EC5.203 Communication Theory I (3-1-0-4):

Lecture 20:
Optimal Demodulation-3

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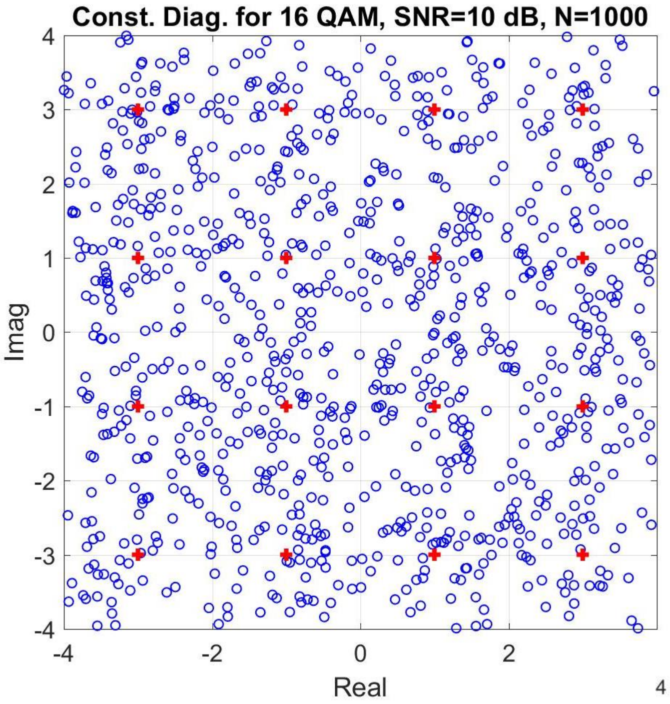
INTERNATIONAL INSTITUTE OF
INFORMATION TECHNOLOGY

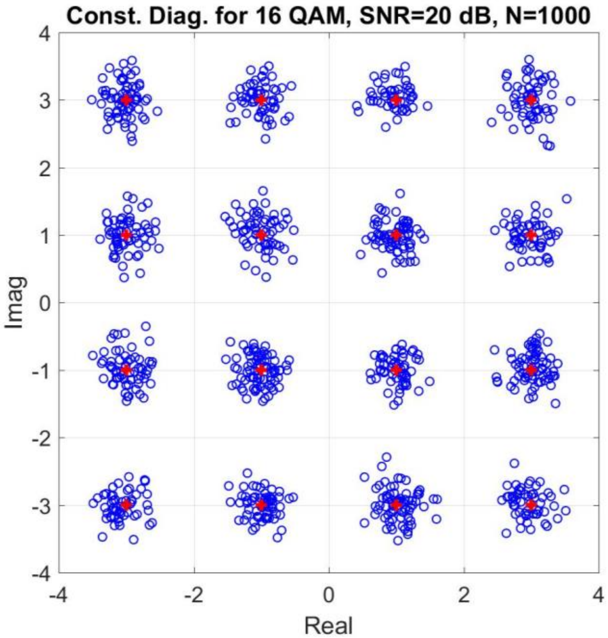
H Y D E R A B A D

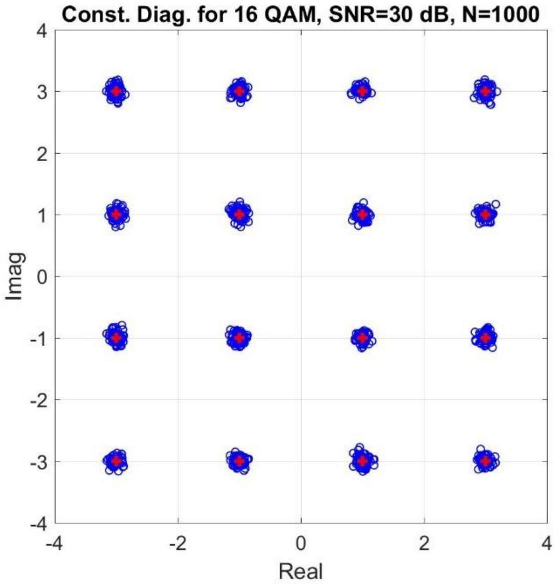
References

- Chap. 6 (Madhow)

Example: 16 QAM in AWGN







Optimal Demodulation

- In 16 QAM, one of 16 passband waveforms corresponding to 16 symbols is sent, where each passband waveform is given by

$$s_i(t) = s_{b_c,c_s} = b_c p(t)\cos(2\pi f_c t) - b_s p(t)\sin(2\pi f_c t)$$

  where b_c, b_s each takes value in $\{\pm 1, \pm 3\}$.

- At the receiver, we are faced with a hypothesis testing problem: we have M possible hypotheses about which signal was sent.

- Based on the observations

$$y(t) = s_i(t) + \boxed{n(t)} \quad \text{AWGN}$$

  we are interested in finding a decision rule to make a best guess which hypothesis was sent.

- For communications applications, performance criteria is to minimize the probability of error (i.e., the probability of making a wrong guess).

Example 5.6.3

- **Binary on-off keying in Gaussian noise**

$$\begin{aligned} Y &= m + n \quad \text{if 1 is sent} \\ Y &= n \quad \text{if 0 is sent} \end{aligned}$$

  Here Y is the received sample, $m > 0$ is some constant and n is AWGN sample with $\mathcal{N}(0, v^2)$.

- At the receiver, the detection strategy is

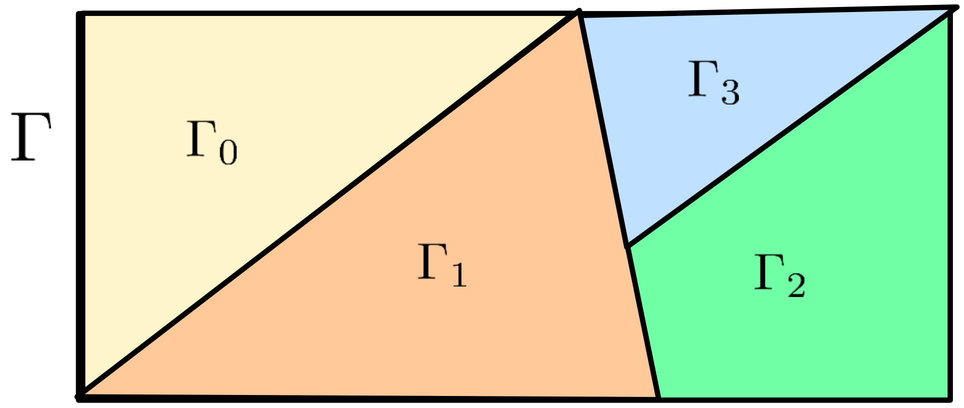
$$\begin{aligned} Y &> m/2 \quad \text{Decide 1 is sent} \\ Y &\leq m/2 \quad \text{Decide 0 is sent} \end{aligned}$$

- Assuming that both 0 and 1 are equally likely,
  - Find the average signal power
  - Find the conditional probability of error conditioned on 0 being sent
  - Find the conditional probability of error conditioned on 1 being sent
  - Find average error probability
  - Find the probability of error for SNR of 13 dB?

Ingredients of Hypothesis Testing Framework

- Hypotheses $H_0, H_1, \ldots, H_{M-1}$
- Observation $Y \in \Gamma$
- Conditional densities $p(y|i)$ for $i = 0, 1, \ldots, M-1$
- Prior probabilities $\pi_i = P(H_i)$ with $\sum_i \pi_i = 1$
- Decision rule $\delta : \Gamma \to \{0, 1, M-1\}$
- Decision region $\Gamma_i : \{y \in \Gamma_i : \delta(y) = i\}$ for $i = 0, 1, M-1$

Example of Decision regions for M=4



Error Probabilities

- Conditional error probabilities, conditioned on H_i, is

$$\begin{aligned}
P_{e|i} &= P(\text{decide } j \text{ for some } j \neq i | H_i \text{ is true}) \\
&= \sum_{j \neq i} P(Y \in \Gamma_j | H_i) \\
&= 1 - P(Y \in \Gamma_i | H_i)
\end{aligned}$$

- Conditional probabilities of correct detection, conditioned on H_i, is

$$\begin{aligned}
P_{c|i} &= P(Y \in \Gamma_i | H_i) \\
&= 1 - P_{e|i}
\end{aligned}$$

- Average error probability

$$P_e = \sum_{i=1}^{M} \pi_i P_{e|i}$$

- Average probability of correct detection

$$P_c = \sum_{i=1}^{M} \pi_i P_{c|i}$$

Ingredients of Hypothesis Testing Framework

- Hypotheses $H_0, H_1, \ldots, H_{M-1}$
- Observation $Y \in \Gamma$
- Conditional densities $p(y|i)$ for $i = 0, 1, \ldots, M-1$
- Prior probabilities $\pi_i = P(H_i)$ with $\sum_i \pi_i = 1$
- Decision rule $\delta : \Gamma \to \{0, 1, M-1\}$
- Decision region $\Gamma_i : \{y \in \Gamma_i : \delta(y) = i\}$ for $i = 0, 1, M-1$

- In earlier example
  - Hypotheses H_0, H_1, Observation $Y \in \Gamma = \mathcal{R}$
  - Conditional densities $p(y|0)$ and $p(y|1)$
  - Prior probabilities π_0 and π_1
  - Decision rule δ:

$$\delta(y) = \begin{cases} 0, & y \leq m/2 \\ 1, & y > m/2 \end{cases}$$

  - Decision regions: $\Gamma_0 = (-\infty, m/2]$ and $\Gamma_1 = (m/2, \infty)$

MAP rule

- Definitions:
  - *A priori* probability: Before the data is observed : $P(H_i) = \pi_i$
  - *A posteriori* probability: After the data is observed: $P(H_i|y)$
- Maximum *a posteriori* probability (MAP) rule:

$$\delta_{\text{MAP}}(y) = \arg\max_i P(H_i|Y = y)$$

where $i = 0, 1, \ldots, M - 1$

- Using Bayes rule $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$, MAP rule can be rewritten as

$$\begin{aligned}
\delta_{\text{MAP}}(y) &= \arg\max_i \frac{P(Y = y|H_i)P(H_i)}{P(Y = y)} \\
&= \arg\max_i \frac{p(y|i)\pi_i}{p(y)} \\
&= \arg\max_i p(y|i)\pi_i \\
&= \arg\max_i \log \pi_i + \log p(y|i)
\end{aligned}$$

Optimality of MAP (or MPE) rule

- Optimality of MAP rule: The MAP rule minimizes the probability of error. Proof!

ML rule

- Definitions:
  - Likelihood function: $p(y|i) = P(Y = y|H_i)$
- Maximum likelihood (ML) rule

$$\delta_{\mathrm{ML}}(y) = \arg\max_i p(y|i)$$

where $i = 0, 1, \ldots, M-1$

- Equivalently,

$$\delta_{\mathrm{ML}}(y) = \arg\max_i \log p(y|i)$$

- ML is equivalent of MAP for equal prior probabilities, i.e., $\pi_i = \frac{1}{M}$, we have

$$\begin{aligned}
\delta_{\mathrm{MAP}}(y) &= \arg\max_i \log \pi_i + \log p(y|i) \\
&= \arg\max_i \log \frac{1}{M} + \log p(y|i) \\
&= \arg\max_i \log p(y|i)
\end{aligned}$$

Binary Hypothesis Testing Problem

- For two hypotheses case, ML decision rule is

$$\begin{aligned}\delta_{\mathrm{ML}}(y) &= \arg\max_i p(y|i) \\ &= \arg\max_i \{p(y|0), p(y|1)\}\end{aligned}$$

- Equivalently,

$$\begin{aligned} p(y|0) > p(y|1) \quad &\rightarrow \quad \delta_{\mathrm{ML}}(y) = 0 \\ p(y|1) > p(y|0) \quad &\rightarrow \quad \delta_{\mathrm{ML}}(y) = 1 \end{aligned}$$

- This can be written as

$$p(y|1) \underset{H_0}{\overset{H_1}{\gtrless}} p(y|0)$$

- Similarly, MAP or MPE rule for binary hypothesis testing problem can be written as

$$\pi_1 p(y|1) \underset{H_0}{\overset{H_1}{\gtrless}} \pi_0 p(y|0)$$

$$P(H_1|Y = y) \underset{H_0}{\overset{H_1}{\gtrless}} P(H_0|Y = y)$$

Likelihood Ratio

- For two hypotheses case, ML decision rule can be written as

$$p(y|1) \underset{H_0}{\overset{H_1}{\gtrless}} p(y|0)$$

- Equivalently,

$$\frac{p(y|1)}{p(y|0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

  This ratio of likelihood functions is called likelihood ratio(LR) and denoted by $L(y)$ and the test is called as likelihood ratio test (LRT)

- Taking log of both sides

$$\log \frac{p(y|1)}{p(y|0)} \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

  The test statistic in this case is Log of LR and is called log-likelihood ratio (LLR) while the test is called as LLRT

Likelihood Ratio: MAP

- For two hypotheses case, MAP decision rule is

$$\pi_1 p(y|1) \underset{H_0}{\overset{H_1}{\gtrless}} \pi_0 p(y|0)$$

- In terms of LR, the LRT is

$$L(y) = \frac{p(y|1)}{p(y|0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

- Taking log of both sides, the test is given in terms of LLR

$$\log L(y) \underset{H_0}{\overset{H_1}{\gtrless}} \log \frac{\pi_0}{\pi_1}$$

Today’s Class:
Signal Space Concepts

Receiver design as hypothesis testing

- Consider the multiple hypothesis testing

$$H_i \; : \; y(t) = s_i(t) + n(t) \quad i = 0, 1, \ldots, M-1$$

  where $n(t)$ is WGN with $S_n(f) = \frac{N_0}{2} = \sigma^2$

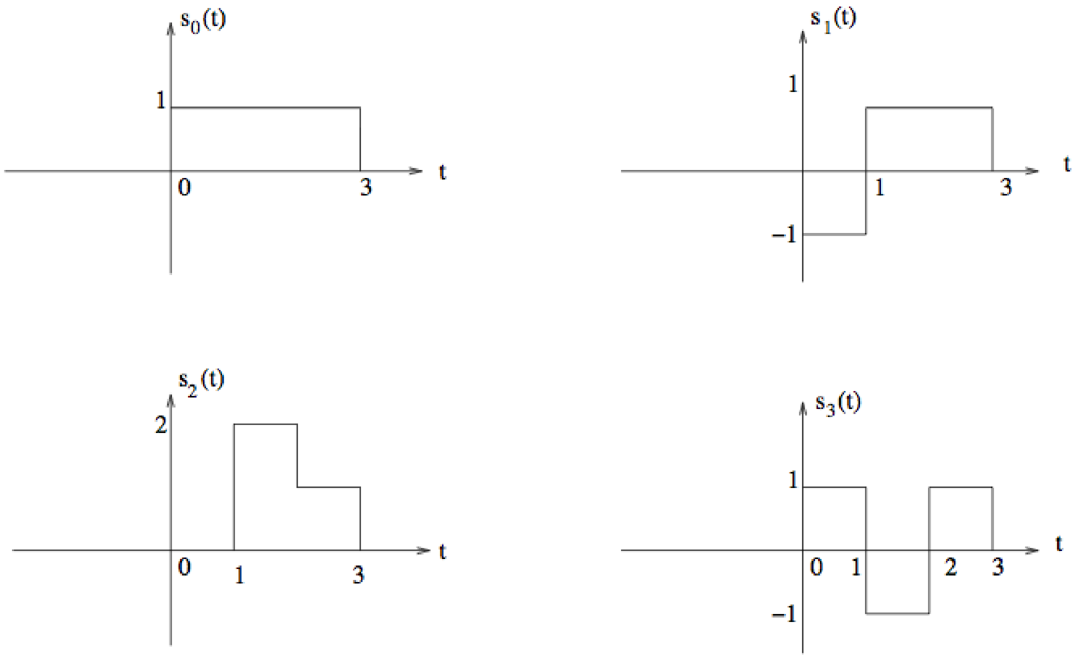
- Strategy:

  - Show that we can reduce the continuous time received signal to a finite-dimensional vector without losing information
  - Derive the optimal receiver based on the finite-dimensional vector observation
  - Map the optimal receiver back to continuous time

- This approach is based on signal space concepts: even though the received signal lives in an infinite-dimensional space, we can restrict attention to the subspace spanned by the signals that could have been transmitted

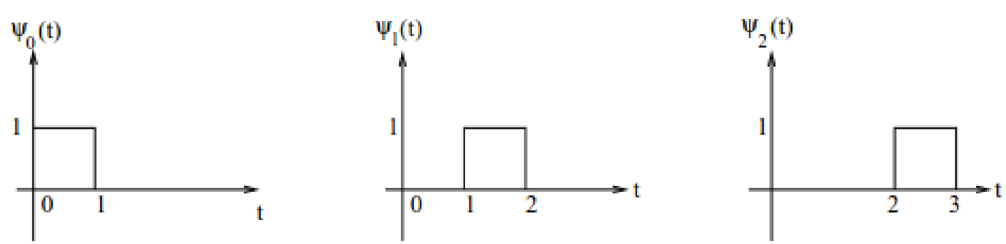
Signal space concepts: an example

- Four possible transmitted signals living in a 3-dimensional space

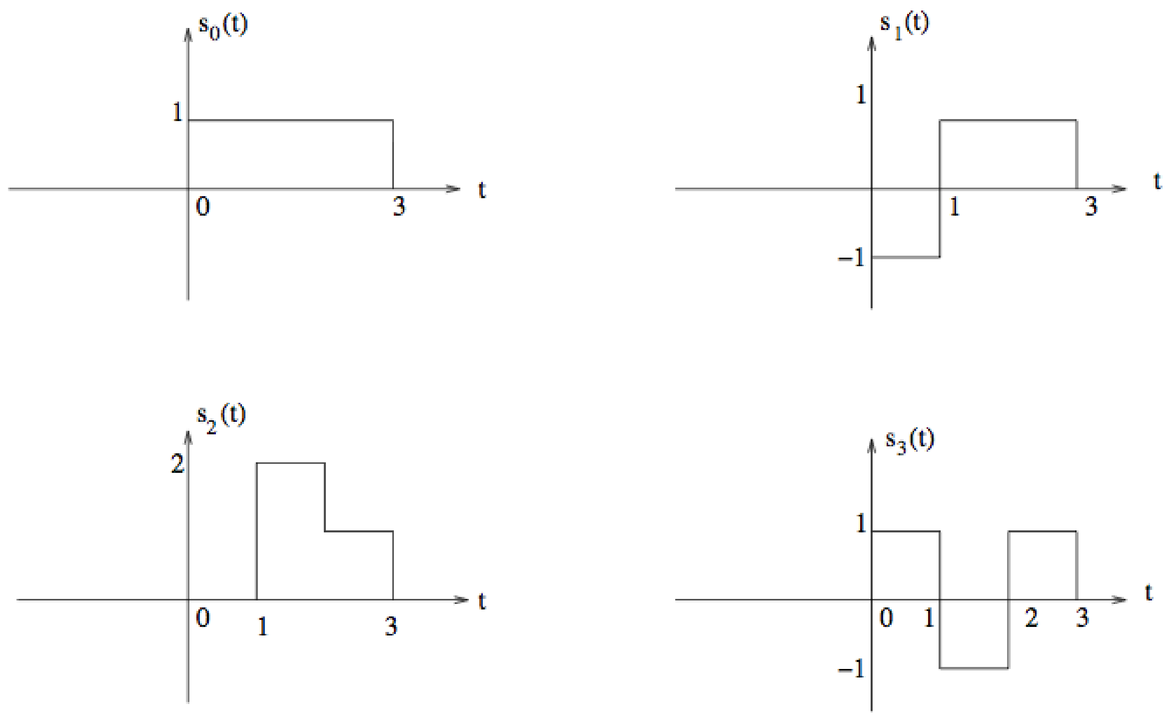


- Orthonormal basis (by inspection)

$$\psi_0(t) = I_{[0,1]}(t),\ \psi_1(t) = I_{[1,2]}(t),\ \psi_2(t) = I_{[2,3]}(t),$$



Example (continued)



- Expand with respect to basis $s_i(t) = \sum_{k=0}^{2} s_i[k]\psi_k(t)$
- Vectors of basis coefficient

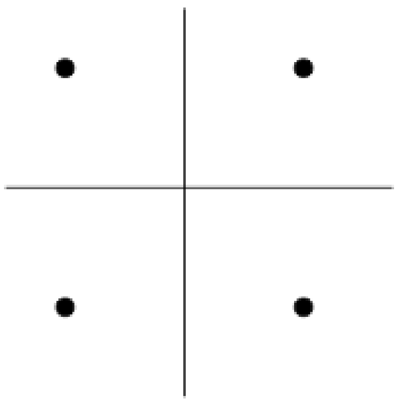
$$\mathbf{s}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad \mathbf{s}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}; \quad \mathbf{s}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}; \quad \mathbf{s}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix};$$

Another example: 2D modulation

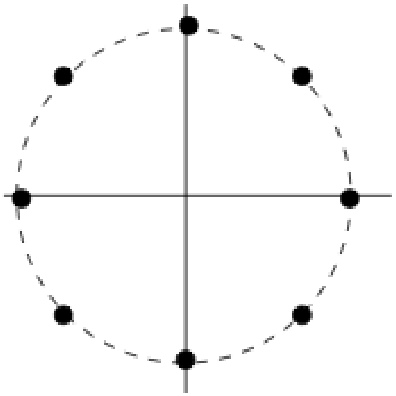
- 2D constellation for passband signaling: signal transmitted over a single symbol interval

$$s_i(t) = s_{b_c,b_s}(t) = b_c p(t)\cos 2\pi f_c t - b_s p(t)\sin 2\pi f_c t$$

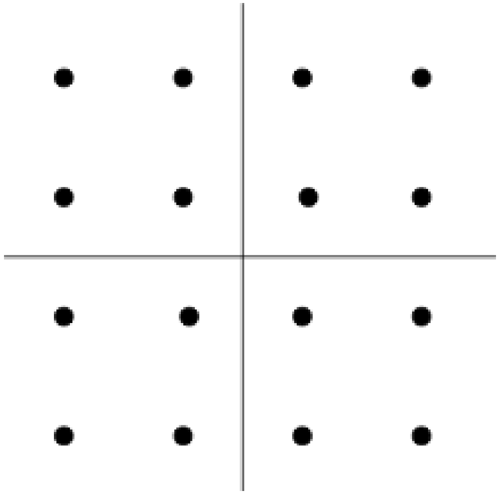
QPSK/4PSK/4QAM



8PSK



16QAM



- Signal is spanned by the following two signals

$$\phi_c(t) = p(t)\cos 2\pi f_c t; \quad \phi_s(t) = -p(t)\sin 2\pi f_c t$$

Another example: 2D modulation…

- Signal is spanned by the following two orthogonal signals

$$\phi_c(t) = p(t) \cos 2\pi f_c t; \quad \phi_s(t) = -p(t) \sin 2\pi f_c t$$

- Noting that $||\phi_c||^2 = ||\phi_s||^2 = \frac{1}{2}||p||^2$, corresponding orthonormal basis functions are

$$\psi_c(t) = \frac{\phi_c(t)}{||\phi_c||}; \quad \psi_s(t) = \frac{\phi_s(t)}{||\phi_s||};$$

- Expansion with respect to orthonormal basis is

$$s_{b_c,b_s}(t) = \frac{1}{\sqrt{2}}||p||b_c\psi_c(t) + \frac{1}{\sqrt{2}}||p||b_s\psi_s(t)$$

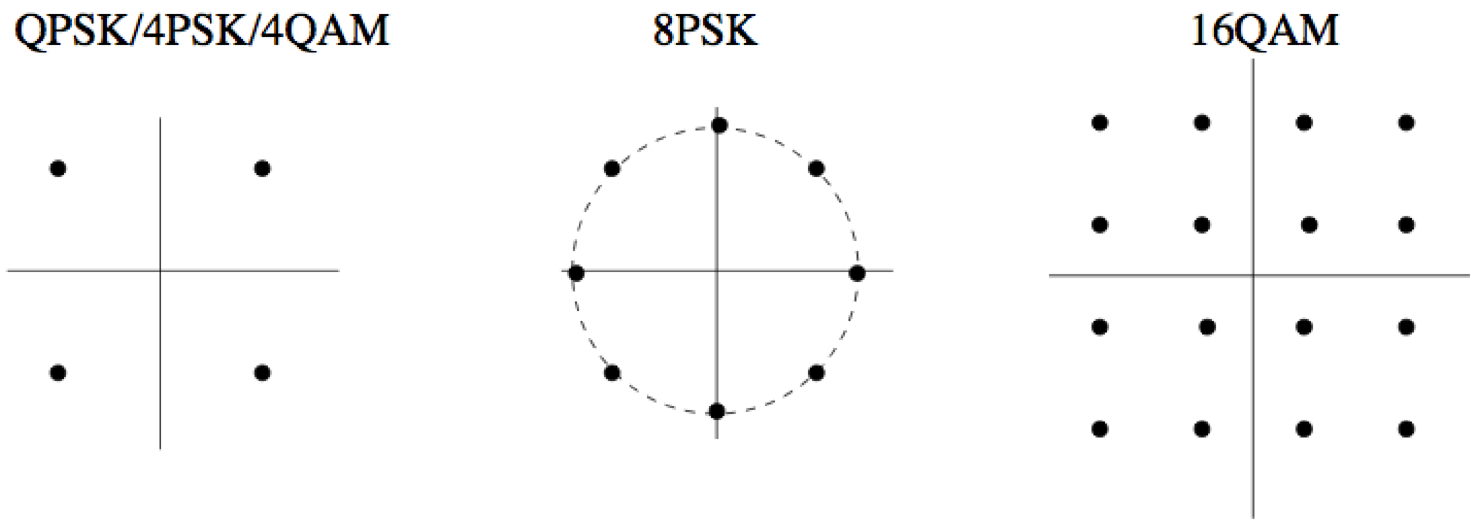
Basis expansion ⇔ constellation

- 2D constellation for passband signaling: signal transmitted over a single symbol interval

$$s_i(t) = s_{b_c,b_s}(t) = b_c p(t) \cos 2\pi f_c t - b_s p(t) \sin 2\pi f_c t$$

- Vector basis coefficients are given by

$$\mathbf{s}_i = \frac{1}{\sqrt{2}} ||p|| \begin{pmatrix} b_c \\ b_s \end{pmatrix}$$



From Signals to Vectors

- Signal space consists of all possible linear combinations of $s_0(t)$, $s_1(t)$, ..., $s_{M-1}(t)$.
- We can always find an orthonormal basis for the signal space, i.e., $\psi_0(t)$, $\psi_1(t)$, ..., $\psi_{n-1}(t)$ where $n \leq M$
- We can express each signal as a vector of basic coefficients

$$s_i(t) = \sum_{k=0}^{n-1} s_i[k]\psi_k(t)$$

  where $s_i[k] =< s_i, \psi_k >$
- Using $s_i[k]$, we can now express $s_i(t)$ as a $n \times 1$ vector $\mathbf{s}_i = [s_i[0] \; s_i[1] \; \ldots s_i[n-1]]^T$
- Note that finite-dimensional basis always exists
- If it is not possible to find natural basis by inspection, then we can always use Gram-Schmidt orthoginalization procedure

Gram-Schmit Orthogonalization Process

Orthogonal Vectors | **Orthonormal Vectors**

$$\mathbf{u}_1 = \mathbf{v}_1, \qquad \mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2), \qquad \mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|}$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_3), \qquad \mathbf{e}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|}$$

$$\mathbf{u}_4 = \mathbf{v}_4 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_3}(\mathbf{v}_4), \qquad \mathbf{e}_4 = \frac{\mathbf{u}_4}{\|\mathbf{u}_4\|}$$

$$\vdots \qquad \vdots$$

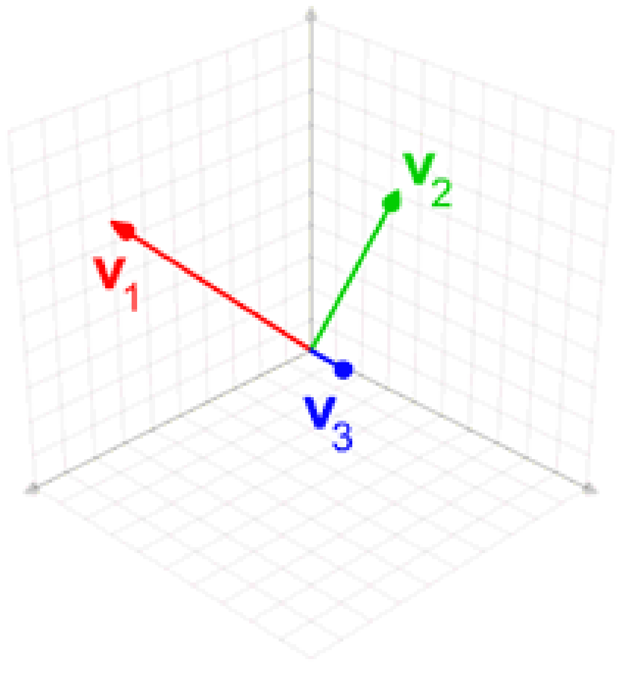
$$\mathbf{u}_k = \mathbf{v}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{u}_j}(\mathbf{v}_k), \qquad \mathbf{e}_k = \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}.$$

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u},$$

This operator projects the vector **v** orthogonally onto the line spanned by vector **u**.

Source: Wikipedia http://en.wikipedia.org/wiki/Gram%E2%80%93Schmidt_process

Gram-Schmit Orthogonalization Process



$$\mathrm{proj}_{\mathbf{u}}\,(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u},$$

Source: Wikipedia http://en.wikipedia.org/wiki/Gram%E2%80%93Schmidt_process

Gram-Schmidt Orthogonalization Process

- **Step 0 (Initialization)**: Let $\phi_0 = s_0$. If $\phi_0 \neq 0$, then set $\psi_0 = \frac{\phi_0}{||\phi_0||}$. Note that ψ_0 provides a basis function for $\mathcal{S}_0$.

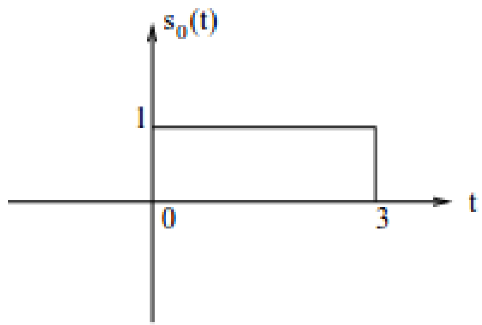
- **Step k**: Suppose that we have constructed an orthornormal basis $\mathcal{B}_{k-1} = \{\psi_0, \ldots, \psi_{m-1}\}$ for subspace $\mathcal{S}_{k-1}$ spanned by the first k signals $s_0, \ldots, s_{k-1}$. Note that $m \leq k$. Define

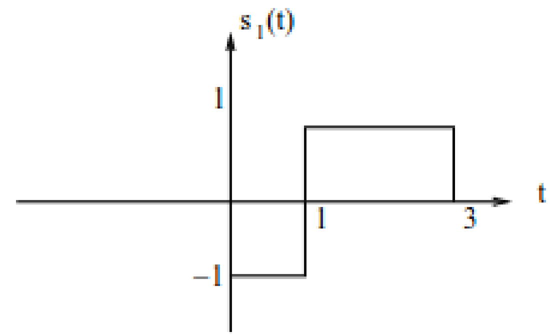
$$\phi_k(t) = s_k(t) - \sum_{i=0}^{m-1} < s_k, \psi_i > \psi_i(t)$$

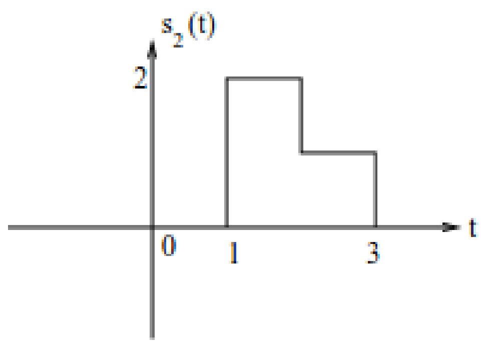
- The signal $\phi_k(t)$ is the component of $s_k(t)$ orthogonal to the subspace $\mathcal{S}_{k-1}$. If $\phi_k \neq 0$, define a new basis function $\psi_m = \frac{\phi_k(t)}{||\phi_k||}$ and the basis as $\mathcal{B}_k = \{\psi_0, \ldots, \psi_m\}$. If $\phi_k = 0$, then $s_k \in \mathcal{S}_{k-1}$ and it is not necessary to update the basis. In this case $\mathcal{B}_k = \mathcal{B}_{k-1} = \{\psi_0, \ldots, \psi_{m-1}\}$.

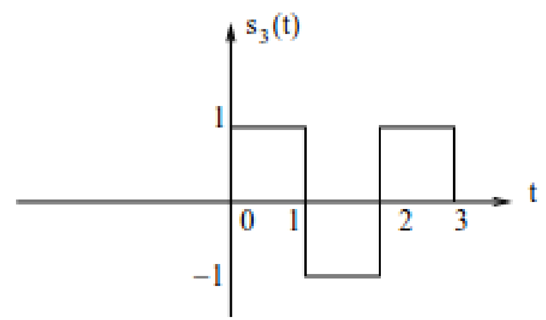
Example: Gram-Schmidt Orthogonalization

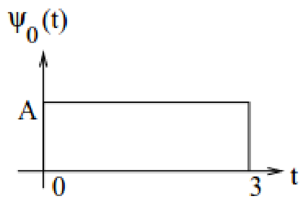
- Find orthonormal basis set for these signals.

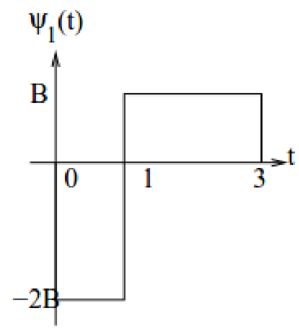


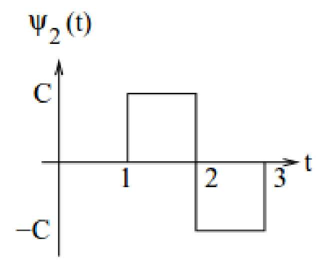












From signals to vectors

- Signal space consists of all possible linear combinations of $s_0(t), s_1(t), \ldots, s_{M-1}(t)$.
- We can always find an orthonormal basis for the signal space, i.e., $\psi_0(t)$, $\psi_1(t), \ldots, \psi_{n-1}(t)$ where $n \leq M$

We can express each signal as a vector of basis coefficients

$$s_i(t) = \sum_{l=0}^{n-1} s_i[k]\psi_k(t), \text{ where } s_i[k] = \langle s_i, \psi_k \rangle \quad \leftrightarrow \mathbf{s}_i = \begin{pmatrix} s_i[0] \\ s_i[1] \\ . \\ . \\ . \\ s_i[n-1] \end{pmatrix}$$

Inner products are preserved

- Performance of M-ary signaling in AWGN depends only on the inner products between the signals if the noise PSD is fixed.
- When mapping the CT hypothesis testing problem to DT, it is important to check that these inner products are preserved when projecting them to signal space.
- Check the following

$$\langle s_i, s_j \rangle = \int s_i(t) s_j(t) dt = \sum_{k=0}^{n-1} s_i[k] s_j[k] = \langle \mathbf{s}_i, \mathbf{s}_j \rangle$$

Inner products are preserved from CT to DT and vice versa
➔ norms, energies, distances are preserved

Modeling WGN in signal space

- What about projection of noise on the signal subspace?
- The noise projection onto the i^{th} basis function as

$$N[i] = \langle \boxed{n}, \psi_i \rangle = \int n(t)\psi_i(t)$$

Noise

for $i = 0, 1, \ldots, \boxed{n} - 1$

Dimension of signal space

- We can write noise $n(t)$ as follows

$$n(t) = \boxed{\sum_{i=0}^{n-1} N[i]\psi_i(t)} + \boxed{n^{\perp}(t)}$$

where $n^{\perp}(t)$ is the projection of noise orthogonal noise subspace.