Maxwell's Equations

(1) (a) Gauss's Law States that the total Electric flux over a closed Surface is $\frac{1}{\epsilon_0}$ times the net charge enclosed within the surface. $\boxed{\phi_E = \frac{1}{\epsilon_0} q}$

$$\oint_S \vec{E} \cdot ds = \frac{Q_{enclosed}}{\epsilon_0} \quad \text{— (1)}$$

Volume Charge density

$$\rho = \frac{dq}{dv}$$

$$q = \int_V \rho \, dv$$

$$\oint_S \vec{E} \cdot ds = \frac{1}{\epsilon_0} \int_V \rho \, dv \quad \text{— (2)}$$

From Divergence theorem

$$\left[\int_V \nabla \cdot \vec{A} \, dv = \oint_S \vec{A} \, ds \right]$$

From Equation (2) using Divergence theorem

$$\int_V \nabla \cdot \vec{E} \, dv = \frac{1}{\epsilon_0} \int_V \rho \, dv$$

This Equation is true for all arbitrary volume so the integral must be equal

$$\boxed{\nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \rho} \quad \text{— (3)}$$

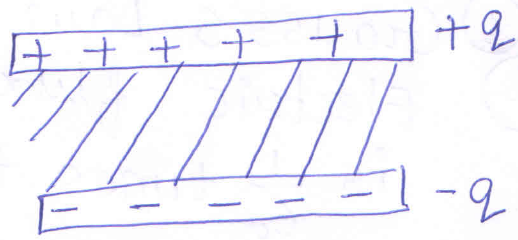
Equation (1) is the integral form of Guass's Law & Equation (2) is the Differential form of Gauss Law.

(2)

3.2

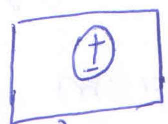
(b) Maxwell First Equation in the presence of Dielectric medium →

Consider a dielectric medium between two parallel charged plates.



These charges are seperated by a distance so that there is potential difference and Corresponding electric field.

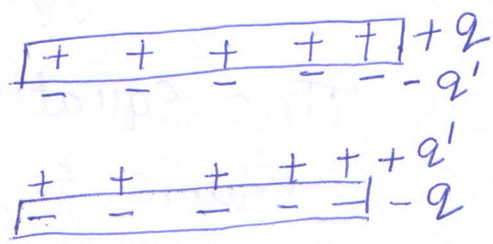
Case 1. In the dielectric medium if there is no external electric field.



$\vec{E} = 0$

The positive and negative charges in the dielectric remain coincident (i.e do not separate or align)

Case 2. In the presence of External Electric field these positive and negative charges get displaced to some distance that will produce the dipoles and therefore a induced dipole moment $(\vec{P})$



Polarization charge density

$$\rho_p = -\vec{\nabla}\cdot\vec{P}$$

$\vec{P}$ = Polarisation vector

$q' \Rightarrow$ bound charges & polarisation charges

Polarisation $\vec{P} = \dfrac{\text{induce dipole moment}}{\text{Volume}}$

$q \rightarrow$ Free charge

$\rho_p \rightarrow$ Polarization charge density

$\rho_F \rightarrow$ Free charge density

Total charge density $= \rho_F + \rho_P$

Now the Guass's law $\rightarrow$

$$\oint_S \vec{E}\cdot ds = \frac{1}{\varepsilon_0}\int_V (\rho_F + \rho_P)\, dv$$

$\Downarrow$ Divergence theorem

$$\int_V \nabla\cdot\vec{E}\, dv = \frac{1}{\varepsilon_0}\int_V (\rho_F - \vec{\nabla}\cdot\vec{P})\, dv$$

$$\varepsilon_0\int_V \vec{\nabla}\cdot\vec{E}\, dv = \int_V \rho_F\, dv - \int_V \vec{\nabla}\cdot\vec{P}\, dv$$

$$\int_V \vec{\nabla}\cdot(\varepsilon_0\vec{E} + \vec{P})\, dv = \int_V \rho_F\, dv$$

Electric Displacement vector

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

$$\int_V (\vec{\nabla}\cdot\vec{D}) = \int_V \rho_F\, dv$$

$$\boxed{\vec{\nabla}\cdot\vec{D} = \rho_F}$$

Guass's law Differential form in the presence of Dielectric medium.

(2) Maxwell's Second Equation: Gauss's law in magnetostatics. $\Rightarrow$ It states that the total magnetic flux over any closed surface is zero. (3.4)

$$\phi_B = 0 \quad \text{—— (1)}$$

$$\oint_S \vec{B}\cdot\vec{ds} = 0 \quad \text{—— (2)}$$

[Using Divergence theorem

$$\int_V \nabla\cdot\vec{A}\, dv = \int_S \vec{A}\cdot ds$$]

$$\int_V (\vec{\nabla}\cdot\vec{B})\, dv = 0$$

$\therefore$ To fullfill the above condition

$$\boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

It shows magnetic monopoles do not exist.

Third Equation:

The induced EMF in any closed loop of wire is equal to the rate of change of magnetic flux Linked with it.

$$\text{Induced EMF } e = -\frac{d\phi_B}{dt}$$

$$\oint_l \vec{E}\cdot dl = -\frac{d}{dt}\int_s \vec{B}\cdot ds$$

[Apply Stoke's Theorem

$$\int_s \nabla \times \vec{A}\, ds = \oint_l \vec{A}\cdot dl$$
]

$$\int_s \vec{\nabla}\times\vec{E}\, \vec{ds} = -\int_s \frac{\partial \vec{B}}{\partial t}\, \vec{ds}$$

$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

It shows "Electric field can also be generated by time varing magnetic field.

Fourth Equation :- Ampere's law ⇒ (3.6)

The Line integral of the magnetic field around any closed path is equal to μ_0 times the current enclosed with in the path.

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I \quad \text{--- (1)}$$

$$= \mu_0 \int_S J \, ds$$

Apply Stoke's theorem

$$\int_S \vec{\nabla} \times \vec{B} \, ds = \mu_0 \int_S J \, ds$$

It shows

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}} \quad \text{--- (2)}$$

$$\vec{B} = \mu_0 \vec{H}$$

$$\boxed{\nabla \times \vec{H} = \vec{J}} \quad \text{--- (3)}$$

hold good for steady current only

from (2) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Taking divergence

$$\vec{\nabla} \cdot (\nabla \times \vec{B}) = \vec{\nabla} \cdot (\mu_0 \vec{J})$$

[Div of curl = 0]

$$0 = \mu_0 (\vec{\nabla} \cdot \vec{J})$$

$$\boxed{\vec{\nabla} \cdot \vec{J} = 0}$$ hold good only for Steady current

We need a correction in this equation for (3.7) time varying field

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

from maxwell's first Equation

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\rho = \varepsilon_0 (\vec{\nabla} \cdot \vec{E})$$

$$\nabla \cdot \vec{J} = -\frac{\partial}{\partial t}(\varepsilon_0 \vec{\nabla} \cdot \vec{E})$$

$$\nabla \cdot \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = 0$$

$$\varepsilon_0 \frac{\partial \vec{E}}{\partial t} = J_D$$

Displacement current density due to time varying electric field

Then

$$\nabla \times \vec{B} = \mu_0 (J + J_D)$$

$$\boxed{\nabla \times \vec{B} = \mu_0 \left(J + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)}$$

$$\boxed{\nabla \times \vec{B} = \mu_0 \left[J + \frac{\partial \vec{D}}{\partial t}\right]}$$

in free space $\vec{D} = \varepsilon_0 \vec{E}$

D = Electric displacement vector

$$B = \mu_0 H$$

$$\boxed{\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$