EC5.203 Communication Theory (3-1-0-4)

Lectures 16 and 17:
Noise Modelling,
And
Linear Operations on Random Process

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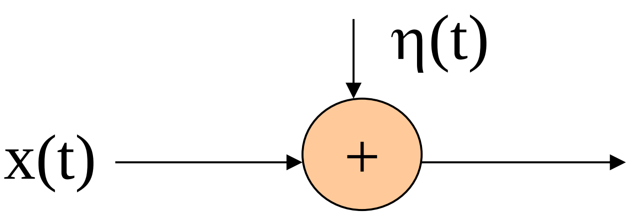
References

- Chap. 5 (Madhow)
  - Sections 5.1-5.7: Probability theory and Random Process (Self-Study, Not part of syllabus)
  - Section 5.8: Noise Modeling
  - Section 5.9: Linear operation on Random Processes

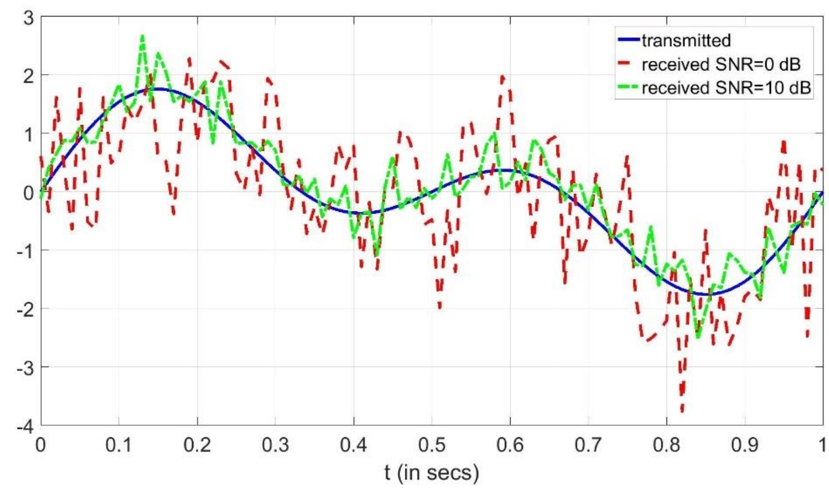
Noise Modeling

Noise in Communication Systems

- Noise is any undesired signal corrupting the desired signal
- It can be man-made or naturally occuring
  - Naturally occuring: thermal noise and shot noise in the receiver, EMI, atmospheric noise, cosmic noise
  - Man-made: Microwave, power-line interference, electric motors, ignition systems, interference from other signal in same band
- Mostly thermal noise is dominant!
- Generally additive nature is assumed

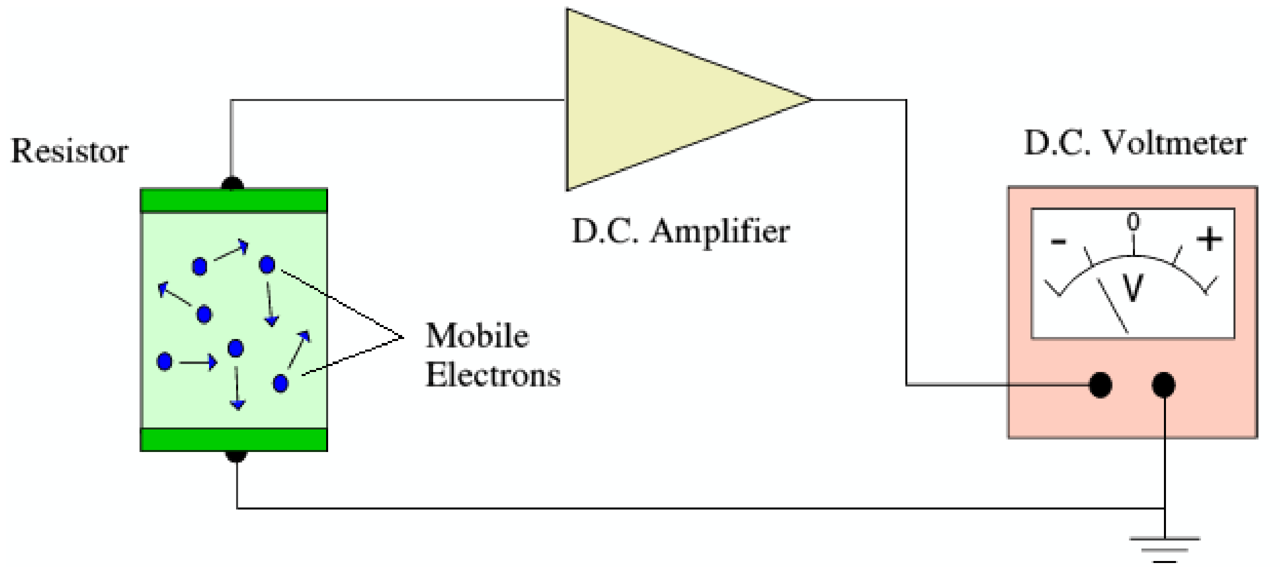


$$y(t) = x(t) + \eta(t)$$



Thermal Noise

- It is also called as Johnson-Nyquist noise
- It is the electronic noise generated by the thermal agitation of the charge carriers (usually the electrons) inside an electrical conductor at equilibrium, which happens regardless of any applied voltage
- This is type of intrinsic noise



https://www.st-andrews.ac.uk/~www_pa/Scots_Guide/iandm/part3/page1.html

Thermal Noise..

- The mean square voltage across a resistor R at temperature $\mathcal{T}$ degree corresponding to bandwidth B is

$$\overline{v_n^2} = 4Rk_B\mathcal{T}B$$

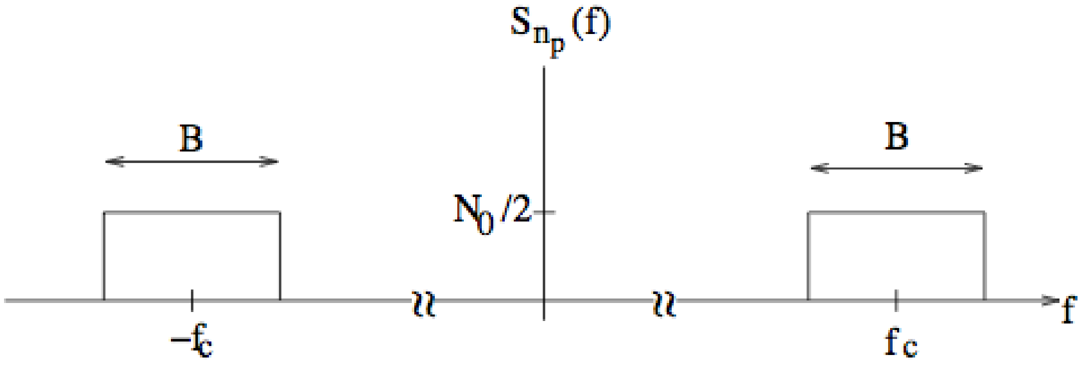
where $k_B = 1.38 \times 10^{-23}$ J/K is the Boltzmann's constant

- If we connect the noise source to a matched load of impedance R, the mean squared power delivered is

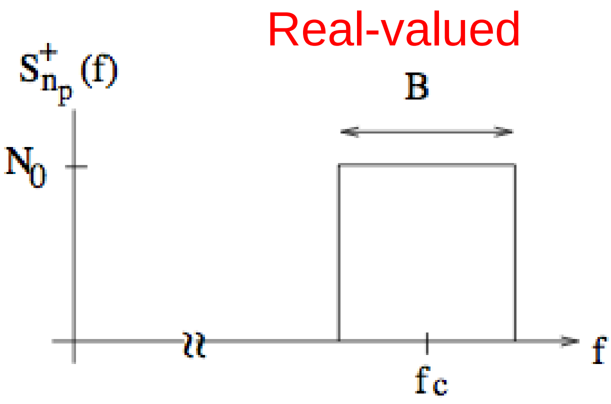
$$\overline{P_n^2} = \frac{\overline{(v_n/2)^2}}{R} = k_B\mathcal{T}B = N_0 B$$

- PSD is given by $N_0 = k_B\mathcal{T}$, a constant for a given $\mathcal{T}$

Modeling Noise: *Passband model*



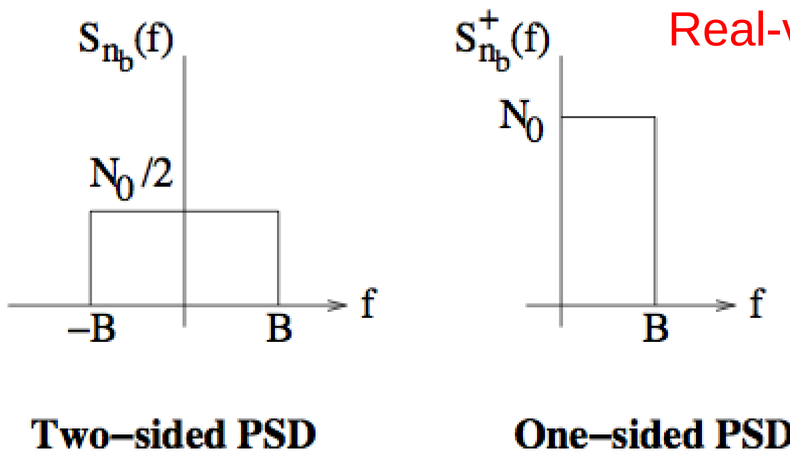
Two–sided PSD



One–sided PSD

- Receiver noise is modeled as random process with zero DC value and with a flat PSD or white over a band of interest.
- Two-sided PSD is $N_0/2$ while one-sided is N_0 so that noise power in Bandwidth is N_0B.
- Key noise mechanisms in communication systems such as thermal and shot noise are both white.

Modeling Noise: *Baseband model*

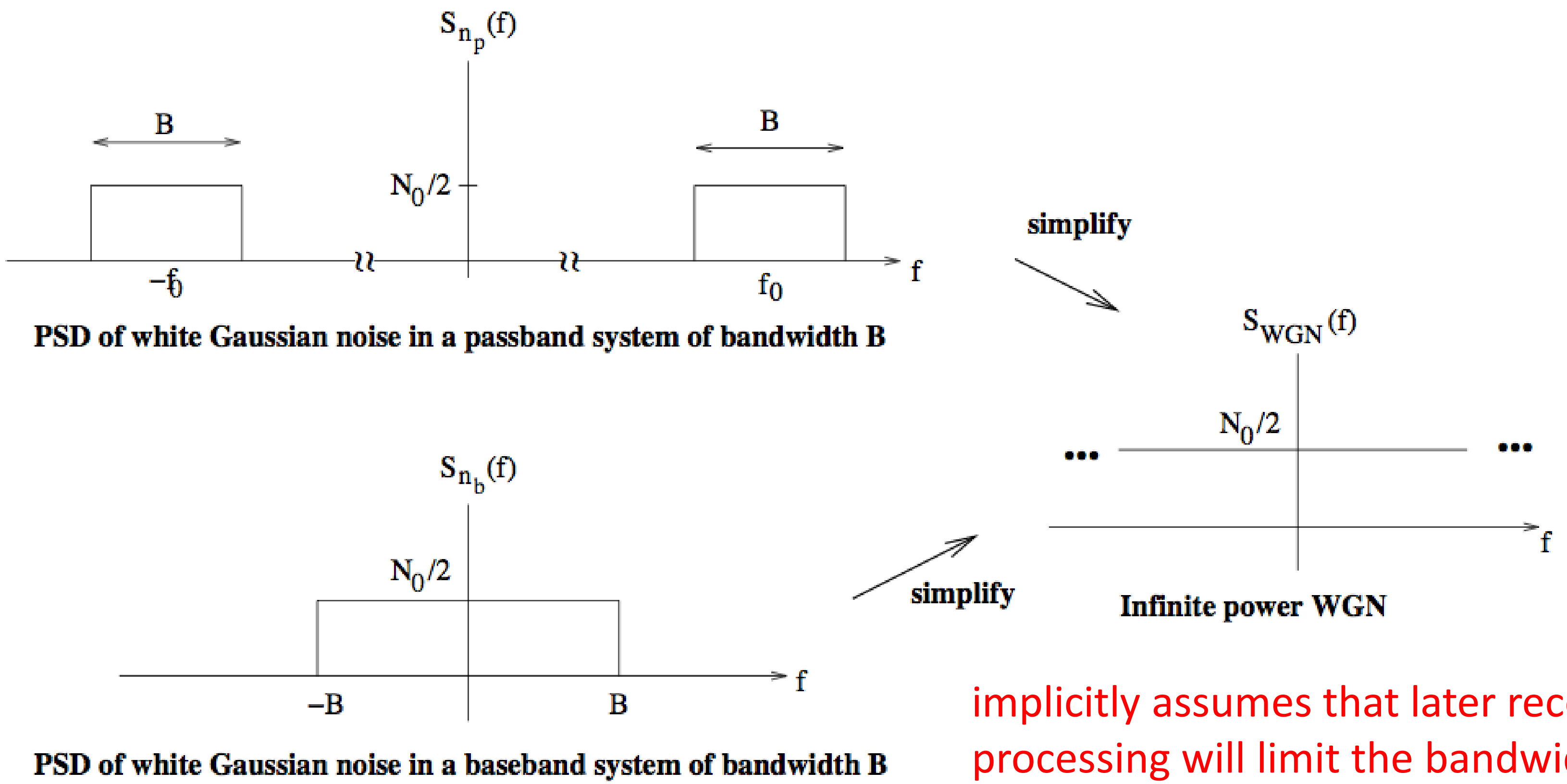


Real-valued

- Receiver noise is modeled as random process with zero DC value and with a flat PSD or white over a band of interest.
- Two-sided PSD is $N_0/2$ while one-sided is N_0 so that noise power in Bandwidth is N_0B.

Unified Model: *White Noise*

- Simplify the model by removing band limitation so that white applies to all systems, passband and baseband.



implicitly assumes that later receiver processing will limit the bandwidth

Modeling Noise as Gaussian

- Assume that the noise is Gaussian
- For example, thermal noise can be modeled as Gaussian: random motion of large number of charge carriers $\rightarrow$ Gaussianity due to central limit theorem

White Gaussian Noise (WGN)

- Combining Gaussian and white assumptions, we get white Gaussian noise (WGN) model.
- One of the widely used models in communications
- Also called additive white Gaussian noise (AWGN) since WGN is additive in nature.
- Real-valued WGN is a zero mean process, WSS Gaussian random process

$$S_n(f) = \frac{N_0}{2} = \sigma^2 \leftrightarrow R_n(\tau) = \frac{N_0}{2}\delta(\tau) = \sigma^2\delta(\tau)$$

where σ^2 is two-sided noise PSD (also called noise variance per dimension)

Example computation: Tutorial

5 GHz WLAN with receiver bandwidth 20 MHz and receiver noise figure 6 dB. What is the noise power?

$$P_n = N_0 B = kT_0 10^{F/10} B = (1.38 \times 10^{-23})(290)(10^{6/10})(20 \times 10^6)$$
$$= 3.2 \times 10^{-13} \text{ Watts} = 3.2 \times 10^{-10} \text{ milliWatts (mW)}$$

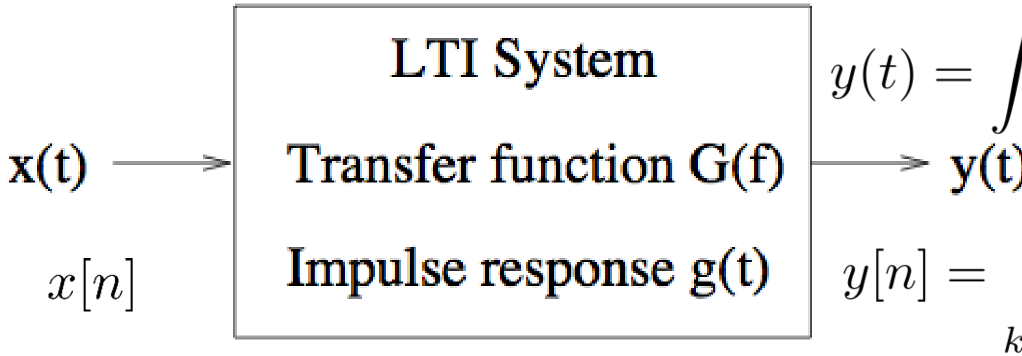
$\downarrow$ convert to dBm

$$P_{n,\text{dBm}} = 10 \log_{10} P_n(\text{mW}) = -95\text{dBm}$$

Can do this computation directly in the dB domain.

Filtering

Filtered Random Processes

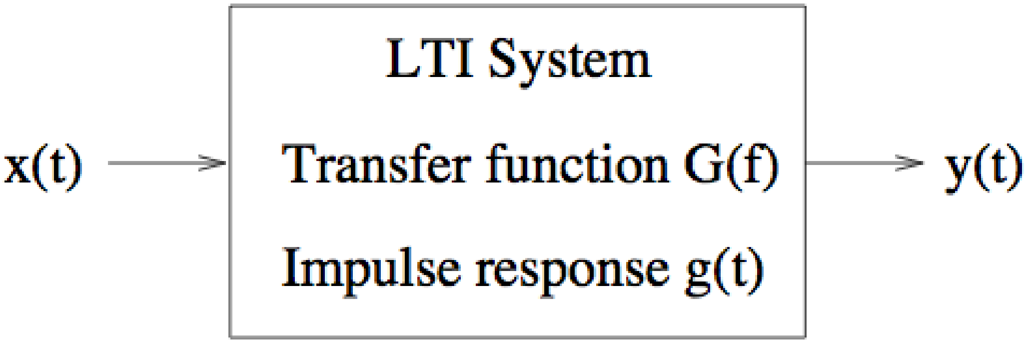


$$y(t) = \int_{-\infty}^{\infty} x(t-\tau]h(\tau)d\tau$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

- What is distribution of $y(t)$ if $x(t)$ is Gaussian distributed?

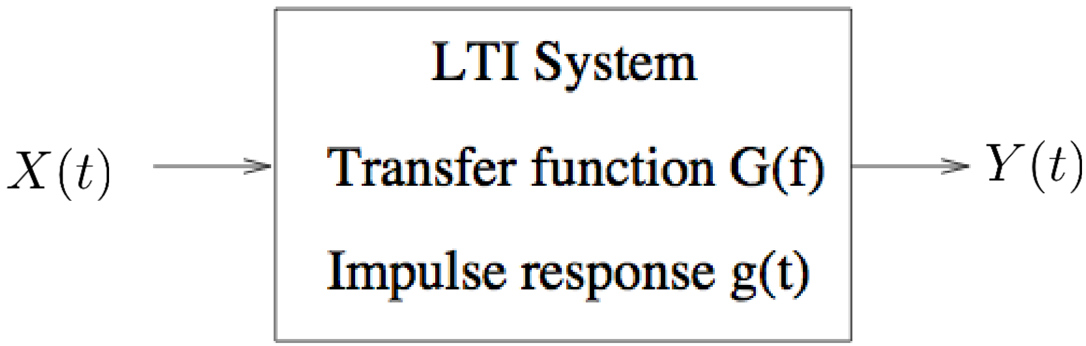
Filtered Random Processes



- Show that the output PSD is given by

$$S_y(f) = S_x(f)|G(f)|^2$$

WSS Filtered Random Processes



- For random processes $X(t)$ and $Y(t)$, show that $Y(t)$ is WSS if $X(t)$ is WSS.
- Also, show that

$$S_Y(f) = S_X(f)|G(f)|^2$$

$$\updownarrow$$

$$R_Y(\tau) = R_X(\tau) * g(\tau) * g^*(-\tau)$$

Wide Sense Stationarity

- A random process X is said to be WSS if

$$m_X(t) = \text{constant} \quad \forall t$$
$$R_X(t_1, t_2) = R_X(t_1 - t_2, 0) \quad \forall t_1, t_2$$

- A WSS random process has shift-variant second order statistics

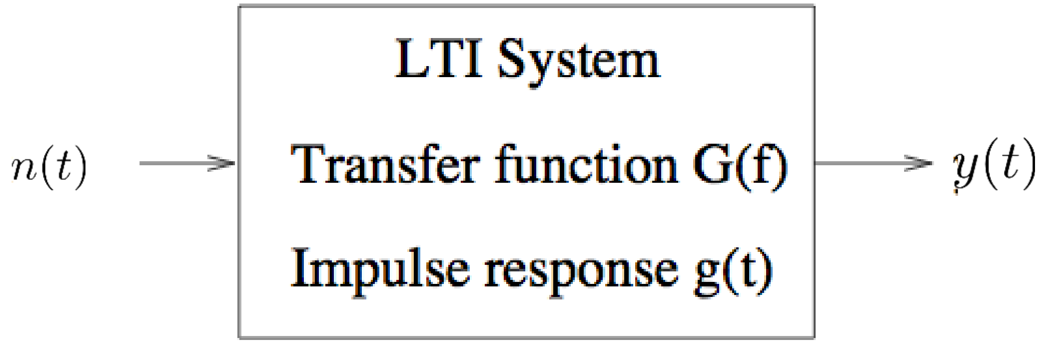
- Expressing the autocorrelation function as a function of $\tau = t_1 - t_2$,

$$R_X(\tau) = E[X(t)X^*(t+\tau)]$$

while the autocovariance is given by

$$C_X(\tau) = R_X(\tau) - |m_X|^2$$

Example



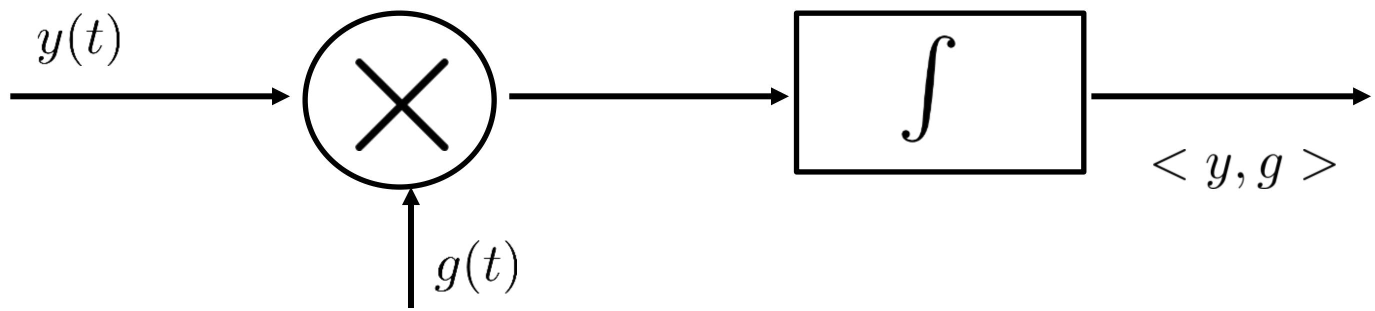
- White noise $n(t)$ with PSD $S_n(f) = \frac{N_0}{2}$ is passed through an LTI system with impulse response $g(t)$. Find the PSD, autocorrelation function and power of the output $y(t) = n(t) * g(t)$.

Example: Tutorial

- Suppose that WGN $n(t)$ with PSD $\sigma^2 = \frac{N_0}{2} = \frac{1}{4}$ is passed through an LTI system with impulse response $g(t) = I_{[0,2]}(t)$ to obtain the output $y(t) = n(t) * g(t)$.
  - Find the autocorrelation function and PSD of y.
  - Find $E[y^2(100)]$.
  - Is y a stationary random process?
  - Are $y(100)$ and $y(101)$ independent random variables?
  - Are $y(100)$ and $y(102)$ independent random variables?

Correlation

Definition and Motivation



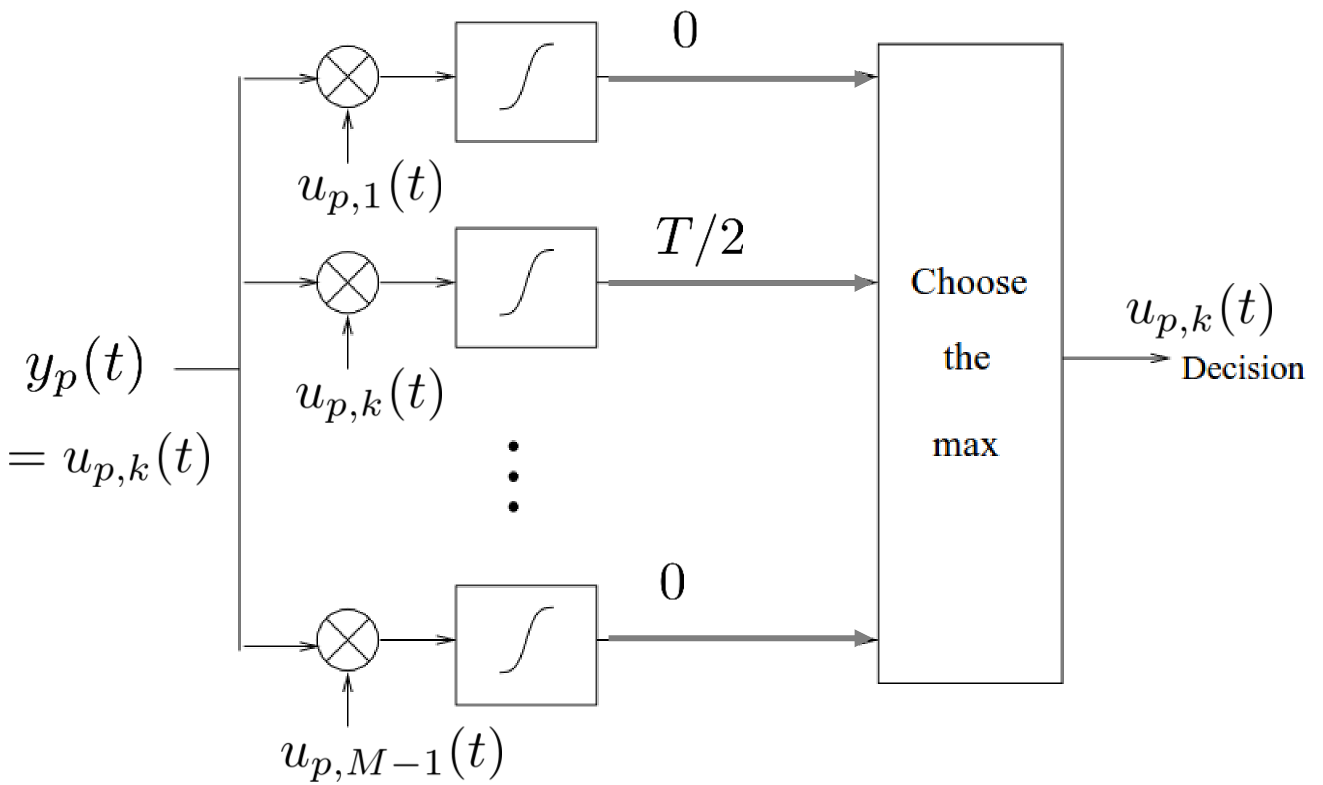
- Correlation between signals $y(t)$ and $g(t)$ is the inner product between the two and is given by

$$< y, g >= \int_{-\infty}^{\infty} y(t) g^*(t) dt$$

- One of the most common operations in communications.

- Used in demodulation of orthogonal waveforms.

Recap: *FSK coherent demodulation*



- Correlate the incoming signal with all M reference sinusoidal tones (passband signal) given by

$$u_{p,k} = \cos(2\pi(f_c + k\Delta f)t) \quad 0 \leq t \leq T$$

for $k = 0, 1, \ldots, M-1$.

- Choose $u_{p,m}$ for which the output is maximum among all the correlated outputs.

SNR

- Consider that the received signal

$$y(t) = s(t) + n(t)$$

  where $s(t)$ is a deterministic signal, corresponding to specific choice of transmitted symbols and $n(t)$ is zero-mean white noise with PSD $S_n(f) = \frac{N_0}{2}$.

- Considering real-valued signals, show that SNR at the output of correlator is given by

$$\text{SNR} = \frac{| < s, g > |^2}{\frac{N_0}{2} ||g||^2}$$

- Find $g(t)$ which will maximize the output SNR? Also find the maximum SNR!

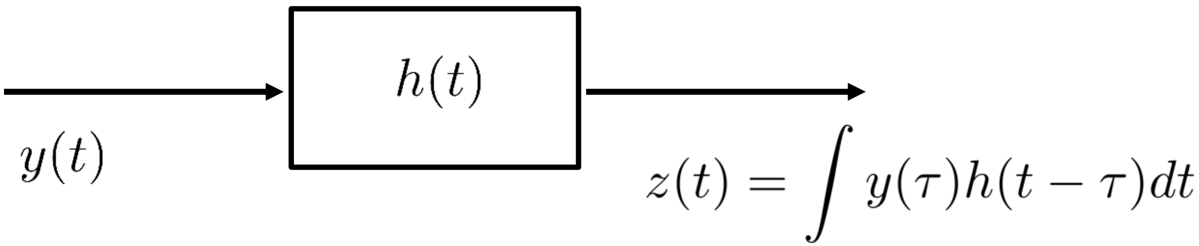
- Also valid for complex signal and noise scenarios with $g(t) = cs^*(t)$

Theorem 5.9.1

- For linear processing of a signal $s(t)$ corrupted by white noise, the output SNR is maximized by correlating against $s(t)$. The resulting SNR is given by

$$\mathrm{SNR}_{\max} = \frac{2||s||^2}{N_0}$$

Filter as Correlator



- Any correlation example can be implemented using a filter and a sampler:
  - For $h(\tau) = g^*(-\tau)$,

$$z(t) = \int y(\tau)h(t-\tau)d\tau = \int y(\tau)g^*(\tau - t)d\tau$$

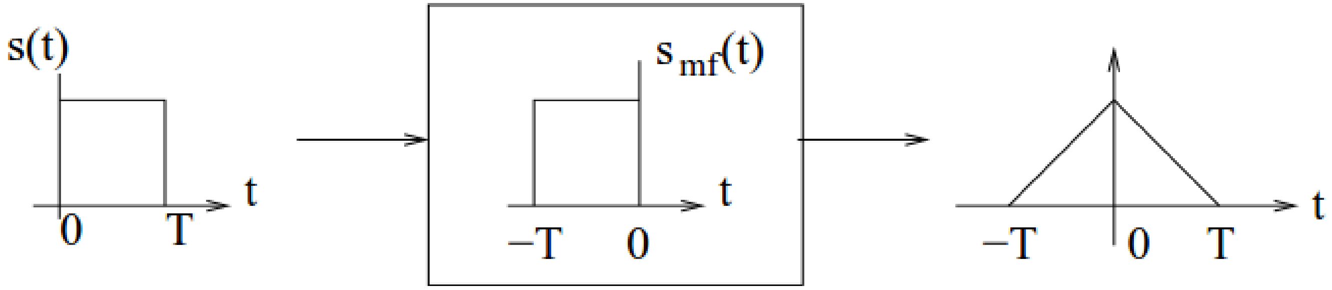
  - Sampled at $t = 0$, we get

$$z(0) = \int y(\tau)g^*(\tau)d\tau = < y, g >$$

- If $y(t) = s(t) + n(t)$, then choosing $g(t) = s^*(-t)$ maximizes SNR at the output of filter.

Matched Filter

- **Theorem**: For linear processing of a signal $s(t)$ corrupted by white noise, the output SNR is maximized by employing a matched filter with impulse response $s_{MF}(t) = s^*(-t)$, sampled at $t = 0$.



- When the received signal $y(t) = s(t) + n(t)$, optimum sampling time is $t = 0$. When the signal is delayed by $t = t_0$, the peak occurs at $t = t_0$, which now becomes the sampling time.

- Thus matched filter enables us to implement an infinite bank of correlators, each corresponding to a version of our signal template at a different delay.