Oscillations Answer Set

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Answer 1 (3)

Answer 1:

1. The restoring force is Hookes law,

$$F = -kx = m\ddot{x}.$$

Hence

$$m\ddot{x} + kx = 0 \quad \Rightarrow \quad \ddot{x} + \omega^2 x = 0, \qquad \omega = \sqrt{\frac{k}{m}}.$$

Thus the motion is SHM with

$$\omega = \sqrt{\frac{50}{0.40}} = \sqrt{125} \approx 11.2 \text{ rad s}^{-1}, \quad T = \frac{2\pi}{\omega} \approx \frac{2\pi}{11.2} \approx 5.6 \times 10^{-1} \text{ s}.$$

$$\boxed{\omega \approx 1.12 \times 10^{1} \text{ rad s}^{-1}}, \qquad \boxed{T \approx 5.6 \times 10^{-1} \text{ s}}$$

2. For SHM, the speed as a function of displacement is

$$v = \omega\sqrt{A^2 - x^2}.$$

Here $A = 0.12$ m and $x = 0.060$ m:

$$v = 11.2\sqrt{0.12^2 - 0.060^2} = 11.2\sqrt{0.0144 - 0.0036} = 11.2\sqrt{0.0108} \approx 1.2 \text{ m s}^{-1}.$$

$$\boxed{v(x = 0.060 \text{ m}) \approx 1.2 \text{ m s}^{-1}}$$

3. Total energy

$$E = \frac{1}{2}kA^2 = \frac{1}{2} \cdot 50 \cdot 0.12^2 = 25 \cdot 0.0144 = 0.36 \text{ J}.$$

At $x = 0.060$ m,

$$U = \frac{1}{2}kx^2 = 25 \cdot 0.060^2 = 25 \cdot 0.0036 = 0.09 \text{ J},$$

so

$$K = E - U = 0.36 - 0.09 = 0.27 \text{ J}.$$

Hence

$$\frac{K}{U} = \frac{0.27}{0.09} = 3.$$

$$\boxed{E = 0.36 \text{ J}}, \qquad \boxed{\frac{K}{U} = 3 : 1 \text{ at } x = 0.060 \text{ m}}$$

Answer 2 (3)

Answer 2:

1. The exact equation of motion is

$$mL\ddot{\theta} + mg\sin\theta = 0.$$

For small angles, $\sin\theta \simeq \theta$, so

$$mL\ddot{\theta} + mg\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \frac{g}{L}\theta = 0,$$

which is SHM with

$$\omega_0 = \sqrt{\frac{g}{L}}.$$

$$\boxed{\omega_0 = \sqrt{\frac{g}{L}}}$$

2. The ideal period at g_0 is $T_0 = 2\pi\sqrt{L_0/g_0}$. At the mountain (g), with the same length L_0 the actual period is

$$T_m = 2\pi\sqrt{\frac{L_0}{g}} = T_0\sqrt{\frac{g_0}{g}}.$$

In real time t the number of swings is $N = t/T_m$, so the clock shows

$$t_{\text{clock}} = NT_0 = t\frac{T_0}{T_m} = t\sqrt{\frac{g}{g_0}}.$$

Hence the error after one day ($t = 86400$ s) is

$$\Delta t = t - t_{\text{clock}} = t\left(1 - \sqrt{\frac{g}{g_0}}\right).$$

With $g = 9.79$, $g_0 = 9.81$,

$$\sqrt{\frac{g}{g_0}} \approx \sqrt{\frac{9.79}{9.81}} \approx 0.99898,$$

so

$$\Delta t \approx 86400(1 - 0.99898)\text{ s} \approx 8.8 \times 10^1\text{ s}.$$

The clock loses time (runs slow) by about 88 s per day.

$$\boxed{\Delta t \approx 8.8 \times 10^1\text{ s lost per day}}$$

3. To restore the original period at g, we require

$$T_0 = 2\pi\sqrt{\frac{L'}{g}} \quad \Rightarrow \quad L' = L_0 \frac{g}{g_0}.$$

Hence the required change in length is

$$\Delta L = L' - L_0 = L_0\left(\frac{g}{g_0} - 1\right) \approx L_0(-2.0 \times 10^{-3}).$$

So the pendulum must be shortened by about 0.20%, i.e. by ≈ 2.0 mm for each metre of length.

$$\boxed{\Delta L \simeq -2.0 \times 10^{-3} L_0 \ \ (\approx -2.0 \text{ mm per metre})}$$

Answer 3 (4)

Answer 3:

1. For an underdamped oscillator,

$$x_n = A_0 e^{-\gamma t_n}, \qquad t_{n+1} - t_n = T_d,$$

so

$$\frac{x_n}{x_{n+1}} = \frac{A_0 e^{-\gamma t_n}}{A_0 e^{-\gamma(t_n+T_d)}} = e^{\gamma T_d}.$$

Defining the logarithmic decrement Λ by

$$\Lambda = \ln\left(\frac{x_n}{x_{n+1}}\right),$$

we see that

$$\Lambda = \gamma T_d.$$

$$\boxed{\Lambda = \ln\left(\frac{x_n}{x_{n+1}}\right)}, \qquad \boxed{\gamma = \frac{\Lambda}{T_d}}$$

2. Using x_1 and x_5,

$$\frac{x_1}{x_5} = e^{4\Lambda} \quad \Rightarrow \quad \Lambda = \frac{1}{4}\ln\left(\frac{x_1}{x_5}\right) = \frac{1}{4}\ln\left(\frac{4.0}{1.5}\right) \approx 0.245.$$

The undamped natural frequency is

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{16}{0.20}} = \sqrt{80} \approx 8.94 \text{ rad s}^{-1},$$

so the undamped period is

$$T_0 = \frac{2\pi}{\omega_0} \approx \frac{2\pi}{8.94} \approx 0.70 \text{ s}.$$

For weak damping we take $T_d \simeq T_0$, hence

$$\gamma \simeq \frac{\Lambda}{T_d} \approx \frac{0.245}{0.70} \approx 0.35 \text{ s}^{-1}.$$

With $\gamma = b/(2m)$,

$$b = 2m\gamma = 2(0.20)(0.35) \approx 0.14 \text{ kg s}^{-1}.$$

$$\boxed{\gamma \approx 3.5 \times 10^{-1} \text{ s}^{-1}}, \qquad \boxed{b \approx 1.4 \times 10^{-1} \text{ kg s}^{-1}}$$

3. The undamped natural frequency is

$$\boxed{\omega_0 \approx 8.94 \text{ rad s}^{-1}}.$$

The damped frequency is

$$\omega_d = \sqrt{\omega_0^2 - \gamma^2} \approx \sqrt{80 - 0.35^2} \approx \sqrt{79.9} \approx 8.94 \text{ rad s}^{-1}.$$

Since $\gamma \ll \omega_0$, the oscillator is clearly underdamped.

$$\boxed{\omega_d \approx 8.9 \text{ rad s}^{-1}}, \qquad \boxed{\text{oscillator is underdamped}}$$

4. The quality factor is

$$Q = \frac{\omega_0}{2\gamma} \approx \frac{8.94}{2 \times 0.35} \approx \frac{8.94}{0.70} \approx 12.8.$$

$$\boxed{Q \approx 1.3 \times 10^1}$$

Answer 4 (4)

Answer 4:

1. Assume a steady-state solution $x(t) = A\cos(\omega t - \phi)$. Substituting into

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t)$$

and equating cosine and sine components leads to

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}},$$

where $\omega_0^2 = k/m$ and $\beta = b/(2m)$.

$$\boxed{A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}}}$$

2. The amplitude is maximal when

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2\beta^2}.$$

Here

$$\omega_0^2 = \frac{100}{0.50} = 200, \qquad \beta = \frac{b}{2m} = \frac{4.0}{1.0} = 4.0\ \text{s}^{-1},$$

so

$$\omega_{\text{res}} = \sqrt{200 - 2(4.0)^2} = \sqrt{200 - 32} = \sqrt{168} \approx 13.0\ \text{rad s}^{-1}.$$

$$\boxed{\omega_{\text{res}} \approx 1.3 \times 10^1\ \text{rad s}^{-1}}$$

3. First compute F_0/m:

$$\frac{F_0}{m} = \frac{2.0}{0.50} = 4.0\ \text{m s}^{-2}.$$

For $\omega = 0.5\,\omega_0$, note $\omega_0 = \sqrt{200} \approx 14.1$ rad s^{-1}, so

$$\omega_1 = 0.5\omega_0 \approx 7.07\ \text{rad s}^{-1}, \quad \omega_1^2 = 0.25\omega_0^2 = 50.$$

Then

$$\omega_0^2 - \omega_1^2 = 200 - 50 = 150, \quad 2\beta\omega_1 = 2(4.0)(7.07) \approx 56.6.$$

Hence

$$A_1 = \frac{4.0}{\sqrt{150^2 + 56.6^2}} \approx \frac{4.0}{1.60 \times 10^2} \approx 2.5 \times 10^{-2}\ \text{m}.$$

The phase lag satisfies

$$\tan\phi_1 = \frac{2\beta\omega_1}{\omega_0^2 - \omega_1^2} = \frac{56.6}{150} \approx 0.38 \;\Rightarrow\; \phi_1 \approx 0.36\ \text{rad}\ (\approx 21^\circ).$$

$$\boxed{A_1 \approx 2.5 \times 10^{-2}\ \text{m}}, \qquad \boxed{\phi_1 \approx 0.36\ \text{rad}}$$

For $\omega = 2\omega_0$,

$$\omega_2 = 2\omega_0 \approx 28.3\ \text{rad s}^{-1}, \quad \omega_2^2 = 4\omega_0^2 = 800.$$

Then

$$\omega_0^2 - \omega_2^2 = 200 - 800 = -600, \quad 2\beta\omega_2 = 2(4.0)(28.3) \approx 2.26 \times 10^2.$$

Hence

$$A_2 = \frac{4.0}{\sqrt{(-600)^2 + 226^2}} \approx \frac{4.0}{6.41 \times 10^2} \approx 6.2 \times 10^{-3}\ \text{m}.$$

The phase lag:

$$\tan\phi_2 = \frac{2\beta\omega_2}{\omega_0^2 - \omega_2^2} \approx \frac{226}{-600} = -0.38,$$

so ϕ_2 lies in the second quadrant,

$$\phi_2 \approx \pi - 0.36 \approx 2.78\ \text{rad}\ (\approx 159^\circ).$$

$$\boxed{A_2 \approx 6.2 \times 10^{-3}\ \text{m}}, \qquad \boxed{\phi_2 \approx 2.78\ \text{rad}}$$

4. Velocity amplitude is $V(\omega) = \omega A(\omega)$.

   For $\omega_1 = 0.5\omega_0$,

$$V_1 = \omega_1 A_1 \approx 7.07 \times 2.5 \times 10^{-2} \approx 1.8 \times 10^{-1}\ \text{m s}^{-1}.$$

   For $\omega_2 = 2\omega_0$,

$$V_2 = \omega_2 A_2 \approx 28.3 \times 6.2 \times 10^{-3} \approx 1.8 \times 10^{-1}\ \text{m s}^{-1}.$$

   Numerically these are equal; indeed the symmetry of the expression for $V(\omega)$ about ω_0 implies

$$V(\tfrac{1}{2}\omega_0) = V(2\omega_0).$$

$$\boxed{V_1 \approx V_2 \approx 1.8 \times 10^{-1}\ \text{m s}^{-1}\ \text{(equal for the two chosen frequencies)}}$$

Answer 5 (5)

Answer 5:

1. Let x_1, x_2 be displacements to the right from equilibrium.

   For mass m_1: Left spring force: $-kx_1$; middle spring force: $-k(x_1 - x_2)$ (restoring toward equilibrium). Thus

   $$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2) = -2kx_1 + kx_2.$$

   For mass m_2: Right spring: $-kx_2$; middle spring: $-k(x_2 - x_1)$. Hence

   $$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1) = kx_1 - 2kx_2.$$

   $$\boxed{m\ddot{x}_1 = -2kx_1 + kx_2, \qquad m\ddot{x}_2 = kx_1 - 2kx_2}$$

2. Take normal-mode solutions $x_j(t) = A_j \cos(\omega t)$. Then

   $$-m\omega^2 A_1 = -2kA_1 + kA_2, \qquad -m\omega^2 A_2 = kA_1 - 2kA_2.$$

   This gives the homogeneous system

   $$\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \mathbf{0}.$$

   For nontrivial solutions, the determinant must vanish:

   $$(2k - m\omega^2)^2 - k^2 = 0 \quad \Rightarrow \quad 2k - m\omega^2 = \pm k.$$

   Hence

   $$2k - m\omega^2 = k \Rightarrow m\omega^2 = k, \qquad 2k - m\omega^2 = -k \Rightarrow m\omega^2 = 3k,$$

   giving

   $$\boxed{\omega_1 = \sqrt{\frac{k}{m}}}, \qquad \boxed{\omega_2 = \sqrt{\frac{3k}{m}}}.$$

3. For $\omega_1^2 = k/m$:

   $$-m\omega_1^2 A_1 = -2kA_1 + kA_2 \Rightarrow -kA_1 = -2kA_1 + kA_2 \Rightarrow kA_1 = kA_2 \Rightarrow A_2 = A_1.$$

   Thus both masses move in phase with equal amplitude (rigid translation of the centre spring).

   For $\omega_2^2 = 3k/m$:

   $$-3kA_1 = -2kA_1 + kA_2 \Rightarrow -kA_1 = kA_2 \Rightarrow A_2 = -A_1.$$

   Here the masses move with equal amplitude but in opposite directions about the centre.

   $$\boxed{\omega_1 : \ A_2/A_1 = +1 \text{ (in phase)}}, \qquad \boxed{\omega_2 : \ A_2/A_1 = -1 \text{ (out of phase)}}$$

4. Superpose the two normal modes. Choose mode amplitudes C_1 and C_2 so that

$$x_1(t) = C_1\cos(\omega_1 t) + C_2\cos(\omega_2 t),$$

$$x_2(t) = C_1\cos(\omega_1 t) - C_2\cos(\omega_2 t),$$

consistent with the amplitude ratios above.

Initial conditions at $t = 0$:

$$x_1(0) = C_1 + C_2 = X_0, \qquad x_2(0) = C_1 - C_2 = 0,$$

giving $C_1 = C_2 = X_0/2$. Velocities vanish automatically since $\dot{x}_j(0) \propto \sin(0) = 0$. Therefore

$$x_1(t) = \frac{X_0}{2}\big[\cos(\omega_1 t) + \cos(\omega_2 t)\big], \qquad x_2(t) = \frac{X_0}{2}\big[\cos(\omega_1 t) - \cos(\omega_2 t)\big].$$

Using the identities

$$\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2}, \quad \cos a - \cos b = -2\sin\frac{a+b}{2}\sin\frac{a-b}{2},$$

we obtain

$$x_1(t) = X_0\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right),$$

$$x_2(t) = -X_0\sin\left(\frac{\omega_1+\omega_2}{2}t\right)\sin\left(\frac{\omega_1-\omega_2}{2}t\right).$$

The fast oscillation at average frequency $(\omega_1+\omega_2)/2$ is modulated by a slow envelope with beat frequency $|\omega_1-\omega_2|/(2\pi)$, corresponding to periodic transfer of energy between the two masses.

$$\boxed{\begin{aligned} x_1(t) &= X_0\cos\left(\tfrac{\omega_1+\omega_2}{2}t\right)\cos\left(\tfrac{\omega_1-\omega_2}{2}t\right), \\ x_2(t) &= -X_0\sin\left(\tfrac{\omega_1+\omega_2}{2}t\right)\sin\left(\tfrac{\omega_1-\omega_2}{2}t\right) \end{aligned}}$$

Answer 6 (3)

For a torsional pendulum,

$$\omega = \sqrt{\frac{\kappa}{I}} \quad \Rightarrow \quad \kappa = I\omega^2.$$

For a uniform solid cylinder about its central axis,

$$I = \frac{1}{2}MR^2 = \frac{1}{2}(1.8)(0.12)^2 = \frac{1}{2}(1.8)(0.0144) = 0.01296\,\mathrm{kg\,m^2}.$$

Given $\omega = 4.6\,\mathrm{rad\,s^{-1}}$,

$$\kappa = 0.01296 \times (4.6)^2 = 0.01296 \times 21.16 \approx 2.74 \times 10^{-1}\,\mathrm{N\,m\,rad^{-1}}.$$

$$\boxed{\kappa \approx 2.7 \times 10^{-1}\,\mathrm{N\,m\,rad^{-1}}}$$

Answer 7 (4)

The period of the pendulum is

$$T = \frac{\tau}{N} = \frac{402\,\mathrm{s}}{200} = 2.01\,\mathrm{s}.$$

1. For a simple pendulum,

$$T = 2\pi\sqrt{\frac{L}{g}} \quad \Rightarrow \quad g = \frac{4\pi^2 L}{T^2}.$$

   Substituting $T = 2.01\,\mathrm{s}$,

$$g = \frac{4\pi^2 L}{(2.01)^2} \approx 9.77\,L\ \mathrm{m\,s^{-2}\,m^{-1}},$$

   so in terms of the length L (in metres),

$$\boxed{g \approx 9.8\left(\frac{L}{1\,\mathrm{m}}\right)\ \mathrm{m\,s^{-2}}}.$$

2. Increasing the length by 3% gives $L' = 1.03L$. Then

$$T' = 2\pi\sqrt{\frac{L'}{g}} = 2\pi\sqrt{\frac{1.03L}{g}} = T\sqrt{1.03}.$$

   Thus the new time for 200 oscillations is

$$\tau' = 200T' = 200T\sqrt{1.03} = 402\sqrt{1.03} \approx 4.08 \times 10^2\,\mathrm{s}.$$

$$\boxed{\tau' \approx 4.08 \times 10^2\,\mathrm{s}\ (\approx 408\,\mathrm{s})}$$

Answer 8 (5)

Given $L = 0.85\,\mathrm{m}$, $g = 9.81\,\mathrm{m\,s^{-2}}$ and $\theta_0 = 28°$.

First convert the angle to radians:

$$\theta_0 = 28° \times \frac{\pi}{180°} \approx 0.489\,\mathrm{rad}.$$

The small-angle period is

$$T_0 = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{0.85}{9.81}} \approx 1.85\,\mathrm{s}.$$

1. Using the given correction,

$$T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{\theta_0^2}{16}\right) = T_0\left(1 + \frac{\theta_0^2}{16}\right).$$

   Compute

$$\frac{\theta_0^2}{16} \approx \frac{(0.489)^2}{16} \approx 0.015.$$

   Hence

$$T \approx 1.85 \times (1 + 0.015) \approx 1.88\,\mathrm{s}.$$

$$\boxed{T \approx 1.88\,\mathrm{s}}$$

2. The percentage error if the small-angle formula is used is

$$\%\text{ error} = \frac{T - T_0}{T} \times 100\% \approx \frac{1.88 - 1.85}{1.88} \times 100\% \approx 1.5\%.$$

$$\boxed{\text{Using the small-angle formula underestimates } T \text{ by } \approx 1.5\%}$$

Answer 9 (4)

For a simple pendulum of length $L = 0.65\,\mathrm{m}$ released from $\theta_0 = 18°$:

The vertical drop of the bob from release to the lowest point is

$$h = L(1 - \cos\theta_0).$$

1. Conservation of energy between the release point (all GPE) and the bottom (all KE) gives

$$mgh = \frac{1}{2}mv_{\max}^2 \quad \Rightarrow \quad v_{\max} = \sqrt{2gL(1-\cos\theta_0)}.$$

Numerically,

$$v_{\max} = \sqrt{2(9.81)(0.65)\bigl(1-\cos 18°\bigr)} \approx 0.79\,\mathrm{m\,s^{-1}}.$$

$$\boxed{v_{\max} \approx 0.79\,\mathrm{m\,s^{-1}}}$$

2. Let θ be the angle when $v = \frac{1}{2}v_{\max}$. Conservation of energy gives

$$mgL(1-\cos\theta_0) = mgL(1-\cos\theta) + \frac{1}{2}mv^2.$$

Divide by mgL and substitute $v = v_{\max}/2$:

$$1-\cos\theta_0 = 1-\cos\theta + \frac{1}{2}\frac{(v_{\max}/2)^2}{gL} = 1-\cos\theta + \frac{1}{4}\frac{v_{\max}^2}{2gL}.$$

But $v_{\max}^2 = 2gL(1-\cos\theta_0)$, so

$$\frac{v_{\max}^2}{2gL} = 1-\cos\theta_0,$$

and therefore

$$1-\cos\theta_0 = 1-\cos\theta + \frac{1}{4}(1-\cos\theta_0).$$

Rearrange:

$$1-\cos\theta = \frac{3}{4}(1-\cos\theta_0) \quad \Rightarrow \quad \cos\theta = 1-\frac{3}{4}(1-\cos\theta_0).$$

With $\theta_0 = 18°$,

$$\cos\theta \approx 1-\frac{3}{4}\bigl(1-\cos 18°\bigr) \approx 0.963,$$

so

$$\theta \approx \cos^{-1}(0.963) \approx 15.6°.$$

To the nearest degree,

$$\boxed{\theta \approx 16°}$$

Answer 10 (5)

A uniform rod of length $L = 0.90\,\mathrm{m}$ and mass $M = 2.2\,\mathrm{kg}$ is pivoted at a point $0.15\,\mathrm{m}$ from one end.

1. The centre of mass is at $L/2 = 0.45\,\mathrm{m}$ from either end, so the distance from the pivot to the centre of mass is
$$d = 0.45 - 0.15 = 0.30\,\mathrm{m}.$$
The moment of inertia of a uniform rod about its centre is
$$I_{\mathrm{cm}} = \frac{1}{12}ML^2 = \frac{1}{12}(2.2)(0.90)^2 \approx 0.1485\,\mathrm{kg\,m^2}.$$
Using the parallel-axis theorem,
$$I = I_{\mathrm{cm}} + Md^2 = 0.1485 + 2.2(0.30)^2 = 0.1485 + 0.198 \approx 0.3465\,\mathrm{kg\,m^2}.$$
$$\boxed{I \approx 3.47 \times 10^{-1}\,\mathrm{kg\,m^2}}$$

2. For a physical pendulum,
$$\omega^2 = \frac{Mgd}{I} \quad \Rightarrow \quad \omega = \sqrt{\frac{Mgd}{I}}.$$
Substitute $M = 2.2\,\mathrm{kg}$, $g = 9.81\,\mathrm{m\,s^{-2}}$, $d = 0.30\,\mathrm{m}$ and $I = 0.3465\,\mathrm{kg\,m^2}$:
$$\omega = \sqrt{\frac{(2.2)(9.81)(0.30)}{0.3465}} \approx 4.32\,\mathrm{rad\,s^{-1}}.$$
The period is
$$T = \frac{2\pi}{\omega} \approx \frac{2\pi}{4.32} \approx 1.45\,\mathrm{s}.$$
$$\boxed{\omega \approx 4.3\,\mathrm{rad\,s^{-1}}, \quad T \approx 1.45\,\mathrm{s}}$$

3. For a simple pendulum with the same period,
$$\omega^2 = \frac{g}{L_{\mathrm{eq}}} \quad \Rightarrow \quad L_{\mathrm{eq}} = \frac{g}{\omega^2}.$$
Thus
$$L_{\mathrm{eq}} = \frac{9.81}{(4.32)^2} \approx 0.525\,\mathrm{m}.$$
$$\boxed{L_{\mathrm{eq}} \approx 0.53\,\mathrm{m}}$$

Answer 11 (4)

1. For a simple pendulum,

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \quad \Rightarrow \quad L = \frac{g}{(2\pi f)^2} = \frac{g}{4\pi^2 f^2}.$$

   With $f = 0.32\,\mathrm{Hz}$ and $g = 9.81\,\mathrm{m\,s^{-2}}$,

$$L = \frac{9.81}{4\pi^2(0.32)^2} \approx 2.43\,\mathrm{m}.$$

$$\boxed{L \approx 2.4\,\mathrm{m}}$$

2. The angular frequency is

$$\omega = 2\pi f = 2\pi(0.32) \approx 2.01\,\mathrm{rad\,s^{-1}}.$$

   With amplitude $A = 0.80\,\mathrm{m}$, the maximum speed is

$$v_{\max} = \omega A \approx (2.01)(0.80) \approx 1.61\,\mathrm{m\,s^{-1}}.$$

   The maximum kinetic energy is then

$$K_{\max} = \frac{1}{2}mv_{\max}^2 = \frac{1}{2}(5.0\times10^5)(1.61)^2 \approx 6.5\times10^5\,\mathrm{J}.$$

$$\boxed{K_{\max} \approx 6.5\times10^5\,\mathrm{J}}$$

Answer 12 (3)

Answer 12:

1. The undamped natural angular frequency is

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{18}{0.20}} = \sqrt{90} \approx 9.5 \text{ rad s}^{-1}.$$

$$\boxed{\omega_0 \approx 9.5 \text{ rad s}^{-1}}$$

2. The damping ratio is

$$\zeta = \frac{b}{2\sqrt{mk}} = \frac{1.2}{2\sqrt{0.20 \times 18}} = \frac{1.2}{2\sqrt{3.6}} \approx \frac{1.2}{3.79} \approx 0.32.$$

Since $0 < \zeta < 1$, the motion is underdamped.

$$\boxed{\zeta \approx 0.32 \ (< 1, \ \text{underdamped})}$$

3. For an underdamped oscillator the displacement is of the form

$$x(t) = Ae^{-\gamma t} \cos(\omega' t + \phi), \quad \gamma = \frac{b}{2m},$$

so the system oscillates with a frequency slightly less than ω_0 and an exponentially decaying amplitude.

$$\boxed{\text{Oscillatory motion with amplitude decaying exponentially in time}}$$

Answer 13 (4)

Answer 13:

1.
$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{1.00} = 2\pi \approx 6.28 \text{ rad s}^{-1},$$
$$\omega_d = \frac{2\pi}{T_d} = \frac{2\pi}{1.10} \approx 5.71 \text{ rad s}^{-1}.$$
$$\boxed{\omega_0 \approx 6.28 \text{ rad s}^{-1}}, \qquad \boxed{\omega_d \approx 5.71 \text{ rad s}^{-1}}$$

2. Using
$$\omega_d^2 = \omega_0^2 - \left(\frac{b}{2m}\right)^2 \quad \Rightarrow \quad \left(\frac{b}{2m}\right)^2 = \omega_0^2 - \omega_d^2,$$
we have
$$\omega_0^2 \approx (2\pi)^2 \approx 39.48, \quad \omega_d^2 \approx 32.63,$$
so
$$\left(\frac{b}{2m}\right)^2 \approx 39.48 - 32.63 = 6.85,$$
$$\frac{b}{2m} \approx \sqrt{6.85} \approx 2.62 \text{ s}^{-1}.$$
With $m = 0.40$ kg,
$$b = 2m\frac{b}{2m} = 0.80 \times 2.62 \approx 2.1 \text{ kg s}^{-1}.$$
$$\boxed{b \approx 2.1 \text{ kg s}^{-1}}$$

3. The damping ratio is
$$\zeta = \frac{b}{2\sqrt{mk}} = \frac{2.1}{2\sqrt{0.40 \times 50}} \approx \frac{2.1}{8.94} \approx 0.23 < 1,$$
so the oscillator is lightly underdamped.
$$\boxed{\zeta \approx 0.23 \ \Rightarrow \ \text{lightly underdamped}}$$

Answer 14 (4)

Answer 14:

1. The amplitude decays as
$$A(t) = Ae^{-\gamma t}.$$
We want $A(t) = 0.25A$, so
$$e^{-\gamma t} = 0.25 = \frac{1}{4} \quad \Rightarrow \quad -\gamma t = \ln\frac{1}{4} = -\ln 4,$$
giving
$$t = \frac{\ln 4}{\gamma} = \frac{1.386}{0.25} \approx 5.5 \text{ s.}$$
$$\boxed{t \approx 5.5 \text{ s}}$$

2. The energy is proportional to A^2, so
$$\frac{E}{E_0} = \left(\frac{A(t)}{A}\right)^2 = e^{-2\gamma t}.$$
At the time found above $A(t)/A = 0.25$, so
$$\frac{E}{E_0} = (0.25)^2 = 0.0625.$$
$$\boxed{\frac{E}{E_0} = 6.25 \times 10^{-2}}$$

3. The damped period is
$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{9.0} \approx 0.70 \text{ s.}$$
The number of oscillations in time t is
$$N = \frac{t}{T_d} \approx \frac{5.5}{0.70} \approx 8.$$
$$\boxed{N \approx 8 \text{ oscillations}}$$

Answer 15 (5)

Answer 15:

1. For a driven, damped oscillator the steadystate amplitude is

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}},$$

where $\omega_0 = \sqrt{k/m}$ and $\beta = b/(2m)$.

$$\boxed{A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}}}$$

2.

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{60}{0.30}} = \sqrt{200} \approx 14.1 \text{ rad s}^{-1},$$

$$\beta = \frac{b}{2m} = \frac{2.4}{0.60} = 4.0 \text{ s}^{-1}.$$

$$\boxed{\omega_0 \approx 1.41 \times 10^1 \text{ rad s}^{-1}}, \qquad \boxed{\beta = 4.0 \text{ s}^{-1}}$$

3. The resonance (maximumamplitude) angular frequency is

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2\beta^2} = \sqrt{200 - 2 \times 16} = \sqrt{168} \approx 13.0 \text{ rad s}^{-1}.$$

$$\boxed{\omega_{\text{res}} \approx 1.3 \times 10^1 \text{ rad s}^{-1}}$$

4. With $F_0/m = 1.5/0.30 = 5.0 \text{ m s}^{-2}$ and $\omega = \omega_{\text{res}}$,

$$\omega_0^2 - \omega_{\text{res}}^2 = 200 - 168 = 32,$$

$$2\beta\omega_{\text{res}} = 2(4.0)(12.96) \approx 1.04 \times 10^2.$$

Hence

$$A_{\text{res}} = \frac{5.0}{\sqrt{32^2 + (2\beta\omega_{\text{res}})^2}} \approx \frac{5.0}{\sqrt{1024 + 1.08 \times 10^4}} \approx \frac{5.0}{1.09 \times 10^2} \approx 4.6 \times 10^{-2} \text{ m}.$$

$$\boxed{A_{\text{res}} \approx 4.6 \times 10^{-2} \text{ m}}$$

5. The phase lag ϕ between displacement and driving force satisfies

$$\tan\phi = \frac{2\beta\omega}{\omega_0^2 - \omega^2}.$$

At $\omega = \omega_{\text{res}}$ we have $\omega_0^2 - \omega_{\text{res}}^2 = 2\beta^2$, so

$$\tan\phi_{\text{res}} = \frac{2\beta\omega_{\text{res}}}{2\beta^2} = \frac{\omega_{\text{res}}}{\beta} \approx \frac{12.96}{4.0} \approx 3.24,$$

giving

$$\phi_{\text{res}} \approx \tan^{-1}(3.24) \approx 1.27 \text{ rad} \approx 73^\circ.$$

$$\boxed{\phi_{\text{res}} \approx 1.3 \text{ rad (displacement lags driving force by } \approx 73^\circ)}$$

Answer 16 (5)

Answer 16:

1. Let $x_1(t)$ and $x_2(t)$ be the displacements of m_1 and m_2 from equilibrium (positive to the right).

   For m_1 the left spring exerts $-kx_1$, and the middle spring exerts $-k(x_1 - x_2)$, so

   $$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2) = -2kx_1 + kx_2.$$

   For m_2 the right spring exerts $-kx_2$, and the middle spring exerts $-k(x_2 - x_1)$, giving

   $$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1) = kx_1 - 2kx_2.$$

   Thus the coupled equations are

   $$\boxed{m\ddot{x}_1 = -2kx_1 + kx_2, \qquad m\ddot{x}_2 = kx_1 - 2kx_2,}$$

   which form a 2×2 eigenvalue problem and hence support two normal modes.

2. Seek solutions of the form $x_j(t) = A_j \cos(\omega t)$. Then

   $$-m\omega^2 A_1 = -2kA_1 + kA_2, \qquad -m\omega^2 A_2 = kA_1 - 2kA_2.$$

   This can be written as

   $$\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0.$$

   Nontrivial solutions require

   $$(2k - m\omega^2)^2 - k^2 = 0 \quad \Rightarrow \quad 2k - m\omega^2 = \pm k.$$

   Hence

   $$2k - m\omega^2 = k \Rightarrow m\omega^2 = k, \quad 2k - m\omega^2 = -k \Rightarrow m\omega^2 = 3k,$$

   giving

   $$\boxed{\omega_1 = \sqrt{\frac{k}{m}}, \qquad \omega_2 = \sqrt{\frac{3k}{m}}.}$$

3. For ω_1:

   $$-m\omega_1^2 A_1 = -2kA_1 + kA_2 \Rightarrow -kA_1 = -2kA_1 + kA_2 \Rightarrow kA_1 = kA_2 \Rightarrow A_2 = A_1,$$

   so the two masses move in phase with equal amplitude (both left and right together).

   For ω_2:

   $$-3kA_1 = -2kA_1 + kA_2 \Rightarrow -kA_1 = kA_2 \Rightarrow A_2 = -A_1,$$

   so the masses move with equal amplitude but in opposite directions (out of phase by π).

$$\boxed{\omega_1 : \; A_2 = A_1 \text{ (in phase)}, \qquad \omega_2 : \; A_2 = -A_1 \text{ (out of phase)}}$$

4. With $m = 0.50$ kg and $k = 200$ N m^{-1},

$$\omega_1 = \sqrt{\frac{k}{m}} = \sqrt{\frac{200}{0.50}} = \sqrt{400} = 20 \text{ rad s}^{-1},$$

$$\omega_2 = \sqrt{\frac{3k}{m}} = \sqrt{\frac{600}{0.50}} = \sqrt{1200} \approx 34.6 \text{ rad s}^{-1}.$$

Corresponding frequencies:

$$f_1 = \frac{\omega_1}{2\pi} \approx \frac{20}{2\pi} \approx 3.2 \text{ Hz}, \quad f_2 = \frac{\omega_2}{2\pi} \approx \frac{34.6}{2\pi} \approx 5.5 \text{ Hz}.$$

$$\boxed{\omega_1 = 20 \text{ rad s}^{-1}\ (f_1 \approx 3.2 \text{ Hz})}, \qquad \boxed{\omega_2 \approx 34.6 \text{ rad s}^{-1}\ (f_2 \approx 5.5 \text{ Hz})}$$

Answer 17 (3)

For an LC circuit,

$$\omega = \frac{1}{\sqrt{LC}}.$$

With $L = 2.5 \times 10^{-3}\,\mathrm{H}$ and $C = 8.0 \times 10^{-9}\,\mathrm{F}$,

$$\omega = \frac{1}{\sqrt{(2.5 \times 10^{-3})(8.0 \times 10^{-9})}} \approx 2.24 \times 10^{5}\,\mathrm{rad\,s^{-1}}.$$

$$\boxed{\omega \approx 2.2 \times 10^{5}\,\mathrm{rad\,s^{-1}}}$$

The period is

$$T = \frac{2\pi}{\omega} \approx \frac{2\pi}{2.24 \times 10^{5}} \approx 2.8 \times 10^{-5}\,\mathrm{s} = 28\,\mu\mathrm{s},$$

and the frequency is

$$f = \frac{1}{T} \approx 3.6 \times 10^{4}\,\mathrm{Hz}.$$

$$\boxed{T \approx 2.8 \times 10^{-5}\,\mathrm{s}}, \qquad \boxed{f \approx 3.6 \times 10^{4}\,\mathrm{Hz}}$$

Answer 18 (4)

The time between adjacent voltage peaks is the period:

$$T = (6.4\,\mathrm{div}) \times (0.50 \times 10^{-3}\,\mathrm{s/div}) = 3.2 \times 10^{-3}\,\mathrm{s}.$$

Thus

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{3.2 \times 10^{-3}} \approx 1.96 \times 10^{3}\,\mathrm{rad\,s^{-1}}.$$

For an LC circuit,

$$\omega = \frac{1}{\sqrt{LC}} \quad \Rightarrow \quad L = \frac{1}{\omega^2 C}.$$

With $C = 12.0 \times 10^{-9}\,\mathrm{F}$,

$$L = \frac{1}{(1.96 \times 10^{3})^2 (12.0 \times 10^{-9})} \approx 2.2 \times 10^{1}\,\mathrm{H}.$$

$$\boxed{L \approx 2.2 \times 10^{1}\,\mathrm{H}\ (\approx 22\,\mathrm{H})}$$

Answer 19 (4)

(i) The maximum current is obtained from energy conservation:

$$\frac{1}{2}LI_{\max}^2 = \frac{1}{2}\frac{Q^2}{C} \quad \Rightarrow \quad I_{\max} = \sqrt{\frac{Q^2}{LC}} = \frac{Q}{\sqrt{LC}}.$$

With $Q = 4.0 \times 10^{-9}\,\mathrm{C}$, $C = 150 \times 10^{-12}\,\mathrm{F}$ and $L = 0.25\,\mathrm{H}$,

$$I_{\max} = \frac{4.0 \times 10^{-9}}{\sqrt{(0.25)(150 \times 10^{-12})}} \approx 6.5 \times 10^{-4}\,\mathrm{A} = 0.65\,\mathrm{mA}.$$

$$\boxed{I_{\max} \approx 6.5 \times 10^{-4}\,\mathrm{A}\ (0.65\,\mathrm{mA})}$$

(ii) At a general time, the energy relation

$$\frac{1}{2}LI^2 + \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}\frac{Q^2}{C}$$

gives

$$I^2 = \frac{Q^2 - q^2}{LC} \quad \Rightarrow \quad I = \pm\sqrt{\frac{Q^2 - q^2}{LC}}.$$

For $q = 2.0 \times 10^{-9}\,\mathrm{C}$,

$$I = \pm\sqrt{\frac{(4.0 \times 10^{-9})^2 - (2.0 \times 10^{-9})^2}{(0.25)(150 \times 10^{-12})}} \approx \pm 5.7 \times 10^{-4}\,\mathrm{A} = \pm 0.57\,\mathrm{mA}.$$

$$\boxed{I \approx \pm 5.7 \times 10^{-4}\,\mathrm{A}\ (\pm 0.57\,\mathrm{mA})}$$

Answer 20 (5)

For an LRC series circuit, the damped angular frequency is

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}.$$

Critical damping occurs when $\omega' = 0$:

$$\frac{1}{LC} - \frac{R_c^2}{4L^2} = 0 \quad \Rightarrow \quad R_c = 2\sqrt{\frac{L}{C}}.$$

1. With $L = 0.40\,\mathrm{H}$ and $C = 220 \times 10^{-6}\,\mathrm{F}$,

$$R_c = 2\sqrt{\frac{0.40}{220 \times 10^{-6}}} \approx 8.5 \times 10^1\,\Omega.$$

$$\boxed{R_c \approx 8.5 \times 10^1\,\Omega\ (\approx 85\,\Omega)}$$

2. Given $R = 6.0\,\Omega$, we see that $R < R_c$, so

$$\boxed{\text{the circuit is underdamped.}}$$

3. Since underdamped, we use

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}.$$

Compute

$$\frac{1}{LC} = \frac{1}{(0.40)(220 \times 10^{-6})} \approx 1.07 \times 10^2\,\mathrm{s}^{-2},$$

$$\frac{R^2}{4L^2} = \frac{6.0^2}{4(0.40)^2} \approx 5.6 \times 10^0\,\mathrm{s}^{-2}.$$

Thus

$$\omega' = \sqrt{1.07 \times 10^2 - 5.6} \approx 1.06 \times 10^2\,\mathrm{rad\,s^{-1}}.$$

(Period $T' = 2\pi/\omega' \approx 5.9 \times 10^{-2}\,\mathrm{s}$.)

$$\boxed{\omega' \approx 1.1 \times 10^2\,\mathrm{rad\,s^{-1}}}$$

Answer 21 (4)

For a series LC tuning circuit,

$$\omega = 2\pi f = \frac{1}{\sqrt{LC}} \quad \Rightarrow \quad C = \frac{1}{L(2\pi f)^2}.$$

1. With $L = 1.6 \times 10^{-6}\,\mathrm{H}$ and $f = 92.0 \times 10^{6}\,\mathrm{Hz}$,

$$C = \frac{1}{(1.6 \times 10^{-6})[2\pi(92.0 \times 10^{6})]^2} \approx 1.9 \times 10^{-12}\,\mathrm{F} = 1.9\,\mathrm{pF}.$$

$$\boxed{C \approx 1.9 \times 10^{-12}\,\mathrm{F}\ (\approx 1.9\,\mathrm{pF})}$$

2. The tuning frequency is

$$f = \frac{1}{2\pi\sqrt{LC}}.$$

For fixed L, this gives

$$f \propto \frac{1}{\sqrt{C}}.$$

Thus, if C is increased by a factor k, the frequency changes by $f \to f/\sqrt{k}$. In particular, increasing C decreases the tuning frequency.

$$\boxed{f = \frac{1}{2\pi\sqrt{LC}} \text{ so larger } C \Rightarrow \text{smaller } f\ (\propto 1/\sqrt{C})}$$

Answer 22 (5)

1. Immediately after switch-on, the energy is purely magnetic:
$$E_0 = \frac{1}{2}LI_0^2,$$
with $L = 0.50\,\mathrm{H}$ and $I_0 = 0.80\,\mathrm{A}$:
$$E_0 = \frac{1}{2}(0.50)(0.80)^2 = 0.16\,\mathrm{J}.$$
$$\boxed{E_0 = 0.16\,\mathrm{J}}$$

2. The instantaneous power loss in the resistor is
$$P = I^2R.$$
At the moment of maximum current ($I = I_0$),
$$P = (0.80)^2(12) = 7.68\,\mathrm{W}.$$
The energy is decreasing at rate $dE/dt = -7.68\,\mathrm{J\,s^{-1}}$.
$$\boxed{P = 7.7\,\mathrm{W}\ (\text{energy loss rate } dE/dt = -7.7\,\mathrm{J\,s^{-1}})}$$

3. For a lightly damped LRC series circuit, the current amplitude decays as
$$I(t) \propto e^{-\frac{R}{2L}t}.$$
Since energy $E \propto I^2$, we have
$$E(t) = E_0e^{-\frac{R}{L}t}.$$
Setting $E(t) = \frac{1}{2}E_0$ gives
$$\frac{1}{2}E_0 = E_0e^{-\frac{R}{L}t} \quad\Rightarrow\quad e^{-\frac{R}{L}t} = \frac{1}{2} \quad\Rightarrow\quad t = \frac{L}{R}\ln 2.$$
With $L = 0.50\,\mathrm{H}$ and $R = 12\,\Omega$,
$$t = \frac{0.50}{12}\ln 2 \approx 2.9 \times 10^{-2}\,\mathrm{s}.$$
$$\boxed{t_{1/2} \approx 2.9 \times 10^{-2}\,\mathrm{s}\ (\approx 0.029\,\mathrm{s})}$$

Answer 23 (4)

Answer 23:

1. The force is

$$F(r) = -\frac{dU}{dr} = -U_0\left[-12\frac{R_0^{12}}{r^{13}} + 12\frac{R_0^6}{r^7}\right] = \frac{12U_0}{R_0}\left[\left(\frac{R_0}{r}\right)^{13} - \left(\frac{R_0}{r}\right)^{7}\right].$$

Put $r = R_0 + x$ and assume $|x| \ll R_0$. Then

$$\left(\frac{R_0}{R_0+x}\right)^n = \left(1 + \frac{x}{R_0}\right)^{-n} \approx 1 - n\frac{x}{R_0}.$$

Hence

$$F \simeq \frac{12U_0}{R_0}\left[\left(1 - 13\frac{x}{R_0}\right) - \left(1 - 7\frac{x}{R_0}\right)\right] = \frac{12U_0}{R_0}\left(-6\frac{x}{R_0}\right) = -\frac{72U_0}{R_0^2}x.$$

Comparing with $F = -kx$ we obtain

$$\boxed{k = \frac{72U_0}{R_0^2}}.$$

2. For motion of the relative coordinate x the effective mass is the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

The equation of motion is

$$\mu\ddot{x} = -kx,$$

so

$$\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{72U_0}{R_0^2\mu}}, \qquad f = \frac{\omega}{2\pi}.$$

$$\boxed{\omega = \sqrt{\frac{72U_0}{R_0^2\mu}}, \quad f = \frac{1}{2\pi}\sqrt{\frac{72U_0}{R_0^2\mu}}}$$

3. With $U_0 = 1.6 \times 10^{-21}$ J and $R_0 = 3.8 \times 10^{-10}$ m,

$$k = \frac{72 \times 1.6 \times 10^{-21}}{(3.8 \times 10^{-10})^2} \approx 0.80 \text{ N m}^{-1}.$$

Each mass is $40u$ with $u = 1.66 \times 10^{-27}$ kg, so

$$m_1 = m_2 \approx 6.64 \times 10^{-26} \text{ kg}, \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m}{2} \approx 3.32 \times 10^{-26} \text{ kg}.$$

Thus

$$\omega = \sqrt{\frac{0.80}{3.32 \times 10^{-26}}} \approx 4.9 \times 10^{12} \text{ rad s}^{-1},$$

$$f = \frac{\omega}{2\pi} \approx \frac{4.9 \times 10^{12}}{6.28} \approx 7.8 \times 10^{11} \text{ Hz}.$$

$$\boxed{f \approx 8 \times 10^{11} \text{ Hz}}$$

Answer 24 (4)

Answer 24:

1. $$F(z) = -\frac{dU}{dz} = -\left(az + \frac{1}{2}bz^2\right) = -az - \frac{1}{2}bz^2.$$
$$\boxed{F(z) = -az - \frac{1}{2}bz^2}$$

2. For sufficiently small z, the quadratic term in F (cubic in U) is much smaller than the linear term, so
$$F(z) \simeq -az.$$
With mass m the equation of motion is
$$m\ddot{z} = -az \quad \Rightarrow \quad \ddot{z} + \frac{a}{m}z = 0,$$
which is SHM with
$$\boxed{\omega = \sqrt{\frac{a}{m}}}.$$

3. The magnitude of the cubic correction relative to the linear term is
$$\frac{|F_{\text{cubic}}|}{|F_{\text{linear}}|} = \frac{\frac{1}{2}|b|z^2}{|a||z|} = \frac{|b|}{2|a|}\,|z|.$$
Define a characteristic amplitude z_{max} where the two terms are equal:
$$\frac{|b|}{2|a|}\,z_{\text{max}} = 1 \quad \Rightarrow \quad z_{\text{max}} = \frac{2|a|}{|b|}.$$
At $z = 0.05\,z_{\text{max}}$ this ratio is
$$\frac{|F_{\text{cubic}}|}{|F_{\text{linear}}|} = \frac{|b|}{2|a|}(0.05 z_{\text{max}}) = 0.05.$$
So the cubic term provides a 5% correction to the restoring force.
$$\boxed{\text{Fractional correction at } z = 0.05\,z_{\text{max}} \text{ is } 5\%}$$

Answer 25 (4)

Answer 25:

1. When the liquid is displaced so that one side is higher by y and the other lower by y, the height difference is $2y$. The extra mass on one side is

   $$m_{\text{ex}} = \rho A(2y),$$

   giving a restoring force

   $$F = -m_{\text{ex}} g = -2\rho A y g.$$

   The total mass of fluid is

   $$m = \rho A(2L) = 2\rho A L.$$

   Newtons second law:

   $$m\ddot{y} = -2\rho A y g \Rightarrow 2\rho A L \ddot{y} = -2\rho A y g \Rightarrow \ddot{y} + \frac{g}{L} y = 0.$$

   $$\boxed{\ddot{y} + \frac{g}{L} y = 0}$$

2. Comparing with $\ddot{y} + \omega^2 y = 0$ gives

   $$\omega = \sqrt{\frac{g}{L}}, \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}.$$

   $$\boxed{\omega = \sqrt{\frac{g}{L}}, \quad T = 2\pi\sqrt{\frac{L}{g}}}$$

3. For $L = 0.25$ m,

   $$\omega = \sqrt{\frac{9.81}{0.25}} = \sqrt{39.24} \approx 6.26 \text{ rad s}^{-1},$$

   $$T = \frac{2\pi}{\omega} \approx \frac{6.28}{6.26} \approx 1.0 \text{ s}.$$

   $$\boxed{T \approx 1.0 \text{ s}}$$

Answer 26 (5)

Answer 26:

1. Take the ball as a point mass moving on a circular arc of radius R. Its speed is

$$v = R\dot{\theta},$$

so kinetic energy

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mR^2\dot{\theta}^2.$$

Take potential energy zero at the lowest point. For displacement θ,

$$U = mgR(1 - \cos\theta).$$

Total energy $E = T + U$ is constant:

$$E = \frac{1}{2}mR^2\dot{\theta}^2 + mgR(1 - \cos\theta).$$

Differentiating with respect to time and setting $dE/dt = 0$,

$$mR^2\dot{\theta}\ddot{\theta} + mgR\sin\theta\,\dot{\theta} = 0.$$

For $\dot{\theta} \neq 0$ this gives

$$R^2\ddot{\theta} + gR\sin\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \frac{g}{R}\sin\theta = 0.$$

For small angles, $\sin\theta \simeq \theta$, so

$$\boxed{\ddot{\theta} + \frac{g}{R}\theta = 0}.$$

2. Comparing with SHM, $\ddot{\theta} + \omega^2\theta = 0$, gives

$$\omega = \sqrt{\frac{g}{R}}, \qquad T = 2\pi\sqrt{\frac{R}{g}}.$$

$$\boxed{T = 2\pi\sqrt{\frac{R}{g}}}$$

3. For $R = 0.80$ m,

$$T = 2\pi\sqrt{\frac{0.80}{9.81}} = 2\pi\sqrt{0.0816} \approx 2\pi \times 0.286 \approx 1.8 \text{ s}.$$

$$\boxed{T \approx 1.8 \text{ s}}$$

4. For larger amplitudes the small-angle approximation $\sin\theta \simeq \theta$ fails, and the exact equation

$$\ddot{\theta} + \frac{g}{R}\sin\theta = 0$$

is nonlinear. The oscillations remain periodic, but the period increases with amplitude and the motion is no longer strictly simple harmonic.

$$\boxed{\text{At large amplitudes the motion is nonlinear and the period becomes amplitude dependent}}$$

Answer 27 (5)

Answer 27:

1. $$F(x) = -\frac{dU}{dx} = -(2\alpha x + 4\beta x^3) = -2\alpha x - 4\beta x^3.$$
$$\boxed{F(x) = -2\alpha x - 4\beta x^3}$$

2. For sufficiently small x, the cubic term in F can be neglected, so
$$F \simeq -2\alpha x.$$
With mass m,
$$m\ddot{x} = -2\alpha x \quad \Rightarrow \quad \ddot{x} + \frac{2\alpha}{m}x = 0,$$
which is SHM with
$$\boxed{\omega = \sqrt{\frac{2\alpha}{m}}}.$$

3. We require the quartic contribution to be 10% of the harmonic restoring force:
$$\frac{|F_{\text{quartic}}|}{|F_{\text{quadratic}}|} = \frac{4\beta|x|^3}{2\alpha|x|} = \frac{2\beta x^2}{\alpha} = 0.10.$$
Solving for x^2,
$$x^2 = \frac{0.10\,\alpha}{2\beta} = 0.05\,\frac{\alpha}{\beta},$$
so
$$\boxed{x_{10\%} = \sqrt{0.05\,\frac{\alpha}{\beta}}}.$$

4. Because $\beta > 0$, the potential becomes steeper than purely quadratic at large $|x|$, giving a stronger restoring force than a simple harmonic oscillator. As the amplitude increases, the oscillator spends less time near the turning points and the period decreases slightly with amplitude; equivalently, the oscillation frequency increases with amplitude (a hardening anharmonic oscillator).

$$\boxed{\text{Quartic term makes the oscillator stiffer, so at large amplitudes the period decreases and the frequency increases}}$$