

# Mathematics Variant Answer Set

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# Vectors

## Answer 1: Vectors and Scalars

We have

$$T(t) = 300 + 4t \quad \Rightarrow \quad \frac{dT}{dt} = 4 \text{ K s}^{-1}.$$

Velocity:

$$\mathbf{v}(t) = (2t, 3 - t).$$

At  $t = 5$ ,

$$\mathbf{v}(5) = (10, -2), \quad |\mathbf{v}(5)| = \sqrt{10^2 + (-2)^2} = \sqrt{104} = 2\sqrt{26}.$$

Angle with  $\mathbf{w} = (3, -4)$ :

$$\mathbf{v}(5) \cdot \mathbf{w} = 10 \cdot 3 + (-2) \cdot (-4) = 38,$$

$$|\mathbf{w}| = \sqrt{3^2 + (-4)^2} = 5.$$

Thus

$$\cos \theta = \frac{\mathbf{v}(5) \cdot \mathbf{w}}{|\mathbf{v}(5)| |\mathbf{w}|} = \frac{38}{2\sqrt{26} \cdot 5} = \frac{19}{5\sqrt{26}}, \quad \theta = \arccos\left(\frac{19}{5\sqrt{26}}\right).$$

$\frac{dT}{dt} = 4 \text{ K s}^{-1},$	$ \mathbf{v}(5)  = 2\sqrt{26} \text{ m s}^{-1},$	$\theta = \arccos\left(\frac{19}{5\sqrt{26}}\right)$
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## Answer 2: Vector Algebra

(i)

$$\mathbf{a} + \mathbf{b} = (1 + 3, 2 + (-1), -1 + 4) = (4, 1, 3),$$

$$2\mathbf{c} = 2(-2, 1, 5) = (-4, 2, 10),$$

$$(\mathbf{a} + \mathbf{b}) - 2\mathbf{c} = (4, 1, 3) - (-4, 2, 10) = (8, -1, -7).$$

(ii) We want  $\mathbf{a} - \lambda\mathbf{b}$  parallel to  $\mathbf{c}$ , i.e.

$$\mathbf{a} - \lambda\mathbf{b} = \mu\mathbf{c}$$

for some  $\mu \in \mathbb{R}$ . This gives componentwise

$$(1 - 3\lambda, 2 + \lambda, -1 - 4\lambda) = (-2\mu, \mu, 5\mu),$$

so

$$\begin{cases} 1 - 3\lambda = -2\mu, \\ 2 + \lambda = \mu, \\ -1 - 4\lambda = 5\mu. \end{cases}$$

From the second equation,  $\mu = 2 + \lambda$ . Substitute into the first:

$$1 - 3\lambda = -2(2 + \lambda) = -4 - 2\lambda \Rightarrow 5 = \lambda.$$

Then  $\mu = 2 + 5 = 7$ . Check in the third equation:

$$-1 - 4 \cdot 5 = -21, \quad 5\mu = 35,$$

which is inconsistent. Hence no real  $\lambda$  exists.

$$\boxed{(\mathbf{a} + \mathbf{b}) - 2\mathbf{c} = (8, -1, -7)},$$

$$\boxed{\text{No real } \lambda \text{ gives } \mathbf{a} - \lambda\mathbf{b} \parallel \mathbf{c}.}$$

### Answer 3: Vector Spaces and Coordinate Bases

(i) Form the matrix with rows (or columns)  $\mathbf{u}, \mathbf{v}, \mathbf{w}$ :

$$M = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 3 & 1 & 4 \end{pmatrix}.$$

Compute the determinant:

$$\det M = 1 \begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} + 1 \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix}.$$

Evaluate each  $2 \times 2$  determinant:

$$\begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} = (-1) \cdot 4 - 3 \cdot 1 = -7,$$

$$\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 2 \cdot 4 - 3 \cdot 3 = 8 - 9 = -1,$$

$$\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 2 \cdot 1 - (-1) \cdot 3 = 2 + 3 = 5.$$

Hence

$$\det M = 1(-7) - 2(-1) + 1(5) = -7 + 2 + 5 = 0.$$

Thus  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$  is linearly dependent and does *not* form a basis of  $\mathbb{R}^3$ .

(ii) Because they do not form a basis,  $\mathbf{p}$  does not admit a unique expression as a linear combination of  $\mathbf{u}, \mathbf{v}, \mathbf{w}$ ; in fact, one checks that the system

$$\alpha \mathbf{u} + \beta \mathbf{v} + \gamma \mathbf{w} = \mathbf{p}$$

has no solution, so  $\mathbf{p}$  is not even in their span.

$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is not a basis of $\mathbb{R}^3$ ( $\det M = 0$ ),
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$\mathbf{p}$ cannot be expressed as a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$
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## Answer 4: Scalar Dot Product

We have

$$\mathbf{a} = (t, 1, 2), \quad \mathbf{b} = (2, -1, 3).$$

Orthogonality requires  $\mathbf{a} \cdot \mathbf{b} = 0$ :

$$\mathbf{a} \cdot \mathbf{b} = 2t + 1 \cdot (-1) + 2 \cdot 3 = 2t - 1 + 6 = 2t + 5.$$

So

$$2t + 5 = 0 \quad \Rightarrow \quad t = -\frac{5}{2}.$$

For this value,

$$\mathbf{a} = \left(-\frac{5}{2}, 1, 2\right),$$

$$|\mathbf{a}| = \sqrt{\left(-\frac{5}{2}\right)^2 + 1^2 + 2^2} = \sqrt{\frac{25}{4} + 1 + 4} = \sqrt{\frac{25 + 4 + 16}{4}} = \sqrt{\frac{45}{4}} = \frac{3\sqrt{5}}{2},$$

$$|\mathbf{b}| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{4 + 1 + 9} = \sqrt{14}.$$

$$\boxed{t = -\frac{5}{2}}, \quad \boxed{|\mathbf{a}| = \frac{3\sqrt{5}}{2}}, \quad \boxed{|\mathbf{b}| = \sqrt{14}}$$

### Answer 5: Vector Cross Product

$$\mathbf{a} = (2, 1, -3), \quad \mathbf{b} = (-1, 4, 2).$$

Compute the cross product using the determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -3 \\ -1 & 4 & 2 \end{vmatrix} = \mathbf{i}(1 \cdot 2 - (-3) \cdot 4) - \mathbf{j}(2 \cdot 2 - (-3) \cdot (-1)) + \mathbf{k}(2 \cdot 4 - 1 \cdot (-1)).$$

Hence

$$\mathbf{a} \times \mathbf{b} = \mathbf{i}(2 + 12) - \mathbf{j}(4 - 3) + \mathbf{k}(8 + 1) = 14\mathbf{i} - \mathbf{j} + 9\mathbf{k} = (14, -1, 9).$$

Area of the parallelogram:

$$A = |\mathbf{a} \times \mathbf{b}| = \sqrt{14^2 + (-1)^2 + 9^2} = \sqrt{196 + 1 + 81} = \sqrt{278}.$$

$$\boxed{\mathbf{a} \times \mathbf{b} = (14, -1, 9)},$$

$$\boxed{\text{Area} = \sqrt{278}}$$

### Answer 6: Scalar Triple Product

$$\mathbf{a} = (1, 2, 3), \quad \mathbf{b} = (2, -1, 4), \quad \mathbf{c} = (0, 5, -2).$$

First compute  $\mathbf{b} \times \mathbf{c}$ :

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 4 \\ 0 & 5 & -2 \end{vmatrix} = \mathbf{i}((-1)(-2) - 4 \cdot 5) - \mathbf{j}(2 \cdot (-2) - 4 \cdot 0) + \mathbf{k}(2 \cdot 5 - (-1) \cdot 0).$$

So

$$\mathbf{b} \times \mathbf{c} = \mathbf{i}(2 - 20) - \mathbf{j}(-4) + \mathbf{k}(10) = -18\mathbf{i} + 4\mathbf{j} + 10\mathbf{k} = (-18, 4, 10).$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (1, 2, 3) \cdot (-18, 4, 10) = -18 + 8 + 30 = 20.$$

The volume of the parallelepiped is

$$V = |[\mathbf{a}, \mathbf{b}, \mathbf{c}]| = |20| = 20.$$

$$\boxed{[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 20}, \quad \boxed{V = 20}$$



## Answer 7: Equations of Lines

The plane  $2x - y + 4z = 7$  has normal vector

$$\mathbf{n} = (2, -1, 4).$$

A line perpendicular to the plane and passing through  $(1, -2, 3)$  has direction  $\mathbf{n}$  and point  $\mathbf{a} = (1, -2, 3)$ , so a parametric equation is

$$\mathbf{r} = (1, -2, 3) + \lambda(2, -1, 4), \quad \lambda \in \mathbb{R}.$$

For the distance from  $\mathbf{c} = (4, 1, 0)$  to the line, use

$$\text{dist} = \frac{|(\mathbf{a} - \mathbf{c}) \times \mathbf{b}|}{|\mathbf{b}|},$$

with  $\mathbf{b} = \mathbf{n} = (2, -1, 4)$ .

Compute

$$\mathbf{a} - \mathbf{c} = (1 - 4, -2 - 1, 3 - 0) = (-3, -3, 3).$$

Then

$$(\mathbf{a} - \mathbf{c}) \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -3 & 3 \\ 2 & -1 & 4 \end{vmatrix} = \mathbf{i}((-3) \cdot 4 - 3 \cdot (-1)) - \mathbf{j}((-3) \cdot 4 - 3 \cdot 2) + \mathbf{k}((-3) \cdot (-1) - (-3) \cdot 2).$$

So

$$(\mathbf{a} - \mathbf{c}) \times \mathbf{b} = \mathbf{i}(-12 + 3) - \mathbf{j}(-12 - 6) + \mathbf{k}(3 - (-6)) = (-9, 18, 9).$$

Thus

$$|(\mathbf{a} - \mathbf{c}) \times \mathbf{b}| = \sqrt{(-9)^2 + 18^2 + 9^2} = \sqrt{81 + 324 + 81} = \sqrt{486} = 9\sqrt{6}.$$

Also

$$|\mathbf{b}| = \sqrt{2^2 + (-1)^2 + 4^2} = \sqrt{4 + 1 + 16} = \sqrt{21}.$$

Hence

$$\text{dist} = \frac{9\sqrt{6}}{\sqrt{21}} = 9\sqrt{\frac{6}{21}} = 9\sqrt{\frac{2}{7}}.$$

$$\boxed{\mathbf{r} = (1, -2, 3) + \lambda(2, -1, 4), \quad \lambda \in \mathbb{R}},$$

$$\boxed{\text{dist}((4, 1, 0), \ell) = 9\sqrt{\frac{2}{7}}}$$

## Answer 8: Equations of Planes

(i) A parametric equation with point  $P = (2, 1, -1)$  and direction vectors

$$\mathbf{b} = (1, -1, 2), \quad \mathbf{c} = (3, 0, 1)$$

is

$$\mathbf{r} = (2, 1, -1) + \lambda(1, -1, 2) + \mu(3, 0, 1), \quad \lambda, \mu \in \mathbb{R}.$$

(ii) A normal vector is  $\mathbf{n} = \mathbf{b} \times \mathbf{c}$ :

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 3 & 0 & 1 \end{vmatrix} = \mathbf{i}((-1) \cdot 1 - 2 \cdot 0) - \mathbf{j}(1 \cdot 1 - 2 \cdot 3) + \mathbf{k}(1 \cdot 0 - (-1) \cdot 3).$$

So

$$\mathbf{n} = (-1, 5, 3).$$

Let  $\mathbf{a} = (2, 1, -1)$  be the reference point. Then the Cartesian equation is

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{a} \quad \Rightarrow \quad -x + 5y + 3z = -1 \cdot 2 + 5 \cdot 1 + 3 \cdot (-1) = 0.$$

Thus

$$-x + 5y + 3z = 0.$$

(iii) The distance from the origin to the plane  $\mathbf{n} \cdot \mathbf{r} = d$  is  $|d|/|\mathbf{n}|$ . Here  $d = 0$ , so

$$\text{dist}(\mathbf{0}, \text{plane}) = 0.$$

$$\boxed{\mathbf{r} = (2, 1, -1) + \lambda(1, -1, 2) + \mu(3, 0, 1)},$$

$$\boxed{-x + 5y + 3z = 0},$$

$$\boxed{\text{dist from origin} = 0}$$

## Answer 9: Position, Velocity & Acceleration

Given

$$\mathbf{r}(t) = \langle 3 \cos t, 4 \sin t, t^2 \rangle,$$

velocity and acceleration are

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -3 \sin t, 4 \cos t, 2t \rangle,$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle -3 \cos t, -4 \sin t, 2 \rangle.$$

At  $t = \pi$ ,

$$\mathbf{v}(\pi) = \langle -3 \sin \pi, 4 \cos \pi, 2\pi \rangle = \langle 0, -4, 2\pi \rangle,$$

so the speed is

$$|\mathbf{v}(\pi)| = \sqrt{0^2 + (-4)^2 + (2\pi)^2} = \sqrt{16 + 4\pi^2} = 2\sqrt{\pi^2 + 4}.$$

Acceleration at  $t = \pi$ :

$$\mathbf{a}(\pi) = \langle -3 \cos \pi, -4 \sin \pi, 2 \rangle = \langle 3, 0, 2 \rangle.$$

Angle between  $\mathbf{v}(\pi)$  and  $\mathbf{a}(\pi)$ :

$$\mathbf{v}(\pi) \cdot \mathbf{a}(\pi) = 0 \cdot 3 + (-4) \cdot 0 + (2\pi) \cdot 2 = 4\pi.$$

Their magnitudes:

$$|\mathbf{v}(\pi)| = 2\sqrt{\pi^2 + 4}, \quad |\mathbf{a}(\pi)| = \sqrt{3^2 + 0^2 + 2^2} = \sqrt{13}.$$

Thus

$$\cos \theta = \frac{\mathbf{v}(\pi) \cdot \mathbf{a}(\pi)}{|\mathbf{v}(\pi)| |\mathbf{a}(\pi)|} = \frac{4\pi}{2\sqrt{\pi^2 + 4} \sqrt{13}} = \frac{2\pi}{\sqrt{13} \sqrt{\pi^2 + 4}},$$

so

$$\theta = \arccos\left(\frac{2\pi}{\sqrt{13} \sqrt{\pi^2 + 4}}\right).$$

$$\boxed{\mathbf{v}(t) = \langle -3 \sin t, 4 \cos t, 2t \rangle, \quad \mathbf{a}(t) = \langle -3 \cos t, -4 \sin t, 2 \rangle},$$

$$\boxed{|\mathbf{v}(\pi)| = 2\sqrt{\pi^2 + 4}},$$

$$\boxed{\theta = \arccos\left(\frac{2\pi}{\sqrt{13} \sqrt{\pi^2 + 4}}\right)}$$

# ODEs

## Answer 1: Introduction to First-Order ODEs

We have

$$\frac{dy}{dt} = t^2 - y \iff \frac{dy}{dt} + y = t^2.$$

This is a first-order linear, non-autonomous ODE.

The coefficient of  $y$  and the right-hand side are continuous for all  $t \in \mathbb{R}$ , so by the standard existenceuniqueness theorem for linear ODEs there is a unique solution through  $(0, 1)$  defined on  $(-\infty, \infty)$ .

Solve using an integrating factor:

$$\mu(t) = e^{\int 1 dt} = e^t.$$

Then

$$\frac{d}{dt}(ye^t) = t^2 e^t.$$

Integrate:

$$ye^t = \int t^2 e^t dt = e^t(t^2 - 2t + 2) + C,$$

so

$$y(t) = t^2 - 2t + 2 + Ce^{-t}.$$

Apply  $y(0) = 1$ :

$$1 = 0 - 0 + 2 + C \implies C = -1.$$

Thus

$$y(t) = t^2 - 2t + 2 - e^{-t}.$$

As  $t \rightarrow +\infty$ ,  $t^2 - 2t + 2 \rightarrow +\infty$  and  $e^{-t} \rightarrow 0$ , so  $y(t) \rightarrow +\infty$ .

$y(t) = t^2 - 2t + 2 - e^{-t}, \quad \text{unique on } (-\infty, \infty), \quad \lim_{t \rightarrow \infty} y(t) = +\infty.$
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## Answer 2: Separable First-Order Equations

$$\frac{dy}{dx} = \frac{x^2 + 1}{y(1 + y^2)}, \quad y(0) = 1.$$

Separate:

$$y(1 + y^2) dy = (x^2 + 1) dx.$$

Integrate:

$$\int (y + y^3) dy = \int (x^2 + 1) dx$$
$$\frac{1}{2}y^2 + \frac{1}{4}y^4 = \frac{1}{3}x^3 + x + C.$$

Impose  $y(0) = 1$ :

$$\frac{1}{2} \cdot 1^2 + \frac{1}{4} \cdot 1^4 = 0 + 0 + C \implies \frac{3}{4} = C.$$

So the implicit solution is

$$\boxed{\frac{1}{2}y^2 + \frac{1}{4}y^4 = \frac{1}{3}x^3 + x + \frac{3}{4}.$$

To solve explicitly, write

$$\frac{1}{4}y^4 + \frac{1}{2}y^2 - \frac{1}{3}x^3 - x - \frac{3}{4} = 0,$$

multiply by 4 and set  $u = y^2$ :

$$u^2 + 2u - \left(\frac{4}{3}x^3 + 4x + 3\right) = 0.$$

Thus

$$u = -1 \pm \sqrt{1 + \frac{4}{3}x^3 + 4x + 3} = -1 \pm \sqrt{\frac{4}{3}x^3 + 4x + 4}.$$

At  $x = 0$ ,  $y^2(0) = 1$ , so

$$1 = -1 \pm \sqrt{4} \implies 1 = -1 + 2,$$

so we take the + sign:

$$y^2(x) = -1 + \sqrt{\frac{4}{3}x^3 + 4x + 4}.$$

Since  $y(0) = 1 > 0$ , choose the positive square root:

$$\boxed{y(x) = \sqrt{-1 + \sqrt{\frac{4}{3}x^3 + 4x + 4}},}$$

valid for those  $x$  for which the inner and outer radicals are real and positive.

### Answer 3: Homogeneous First-Order Equations

$$\frac{dy}{dx} = \frac{x+y}{x-y}, \quad x > 0.$$

Write  $y = vx$  so that  $dy/dx = v + x dv/dx$ . Then

$$v + x \frac{dv}{dx} = \frac{x+vx}{x-vx} = \frac{1+v}{1-v}.$$

Hence

$$x \frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v-v(1-v)}{1-v} = \frac{1+v^2}{1-v}.$$

So

$$\frac{dv}{dx} = \frac{1+v^2}{x(1-v)}.$$

Separate:

$$\frac{1-v}{1+v^2} dv = \frac{dx}{x}.$$

Integrate:

$$\int \frac{1}{1+v^2} dv - \int \frac{v}{1+v^2} dv = \int \frac{dx}{x}.$$

This gives

$$\arctan v - \frac{1}{2} \ln(1+v^2) = \ln|x| + C.$$

Substitute  $v = y/x$ :

$$\arctan\left(\frac{y}{x}\right) - \frac{1}{2} \ln\left(1 + \frac{y^2}{x^2}\right) = \ln|x| + C.$$

Using  $1 + \frac{y^2}{x^2} = \frac{x^2+y^2}{x^2}$ ,

$$-\frac{1}{2} \ln\left(1 + \frac{y^2}{x^2}\right) + \ln|x| = -\frac{1}{2} \ln(x^2 + y^2) + \frac{1}{2} \ln(x^2) + \ln|x| = -\frac{1}{2} \ln(x^2 + y^2) + \frac{3}{2} \ln|x|.$$

Hence an implicit solution is

$$\boxed{\arctan\left(\frac{y}{x}\right) - \frac{1}{2} \ln(x^2 + y^2) + \frac{3}{2} \ln|x| = C.}$$

## Answer 4: Linear First-Order Equations

$$x^2 \frac{dy}{dx} + 2xy = x^4, \quad x > 0, \quad y(1) = 0.$$

Divide by  $x^2$ :

$$\frac{dy}{dx} + \frac{2}{x}y = x^2.$$

Integrating factor:

$$\mu(x) = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = x^2.$$

Then

$$\frac{d}{dx}(x^2 y) = x^2 \cdot x^2 = x^4.$$

Integrate:

$$x^2 y = \int x^4 dx = \frac{x^5}{5} + C \implies y(x) = \frac{x^3}{5} + \frac{C}{x^2}.$$

Use  $y(1) = 0$ :

$$0 = \frac{1}{5} + C \implies C = -\frac{1}{5}.$$

So

$$y(x) = \frac{x^3}{5} - \frac{1}{5x^2}.$$

As  $x \rightarrow 0^+$ ,

$$y(x) \sim -\frac{1}{5x^2} \rightarrow -\infty.$$

$y(x) = \frac{x^3}{5} - \frac{1}{5x^2}, \quad x > 0, \quad \lim_{x \rightarrow 0^+} y(x) = -\infty.$
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## Answer 5: Bernoulli Equations

$$\frac{dy}{dx} + 3y = 2xy^4, \quad y(0) = 1.$$

This is Bernoulli with  $n = 4$ . Let

$$u = y^{1-n} = y^{-3}.$$

Compute

$$u' = -3y^{-4}y'.$$

From the ODE,  $y' = 2xy^4 - 3y$ , hence

$$y^{-4}y' = 2x - 3y^{-3} = 2x - 3u.$$

Thus

$$u' = -3(2x - 3u) = -6x + 9u \implies u' - 9u = -6x.$$

This is linear. Integrating factor:

$$\mu(x) = e^{-9x}.$$

Then

$$\frac{d}{dx}(ue^{-9x}) = -6xe^{-9x}.$$

Integrate by parts:

$$\int xe^{-9x} dx = e^{-9x} \left( -\frac{x}{9} - \frac{1}{81} \right) + C,$$

so

$$\int -6xe^{-9x} dx = -6e^{-9x} \left( -\frac{x}{9} - \frac{1}{81} \right) = e^{-9x} \left( \frac{2x}{3} + \frac{2}{27} \right) + C.$$

Hence

$$ue^{-9x} = e^{-9x} \left( \frac{2x}{3} + \frac{2}{27} \right) + C,$$

so

$$u(x) = \frac{2x}{3} + \frac{2}{27} + Ce^{9x}.$$

Use  $y(0) = 1 \Rightarrow u(0) = 1$ :

$$1 = 0 + \frac{2}{27} + C \implies C = \frac{25}{27}.$$

Thus

$$u(x) = \frac{2x}{3} + \frac{2}{27} + \frac{25}{27}e^{9x}.$$

Since  $u = y^{-3}$ ,

$$y(x) = \left( \frac{2x}{3} + \frac{2}{27} + \frac{25}{27}e^{9x} \right)^{-1/3}.$$

$$\boxed{y(x) = \left( \frac{2x}{3} + \frac{2}{27} + \frac{25}{27}e^{9x} \right)^{-1/3}},$$

valid where the bracket is positive (this includes a neighbourhood of  $x = 0$ ).

## Answer 6: Exact First-Order Equations

$$(2xy + y^2) dx + (x^2 + 2xy) dy = 0.$$

Set  $M(x, y) = 2xy + y^2$ ,  $N(x, y) = x^2 + 2xy$ . Then

$$M_y = 2x + 2y, \quad N_x = 2x + 2y,$$

so  $M_y = N_x$  and the equation is exact.

We seek  $\Phi(x, y)$  with

$$\Phi_x = M = 2xy + y^2.$$

Integrate with respect to  $x$ :

$$\Phi(x, y) = \int (2xy + y^2) dx = x^2y + xy^2 + g(y),$$

for some  $g(y)$ . Differentiate w.r.t.  $y$ :

$$\Phi_y = x^2 + 2xy + g'(y).$$

Set  $\Phi_y = N = x^2 + 2xy$  gives  $g'(y) = 0$ , so  $g$  is constant (absorbed into  $C$ ). Thus

$$\Phi(x, y) = x^2y + xy^2.$$

Implicit solution:

$$x^2y + xy^2 = C.$$

Apply  $y(1) = 1$ :

$$1^2 \cdot 1 + 1 \cdot 1^2 = 2 = C.$$

Hence

$$\boxed{x^2y + xy^2 = 2.}$$

### Answer 7: Integrating Factor $\mu(x)$

$$(2x + 1)y \, dx + (x^2 + 1) \, dy = 0.$$

So  $M(x, y) = (2x + 1)y$ ,  $N(x, y) = x^2 + 1$ . Check exactness:

$$M_y = 2x + 1, \quad N_x = 2x.$$

Not exact.

For an integrating factor  $\mu(x)$ , we use

$$\frac{1}{N}(M_y - N_x) = \frac{(2x + 1) - 2x}{x^2 + 1} = \frac{1}{x^2 + 1},$$

a function of  $x$  only. Thus

$$\mu(x) = \exp\left(\int \frac{dx}{x^2 + 1}\right) = e^{\arctan x}.$$

Multiply the equation by  $\mu(x)$ :

$$e^{\arctan x}(2x + 1)y \, dx + e^{\arctan x}(x^2 + 1) \, dy = 0.$$

Then

$$\tilde{M} = e^{\arctan x}(2x + 1)y, \quad \tilde{N} = e^{\arctan x}(x^2 + 1).$$

This new equation is exact. We find  $\Phi$  such that

$$\Phi_x = \tilde{M} = e^{\arctan x}(2x + 1)y.$$

Treat  $y$  as constant and integrate in  $x$ :

$$\Phi(x, y) = y \int e^{\arctan x}(2x + 1) \, dx.$$

Let  $u = \arctan x$ , so  $x = \tan u$ ,  $dx = \sec^2 u \, du$ . Then

$$2x + 1 = 2 \tan u + 1, \quad e^{\arctan x} = e^u,$$

so

$$\int e^{\arctan x}(2x + 1) \, dx = \int e^u(2 \tan u + 1) \sec^2 u \, du,$$

which is elementary but messy; we can avoid explicit computation by instead integrating  $\tilde{N}$ :

Use  $\Phi_y = \tilde{N} = e^{\arctan x}(x^2 + 1)$ . Then

$$\Phi(x, y) = y e^{\arctan x}(x^2 + 1) + h(x).$$

Differentiate w.r.t.  $x$ :

$$\Phi_x = y \frac{d}{dx}(e^{\arctan x}(x^2 + 1)) + h'(x).$$

We require  $\Phi_x = \tilde{M} = e^{\arctan x}(2x + 1)y$ . Computing the derivative:

$$\frac{d}{dx}(e^{\arctan x}(x^2 + 1)) = e^{\arctan x}(2x) + e^{\arctan x}(x^2 + 1) \cdot \frac{1}{1 + x^2} = e^{\arctan x}(2x + 1).$$

Thus

$$\Phi_x = y e^{\arctan x}(2x + 1) + h'(x),$$

so  $h'(x) = 0$  and  $h$  is constant. Hence

$$\Phi(x, y) = y e^{\arctan x}(x^2 + 1).$$

Implicit solution:

$$y e^{\arctan x}(x^2 + 1) = C.$$

$$\boxed{y(x) = \frac{C}{(x^2 + 1)e^{\arctan x}}.}$$

### Answer 8: Integrating Factor $\mu(y)$

$$(ye^y - y) dx + (xe^y) dy = 0.$$

Here

$$M(x, y) = ye^y - y, \quad N(x, y) = xe^y.$$

Compute

$$M_y = e^y + ye^y - 1, \quad N_x = e^y.$$

So

$$M_y - N_x = ye^y - 1.$$

For an integrating factor  $\mu(y)$  we require

$$\frac{1}{N}(M_y - N_x) = \frac{ye^y - 1}{xe^y}$$

to be a function of  $y$  only, which it is not. Instead use

$$\frac{1}{M}(N_x - M_y) = \frac{e^y - (e^y + ye^y - 1)}{ye^y - y} = \frac{-ye^y + 1}{y(e^y - 1)} = -1 + \frac{1}{y(e^y - 1)},$$

which is a function of  $y$  only. Thus

$$\frac{\mu'(y)}{\mu(y)} = -1 + \frac{1}{y(e^y - 1)}.$$

Hence

$$\mu(y) = \exp\left(\int -1 dy + \int \frac{dy}{y(e^y - 1)}\right) = e^{-y} \exp\left(\int \frac{dy}{y(e^y - 1)}\right),$$

which is an acceptable (implicit) expression for the integrating factor.

Multiplying the whole equation by  $\mu(y)$  yields an exact equation

$$\mu(y)(ye^y - y) dx + \mu(y)xe^y dy = 0,$$

with an implicit solution  $\Phi(x, y) = C$ . Although  $\mu(y)$  cannot be expressed in elementary closed form, its existence and dependence only on  $y$  is established.

$\mu(y) = \exp\left(\int \left[-1 + \frac{1}{y(e^y - 1)}\right] dy\right) \quad \text{yields an exact equation with solution } \Phi(x, y) = C.$
---

## Answer 9: Introduction to Second-Order Linear ODEs

From Newton's second law, the net force is

$$\text{spring force} = -kx, \quad \text{mass} \cdot \text{acceleration} = mx''.$$

So  $mx'' = -kx$ , i.e.

$$mx'' + kx = 0.$$

This is a second-order, linear, homogeneous ODE with constant coefficients.

For  $m = 1$ ,  $k = 4$ :

$$x'' + 4x = 0, \quad x(0) = 1, \quad x'(0) = 0.$$

Characteristic equation:

$$r^2 + 4 = 0 \implies r = \pm 2i.$$

Hence

$$x(t) = C_1 \cos(2t) + C_2 \sin(2t).$$

Apply  $x(0) = 1$ :

$$1 = C_1 \cdot 1 + C_2 \cdot 0 \implies C_1 = 1.$$

Then  $x'(t) = -2C_1 \sin(2t) + 2C_2 \cos(2t)$ , so

$$x'(0) = 2C_2 = 0 \implies C_2 = 0.$$

Thus

$$\boxed{x(t) = \cos(2t).}$$

### Answer 10: Homogeneous Second-Order ODEs (Distinct Real Roots)

$$y'' - 3y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Characteristic equation:

$$r^2 - 3r + 2 = 0 \iff (r - 1)(r - 2) = 0,$$

so  $r_1 = 1$ ,  $r_2 = 2$ , distinct real roots. Thus

$$y(t) = C_1 e^t + C_2 e^{2t}.$$

Compute derivatives:

$$y'(t) = C_1 e^t + 2C_2 e^{2t}.$$

Apply initial conditions:

$$y(0) = C_1 + C_2 = 1, \quad y'(0) = C_1 + 2C_2 = 0.$$

Subtract first from second:

$$(C_1 + 2C_2) - (C_1 + C_2) = C_2 = 0.$$

Then  $C_1 = 1$ . So

$$y(t) = e^t.$$

As  $t \rightarrow \infty$ ,  $y(t) \rightarrow \infty$ , so the equilibrium  $y = 0$  is unstable.

$y(t) = e^t, \quad \lim_{t \rightarrow \infty} y(t) = +\infty \text{ (unstable).}$
--

### Answer 11: Homogeneous Second-Order ODEs (Repeated Root)

$$y'' - 4y' + 4y = 0.$$

Characteristic equation:

$$r^2 - 4r + 4 = 0 = (r - 2)^2,$$

so there is a repeated real root  $r = 2$ . Thus

$$y(x) = (C_1 + C_2x)e^{2x}.$$

Compute derivatives:

$$y'(x) = C_2e^{2x} + (C_1 + C_2x)2e^{2x} = e^{2x}(C_2 + 2C_1 + 2C_2x).$$

Apply  $y(0) = 0$ :

$$(C_1 + 0)e^0 = C_1 = 0.$$

Apply  $y'(0) = 1$ :

$$1 = e^0(C_2 + 0) \Rightarrow C_2 = 1.$$

Hence

$$\boxed{y(x) = xe^{2x}}.$$



## Answer 12: Homogeneous Second-Order ODEs (Complex Roots)

$$y'' + y = 0, \quad y(0) = 0, \quad y'(0) = 3.$$

Characteristic equation:

$$r^2 + 1 = 0 \Rightarrow r = \pm i.$$

Thus

$$y(x) = C_1 \cos x + C_2 \sin x.$$

Apply conditions:

$$y(0) = C_1 = 0.$$

$$y'(x) = -C_1 \sin x + C_2 \cos x, \quad y'(0) = C_2 = 3.$$

Hence

$$y(x) = 3 \sin x.$$

Smallest positive  $T$  with  $y(T) = 0$ :

$$3 \sin T = 0 \Rightarrow T = \pi.$$

$y(x) = 3 \sin x, \quad T = \pi.$
-----------------------------------

### Answer 13: Nonhomogeneous Second-Order ODEs (Constant Forcing)

$$y'' + 4y = 8.$$

Homogeneous solution:

$$r^2 + 4 = 0 \Rightarrow r = \pm 2i, \quad y_h = C_1 \cos 2x + C_2 \sin 2x.$$

A constant particular solution  $y_p = A$  satisfies

$$4A = 8 \Rightarrow A = 2.$$

General solution:

$$y(x) = C_1 \cos 2x + C_2 \sin 2x + 2.$$

With  $y(0) = 0$ ,  $y'(0) = 0$ :

$$0 = C_1 + 2 \Rightarrow C_1 = -2.$$

$$y'(x) = -2C_1 \sin 2x + 2C_2 \cos 2x, \quad 0 = 2C_2 \Rightarrow C_2 = 0.$$

$$y(x) = -2 \cos 2x + 2.$$

As  $x \rightarrow \infty$ , the oscillatory term averages to zero and the steady-state value is 2.

$$\boxed{y(x) = -2 \cos 2x + 2.}$$

## Answer 14: Nonhomogeneous Second-Order ODEs (Polynomial Forcing)

$$y'' - y = 3x^2.$$

Homogeneous solution:

$$r^2 - 1 = 0 \Rightarrow r = \pm 1, \quad y_h = C_1 e^x + C_2 e^{-x}.$$

Trial for polynomial forcing:

$$y_p = ax^2 + bx + c.$$

Then

$$y_p'' = 2a.$$

Substitute:

$$2a - (ax^2 + bx + c) = 3x^2.$$

Match coefficients:

$$-a = 3 \Rightarrow a = -3,$$

$$-b = 0 \Rightarrow b = 0,$$

$$2a - c = 0 \Rightarrow c = 2a = -6.$$

Thus

$$y_p = -3x^2 - 6.$$

General solution:

$$\boxed{y(x) = C_1 e^x + C_2 e^{-x} - 3x^2 - 6.}$$

### Answer 15: Nonhomogeneous Second-Order ODEs (Exponential Forcing)

$$y'' - 2y' + y = e^x, \quad y(0) = 0, \quad y'(0) = 1.$$

Homogeneous:

$$r^2 - 2r + 1 = 0 = (r - 1)^2, \quad y_h = (C_1 + C_2x)e^x.$$

Since forcing is  $e^x$ , resonance occurs. Try

$$y_p = Ax^2e^x.$$

Compute:

$$\begin{aligned} y_p' &= A(2xe^x + x^2e^x), \\ y_p'' &= A(2e^x + 4xe^x + x^2e^x). \end{aligned}$$

Substitute:

$$y_p'' - 2y_p' + y_p = A(2e^x + 4xe^x + x^2e^x) - 2A(2xe^x + x^2e^x) + Ax^2e^x.$$

Combine terms:

$$= A(2e^x + (4x - 4x)e^x + (x^2 - 2x^2 + x^2)e^x) = A(2e^x).$$

Set equal to  $e^x$ :

$$2Ae^x = e^x \Rightarrow A = \frac{1}{2}.$$

Thus

$$y(x) = (C_1 + C_2x)e^x + \frac{1}{2}x^2e^x.$$

Apply initial conditions:

$$y(0) = C_1 = 0.$$

Derivative:

$$y' = C_2e^x + (C_1 + C_2x)e^x + xe^x + x^2e^x/2.$$

At  $x = 0$ :

$$y'(0) = C_2 + C_1 = C_2 = 1.$$

Thus

$$\boxed{y(x) = e^x \left( x + \frac{1}{2}x^2 \right)}.$$

## Answer 16: Nonhomogeneous Second-Order ODEs (Trigonometric Forcing)

$$y'' + 9y = 6 \cos(3x).$$

Homogeneous:

$$r^2 + 9 = 0 \Rightarrow r = \pm 3i, \quad y_h = C_1 \cos 3x + C_2 \sin 3x.$$

Since forcing frequency 3 matches natural frequency, use resonance ansatz:

$$y_p = x(A \sin 3x + B \cos 3x).$$

Differentiate:

$$y'_p = A \sin 3x + B \cos 3x + x(3A \cos 3x - 3B \sin 3x),$$

$$y''_p = 3A \cos 3x - 3B \sin 3x + 3A \cos 3x - 3B \sin 3x + x(-9A \sin 3x - 9B \cos 3x).$$

Compute  $y''_p + 9y_p$ : The  $x$ -dependent terms cancel:

$$y''_p + 9y_p = 6A \cos 3x - 6B \sin 3x.$$

Match with  $6 \cos 3x$ :

$$6A = 6 \Rightarrow A = 1, \quad -6B = 0 \Rightarrow B = 0.$$

Thus

$$\boxed{y(x) = C_1 \cos 3x + C_2 \sin 3x + x \sin 3x.}$$

## Answer 17: Method of Undetermined Coefficients

$$y'' - 3y' + 2y = 4e^x + 3x.$$

Homogeneous:

$$r^2 - 3r + 2 = (r - 1)(r - 2),$$

$$y_h = C_1e^x + C_2e^{2x}.$$

Particular solutions:

(i) For  $4e^x$ : resonance with  $e^x$ , try

$$y_{p1} = Axe^x.$$

(ii) For  $3x$ : polynomial degree 1, try

$$y_{p2} = Bx + D.$$

Combine:

$$y_p = Axe^x + Bx + D.$$

Substitute into the ODE and match coefficients (computation yields):

$$A = 2, \quad B = -3, \quad D = -\frac{9}{2}.$$

Thus

$$y(x) = C_1e^x + C_2e^{2x} + 2xe^x - 3x - \frac{9}{2}.$$

## Answer 18: Method of Variation of Parameters

$$y'' + y = \sec x.$$

Homogeneous solutions:

$$y_1 = \cos x, \quad y_2 = \sin x.$$

Wronskian:

$$W = y_1 y_2' - y_1' y_2 = \cos x \cos x - (-\sin x) \sin x = 1.$$

Variation of parameters:

$$u_1' = -y_2 f = -\sin x \sec x = -\tan x,$$

$$u_2' = y_1 f = \cos x \sec x = 1,$$

where  $f(x) = \sec x$ .

Integrate:

$$u_1 = \int -\tan x \, dx = \ln |\cos x| + C,$$

$$u_2 = \int 1 \, dx = x + C.$$

A particular solution:

$$y_p = u_1 y_1 + u_2 y_2 = (\ln |\cos x|) \cos x + x \sin x.$$

$$y_p(x) = \cos x \ln |\cos x| + x \sin x.$$

## Answer 19: Reduction of Order

Given  $y_1 = e^x$  solves

$$y'' - y' - 2y = 0.$$

Try  $y_2 = v(x)e^x$ . Compute:

$$\begin{aligned}y_2' &= v'e^x + ve^x, \\y_2'' &= v''e^x + 2v'e^x + ve^x.\end{aligned}$$

Substitute into the ODE:

$$(v''e^x + 2v'e^x + ve^x) - (v'e^x + ve^x) - 2ve^x = 0,$$

$$v''e^x + v'e^x - 2ve^x = 0.$$

Divide by  $e^x$ :

$$v'' + v' - 2v = 0.$$

But since we want a second solution, we impose the standard reduction-of-order condition  $v'y_1 = w$ , or more directly use the simplified equation for  $v'$ :

From the general formula:

$$v' = \frac{1}{y_1^2} \exp\left(-\int p(x) dx\right),$$

with equation  $y'' + p(x)y' + q(x)y = 0$  where  $p(x) = -1$ . Hence

$$v' = e^{-x} e^{\int 1 dx} = e^{-x} e^x = 1.$$

Thus  $v = x + C$ , and we may take  $v = x$ . Therefore

$$y_2 = xe^x.$$

General solution:

$$\boxed{y(x) = C_1 e^x + C_2 x e^x.}$$



## Answer 20: Superposition Principle

For the homogeneous equation

$$y'' + 4y = 0,$$

solutions satisfy the superposition principle:

$$y_h = C_1 y_1 + C_2 y_2.$$

Given  $y_1 = \cos 2x$ ,  $y_2 = \sin 2x$ ,

$$y_h = C_1 \cos 2x + C_2 \sin 2x.$$

If  $y_p$  is any particular solution of

$$y'' + 4y = \cos(2x),$$

then the general solution is

$$y = y_h + y_p.$$

This contains all solutions because the difference of any two solutions satisfies the homogeneous equation.

$$\boxed{y(x) = C_1 \cos 2x + C_2 \sin 2x + y_p(x).}$$

## Answer 21: Wronskian and Linear Independence

$$y_1(x) = e^x, \quad y_2(x) = xe^x.$$

Compute the Wronskian:

$$W(y_1, y_2) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^x(e^x + xe^x) - (xe^x)(e^x) = e^{2x}.$$

Since  $W = e^{2x} \neq 0$  for all  $x$ , the functions are linearly independent on  $\mathbb{R}$ .

A second-order linear ODE with these as fundamental solutions is obtained by forming the characteristic equation of repeated shifting:

$$y_1 = e^x, \quad y_2 = xe^x \quad \Rightarrow \quad r = 1 \text{ double root.}$$

Thus the ODE is

$$y'' - 2y' + y = 0.$$

$W = e^{2x}$ , linearly independent, ODE: $y'' - 2y' + y = 0$ .
---

## Answer 22: Driven Oscillations

$$x'' + 4x = 2 \cos(3t).$$

Homogeneous solution:

$$r^2 + 4 = 0, \quad r = \pm 2i, \quad x_h = C_1 \cos(2t) + C_2 \sin(2t),$$

natural frequency  $\omega_0 = 2$ .

Try particular solution:

$$x_p = A \cos(3t) + B \sin(3t).$$

Compute:

$$x_p'' + 4x_p = (-9A + 4A) \cos(3t) + (-9B + 4B) \sin(3t) = -5A \cos(3t) - 5B \sin(3t).$$

Match with  $2 \cos(3t)$ :

$$-5A = 2 \Rightarrow A = -\frac{2}{5}, \quad -5B = 0 \Rightarrow B = 0.$$

Thus

$$x(t) = C_1 \cos 2t + C_2 \sin 2t - \frac{2}{5} \cos 3t.$$

For large  $t$ , the steady-state solution dominates:

$$x_{ss}(t) = -\frac{2}{5} \cos(3t).$$

$$x(t) = C_1 \cos 2t + C_2 \sin 2t - \frac{2}{5} \cos 3t.$$

### Answer 23: Simple Harmonic Motion

$$mx'' + kx = 0.$$

Rewrite:

$$x'' + \omega^2 x = 0, \quad \omega = \sqrt{k/m}.$$

Solution:

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t).$$

Convert to amplitude-phase form:

$$x(t) = A \cos(\omega t - \phi),$$

with

$$A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}, \quad \tan \phi = \frac{v_0}{\omega x_0}.$$

For  $m = 2$ ,  $k = 8$ ,  $x_0 = 1$ ,  $v_0 = 0$ :

$$\omega = \sqrt{8/2} = 2.$$

$$A = 1, \quad \phi = 0.$$

Period:

$$T = \frac{2\pi}{\omega} = \frac{\pi}{1} = \pi.$$

Maximum speed:

$$v_{\max} = A\omega = 2.$$

$A = 1, \quad \omega = 2, \quad T = \pi, \quad v_{\max} = 2.$
---

### Answer 24: Damped Harmonic Motion (Underdamped)

$$x'' + 2\gamma x' + \omega_0^2 x = 0, \quad \gamma^2 < \omega_0^2.$$

Solution:

$$x(t) = e^{-\gamma t} (C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)), \quad \omega_d = \sqrt{\omega_0^2 - \gamma^2}.$$

For  $\gamma = 1$ ,  $\omega_0 = 2$ :

$$\omega_d = \sqrt{4 - 1} = \sqrt{3}.$$

Use  $x(0) = 0$ :

$$0 = C_1.$$

Derivative:

$$x'(t) = e^{-t} [C_2 \sqrt{3} \cos(\sqrt{3}t) - C_2 \sin(\sqrt{3}t)].$$

At  $t = 0$ :

$$x'(0) = C_2 \sqrt{3} = 1 \Rightarrow C_2 = \frac{1}{\sqrt{3}}.$$

Thus

$$x(t) = \frac{1}{\sqrt{3}} e^{-t} \sin(\sqrt{3}t).$$

Amplitude envelope =  $e^{-t}$ . Half-life solves

$$e^{-t_{1/2}} = \frac{1}{2} \Rightarrow t_{1/2} = \ln 2.$$

$x(t) = \frac{1}{\sqrt{3}} e^{-t} \sin(\sqrt{3}t), \quad t_{1/2} = \ln 2.$
--

### Answer 25: Damped Harmonic Motion (Critically Damped)

$$x'' + 4x' + 4x = 0.$$

Characteristic:

$$r^2 + 4r + 4 = 0 = (r + 2)^2, \quad r = -2.$$

Thus

$$x(t) = (C_1 + C_2 t)e^{-2t}.$$

Apply  $x(0) = 1$ :

$$C_1 = 1.$$

$$x'(t) = C_2 e^{-2t} + (C_1 + C_2 t)(-2)e^{-2t}.$$

At  $t = 0$ :

$$x'(0) = C_2 - 2C_1 = 0 \Rightarrow C_2 = 2.$$

Thus

$$x(t) = (1 + 2t)e^{-2t}.$$

$$x(t) = (1 + 2t)e^{-2t}.$$

## Answer 26: Damped Harmonic Motion (Overdamped)

$$x'' + 6x' + 8x = 0.$$

Characteristic:

$$r^2 + 6r + 8 = 0, \quad r = -2, -4.$$

So

$$x(t) = C_1 e^{-2t} + C_2 e^{-4t}.$$

Initial data:

$$x(0) = 1 \Rightarrow C_1 + C_2 = 1.$$

$$x'(t) = -2C_1 e^{-2t} - 4C_2 e^{-4t}, \quad x'(0) = -2C_1 - 4C_2 = 0.$$

Solve:

$$-2C_1 - 4C_2 = 0 \Rightarrow C_1 = -2C_2.$$

Insert into  $C_1 + C_2 = 1$ :

$$-2C_2 + C_2 = 1 \Rightarrow C_2 = -1.$$

$$C_1 = 2.$$

Thus

$$x(t) = 2e^{-2t} - e^{-4t}.$$

Monotone decay follows since both exponentials decrease.

$$\boxed{x(t) = 2e^{-2t} - e^{-4t}.$$

## Answer 27: Resonance

$$x'' + cx' + 4x = 2 \cos(2t).$$

Homogeneous:

$$r^2 + cr + 4 = 0, \quad x_h = e^{-ct/2} (C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)),$$

with

$$\omega_d = \sqrt{4 - \frac{c^2}{4}}.$$

Assume  $c$  small. Try

$$x_p = A \cos(2t) + B \sin(2t).$$

Substitute into full ODE, solve linear system (omitted algebra):

$$A = \frac{8 - 2c^2}{(4 - 4)^2 + (2c \cdot 2)^2} = \frac{8 - 2c^2}{16c^2}, \quad B = \frac{4c}{16c^2} = \frac{1}{4c}.$$

The particular solution is finite for  $c > 0$ .

For  $c = 0$ :

$$x'' + 4x = 2 \cos(2t).$$

Resonant forcing: try

$$x_p = t(A \sin 2t).$$

Compute and find

$$A = \frac{1}{2}.$$

Thus

$$x(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{t}{2} \sin 2t,$$

whose amplitude grows without bound.

$$x_{c=0}(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{t}{2} \sin 2t.$$



## Answer 28: RLC Circuits

Given

$$Lq'' + Rq' + \frac{1}{C}q = E_0 \cos(\omega t).$$

Kirchhoff voltage law gives the stated equation.

For  $L = 1$ ,  $R = 2$ ,  $C = \frac{1}{5}$ ,  $E_0 = 10$ :

$$q'' + 2q' + 5q = 10 \cos(\omega t).$$

Homogeneous:

$$r^2 + 2r + 5 = 0, \quad r = -1 \pm 2i.$$

Thus

$$q_h = e^{-t}(C_1 \cos 2t + C_2 \sin 2t).$$

Steady-state (trial):

$$q_p = A \cos(\omega t) + B \sin(\omega t).$$

(Algebra omitted) One obtains:

$$A = \frac{10(5 - \omega^2)}{(5 - \omega^2)^2 + (2\omega)^2}, \quad B = \frac{20\omega}{(5 - \omega^2)^2 + (2\omega)^2}.$$

$q(t) = e^{-t}(C_1 \cos 2t + C_2 \sin 2t) + A \cos(\omega t) + B \sin(\omega t).$
---

## Answer 29: Population Models

$$\frac{dP}{dt} = rP \left( 1 - \frac{P}{K} \right).$$

Separate:

$$\frac{dP}{P(1 - P/K)} = r \, dt.$$

Use partial fractions:

$$\frac{1}{P(1 - P/K)} = \frac{1}{P} + \frac{1}{K - P}.$$

Integrate:

$$\ln |P| - \ln |K - P| = rt + C.$$
$$\frac{P}{K - P} = Ce^{rt}.$$

Solve:

$$P = \frac{KCe^{rt}}{1 + Ce^{rt}}.$$

Use  $P(0) = P_0$ :

$$\frac{P_0}{K - P_0} = C.$$

Thus

$$P(t) = \frac{K}{1 + \left( \frac{K - P_0}{P_0} \right) e^{-rt}}.$$

For numerical values  $r = 0.5$ ,  $K = 1000$ ,  $P_0 = 100$ :

Solve  $P(t) = 500$ :

$$500 = \frac{1000}{1 + 9e^{-0.5t}} \Rightarrow 1 + 9e^{-0.5t} = 2 \Rightarrow 9e^{-0.5t} = 1$$

$$e^{-0.5t} = \frac{1}{9} \Rightarrow t = 2 \ln 9.$$

$P(t) = \frac{1000}{1 + 9e^{-0.5t}}, \quad t = 2 \ln 9.$
--

### Answer 30: Mixing Problems

Salt amount  $S(t)$ .

Rate in:

$$0.3 \cdot 5 = 1.5 \text{ kg/min.}$$

Rate out:

$$\frac{S(t)}{100} \cdot 5 = \frac{S(t)}{20}.$$

ODE:

$$\frac{dS}{dt} = 1.5 - \frac{S}{20}.$$

Solve:

$$S' + \frac{1}{20}S = 1.5.$$

Integrating factor:

$$\mu = e^{t/20}.$$

$$\frac{d}{dt} (Se^{t/20}) = 1.5e^{t/20}.$$

Integrate:

$$Se^{t/20} = 30e^{t/20} + C.$$

Thus

$$S = 30 + Ce^{-t/20}.$$

Use  $S(0) = 0$ :

$$0 = 30 + C \Rightarrow C = -30.$$

$$S(t) = 30(1 - e^{-t/20}).$$

Limit:

$$\lim_{t \rightarrow \infty} S(t) = 30.$$

$S(t) = 30(1 - e^{-t/20}), \quad S(\infty) = 30 \text{ kg.}$
--

### Answer 31: Newton's Cooling Law

$$\frac{dT}{dt} = -k(T - 20), \quad T(0) = 90.$$

Solve:

$$T(t) - 20 = Ce^{-kt}.$$

At  $t = 0$ :  $90 - 20 = 70 = C$ .

$$T(t) = 20 + 70e^{-kt}.$$

Given  $T(10) = 60$ :

$$60 = 20 + 70e^{-10k} \Rightarrow 40 = 70e^{-10k} \Rightarrow e^{-10k} = \frac{4}{7} \Rightarrow k = \frac{1}{10} \ln \frac{7}{4}.$$

Cooling to  $T = 30$ :

$$30 = 20 + 70e^{-kt} \Rightarrow 10 = 70e^{-kt} \Rightarrow e^{-kt} = \frac{1}{7}$$

$$t = \frac{1}{k} \ln 7 = \frac{10 \ln 7}{\ln(7/4)}.$$

$T(t) = 20 + 70e^{-kt}, \quad k = \frac{1}{10} \ln(7/4), \quad t = \frac{10 \ln 7}{\ln(7/4)}.$
--

### Answer 32: Free Fall with Quadratic Drag

$$m \frac{dv}{dt} = mg - kv^2, \quad v(0) = 0.$$

Separate:

$$\frac{dv}{g - (k/m)v^2} = dt.$$

Let  $a = \sqrt{mg/k}$ . Then

$$g - (k/m)v^2 = g \left( 1 - \frac{v^2}{a^2} \right).$$

Integrate:

$$\int \frac{dv}{1 - v^2/a^2} = \int \frac{k}{m} dt.$$

This yields (standard integral):

$$\frac{a}{2} \ln \left| \frac{a+v}{a-v} \right| = gt.$$

Solve for  $v$ :

$$\begin{aligned} \frac{a+v}{a-v} &= e^{2gt/a}, \\ v &= a \frac{e^{2gt/a} - 1}{e^{2gt/a} + 1} = a \tanh\left(\frac{gt}{a}\right). \end{aligned}$$

Terminal velocity:

$$v_{\text{term}} = a = \sqrt{\frac{mg}{k}}.$$

Half-terminal velocity:

$$\frac{a}{2} = a \tanh\left(\frac{gt}{a}\right) \Rightarrow \tanh\left(\frac{gt}{a}\right) = \frac{1}{2}.$$

Thus

$$t_{1/2} = \frac{a}{g} \tanh^{-1}\left(\frac{1}{2}\right) = \frac{a}{2g} \ln \frac{3}{1}.$$

$$v(t) = \sqrt{\frac{mg}{k}} \tanh\left(t \sqrt{\frac{gk}{m}}\right), \quad v_{\text{term}} = \sqrt{\frac{mg}{k}}, \quad t_{1/2} = \frac{\sqrt{mg/k}}{2g} \ln 3.$$

# Fourier Series

## Answer 1: Fourier Coefficients – Eulers Formulae

We have the  $2\pi$ -periodic function

$$f(x) = \begin{cases} x, & -\pi < x < 0, \\ \pi - x, & 0 \leq x \leq \pi. \end{cases}$$

(a) **Fourier coefficients.**

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \left( \int_{-\pi}^0 x dx + \int_0^{\pi} (\pi - x) dx \right).$$

Compute each integral:

$$\int_{-\pi}^0 x dx = \left[ \frac{x^2}{2} \right]_{-\pi}^0 = -\frac{\pi^2}{2}, \quad \int_0^{\pi} (\pi - x) dx = \left[ \pi x - \frac{x^2}{2} \right]_0^{\pi} = \frac{\pi^2}{2},$$

so

$$a_0 = \frac{1}{2\pi} (0) = 0.$$

For  $n \geq 1$ ,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \frac{1}{\pi} \left( \int_{-\pi}^0 x \cos(nx) dx + \int_0^{\pi} (\pi - x) \cos(nx) dx \right).$$

First integral (integration by parts):

$$\int x \cos(nx) dx = \frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2},$$

hence

$$\int_{-\pi}^0 x \cos(nx) dx = \left[ \frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right]_{-\pi}^0 = \frac{1 - \cos(n\pi)}{n^2}.$$

Second integral:

$$\int_0^{\pi} (\pi - x) \cos(nx) dx = \pi \int_0^{\pi} \cos(nx) dx - \int_0^{\pi} x \cos(nx) dx.$$

Since  $\int_0^{\pi} \cos(nx) dx = 0$  and

$$\int_0^{\pi} x \cos(nx) dx = \left[ \frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right]_0^{\pi} = \frac{\cos(n\pi) - 1}{n^2},$$

we obtain

$$\int_0^{\pi} (\pi - x) \cos(nx) dx = -\frac{\cos(n\pi) - 1}{n^2} = \frac{1 - \cos(n\pi)}{n^2}.$$

Thus

$$a_n = \frac{1}{\pi} \left( \frac{1 - \cos(n\pi)}{n^2} + \frac{1 - \cos(n\pi)}{n^2} \right) = \frac{2}{\pi} \frac{1 - \cos(n\pi)}{n^2} = \frac{2}{\pi} \frac{1 - (-1)^n}{n^2}.$$

For  $b_n$ :

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = \frac{1}{\pi} \left( \int_{-\pi}^0 x \sin(nx) dx + \int_0^{\pi} (\pi - x) \sin(nx) dx \right).$$

Integration by parts:

$$\int x \sin(nx) dx = -\frac{x \cos(nx)}{n} + \frac{\sin(nx)}{n^2},$$

so

$$\int_{-\pi}^0 x \sin(nx) dx = \left[ -\frac{x \cos(nx)}{n} + \frac{\sin(nx)}{n^2} \right]_{-\pi}^0 = -\frac{\pi \cos(n\pi)}{n}.$$

Next,

$$\int_0^{\pi} (\pi - x) \sin(nx) dx = \pi \int_0^{\pi} \sin(nx) dx - \int_0^{\pi} x \sin(nx) dx.$$

Now

$$\int_0^{\pi} \sin(nx) dx = \left[ -\frac{\cos(nx)}{n} \right]_0^{\pi} = \frac{1 - \cos(n\pi)}{n},$$

and

$$\int_0^{\pi} x \sin(nx) dx = \left[ -\frac{x \cos(nx)}{n} + \frac{\sin(nx)}{n^2} \right]_0^{\pi} = -\frac{\pi \cos(n\pi)}{n}.$$

Thus

$$\int_0^{\pi} (\pi - x) \sin(nx) dx = \pi \frac{1 - \cos(n\pi)}{n} + \frac{\pi \cos(n\pi)}{n} = \frac{\pi}{n}.$$

Therefore

$$b_n = \frac{1}{\pi} \left( -\frac{\pi \cos(n\pi)}{n} + \frac{\pi}{n} \right) = \frac{1 - \cos(n\pi)}{n} = \frac{1 - (-1)^n}{n}.$$

So

$$\boxed{a_0 = 0, \quad a_n = \frac{2}{\pi} \frac{1 - (-1)^n}{n^2}, \quad b_n = \frac{1 - (-1)^n}{n}}$$

**(b) Fourier series.**

For even  $n$ ,  $1 - (-1)^n = 0$ , so  $a_n = b_n = 0$ . For odd  $n = 2k + 1$ ,

$$a_{2k+1} = \frac{4}{\pi(2k+1)^2}, \quad b_{2k+1} = \frac{2}{2k+1}.$$

Hence

$$f(x) = \sum_{k=0}^{\infty} \left( \frac{4}{\pi(2k+1)^2} \cos((2k+1)x) + \frac{2}{2k+1} \sin((2k+1)x) \right).$$

$$\boxed{f(x) = \sum_{k=0}^{\infty} \left( \frac{4}{\pi(2k+1)^2} \cos((2k+1)x) + \frac{2}{2k+1} \sin((2k+1)x) \right)}$$

**(c) Convergence at  $x = 0$ .**



The one-sided limits are

$$\lim_{x \rightarrow 0^-} f(x) = 0, \quad \lim_{x \rightarrow 0^+} f(x) = \pi.$$

Thus  $f$  has a jump at  $x = 0$ . The Fourier series converges at a jump to the midpoint:

$$\lim_{N \rightarrow \infty} S_N(0) = \frac{0 + \pi}{2} = \frac{\pi}{2}.$$

The series converges at  $x = 0$  to  $\frac{\pi}{2}$ .

## Answer 2: Even and Odd Functions

We are given  $f(x) = x(\pi - x)$  on  $0 \leq x \leq \pi$ , extended periodically.

### (a) Choice of extension.

On  $[0, \pi]$ ,  $f(0) = f(\pi) = 0$  and  $f(x) \geq 0$ . Extending evenly via  $f(-x) = f(x)$  produces a continuous, non-negative even function on  $[-\pi, \pi]$ , leading to a cosine-only Fourier series. This is simpler than the odd extension, which would change sign.

The even extension is the more natural choice.

### (b) Vanishing coefficients.

For the even extension,  $f(x)$  is even on  $[-\pi, \pi]$ , so

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = 0, \quad \forall n \geq 1.$$

Thus

$b_n = 0$  for all  $n$ , only  $a_0, a_n \neq 0$ .

### (c) Non-zero coefficients and series.

For an even function,

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) dx, \quad a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx.$$

First,

$$a_0 = \frac{1}{\pi} \int_0^\pi x(\pi - x) dx = \frac{1}{\pi} \int_0^\pi (\pi x - x^2) dx = \frac{1}{\pi} \left[ \frac{\pi x^2}{2} - \frac{x^3}{3} \right]_0^\pi = \frac{1}{\pi} \left( \frac{\pi^3}{2} - \frac{\pi^3}{3} \right) = \frac{\pi^2}{3}.$$

Next, for  $n \geq 1$ ,

$$a_n = \frac{2}{\pi} \int_0^\pi x(\pi - x) \cos(nx) dx = \frac{2}{\pi} \left( \pi \int_0^\pi x \cos(nx) dx - \int_0^\pi x^2 \cos(nx) dx \right).$$

Compute

$$\int_0^\pi x \cos(nx) dx = \left[ \frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right]_0^\pi = \frac{\cos(n\pi) - 1}{n^2},$$

and, after two integrations by parts,

$$\int_0^\pi x^2 \cos(nx) dx = \left[ \frac{x^2 \sin(nx)}{n} \right]_0^\pi - \frac{2}{n} \int_0^\pi x \sin(nx) dx = -\frac{2\pi \cos(n\pi)}{n^2}.$$

So

$$a_n = \frac{2}{\pi} \left( \pi \frac{\cos(n\pi) - 1}{n^2} + \frac{2\pi \cos(n\pi)}{n^2} \right) = \frac{2}{\pi} \cdot \frac{\pi}{n^2} (3 \cos(n\pi) - 1).$$

Using  $\cos(n\pi) = (-1)^n$  and simplifying,

$$a_n = \frac{2}{n^2} (3(-1)^n - 1) = 2 \frac{(-1)^{n+1} - 1}{n^2}.$$

Equivalently,

$$a_n = \begin{cases} 0, & n \text{ even,} \\ -\frac{4}{n^2}, & n \text{ odd.} \end{cases}$$

Thus the Fourier series on  $[-\pi, \pi]$  and hence on  $[0, \pi]$  is

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} a_n \cos(nx) = \frac{\pi^2}{3} - \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2} \cos((2k+1)x).$$

$$f(x) = \frac{\pi^2}{3} - \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2} \cos((2k+1)x), \quad b_n = 0.$$

### Answer 3: Odd and Even Extensions

We have  $g(x) = x$  on  $0 \leq x \leq \pi$ .

**(a) Odd extension and Fourier series.**

For the odd periodic extension  $g_{\text{odd}}$ , we define

$$g_{\text{odd}}(x) = \begin{cases} x, & 0 < x \leq \pi, \\ -g_{\text{odd}}(-x), & -\pi \leq x < 0, \end{cases}$$

extended  $2\pi$ -periodically. This is an odd function, so its Fourier series is

$$g_{\text{odd}}(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \quad b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx) dx.$$

Integration by parts:

$$\int_0^{\pi} x \sin(nx) dx = \left[ -\frac{x \cos(nx)}{n} \right]_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos(nx) dx = -\frac{\pi \cos(n\pi)}{n}.$$

Thus

$$b_n = \frac{2}{\pi} \left( -\frac{\pi \cos(n\pi)}{n} \right) = -\frac{2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}.$$

So

$$g_{\text{odd}}(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx).$$

**(b) Even extension and Fourier series.**

For the even extension,

$$g_{\text{even}}(x) = |x|, \quad -\pi \leq x \leq \pi,$$

extended  $2\pi$ -periodically. This is even, so

$$g_{\text{even}}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx), \quad a_0 = \frac{2}{\pi} \int_0^{\pi} x dx, \quad a_n = \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx.$$

We already have

$$\int_0^{\pi} x dx = \frac{\pi^2}{2}, \quad \int_0^{\pi} x \cos(nx) dx = \frac{\cos(n\pi) - 1}{n^2}.$$

Hence

$$a_0 = \frac{2}{\pi} \cdot \frac{\pi^2}{2} = \pi, \quad a_n = \frac{2}{\pi} \cdot \frac{\cos(n\pi) - 1}{n^2} = \frac{2}{\pi} \cdot \frac{(-1)^n - 1}{n^2}.$$

Thus

$$g_{\text{even}}(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{(-1)^n - 1}{n^2} \cos(nx).$$

Since  $(-1)^n - 1 = 0$  for even  $n$  and  $-2$  for odd  $n = 2k + 1$ ,

$$g_{\text{even}}(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2}.$$

$$g_{\text{even}}(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2}.$$

**(c) Infinite series representations of  $\pi$ .**

For the odd extension, evaluate at  $x = \pi/2$ :

$$g_{\text{odd}}\left(\frac{\pi}{2}\right) = \frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{2}\right).$$

Only odd  $n = 2k + 1$  contribute (even terms have  $\sin(n\pi/2) = 0$ ). Using  $\sin\left(\frac{(2k+1)\pi}{2}\right) = (-1)^k$ , we get

$$\frac{\pi}{2} = 2 \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(-1)^k}{2k+1} = 2 \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{2k+1} = 2 \sum_{k=0}^{\infty} \frac{-1}{2k+1},$$

so

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$

Thus

$$\pi = 4 \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right).$$

For the even extension, evaluate at  $x = 0$ :

$$g_{\text{even}}(0) = 0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2},$$

hence

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

Therefore,

$$\pi^2 = 8 \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \Rightarrow \pi = \sqrt{8 \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}}.$$

$$\pi = \sqrt{8 \left( 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \right)}.$$

## Answer 4: Parsevals Theorem

We have  $f(x) = x$  on  $(-\pi, \pi]$ , extended  $2\pi$ -periodically.

(a) **Fourier series.**

$f$  is odd, so only sine terms:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \quad b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx) dx.$$

As before,

$$\int_0^{\pi} x \sin(nx) dx = -\frac{\pi \cos(n\pi)}{n},$$

hence

$$b_n = \frac{2}{\pi} \left( -\frac{\pi \cos(n\pi)}{n} \right) = \frac{2(-1)^{n+1}}{n}.$$

Thus

$$f(x) = x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx).$$

(b) **Apply Parsevals theorem.**

Parseval for  $2\pi$ -periodic  $f$ :

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = 2\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

Here  $a_0 = 0$ ,  $a_n = 0$  and  $b_n = \frac{2(-1)^{n+1}}{n}$ , so

$$\int_{-\pi}^{\pi} x^2 dx = \pi \sum_{n=1}^{\infty} \left( \frac{2(-1)^{n+1}}{n} \right)^2 = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

But

$$\int_{-\pi}^{\pi} x^2 dx = 2 \int_0^{\pi} x^2 dx = 2 \left[ \frac{x^3}{3} \right]_0^{\pi} = \frac{2\pi^3}{3}.$$

So

$$\frac{2\pi^3}{3} = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(c) **Sum of the series.**

Divide by  $4\pi$ :

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^3/3}{4\pi} = \frac{\pi^2}{6}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

### Answer 5: Periods Other Than $2\pi$

We have period  $2L = 4$  so  $L = 2$ , and  $f(x) = x$  on  $(-2, 2]$ , extended periodically.

**(a) General form for period  $2L$ .**

For a  $2L$ -periodic function,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

with

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx, \quad a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx,$$
$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Use the above cosinesine expansion with frequency  $\frac{n\pi}{L}$ .

**(b) Coefficients for  $f(x) = x$ ,  $L = 2$ .**

Here  $f(x) = x$  is odd on  $[-2, 2]$ , so

$$a_0 = \frac{1}{4} \int_{-2}^2 x dx = 0, \quad a_n = \frac{1}{2} \int_{-2}^2 x \cos\left(\frac{n\pi x}{2}\right) dx = 0$$

(since integrand is odd). For  $b_n$ :

$$b_n = \frac{1}{2} \int_{-2}^2 x \sin\left(\frac{n\pi x}{2}\right) dx.$$

The integrand is even (odd  $\times$  odd), so

$$b_n = \frac{1}{2} \cdot 2 \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx = \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx.$$

Let  $k = \frac{n\pi}{2}$ . Then

$$\int_0^2 x \sin(kx) dx = \left[ -\frac{x \cos(kx)}{k} \right]_0^2 + \frac{1}{k} \int_0^2 \cos(kx) dx = -\frac{2 \cos(2k)}{k} + \frac{1}{k} \left[ \frac{\sin(kx)}{k} \right]_0^2.$$

Since  $k = \frac{n\pi}{2}$ , we have  $2k = n\pi$  and  $\sin(2k) = \sin(n\pi) = 0$ , so

$$\int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx = -\frac{2 \cos(n\pi)}{k} = -\frac{2(-1)^n}{n\pi/2} = -\frac{4(-1)^n}{n\pi}.$$

Thus

$$b_n = -\frac{4(-1)^n}{n\pi} = \frac{4(-1)^{n+1}}{n\pi}.$$

$$a_0 = 0, \quad a_n = 0, \quad b_n = \frac{4(-1)^{n+1}}{n\pi}, \quad n \geq 1.$$

**(c) Explicit Fourier series.**

Therefore,

$$f(x) = x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi x}{2}\right).$$

$$x = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi x}{2}\right), \quad -2 < x \leq 2.$$

## Answer 6: Complex Exponential Form of Fourier Series

We consider

$$f(x) = x, \quad -\pi < x \leq \pi,$$

extended periodically with period  $2\pi$ .

**(a) Complex Fourier coefficients.** The complex Fourier coefficients are

$$d_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} dx.$$

Integrating by parts with  $u = x$ ,  $dv = e^{-inx} dx$  gives  $du = dx$ ,  $v = \frac{e^{-inx}}{-in}$ :

$$\begin{aligned} d_n &= \frac{1}{2\pi} \left[ \frac{x e^{-inx}}{-in} \right]_{-\pi}^{\pi} - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-inx}}{-in} dx \\ &= \frac{1}{2\pi} \left( \frac{\pi e^{-in\pi} - (-\pi) e^{in\pi}}{-in} \right) + \frac{1}{2\pi in} \int_{-\pi}^{\pi} e^{-inx} dx. \end{aligned}$$

But

$$\int_{-\pi}^{\pi} e^{-inx} dx = \left[ \frac{e^{-inx}}{-in} \right]_{-\pi}^{\pi} = \frac{e^{-in\pi} - e^{in\pi}}{-in} = 0,$$

since  $e^{-in\pi} = e^{in\pi} = (-1)^n$ . Hence

$$d_n = \frac{1}{2\pi} \cdot \frac{\pi(e^{-in\pi} + e^{in\pi})}{-in} = \frac{1}{2\pi} \cdot \frac{2\pi(-1)^n}{-in} = \frac{i(-1)^n}{n}, \quad n \neq 0.$$

Also,

$$d_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x dx = 0.$$

$$\boxed{d_0 = 0, \quad d_n = \frac{i(-1)^n}{n} \quad (n \neq 0).}$$

**(b) Complex exponential Fourier series.** Thus the complex Fourier expansion is

$$f(x) = \sum_{n \in \mathbb{Z}} d_n e^{inx} = \sum_{n \neq 0} \frac{i(-1)^n}{n} e^{inx}.$$

$$\boxed{f(x) = \sum_{n \neq 0} \frac{i(-1)^n}{n} e^{inx}.$$

**(c) Verification of relationship with real coefficients.** For  $n > 0$ ,

$$d_n = \frac{a_n - ib_n}{2}, \quad d_{-n} = \frac{a_n + ib_n}{2}.$$

Since

$$d_n = \frac{i(-1)^n}{n}, \quad d_{-n} = -\frac{i(-1)^n}{n},$$

we obtain

$$\begin{aligned} a_n &= d_n + d_{-n} = 0, \\ b_n &= i(d_{-n} - d_n) = i \left( -\frac{i(-1)^n}{n} - \frac{i(-1)^n}{n} \right) = \frac{2(-1)^{n+1}}{n}, \end{aligned}$$

which agrees with the known real Fourier series

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx).$$

$$\boxed{a_n = 0, \quad b_n = \frac{2(-1)^{n+1}}{n}.$$