EC5.203 Communication Theory I (3-1-0-4):

Lecture 9:

Analog Communication Techniques: Frequency Modulation - 2

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INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY

HYDERABAD

Recap

Key Concepts

- Two ways of encoding info in complex envelope
  - I and Q: amplitude modulation (several variants)
  - Envelope and phase: angle modulation (constant envelope)
- For example, for a sinusoid carrier

$$A_c(t)\cos(2\pi f_c(t)t + \theta_c(t))$$

where $A_c(t)$, $f_c(t)$, $\theta_c(t)$ are the amplitude, frequency, and the phase of the carrier respectively.

Amplitude Modulation

Frequency Modulation

Phase Modulation

Frequency Modulation

- The transmitted signal is given as

$$u_{\mathrm{FM}}(t) = A_c \cos(2\pi(\underline{f_c + f(t)})t + \phi)$$

- Here $f(t) = k_f m(t)$ is the frequency offset relative to the carrier, $m(t)$ is the message signal while k_f (design constant), A_c (amplitude) and ϕ (phase) are constants.

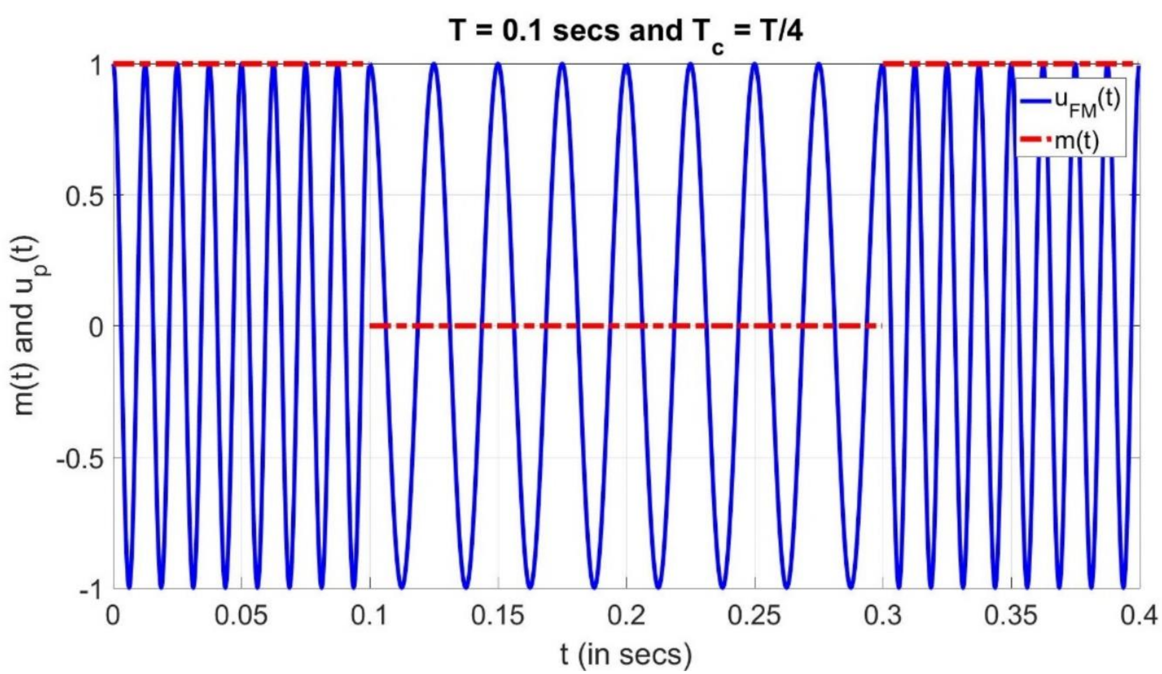
- The instantaneous frequency is given by

$$f_i(t) = f_c + f(t) = f_c + k_f m(t)$$

Example of FM Wave

- The instantaneous frequency is given by

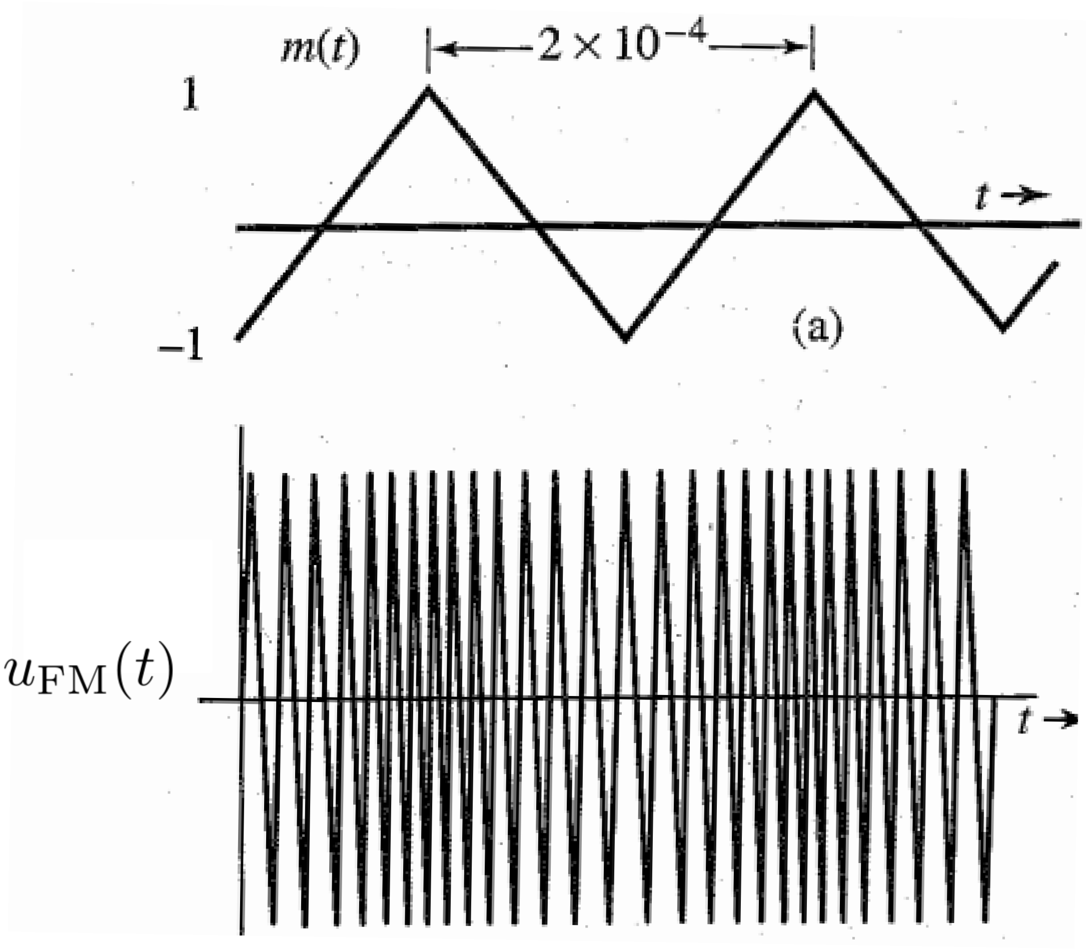
$$f_i(t) = f_c(1 + m(t))$$



Example 2 of FM Wave

- The instantaneous frequency is given by

$$f_i(t) = f_c + f(t) = f_c + k_f m(t)$$



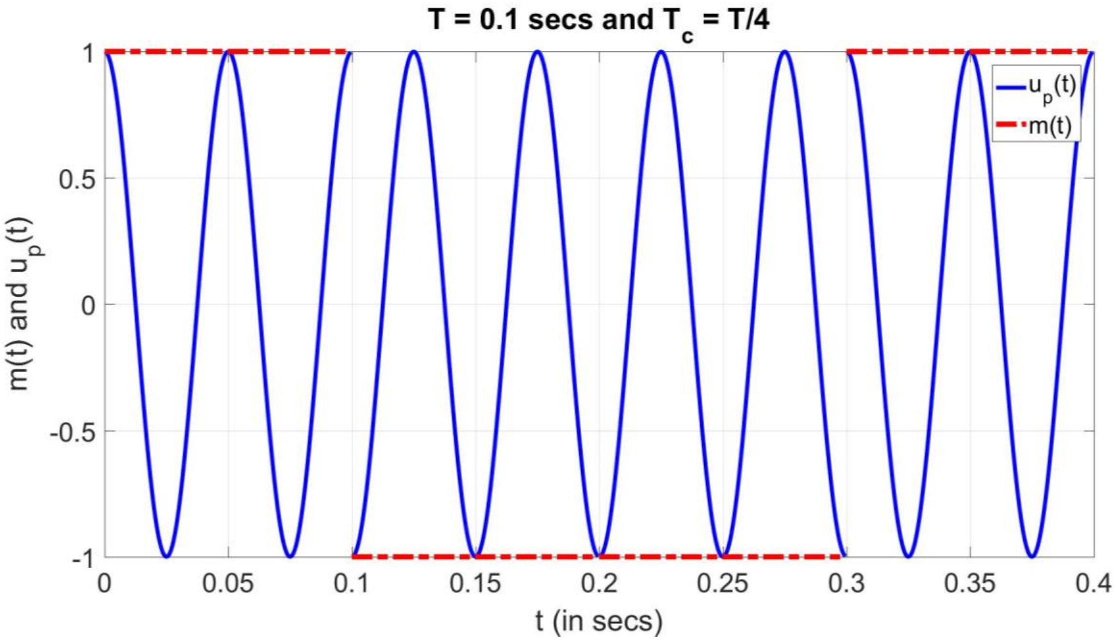
Phase Modulation

- The transmitted signal is given as

$$u_{\mathrm{PM}}(t) = A_c \cos(2\pi f_c t + \boxed{\theta(t)} + \phi)$$

- Here $\theta(t) = k_p m(t)$ while k_p, A_c, ϕ and f_c are constants.

Example of PM Wave



$$u(t) = A_c \cos(2\pi f_c t + 0.5(m(t) - 1)\pi)$$

$$= A_c \cos(2\pi f_c t) \qquad m(t) = 1$$

$$= A_c \cos(2\pi f_c t - \pi) \qquad m(t) = -1$$

Generalized Model: Angle Modulation

- The transmitted signal is given as

$$u_{\mathrm{p}}(t) = A_c \cos(2\pi f_c t + \boxed{\theta(t)})$$

$$\theta(t) = g(m(t))$$

- Angle modulation is a general form
  - Phase modulation

$$\theta(t) = \theta(0) + k_p m(t)$$

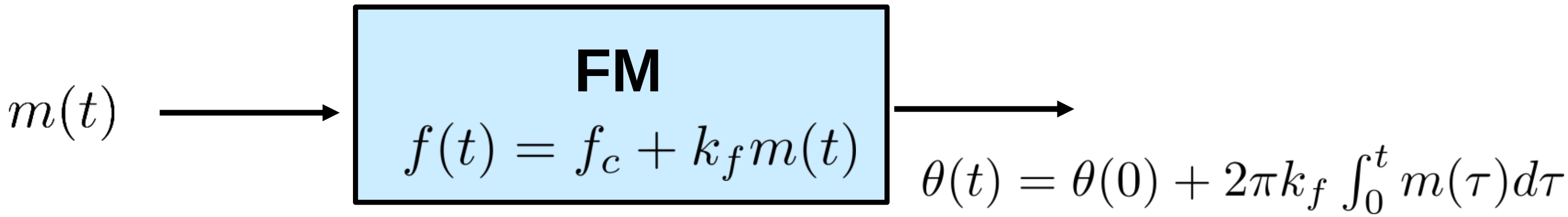
  - Frequency modulation

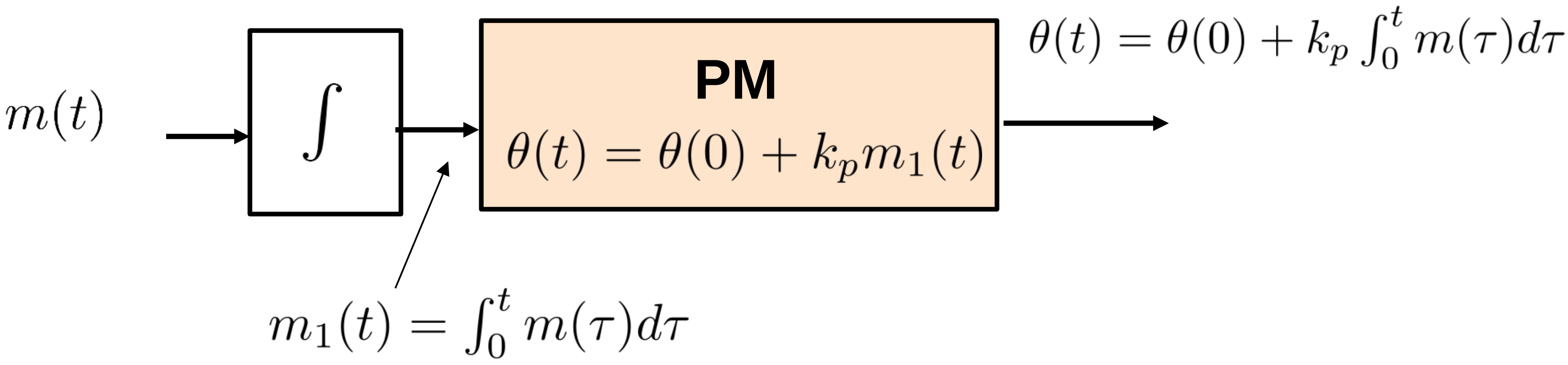
$$\frac{1}{2\pi}\frac{d\theta(t)}{dt} = f(t) = k_f m(t)$$

$$\theta(t) = \theta(0) + 2\pi k_f \int_0^t m(\tau)\, d\tau$$

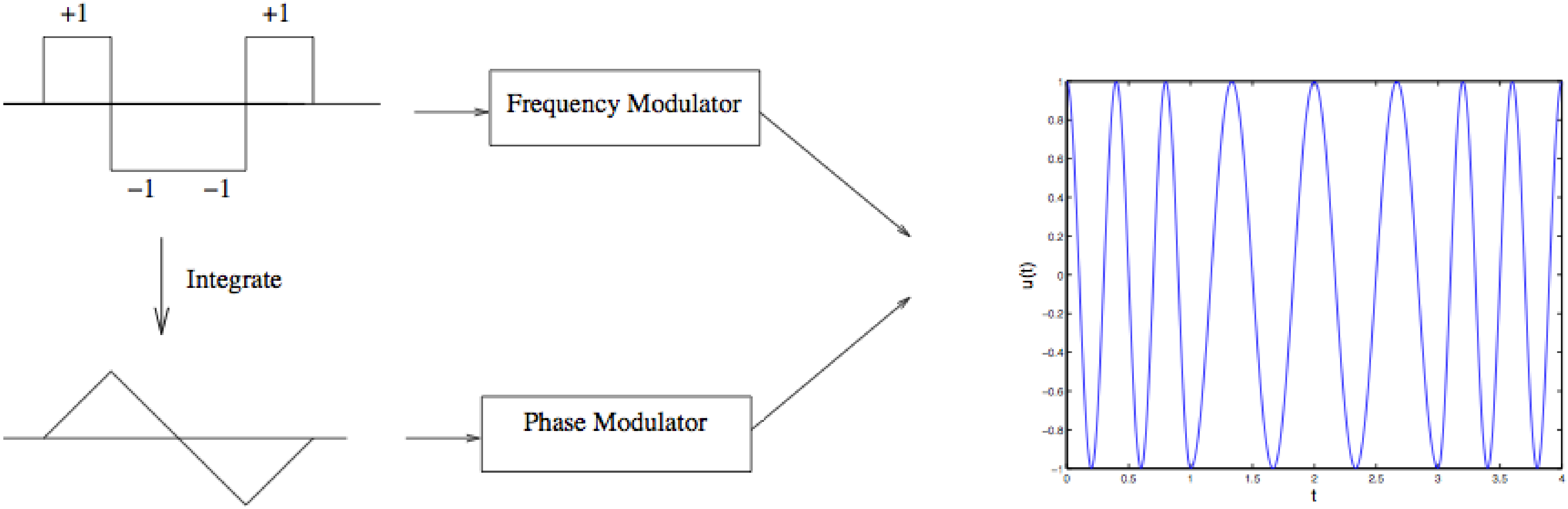
where k_p and k_f are constants while $f(t)$ is the frequency offset relative to the carrier. Also $\phi = \theta(0)$ where $t = 0$ is used as reference point.

Equivalence of PM and FM: *FM using PM*

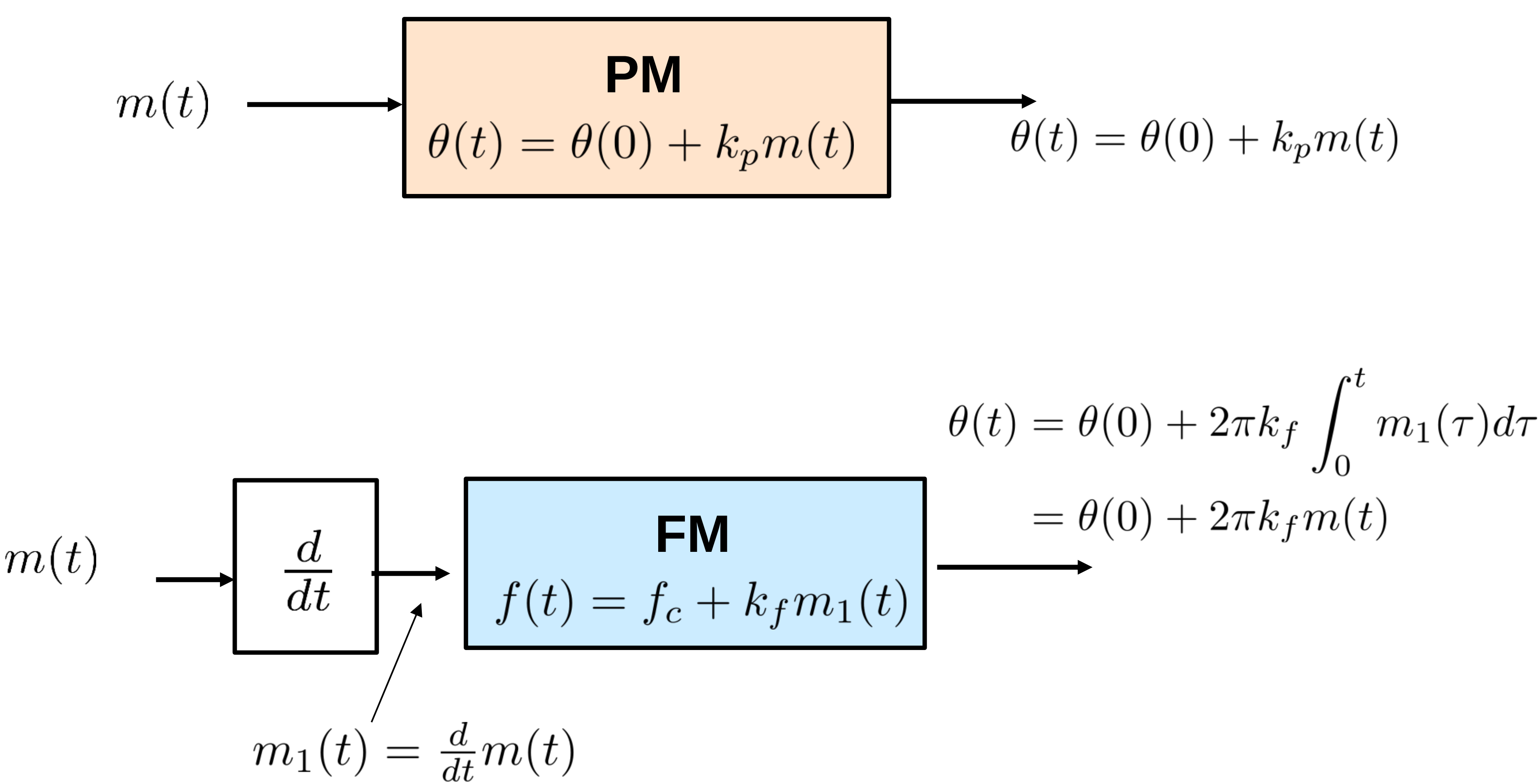




Equivalence of FM and PM: *FM using PM*

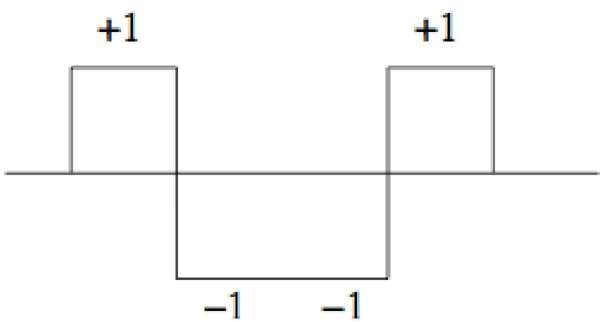


Equivalence of PM and FM: *PM using FM*

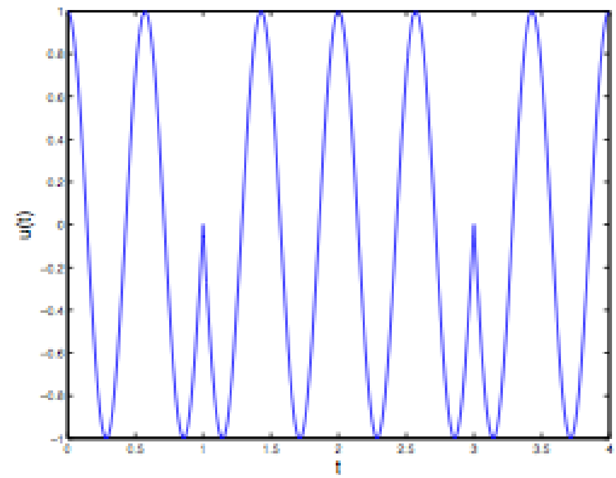


PM versus FM

Digital message



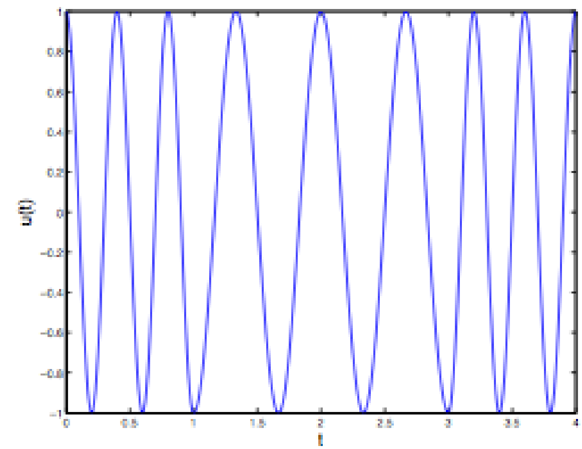
PM
(PSK for digital messages)



Discontinuous phase ➔ vulnerable to nonlinearities, poorer frequency containment

Continuous phase ➔ more robust to nonlinearities, better frequency containment

FM



PM versus FM in practice

- Legacy analog communication → no control over message signal → FM preferred
  - Integration of message prior to phase modulation leads to smooth phase which leads to better bandwidth containment.
  - Most famous application: radio broadcasting
  - FM has been used in 2G GSM (Gaussian MSK, a form of FM); Optimal demodulation more complicated
  - Lately being used in power limited systems: FSK is used in LoRaWAN
- Digital communication → can design message signal → PM (PSK specifically) often preferred
  - Easier to implement optimal demodulator
  - Use bandwidth-efficient pulses rather than rectangular pulses to create smoother signals with better frequency containment
  - Used in modern digital communication systems

Focus on FM in this chapter. PSK studied in Chapter 4 and beyond.

Frequency Modulation

- The transmitted signal is given as

$$u_{\text{FM}}(t) = A_c \cos(2\pi(\underline{f_c + f(t)})t + \phi)$$

- Here $f(t) = k_f m(t)$ is the frequency offset relative to the carrier, $m(t)$ is the message signal while k_f (design constant), A_c (amplitude) and ϕ (phase) are constants.

- The instantaneous frequency is given by

$$f_i(t) = f_c + f(t) = f_c + k_f m(t)$$

FM Modulation Index

- Modulation index for FM is given by

$$\beta = \frac{\Delta f_{\max}}{B}$$

  where the frequency deviation $\Delta f_{\max} = k_f \max_t |m(t)|$ and B is the bandwidth of the signal..

- Narrowband FM: $\beta < 1$
- Wideband FM: $\beta > 1$

- Solve: For sinusoidal message $m(t) = A_m \cos(2\pi f_m t)$, find β. Also find $\theta(t)$ in terms of β assuming $\theta(0) = 0$.

Modulation

- Direct method
  - Voltage controlled oscillator (VCO)
  - Use of varacter diode which provides voltage controlled capacitance in LC tuned circuits
  - Directly generates passband
  - Both narrow and wideband
- Indirect method
  - An alternative method for wideband FM signal generation when direct method is infeasible or costly
  - First generate narrowband signal (using PM modulation) and then increase the frequency shift and frequency by using several stages of multipliers (non-linearity)
  - Not used nowadays as direct FM methods are now feasible and cost-effective.

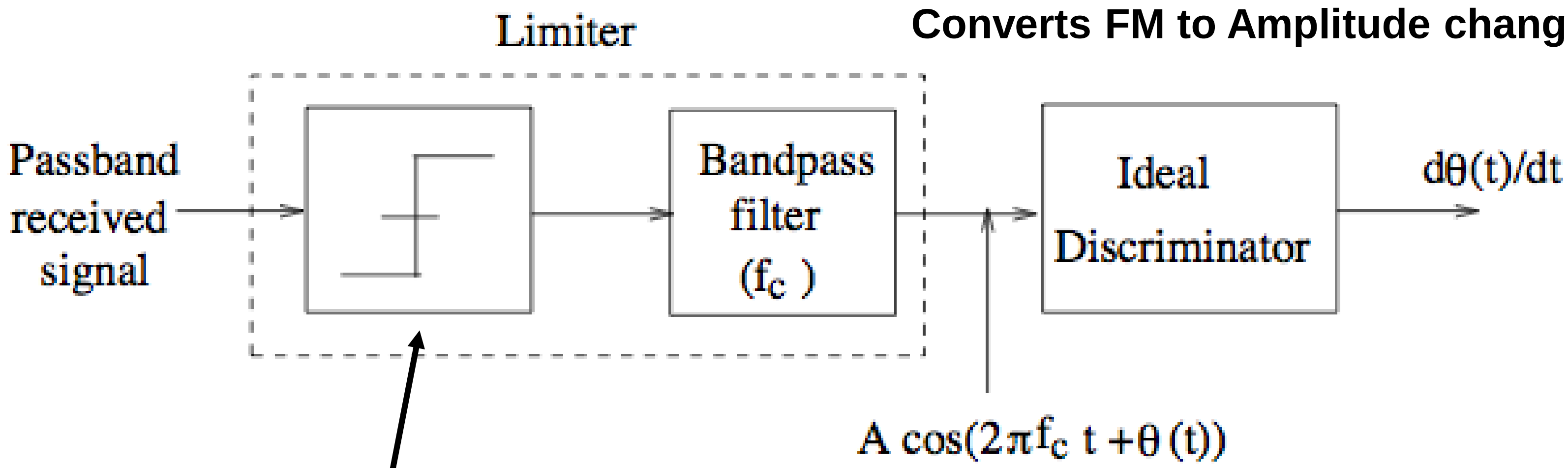
FM Demodulation

- There are several methods
  - Limiter discriminator
  - Phase locked loop (PLL) (in detail later in this chapter)

Limiter Discriminator

Enforces constant envelope

Converts FM to Amplitude changes



Removes amplitude fluctuations caused by noise and channel

Limiting enduces harmonics since it is a non-linear operation

$y(t) = a x(t) + b x^2(t) + c x^3(t) + \ldots$

Today’s Class

Ref Books: U. Madhow and B. P. Lathi

FM Spectrum

FM spectrum

- Narrowband FM
  - Similar to DSB
  - Bandwidth = 2B (where B=message bandwidth)
- Wideband FM
  - Bandwidth dominated by max frequency deviation
- Carson's formula: adds the two components

Recap: *Time domain expressions for a passband signal*

- In terms of I and Q components

$$u_p(t) = u_c(t)\cos(2\pi f_c t) - u_s(t)\sin(2\pi f_c t)$$

- In terms of envelope and phase

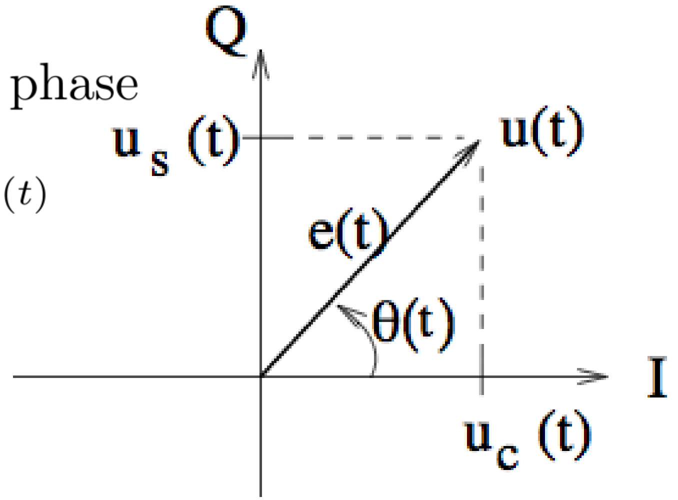
$$u_p(t) = e(t)\cos(2\pi f_c t + \theta(t))$$

- In terms of complex envelope

$$u_p(t) = \mathrm{Re}(u(t)e^{j2\pi f_c t})$$

- Complex baseband in terms of Envelope and phase

$$u(t) = u_c(t) + j u_s(t) = e(t)e^{j\theta(t)}$$



Starting from one representation, can derive the rest based on the relations depicted in the figure

Narrowband FM

- Show that the bandwidth of narrowband of FM is 2B where B is the bandwidth of the signal $m(t)$.

Narrowband FM: Example

- Example: find the bandwidth of FM signal corresponding to a sinusoidal message $m(t) = \cos 2\pi f_m t$.

Questions?

Wideband FM

- Bandwidth is dominated by frequency deviation

$$\Delta f = k_f m(t)$$

- Frequency will swing between $\pm\Delta f_{\max}$ assuming equal positive and negative swings in message.

$$\Delta f_{\max} = k_f \max_t |m(t)|$$

- Bandwidth $B_{\mathrm{FM}} = 2\Delta f_{\max}$

Carson's rule

- Add up estimates for narrowband and wideband FM

$$B_{\mathrm{FM}} \approx 2B + 2\Delta f_{\max} = 2B(\beta + 1)$$

where $\beta = \Delta f_{\max}/B$ is the FM modulation index or the deviation ration.

Bandwidth of Angle Modulated Waveforms

- Prove that Angle Modulated Waveforms have infinite bandwidth theoretically!
- As a special case, derive narrowband FM expression and its bandwidth!

Ref: B.P.Lathi

FM spectrum for periodic messages

- Complex envelope is periodic for periodic messages $\rightarrow$ Fourier series
  - Spectrum of complex envelope is discrete with impulses at integer multiples of fundamental frequency
- Standard example: sinusoidal message
  - But approach is quite general
- Somewhat artifical since most messages (such as speech) are not periodic

(Approximate) FM spectrum

- The bandwidth of narrowband FM ($\beta < 1$) is 2B where B is the bandwidth of the signal $m(t)$.

Assumption $\theta(t)$ is small for narrowband FM! Not valid for general case.

- Add up estimates for narrowband and wideband FM

$$B_{\mathrm{FM}} \approx 2B + 2\Delta f_{\max} = 2B(\beta + 1)$$

where $\beta = \Delta f_{\max}/B$ is the FM modulation index or the deviation ration.

Exact FM spectrum for sinusoidal message

- Prove that FM spectrum for sinusoidal message is given by

$$U(f) = \sum_{n=-\infty}^{\infty} J_n(\beta)\delta(f - nf_m)$$

where $J_n(\beta)$ is the n^{th} order Bessel function given by

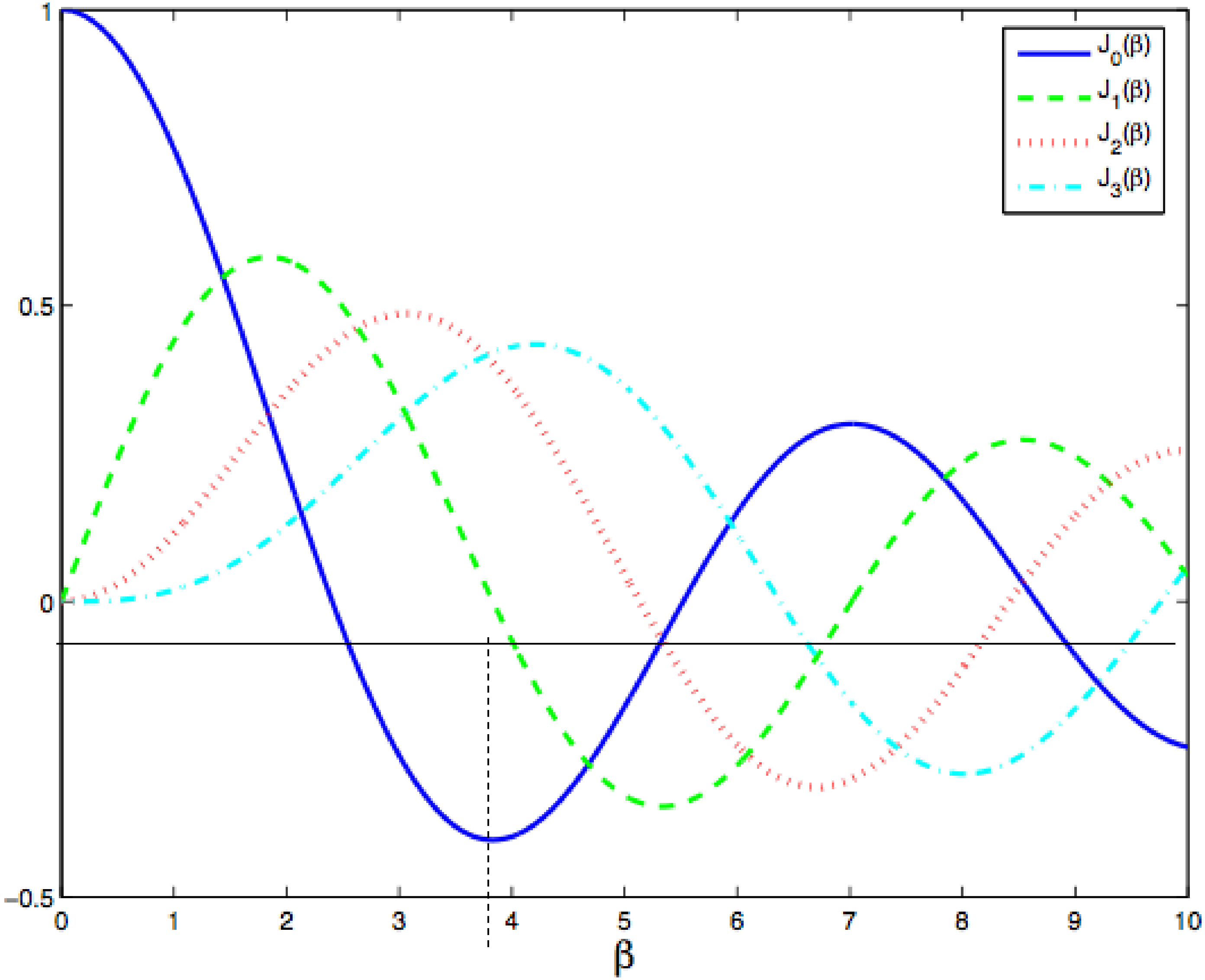
$$J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta\sin x - nx)}\,dx$$

Complex-valued integral

$$= \frac{1}{\pi}\cos(\beta\sin x - nx)\,dx$$

Real Valued

Bessel function plots



$$J_n(-\beta) = (-1)^n J_n(\beta)$$

Bessel function properties

- Note that the Bessel function $J_n(\beta)$ is real

$$J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta\sin x - nx)}\,dx = \frac{1}{\pi}\int_{0}^{\pi}\cos(\beta\sin x - nx)\,dx$$

Properties of Bessel function

- $J_n(\beta) = (-1)^n J_{-n}(\beta) = (-1)^n J_n(-\beta)$
- For fixed β, $J_n(\beta) \to 0$ as $n \to \infty$.
  Generally, $J_n(\beta) \approx 0$ for $|n| > \beta + 1$
  $B_{\mathrm{FM}} \approx 2(\beta+1)f_m$
  This is consistent with Carson's rule.

This is only an approximation

- For fixed n, $J_n(\beta)$ vanishes for specific values of β. This is useful in spectral shaping.

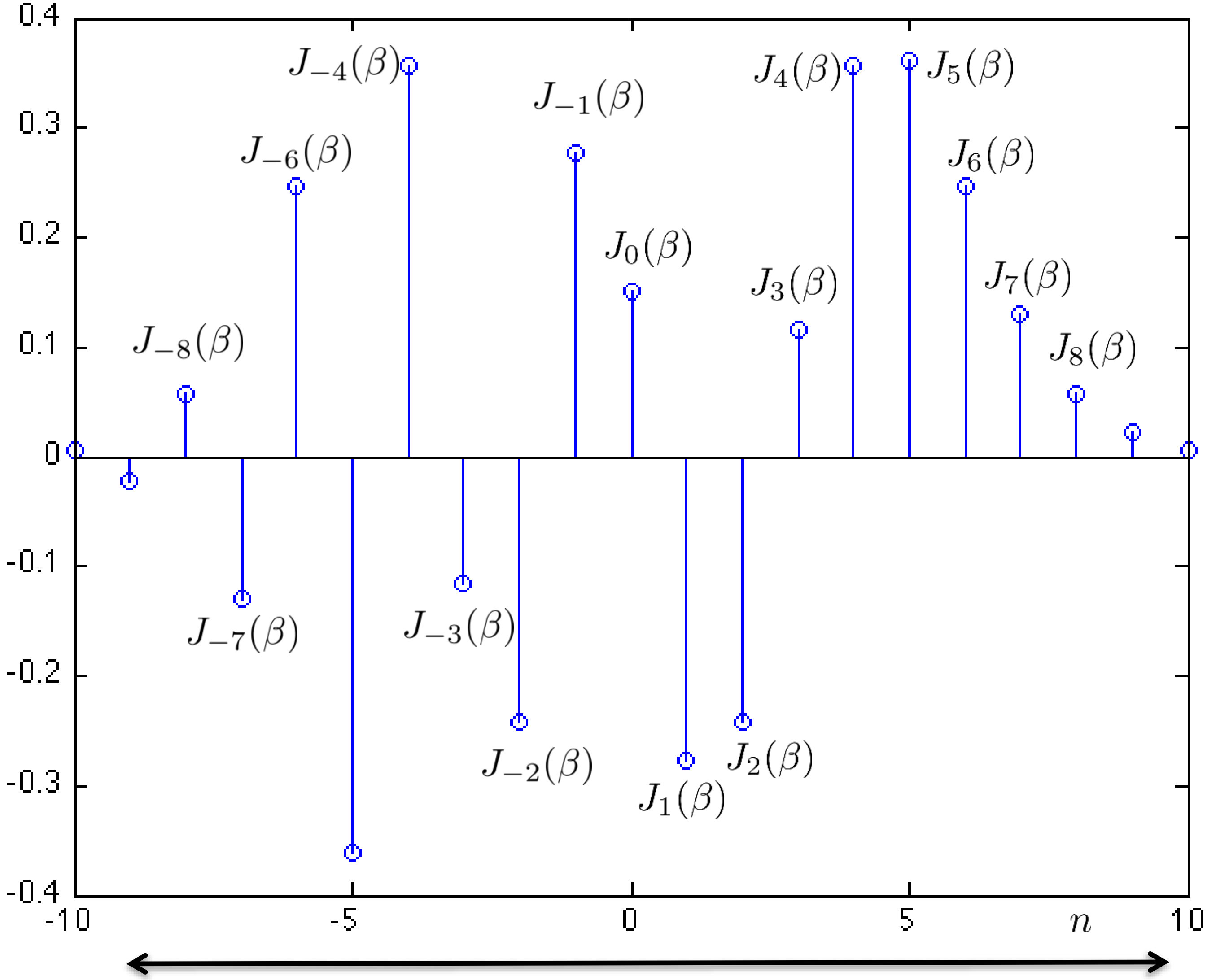
Modulation index and Power in Sidebands

Modulation index	Sideband																
	Carrier	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0.00	1.00																
0.25	0.98	0.12															
0.5	0.94	0.24	0.03														
1.0	0.77	0.44	0.11	0.02													
1.5	0.51	0.56	0.23	0.06	0.01												
2.0	0.22	0.58	0.35	0.13	0.03												
2.41	0	0.52	0.43	0.20	0.06	0.02											
2.5	−0.05	0.50	0.45	0.22	0.07	0.02	0.01										
3.0	−0.26	0.34	0.49	0.31	0.13	0.04	0.01										
4.0	−0.40	−0.07	0.36	0.43	0.28	0.13	0.05	0.02									
5.0	−0.18	−0.33	0.05	0.36	0.39	0.26	0.13	0.05	0.02								
5.53	0	−0.34	−0.13	0.25	0.40	0.32	0.19	0.09	0.03	0.01							
6.0	0.15	−0.28	−0.24	0.11	0.36	0.36	0.25	0.13	0.06	0.02							
7.0	0.30	0.00	−0.30	−0.17	0.16	0.35	0.34	0.23	0.13	0.06	0.02						
8.0	0.17	0.23	−0.11	−0.29	−0.10	0.19	0.34	0.32	0.22	0.13	0.06	0.03					
8.65	0	0.27	0.06	−0.24	−0.23	0.03	0.26	0.34	0.28	0.18	0.10	0.05	0.02				
9.0	−0.09	0.25	0.14	−0.18	−0.27	−0.06	0.20	0.33	0.31	0.21	0.12	0.06	0.03	0.01			
10.0	−0.25	0.04	0.25	0.06	−0.22	−0.23	−0.01	0.22	0.32	0.29	0.21	0.12	0.06	0.03	0.01		
12.0	0.05	−0.22	−0.08	0.20	0.18	−0.07	−0.24	−0.17	0.05	0.23	0.30	0.27	0.20	0.12	0.07	0.03	0.01

https://en.wikipedia.org/wiki/Frequency_modulation#Bessel_functions

Fourier coefficients for complex envelope

Plot of $J_n(\beta)$ as a function of n for a given $\beta = 6$



$$J_{-n}(\beta) = (-1)^n J_n(\beta)$$

Passband Bandwidth (frequency unit = message frequency f_m)

Fractional power containment BW

- Parseval's theorem: Power = sum of magnitude of Fourier series coefficients

$$1 = |u(t)|^2 = \overline{|u(t)|^2} = \sum_{n=-\infty}^{\infty} J_n^2(\beta) = J_0^2(\beta) + 2\sum_{n=1}^{\infty} J_n^2(\beta)$$

- Fractional power containment bandwidth for fraction α is $2Kf_m$ with K given by

$$J_0^2(\beta) + 2\sum_{n=1}^{K} J_n^2(\beta) \geq \alpha$$

Carson's rule

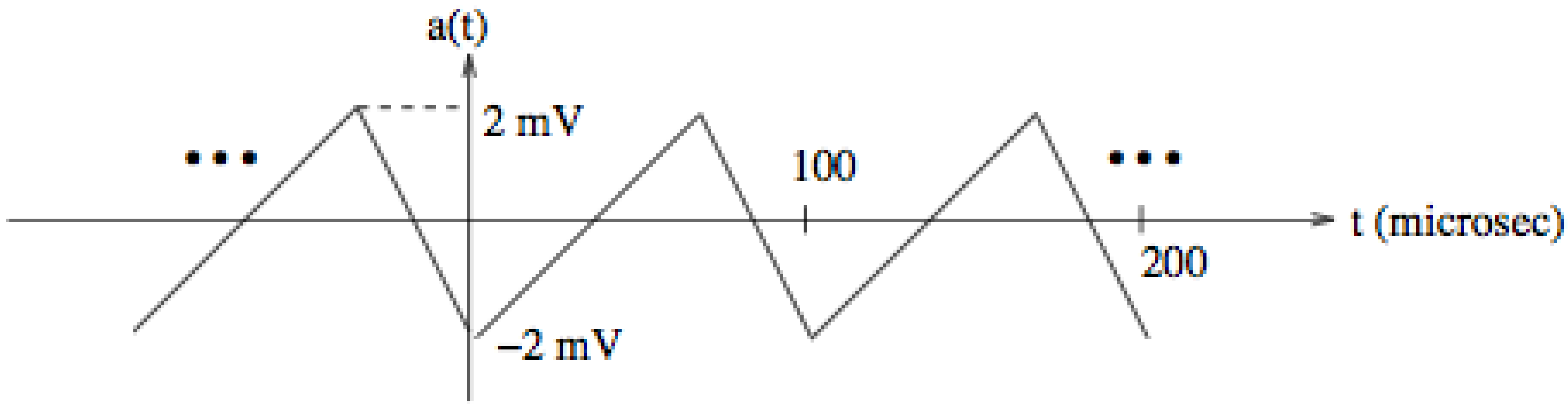
- Add up estimates for narrowband and wideband FM

$$B_{\mathrm{FM}} \approx 2B + 2\Delta f_{\max} = 2B(\beta + 1)$$

  where $\beta = \Delta f_{\max}/B$ is the FM modulation index or the deviation ration.

- Carson's rule uses $\alpha = 0.98$.

Example 3.3.1: Tutorial



- The signal $a(t)$ is fed to a VCO with quiescent frequency 5 MHz and frequency deviation of 25 KHz/mV.
- Give an estimate of the bandwidth of y(t), which is output of VCO. Use only the first harmonic for bandwidth calculation.
- If signal $y(t)$ is passed through an ideal passband filter of bandwidth 5 KHz, centered at 5.005 MHz, then provide the simplest possible expression for the power at the filter output.

Features of Angle Modulated Non-linearities

- Exchanging signal power with bandwidth
  - Bandwidth for AM cannot be changed while it can be changed changed based on Δf.
  - SNR is roughly proportional to square of transmission signal bandwidth.
- Immunity of angle modulation to non-linearities.
  - Non-linearity does not affect FM signal while it does affect AM signal.(Proofs)

Ref: B.P.Lathi