EC5.203 Communication Theory (3-1-0-4):

Lecture 14:
Digital Modulation - 3

10 March 2025

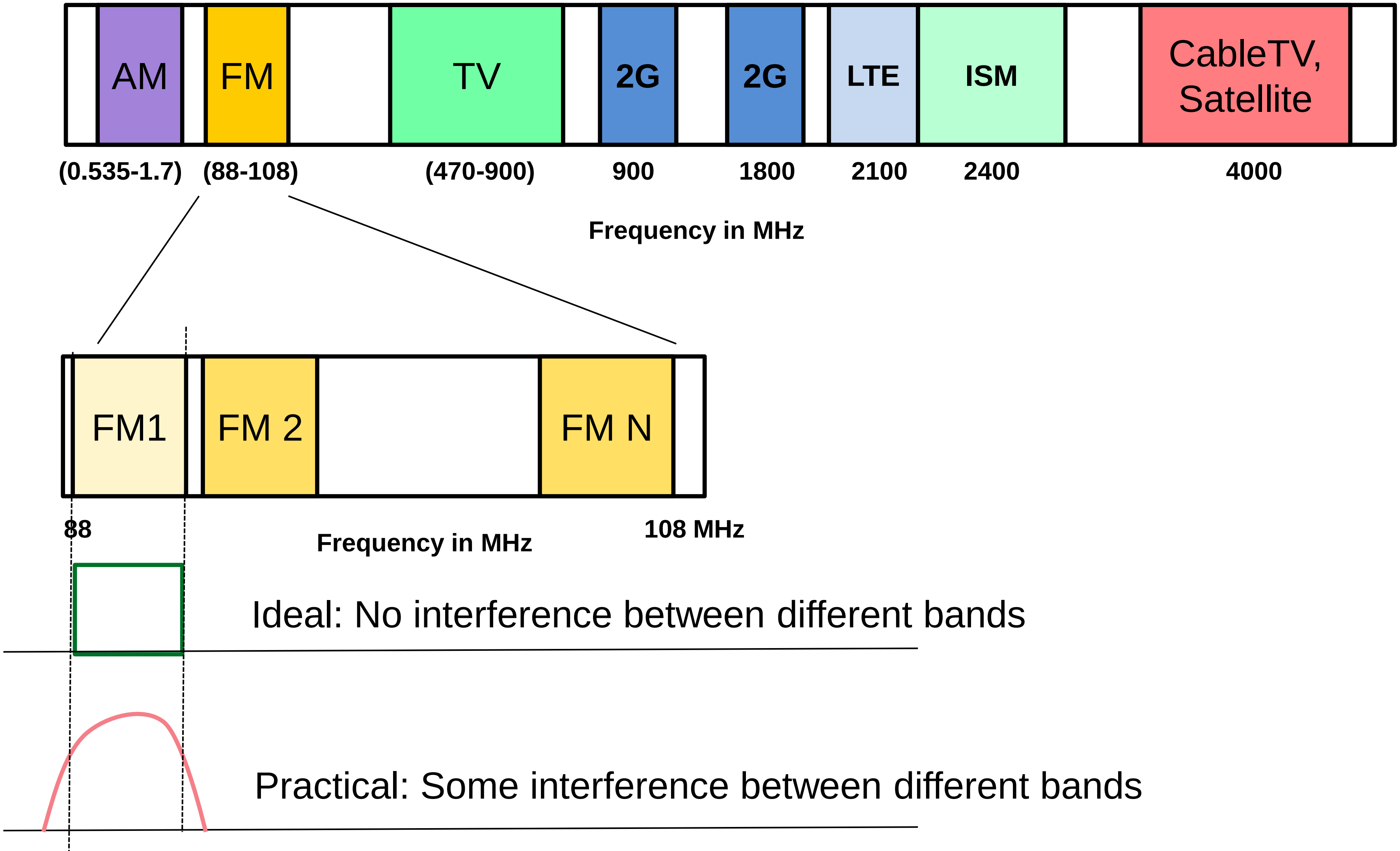
INTERNATIONAL INSTITUTE OF
INFORMATION TECHNOLOGY
HYDERABAD

References

- Chap. 4 (Madhow)

Recap: *Bandwidth Occupancy*

Motivation



AM
FM
TV
2G
2G
LTE
ISM
CableTV, Satellite
(0.535-1.7)
(88-108)
(470-900)
900
1800
2100
2400
4000
Frequency in MHz
FM1
FM 2
FM N
88
Frequency in MHz
108 MHz
Ideal: No interference between different bands
Practical: Some interference between different bands

Modeling bandwidth occupancy

- Consider the complex envelope of a linearly modulated signal

$$u(t) = \sum_n b[n]p(t - nT)$$

  where $\{b[n]\}$ is sequence of symbols and $p(t)$ is modulating pulse for T seconds.

- $\{b[n]\}$ is modeled as random at the transmitter as well as receiver.

- However for characterizing the bandwidth occupancy of digitally modulated signal u, we define the quantities of interest in terms of average across time.

- We treat $u(t)$ as a finite power signal that can be modeled as a deterministic sequences once $\{b[n]\}$ is fixed.

- Bandwidth is then defined in terms of power spectral density.

PSD of Linearly Modulated Signal

- Theorem 4.2.1: Consider a linearly modulated signal where the symbol sequence $\{b[n]\}$ is zero mean and uncorrelated with average symbol energy

$$\sigma_b^2 = \overline{|b[n]|^2} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |b[n]|^2,$$

then the PSD is given by

$$S_u(f) = \frac{|P(f)|^2}{T}\sigma_b^2$$

and the power of the modulated signal is

$$P_u = \frac{\sigma_b^2 ||p||^2}{T}$$

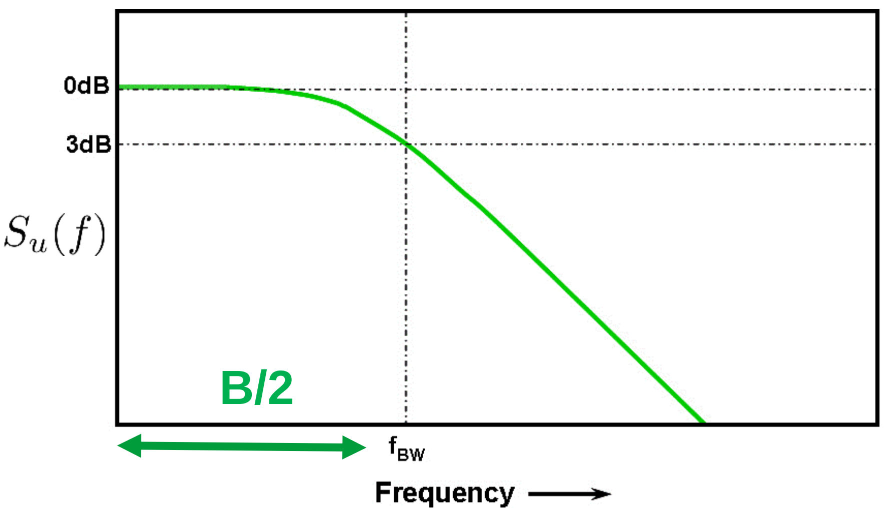
where $||p||^2$ denotes the energy of the modulating pulse. Proof.

- Assumptions:
  - The symbols have zero DC value: $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} b[n] = 0.$
  - The symbols are uncorrelated: $\lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} b[n]b^*[n-k] = 0 \quad \text{for } k \neq 0.$

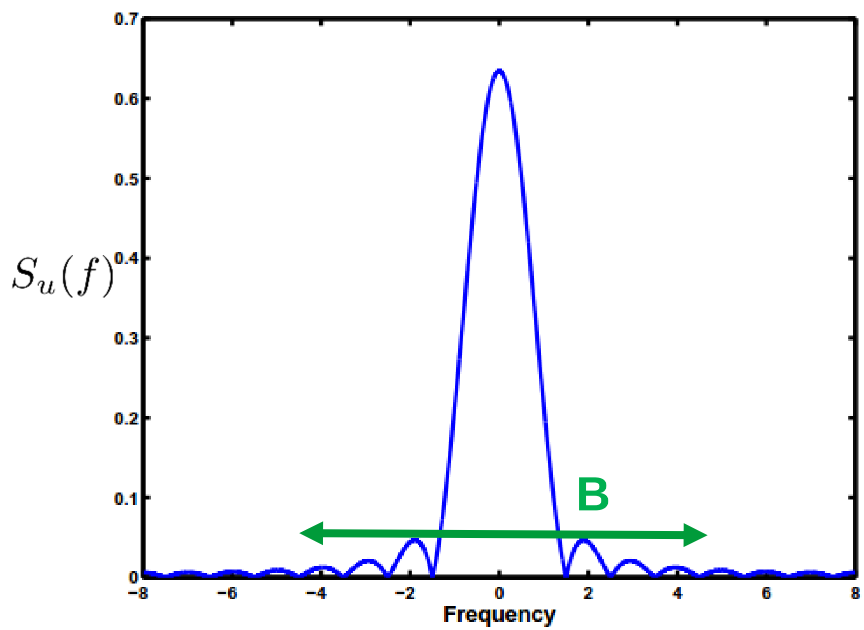
Bandwidth based on PSD

- 3-dB bandwidth
- Fractional power-containment bandwidth: This is the smallest interval that contains a given fraction of the power

$$\int_{-B/2}^{B/2} S_u(f)df = \gamma P_u = \gamma \int_{-\infty}^{\infty} S_u(f)df$$



$$S_u(B_{3dB}/2) = S_u(-B_{3dB}/2) = S_u(0)/2$$



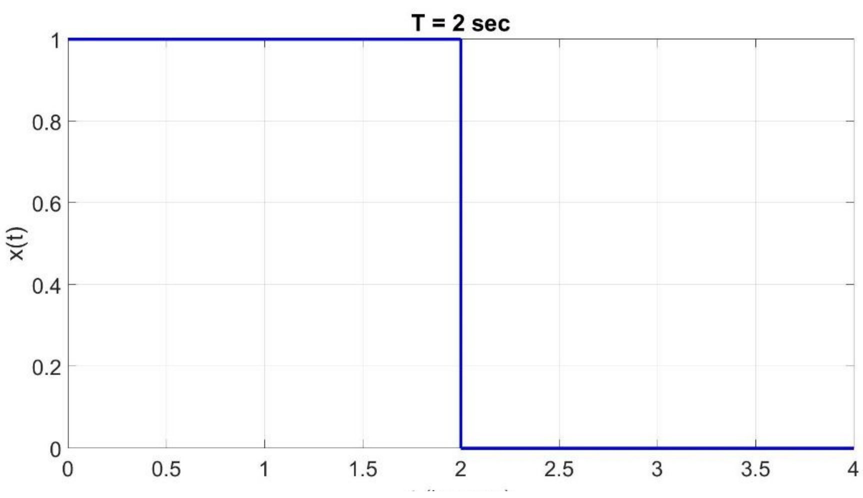
We will focus mostly on this!

Time Frequency Normalization: *Time Domain*

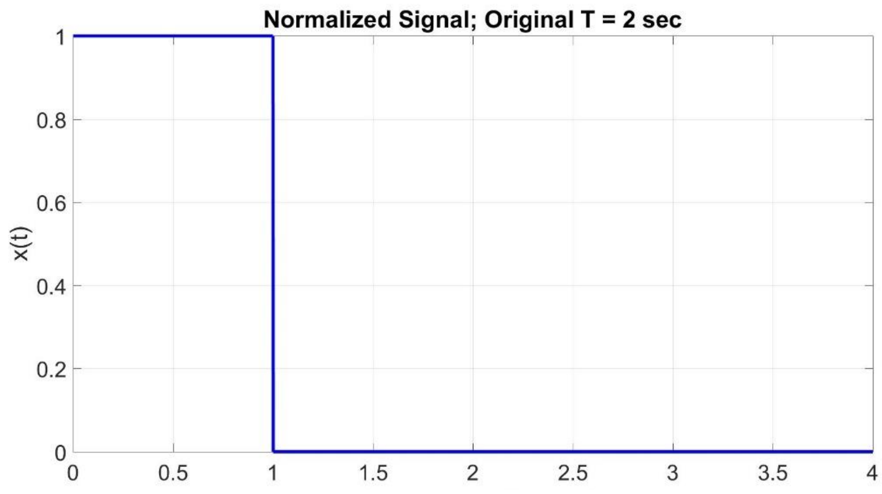
- If we are sending one symbol every T time units, then the symbol rate is $1/T$ in units of symbols/time unit.
- If we normalize the system for the symbol rate of 1, where the unit of time is T. This implies unit of frequency is $1/T$. In terms of new unit, the linearly modulated signals can be written as

$$u_1(t) = \sum_n b[n]p_1(t-n)$$

where $p_1(t)$ is the modulation pulse for the normalized system.



t in secs

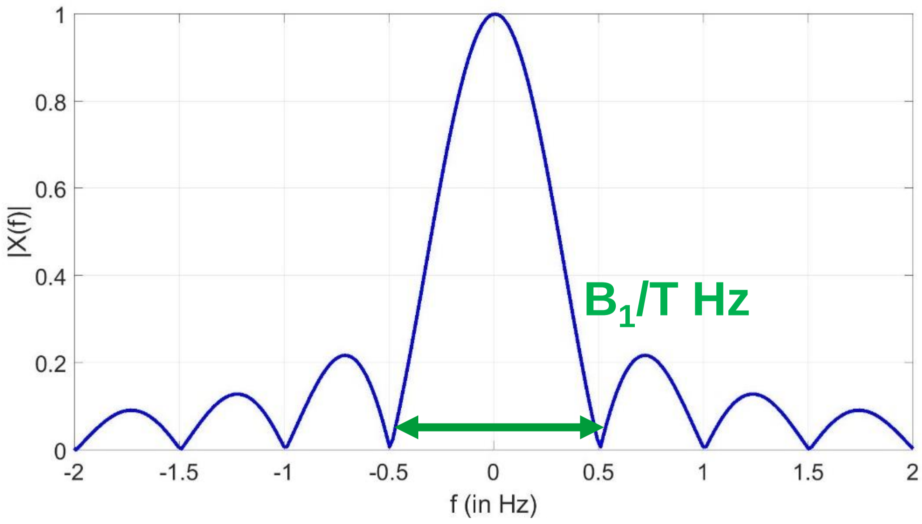


$t_0 = t/T$ Time Unit = T secs

Time Frequency Normalization: *Freq. Domain*

- Let us denote B_1 to be the bandwidth of the normalized system, then the bandwidth of the original system is B_1/T.
- In terms of determining the bandwidth occupancy, we can work, without loss of generality, with the normalized system.
- In essesence, we are working in the original system with the normalized time domain t/T and normalized frequency fT.

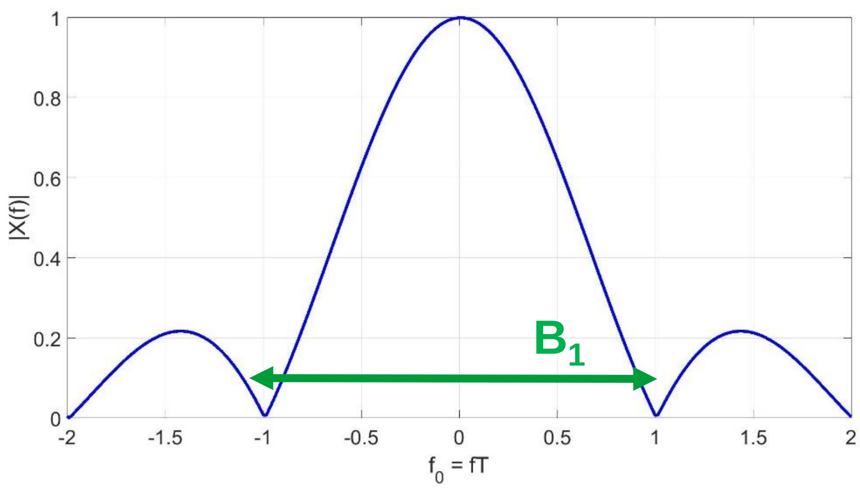
$$|P(f)| = |\mathrm{sinc(fT)}|$$



f

Frequency unit = 1 cycle/sec

$$|P(f)| = |\mathrm{sinc(f)}|$$ **normalized**



$f_0 = fT$

Frequency unit = 1/T

Bandwidth computation for Rectangular Pulse

- Consider normalized system with $p_1(t) = I_{[0,1]}(t)$ for which

$$P_1(f) = \text{sinc}(f)e^{j\pi f}$$

- For $\{b[n]\}$ iid, taking values between ± 1 with equal probability, $\sigma_b^2 = 1$, we get $S_{u_1}(f) = \text{sinc}^2(f)$. Real and Positive

- For a fractional power-containment bandwidth with fraction γ

$$\int_{-B_1/2}^{B_1/2} S_{u_1}(f)\,df = \int_{-B_1/2}^{B_1/2} \text{sinc}^2(f)\,df = \gamma \int_{-\infty}^{\infty} \text{sinc}^2(f)\,df$$
$$= \gamma \int_{-\infty}^{\infty} 1^2(t)\,dt = \gamma$$

Parseval

$$\int_{-B_1/2}^{B_1/2} S_{u_1}(f)\,df = \gamma/2$$

Symmetry of PSD

- For $\gamma = 0.99$, we obtain $B_1 = 10.2$ while for For $\gamma = 0.9$, we obtain $B_1 = 0.85$.

Design for Bandlimited Channels:

Nyquist Criteria for pulse shaping!

Nyquist Sampling Theorem (Book Notations!!!)

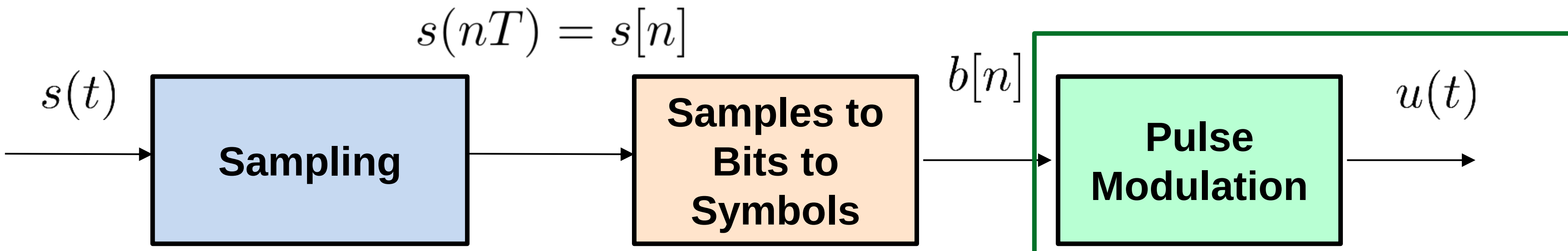
- Any signal $s(t)$ bandlimited to $[-W/2, W/2]$ can be described completely by its samples $\{s(n/W)\}$ at rate W. The signal $s(t)$ can be recovered from its samples using the following interpolation formula

$$s(t) = \sum_{n=-\infty}^{\infty} s\left(\frac{n}{W}\right) g\left(t - \frac{n}{W}\right)$$

  where $g(t) = \mathrm{sinc}(Wt)$.

- Book uses $p(t)$ here but I have used $g(t)$ on purpose to differentiate it from the modulating pulse $p(t)$.

Design for Bandlimited Channels



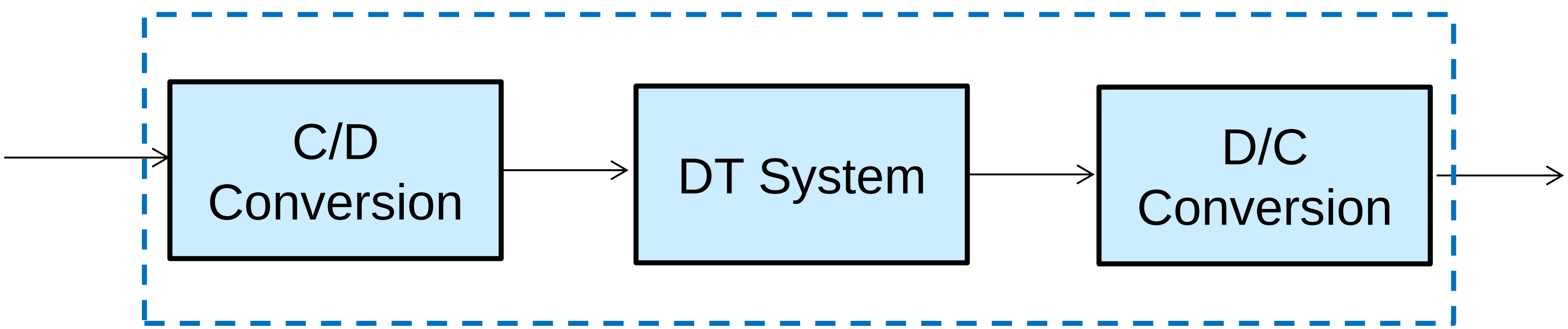
$$s(t) = \sum_{n=-\infty}^{\infty} s\left(\frac{n}{W}\right) g\left(t - \frac{n}{W}\right)$$

$$u(t) = \sum_{n} b[n]p(t - nT)$$

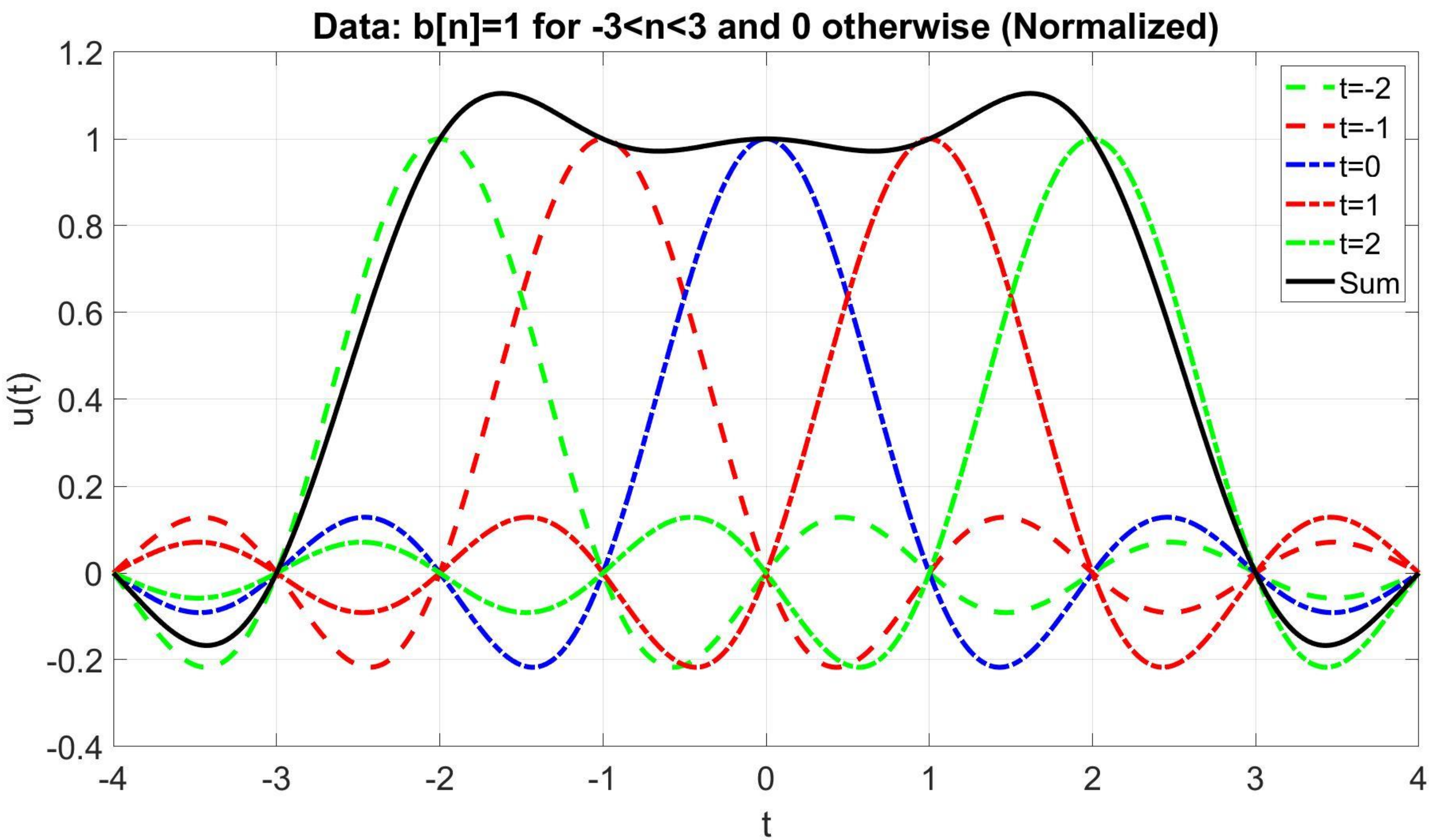
$$S_u(f) = \frac{|P(f)|^2}{T}\sigma_b^2$$

Minimum bandwidth dictated by Sampling
W=1/T

Bandwidth occupancy can be designed based on P(f) and independent of S(f) (FT of s(t)).



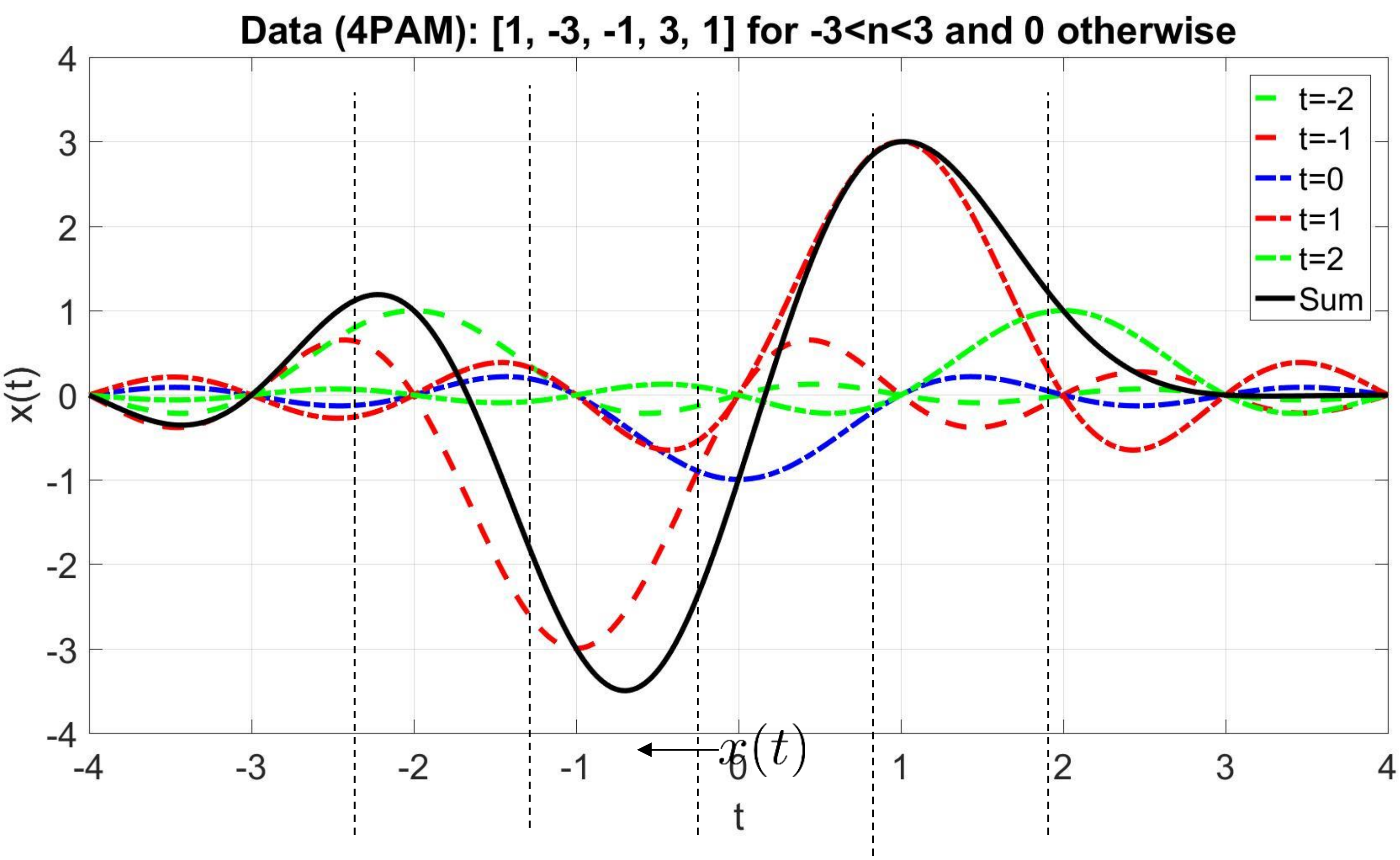
Sum of Sinc Pulses: Ex. 1



$$u(t) = \sum_{n=-\infty}^{\infty} u\left(\frac{n}{W}\right)\mathrm{sinc}\left(t - \frac{n}{W}\right) = \sum_{n=-\infty}^{\infty} b[n]\mathrm{sinc}\left(t - \frac{n}{W}\right)$$

- Normalized W=1=1/T
- *p(t)* is sinc function

Signal as Sum of Sinc Pulses: Ex. 2



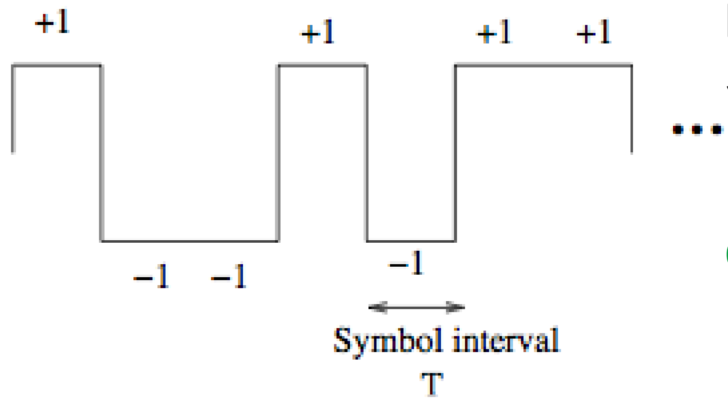
$$u(t) = \sum_{n=-\infty}^{\infty} u\left(\frac{n}{W}\right)\text{sinc}\left(t - \frac{n}{W}\right) = \sum_{n=-\infty}^{\infty} b[n]\text{sinc}\left(t - \frac{n}{W}\right)$$

- Normalized W=1=1/T
- *p(t)* is sinc function

What is Inter Symbol Interference (ISI)?

Time Domain *p(t)*

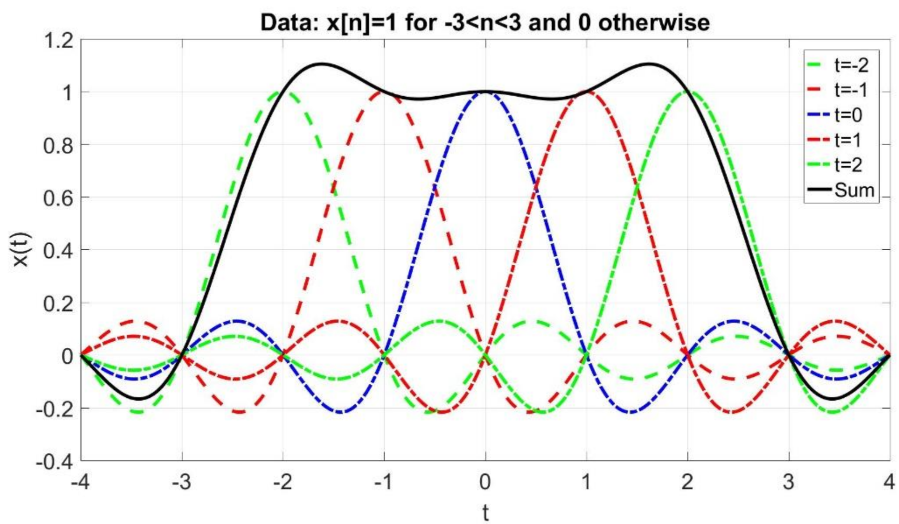
Rectangular Pulse



Freq. Domain *P(f)*
Sinc

No ISI=> SmallTiming offset does not cause issues

Sinc Pulse



Freq. Domain *P(f)*
Rect. Pulse

ISI=> SmallTiming offset does cause issues

No ISI at sampling instances though!

Nyquist Criterion for ISI avoidance

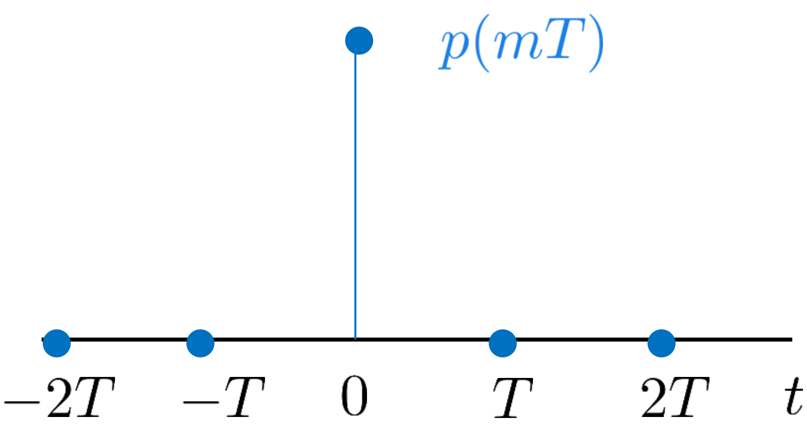
- The pulse $p(t) \leftrightarrow P(f)$ is Nyquist for sampling rate $1/T$ if

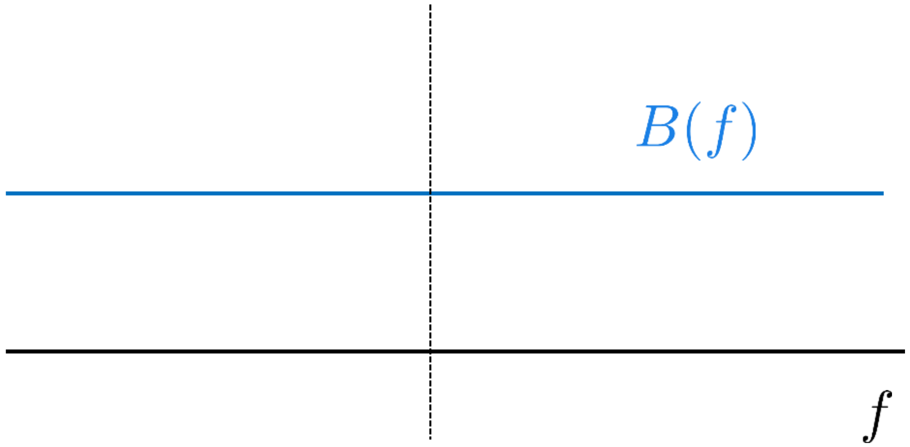
$$p(mT) = \delta_{m0} = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases}$$

or equivalently

DT Fourier Transform Pair

$$B(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} P\left(f + \frac{k}{T}\right) = 1 \quad \forall f$$





Nyquist Criterion for ISI avoidance

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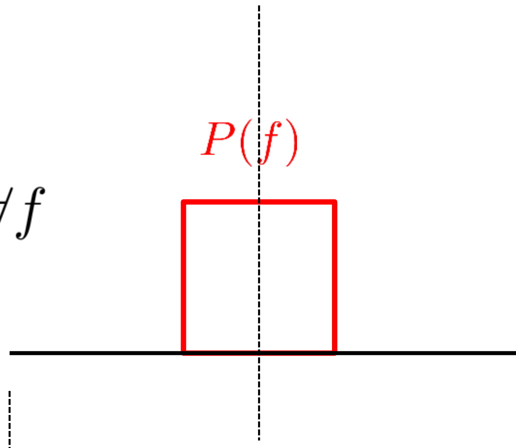
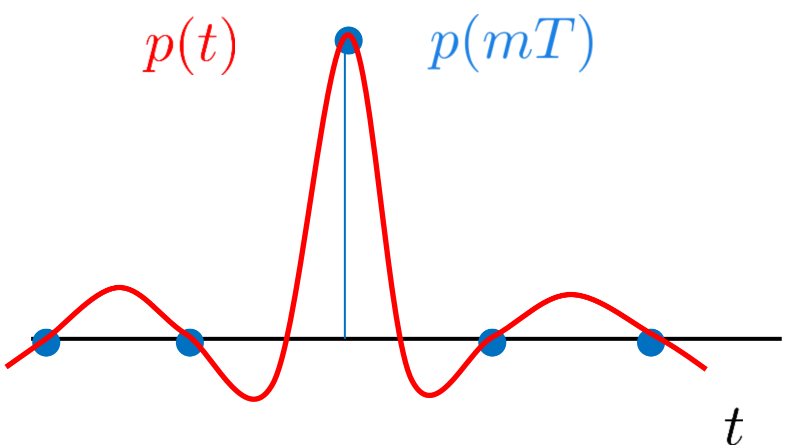
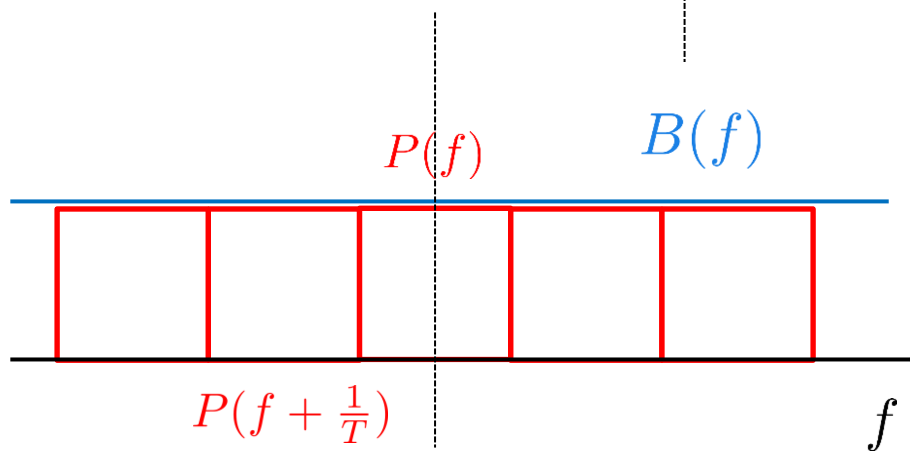
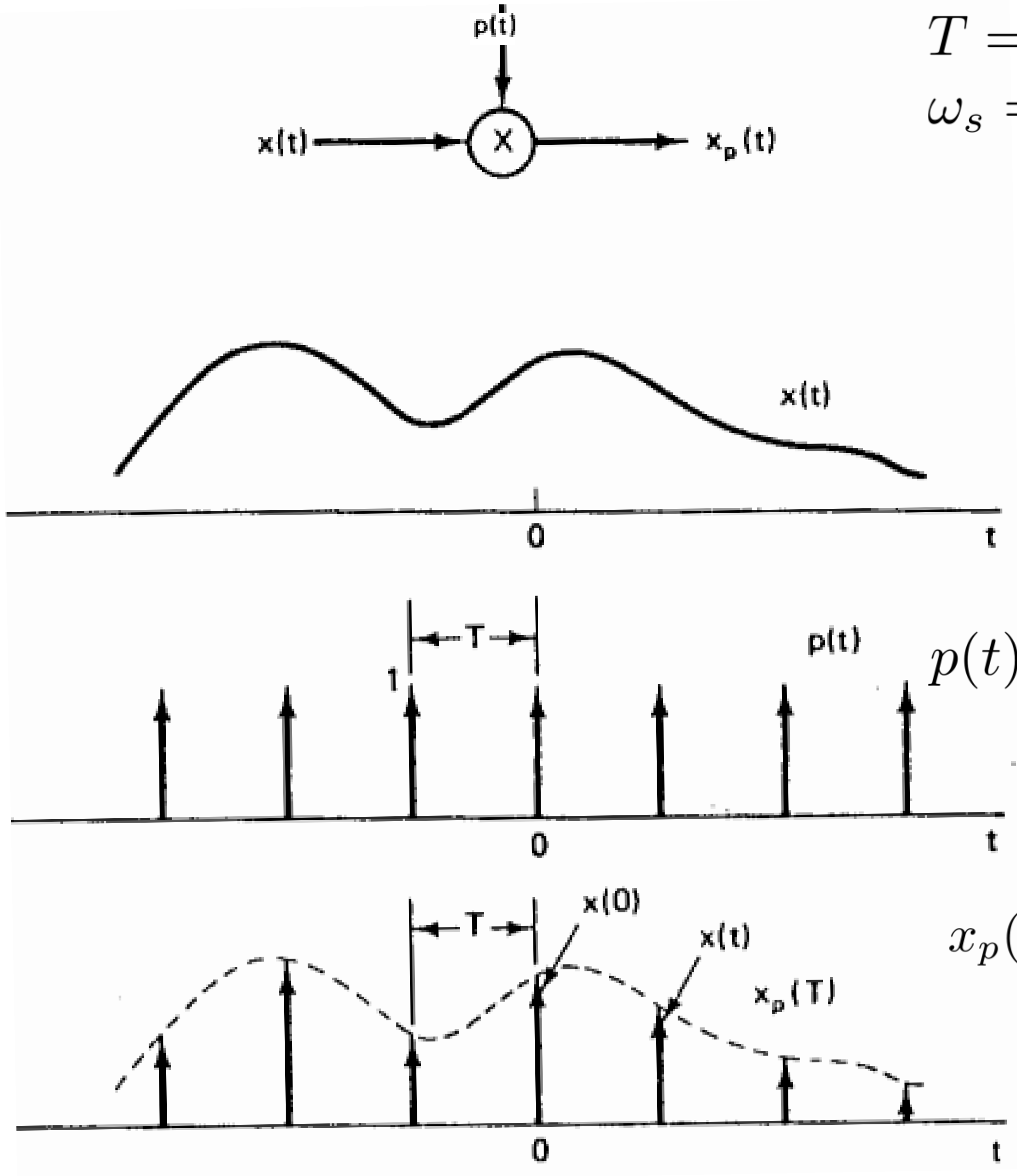






S&S Recap: *Impulse Train Sampling (Time)*



T = sampling time

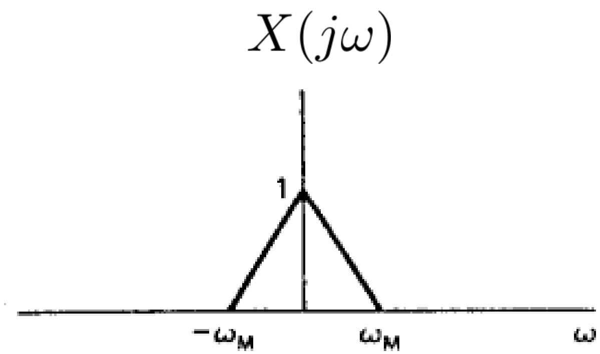
$\omega_s = 2\pi/T$ = sampling frequency

$$p(t) = \textstyle\sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT)$$

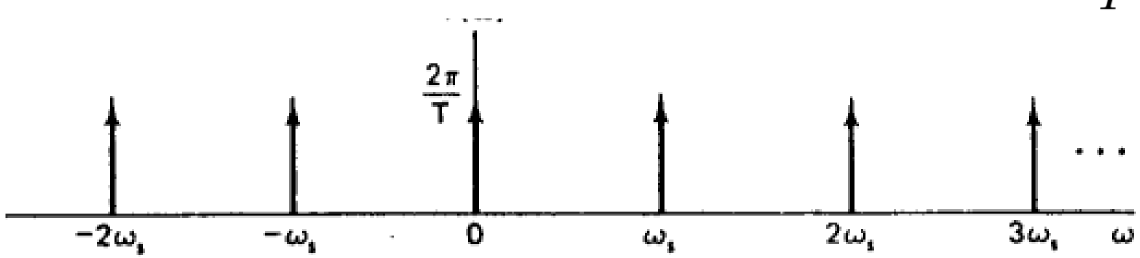
$$= \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

S&S Recap: *Impulse Train Sampling (Freq.)*



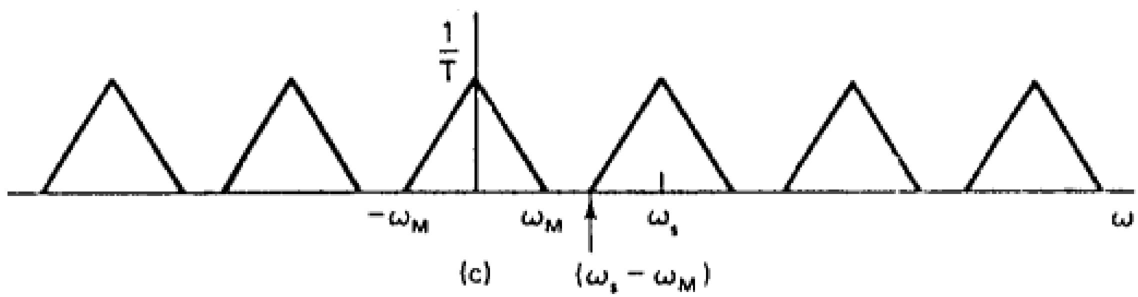
(a)

$$P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{k=\infty} \delta(\omega - k\omega_s)$$



(b)

$$\begin{aligned} X_p(j\omega) &= \frac{1}{2\pi} X(j\omega) * P(j\omega) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} P(j\theta) X(j(w-\theta))\, d\theta \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \end{aligned}$$



(c)

Today's Class

Theorem 4.5.1: Sampling

- Theorem (Sampling): Consider a signal $s(t)$, sampled at rate $1/T_s$. Let $S(f)$ denote the spectrum of $s(t)$, and let

$$B(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} S(f + \frac{k}{T_s})$$

denotes the sum of translates of the spectrum. Then the following observations hold

1. $B(f)$ is periodic with period $1/T_s$.
2. The samples $s(nT_s)$ are Fourier series for $B(f)$, satisfying

$$s(nT_s) = T_s \int_{-\frac{1}{2T_s}}^{\frac{1}{2T_s}} B(f) e^{j2\pi f n T_s} df$$

$$B(f) = \sum_{n=-\infty}^{\infty} s(nT_s) e^{-j2\pi f n T_s}$$

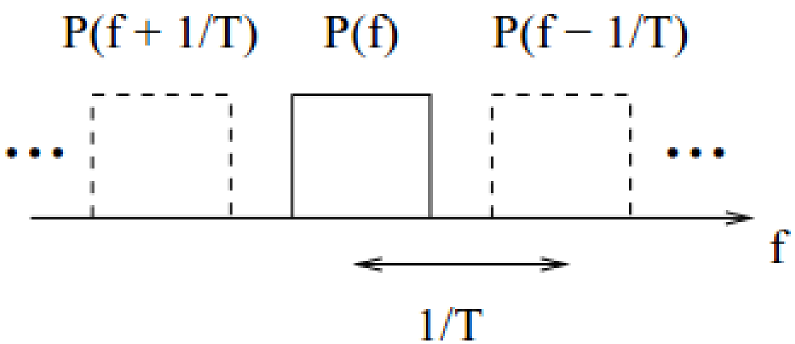
Significance of Nyquist Criterion for ISI avoidance

- It provides freedom to expand the modulation in time beyond the symbol duration so that bandwidth containment is better in frequency domain while ensuring that there is no ISI at appropriately chosen sampling intervals despite the significant overlap between successive pulses.

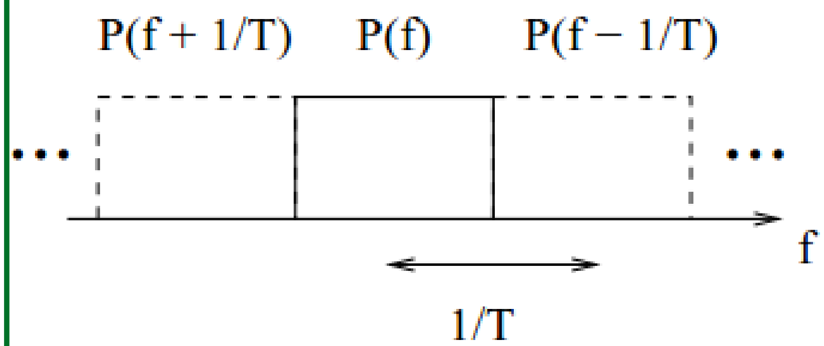
Nyquist Pulses?

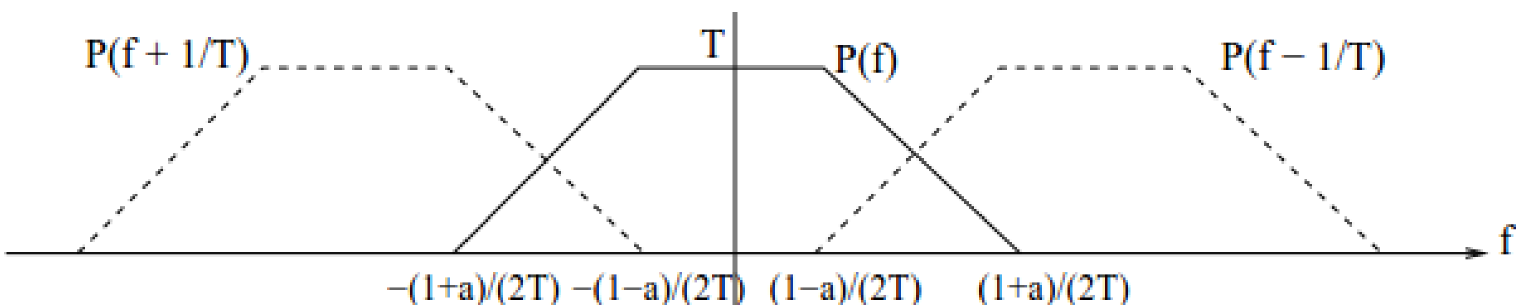
Corresponds to sinc in time domain

Not Nyquist



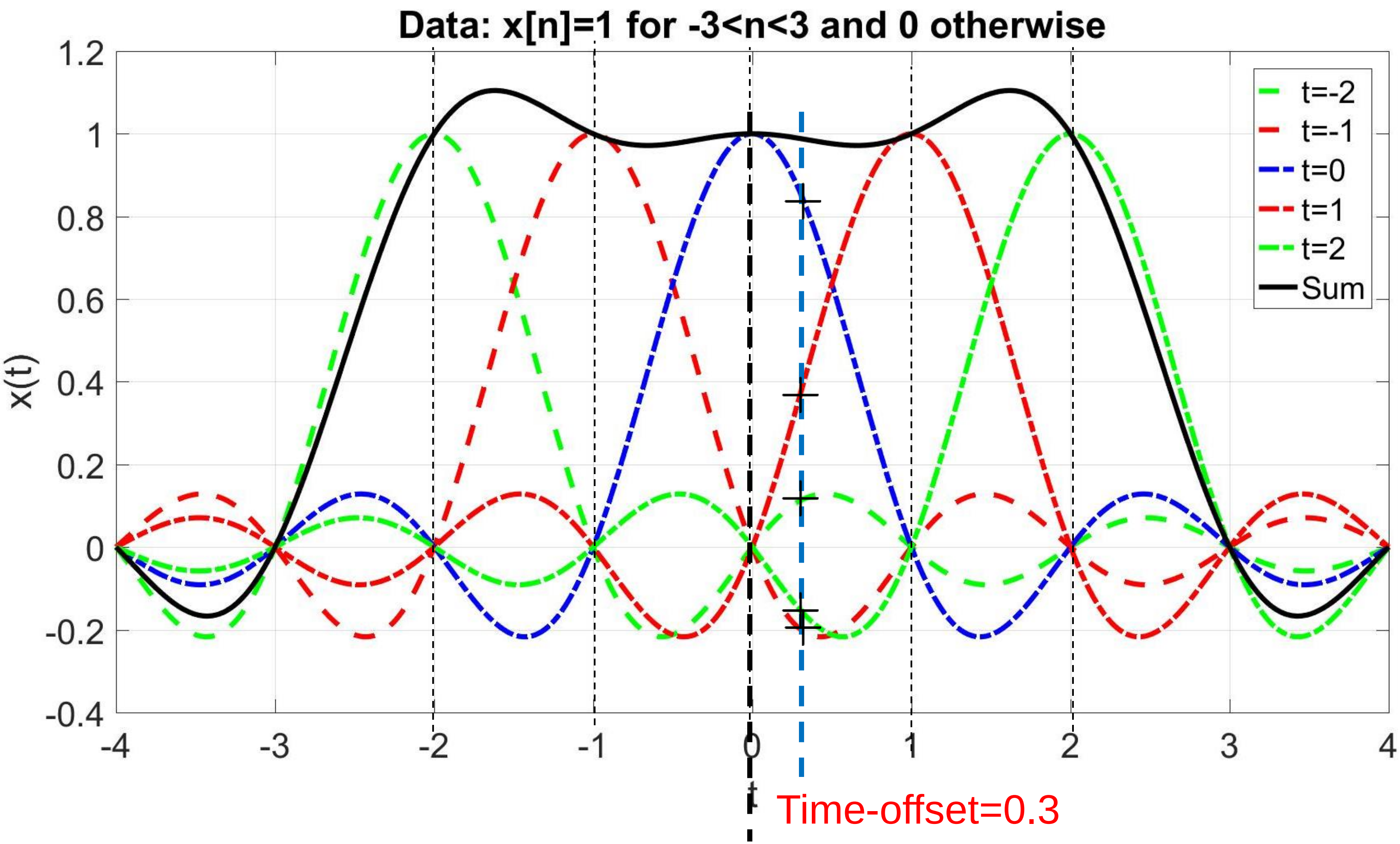
Nyquist pulse with
minimum bandwidth 1/T





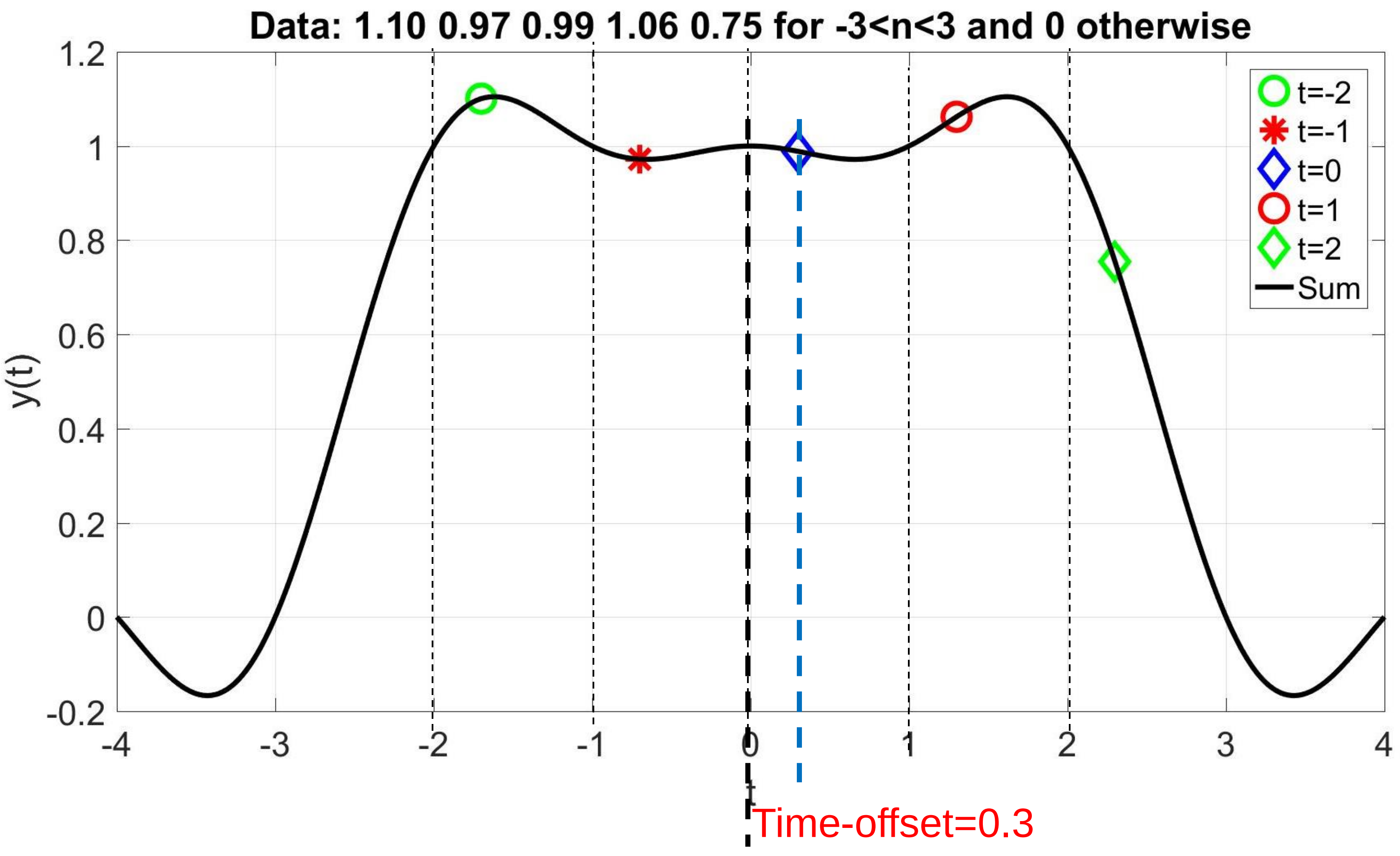
Nyquist

Problem With Sinc Pulse: *ISI*



- Sinc pulse decays as $1/|t|$ and the divergence of harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ implies significant interference from distant symbols.

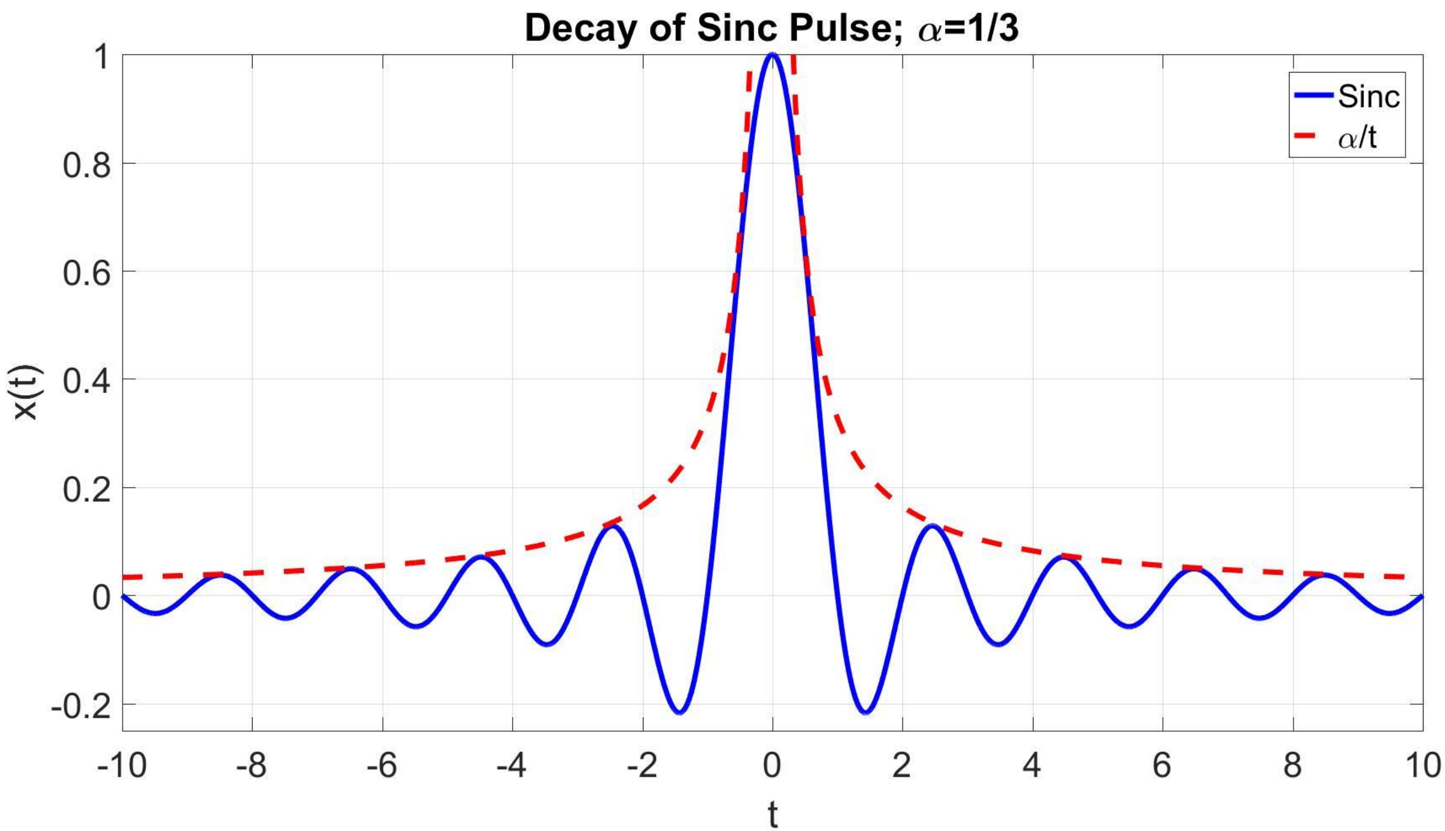
Problem with Sinc Pulse



Data: All ones were sent for -3<n<3 and 0 otherwise.

Here interference considered only from neighbouring 4 pulses

Problems with Sinc Pulse: *Slow Decay*



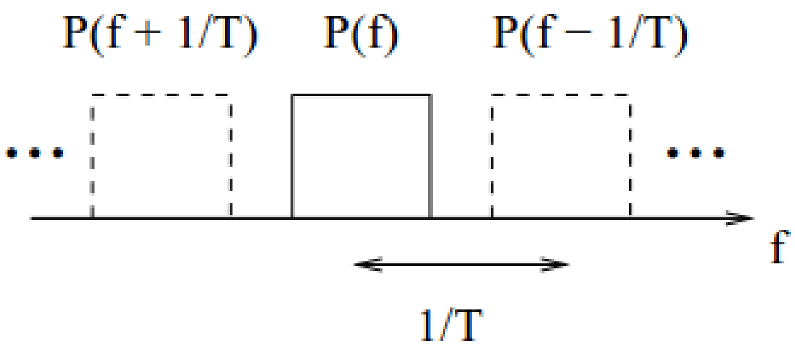
- Sinc pulse is given by

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}.$$

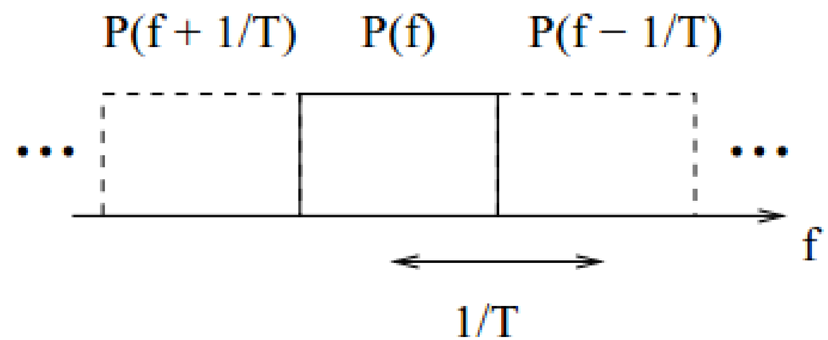
- Sinc pulse decays as $1/|t|$.

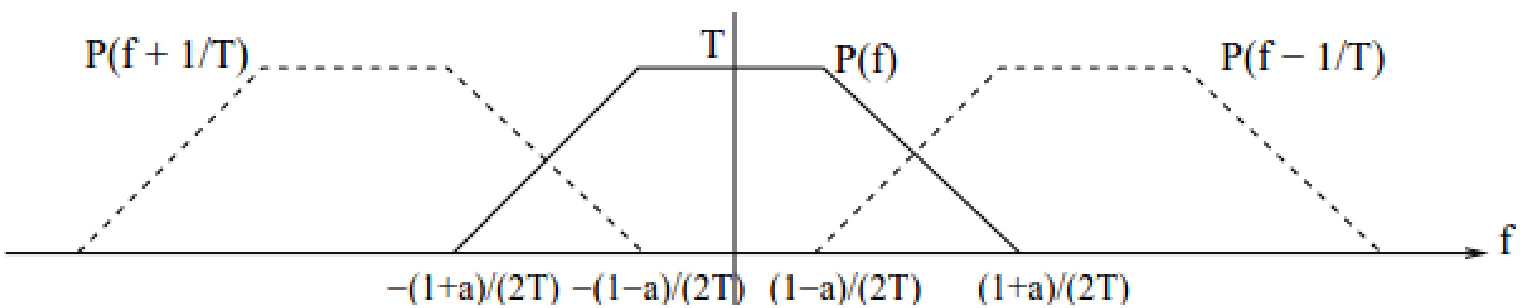
Nyquist Pulses: Excess Bandwidth

Not Nyquist



Nyquist pulse with minimum bandwidth





Nyquist pulse with excess bandwidth

Need of Excess Bandwidth!

- Sinc pulse decays as $1/|t|$ and the divergence of harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ implies significant interference from distant symbols.

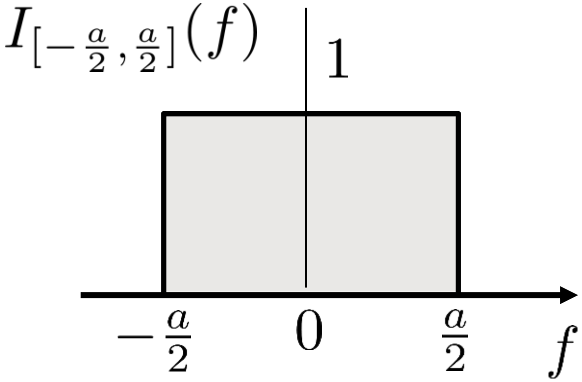
- However a pulse decaying as $1/|t|^b$ with $b > 1$ should work as $\sum_{n=1}^{\infty} \frac{1}{n^b}$ converges for $b > 1$.

- A faster decay in time requires slower decay in frequency. Thus we need excess bandwidth.

- Excess bandwidth is defined as the fraction of the bandwidth over the minimum required for ISI avoidance at a given symbol rate.

Trapezoidal Pulse

- Trapeziodal pulse is convolution of two rectangular pulses!



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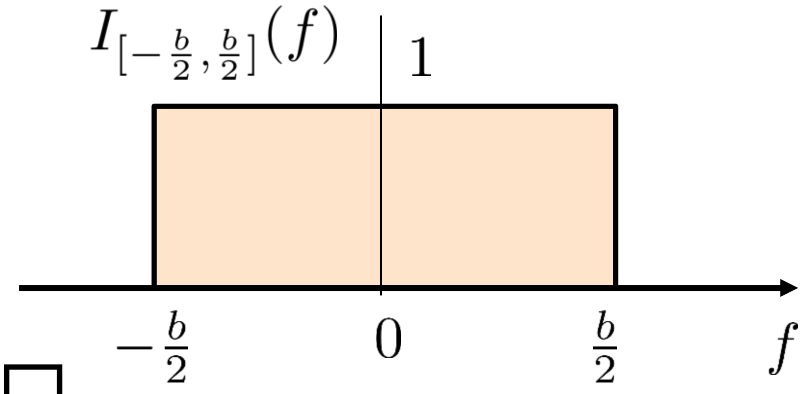
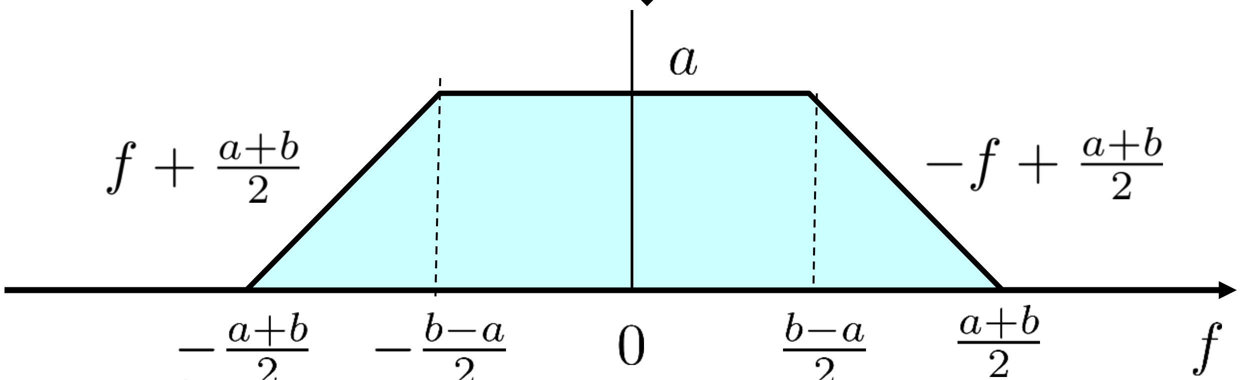


$I_{[-\frac{a}{2},\frac{a}{2}]}(f) \leftrightarrow a\,\mathrm{sinc}(at)$

Assumption: $b \geq a$



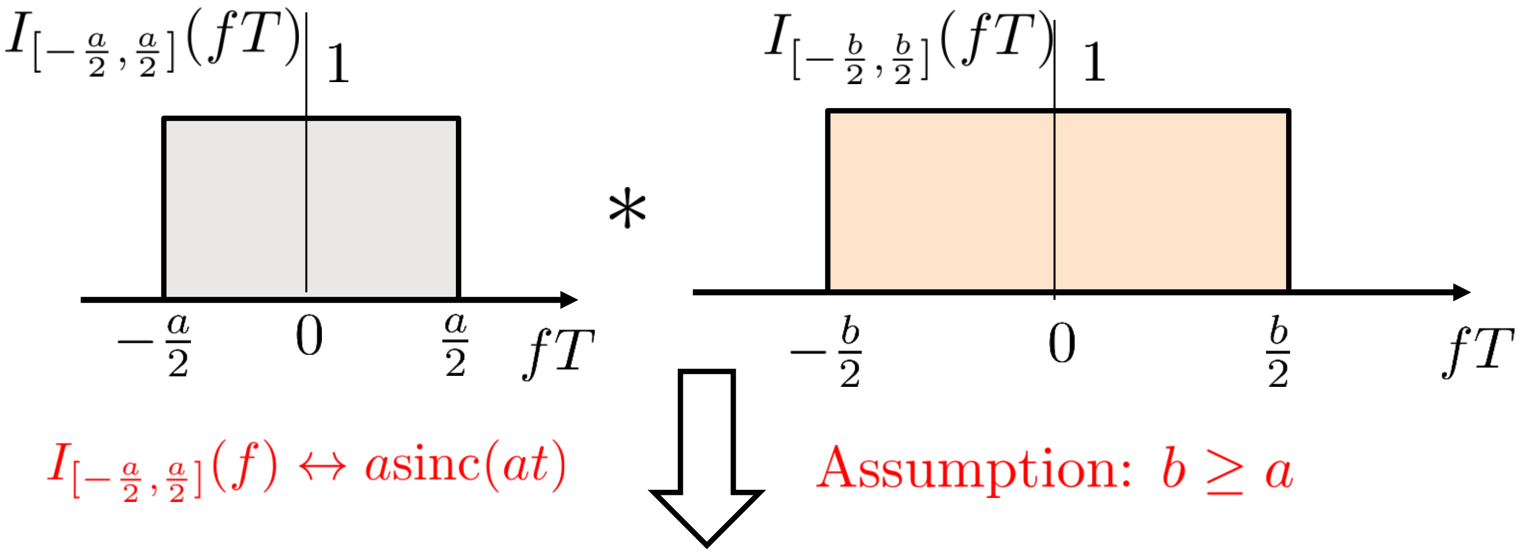
X(f) Y(f)

Z(f) $I_{[-\frac{a}{2},\frac{a}{2}]}(f) * I_{[-\frac{b}{2},\frac{b}{2}]}(f) \longleftrightarrow a\,b\,\mathrm{sinc}(at)\,\mathrm{sinc}(bt)$

Questions?

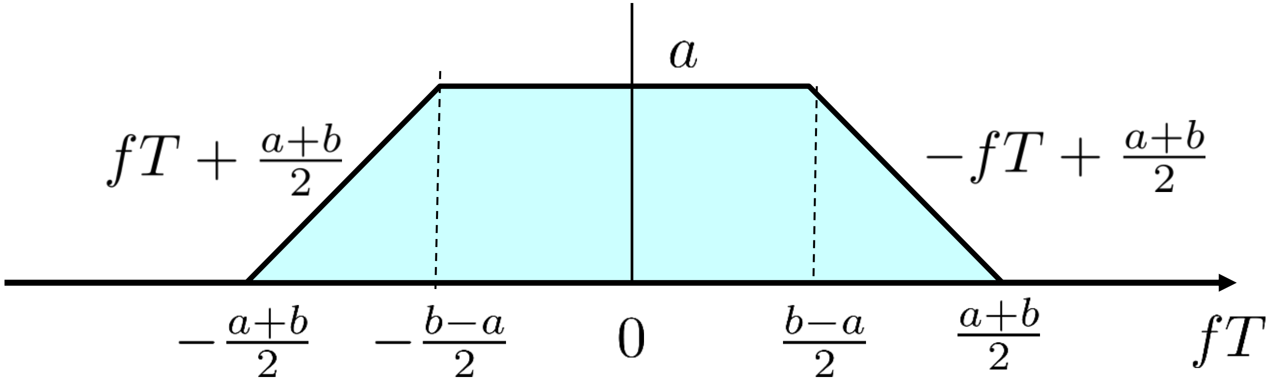
Trapezoidal Pulse: Normalized Freq.

- Trapeziodal pulse is convolution of two rectangular pulses!



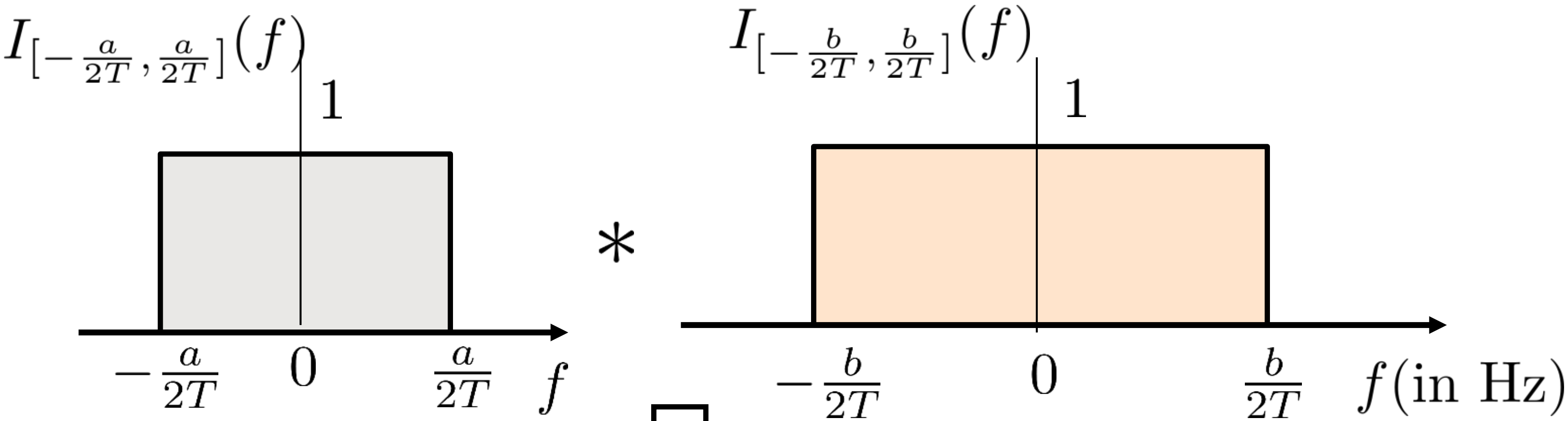
$I_{[-\frac{a}{2},\frac{a}{2}]}(f) \leftrightarrow a\mathrm{sinc}(at)$

Assumption: $b \geq a$

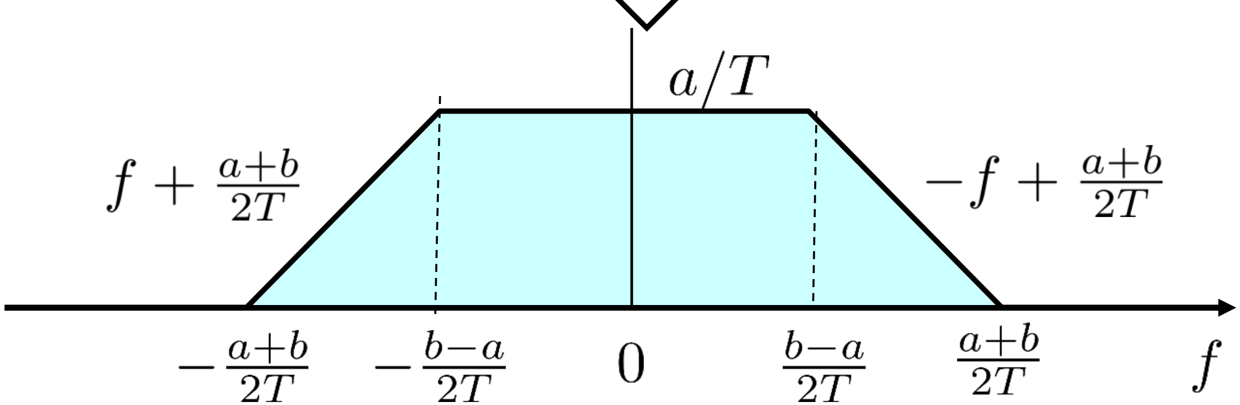


$$I_{[-\frac{a}{2},\frac{a}{2}]}(fT) * I_{[-\frac{b}{2},\frac{b}{2}]}(fT) \longleftrightarrow a\, b\, \mathrm{sinc}(at)\, \mathrm{sinc}(bt)$$

Trapezoidal Pulse: Non-normalized

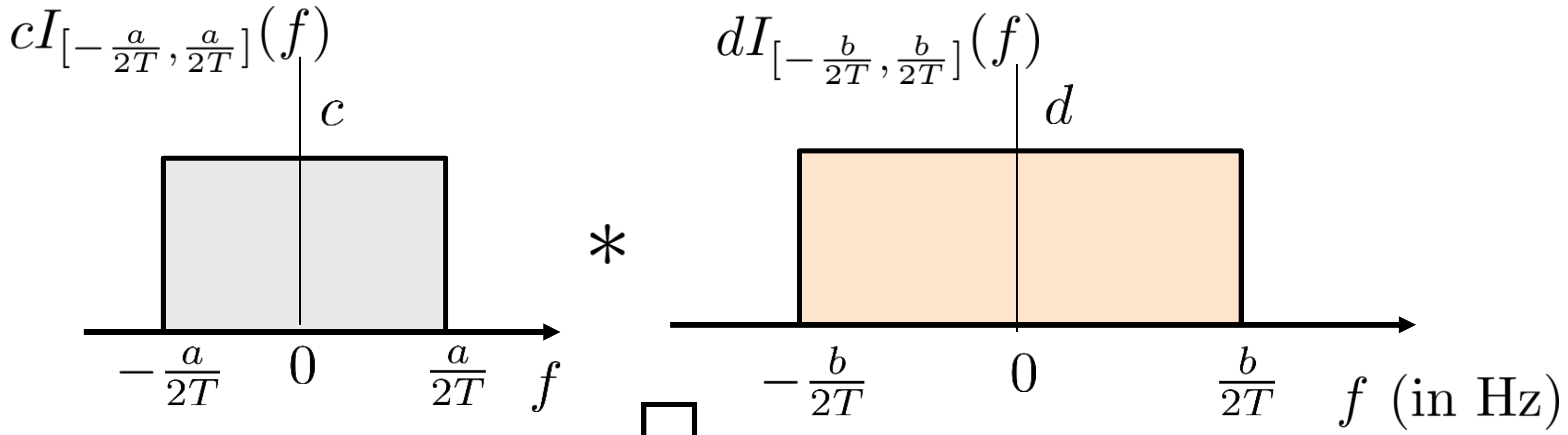


$$I_{[-\frac{a}{2T},\frac{a}{2T}]}(f) \leftrightarrow \frac{a}{T}\mathrm{sinc}\left(\frac{at}{T}\right)$$

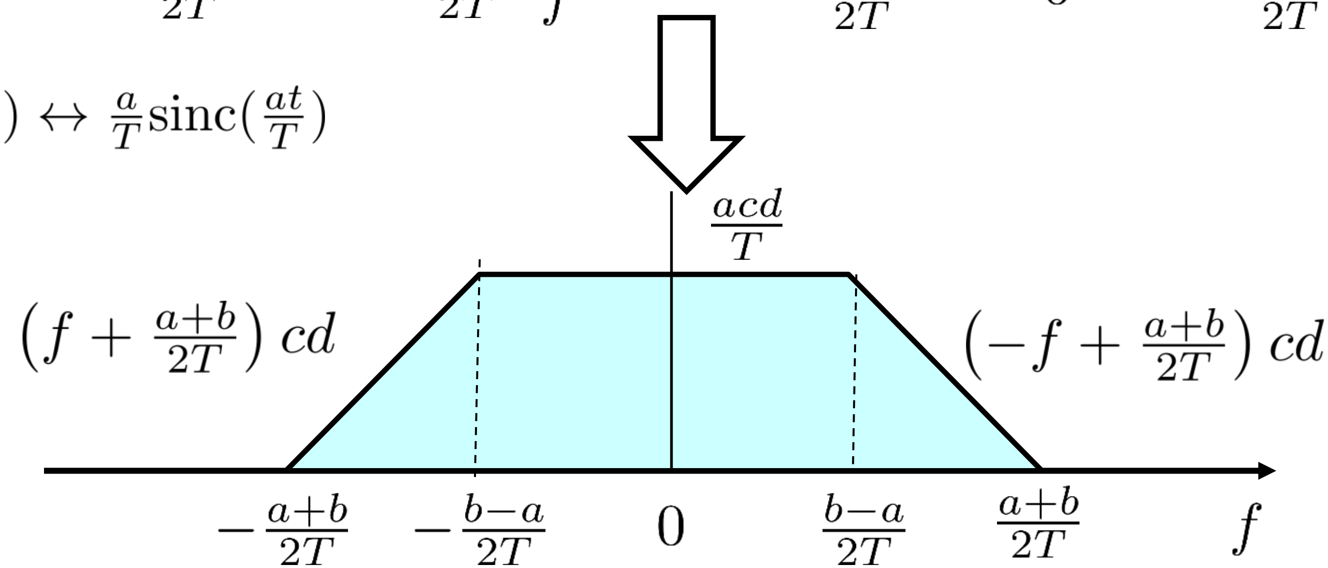


$$I_{[-\frac{a}{2T},\frac{a}{2T}]}(f) * I_{[-\frac{b}{2T},\frac{b}{2T}]}(f) \longleftrightarrow \frac{a\,b}{T^2}\,\mathrm{sinc}\left(\frac{at}{T}\right)\mathrm{sinc}\left(\frac{bt}{T}\right)$$

Trapezoidal Pulse: General Expressions with scaling



$$I_{[-\frac{a}{2T},\frac{a}{2T}]}(f) \leftrightarrow \frac{a}{T}\text{sinc}(\frac{at}{T})$$



$$c\, I_{[-\frac{a}{2T},\frac{a}{2T}]}(f) * d\, I_{[-\frac{b}{2T},\frac{b}{2T}]}(f) \longleftrightarrow \frac{a\,b\,c\,d}{T^2}\,\text{sinc}(\frac{at}{T})\,\text{sinc}(\frac{bt}{T})$$

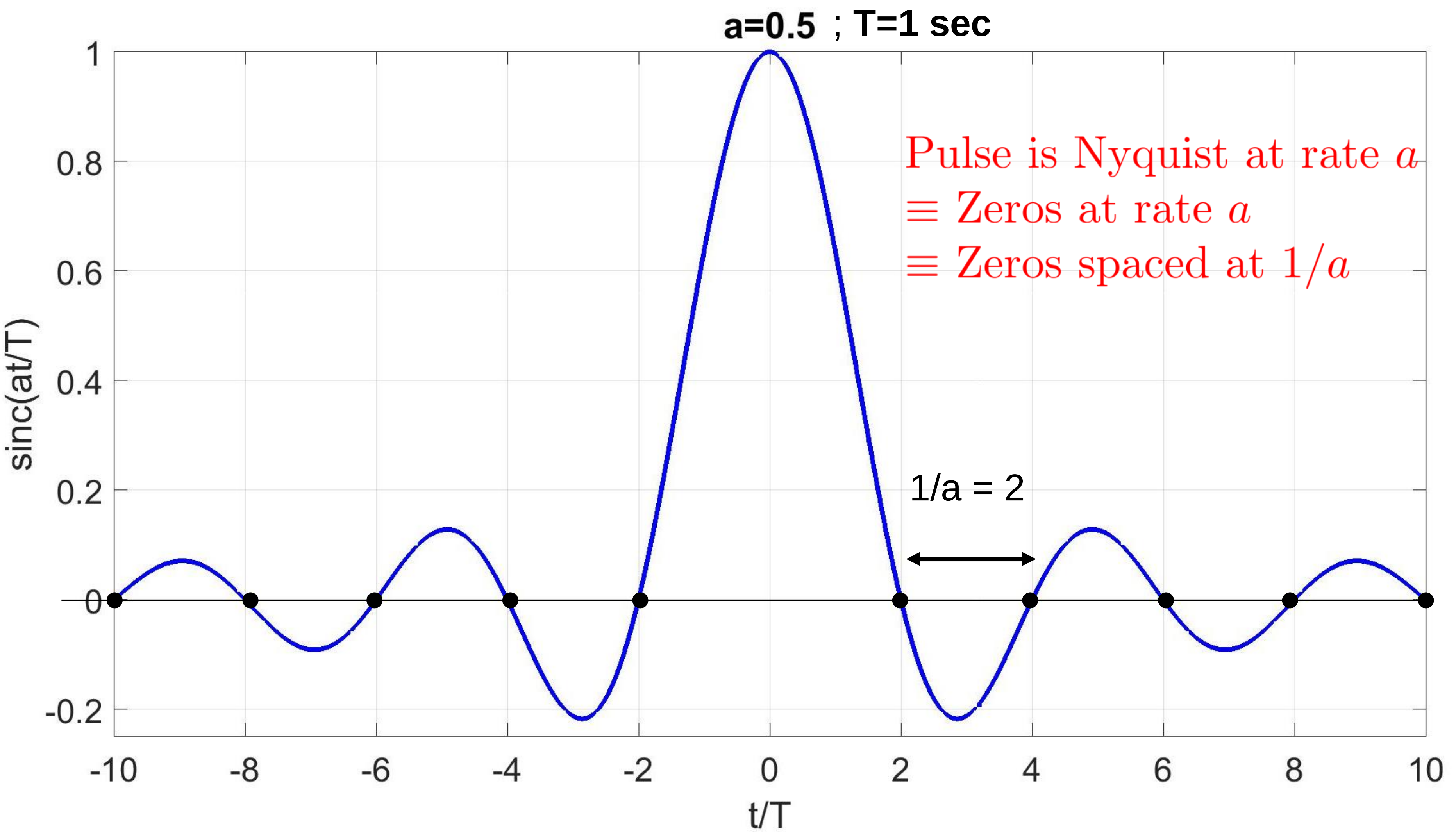
Some Interesting Properties of Nyquist Pulses

- For trapezoidal and the raised-cosine waveforms, the time-domain pulse has a sinc(at) term that provides zeros at the integer multiples of $1/a$. This means that pulse is Nyquist at rate a. In other words, a time domain factor that provides *zeros at rate a* enables Nyquist signaling at rate a.

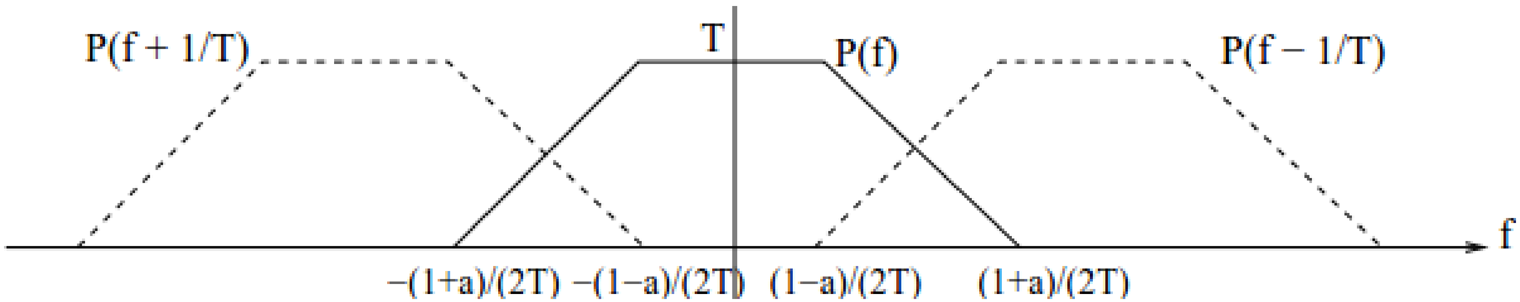
- A pulse that is trapedoizal has a time-domain pulse of the form sinc(at)sinc(bt), which provides zeros at rate a and b. Thus this is Nyquist at rate a and rate b.

- Once we have zeros at integer multiples of T, we also have zeros at integer multiples of KT where K is any positive integer. In other words, if a pulse is Nyquist at rate $1/T$, then it is also Nyquist at integer submultiples of this rate, i.e., $1/KT$.

Sinc Pulse and Nyquist Rate

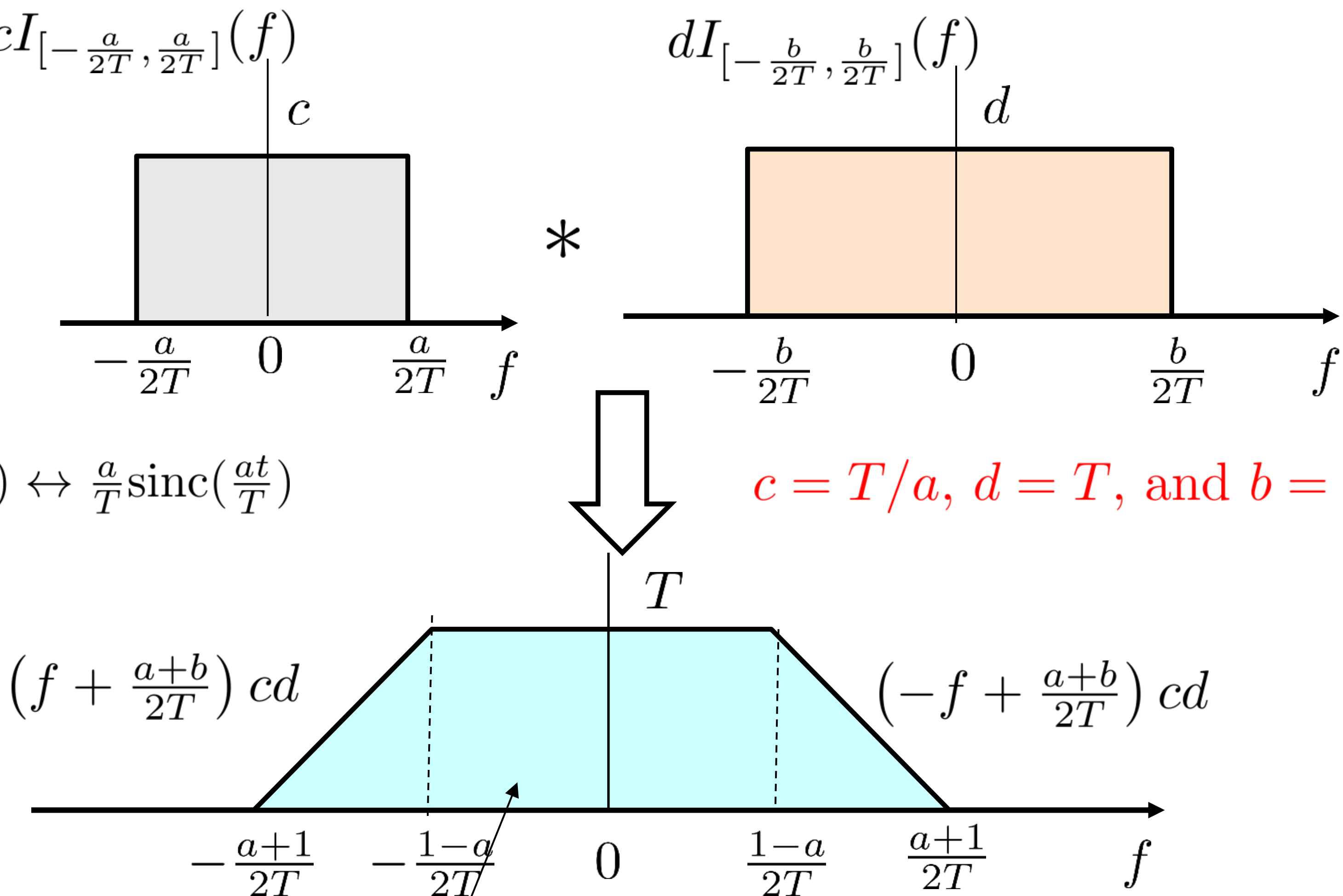


Example: Tutorial



- Consider the trapezoidal pulse of excess bandwidth a shown in figure above.
  - Find an explicit expression for the time-domain pulse $p(t)$.
  - What is the bandwidth required for a passband system using this pulse operating at 120 Mbps using 64 QAM with an excess bandwidth of 25%?

Trapezoidal Pulse: Example 4.1



$$I_{[-\frac{a}{2T},\frac{a}{2T}]}(f) \leftrightarrow \frac{a}{T}\mathrm{sinc}\left(\frac{at}{T}\right)$$

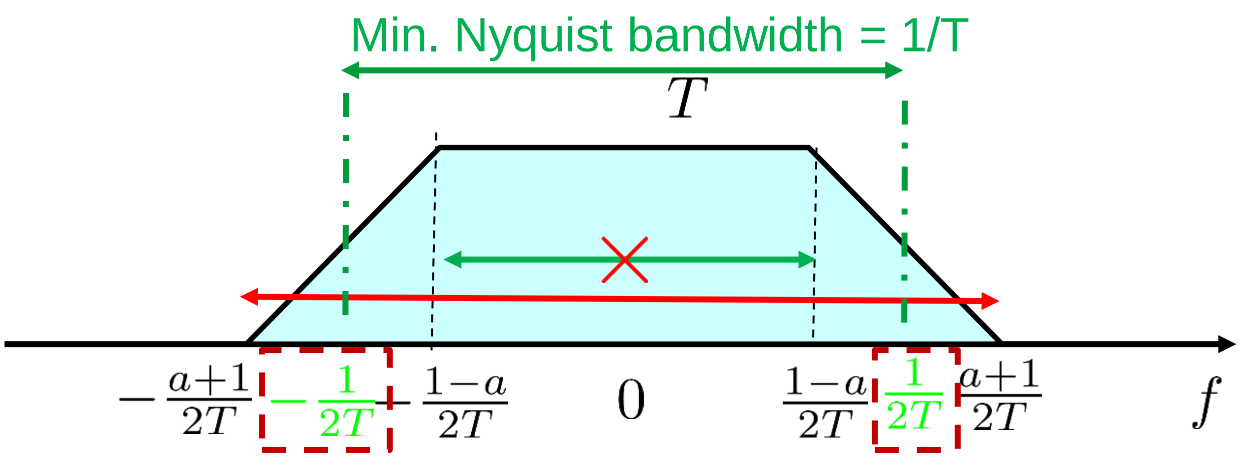
$c = T/a$, $d = T$, and $b = 1$

$$\frac{T^2}{a}\, I_{[-\frac{a}{2T},\frac{a}{2T}]}(f) * I_{[-\frac{1}{2T},\frac{1}{2T}]}(f) \longleftrightarrow \boxed{\mathrm{sinc}\left(\frac{at}{T}\right)\mathrm{sinc}\left(\frac{t}{T}\right)}$$

Trapezoidal Pulse: Example 4.1

- Given bit rate is 120 Mbps and M=64 corresponding to to 64 QAM.
- Symbol rate is given by

$$\frac{1}{T} = \frac{\text{Bit Rate}}{\log_2 M} = \frac{120}{6} = 20 \text{ MHz}$$



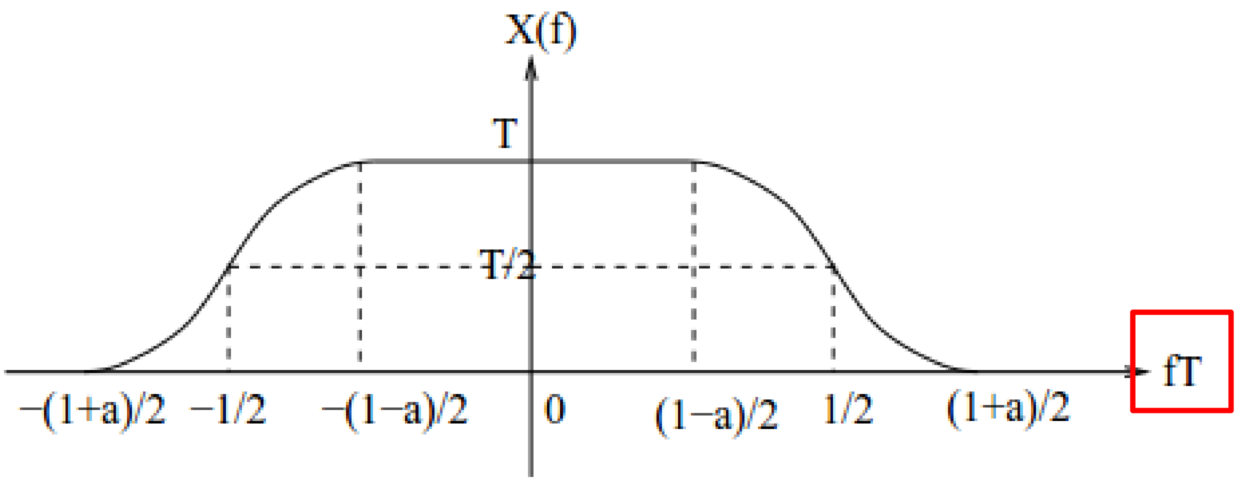
Excessive Nyquist bandwidth $= \frac{1+a}{T}$ with a being fraction of excessive bandwidth, i.e., $a < 1$

- With excess bandwidth of 25%,i.e., $a = 0.25$, we need 1.25*20 = 25 MHz.

Raised-cosine pulse: *Freq. Domain*

- Raise cosine pulse, which has a decay rate of $1/t^3$ in time domain, is given by

$$P(f) = \begin{cases} T, & |f| \leq \frac{1-a}{2T} \\ \frac{T}{2}\left(1 + \cos\left(\left(|f| - \frac{1-a}{2T}\right)\frac{\pi T}{a}\right)\right), & \frac{1-a}{2T} \leq |f| \leq \frac{1+a}{2T} \\ 0, & |f| > \frac{1+a}{2T} \end{cases}$$

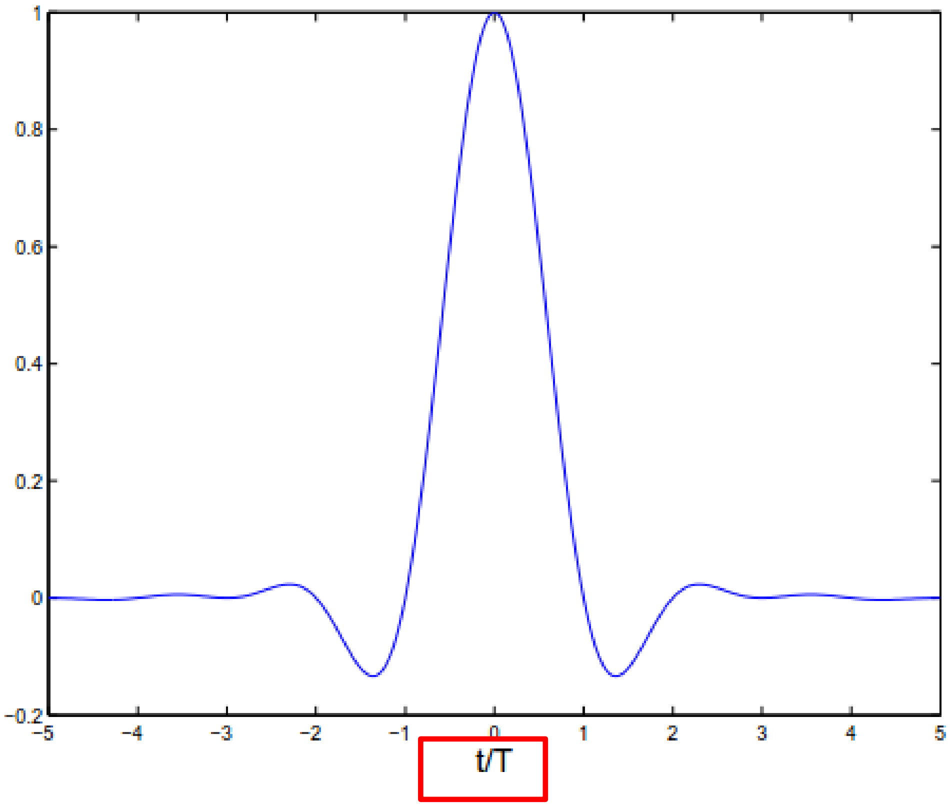


(a) Frequency domain raised cosine

Raised-cosine pulse: *Time Domain*

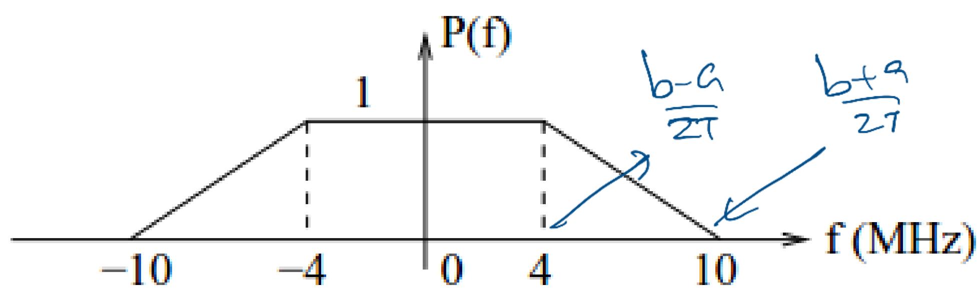
- The time domain pulse $p(t)$ is given by

$$p(t) = \mathrm{sinc}\left(\frac{t}{T}\right)\frac{\cos(\pi a t/T)}{1-(2at/T)^2} \qquad \text{overall decay of } 1/t^3$$



(b) Time domain pulse (excess bandwidth $a = 0.5$)

Example: Tutorial

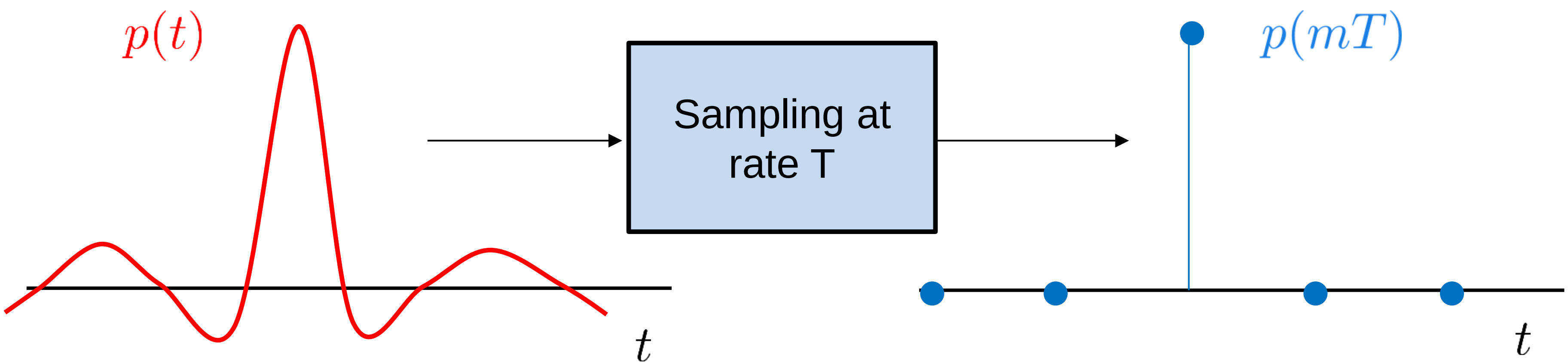


$\frac{b-a}{2T}$ $\frac{b+a}{2T}$

$\frac{acd}{T} = 1 \Rightarrow c = T/a$

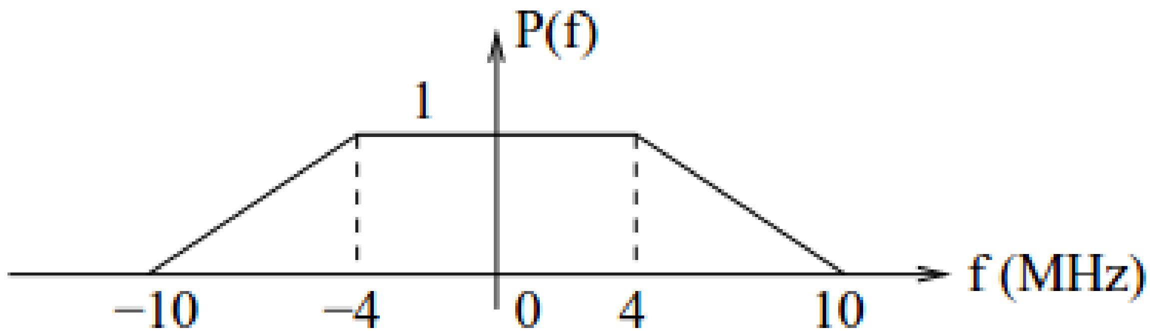
$d = T.$

- Consider passband linear modulation using the bandlimited pulse shown in Fig. above.
- Can the pulse $p(t)$ be used for Nyquist sampling while using bit rate of 56 Mbps and 16 QAM constellation?



- The question is if we sample $p(t)$ at sample rate of $1/T$ or sampling intervals of T, would the resulting sequence $p(mT)$ be Nyquist with $p(mT) = 1$ for $m = 0$ and $p(mT) = 0$ $m \neq 0$.

Example continues: Tutorial



- Can the pulse $p(t)$ be used for Nyquist sampling while using the combination of given bit rate and constellation?
  - 21 Mbps and 8 PSK
  - 18 Mbps and 8 PSK
  - 25 Mbps and QPSK

Bandwidth Efficiency

- The bandwidth efficiency of linear modulation with an $M-$ary alphabet is given by

$$\eta_B = \log_2 M \text{ bits/symbol}$$

- Knowing the bit rate R_b and bandwidth efficiency we can determine symbol rate and hence the minimum required bandwidth $B_{\min}$

$$B_{\min} = \frac{R_b}{\eta_B}$$

- In terms of excessive bandwidth a,

$$B = (1+a)B_{\min} = (1+a)\frac{R_b}{\eta_B}$$

Questions?