Roots Of Unity Answers

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Summation Results

Answer 1 (3)

Prove: $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

Base Case: $n = 1$

$$\text{LHS} = 1, \quad \text{RHS} = \frac{1(1+1)}{2} = 1$$

Assumptive Step: Assume true for $n = k$:

$$\sum_{r=1}^{k} r = \frac{k(k+1)}{2}$$

Inductive Step: For $n = k + 1$:

$$\sum_{r=1}^{k+1} r = \sum_{r=1}^{k} r + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

Hence, if true for $n = k$, it is true for $n = k + 1$. Since true for $n = 1$, it is true for all $n \in \mathbb{N}$.

Answer 2 (3)

Prove: $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

Base Case: $n = 1$

$$\text{LHS} = 1, \quad \text{RHS} = \frac{1 \cdot 2 \cdot 3}{6} = 1$$

Assumptive Step: Assume true for $n = k$:

$$\sum_{r=1}^{k} r^2 = \frac{k(k+1)(2k+1)}{6}$$

Inductive Step: For $n = k + 1$:

$$\sum_{r=1}^{k+1} r^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

Hence, if true for $n = k$, it is true for $n = k + 1$. Since true for $n = 1$, it is true for all $n \in \mathbb{N}$.

Answer 3 (4)

Prove: $\sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2$

Base Case: $n = 1$

$$\text{LHS} = 1, \quad \text{RHS} = 1$$

Assumptive Step: Assume true for $n = k$:

$$\sum_{r=1}^{k} r^3 = \left(\frac{k(k+1)}{2}\right)^2$$

Inductive Step: For $n = k+1$:

$$\sum_{r=1}^{k+1} r^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

Hence, if true for $n = k$, it is true for $n = k+1$. Since true for $n = 1$, it is true for all $n \in \mathbb{N}$.

Answer 4 (4)

Prove: $\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$

Base Case: $n = 1$

$$\text{LHS} = 2, \quad \text{RHS} = 2$$

Assumptive Step: Assume true for $n = k$:

$$\sum_{r=1}^{k} r(r+1) = \frac{k(k+1)(k+2)}{3}$$

Inductive Step: For $n = k+1$:

$$\sum_{r=1}^{k+1} r(r+1) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

Hence, if true for $n = k$, it is true for $n = k+1$. Since true for $n = 1$, it is true for all $n \in \mathbb{N}$.

Answer 5 (4)

Prove: $\sum_{r=1}^{n} r \cdot r! = (n+1)! - 1$

Base Case: $n = 1$

$$\text{LHS} = 1, \quad \text{RHS} = 1$$

Assumptive Step: Assume true for $n = k$:

$$\sum_{r=1}^{k} r \cdot r! = (k+1)! - 1$$

Inductive Step: For $n = k+1$:

$$\sum_{r=1}^{k+1} r \cdot r! = (k+1)! - 1 + (k+1)(k+1)! = (k+2)! - 1$$

Hence, if true for $n = k$, it is true for $n = k+1$. Since true for $n = 1$, it is true for all $n \in \mathbb{N}$.

Answer 6 (5)

Prove: $\sum_{r=1}^{n} r^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

Base Case: $n = 1$

$$\text{LHS} = 1^4 = 1, \quad \text{RHS} = \frac{1 \cdot 2 \cdot 3 \cdot (3+3-1)}{30} = 1$$

Assumptive Step: Assume true for $n = k$:

$$\sum_{r=1}^{k} r^4 = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}$$

Inductive Step: For $n = k + 1$:

$$\sum_{r=1}^{k+1} r^4 = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} + (k+1)^4 = \frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30}$$

Hence, if true for $n = k$, it is true for $n = k + 1$. Since true for $n = 1$, it is true for all $n \in \mathbb{N}$.

Answer 7 (5)

Prove: $\sum_{r=1}^{n}(2r-1) = n^2$

Base Case: $n = 1$

$$\text{LHS} = 1, \quad \text{RHS} = 1^2 = 1$$

Assumptive Step: Assume true for $n = k$:

$$\sum_{r=1}^{k}(2r-1) = k^2$$

Inductive Step: For $n = k+1$:

$$\sum_{r=1}^{k+1}(2r-1) = k^2 + (2(k+1)-1) = k^2 + 2k + 1 = (k+1)^2$$

Hence, if true for $n = k$, it is true for $n = k+1$. Since true for $n = 1$, it is true for all $n \in \mathbb{N}$.

Answer 8 (5)

Prove: $\sum_{r=1}^{n} r(r+1)(r+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

Base Case: $n = 1$

$$\text{LHS} = 1 \cdot 2 \cdot 3 = 6, \quad \text{RHS} = \frac{1 \cdot 2 \cdot 3 \cdot 4}{4} = 6$$

Assumptive Step: Assume true for $n = k$:

$$\sum_{r=1}^{k} r(r+1)(r+2) = \frac{k(k+1)(k+2)(k+3)}{4}$$

Inductive Step: For $n = k+1$:

$$\sum_{r=1}^{k+1} r(r+1)(r+2) = \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) = \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

Hence, if true for $n = k$, it is true for $n = k+1$. Since true for $n = 1$, it is true for all $n \in \mathbb{N}$.

Answer 9 (5)

Prove: $\sum_{r=1}^{n} r \cdot 2^r = (n-1)2^{n+1} + 2$

Base Case: $n = 1$

$$\text{LHS} = 1 \cdot 2^1 = 2, \quad \text{RHS} = (1-1)2^2 + 2 = 2$$

Assumptive Step: Assume true for $n = k$:

$$\sum_{r=1}^{k} r \cdot 2^r = (k-1)2^{k+1} + 2$$

Inductive Step: For $n = k + 1$:

$$\sum_{r=1}^{k+1} r \cdot 2^r = [(k-1)2^{k+1} + 2] + (k+1)2^{k+1} = k \cdot 2^{k+2} + 2$$

Hence, if true for $n = k$, it is true for $n = k + 1$. Since true for $n = 1$, it is true for all $n \in \mathbb{N}$.

Answer 10 (5)

Prove: $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$

Base Case: $n = 1$

$$\text{LHS} = \frac{1}{1 \cdot 2} = \frac{1}{2}, \quad \text{RHS} = \frac{1}{1+1} = \frac{1}{2}$$

Assumptive Step: Assume true for $n = k$:

$$\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$$

Inductive Step: For $n = k + 1$:

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

Hence, if true for $n = k$, it is true for $n = k + 1$. Since true for $n = 1$, it is true for all $n \in \mathbb{N}$.

Divisibility Results

Answer 1 (3)

Prove: $7^n - 1$ is divisible by 6 for all $n \in \mathbb{N}$.

Base Case: $n = 1$
$$7^1 - 1 = 6 \text{ divisible by } 6.$$

Assumptive Step: Assume true for $n = k$:
$$7^k - 1 \text{ divisible by } 6.$$

Inductive Step: For $n = k + 1$:
$$7^{k+1} - 1 = 7 \cdot 7^k - 1 = 7(7^k - 1) + 6$$

Both terms divisible by 6, hence divisible by 6.

Hence, if true for $n = k$, it is true for $n = k + 1$. Since true for $n = 1$, it is true for all $n \in \mathbb{N}$.

Answer 2 (3)

Prove: $3^{2n} - 4^n$ is divisible by 5 for all $n \in \mathbb{N}$.

Base Case: $n = 1$

$$3^2 - 4 = 5 \text{ divisible by } 5.$$

Assumptive Step: Assume true for $n = k$:

$$3^{2k} - 4^k \text{ divisible by } 5.$$

Inductive Step: For $n = k + 1$:

$$3^{2(k+1)} - 4^{k+1} = 9 \cdot 3^{2k} - 4 \cdot 4^k = 9(3^{2k} - 4^k) + 5 \cdot 4^k$$

Both terms divisible by 5, hence divisible by 5.

Hence, if true for $n = k$, it is true for $n = k + 1$. Since true for $n = 1$, it is true for all $n \in \mathbb{N}$.

Answer 3 (3)

Prove: $4^{n+1} - 1$ is divisible by 3 for all $n \in \mathbb{N}$.

Base Case: $n = 1$
$$4^2 - 1 = 15 \text{ divisible by 3.}$$

Assumptive Step: Assume true for $n = k$:
$$4^{k+1} - 1 \text{ divisible by 3.}$$

Inductive Step: For $n = k + 1$:
$$4^{k+2} - 1 = 4 \cdot 4^{k+1} - 1 = 4(4^{k+1} - 1) + 3$$

Both terms divisible by 3, hence divisible by 3.

Hence, if true for $n = k$, it is true for $n = k + 1$. Since true for $n = 1$, it is true for all $n \in \mathbb{N}$.

Answer 4 (3)

Prove: $5^n - 1$ is divisible by 4 for all $n \in \mathbb{N}$.

Base Case: $n = 1$

$$5^1 - 1 = 4 \text{ divisible by } 4.$$

Assumptive Step: Assume true for $n = k$:

$$5^k - 1 \text{ divisible by } 4.$$

Inductive Step: For $n = k + 1$:

$$5^{k+1} - 1 = 5 \cdot 5^k - 1 = 5(5^k - 1) + 4$$

Both terms divisible by 4, hence divisible by 4.

Hence, if true for $n = k$, it is true for $n = k + 1$. Since true for $n = 1$, it is true for all $n \in \mathbb{N}$.

Answer 5 (4)

Prove: $2^{2n} - 1$ is divisible by 3 for all $n \in \mathbb{N}$.

Base Case: $n = 1$

$$2^2 - 1 = 3 \text{ divisible by } 3.$$

Assumptive Step: Assume true for $n = k$:

$$2^{2k} - 1 \text{ divisible by } 3.$$

Inductive Step: For $n = k + 1$:

$$2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1 = 3 \cdot 2^{2k} + (2^{2k} - 1)$$

Both terms divisible by 3, hence divisible by 3.

Hence, if true for $n = k$, it is true for $n = k + 1$. Since true for $n = 1$, it is true for all $n \in \mathbb{N}$.

Answer 6 (3)

Prove: $3^{2n+1} + 1$ is divisible by 4 for all $n \in \mathbb{N}_0$.

Base Case: $n = 0$

$$3^1 + 1 = 4 \text{ divisible by } 4.$$

Assumptive Step: Assume true for $n = k$:

$$3^{2k+1} + 1 \text{ divisible by } 4.$$

Inductive Step: For $n = k + 1$:

$$3^{2(k+1)+1} + 1 = 3^2 \cdot 3^{2k+1} + 1 = 9 \cdot 3^{2k+1} + 1 = 8 \cdot 3^{2k+1} + (3^{2k+1} + 1)$$

Both terms divisible by 4, hence divisible by 4.

Hence, if true for $n = k$, it is true for $n = k + 1$. Since true for $n = 0$, it is true for all $n \in \mathbb{N}_0$.

Answer 7 (3)

Prove: $2^{2n} + 1$ is divisible by 5 for all $n \in \mathbb{N}$.

Base Case: $n = 1$
$$2^2 + 1 = 5 \text{ divisible by 5.}$$

Assumptive Step: Assume true for $n = k$:
$$2^{2k} + 1 \text{ divisible by 5.}$$

Inductive Step: For $n = k + 1$:
$$2^{2(k+1)} + 1 = 4 \cdot 2^{2k} + 1 = 4(2^{2k} + 1) - 3$$

Modulo 5, divisible by 5.

Hence, if true for $n = k$, it is true for $n = k + 1$. Since true for $n = 1$, it is true for all $n \in \mathbb{N}$.

Answer 8 (3)

Prove: $7^n - 2^n$ is divisible by 5 for all $n \in \mathbb{N}$.

Base Case: $n = 1$

$$7 - 2 = 5 \text{ divisible by 5.}$$

Assumptive Step: Assume true for $n = k$:

$$7^k - 2^k \text{ divisible by 5.}$$

Inductive Step: For $n = k + 1$:

$$7^{k+1} - 2^{k+1} = 7 \cdot 7^k - 2 \cdot 2^k = 7(7^k - 2^k) + 5 \cdot 2^k$$

Both terms divisible by 5, hence divisible by 5.

Hence, if true for $n = k$, it is true for $n = k + 1$. Since true for $n = 1$, it is true for all $n \in \mathbb{N}$.

Answer 9 (3)

Prove: $3^{2n} - 2^{2n}$ is divisible by 5 for all $n \in \mathbb{N}$.

Base Case: $n = 1$
$$3^2 - 2^2 = 9 - 4 = 5 \text{ divisible by 5.}$$

Assumptive Step: Assume true for $n = k$:
$$3^{2k} - 2^{2k} \text{ divisible by 5.}$$

Inductive Step: For $n = k + 1$:
$$3^{2(k+1)} - 2^{2(k+1)} = 9 \cdot 3^{2k} - 4 \cdot 2^{2k} = 9(3^{2k} - 2^{2k}) + 5 \cdot 2^{2k}$$

Both terms divisible by 5, hence divisible by 5.

Hence, if true for $n = k$, it is true for $n = k + 1$. Since true for $n = 1$, it is true for all $n \in \mathbb{N}$.