{1}{2}bz^2$$

2. For sufficiently small  $z$ , the quadratic term in  $F$  (cubic in  $U$ ) is much smaller than the linear term, so

$$F(z) \simeq -az.$$

With mass  $m$  the equation of motion is

$$m\ddot{z} = -az \quad \Rightarrow \quad \ddot{z} + \frac{a}{m}z = 0,$$

which is SHM with

$$\omega = \sqrt{\frac{a}{m}}.$$

3. The magnitude of the cubic correction relative to the linear term is

$$\frac{|F_{\text{cubic}}|}{|F_{\text{linear}}|} = \frac{\frac{1}{2}|b|z^2}{|a||z|} = \frac{|b|}{2|a|}|z|.$$

Define a characteristic amplitude  $z_{\text{max}}$  where the two terms are equal:

$$\frac{|b|}{2|a|} z_{\text{max}} = 1 \quad \Rightarrow \quad z_{\text{max}} = \frac{2|a|}{|b|}.$$

At  $z = 0.05 z_{\text{max}}$  this ratio is

$$\frac{|F_{\text{cubic}}|}{|F_{\text{linear}}|} = \frac{|b|}{2|a|}(0.05 z_{\text{max}}) = 0.05.$$

So the cubic term provides a 5% correction to the restoring force.

$$\text{Fractional correction at } z = 0.05 z_{\text{max}} \text{ is } 5\%$$

## Answer 25 (4)

Answer 25:

1. When the liquid is displaced so that one side is higher by  $y$  and the other lower by  $y$ , the height difference is  $2y$ . The extra mass on one side is

$$m_{\text{ex}} = \rho A(2y),$$

giving a restoring force

$$F = -m_{\text{ex}}g = -2\rho A y g.$$

The total mass of fluid is

$$m = \rho A(2L) = 2\rho AL.$$

Newtons second law:

$$m\ddot{y} = -2\rho A y g \Rightarrow 2\rho AL\ddot{y} = -2\rho A y g \Rightarrow \ddot{y} + \frac{g}{L}y = 0.$$

$$\boxed{\ddot{y} + \frac{g}{L}y = 0}$$

2. Comparing with  $\ddot{y} + \omega^2 y = 0$  gives

$$\omega = \sqrt{\frac{g}{L}}, \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}.$$

$$\boxed{\omega = \sqrt{\frac{g}{L}}, \quad T = 2\pi\sqrt{\frac{L}{g}}}$$

3. For  $L = 0.25$  m,

$$\omega = \sqrt{\frac{9.81}{0.25}} = \sqrt{39.24} \approx 6.26 \text{ rad s}^{-1},$$

$$T = \frac{2\pi}{\omega} \approx \frac{6.28}{6.26} \approx 1.0 \text{ s.}$$

$$\boxed{T \approx 1.0 \text{ s}}$$



## Answer 26 (5)

Answer 26:

1. Take the ball as a point mass moving on a circular arc of radius  $R$ . Its speed is

$$v = R\dot{\theta},$$

so kinetic energy

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mR^2\dot{\theta}^2.$$

Take potential energy zero at the lowest point. For displacement  $\theta$ ,

$$U = mgR(1 - \cos \theta).$$

Total energy  $E = T + U$  is constant:

$$E = \frac{1}{2}mR^2\dot{\theta}^2 + mgR(1 - \cos \theta).$$

Differentiating with respect to time and setting  $dE/dt = 0$ ,

$$mR^2\dot{\theta}\ddot{\theta} + mgR \sin \theta \dot{\theta} = 0.$$

For  $\dot{\theta} \neq 0$  this gives

$$R^2\ddot{\theta} + gR \sin \theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \frac{g}{R} \sin \theta = 0.$$

For small angles,  $\sin \theta \simeq \theta$ , so

$$\boxed{\ddot{\theta} + \frac{g}{R}\theta = 0}.$$

2. Comparing with SHM,  $\ddot{\theta} + \omega^2\theta = 0$ , gives

$$\omega = \sqrt{\frac{g}{R}}, \quad T = 2\pi\sqrt{\frac{R}{g}}.$$

$$\boxed{T = 2\pi\sqrt{\frac{R}{g}}}$$

3. For  $R = 0.80$  m,

$$T = 2\pi\sqrt{\frac{0.80}{9.81}} = 2\pi\sqrt{0.0816} \approx 2\pi \times 0.286 \approx 1.8 \text{ s}.$$

$$\boxed{T \approx 1.8 \text{ s}}$$

4. For larger amplitudes the small-angle approximation  $\sin \theta \simeq \theta$  fails, and the exact equation

$$\ddot{\theta} + \frac{g}{R} \sin \theta = 0$$

is nonlinear. The oscillations remain periodic, but the period increases with amplitude and the motion is no longer strictly simple harmonic.

At large amplitudes the motion is nonlinear and the period becomes amplitude dependent

## Answer 27 (5)

Answer 27:

1.

$$F(x) = -\frac{dU}{dx} = -(2\alpha x + 4\beta x^3) = -2\alpha x - 4\beta x^3.$$

$$\boxed{F(x) = -2\alpha x - 4\beta x^3}$$

2. For sufficiently small  $x$ , the cubic term in  $F$  can be neglected, so

$$F \simeq -2\alpha x.$$

With mass  $m$ ,

$$m\ddot{x} = -2\alpha x \quad \Rightarrow \quad \ddot{x} + \frac{2\alpha}{m}x = 0,$$

which is SHM with

$$\boxed{\omega = \sqrt{\frac{2\alpha}{m}}}.$$

3. We require the quartic contribution to be 10% of the harmonic restoring force:

$$\frac{|F_{\text{quartic}}|}{|F_{\text{quadratic}}|} = \frac{4\beta|x|^3}{2\alpha|x|} = \frac{2\beta x^2}{\alpha} = 0.10.$$

Solving for  $x^2$ ,

$$x^2 = \frac{0.10 \alpha}{2\beta} = 0.05 \frac{\alpha}{\beta},$$

so

$$\boxed{x_{10\%} = \sqrt{0.05 \frac{\alpha}{\beta}}}.$$

4. Because  $\beta > 0$ , the potential becomes steeper than purely quadratic at large  $|x|$ , giving a stronger restoring force than a simple harmonic oscillator. As the amplitude increases, the oscillator spends less time near the turning points and the period decreases slightly with amplitude; equivalently, the oscillation frequency increases with amplitude (a hardening anharmonic oscillator).

Quartic term makes the oscillator stiffer, so at large amplitudes the period decreases and the frequency increases