Fourier Series Questions

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Question 1 (3)

Find the Fourier series for the function defined on $(-\pi, \pi)$ by

$$f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0, \\ 2 & \text{if } 0 < x < \pi. \end{cases}$$

Determine the value to which the series converges at $x = 0$ and $x = \pi$.

Question 2 (4)

Consider the function $f(x) = x|\sin x|$ defined on the interval $(-\pi, \pi)$.

1. Determine whether $f(x)$ is even, odd, or neither.
2. Find the Fourier series representation of $f(x)$.
3. Calculate the sum $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}$ using your Fourier series by evaluating at an appropriate point.

Question 3 (3)

Find the Fourier series for the periodic extension of the function

$$f(x) = \pi^2 - x^2 \quad \text{for } -\pi < x < \pi.$$

Hence show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

Question 4 (4)

A function $g(x)$ is defined on $(-\pi, \pi)$ by

$$g(x) = \begin{cases} x^2 & \text{if } -\pi < x < 0, \\ \pi x & \text{if } 0 < x < \pi. \end{cases}$$

1. Compute the Fourier coefficients a_0, a_n, and b_n for $g(x)$.
2. Write down the complete Fourier series representation.
3. Determine the value to which the Fourier series converges at $x = -\pi$, $x = 0$, and $x = \pi$.

Question 5 (5)

Let $f(x) = e^{ax}$ where $a \in \mathbb{R}$ and $a \neq 0$, defined on $(-\pi, \pi)$.

1. Derive the Fourier series for $f(x)$, expressing your answer in terms of a.
2. Show that your series satisfies Parseval's identity:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

3. Use Parseval's identity to evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a^2 + n^2}$ in terms of a.

Question 6 (3)

Find the Fourier series of the function

$$f(x) = \sin^3 x \quad \text{for } -\pi < x < \pi$$

by first expressing $\sin^3 x$ as a sum of sines using trigonometric identities, and verify your result by direct calculation of the Fourier coefficients.

Question 7 (4)

A periodic function $h(x)$ with period 2π is defined on one period by

$$h(x) = \begin{cases} 0 & \text{if } -\pi < x < -\frac{\pi}{2}, \\ 1 & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ 0 & \text{if } \frac{\pi}{2} < x < \pi. \end{cases}$$

1. Find the Fourier series representation of $h(x)$.
2. Sketch the graph of the sum of the Fourier series over the interval $[-3\pi, 3\pi]$.
3. Use the Fourier series to evaluate $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$.

Question 8 (3)

A periodic function $g(x)$ with period 2π is defined on one period by

$$g(x) = \begin{cases} 2 - x & \text{if } -\pi \leq x < 0, \\ 2 + x & \text{if } 0 \leq x < \pi. \end{cases}$$

Determine whether $g(x)$ is even or odd, sketch the function over the interval $[-3\pi, 3\pi]$, and find its Fourier series representation.

Question 9 (4)

Consider the function $f(x) = x^2$ defined on the interval $(0, \pi)$.

1. Construct the even extension $f_e(x)$ of $f(x)$ to the interval $(-\pi, \pi)$ and sketch it.
2. Find the Fourier cosine series for the even extension.
3. Construct the odd extension $f_o(x)$ of $f(x)$ to the interval $(-\pi, \pi)$ and sketch it.
4. Find the Fourier sine series for the odd extension.

Question 10 (4)

A function $h(x) = \cos x$ is defined on the interval $(0, \pi)$.

1. Determine the even periodic extension of $h(x)$ to $(-\pi, \pi)$ and calculate its Fourier series (cosine series only).

2. Determine the odd periodic extension of $h(x)$ to $(-\pi, \pi)$ and calculate its Fourier series (sine series only).

3. Evaluate the sum $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{4n^2 - 1}$ using the appropriate series at $x = \pi$.

Question 11 (5)

A periodic function $p(x)$ with period 2π is defined on the interval $[-\pi, \pi)$ by

$$p(x) = \begin{cases} -2 + |x| & \text{if } -\pi \leq x < 0, \\ 2 - |x| & \text{if } 0 \leq x < \pi. \end{cases}$$

1. Simplify the definition of $p(x)$ by removing the absolute value signs and sketch the function over two complete periods.

2. Determine whether $p(x)$ is even, odd, or neither.

3. Find the complete Fourier series for $p(x)$.

4. Use your Fourier series to evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ by choosing an appropriate value of x.

Question 12 (3)

Find the Fourier series for the function $f(x) = 2x + 1$ defined on the interval $(0, 3)$, and sketch the corresponding periodic extension over the interval $[-6, 9]$. Determine the value to which the series converges at $x = 0$ and $x = 3$.

Question 13 (4)

Consider the piecewise function defined on $(-2, 2)$ by

$$f(x) = \begin{cases} 2 & \text{if } -2 < x < -1, \\ 0 & \text{if } -1 < x < 1, \\ 3 & \text{if } 1 < x < 2. \end{cases}$$

1. Find the Fourier series for $f(x)$ with period 4.
2. Sketch the periodic extension of $f(x)$ over three complete periods.
3. Use your Fourier series to evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n}$.

Question 14 (4)

A continuous function $g(x) = x^2$ is defined on the interval $[0, L]$ where $L > 0$.

1. Construct and sketch the odd extension $g_{\text{odd}}(x)$ of $g(x)$ to the interval $[-L, L]$.
2. Show that the odd extension has a Fourier sine series of the form

$$g_{\text{odd}}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

3. Calculate the coefficients b_n explicitly.
4. Verify your result by evaluating the series at $x = \frac{L}{2}$ and comparing with $g\left(\frac{L}{2}\right)$.

Question 15 (5)

A continuous triangular wave function is defined on $[0, 2L]$ by

$$f(x) = \begin{cases} \frac{2x}{L} & \text{if } 0 \leq x \leq L, \\ \frac{4L-2x}{L} & \text{if } L < x \leq 2L. \end{cases}$$

1. Sketch $f(x)$ over the interval $[0, 2L]$ and verify that $f(0) = f(2L) = 0$.
2. Construct the odd extension $f_{\text{odd}}(x)$ to $[-2L, 2L]$ and sketch it.
3. Determine the Fourier sine series representation

$$f_{\text{odd}}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

4. Show that $b_n = 0$ for all even values of n.
5. Use Parseval's identity to evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$.