Lecture 3

Del Operator, Maxwell Equations

Reference book

Introduction to Electrodynamics by David J Griffiths

Del Operator (∇) in Electromagnetism

The **Del operator (∇)**, also known as the **nabla operator**, is a fundamental vector differential operator used in **electromagnetism** to describe various field properties. It plays a key role in **Maxwell's equations** and vector calculus.

Definition of Del Operator (∇)

In Cartesian coordinates, the **Del operator** is defined as:

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

where:

- $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors in the **x, y,** and **z** directions.
- $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are the partial derivatives in those directions.

The **del operator** is used in different ways to describe physical phenomena in electromagnetism.

When $\vec{\nabla}$ operates on

(i) Scalar function $\longrightarrow$ Gradient

(ii) Vector function via dot product $\longrightarrow$ Divergence

(iii) Vector function via cross product $\longrightarrow$ Curl

1. Del (∇) Operating on a Scalar Field

Gradient (∇ϕ)

When the **Del operator (∇)** acts on a **scalar function** $\phi(x, y, z)$, it produces a **vector field**, known as the **gradient**.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Physical Meaning:

- The **gradient** points in the direction of the **maximum rate of increase** of the scalar function.
- The magnitude of the gradient represents how fast the scalar quantity is changing.

Example: Temperature gradient, Potential gradient

2. Del (∇) Operating on a Vector Field

When the **Del operator (∇)** acts on a **vector field**, it produces **different results** depending on how it is applied.

(A) Divergence (∇ · A) – Scalar Result

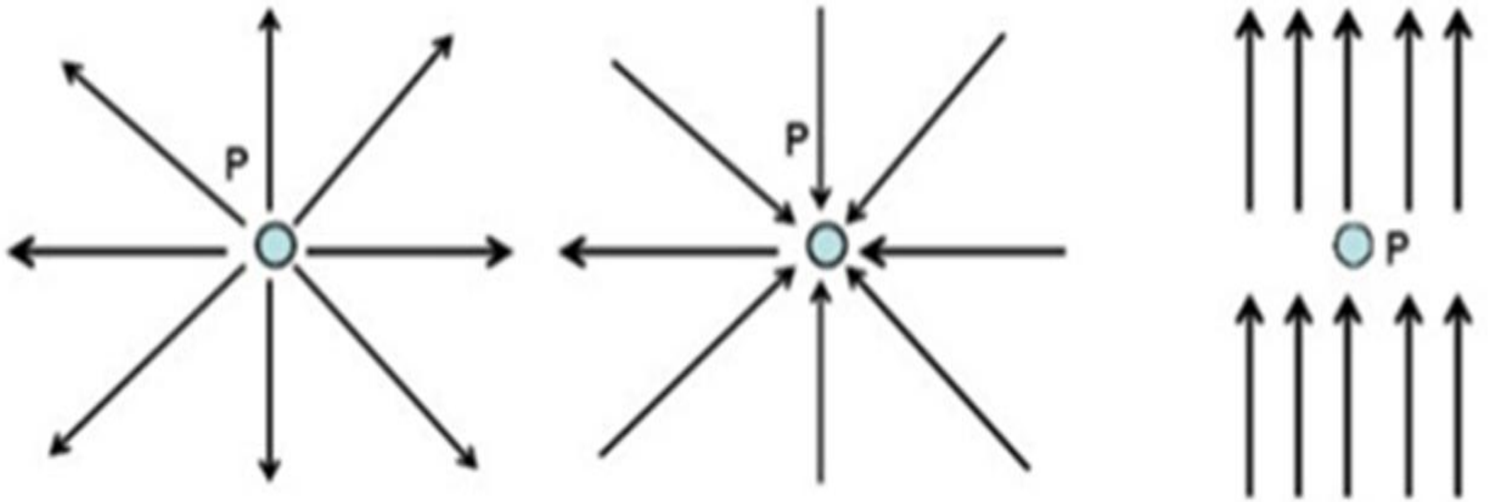
The **divergence** of a vector field $\mathbf{A}$ measures the **rate of change of the field's flux** in a given region.

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Physical Meaning:

- It tells us whether a field is **spreading out (positive divergence)** or **converging (negative divergence)**.
- A positive divergence means a **source** is present (e.g., charge creating an electric field).
- A zero divergence indicates the field is **solenoidal** (no sources).

Illustration of the divergence of a vector field at point P:



Positive Divergence | Negative Divergence | Zero Divergence

(B) Curl (∇ × A) – Vector Result

The **curl** of a vector field $\mathbf{A}$ gives another **vector field**, representing the **rotation (circulation) of the field**.

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Physical Meaning:

- The curl measures the **tendency of a field to circulate around a point**.
- A **zero curl** means the field has no rotation (it is **irrotational**).

Example in Electromagnetism:

1. **Faraday's Law of Induction:**

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

- A **changing magnetic field** creates an **electric field** that circulates around it.
- This is the principle behind **electrical generators**.

Laplacian Operator

For a scalar function $f(x, y, z)$, the Laplacian is defined as:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

It is the **sum of second partial derivatives** of f with respect to all spatial coordinates.

In **vector notation**, the Laplacian operator is written as:

$$\nabla^2 f = \Delta f$$

where Δ (delta) is the Laplace operator.

Physical Interpretation

The Laplacian tells us how much the function's value at a point **differs from its surroundings**:

- If $\nabla^2 f > 0$, the function has a **local minimum** (valley).
- If $\nabla^2 f < 0$, the function has a **local maximum** (peak).
- If $\nabla^2 f = 0$, the function is in an **equilibrium state**.

Divergence Theorem (Gauss's Theorem) Statement

The **Divergence Theorem** (also called **Gauss's Theorem**) states that:

> **The total outward flux of a vector field $\mathbf{F}$ through a closed surface S is equal to the volume integral of the divergence of $\mathbf{F}$ over the region V enclosed by S.**

Mathematical Form:

$$\oint_S \mathbf{F} \cdot d\mathbf{S} = \int_V (\nabla \cdot \mathbf{F}) dV$$

Meaning of the Terms:

- $\mathbf{F}$ → Vector field (e.g., velocity, electric field, etc.)
- S → **Closed surface** enclosing a volume V (e.g., a sphere, cube)
- $d\mathbf{S}$ → Infinitesimal surface element **pointing outward**
- $\nabla \cdot \mathbf{F}$ → **Divergence** of $\mathbf{F}$, measuring how much the field spreads
- $\oint_S \mathbf{F} \cdot d\mathbf{S}$ → **Flux** (amount of field flowing out of S)
- $\int_V (\nabla \cdot \mathbf{F}) dV$ → **Sum of all sources and sinks inside V**

- **Physical Interpretation:**
- The left-hand side represents the total flux leaving the closed surface.
- The right-hand side represents the sum of all divergence (sources and sinks) inside the volume.
- If $\nabla \cdot F > 0$, more field is leaving (source).
- If $\nabla \cdot F < 0$, more field is entering (sink).

Stokes' Theorem Statement:

Stokes' theorem relates the **surface integral** of the **curl** of a vector field over a surface to the **line integral** of the vector field along the boundary of that surface. Mathematically, it is given by:

$$\oint_C \mathbf{F} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

Explanation:

- $\mathbf{F}$ is a vector field.
- C is a **closed** curve (boundary of the surface S).
- $d\mathbf{l}$ is the **line element** along C.
- S is an **open** surface whose boundary is C.
- $d\mathbf{S}$ is the **vector surface element**.
- $\nabla \times \mathbf{F}$ is the **curl** of $\mathbf{F}$.

Interpretation:

Stokes' theorem states that the **circulation** of a vector field around a closed curve C is equal to the **sum of the curls** of the field over the surface S enclosed by C. This helps convert a **surface integral** into a **line integral** and is widely used in **electromagnetic theory** (e.g., deriving Maxwell's equations in differential form).

Equation of Continuity:

The **continuity equation** is a fundamental principle in [redacted] expresses the conservation of a quantity (such as mass or charge) in a given system. It states that the rate of change of a conserved quantity within a volume is equal to the net flux of that quantity across the boundary.

1. General Form (Differential Form)

In fluid dynamics and electromagnetism, the continuity equation is written as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

where:

- ρ = density of the conserved quantity (e.g., mass density, charge density)
- $\mathbf{J}$ = flux or current density vector (e.g., mass flux, current density)
- $\nabla \cdot \mathbf{J}$ = divergence of the flux
- $\frac{\partial \rho}{\partial t}$ = time rate of change of density