EC5.203 Communication Theory (3-1-0-4):

Lecture 22:
Optimal Demodulation-5

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References

- Chap. 6 (Madhow)

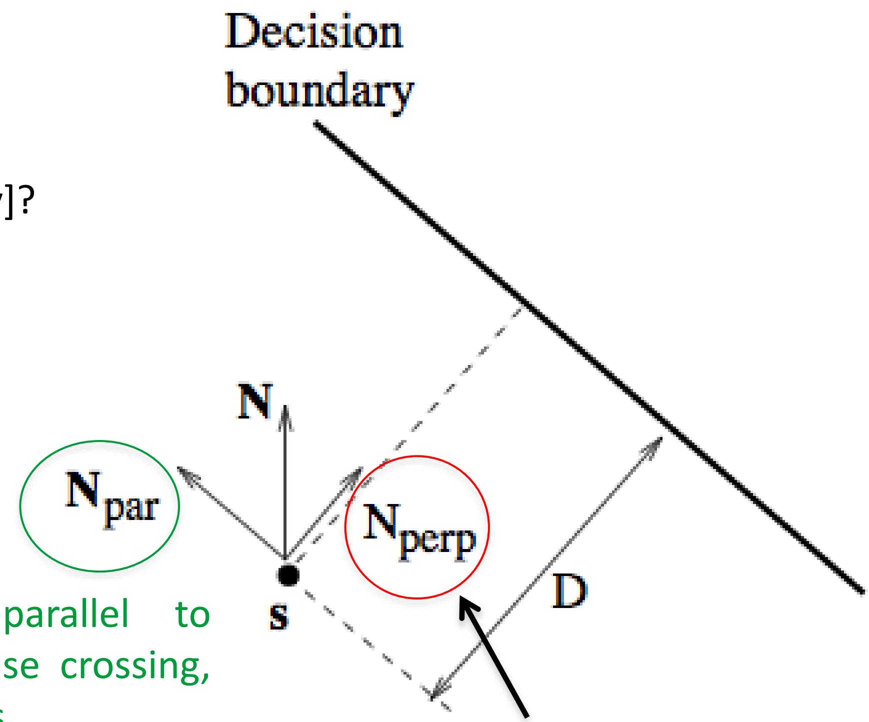
Performance Analysis of ML Reception

Performance Analysis of ML Reception

- Performance analysis of ML under the assumption of equiprobable priors.
- Although performance analysis of MPE or MAP is skipped, it is simple extension of ML case.

The Geometry of Errors

Basic building block:
Send signal **s.**
Noise **N** gets added.
P[crossing a given boundary]?



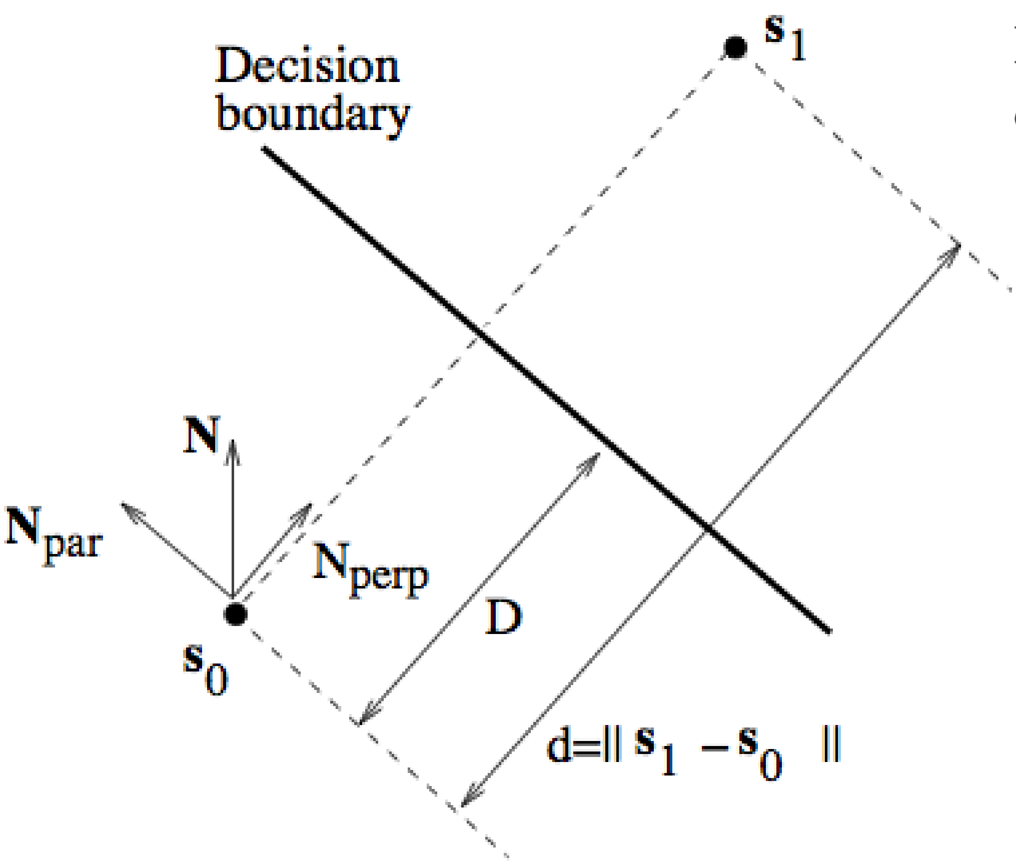
Noise component parallel to boundary cannot cause crossing, no matter how big it is.
In *n*-dimensional space, this component is *(n-1)*-dimensional

Boundary crossing determined by length and sign of noise component perpendicular to boundary. 1-dimensional, regardless of the signal space dimension

$$N_{perp} \sim N(0, \sigma^2)$$

$$\boxed{P[\text{cross a boundary at distance } D] = P[N_{perp} > D] = Q\left(\frac{D}{\sigma}\right)}$$

Geometry for binary signaling



Distance between signals
$d = ||\mathbf{s}_1 - \mathbf{s}_0||$

Distance from boundary
(starting from either signal)
$D = d/2 = ||\mathbf{s}_1 - \mathbf{s}_0||/2$

$$P[\text{cross ML boundary between } \mathbf{s}_0 \text{ and } \mathbf{s}_1] = Q\left(\frac{||\mathbf{s}_1 - \mathbf{s}_0||}{2\sigma}\right) = \boxed{Q\left(\frac{||s_1 - s_0||}{2\sigma}\right)}$$

(starting from either signal)

$$P_{e|0} = P_{e|1} = P_e$$

Important Note

- As the equivalence between CT signal s_i and vector $\mathbf{s}_i$ has been already established, we drop the boldface notation, using y, s_i, and n to denote the received signal, the transmitted signal, and the noise, respectively, in both the settings

Performance with binary signaling: *Algebraic*

- For the binary hypothesis test

$$H_0 : y(t) = n(t)$$
$$H_1 : y(t) = s(t) + n(t)$$

show that

$$P_e = P_{e|1} = P_{e|0} = Q\left(\frac{||s||}{2\sigma}\right) = Q\left(\frac{d}{2\sigma}\right)$$

- Moreover for following binary hypothesis test

$$H_0 : y(t) = s_0(t) + n(t)$$
$$H_1 : y(t) = s_1(t) + n(t)$$

show that

$$P_e = P_{e|1} = P_{e|0} = Q\left(\frac{||s_1 - s_0||}{2\sigma}\right) = Q\left(\frac{d}{2\sigma}\right)$$

Importance of Scale Invariance

- The error probability is given by

$$P_e = P_{e|1} = P_{e|0} = Q\left(\frac{||s_1 - s_0||}{2\sigma}\right) = Q\left(\frac{d}{2\sigma}\right)$$

- Same scaling in the signal and noise does not change the performance
- The performance depends on ratio rather than individually on the signal and noise strengths

Some Standard Measures

- Energy Bit E_b: For binary signaling it is given by

$$E_b = \frac{1}{2}(||s_0||^2 + ||s_1||^2)$$

  assuming that 0 and 1 are equally likely

- Scale invariant parameter

$$\eta_p = \frac{d^2}{E_b}$$

  Note that this does not change if we scale both s_1 and s_0 by A

- In terms of energy per bit and scale-invariant parameter, the probability of error for binary signaling is given by

$$\begin{aligned} P_e &= Q\left(\frac{d}{2\sigma}\right) \\ &= Q\left(\sqrt{\frac{\eta_p E_b}{2N_0}}\right) \end{aligned}$$

  where $\sigma^2 = N_0/2$

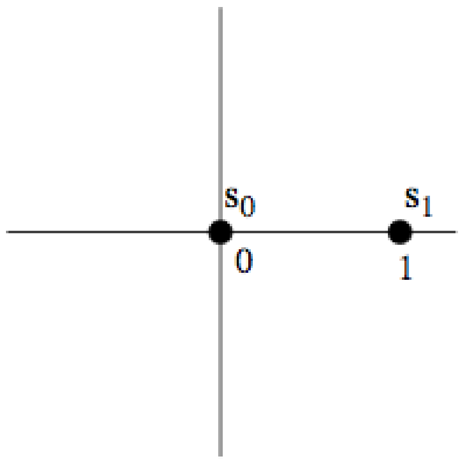
ML Binary Signaling Performance..

- In terms of energy per bit and scale-invariant parameter, the probability of error for binary signaling is given by

$$P_e = Q\left(\sqrt{\frac{\eta_p E_b}{2N_0}}\right)$$

- Important observations
  - Performance depends on the signal-to-noise ratio
  - For a fixed E_b/N_0, the performance is better for higher for higher value of η_p. The parameter η_p is also called as power efficiency of the constellation

Performance for different binary schemes



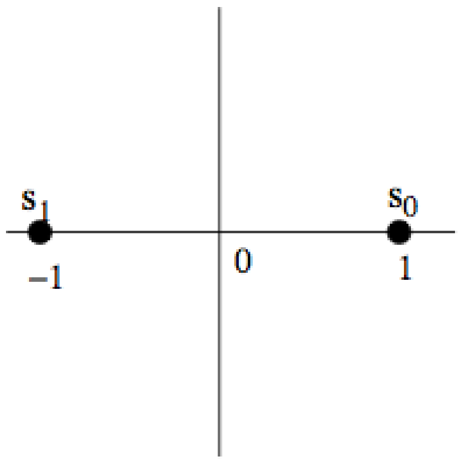
On–off keying

$$d = 1$$

$$E_b = (0^2 + 1^2)/2 = 1/2$$

$$\eta_P = d^2 / E_b = 2$$

$$P_{e,ML} = Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$



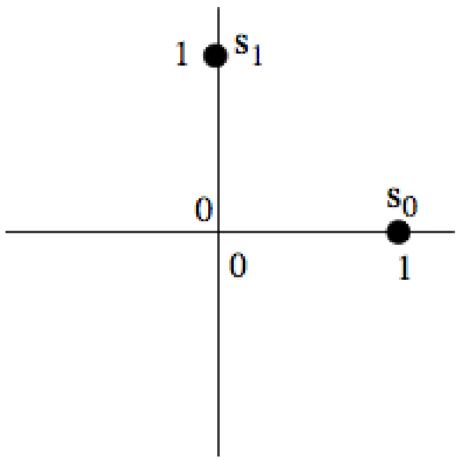
Antipodal signaling

$$d = 2$$

$$E_b = ((-1)^2 + 1^2)/2 = 1$$

$$\eta_P = d^2 / E_b = 4$$

$$P_{e,ML} = Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$



Equal energy, orthogonal signaling

$$d = \sqrt{2}$$

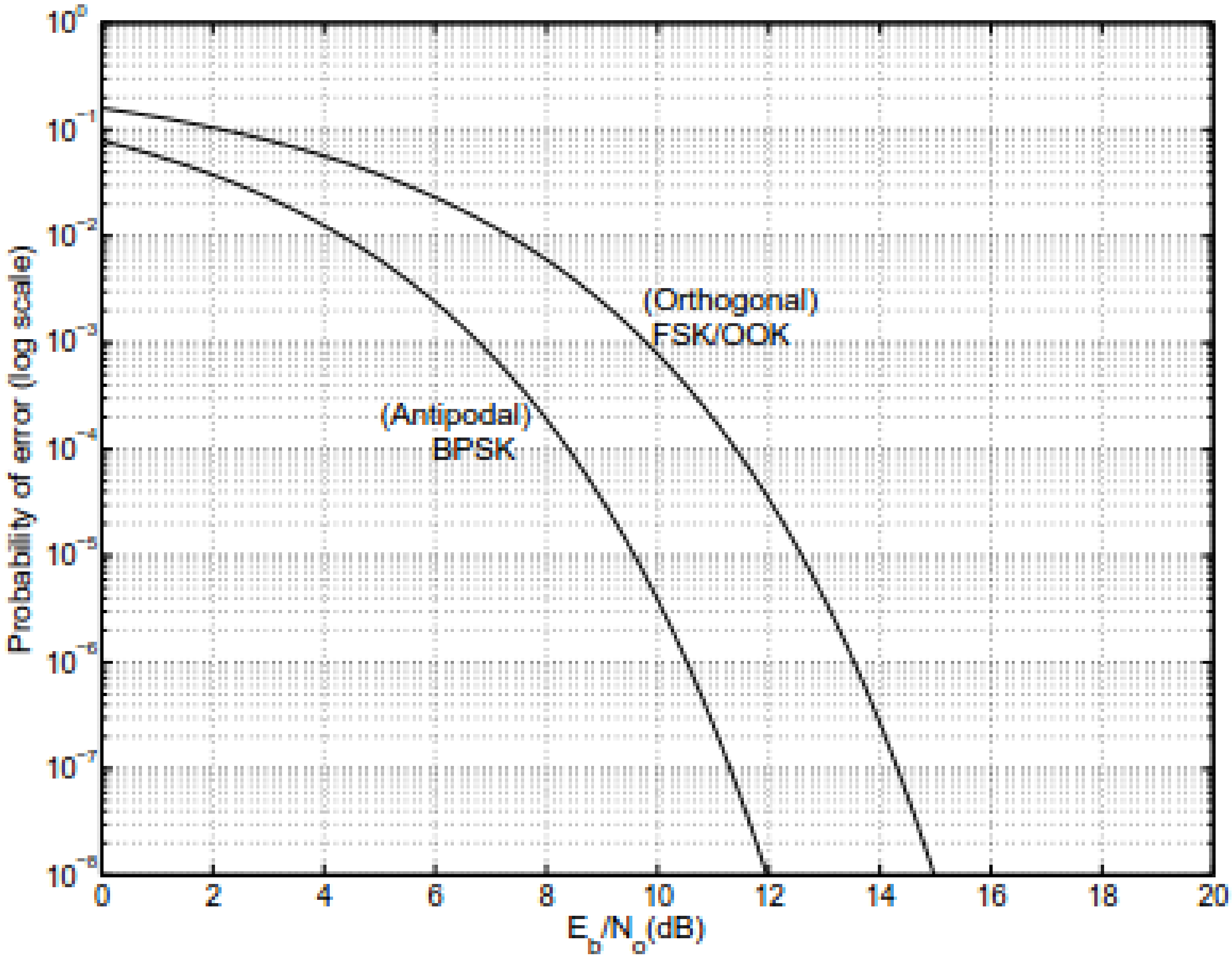
$$E_b = (1^2 + 1^2)/2 = 1$$

$$\eta_P = d^2 / E_b = 2$$

$$P_{e,ML} = Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

- OOK and equal energy orthogonal signaling have same power efficiency. Antipodal is 3 dB better.

Performance for different binary schemes..



- OOK and equal energy orthogonal signaling have same power efficiency. Antipodal is 3 dB better.