

Limits Answers

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Answer 1 (2)

Since the function $f(x) = 2x^3 - 5x^2 + 7x - 1$ is a polynomial, nothing special happens at $x = 3$. We can directly substitute:

$$\lim_{x \rightarrow 3} (2x^3 - 5x^2 + 7x - 1) = 2(3)^3 - 5(3)^2 + 7(3) - 1$$

$$= 2(27) - 5(9) + 21 - 1 = 54 - 45 + 21 - 1 = 29$$

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Answer 2 (3)

Direct substitution gives $\frac{0}{0}$. Factor numerator and denominator:

$$\lim_{x \rightarrow 2} \frac{x^3 - 3x^2 + 4}{x^2 - 4}$$

Note that $x^2 - 4 = (x - 2)(x + 2)$. For the numerator, we test if $x = 2$ is a root:

$$2^3 - 3(2)^2 + 4 = 8 - 12 + 4 = 0$$

So $(x - 2)$ is a factor. By polynomial division: $x^3 - 3x^2 + 4 = (x - 2)(x^2 - x - 2) = (x - 2)(x - 2)(x + 1) = (x - 2)^2(x + 1)$

Therefore:

$$\lim_{x \rightarrow 2} \frac{(x - 2)^2(x + 1)}{(x - 2)(x + 2)} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 1)}{(x + 2)} = \frac{0 \cdot 3}{4} = 0$$

0

Answer 3 (4)

Since $-1 \leq \cos\left(\frac{1}{x^2}\right) \leq 1$ for all $x \neq 0$, we have:

$$-x^2 \leq x^2 \cos\left(\frac{1}{x^2}\right) \leq x^2$$

Let $g(x) = -x^2$ and $h(x) = x^2$. Then:

$$\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} (-x^2) = 0$$

$$\lim_{x \rightarrow 0} h(x) = \lim_{x \rightarrow 0} x^2 = 0$$

By the Pinching Theorem, since $g(x) \leq x^2 \cos\left(\frac{1}{x^2}\right) \leq h(x)$ and both $g(x)$ and $h(x)$ approach 0:

$$\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0$$

[0]

Answer 4 (3)

We are given that $1 - x^2 \leq f(x) \leq 1 + x^2$ for all x near 0.

Let $g(x) = 1 - x^2$ and $h(x) = 1 + x^2$. Then:

$$\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} (1 - x^2) = 1 - 0 = 1$$

$$\lim_{x \rightarrow 0} h(x) = \lim_{x \rightarrow 0} (1 + x^2) = 1 + 0 = 1$$

Since both bounding functions have the same limit $L = 1$, by the Pinching Theorem:

$$\lim_{x \rightarrow 0} f(x) = 1$$

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Answer 5 (4)

We rewrite the expression to use the known limit $\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 2x} &= \lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 2x} \cdot \frac{5x}{5x} \cdot \frac{2x}{2x} \\&= \lim_{x \rightarrow 0} \frac{5x \cdot \sin 5x}{5x} \cdot \frac{2x}{2x \cdot \sin 2x} = \lim_{x \rightarrow 0} \frac{5 \sin 5x}{5x} \cdot \frac{2x}{\sin 2x}\end{aligned}$$

By Calculus of Limits Theorem and Change of Variables (with $y = 5x$ and $z = 2x$):

$$\begin{aligned}&= 5 \cdot \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} \cdot \frac{1}{2} \cdot \lim_{x \rightarrow 0} \frac{2x}{\sin 2x} \\&= 5 \cdot 1 \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{5}{2} \\&\boxed{\frac{5}{2}}\end{aligned}$$

Answer 6 (5)

Using the double angle formula $\cos 3x = 1 - 2 \sin^2 \left(\frac{3x}{2}\right)$, or more directly, $1 - \cos 3x = 2 \sin^2 \left(\frac{3x}{2}\right)$:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1 - \cos 3x}{x^2} &= \lim_{x \rightarrow 0} \frac{2 \sin^2 \left(\frac{3x}{2}\right)}{x^2} \\&= \lim_{x \rightarrow 0} 2 \cdot \frac{\sin^2 \left(\frac{3x}{2}\right)}{x^2} = \lim_{x \rightarrow 0} 2 \cdot \left(\frac{\sin \left(\frac{3x}{2}\right)}{\frac{3x}{2}} \right)^2 \cdot \left(\frac{3}{2}\right)^2\end{aligned}$$

By Change of Variables with $y = \frac{3x}{2}$ and Calculus of Limits Theorem:

$$= 2 \cdot 1^2 \cdot \frac{9}{4} = 2 \cdot \frac{9}{4} = \frac{9}{2}$$

$$\boxed{\frac{9}{2}}$$

Answer 7 (4)

Divide both numerator and denominator by x^3 :

$$\lim_{x \rightarrow \infty} \frac{5x^3 - 2x^2 + 7}{2x^3 + 3x - 1} = \lim_{x \rightarrow \infty} \frac{5 - \frac{2}{x} + \frac{7}{x^3}}{2 + \frac{3}{x^2} - \frac{1}{x^3}}$$

As $x \rightarrow \infty$: $\frac{2}{x} \rightarrow 0$, $\frac{7}{x^3} \rightarrow 0$, $\frac{3}{x^2} \rightarrow 0$, $\frac{1}{x^3} \rightarrow 0$

By Calculus of Limits Theorem:

$$= \frac{5 - 0 + 0}{2 + 0 - 0} = \frac{5}{2}$$

$$\boxed{\frac{5}{2}}$$

Answer 8 (5)

Since $-1 \leq \sin x \leq 1$, for large x we have $x^2 + \sin x \approx x^2$. Divide numerator and denominator by x^2 :

$$\lim_{x \rightarrow \infty} \frac{3x^2 + 2x}{x^2 + \sin x} = \lim_{x \rightarrow \infty} \frac{3 + \frac{2}{x}}{1 + \frac{\sin x}{x^2}}$$

As $x \rightarrow \infty$: $\frac{2}{x} \rightarrow 0$

For $\frac{\sin x}{x^2}$: Since $-1 \leq \sin x \leq 1$, we have $-\frac{1}{x^2} \leq \frac{\sin x}{x^2} \leq \frac{1}{x^2}$

Both bounds approach 0, so by the Pinching Theorem: $\lim_{x \rightarrow \infty} \frac{\sin x}{x^2} = 0$

Therefore, by Calculus of Limits Theorem:

$$= \frac{3 + 0}{1 + 0} = 3$$

$$\boxed{3}$$

Answer 9 (4)

Using $\tan 7x = \frac{\sin 7x}{\cos 7x}$:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan 7x}{x} &= \lim_{x \rightarrow 0} \frac{\sin 7x}{x \cos 7x} \\ &= \lim_{x \rightarrow 0} \frac{\sin 7x}{x} \cdot \frac{1}{\cos 7x} = \lim_{x \rightarrow 0} \frac{7 \sin 7x}{7x} \cdot \frac{1}{\cos 7x} \end{aligned}$$

By Change of Variables $y = 7x$) and Calculus of Limits Theorem:

$$\begin{aligned} &= 7 \cdot \lim_{y \rightarrow 0} \frac{\sin y}{y} \cdot \lim_{x \rightarrow 0} \frac{1}{\cos 7x} \\ &= 7 \cdot 1 \cdot \frac{1}{\cos 0} = 7 \cdot 1 \cdot 1 = 7 \end{aligned}$$

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Answer 10 (5)

Using the exponential series $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots$, with $u = 2x$:

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots = 1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{6} + \dots$$

Therefore:

$$\frac{e^{2x} - 1}{x} = \frac{2x + 2x^2 + \frac{4x^3}{3} + \dots}{x} = 2 + 2x + \frac{4x^2}{3} + \dots$$

Taking the limit as $x \rightarrow 0$, each term with positive power of x vanishes:

$$\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x} = \lim_{x \rightarrow 0} \left(2 + 2x + \frac{4x^2}{3} + \dots \right) = 2$$

By Calculus of Limits Theorem, since all terms except the constant approach 0.

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