Lecture 6

EM Wave Continue….

Wave Equation in a Dielectric Medium

- A dielectric medium is an insulating material with no free charges and no free currents. Examples include glass, plastic, and air.

Maxwell's Equations in a Dielectric Medium

$$\nabla \cdot \mathbf{E} = \frac{\rho_f}{\varepsilon} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

Taking the curl of **Faraday's Law**:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$$

Using **Ampère's Law**:

$$\nabla \times \mathbf{B} = \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

Substituting:

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Using the vector identity:

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

Substituting:

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Using the vector identity:

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

Since $\nabla \cdot \mathbf{E} = 0$ in a dielectric:

$$\nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

A similar equation holds for **magnetic field**:

$$\nabla^2 \mathbf{B} = \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

From the wave equation, the wave speed is:

$$v = \frac{1}{\sqrt{\mu\varepsilon}}$$

The total energy density of an electromagnetic (EM) wave

Electromagnetic Waves carry energy as they travel through space. Thus Energy is contained in oscillating Electric and Magnetic Field.

Equal amount of Energy is contributed by Electric and Magnetic Field.

$$\mu = \mu_E + \mu_B$$

Where μE and μB are the energy densities due to Electric and magnetic fields

Energy density of Electric Field (μ_E)

1. The energy stored in a capacitor:

$$U = \frac{1}{2}CV^2$$

Since:

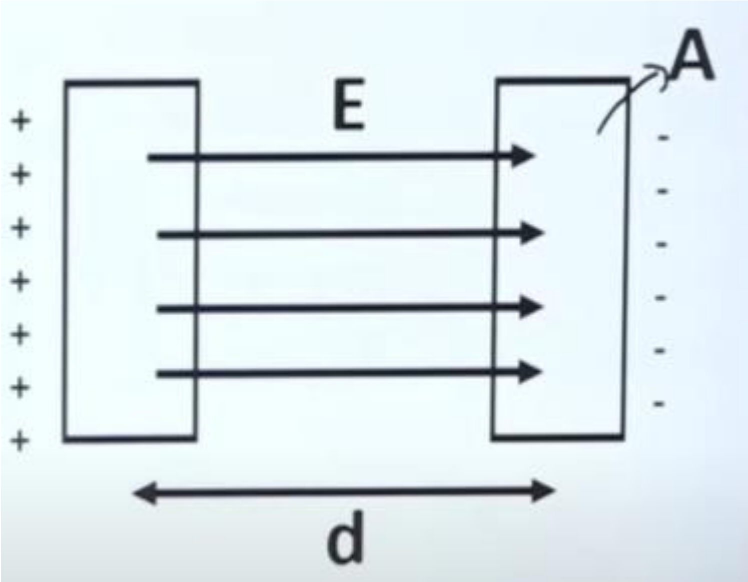
$$C = \epsilon_0 \frac{A}{d}$$

Substituting, we get:

$$U = \frac{1}{2}\frac{\epsilon_0 A}{d}(E\,d)^2$$

2. Simplifying the expression:

$$U = \frac{1}{2}\epsilon_0 \frac{A}{d} E^2 d^2$$



E.d= Ed cos 0=Ed

$$U = \frac{1}{2}\epsilon_0 E^2\, V$$

$$\frac{U}{V} = \frac{1}{2}\epsilon_0 E^2$$

$$\mu_E = \frac{1}{2}\epsilon_0 E^2$$

This is the Energy stored in Electric field

But for EM wave, we know

$$E = E_0 \sin(kx - \omega t)$$

$E \rightarrow$ variable, $\mu_E \rightarrow$ Avg.

$$\mu_E = \frac{1}{2}\epsilon_0 E^2$$

$$\mu_E = \frac{1}{2}\epsilon_0 E_0^2 \sin^2(kx - \omega t)$$

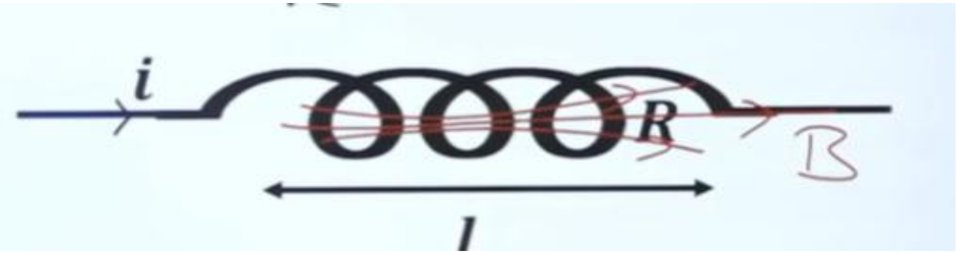
$$\overline{\mu_E} = \frac{1}{2}\epsilon_0 E_0^2 \sin^2(kx - \omega t)$$

$$(\mu_E)_{avg} = \frac{1}{2}\epsilon_0 E_0^2 \times \frac{1}{2} = \frac{1}{4}\epsilon_0 E_0^2$$

$$(\mu_E)_{avg} = \frac{1}{4}\epsilon_0 E_0^2$$

This value is Energy per unit volume due to Electric field.

Energy density of Magnetic Field (μ_B)



The formula shown in the image is as follows:

$$U = \frac{1}{2} L i^2$$

Expanding the inductance L:

$$L = \mu_0 n^2 \pi R^2 \ell$$

$$U = \frac{1}{2} \left(\mu_0 n^2 \pi R^2 \ell\right) \left(\frac{B}{\mu_0 n}\right)^2$$

Here:

- L = inductance
- μ_0 = permeability of free space
- n = number of turns per unit length
- R = radius of the coil
- ℓ = length of the solenoid

Simplifying further:

$$U = \frac{1}{2} \mu_0 n^2 \pi R^2 \ell \frac{B^2}{\mu_0^2 n^2}$$

1. Energy density formula:

$$U = \frac{1}{2}\frac{B^2}{\mu_0}(\pi R^2 \ell)$$

2. General energy density expression:

$$U = \frac{1}{2}\frac{B^2}{\mu_0}V$$

3. Energy density per unit volume:

$$\frac{U}{V} = \frac{1}{2}\frac{B^2}{\mu_0}$$

$$B = \mu_0 n i$$

Here:

- B = Magnetic field inside a solenoid
- μ_0 = Permeability of free space
- n = Number of turns per unit length (turns per meter)
- i = Current flowing through the solenoid

3. Energy density per unit volume:

$$\frac{U}{V} = \frac{1}{2} \frac{B^2}{\mu_0}$$

4. Magnetic energy density (μ_B):

$$\mu_B = \frac{1}{2} \frac{B^2}{\mu_0}$$

Here:

- U = Magnetic energy
- V = Volume
- μ_0 = Permeability of free space
- B = Magnetic flux density (magnetic field)
- μ_B = Magnetic energy density

1. Magnetic field in a wave:

$$B = B_0 \sin(kx - \omega t)$$

Here:

- B = Magnetic field
- B_0 = Amplitude of the magnetic field
- k = Wave number
- x = Position
- ω = Angular frequency
- t = Time

2. Magnetic energy density (μ_B):

$$\mu_B = \frac{1}{2\mu_0} B_0^2 \sin^2(kx - \omega t)$$

3. Average magnetic energy density (μ_B_{avg}):

$$(\mu_B)_{avg} = \frac{1}{2\mu_0} \langle B_0^2 \sin^2(kx - \omega t) \rangle$$

Since the average value of $\sin^2$ over a period is $\frac{1}{2}$, the final average magnetic energy density becomes:

$$(\mu_B)_{avg} = \frac{1}{2\mu_0} \frac{B_0^2}{2} = \frac{1}{4\mu_0} B_0^2$$

To Show:

$$(\mu_B)_{avg} = (\mu_E)_{avg}$$

Electric Field Energy Density (Average):

$$(\mu_E)_{avg} = \frac{1}{4}\epsilon_0 E_0^2$$

Substituting the relation:

$$E_0 = C \cdot B_0$$

$$(\mu_E)_{avg} = \frac{1}{4}\epsilon_0 (B_0 \cdot C)^2$$

$$= \frac{1}{4}\epsilon_0 B_0^2 C^2 \longrightarrow \text{(a)}$$

We know

$$C = \frac{E_0}{B_0} = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Put this value in equation (a)

We will get

$$= \frac{1}{4} \frac{B_0^2}{\mu_0}$$

Which is equal to $(\mu_B)_{avg}$

Thus,

$$(\mu_B)_{avg} = (\mu_E)_{avg}$$

Total Energy Density in Electromagnetic Wave:

The total energy density (μ) in an electromagnetic wave is the sum of the electric and magnetic energy densities:

$$\mu = \mu_E + \mu_B$$

Since we have shown earlier that:

$$\mu_E = \mu_B$$

Therefore:

$$\mu = 2\mu_E$$

Substituting the value of electric energy density:

$$\mu = 2 \times \frac{1}{4}\epsilon_0 E_0^2$$

Substituting the value of electric energy density:

$$\mu = 2 \times \frac{1}{4} \epsilon_0 E_0^2$$

$$\mu = \frac{1}{2} \epsilon_0 E_0^2$$

Alternatively:

$$\mu = 2\mu_B$$

Substituting the value of magnetic energy density:

$$\mu = 2 \times \frac{1}{4\mu_0} B_0^2$$

$$\mu = \frac{1}{2\mu_0} B_0^2$$

Applications of Energy Density in EM Waves

- **Wireless Power Transfer**: Tracks EM energy stored in space.
- **Optical Fiber Communications**: Determines light wave energy in a fiber.
- **Solar Panels**: Energy density of sunlight is used for power conversion.
- **Microwave Heating**: Determines how much EM energy is absorbed by food.

Poynting Vector – Electromagnetic Power Flow

The **Poynting vector** represents the **power flow per unit area** in an electromagnetic (EM) wave. It tells us **how energy moves** in space due to electric ($\mathbf{E}$) and magnetic ($\mathbf{H}$) fields.

The instantaneous **Poynting vector** is given by:

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

where:

- $\mathbf{E}$ is the **electric field** (V/m),
- $\mathbf{H}$ is the **magnetic field** (A/m),
- $\mathbf{S}$ has units of **watts per square meter (W/m^2)**.

🔹 **Interpretation**: The direction of $\mathbf{S}$ gives the **direction of energy flow**, and its magnitude represents the **power density**.

Poynting Vector in Free Space (Plane Wave Case)

For a **uniform plane wave** traveling in the $+z$-direction:

$$\mathbf{E} = E_0 e^{i(kz-\omega t)}\hat{x}, \quad \mathbf{H} = H_0 e^{i(kz-\omega t)}\hat{y}$$

Using $\mathbf{S} = \mathbf{E} \times \mathbf{H}$:

$$\mathbf{S} = (E_0 e^{i(kz-\omega t)}\hat{x}) \times (H_0 e^{i(kz-\omega t)}\hat{y})$$

Since $\hat{x} \times \hat{y} = \hat{z}$, we get:

$$\mathbf{S} = (E_0 H_0) e^{2i(kz-\omega t)}\hat{z}$$

Taking the **time average**:

$$\langle \mathbf{S} \rangle = \frac{1}{2} E_0 H_0 \hat{z}$$

Since **wave impedance** is $Z = E/H$, we can express this as:

$$\langle \mathbf{S} \rangle = \frac{E_0^2}{2Z} \hat{z}$$

- **Key takeaway**: The EM wave transports energy **in the direction of wave propagation**, and the magnitude depends on E_0^2 and the impedance Z.

Applications of the Poynting Vector

- **Antenna Power Radiation**: Measures how power is radiated from an antenna.
- **Optical Fiber Communication**: Tracks energy flow in light waves.
- **Waveguides and Transmission Lines**: Determines power flow in confined structures.
- **Solar Panels**: Calculates power received from sunlight.

Electromagnetic Wave Impedance (Z)

Electromagnetic (EM) wave impedance is a fundamental parameter that describes the relationship between the **electric field** (E) and the **magnetic field** (H) in a medium. It plays a crucial role in **wave propagation, reflection, and transmission**.

The intrinsic **wave impedance** of a medium is given by:

$$Z = \frac{E}{H}$$

where:

- E is the electric field intensity (V/m),
- H is the magnetic field intensity (A/m).

It determines how an EM wave propagates in a given medium.

Wave Impedance in Free Space

For free space (μ_0, ε_0):

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

Using the values:

- $\mu_0 = 4\pi \times 10^{-7}$ H/m,
- $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m,

we get:

$$Z_0 \approx 377\,\Omega$$

This is the **characteristic impedance of free space**, an important constant in EM theory.

Wave Impedance in a Dielectric Medium

For a **lossless dielectric** (ϵ_r, μ_r), the impedance is:

$$Z = \sqrt{\frac{\mu_0 \mu_r}{\epsilon_0 \epsilon_r}}$$

or simply:

$$Z = \frac{Z_0}{\sqrt{\epsilon_r}}$$

- **Example:**

- In **air** ($\epsilon_r \approx 1$), $Z \approx 377\,\Omega$.
- In **water** ($\epsilon_r \approx 80$), $Z \approx 42\,\Omega$ → **wave attenuates quickly.**

Applications of Wave Impedance

- **Antenna Design**: Matching impedance ensures **efficient energy transfer**.
- **RF and Microwave Circuits**: Transmission lines have **characteristic impedance** (e.g., 50Ω for coaxial cables).
- **Medical Imaging (MRI, Ultrasound)**: Different tissues have different impedances, affecting wave propagation.
- **Radar and Stealth Technology**: **Low-impedance coatings** reduce reflections.