Daily Challenge - December 30, 2024

Vector Geometry - Collinearity

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Challenge Problem

Problem: The points O, A, B and C are such that $\overrightarrow{OA} = 6\mathbf{u} - 4\mathbf{v}$, $\overrightarrow{OB} = 3\mathbf{u} - \mathbf{v}$ and $\overrightarrow{OC} = \mathbf{v} - 3\mathbf{u}$, where $\mathbf{u}$ and $\mathbf{v}$ are non-parallel vectors.
The point M is the midpoint of OA and the point N is the point on AB such that $AN : NB = 1 : 2$.
Prove that C, M and N are collinear.

Hint: Find $\overrightarrow{CM}$ and $\overrightarrow{CN}$, then show one is a scalar multiple of the other.

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Solution

We need to prove that points C, M and N are collinear.

Step 1: Find position vector of M

Since M is the midpoint of OA:

$$\begin{aligned}\overrightarrow{OM} &= \frac{1}{2}\overrightarrow{OA} \\ &= \frac{1}{2}(6\mathbf{u} - 4\mathbf{v}) \\ &= 3\mathbf{u} - 2\mathbf{v}\end{aligned}$$

Step 2: Find position vector of N

Since N divides AB in the ratio $1:2$, we have:

$$\begin{aligned}\overrightarrow{ON} &= \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB} \\ &= \overrightarrow{OA} + \frac{1}{3}(\overrightarrow{OB} - \overrightarrow{OA}) \\ &= \frac{2}{3}\overrightarrow{OA} + \frac{1}{3}\overrightarrow{OB} \\ &= \frac{2}{3}(6\mathbf{u} - 4\mathbf{v}) + \frac{1}{3}(3\mathbf{u} - \mathbf{v}) \\ &= 4\mathbf{u} - \frac{8}{3}\mathbf{v} + \mathbf{u} - \frac{1}{3}\mathbf{v} \\ &= 5\mathbf{u} - 3\mathbf{v}\end{aligned}$$

Step 3: Find vector CM

$$\begin{aligned}\overrightarrow{CM} &= \overrightarrow{OM} - \overrightarrow{OC} \\ &= (3\mathbf{u} - 2\mathbf{v}) - (\mathbf{v} - 3\mathbf{u}) \\ &= 3\mathbf{u} - 2\mathbf{v} - \mathbf{v} + 3\mathbf{u} \\ &= 6\mathbf{u} - 3\mathbf{v}\end{aligned}$$

Step 4: Find vector CN

$$\begin{aligned}\overrightarrow{CN} &= \overrightarrow{ON} - \overrightarrow{OC} \\ &= (5\mathbf{u} - 3\mathbf{v}) - (\mathbf{v} - 3\mathbf{u}) \\ &= 5\mathbf{u} - 3\mathbf{v} - \mathbf{v} + 3\mathbf{u} \\ &= 8\mathbf{u} - 4\mathbf{v}\end{aligned}$$

Step 5: Show CM and CN are parallel

We need to check if $\overrightarrow{CM} = k\overrightarrow{CN}$ for some scalar k:

$$6\mathbf{u} - 3\mathbf{v} = k(8\mathbf{u} - 4\mathbf{v})$$

Comparing coefficients of $\mathbf{u}$:

$$6 = 8k \implies k = \frac{6}{8} = \frac{3}{4}$$

Comparing coefficients of $\mathbf{v}$:

$$-3 = -4k \implies k = \frac{3}{4}$$

Both coefficients give $k = \frac{3}{4}$.
Therefore:

$$\overrightarrow{CM} = \frac{3}{4}\overrightarrow{CN}$$

Conclusion

Since $\overrightarrow{CM}$ is a scalar multiple of $\overrightarrow{CN}$, the vectors are parallel and share the common point C.

Therefore, **C, M and N are collinear**. $\boxed{\text{Proven}}$

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