Ordinary Differential Equations Answers

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Answer 1 (2)

Solve

$$\frac{dy}{dx} + 3y = 6e^{-2x}$$

Step 1. Integrating factor:

$$w(x) = e^{\int 3\,dx} = e^{3x}$$

Step 2. Multiply and integrate:

$$\frac{d}{dx}\left(ye^{3x}\right) = 6e^{x}$$

$$ye^{3x} = 6e^{x} + A$$

$$\boxed{y(x) = 6e^{-2x} + Ae^{-3x}}$$

Answer 2 (2)

Solve

$$\frac{dy}{dx} - 2y = e^{3x}$$

Step 1. Integrating factor:

$$w(x) = e^{\int -2\,dx} = e^{-2x}$$

Step 2. Multiply and integrate:

$$\frac{d}{dx}\left(ye^{-2x}\right) = e^{x}$$

$$ye^{-2x} = e^{x} + A$$

$$\boxed{y(x) = e^{3x} + Ae^{2x}}$$

Answer 3 (3)

Solve

$$y'' - 4y' + 4y = 0$$

Step 1. Characteristic equation:

$$r^2 - 4r + 4 = 0 \quad \Longrightarrow \quad (r-2)^2 = 0$$

$$\boxed{y(x) = (A + Bx)e^{2x}}$$

Answer 4 (3)

Solve

$$\frac{dy}{dx} + 2y = y^3 e^{-x}$$

where $f(x) = 2,\ g(x) = e^{-x},\ n = 3.$

Step 1. Divide by y^3:

$$y^{-3}\frac{dy}{dx} + 2y^{-2} = e^{-x}$$

Step 2. Let

$$z = y^{-2}, \qquad \frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$y^{-3}\frac{dy}{dx} = -\tfrac{1}{2}\frac{dz}{dx}$$

Step 3. Substitute:

$$-\tfrac{1}{2}\frac{dz}{dx} + 2z = e^{-x}$$

$$\frac{dz}{dx} - 4z = -2e^{-x}$$

Step 4. Integrating factor:

$$w(x) = e^{\int -4\,dx} = e^{-4x}$$

$$\frac{d}{dx}\big(ze^{-4x}\big) = -2e^{-5x}$$

$$ze^{-4x} = \tfrac{2}{5}e^{-5x} + A$$

$$z = Ae^{4x} + \tfrac{2}{5}e^{-x}$$

Step 5. Back-substitute $z = y^{-2}$:

$$y^{-2} = Ae^{4x} + \tfrac{2}{5}e^{-x}$$

$$\boxed{y(x) = \frac{1}{\sqrt{Ae^{4x} + \tfrac{2}{5}e^{-x}}}}$$

Answer 5 (3)

Solve

$$y'' + 9y = 0$$

Step 1. Characteristic equation:

$$r^2 + 9 = 0 \quad \Longrightarrow \quad r = \pm 3i$$

$$\boxed{y(x) = A\cos(3x) + B\sin(3x)}$$

Answer 6 (4)

Solve

$$y'' - y = e^x$$

Step 1. Homogeneous solution:

$$r^2 - 1 = 0 \quad \Longrightarrow \quad r = \pm 1$$

$$y_h = Ae^x + Be^{-x}$$

Step 2. Particular solution (resonance with e^x): Try $y_p = Cxe^x$.

$$y_p' = Ce^x + Cxe^x, \qquad y_p'' = 2Ce^x + Cxe^x$$

$$y_p'' - y_p = 2Ce^x = e^x \quad \Rightarrow \quad C = \tfrac{1}{2}$$

$$\boxed{y(x) = Ae^x + Be^{-x} + \tfrac{1}{2}xe^x}$$

Answer 7 (4)

Solve

$$y'' + 4y = \cos(2x)$$

Step 1. Homogeneous solution:

$$r^2 + 4 = 0 \quad \Longrightarrow \quad r = \pm 2i$$

$$y_h = A\cos(2x) + B\sin(2x)$$

Step 2. Particular solution (resonance): try

$$y_p = x\big(C\cos(2x) + D\sin(2x)\big)$$

After substitution one obtains

$$4D\cos(2x) - 4C\sin(2x) = \cos(2x)$$

so

$$C = 0, \qquad D = \tfrac{1}{4}$$

$$\boxed{y(x) = A\cos(2x) + B\sin(2x) + \tfrac{1}{4}x\sin(2x)}$$

Answer 8 (4)

Solve

$$\frac{dy}{dx} + y = y^2 e^x$$

where $f(x) = 1,\ g(x) = e^x,\ n = 2.$

Step 1. Divide by y^2:

$$y^{-2}\frac{dy}{dx} + y^{-1} = e^x$$

Step 2. Let

$$z = y^{-1}, \qquad \frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$$

$$-y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$$

Step 3. Substitute:

$$-\frac{dz}{dx} + z = e^x$$

$$\frac{dz}{dx} - z = -e^x$$

Step 4. Integrating factor:

$$w(x) = e^{\int -1\,dx} = e^{-x}$$

$$\frac{d}{dx}\left(ze^{-x}\right) = -1$$

$$ze^{-x} = -x + A$$

$$z = (-x + A)e^x$$

Step 5. Back-substitute $z = y^{-1}$:

$$y = \frac{1}{z} = \frac{1}{(-x + A)e^x} = \frac{e^{-x}}{A - x}$$

$$\boxed{y(x) = \frac{e^{-x}}{A - x}}$$

Answer 9 (5)

Solve

$$y'' - 3y' + 2y = e^{2x}$$

Step 1. Homogeneous solution:

$$r^2 - 3r + 2 = 0 \quad \Longrightarrow \quad r = 1, 2$$

$$y_h = Ae^x + Be^{2x}$$

Step 2. Particular solution (resonance with e^{2x}): try

$$y_p = Cxe^{2x}$$

$$y_p' = Ce^{2x} + 2Cxe^{2x}, \qquad y_p'' = 4Ce^{2x} + 4Cxe^{2x}$$

$$y_p'' - 3y_p' + 2y_p = Ce^{2x} = e^{2x} \quad \Rightarrow \quad C = 1$$

$$\boxed{y(x) = Ae^x + Be^{2x} + xe^{2x}}$$

Answer 10 (5)

Solve

$$\frac{dy}{dx} - y = y^3 e^{-2x}$$

where $f(x) = -1,\ g(x) = e^{-2x},\ n = 3.$

Step 1. Divide by y^3:

$$y^{-3}\frac{dy}{dx} - y^{-2} = e^{-2x}$$

Step 2. Let

$$z = y^{-2}, \qquad \frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$y^{-3}\frac{dy}{dx} = -\tfrac{1}{2}\frac{dz}{dx}$$

Step 3. Substitute:

$$-\tfrac{1}{2}\frac{dz}{dx} - z = e^{-2x}$$

$$\frac{dz}{dx} + 2z = -2e^{-2x}$$

Step 4. Integrating factor:

$$w(x) = e^{\int 2\,dx} = e^{2x}$$

$$\frac{d}{dx}\big(ze^{2x}\big) = -2$$

$$ze^{2x} = -2x + A$$

$$z = (-2x + A)e^{-2x}$$

Step 5. Back-substitute $z = y^{-2}$:

$$y^{-2} = (-2x + A)e^{-2x}$$

$$\boxed{y(x) = \frac{e^{x}}{\sqrt{A - 2x}}}$$

Answer 11 (2)

Solve

$$\frac{dy}{dx} - 4y = 8e^{2x}$$

Step 1. Integrating factor:

$$w(x) = e^{\int -4\,dx} = e^{-4x}$$

Step 2. Multiply and integrate:

$$\frac{d}{dx}\left(ye^{-4x}\right) = 8e^{-2x}$$

$$ye^{-4x} = -4e^{-2x} + A$$

$$\boxed{y(x) = -4e^{2x} + Ae^{4x}}$$

Answer 12 (2)

Solve

$$\frac{dy}{dx} + 5y = 10e^{-3x}$$

Step 1. Integrating factor:

$$w(x) = e^{\int 5\,dx} = e^{5x}$$

Step 2. Multiply and integrate:

$$\frac{d}{dx}\left(ye^{5x}\right) = 10e^{2x}$$

$$ye^{5x} = 5e^{2x} + A$$

$$\boxed{y(x) = 5e^{-3x} + Ae^{-5x}}$$

Answer 13 (3)

Solve

$$y'' + 6y' + 9y = 0$$

Step 1. Characteristic equation:

$$r^2 + 6r + 9 = 0 \quad \Longrightarrow \quad (r+3)^2 = 0$$

$$\boxed{y(x) = (A + Bx)e^{-3x}}$$

Answer 14 (3)

Solve
$$\frac{dy}{dx} + 3y = y^2 e^{-x}$$
where $f(x) = 3,\ g(x) = e^{-x},\ n = 2.$

Step 1. Divide by y^2:
$$y^{-2}\frac{dy}{dx} + 3y^{-1} = e^{-x}$$

Step 2. Let
$$z = y^{-1}$$
$$\frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$$
$$-y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$$

Step 3. Substitute:
$$-\frac{dz}{dx} + 3z = e^{-x}$$
$$\frac{dz}{dx} - 3z = -e^{-x}$$

Step 4. Integrating factor:
$$w(x) = e^{\int -3\,dx} = e^{-3x}$$
$$\frac{d}{dx}\left(ze^{-3x}\right) = -e^{-4x}$$
$$ze^{-3x} = \tfrac{1}{4}e^{-4x} + A$$
$$z = Ae^{3x} + \tfrac{1}{4}e^{-x}$$

Step 5. Back-substitute $z = y^{-1}$:
$$\boxed{y(x) = \frac{1}{Ae^{3x} + \frac{1}{4}e^{-x}}}$$

Answer 15 (3)

Solve

$$y'' - 2y' + 5y = 0$$

Step 1. Characteristic equation:

$$r^2 - 2r + 5 = 0 \quad \Longrightarrow \quad r = 1 \pm 2i$$

$$\boxed{y(x) = e^x\big(A\cos(2x) + B\sin(2x)\big)}$$

Answer 16 (4)

Solve

$$y'' + y = \sin x$$

Step 1. Homogeneous solution:

$$y_h = A\cos x + B\sin x$$

Step 2. Particular solution (resonance): try

$$y_p = x\big(C\cos x + D\sin x\big)$$

Substitution yields

$$-2C\sin x + 2D\cos x = \sin x$$

so

$$C = -\tfrac{1}{2}, \qquad D = 0$$

$$\boxed{y(x) = A\cos x + B\sin x - \tfrac{1}{2}x\cos x}$$

Answer 17 (4)

Solve

$$y'' - 4y = 8e^{2x}$$

Step 1. Homogeneous solution:

$$r^2 - 4 = 0 \quad \Longrightarrow \quad r = \pm 2$$

$$y_h = Ae^{2x} + Be^{-2x}$$

Step 2. Particular solution (resonance): try

$$y_p = Cxe^{2x}$$

$$y_p'' - 4y_p = 2Ce^{2x} = 8e^{2x} \quad \Rightarrow \quad C = 4$$

$$\boxed{y(x) = Ae^{2x} + Be^{-2x} + 4xe^{2x}}$$

Answer 18 (4)

Solve

$$\frac{dy}{dx} - 2y = y^3 e^x$$

where $f(x) = -2,\ g(x) = e^x,\ n = 3.$

Step 1. Divide by y^3:

$$y^{-3}\frac{dy}{dx} - 2y^{-2} = e^x$$

Step 2. Let

$$z = y^{-2}$$

$$\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$y^{-3}\frac{dy}{dx} = -\tfrac{1}{2}\frac{dz}{dx}$$

Step 3. Substitute:

$$-\tfrac{1}{2}\frac{dz}{dx} - 2z = e^x$$

$$\frac{dz}{dx} + 4z = -2e^x$$

Step 4. Integrating factor:

$$w(x) = e^{\int 4\,dx} = e^{4x}$$

$$\frac{d}{dx}\left(ze^{4x}\right) = -2e^{5x}$$

$$ze^{4x} = -\tfrac{2}{5}e^{5x} + A$$

$$z = Ae^{-4x} - \tfrac{2}{5}e^x$$

Step 5. Back-substitute $z = y^{-2}$:

$$\boxed{y(x) = \frac{1}{\sqrt{Ae^{-4x} - \tfrac{2}{5}e^x}}}$$

Answer 19 (5)

Solve

$$y'' + 4y' + 4y = e^{-2x}$$

Step 1. Homogeneous solution:

$$r^2 + 4r + 4 = 0 \quad \Longrightarrow \quad (r+2)^2 = 0$$

$$y_h = (A + Bx)e^{-2x}$$

Step 2. Particular solution (double root, multiply by x^2): Try

$$y_p = Cx^2 e^{-2x}$$

$$y_p'' + 4y_p' + 4y_p = 2Ce^{-2x} = e^{-2x} \quad \Rightarrow \quad C = \tfrac{1}{2}$$

$$\boxed{y(x) = (A + Bx)e^{-2x} + \tfrac{1}{2}x^2 e^{-2x}}$$

Answer 20 (5)

Solve

$$\frac{dy}{dx} + y = y^4 e^{-x}$$

where $f(x) = 1,\ g(x) = e^{-x},\ n = 4.$

Step 1. Divide by y^4:

$$y^{-4}\frac{dy}{dx} + y^{-3} = e^{-x}$$

Step 2. Let

$$z = y^{-3}$$

$$\frac{dz}{dx} = -3y^{-4}\frac{dy}{dx}$$

$$y^{-4}\frac{dy}{dx} = -\tfrac{1}{3}\frac{dz}{dx}$$

Step 3. Substitute:

$$-\tfrac{1}{3}\frac{dz}{dx} + z = e^{-x}$$

$$\frac{dz}{dx} - 3z = -3e^{-x}$$

Step 4. Integrating factor:

$$w(x) = e^{\int -3\,dx} = e^{-3x}$$

$$\frac{d}{dx}\left(ze^{-3x}\right) = -3e^{-4x}$$

$$ze^{-3x} = \tfrac{3}{4}e^{-4x} + A$$

$$z = Ae^{3x} + \tfrac{3}{4}e^{-x}$$

Step 5. Back-substitute $z = y^{-3}$:

$$\boxed{y(x) = \left(Ae^{3x} + \tfrac{3}{4}e^{-x}\right)^{-1/3}}$$