Lecture – 2

Gauss Law: → For Symmetrical Charge distribution.

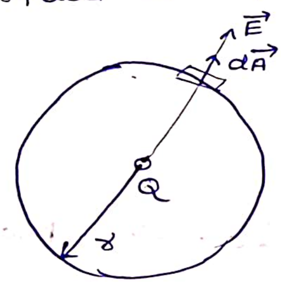
The Electric flux through any closed surface is equal to charge enclosed by permittivity.

Electric flux in an area → The Electric field multiplied by area of surface projected in a plane and perpendicular to it.

$$\oint \vec{E} \cdot \vec{ds} = \frac{1}{\varepsilon_0} \cdot q$$

A positive point charge Q surrounded by an imaginary sphere of radius r.

The total flux through the closed surface of sphere using the equation



$$\phi_E = \oint \vec{E} \cdot \vec{dA}$$

$$= \oint E\, dA \cos\theta$$

$$\theta = 0^\circ \Rightarrow \phi_E = \oint \underbrace{E}_{\text{Constant (Uniform)}} dA$$

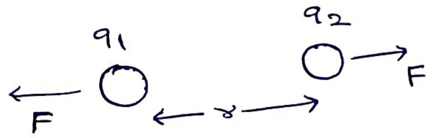
$$= E \oint dA$$

$$= E\, 4\pi r^2$$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

$$\phi_E = \frac{Q}{\varepsilon_0}$$

Coloumb Laws:



$$F = \frac{K q_1 q_2}{r^2}$$

K → Electrostatic constant depends on nature of medium seperating the charges and on system of units.

$$K = \frac{1}{4\pi\varepsilon_0}$$

$$= 9 \times 10^9 \ Nm^2/C^2$$

$$\varepsilon_0 = 8.85 \times 10^{-12}\ C^2/Nm^2$$

↳ Electrical permittivty of free space

- Law applicable only for the point charges at rest.
- Difficult to implement Coulomb's law where charges are in arbitary shape.

: Gauss's Law for Magnetism.

Magnetic monopoles doesn't exist

Total Flux through closed surfaces must be zero.

$$\oint \vec{B} \cdot d\vec{s} = 0.$$

⇒ No of Lines of Magnetic flux enter an enclosed surface will be equal to that of leave an enclosed surface.

→ There are no Sources / Sinks for Magnetic field lines unlike E-Fields.

Faraday's Law:

Faraday's law of Induction states that

Emf induced in a circuit is directly proportional to time rate of change of magnetic flux through circuit.

– Changing Magnetic field induces an electric current in a conductor.

$$\mathcal{E} = -N\frac{d\phi_B}{dt}.$$

Lenz Law

It states that the direction of Electric current induced in a conductor by a changing magnetic field such that magnetic field created by induced current opposes change in initial magnetic field.

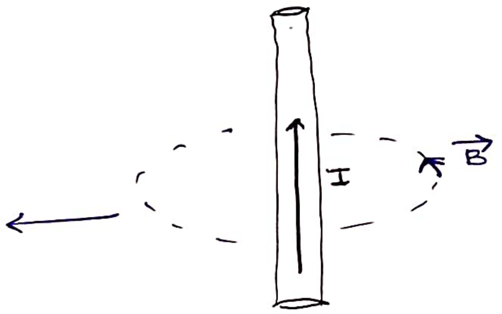
$$\mathcal{E} = -\frac{d\phi_B}{dt}$$

Ampere Law

The magnetic field produced by an electric current is directly proportional to the intensity of electric current and constant of probability (permeability of free space).

The Line Integral of magnetic field surrounding closed loop equal the μ_0 number of times the algebraic sum of current passing through loop.

The line integral of the magnetic field B around any closed loop is μ_0 times the net steady current I enclosed by this path.



$$\oint \vec{B}\cdot\vec{dl} = \mu_0 I$$

$$= \oint B\,dl\,\cancel{\cos\theta}$$

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 I$$

$$\oint B\,dl\cos 0 = \mu_0 I$$

$$B\oint dl = \mu_0 I$$

$$B(2\pi r) = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r}.$$

- Law true for Steady currents.

Maxwell's correction to Ampere Law

- Displacement Current.

As amount of charge on capacitor increase with time. Hence the q charge changes with time produces a current.

$$I_d = dq/dt.$$

For a Capacitor $q = \varepsilon_0 EA \Rightarrow I_d = \frac{dq}{dt} = \varepsilon_0 \frac{d(EA)}{dt}$

$$= \varepsilon_0 \frac{d\phi_E}{dt}$$

Current is produced due to changing electric flux w.r.to time.

(or)

Electric flux is generated due to presence of $\vec{E}$ b/w plates.

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 I_c + \mu_0 \varepsilon_0 \frac{d\phi_E}{dt}$$

Del operator of Electromagnetism.

↳ also known as nabla operator

In Cartesian co-ordinates;

Del operator defined as

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Deloperator on Scalar Field

⇒ Del operator ∇ acts on a scalar function $\phi(x,y,z)$, it produces a vector field, known as gradient.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

physical Meaning

- The gradient points in the direction of the maximum rate of increase of scalar function.
- The magnitude of gradient represents how fast the scalar quantity is changing

Del operator on Vector Field

⇒ Del operator ∇ acts on a vector field, it produces different results depending on how it is applied.

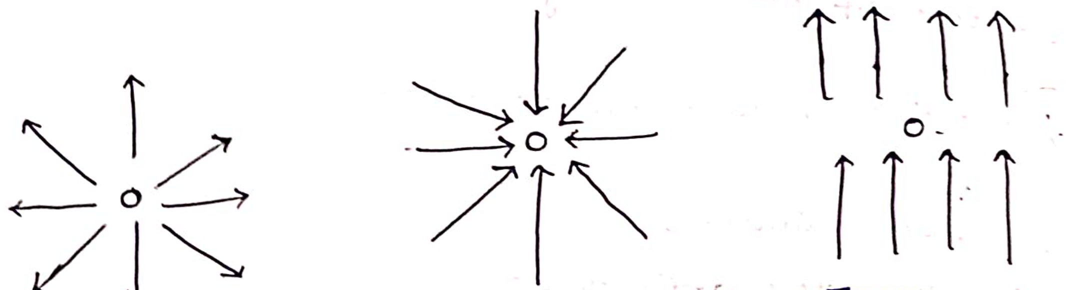
- Divergence $(\nabla \cdot A)$ – Scalar Result

Divergence of a vector field A measures rate of change of field's flux in given region.

$$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

physical Meaning

- It tell us whether a field is spreading out (positive divergence) or converging (negative divergence).
- A positive divergence means source is present.
- A zero divergence indicates the field is solenoidal (no Source)



positive Divergence — Negative Divergence — Zero Divergence.

- Curl ($\nabla \times A$) — Vector Result.
  - Represents rotation of Field

$$\nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

physical Meaning

- The curl measures the tendency of a field to circulate around point.
- A zero curl means field has no rotation.

— Faraday's Law of Induction

$$\nabla \times E = -\frac{dB}{dt}.$$

A changing magnetic field creates an electric field that circulates around it.

Laplacian Operator

For a scalar function $f(x,y,z)$, the Laplacian is defined as

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

In vector notation

$$\nabla^2 f = \underbrace{\Delta}_{\text{Laplace Operator.}} f$$

It tells how much function's value at a point differs from its surroundings.

$\nabla^2 f > 0$ — Local minimum

$\nabla^2 f < 0$ — Local maximum

$\nabla^2 f = 0$ — Equilibrium State.

1 Divergence Theorem Statement — Gauss's Theorem.

The total outward flux of a vector field F through a closed surface S is equal to the volume integral of the divergence of F over the region V enclosed by S.

$$\underbrace{\oint_S F \cdot dS}_{\text{Total Flux leaving closed surface.}} = \underbrace{\int_V \underbrace{\nabla \cdot F}_{\text{divergence.}} \, dV}_{\text{sum of all divergence inside volume}}$$

(sources & Sinks).

$\nabla \cdot F > 0$: more field leaving — source

$\nabla \cdot F < 0$: more field entering. — sink.

1 Strokes Theorem.

Strokes theorem relates the surface integral of the curl of a vector field over a surface to the line integral of the vector field along the boundary of that surface.

$$\oint_C F \cdot dl = \int_S \underbrace{(\nabla \times F)}_{\text{curl of F}} \cdot dS.$$

- The circulation of vector field around a closed curve C is equal to the sums of the curls of the field over the surface S enclosed by C.
- Helps convert a surface integral into line integral

1 Equation of Continuity.

The continuity equation is a fundamental principle in expresses of a quantity in a given system.

The rate of change of a conserved quantitity within a volume is equal to net flux of the quantity across boundary.

1, General Form (Differential Form)

In fluid dynamics and Electromagnetics.

continuity equation is:

Divergence of flux

$$\frac{\partial \rho}{\partial t} + \overline{\nabla \cdot J} = 0 .$$

Density of conserved quantity

mass density, charge density.

J = Flux / current density vector.

Maxwell Equations.

1) Gauss Law.

$$\oint \vec{E} \cdot ds = \frac{Q_{enclosed}}{\epsilon_0}$$

we know that

Volume charge density $\rho = \frac{dq}{dV}$

$$\Rightarrow q = \int_V \rho \, dV$$

$$\therefore \oint \vec{E} \cdot ds = \frac{1}{\epsilon_0} \int_V \rho \, dV \longrightarrow \text{Integral Form of Gauss Law}$$

From Divergence theorem

$$\int_V (\nabla \cdot \vec{A}) dV = \oint \vec{A} \cdot ds$$

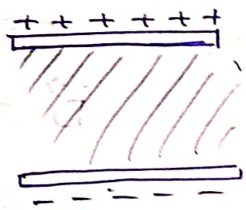
$$\int_V \nabla \cdot \vec{E} \, dV = \oint \vec{E} \cdot ds = \frac{1}{\epsilon_0} \int_V \rho \, dV.$$

This Equation is true for all arbitary volume. So, integral must be equal.

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \longrightarrow \text{Differential form of Gauss Law.}$$

Maxwell first equation in presence of Dielectric medium.

consider a dielectric medium between two parallel charged plates.



Case- 1

No External Field.

The positive & Negative charges in dielectric remains coincident (do not seperate/align).

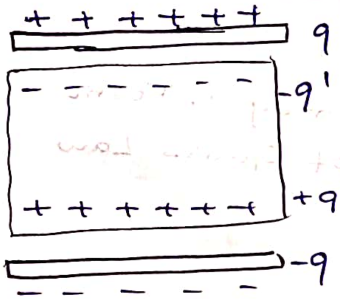
case – 2

~~In presence of External diople Field, those positive and negative charges get displaced to seperate~~

In presence of External Field.

: these positive and negative charges get displaced to some distance that will produce diople

— A induced dipole momen $\vec{P}$.



polarization charge density

$$\rho_{P} = -\nabla\cdot\vec{P}$$

$\downarrow$

polarization vector.

$$\text{total charge density} = \rho_F + \rho_P.$$

Gauss Law

$$\oint \vec{E}\cdot ds = \frac{1}{\varepsilon_0}\int_V (\rho_F + \rho_P)\, dv.$$

$$\int_V \nabla\cdot\vec{E}\, dv = \frac{1}{\varepsilon_0}\int_V (\rho_F - \vec{\nabla}\cdot\vec{P})\, dv$$

$$\varepsilon_0 \int_V \nabla\cdot\vec{E}\, dv = \int_V \rho_F\, dv - \int_V \vec{\nabla}\cdot\vec{P}\, dv$$

$$\int_V \vec{\nabla}(\varepsilon_0\vec{E} + \vec{P})\, dv = \int_V \rho_F\, dv.$$

Electric Displacement vector.

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

$$\int \vec{\nabla}\cdot\vec{D}\, dv = \int_V \rho_F\, dv$$

$$\vec{\nabla}\cdot\vec{D} = \rho_F$$

$\downarrow$

Gauss Law's Differential form in presence of Dieletric medium.

2) Gauss Law in Magnetostatics

$$\phi_B = 0$$

$$\oint_S \vec{B} \cdot ds = 0.$$

Divergence Theorem.

$$\Rightarrow \int_V \nabla \cdot \vec{B}\, dv = 0.$$

to fullfill above condition : $\vec{\nabla} \cdot \vec{B} = 0.$

$\downarrow$

No magnetic monopoles exist.

3) $$\mathcal{E} = -\frac{d\phi_B}{dt}.$$ (Lenz and faraday)

$$\oint_l \vec{E} \cdot dl = -\frac{d}{dt}\int_S \vec{B} \cdot ds.$$

Strokes law.

$$\int_S \nabla \times \vec{A}\, ds = \oint_l \vec{A} \cdot dl.$$

$$\int_S \nabla \times \vec{E}\, ds = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}.$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

- Electric field can ~~be~~ also be generated by time varing magnetic field.

4) Ampere Law.

$$\oint_l \vec{B} \cdot d\vec{l} = \mu_0 I$$

$$= \mu_0 \int_S J\, ds.$$

$$\int_S \vec{\nabla} \times \vec{B}\, ds = \mu_0 \int_S J\, ds.$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}. \longrightarrow 2$$

$$\vec{B} = \mu_0 \vec{H}$$

$$\Rightarrow \vec{\nabla} \times \vec{H} = \vec{J}$$ - good for Steady current only.

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \vec{\nabla} (\mu_0 \vec{J})$$

$\downarrow$

Divergence of Curl = 0.

$$\mu_0 \vec{\nabla} \cdot \vec{J} = 0$$

$$\vec{\nabla} \cdot \vec{J} = 0.$$ — holds good only for steady current.

correction

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}.$$: For time varying field.

From Maxwell first Equation

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\rho = \varepsilon_0 (\vec{\nabla} \cdot \vec{E})$$

$$\nabla \cdot \vec{J} = -\frac{\partial}{\partial t} (\varepsilon_0 \vec{\nabla} \cdot \vec{E})$$

$$\nabla \cdot \Big(\vec{J} + \underbrace{\varepsilon_0 \frac{\partial \vec{E}}{\partial t}}_{\vec{J}_D} \Big) = 0.$$

$\vec{J}_D$: Displacement current density, due to Time varying Electric field.

$$\nabla \times \vec{B} = \mu_0 (\vec{J} + \vec{J}_D)$$

$$= \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

In free space.

$$\vec{D} = \varepsilon_0 \vec{E}.$$

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right)$$

$$\mu_0 \vec{H} = \vec{B}.$$

$$\nabla \times \vec{B} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Other than Lectures
(Extra Stuff).

- Conversion between different Co-ordinates.

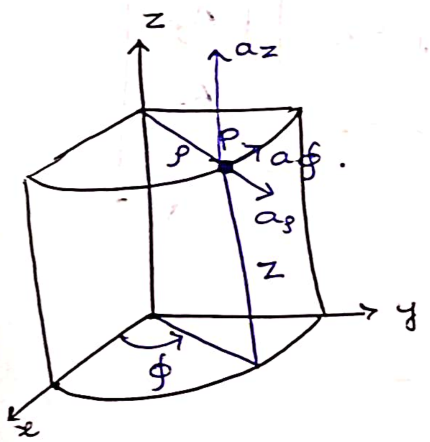
- cartesian co-ordinates (x, y, z).

A vector in cartesian co-ordinate can be written. as.

$$(A_x, A_y, A_z) \text{ or } A_x a_x + A_y a_y + A_z a_z$$

Here a_x, a_y, a_z are unit vectors along x-axis, y-axis, z-axis.

- circular cylindrical Co-ordinates (ρ, ϕ, z).



(A_ρ, A_ϕ, A_z) or
$A_\rho a_\rho + A_\phi a_\phi + A_z a_z$.

Here a_ρ, a_ϕ, a_z are unit vectors. along ρ-, ϕ-, z- direction.

$$\rho = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1}(y/x).$$

$$z = z.$$

$x = \rho\cos\theta \quad y = \rho\sin\theta \quad z = z.$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}.$$

$$a_x = a_\rho \cos\phi - \sin\phi \, a_\phi$$

$$a_y = a_\rho \sin\phi + \cos\phi \, a_\phi$$

$$a_z = a_z$$

$$a_\rho = a_x \cos\phi + \sin\phi \, a_y$$

$$a_\phi = -\sin\phi \, a_x + \cos\phi \, a_y$$

$$a_z = a_z$$

$\cos\phi = \cancel{\rho}$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} a_x \cdot a_\rho & a_x \cdot a_\phi & a_x \cdot a_z \\ a_y \cdot a_\rho & a_y \cdot a_\phi & a_y \cdot a_z \\ a_z \cdot a_\rho & a_z \cdot a_\phi & a_z \cdot a_z \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

- Spherical Co-ordinates. (r, θ, ϕ).

(A_r, A_θ, A_ϕ)

$A_r a_r + A_\theta a_\theta + A_\phi a_\phi$.

Here a_r, a_θ, a_ϕ are unit vectors $r-$, $\theta-$, $\phi-$ directions

$$a_x = a_r \sin\theta \cos\phi + a_\theta \cos\theta \cos\phi - \sin\phi \, a_\phi$$

$$a_y = a_r \sin\theta \sin\phi + \cos\theta \sin\phi \, a_\theta + \cos\phi \, a_\phi$$

$$a_z = \cos\theta \, a_r - \sin\theta \, a_\theta$$

$$a_r = a_x \sin\theta\cos\phi + a_y \sin\theta \sin\phi + \cos\theta \, a_z.$$

$$a_\theta = \cos\theta \cos\phi \, a_x + \cos\theta \sin\phi \, a_y - \sin\theta \, a_z$$

$$a_\phi = -\sin\phi \, a_x + \cos\phi \, a_y.$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi. \\ \cos\theta & -\sin\theta & \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

– Differential Length, Area, Volume.

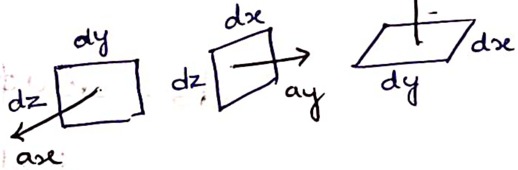
- Cartesian Co-ordinate point.

Differential displacement dl at point S is vector form point $S(x, y, z)$ to point $B(x+dx, y+dy, z+dz)$.

$$dl = dx\, a_x + dy\, a_y + dz\, a_z$$

Differential normal surface area

$$dS = \begin{matrix} dy\,dz\ a_x \\ dx\,dz\ a_y \\ dx\,dy\ a_z \end{matrix}$$



Differential volume

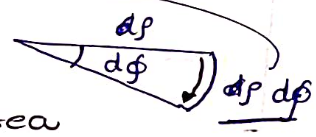
$$dV = dx\,dy\,dz.$$

A scalar.

Cylindrical Co-ordinate Systems.

Differential displacement

$$dl = d\rho\, a_\rho + \rho\, d\phi\, a_\phi + dz\, a_z$$

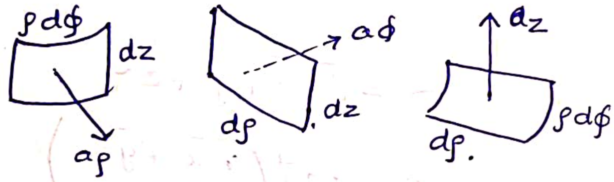


Differential normal Surface area

$$ds = \rho\, d\phi\, dz \quad a_\rho$$

$$= d\rho\, dz \quad a_\phi$$

$$= \rho\, d\rho\, d\phi \quad a_z$$



Differential Volume

$$dV = \rho\, d\rho\, d\phi\, dz$$

Spherical Co-ordinate Systems.

Differential displacement

$$dl = dr\, a_r + r\, d\theta\, a_\theta + r \sin\theta\, d\phi\, a_\phi.$$

$$ds = d\theta\, d\phi\, r^2 \sin\theta \quad a_r$$

$$= dr\, d\phi\, r \sin\theta \quad a_\theta$$

$$= dr\, d\theta\, r\, 6 \quad a_\phi.$$

$$dV = r^2 \sin\theta\, dr\, d\theta\, d\phi.$$