

# Oscillations Question Set

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## Question 1 (3)

A block of mass  $m = 0.40$  kg is attached to a horizontal spring with spring constant  $k = 50$  N m<sup>-1</sup> and oscillates without friction on a smooth surface. At time  $t = 0$  the block is at its maximum displacement  $x = A = 0.12$  m and is released from rest.

1. Write down the equation of motion for the block and show that the motion is simple harmonic. Determine the angular frequency  $\omega$  and period  $T$ .
2. Find the speed of the block when it passes through  $x = 0.060$  m.
3. Determine the total mechanical energy of the oscillator and the ratio of kinetic to potential energy when the block is at  $x = 0.060$  m.

## Question 2 (3)

A simple pendulum of length  $L$  consists of a small bob of mass  $m$  suspended from a light string. For small oscillations, the motion is approximately simple harmonic.

1. Starting from the exact equation of motion  $mL\ddot{\theta} + mg \sin \theta = 0$ , show that for small angles the motion reduces to SHM and find the angular frequency  $\omega_0$ .
2. A pendulum clock using such a pendulum of length  $L_0$  keeps perfect time at a location where the local gravitational acceleration is  $g_0 = 9.81$  m s<sup>-2</sup>. The clock is moved to a mountain where  $g = 9.79$  m s<sup>-2</sup>, but the length is not adjusted. After one full day (24 h of true time), by how many seconds does the clock gain or lose?
3. To restore the correct period at the mountain, by how much (in mm) must the pendulum length be adjusted from  $L_0$ ?

### Question 3 (4)

A damped harmonic oscillator consists of a mass  $m = 0.20$  kg attached to a spring of constant  $k = 16$  N m<sup>-1</sup> and a dashpot providing a damping force  $F_d = -b\dot{x}$ .

When displaced and released, the displacement is observed to be

$$x(t) = A_0 e^{-\gamma t} \cos(\omega_d t),$$

with successive maxima of the displacement envelope measured as  $x_1 = 4.0$  cm and  $x_5 = 1.5$  cm.

1. Express the damping coefficient  $\gamma$  in terms of the logarithmic decrement  $\Lambda$  and show that

$$\Lambda = \ln \left( \frac{x_n}{x_{n+1}} \right).$$

2. Using the measurements  $x_1$  and  $x_5$ , determine  $\gamma$  and hence the damping constant  $b$ .
3. Determine the undamped natural frequency  $\omega_0$  and the damped frequency  $\omega_d$ . Comment briefly on whether the oscillator is underdamped, critically damped, or overdamped.
4. Calculate the quality factor  $Q$  of the oscillator.

## Question 4 (4)

A driven damped oscillator obeys

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t),$$

with  $m = 0.50$  kg,  $k = 100$  N m<sup>-1</sup>,  $b = 4.0$  kg s<sup>-1</sup> and  $F_0 = 2.0$  N.

1. Show that the steady-state amplitude  $A(\omega)$  is given by

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}},$$

where  $\omega_0^2 = k/m$  and  $\beta = b/(2m)$ .

2. Determine the driving frequency  $\omega_{\text{res}}$  at which the amplitude is maximised, and calculate its numerical value.
3. Calculate the steady-state amplitude and phase lag  $\phi$  (defined by  $x(t) = A \cos(\omega t - \phi)$ ) when driven at  $\omega = 0.5\omega_0$  and at  $\omega = 2\omega_0$ .
4. For which of these two driving frequencies does the oscillators velocity amplitude reach a larger value? Justify quantitatively.

## Question 5 (5)

Two identical masses  $m$  are connected by three identical light springs of constant  $k$  in a line between two fixed walls, as shown:

$$\text{wall} - k - m_1 - k - m_2 - k - \text{wall}.$$

All motion is along the line of the springs and friction is negligible.

1. Let  $x_1(t)$  and  $x_2(t)$  be the displacements of  $m_1$  and  $m_2$  from their equilibrium positions. By applying Newton's second law to each mass, derive the coupled equations of motion.
2. Seek normal mode solutions of the form  $x_j(t) = A_j \cos(\omega t)$  and show that the allowed angular frequencies are

$$\omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{3k}{m}}.$$

3. For each normal mode, determine the ratio  $A_2/A_1$  and describe qualitatively the motion of the two masses.
4. The system is released from rest with initial displacements  $x_1(0) = X_0$  and  $x_2(0) = 0$ . Express  $x_1(t)$  and  $x_2(t)$  as superpositions of the two normal modes and discuss briefly the phenomenon of beats that may be observed.

## Question 6 (3)

A torsional pendulum consists of a uniform solid cylinder of mass  $M = 1.8 \text{ kg}$  and radius  $R = 0.12 \text{ m}$  suspended by a thin wire. The measured angular frequency of small oscillations is  $\omega = 4.6 \text{ rad s}^{-1}$ . Determine the torsion constant  $\kappa$  of the wire.

## Question 7 (4)

A simple pendulum of length  $L$  is released from rest at a small angle and undergoes small-amplitude oscillations in a region where the local gravitational acceleration is unknown. The pendulum completes 200 oscillations in 402 s.

1. Determine the value of  $g$  at this location.
2. If the pendulum length were increased by 3.0%, determine the new time required for 200 oscillations.

## Question 8 (5)

A pendulum of length  $L = 0.85$  m is released from rest at a large initial angle of  $28^\circ$ . Using the first two terms of the large-angle correction expansion,

$$T \approx 2\pi \sqrt{\frac{L}{g}} \left( 1 + \frac{1}{16} \theta_0^2 \right),$$

where  $\theta_0$  is in radians,

1. calculate the corrected period,
2. determine the percentage error that would result if the small-angle formula were used instead.

## Question 9 (4)

A simple pendulum of length 0.65 m is released from rest at an initial angle of  $18^\circ$ .

1. Using energy conservation, determine the maximum speed of the bob.
2. At what angle (to the nearest degree) does the speed equal half of this maximum value?

### Question 10 (5)

A physical pendulum is formed by a uniform thin rod of length  $L = 0.90\text{ m}$  and mass  $M = 2.2\text{ kg}$  pivoted about a point  $0.15\text{ m}$  from one end.

1. Determine the moment of inertia of the rod about the pivot.
2. Hence determine the angular frequency and the period of small oscillations.
3. Find the length of a simple pendulum that would have the same period.

### Question 11 (4)

A tuned mass damper in a tall building is modelled as a simple pendulum of effective length  $L$ . The fundamental sway frequency of the building is  $0.32\text{ Hz}$ .

1. Determine the required pendulum length so that the damper is in resonance with the building.
2. If the damper mass is  $5.0 \times 10^5\text{ kg}$  and the maximum horizontal displacement of the mass is  $0.80\text{ m}$ , estimate the maximum kinetic energy of the damper during operation.



### Question 12 (3)

A massspring system with mass  $m = 0.20$  kg and spring constant  $k = 18$  N m<sup>-1</sup> is fitted with a damper providing a resistive force  $F_d = -b\dot{x}$ . The damping constant is  $b = 1.2$  kg s<sup>-1</sup>.

1. Determine the undamped natural angular frequency  $\omega_0$  of the system.
2. Calculate the damping ratio  $\zeta$  and classify the type of motion.
3. If the system is released from rest at  $x = 0.050$  m, state qualitatively how the displacement varies with time.

### Question 13 (4)

A damped oscillator of mass  $m = 0.40$  kg and spring constant  $k = 50$  N m<sup>-1</sup> oscillates with an observed period of  $T_d = 1.10$  s. The undamped period is known to be  $T_0 = 1.00$  s.

1. Determine the angular frequencies  $\omega_0$  and  $\omega_d$ .
2. Use the relation  $\omega_d^2 = \omega_0^2 - (b/2m)^2$  to calculate the damping constant  $b$ .
3. State whether the oscillator is lightly damped, critically damped, or overdamped.

## Question 14 (4)

A lightly damped oscillator has displacement

$$x(t) = Ae^{-\gamma t} \cos(\omega_d t),$$

with  $A = 0.080$  m and  $\gamma = 0.25$  s<sup>-1</sup>.

1. Calculate the time taken for the amplitude to decay to 25% of its initial value.
2. Determine the ratio of the energy after this time to the initial energy.
3. If  $\omega_d = 9.0$  rad s<sup>-1</sup>, how many oscillations occur in this time?

## Question 15 (5)

A driven, damped oscillator obeys

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t),$$

with  $m = 0.30$  kg,  $k = 60$  N m<sup>-1</sup>,  $b = 2.4$  kg s<sup>-1</sup> and  $F_0 = 1.5$  N.

1. Determine the steadystate amplitude  $A(\omega)$  at a general driving frequency  $\omega$ .
2. Calculate the natural angular frequency  $\omega_0$  and the damping parameter  $\beta = b/(2m)$ .
3. Determine the resonant angular frequency  $\omega_{\text{res}}$ .
4. Evaluate the steadystate amplitude at  $\omega = \omega_{\text{res}}$ .
5. State the phase difference between the displacement and driving force at resonance.

## Question 16 (5)

Two masses  $m$  are connected in a line by two identical springs of spring constant  $k$  between rigid walls:

$$\text{wall} - k - m_1 - k - m_2 - k - \text{wall}.$$

1. By writing down the equations of motion for  $x_1(t)$  and  $x_2(t)$ , show that the system supports two normal modes.
2. Determine expressions for the two normalmode angular frequencies.
3. Describe the motion of the masses in each mode.
4. If  $m = 0.50 \text{ kg}$  and  $k = 200 \text{ N m}^{-1}$ , calculate the numerical values of both normalmode frequencies.

## Question 17 (3)

An LC series circuit consists of a capacitor of capacitance  $C = 8.0 \text{ nF}$  and an inductor of inductance  $L = 2.5 \text{ mH}$ .

1. Determine the natural angular frequency of oscillation.
2. Calculate the oscillation period and frequency.

### Question 18 (4)

An oscilloscope connected across a capacitor in an LC circuit shows adjacent voltage maxima separated by 6.4 horizontal divisions. The time-base is set to 0.50 ms/div. The capacitance is  $C = 12.0 \text{ nF}$ . Determine the inductance  $L$  of the circuit.

### Question 19 (4)

A capacitor of capacitance  $C = 150 \text{ pF}$  in an LC circuit is initially charged to a maximum charge of  $Q = 4.0 \text{ nC}$ . It is then connected in series with an inductor of inductance  $L = 250 \text{ mH}$ .

1. Determine the maximum current in the circuit.
2. Find the current when the charge on the capacitor is  $q = 2.0 \text{ nC}$ .

## Question 20 (5)

An LRC series circuit consists of an inductor  $L = 0.40 \text{ H}$ , a capacitor  $C = 220 \mu\text{F}$  and a variable resistor  $R$ .

1. Determine the critical damping resistance.
2. For  $R = 6.0 \Omega$ , determine whether the system is underdamped, critically damped or overdamped.
3. If underdamped, calculate the damped angular frequency.

## Question 21 (4)

A radio tuning circuit is modelled as a series LC circuit. The inductor has fixed inductance  $L = 1.6 \mu\text{H}$ .

1. Determine the capacitance required to tune the circuit to a broadcast frequency of  $92.0 \text{ MHz}$ .
2. If the capacitance is increased, explain quantitatively how the tuning frequency changes.

## Question 22 (5)

A lightly damped LRC circuit has inductance  $L = 0.50 \text{ H}$ , capacitance  $C = 100 \mu\text{F}$  and resistance  $R = 12 \Omega$ . The maximum current immediately after switch-on is  $0.80 \text{ A}$ .

1. Determine the initial total energy stored in the circuit.
2. Calculate the rate of energy loss at the instant when the current is maximal.
3. Estimate the time taken for the total energy to fall to 50% of its initial value.

## Question 23 (4)

Two atoms of masses  $m_1$  and  $m_2$  interact via a LennardJonestype potential

$$U(r) = U_0 \left[ \left( \frac{R_0}{r} \right)^{12} - 2 \left( \frac{R_0}{r} \right)^6 \right],$$

where  $r$  is the separation.

1. Show that for small displacements  $x = r - R_0$  about equilibrium, the force can be approximated by  $F = -kx$  and determine  $k$  in terms of  $U_0$  and  $R_0$ .
2. Derive an expression for the angular frequency of small oscillations in terms of the reduced mass  $\mu$ .
3. For  $U_0 = 1.6 \times 10^{-21} \text{ J}$ ,  $R_0 = 3.8 \times 10^{-10} \text{ m}$  and  $m_1 = m_2 = 40u$ , calculate the oscillation frequency.

## Question 24 (4)

A system has a generalised coordinate  $z$  and potential energy  $U(z)$ . Near equilibrium at  $z = 0$ , the potential is expanded as

$$U(z) = \frac{1}{2}az^2 + \frac{1}{6}bz^3.$$

1. Derive the corresponding expression for the force  $F(z)$ .
2. Show that to first order in  $z$  the motion is simple harmonic and determine the effective angular frequency.
3. Estimate the fractional correction to the force due to the cubic term when  $z = 0.05 z_{\max}$ .

## Question 25 (4)

A vertical U-tube of uniform cross-sectional area  $A$  contains an incompressible fluid of density  $\rho$ . If the fluid is displaced by a small distance  $y$  from equilibrium and released, it oscillates.

1. Show that the equation of motion takes the form  $\ddot{y} + \frac{g}{L}y = 0$ , where  $2L$  is the total length of the fluid column.
2. Hence determine the angular frequency and period of oscillation.
3. For water ( $\rho = 1000 \text{ kg m}^{-3}$ ) and  $L = 0.25 \text{ m}$ , calculate the numerical value of the period.

## Question 26 (5)

A ball of mass  $m$  rolls without slipping inside a smooth spherical bowl of radius of curvature  $R$ .

1. Using energy methods, show that the equation of motion for small angular displacements  $\theta$  is

$$\ddot{\theta} + \frac{g}{R}\theta = 0.$$

2. Hence determine the period of oscillation.
3. For  $R = 0.80$  m, calculate the period.
4. Explain qualitatively why the motion ceases to be simple harmonic for large amplitudes.

## Question 27 (5)

A particle oscillates in a one-dimensional potential well with potential energy near equilibrium given by

$$U(x) = \alpha x^2 + \beta x^4,$$

where  $\alpha > 0$  and  $\beta > 0$ .

1. Derive the expression for the force  $F(x)$ .
2. Show that for sufficiently small  $x$  the motion reduces to simple harmonic motion and determine the angular frequency.
3. Estimate the amplitude at which the quartic term contributes 10% of the restoring force.
4. Discuss briefly how the presence of the  $x^4$  term modifies the period of oscillation at large amplitudes.