Lecture 9

- **Maxwell Equation in Conducting Medium**

Propagation of Electromagnetic Wave in Conducting Medium

- $\rho = 0$ (no free charge density),
- $\vec{J} = \sigma\vec{E}$ (Ohm's Law in conducting medium),
- $\epsilon = \epsilon_0$, $\mu = \mu_0 = \mu$.

Maxwell's Equations in Differential Form:

1. $\nabla \cdot \vec{E} = 0$
2. $\nabla \cdot \vec{B} = 0$
3. $\nabla \times \vec{E} = -\frac{\partial\vec{B}}{\partial t}$
4. $\nabla \times \vec{B} = \mu\vec{J} + \mu\epsilon\frac{\partial\vec{E}}{\partial t}$

Using $\vec{J} = \sigma\vec{E}$, the fourth equation becomes:

$$\nabla \times \vec{B} = \mu\sigma\vec{E} + \mu\epsilon\frac{\partial\vec{E}}{\partial t}$$

Derivation of EM Wave Equation in a Conducting Medium

We begin from Maxwell's curl equation for the electric field:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Taking the curl of both sides:

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

Using the vector identity:

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$$

Since $\nabla \cdot \vec{E} = 0$, the identity reduces to:

$$-\nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

Substituting Maxwell's 4th equation:

$$\nabla \times \vec{B} = \mu\sigma\vec{E} + \mu\epsilon\frac{\partial\vec{E}}{\partial t}$$

Now:

$$-\nabla^2\vec{E} = -\frac{\partial}{\partial t}(\mu\sigma\vec{E} + \mu\epsilon\frac{\partial\vec{E}}{\partial t})$$

$$\Rightarrow \nabla^2\vec{E} = \mu\sigma\frac{\partial\vec{E}}{\partial t} + \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{(Equation 5)}$$

This is the **wave equation for the electric field in a conducting medium.**

Similarly, using the same method, we derive the **wave equation for the magnetic field**:

$$\nabla^2\vec{B} = \mu\sigma\frac{\partial\vec{B}}{\partial t} + \mu\epsilon\frac{\partial^2\vec{B}}{\partial t^2} \quad \text{(Equation 6)}$$

$$\vec{\nabla}^2\vec{E} = \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} + \mu\sigma\frac{\partial\vec{E}}{\partial t} \quad \text{—— (5)} \quad \& \quad \vec{\nabla}^2\vec{B} = \mu\epsilon\frac{\partial^2\vec{B}}{\partial t^2} + \mu\sigma\frac{\partial\vec{B}}{\partial t} \quad \text{—— (6)}$$

one of the possible solution

$$\vec{E}(\vec{r},t) = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \quad \text{—— (7)}$$

If the EM wave is travelling along the z-direction

$$\vec{E}(z,t) = \vec{E}_0\, e^{i(\vec{k}\cdot z-\omega t)} \quad \text{—— (8)}$$

$$\vec{\nabla}^2 = \cancel{\frac{\partial^2}{\partial x^2}} + \cancel{\frac{\partial^2}{\partial y^2}} + \frac{\partial^2}{\partial z^2} \checkmark$$

Now eqn (5) can be written $\dfrac{\partial^2\vec{E}}{\partial z^2} = \mu\epsilon\dfrac{\partial^2\vec{E}}{\partial t^2} + \mu\sigma\dfrac{\partial\vec{E}}{\partial t}$

$$\vec{E}(z,t) = \vec{E}_0\, e^{i(\vec{k}z-\omega t)}$$

$$\frac{\partial E}{\partial z} = (ik)\, E_0\, e^{i(\vec{k}z-\omega t)}$$

$$\frac{\partial^2 E}{\partial z^2} = (ik)$$

$$\vec{E}(z,t) = E_0 e^{i(\vec{k}\cdot z - \omega t)}$$

$$\frac{\partial E}{\partial t} = (-i\omega) E_0 e^{i(\vec{k}\cdot z - \omega t)}$$

$$\frac{\partial^2 E}{\partial t^2} = (-i\omega)^2 E_0 e^{i(\vec{k}z - \omega t)}$$

Now eqn (9) $\frac{\partial^2 \vec{E}}{\partial z^2} = \mu\epsilon \frac{\partial^2 E}{\partial t^2} + \mu\sigma \frac{\partial E}{\partial t}$

$$(i\vec{k})^2 E_0 e^{i(\vec{k}z - \omega t)} = \mu\epsilon (-i\omega)^2 E_0 e^{i(\vec{k}z - \omega t)} + \mu\sigma (-i\omega) E_0 e^{i(\vec{k}z - \omega t)}$$

$$-\vec{k}^2 = -\mu\epsilon\omega^2 - i\mu\sigma\omega$$

$$\vec{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \qquad (10)$$

$\vec{k}$ is a complex quantity, suppose $\vec{k} = \alpha + i\beta$ (11)

$$\vec{k}^2 = \alpha^2 - \beta^2 + i\, 2\alpha\beta \qquad (12)$$

$$\boxed{\alpha^2-\beta^2 = \mu\epsilon\omega^2} \quad \text{--- (13)}$$

$$2\alpha\beta = \mu\sigma\omega$$

or $$\boxed{\beta = \frac{\mu\sigma\omega}{2\alpha}} \quad \text{--- (14)}$$

Now eqn (13) will be

$$\alpha^2 - \left(\frac{\mu\sigma\omega}{2\alpha}\right)^2 = \mu\epsilon\omega^2$$

$$\alpha^2 - \frac{\mu^2\sigma^2\omega^2}{4\alpha^2} - \mu\epsilon\omega^2 = 0$$

$$\alpha^2 - \mu\epsilon\omega^2\cdot\alpha^2 - \frac{\mu^2\sigma^2\omega^2}{4} = 0 \quad \text{--- (15)}$$

$$ax^2 + bx + c = 0$$

$$\boxed{a=1} \quad \boxed{b = -\mu\epsilon\omega^2} \quad \boxed{c = -\frac{\mu^2\sigma^2\omega^2}{4}}$$

$$\alpha^2 = \frac{+\mu\epsilon\omega^2 \pm \sqrt{\mu^2\epsilon^2\omega^4 + \mu^2\sigma^2\omega^2}}{2}$$

$$\alpha^2 = \frac{\mu\epsilon\omega^2 \pm \mu\epsilon\omega^2\sqrt{1 + \frac{\sigma^2}{\epsilon\omega^2}}}{2}$$

$$\alpha^2 = \frac{\mu\epsilon\omega^2 \pm \mu\epsilon\omega^2\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}}{2} \quad \text{--- (16)}$$

Now for good conductor

$$\frac{\sigma}{\epsilon\omega} >> 1$$

$$\alpha^2 = \frac{\mu\epsilon\omega^2 \pm \mu\epsilon\omega^2\left(\frac{\sigma}{\epsilon\omega}\right)}{2}$$

$$\alpha^2 = \frac{\mu\epsilon\omega^2 \pm \mu\sigma\omega}{2}$$

$$\alpha^2 = \frac{\mu\epsilon\omega^2\left(1 \pm \frac{\sigma}{\epsilon\omega}\right)}{2}$$

$\frac{\sigma}{\epsilon\omega} >> 1$ for good conductor

$$\alpha^2 = \frac{\mu\cancel{\epsilon}\omega^{\cancel{2}}}{2} \cdot \frac{\sigma}{\cancel{\epsilon\omega}}$$

$$\alpha^2 = \frac{\mu\sigma\omega}{2}$$

$$\boxed{\alpha = \sqrt{\frac{\mu\sigma\omega}{2}}} \quad \text{——— (17)}$$

From eqn (14)

$$\beta = \frac{\mu\sigma\omega}{2\alpha}$$

$$\beta = \frac{\cancel{\mu\sigma\omega}^{\sqrt{\mu\sigma\omega}}}{\cancel{2}_{\sqrt{2}}} \sqrt{\frac{2}{\cancel{\mu\sigma\omega}}}$$

$$\boxed{\beta = \sqrt{\frac{\mu\sigma\omega}{2}}} \quad \text{——— (18)}$$

for good conductor

$$\boxed{\alpha = \beta = \sqrt{\frac{\mu\sigma\omega}{2}}}$$

Skin Depth or depth of Penetration :-

$$\vec{E}_{(34)} = \vec{E}_0 \, e^{i(\vec{k}z-\omega t)}$$

$$\vec{k} = \alpha + i\beta$$

$$\vec{E} = E_0 \, e^{i[(\alpha+i\beta)z-\omega t]}$$

$$E = E_0 \, e^{-\beta z} \, e^{i(\alpha z-\omega t)}$$

$$E = \frac{E_0 \, e^{i(\alpha z-\omega t)}}{e^{\beta z}}$$

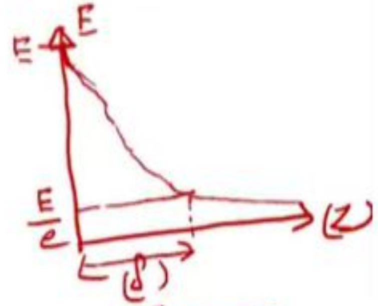
$$\boxed{\vec{E} = \frac{E}{e^{\beta z}}} \quad (19)$$

It shows that EM waves damps exponentially in a conducting medium

If $\boxed{\beta z = 1}$ or $\boxed{z = \frac{1}{\beta} = \delta}$

Eqn (19) $\Rightarrow$ $\boxed{\vec{E} = \frac{E}{e}}$

The distance at which the strength of Electric field reduce by a factor of $\frac{1}{e}$ times of its initial value is called skin depth.



$$\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\omega\mu\sigma}}$$

or $\delta = \sqrt{\frac{\cancel{2}}{\cancel{2}\pi f \cdot \mu\sigma}}$

$$\boxed{\delta = \sqrt{\frac{1}{\pi f \mu\sigma}}}$$

Poor conductor :- From eqn (16)

$$\alpha^2 = \frac{\mu\epsilon\omega^2 \pm \mu\epsilon\omega^2\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}}{2}$$

For poor conductor $\frac{\sigma}{\epsilon\omega} << 1$

$$\alpha^2 = \frac{\mu\epsilon\omega^2 \pm \mu\epsilon\omega^2}{2} = \mu\epsilon\omega^2$$

$$\boxed{\alpha = \omega\sqrt{\mu\epsilon}}$$

we know from eqn (14)

$$\beta = \frac{\mu\sigma\omega}{2\alpha}$$

$$\beta = \frac{\overset{\sqrt{\mu}}{\cancel{\mu}}\sigma\cancel{\omega}}{2}\frac{1}{\cancel{\omega}\sqrt{\cancel{\mu}\epsilon}}$$

$$\boxed{\beta = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}}$$

skin depth of poor conductor

$$\boxed{\delta = \frac{1}{\beta} = \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}}$$

It means that skin depth of poor conductor is independent of frequency.