Mathematics Variant Question Set

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Vectors

Question 1: Vectors and Scalars

A particle has temperature $T(t) = 300 + 4t$ K and velocity $\mathbf{v}(t) = (2t)\,\mathbf{i} + (3 - t)\,\mathbf{j}$ m/s. At $t = 5$ s, compute

1. the instantaneous rate of change of temperature,

2. the speed of the particle,

3. the angle between $\mathbf{v}(5)$ and the vector $\mathbf{w} = 3\mathbf{i} - 4\mathbf{j}$.

Question 2: Vector Algebra

Let $\mathbf{a} = (1, 2, -1)$, $\mathbf{b} = (3, -1, 4)$ and $\mathbf{c} = (-2, 1, 5)$. Evaluate

1. $(\mathbf{a} + \mathbf{b}) - 2\mathbf{c}$,

2. all real numbers λ such that $\mathbf{a} - \lambda\mathbf{b}$ is parallel to $\mathbf{c}$.

Question 3: Vector Spaces and Coordinate Bases

Consider vectors $\mathbf{u} = (1, 2, 1)$, $\mathbf{v} = (2, -1, 3)$ and $\mathbf{w} = (3, 1, 4)$ in $\mathbb{R}^3$.

1. Show whether $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ forms a basis of $\mathbb{R}^3$.

2. If it does, express the vector $\mathbf{p} = (4, 3, 2)$ uniquely as a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

Question 4: Scalar Dot Product

Let $\mathbf{a} = (t, 1, 2)$ and $\mathbf{b} = (2, -1, 3)$.

1. Find all real values of t for which $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

2. For the value(s) of t obtained, compute $|\mathbf{a}|$ and $|\mathbf{b}|$.

Question 5: Vector Cross Product

Let $\mathbf{a} = (2, 1, -3)$ and $\mathbf{b} = (-1, 4, 2)$.

1. Compute $\mathbf{a} \times \mathbf{b}$.

2. Hence find the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

Question 6: Scalar Triple Product

Three vectors are given by $\mathbf{a} = (1, 2, 3)$, $\mathbf{b} = (2, -1, 4)$ and $\mathbf{c} = (0, 5, -2)$.

1. Compute the scalar triple product $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$.

2. Hence determine the volume of the parallelepiped they form.

Question 7: Equations of Lines

A line ℓ passes through the point $(1, -2, 3)$ and is perpendicular to the plane $2x - y + 4z = 7$.

1. Find a parametric equation of ℓ.

2. Determine the shortest distance from the point $(4, 1, 0)$ to the line ℓ.

Question 8: Equations of Planes

A plane contains the point $P = (2, 1, -1)$ and has direction vectors $\mathbf{b} = (1, -1, 2)$ and $\mathbf{c} = (3, 0, 1)$.

1. Write down a parametric equation of the plane.

2. Convert it into Cartesian form $ax + by + cz = d$.

3. Compute the distance from the origin to the plane.

Question 9: Position, Velocity & Acceleration

A particle moves according to

$$\mathbf{r}(t) = \langle\, 3\cos t, \ 4\sin t, \ t^2 \,\rangle.$$

1. Compute the velocity and acceleration vectors.

2. Determine the speed at $t = \pi$.

3. Find the angle between $\mathbf{v}(\pi)$ and $\mathbf{a}(\pi)$.

ODEs

Question 1: Introduction to First-Order ODEs

Consider the initial value problem

$$\frac{dy}{dt} = t^2 - y, \qquad y(0) = 1.$$

1. Classify this differential equation according to *order*, *linearity*, and *autonomous/non-autonomous*.

2. Use an appropriate existence and uniqueness theorem to justify that there is a unique solution through the point $(t, y) = (0, 1)$, and determine the maximal interval of existence.

3. Solve the initial value problem explicitly and determine $\lim_{t \to +\infty} y(t)$.

Question 2: Separable First-Order Equations

Solve the initial value problem

$$\frac{dy}{dx} = \frac{(x^2 + 1)}{y(1 + y^2)}, \qquad y(0) = 1.$$

1. Separate variables and integrate to obtain an implicit relation between x and y.

2. Solve explicitly for $y(x)$.

3. Determine the largest interval containing $x = 0$ on which your explicit solution is valid.

Question 3: Homogeneous First-Order Equations

Consider the first-order ODE

$$\frac{dy}{dx} = \frac{x+y}{x-y}, \qquad x > 0.$$

1. Show that the right-hand side can be written as a function of the ratio $v = y/x$, and hence the equation is homogeneous.

2. Perform the substitution $y = vx$ and derive the resulting differential equation for $v(x)$.

3. Solve the ODE and express the general solution in the form of an implicit relation between x and y.

Question 4: Linear First-Order Equations

Solve the linear first-order initial value problem

$$x^2 \frac{dy}{dx} + 2xy = x^4, \qquad x > 0, \qquad y(1) = 0.$$

1. Rewrite the equation in standard linear form $\frac{dy}{dx} + P(x)y = Q(x)$ and determine the integrating factor.

2. Solve the differential equation to obtain $y(x)$ for $x > 0$.

3. Determine $\lim_{x \to 0^+} y(x)$, or show that the limit does not exist.

Question 5: Bernoulli Equations

Consider the Bernoulli equation

$$\frac{dy}{dx} + 3y = 2xy^4, \qquad y(0) = 1.$$

1. Rewrite the equation in the standard Bernoulli form and perform the substitution $u = y^{1-n}$ for an appropriate power n.

2. Derive and solve the resulting linear differential equation for $u(x)$.

3. Hence obtain the explicit solution $y(x)$ and determine the maximal interval of existence of the solution.

Question 6: Exact First-Order Equations

Consider the differential equation

$$(2xy + y^2)\, dx + (x^2 + 2xy)\, dy = 0.$$

1. Show that the equation is exact by verifying the appropriate condition on the coefficients.

2. Find a potential function $\Phi(x, y)$ such that $d\Phi = 0$ corresponds to the given differential equation.

3. Write down the general implicit solution $\Phi(x, y) = C$ and find the particular solution satisfying $y(1) = 1$.

Question 7: Integrating Factor $\mu(x)$

The differential equation
$$\big((2x+1)y\big)\,dx + (x^2+1)\,dy = 0$$
is not exact.

1. Verify that the equation is not exact by computing the relevant partial derivatives.

2. Show that there exists an integrating factor depending only on x, i.e. $\mu = \mu(x)$, and derive a formula for $\mu(x)$.

3. Find $\mu(x)$ explicitly, multiply the equation by $\mu(x)$ to make it exact, and determine the implicit general solution.

Question 8: Integrating Factor $\mu(y)$

Consider the differential equation
$$\big(ye^y - y\big)\,dx + \big(xe^y\big)\,dy = 0.$$

1. Show that the equation is not exact.

2. Prove that there exists an integrating factor $\mu(y)$ depending only on y, and derive a differential equation for $\mu(y)$.

3. Determine $\mu(y)$ explicitly, obtain an exact equation, and hence find the general implicit solution.

Question 9: Introduction to Second-Order Linear ODEs

A mass–spring system with mass m, spring constant k, and no damping is described by the ODE

$$m\frac{d^2x}{dt^2} + kx = 0.$$

1. Starting from Newton's second law, derive this differential equation for the displacement $x(t)$.

2. Classify the ODE according to order, linearity, homogeneity, and whether it has constant coefficients.

3. Solve the ODE for the case $m = 1\,\text{kg}$, $k = 4\,\text{N/m}$, with initial conditions $x(0) = 1$ and $x'(0) = 0$.

Question 10: Homogeneous Second-Order ODEs (Distinct Real Roots)

Solve the initial value problem

$$y'' - 3y' + 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

1. Find the characteristic equation and its roots, and state why the solution is a linear combination of exponentials.

2. Determine the unique solution $y(t)$ satisfying the initial conditions.

3. Compute $\lim\limits_{t \to +\infty} y(t)$ and interpret briefly in terms of stability of the equilibrium solution $y = 0$.

Question 11: Homogeneous Second-Order ODEs (Repeated Root)

Consider the homogeneous second-order ODE

$$y'' - 4y' + 4y = 0.$$

1. Find the characteristic equation and show that it has a repeated real root.

2. Using an appropriate method (e.g. reduction of order), derive the general solution and show that it can be written in the form

$$y(x) = (C_1 + C_2 x)e^{2x}.$$

3. Determine the particular solution satisfying $y(0) = 0$ and $y'(0) = 1$.

Question 12: Homogeneous Second-Order ODEs (Complex Roots)

Solve the initial value problem

$$y'' + y = 0, \qquad y(0) = 0, \qquad y'(0) = 3.$$

1. Find the characteristic equation and its roots, and express the general solution in real form using trigonometric functions.

2. Determine the constants from the initial conditions and write down the explicit solution.

3. Compute the smallest positive T such that $y(T) = 0$ again, and interpret T as a physical quantity if y represents the displacement of a simple harmonic oscillator.

Question 13: Nonhomogeneous Second-Order ODEs (Constant Forcing)

Consider the equation

$$y'' + 4y = 8.$$

1. Solve the associated homogeneous equation.

2. Find a particular solution corresponding to the constant forcing term, and hence write down the general solution of the nonhomogeneous equation.

3. If $y(0) = 0$ and $y'(0) = 0$, determine the solution and describe its long-term behaviour as $t \to \infty$.

Question 14: Nonhomogeneous Second-Order ODEs (Polynomial Forcing)

Solve the differential equation

$$y'' - y = 3x^2.$$

1. Find the general solution of the homogeneous equation $y'' - y = 0$.

2. Using an appropriate ansatz for a polynomial particular solution, determine a particular solution of the nonhomogeneous equation.

3. Combine your results to obtain the general solution and verify it by substitution.

Question 15: Nonhomogeneous Second-Order ODEs (Exponential Forcing)

Consider the initial value problem

$$y'' - 2y' + y = e^x, \qquad y(0) = 0, \qquad y'(0) = 1.$$

1. Solve the homogeneous equation and discuss the nature of its characteristic roots.

2. Find a particular solution corresponding to the forcing term e^x.

3. Determine the constants in the general solution using the initial conditions.

Question 16: Nonhomogeneous Second-Order ODEs (Trigonometric Forcing)

Solve the differential equation

$$y'' + 9y = 6\cos(3x).$$

1. Solve the associated homogeneous equation and determine the fundamental solutions.

2. Explain why the usual ansatz $y_p = A\cos(3x) + B\sin(3x)$ fails for this forcing term, and propose a modified ansatz.

3. Determine a particular solution using your modified ansatz and hence write down the general solution of the ODE.

Question 17: Method of Undetermined Coefficients

Solve the differential equation

$$y'' - 3y' + 2y = 4e^x + 3x.$$

1. Find the general solution of the associated homogeneous equation.

2. Using the method of undetermined coefficients, construct an appropriate trial function for a particular solution corresponding to the combined forcing $4e^x + 3x$.

3. Determine the coefficients in your trial function and hence write down the general solution of the ODE.

Question 18: Method of Variation of Parameters

Consider the nonhomogeneous equation

$$y'' + y = \sec x, \qquad 0 < x < \frac{\pi}{2}.$$

1. Find a fundamental set of solutions for the associated homogeneous equation.

2. Use the method of variation of parameters to derive a system of equations for the functions $u_1(x)$ and $u_2(x)$ in the representation

$$y_p(x) = u_1(x)\cos x + u_2(x)\sin x.$$

3. Solve for $u_1(x)$ and $u_2(x)$ (up to constants) and obtain an explicit expression for a particular solution $y_p(x)$ as an integral.

Question 19: Reduction of Order

Suppose $y_1(x) = e^x$ is a known solution of the homogeneous equation

$$y'' - y' - 2y = 0.$$

1. Use reduction of order by setting $y_2(x) = v(x)y_1(x)$ to derive a first-order ODE for $v'(x)$.

2. Solve for $v(x)$ and hence obtain a second linearly independent solution $y_2(x)$.

3. Write down the general solution of the differential equation and verify that y_1 and y_2 are linearly independent.

Question 20: Superposition Principle

Let y_1 and y_2 be solutions of the homogeneous equation

$$y'' + 4y = 0$$

and let y_p be one particular solution of the nonhomogeneous equation

$$y'' + 4y = \cos(2x).$$

1. State the superposition principle for linear homogeneous ODEs and explain how it extends to the nonhomogeneous case.

2. Suppose $y_1(x) = \cos(2x)$ and $y_2(x) = \sin(2x)$. Write the most general real-valued solution of the homogeneous equation.

3. Given a particular solution $y_p(x)$, write the general solution of the nonhomogeneous equation and briefly explain why this representation contains all solutions.

Question 21: Wronskian and Linear Independence

Consider the functions
$$y_1(x) = e^x, \qquad y_2(x) = xe^x.$$

1. Compute the Wronskian $W(y_1, y_2)(x)$.

2. Determine whether y_1 and y_2 are linearly independent on $\mathbb{R}$ and justify your answer.

3. Find a second-order linear ODE with continuous coefficients on $\mathbb{R}$ for which y_1 and y_2 form a fundamental set of solutions.

Question 22: Driven Oscillations

A massspring system with mass $m = 1$ and spring constant $k = 4$ is subjected to an external periodic force $F(t) = 2\cos(3t)$. Neglect damping.

$$x'' + 4x = 2\cos(3t).$$

1. Solve the homogeneous equation and find the natural frequency of the system.

2. Find a particular solution corresponding to the driving force.

3. Write down the general solution and describe qualitatively the motion for large t.

Question 23: Simple Harmonic Motion

A particle of mass m is attached to a spring with stiffness k and moves without damping or external forces.

$$mx'' + kx = 0.$$

1. Solve the ODE and show that the motion is simple harmonic.

2. Express the solution in the form $x(t) = A\cos(\omega t - \phi)$ and identify A, ω, and ϕ in terms of the initial conditions $x(0) = x_0$, $x'(0) = v_0$.

3. For $m = 2$, $k = 8$, $x_0 = 1$, and $v_0 = 0$, compute the period of oscillation and the maximum speed of the particle.

Question 24: Damped Harmonic Motion (Underdamped)

A damped massspring system satisfies

$$x'' + 2\gamma x' + \omega_0^2 x = 0,$$

with $\gamma^2 < \omega_0^2$ (underdamped case).

1. Solve the characteristic equation and obtain the general solution $x(t)$ in real form.

2. For $\gamma = 1$, $\omega_0 = 2$, and initial conditions $x(0) = 0$, $x'(0) = 1$, find the specific solution.

3. Determine the time at which the amplitude envelope decays to half of its initial value.

Question 25: Damped Harmonic Motion (Critically Damped)

A critically damped system satisfies

$$x'' + 4x' + 4x = 0.$$

1. Show that the characteristic equation has a repeated real root and find the general solution.

2. Given $x(0) = 1$ and $x'(0) = 0$, determine the specific solution $x(t)$.

3. Compare qualitatively the decay of $x(t)$ with that of an underdamped system having the same $x(0)$ and $x'(0)$.

Question 26: Damped Harmonic Motion (Overdamped)

Consider the overdamped system

$$x'' + 6x' + 8x = 0.$$

1. Find the characteristic roots and write down the general solution.

2. For initial conditions $x(0) = 1$ and $x'(0) = 0$, determine the specific solution.

3. Show that $x(t)$ does not oscillate and tends monotonically to zero as $t \to \infty$.

Question 27: Resonance

A mass‑spring system with mass $m = 1$ and spring constant $k = 4$ is driven by a force $F(t) = 2\cos(2t)$ and has small damping with coefficient $c > 0$:

$$x'' + cx' + 4x = 2\cos(2t).$$

1. Solve the homogeneous equation and determine the damped natural frequency.

2. Assume c is sufficiently small so that the forcing frequency is close to the damped natural frequency. Find a particular solution using an undetermined coefficients ansatz.

3. For $c = 0$, find the general solution and show that the amplitude grows without bound, illustrating the phenomenon of resonance.

Question 28: RLC Circuits

An RLC series circuit with resistance R, inductance L, and capacitance C is driven by a voltage source $E(t) = E_0 \cos(\omega t)$. The charge $q(t)$ on the capacitor satisfies

$$Lq'' + Rq' + \frac{1}{C}q = E_0 \cos(\omega t).$$

1. Derive this differential equation from Kirchhoff's voltage law.

2. For $L = 1$, $R = 2$, $C = \frac{1}{5}$, and $E_0 = 10$, write down the explicit ODE for $q(t)$.

3. Solve the associated homogeneous equation and find the general form of the steady-state (particular) solution corresponding to the forcing term.

Question 29: Population Models

A population $P(t)$ grows according to the logistic model

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right),$$

with $r > 0$, $K > 0$ and initial population $P(0) = P_0$.

1. Classify this ODE (order, linear/nonlinear, autonomous/non-autonomous).

2. Solve the differential equation by separation of variables to obtain an explicit formula for $P(t)$.

3. For $r = 0.5$, $K = 1000$, and $P_0 = 100$, compute the time t at which $P(t)$ reaches 500.

Question 30: Mixing Problems

A tank initially contains $100\,\text{L}$ of pure water. A brine solution containing $0.3\,\text{kg}$ of salt per litre flows into the tank at a rate of $5\,\text{L/min}$, and the well-stirred mixture flows out at the same rate.

1. Let $S(t)$ be the amount of salt (in kg) in the tank at time t minutes. Derive a first-order ODE for $S(t)$.

2. Solve the ODE to find $S(t)$, assuming $S(0) = 0$.

3. Determine the limiting amount of salt in the tank as $t \to \infty$ and interpret physically.

Question 31: Newton's Cooling Law

An object with temperature $T(t)$ cools in a room kept at constant temperature $T_{\text{env}} = 20°\text{C}$. Newton's law of cooling states that

$$\frac{dT}{dt} = -k(T - T_{\text{env}}), \qquad k > 0.$$

Initially, the object has temperature $T(0) = 90°\text{C}$.

1. Solve the differential equation to obtain $T(t)$ in terms of k.

2. Suppose that after 10 minutes the temperature has dropped to $60°\text{C}$. Determine k.

3. Using your value of k, estimate the time required for the object to cool to $30°\text{C}$.

Question 32: Free Fall with Quadratic Drag

A body of mass m is falling vertically under gravity with acceleration g and experiences a quadratic air resistance proportional to the square of its speed. Taking downward as the positive direction, its velocity $v(t)$ satisfies

$$m\frac{dv}{dt} = mg - kv^2, \qquad k > 0, \qquad v(0) = 0.$$

1. Show that the equation can be written in the separable form

$$\frac{dv}{g - (k/m)v^2} = dt.$$

2. Solve the differential equation to obtain $v(t)$ and determine the terminal velocity v_{term}.

3. Find the time $t_{1/2}$ at which the velocity reaches half of the terminal velocity.

Fourier Series

Question 1: Fourier Coefficients – Eulers Formulae

Let $f(x)$ be the 2π-periodic function defined by

$$f(x) = \begin{cases} x, & -\pi < x < 0, \\ \pi - x, & 0 \le x \le \pi. \end{cases}$$

1. Using Eulers formulae, compute a_0, a_n and b_n.

2. Hence write down the full Fourier series for $f(x)$.

3. State whether the resulting series converges at $x = 0$ and justify your answer.

Question 2: Even and Odd Functions

A 2π-periodic function is defined by

$$f(x) = x(\pi - x), \qquad 0 \le x \le \pi,$$

and extended periodically.

1. Determine whether the even or odd extension of this function is more natural for Fourier analysis.

2. Using symmetry arguments, identify which Fourier coefficients must vanish.

3. Compute the non-zero coefficients and hence obtain the Fourier series of $f(x)$.

Question 3: Odd and Even Extensions

The function $g(x)$ is defined on $0 \le x \le \pi$ by

$$g(x) = x.$$

1. Construct the odd periodic extension $g_{\text{odd}}(x)$ of period 2π and find its Fourier series.

2. Construct the even periodic extension $g_{\text{even}}(x)$ of period 2π and find its Fourier series.

3. By evaluating each series at $x = \pi/2$, deduce two different infinite series representations of π.

Question 4: Parsevals Theorem

Let $f(x) = x$ for $-\pi < x \le \pi$, extended periodically with period 2π.

1. Write down the Fourier series for $f(x)$.

2. Apply Parsevals theorem to this series.

3. Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Question 5: Periods Other Than 2π

A function $f(x)$ has period $2L$ with $L = 2$, and is defined by

$$f(x) = x, \qquad -2 < x \leq 2.$$

1. Write down the general form of the Fourier series for a $2L$-periodic function.

2. Compute the coefficients a_0, a_n and b_n for this function.

3. Hence write the Fourier series explicitly in terms of $\sin\left(\frac{n\pi x}{2}\right)$ and $\cos\left(\frac{n\pi x}{2}\right)$.

Question 6: Complex Exponential Form of Fourier Series

Let $f(x) = x$ for $-\pi < x \leq \pi$, extended periodically with period 2π.

1. Compute the complex Fourier coefficients

$$d_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}\, dx.$$

2. Hence write the complex exponential form of the Fourier series for $f(x)$.

3. By comparing with the real Fourier series of $f(x)$, verify the relationship between d_n, a_n and b_n.