EC5.203 Communication Theory I (3-1-0-4):

Lecture 19:

Optimal Demodulation-2

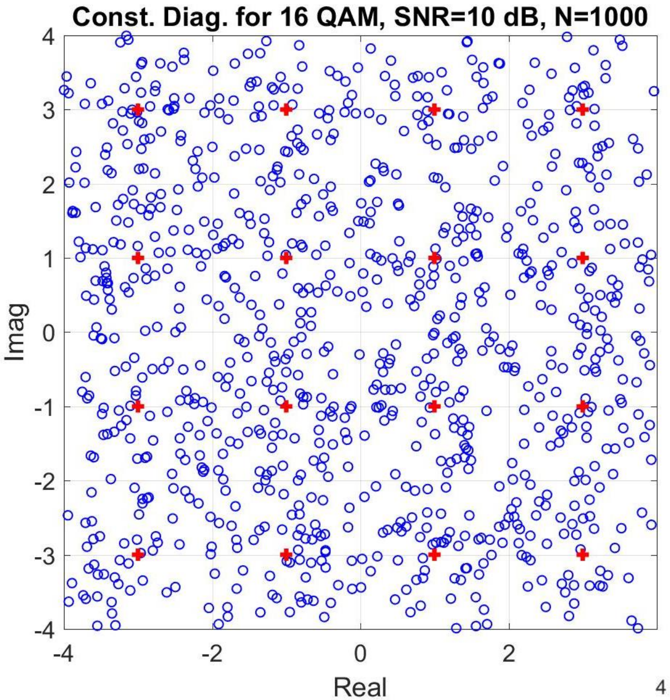
Mar. 27, 2025

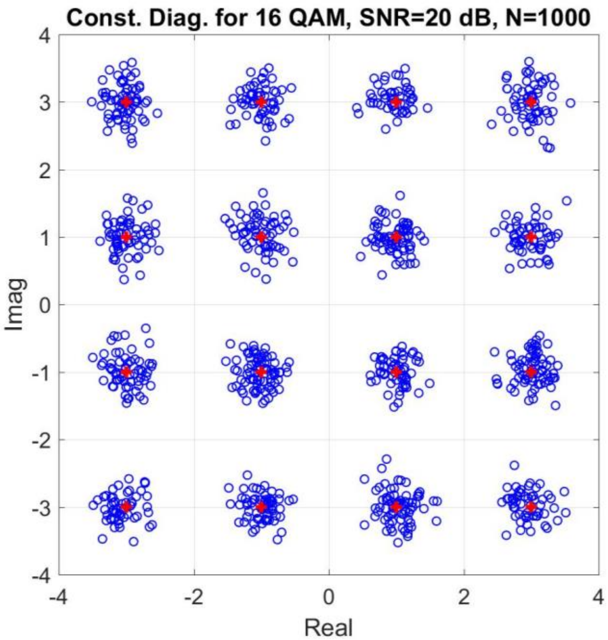
INTERNATIONAL INSTITUTE OF
INFORMATION TECHNOLOGY
HYDERABAD

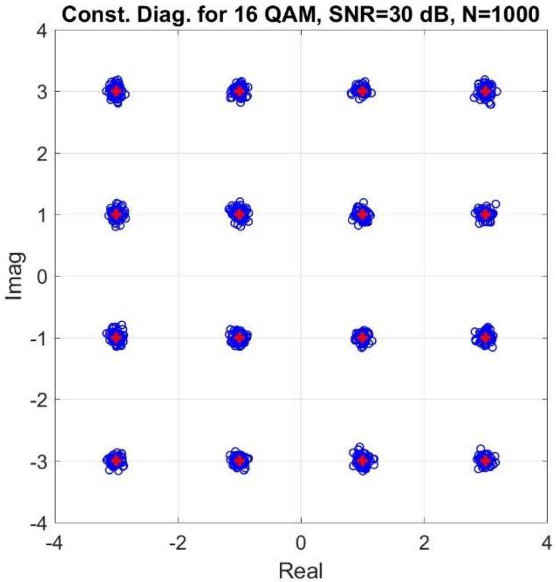
References

- Chap. 6 (Madhow)

Example: 16 QAM in AWGN







Optimal Demodulation

- In 16 QAM, one of 16 passband waveforms corresponding to 16 symbols is sent, where each passband waveform is given by

$$s_i(t) = s_{b_c, c_s} = b_c p(t) \cos(2\pi f_c t) - b_s p(t) \sin(2\pi f_c t)$$

  where b_c, b_s each takes value in $\{\pm 1, \pm 3\}$.

- At the receiver, we are faced with a hypothesis testing problem: we have M possible hypotheses about which signal was sent.

- Based on the observations

$$y(t) = s_i(t) + \boxed{n(t)} \quad \text{AWGN}$$

  we are interested in finding a decision rule to make a best guess which hypothesis was sent.

- For communications applications, performance criteria is to minimize the probability of error (i.e., the probability of making a wrong guess).

Example 5.6.3

- **Binary on-off keying in Gaussian noise**

$$\begin{aligned} Y &= m + n \quad \text{if 1 is sent} \\ Y &= n \qquad\quad \text{if 0 is sent} \end{aligned}$$

  Here Y is the received sample, $m > 0$ is some constant and n is AWGN sample with $\mathcal{N}(0, v^2)$.

- At the receiver, the detection strategy is

$$\begin{aligned} Y &> m/2 \quad \text{Decide 1 is sent} \\ Y &\leq m/2 \quad \text{Decide 0 is sent} \end{aligned}$$

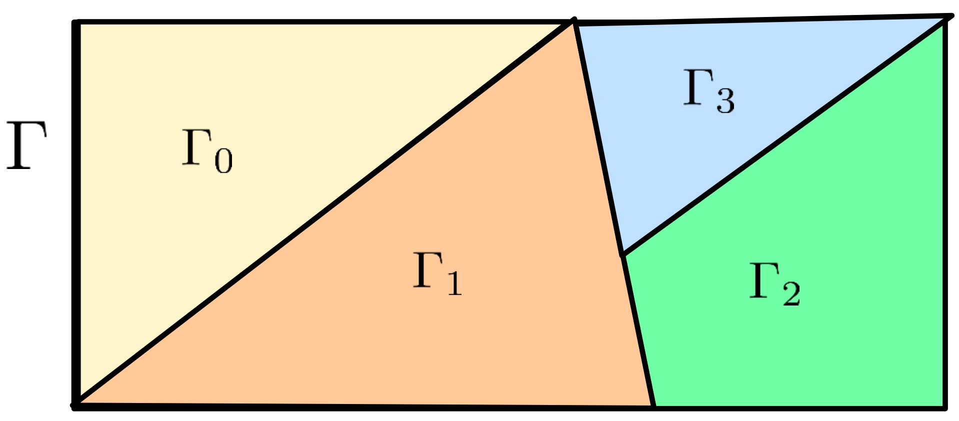
- Assuming that both 0 and 1 are equally likely,
  - Find the average signal power
  - Find the conditional probability of error conditioned on 0 being sent
  - Find the conditional probability of error conditioned on 1 being sent
  - Find average error probability
  - Find the probability of error for SNR of 13 dB?

Today's Class

Ingredients of Hypothesis Testing Framework

- Hypotheses $H_0, H_1, \ldots, H_{M-1}$
- Observation $Y \in \Gamma$
- Conditional densities $p(y|i)$ for $i = 0, 1, \ldots, M-1$
- Prior probabilities $\pi_i = P(H_i)$ with $\sum_i \pi_i = 1$
- Decision rule $\delta : \Gamma \to \{0, 1, M-1\}$
- Decision region $\Gamma_i : \{y \in \Gamma_i : \delta(y) = i\}$ for $i = 0, 1, M-1$

Example of Decision regions for M=4



Error Probabilities

- Conditional error probabilities, conditioned on H_i, is

$$\begin{aligned}
P_{e|i} &= P(\text{decide } j \text{ for some } j \neq i | H_i \text{ is true}) \\
&= \sum_{j \neq i} P(Y \in \Gamma_j | H_i) \\
&= 1 - P(Y \in \Gamma_i | H_i)
\end{aligned}$$

- Conditional probabilities of correct detection, conditioned on H_i, is

$$\begin{aligned}
P_{c|i} &= P(Y \in \Gamma_i | H_i) \\
&= 1 - P_{e|i}
\end{aligned}$$

- Average error probability

$$P_e = \sum_{i=1}^{M} \pi_i P_{e|i}$$

- Average probability of correct detection

$$P_c = \sum_{i=1}^{M} \pi_i P_{c|i}$$

Ingredients of Hypothesis Testing Framework

- Hypotheses $H_0, H_1, \ldots, H_{M-1}$
- Observation $Y \in \Gamma$
- Conditional densities $p(y|i)$ for $i = 0, 1, \ldots, M-1$
- Prior probabilities $\pi_i = P(H_i)$ with $\sum_i \pi_i = 1$
- Decision rule $\delta : \Gamma \to \{0, 1, M-1\}$
- Decision region $\Gamma_i : \{y \in \Gamma_i : \delta(y) = i\}$ for $i = 0, 1, M-1$
- In earlier example
  - Hypotheses H_0, H_1, Observation $Y \in \Gamma = \mathcal{R}$
  - Conditional densities $p(y|0)$ and $p(y|1)$
  - Prior probabilities π_0 and π_1
  - Decision rule δ:
    $$\delta(y) = \begin{cases} 0, & y \leq m/2 \\ 1, & y > m/2 \end{cases}$$
  - Decision regions: $\Gamma_0 = (-\infty, m/2]$ and $\Gamma_1 = (m/2, \infty)$

MAP rule

- Definitions:
  - *A priori* probability: Before the data is observed : $P(H_i) = \pi_i$
  - *A posteriori* probability: After the data is observed: $P(H_i|y)$
- Maximum *a posteriori* probability (MAP) rule:

$$\delta_{\text{MAP}}(y) = \arg\max_i P(H_i|Y = y)$$

  where $i = 0, 1, \ldots, M-1$
- Using Bayes rule $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$, MAP rule can be rewritten as

$$\begin{aligned}
\delta_{\text{MAP}}(y) &= \arg\max_i \frac{P(Y = y|H_i)P(H_i)}{P(Y = y)} \\
&= \arg\max_i \frac{p(y|i)\pi_i}{p(y)} \\
&= \arg\max_i p(y|i)\pi_i \\
&= \arg\max_i \log \pi_i + \log p(y|i)
\end{aligned}$$

Optimality of MAP (or MPE) rule

- Optimality of MAP rule: The MAP rule minimizes the probability of error. Proof!

ML rule

- Definitions:
  - Likelihood function: $p(y|i) = P(Y = y|H_i)$
- Maximum likelihood (ML) rule

$$\delta_{\mathrm{ML}}(y) = \arg\max_i p(y|i)$$

where $i = 0, 1, \ldots, M-1$

- Equivalently,

$$\delta_{\mathrm{ML}}(y) = \arg\max_i \log p(y|i)$$

- ML is equivalent of MAP for equal prior probabilities, i.e., $\pi_i = \frac{1}{M}$, we have

$$\begin{aligned}
\delta_{\mathrm{MAP}}(y) &= \arg\max_i \log \pi_i + \log p(y|i) \\
&= \arg\max_i \log \frac{1}{M} + \log p(y|i) \\
&= \arg\max_i \log p(y|i)
\end{aligned}$$

Binary Hypothesis Testing Problem

- For two hypotheses case, ML decision rule is

$$\begin{aligned}\delta_{\mathrm{ML}}(y) &= \arg\max_i p(y|i) \\ &= \arg\max_i \{p(y|0), p(y|1)\}\end{aligned}$$

- Equivalently,

$$\begin{aligned} p(y|0) > p(y|1) \quad &\rightarrow \quad \delta_{\mathrm{ML}}(y) = 0 \\ p(y|1) > p(y|0) \quad &\rightarrow \quad \delta_{\mathrm{ML}}(y) = 1 \end{aligned}$$

- This can be written as

$$p(y|1) \underset{H_0}{\overset{H_1}{\gtrless}} p(y|0)$$

- Similarly, MAP or MPE rule for binary hypothesis testing problem can be written as

$$\pi_1 p(y|1) \underset{H_0}{\overset{H_1}{\gtrless}} \pi_0 p(y|0)$$

$$P(H_1|Y = y) \underset{H_0}{\overset{H_1}{\gtrless}} P(H_0|Y = y)$$

Example of exponential distribution

- A binary hypothesis testing problem is specified as follows

$$H_0 : Y \sim \mathcal{E}(1)$$
$$H_1 : Y \sim \mathcal{E}(1/4)$$

where $\mathcal{E}(\mu)$ denotes an exponential density $\mu e^{-\mu y}$ and CDF $1 - e^{-\mu y}$ where $y \geq 0$. Note that the mean of $\mathcal{E}(\mu)$ is $1/\mu$.

- Find the ML rule and the corresponding error probabilities.

- Find the MAP rule when the prior probability of H_1 is $1/5$. Also find the conditional and average error probabilities.

Likelihood Ratio

- For two hypotheses case, ML decision rule can be written as

$$p(y|1) \underset{H_0}{\overset{H_1}{\gtrless}} p(y|0)$$

- Equivalently,

$$\frac{p(y|1)}{p(y|0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

  This ratio of likelihood functions is called likelihood ratio(LR) and denoted by $L(y)$ and the test is called as likelihood ratio test (LRT)

- Taking log of both sides

$$\log \frac{p(y|1)}{p(y|0)} \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

  The test statistic in this case is Log of LR and is called log-likelihood ratio (LLR) while the test is called as LLRT

Likelihood Ratio: MAP

- For two hypotheses case, MAP decision rule is

$$\pi_1 p(y|1) \underset{H_0}{\overset{H_1}{\gtrless}} \pi_0 p(y|0)$$

- In terms of LR, the LRT is

$$L(y) = \frac{p(y|1)}{p(y|0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

- Taking log of both sides, the test is given in terms of LLR

$$\log L(y) \underset{H_0}{\overset{H_1}{\gtrless}} \log \frac{\pi_0}{\pi_1}$$