Kostant 78, On Whittaker Vectors and Representation Theory

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Todo list

1 Section

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and root system Φ.

In section 1.1, we defined $\mathfrak{n} \subset \mathfrak{g}$ to be the nilradical of $\mathfrak{b} = \mathfrak{h} + \sum_{\phi \in \Delta_+} \mathfrak{g}_\phi$. $\mathfrak{n}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$. We also defined $\mathfrak{n}^-$ to be the nilradical of the opposite Borel subalgebra $\mathfrak{b}^- = \mathfrak{h} + \sum_{\phi \in \Delta_-} \mathfrak{g}_\phi$.

In section 2.3, we defined a Lie algebra homomorphism $\eta : \mathfrak{n} \to \mathbb{C}$ to be non-singular if the constants $c_i = \eta(e_{\alpha_i})$ are non-zero for all $\alpha_i \in \Delta_+$. This extends to a homomorphism $\eta : U(\mathfrak{n}) \to \mathbb{C}$, and let $U_\eta(\mathfrak{n})$ be the kernel of this homomorphism. We also know that $U(\mathfrak{n}) = \mathbb{C} \oplus U_\eta(\mathfrak{n})$ and since $\mathfrak{g} = \overline{\mathfrak{b}} \oplus \mathfrak{n}$, we have $U = U(\overline{\mathfrak{b}}) \oplus UU_\eta(\mathfrak{n})$. And now let ρ_η be the projection map form U to $U(\overline{\mathfrak{b}})$ with kernel $UU_\eta(\mathfrak{n})$.

Read section 2.5 in paper when needed.

Definition 1.1 – Whittaker Vector and Whittaker Module A vector $w \in V$ is called a Whittaker vector with respect to η if $xw = \eta(x)w$ for all $x \in \mathfrak{n}$. A $U-$module V is called a Whittaker module if it contains a cyclic ($Uw = V$) Whittaker vector.

Remark 1.1 This definition seems to mimic the ideas from Cartan subalgebra? We have a maximal nilpotent subalgebra and are looking at vectors which are eigenvectors w.r.t. this subalgebra with η representing the roots. A cyclic Whittaker vector resembles a highest weight vector.

If V is a Whittaker module with w as a cyclic Whittaker vector, and if U_w and U_V are the annihilators of w and V respectively, then one has $V \cong U/U_w$ as $U-$modules, $UU_\eta(\mathfrak{n}) \subseteq U_w$, and $UZ_V \subseteq U_w$ where $Z_V = Z \cap U_V$. This means that U_w contains the kernel of the projection map ρ_η and hence U_w is stable.

Also note U_w is a left ideal and U_V is a two-sided ideal.

Remark 1.2 Since V is cyclic, we have $V \cong U/U_w$ as $U-$modules, so V is determined upto equivalence by U_w.

Theorem 1.1 – U_w decomposition Let V be any $U-$module which admits a cyclic Whittaker vector w then, $U_w = UZ_V + UU_\eta(\mathfrak{n})$.

To prove this we will need the following lemma.

Definition 1.2 – η-reduced action If $x \circ v^\eta = (xv)^\eta$ then the η-reduced action of $\mathfrak{n}$ on $U(\bar{\mathfrak{b}})$ is given by $x \cdot v = x \circ v - \eta(x)v$.

Read 2.6 in paper

But before that, as mentioned earlier, U_w is stable under the projection map ρ_η. And for $X \subseteq U$, denote $X^\eta = \rho_\eta(X)$. From section 2.4.11 in the paper, we know that ρ_η induces an isomorphism $Z \to W(\bar{\mathfrak{b}}) = Z^\eta$. If Z_* is an ideal in Z, then $W_*(\bar{\mathfrak{b}}) = Z_*^\eta$ is an isomorphic ideal in $W(\bar{\mathfrak{b}})$. Thus, we have $(UZ_*)^\eta = U(\bar{\mathfrak{b}})W_*(\bar{\mathfrak{b}}) = \tilde{A} \otimes W_*(\bar{\mathfrak{b}})$. This is due to Remark 2.3 in paper which says $(uv)^\eta = u^\eta v^\eta$, and Theorem 2.5 in paper which gives the last equality. And finally by 2.3.5 we have

Read 2.4.11 in paper

Read Remark 2.3 in paper

Read Theorem 2.5 in paper, what the hell is happening over there?

$$UZ_* + UU_\eta(\mathfrak{n}) = (\tilde{A} \otimes W_*(\bar{\mathfrak{b}})) \oplus UU_\eta(\mathfrak{n}) \tag{1}$$

Lemma 1.1 – Technical Lemma Let $X = \{v \in U(\bar{\mathfrak{b}}) \mid (x \cdot v)w = 0 \text{ for all } x \in \mathfrak{n}\}$. Then $X = (\tilde{A} \otimes W_V(\bar{\mathfrak{b}})) \oplus UU_\eta(\mathfrak{n})$ where $W_V(\bar{\mathfrak{b}}) = (Z_V)^\eta$. We also have $U_w(\bar{\mathfrak{b}}) \subseteq X$ and $U_w(\bar{\mathfrak{b}}) = \tilde{A} \otimes W_V(\bar{\mathfrak{b}})$ where $U_w(\bar{\mathfrak{b}}) = U_w \cap U(\bar{\mathfrak{b}})$.

Proof of lemma. The proof is technical, best to come back to them later and focus now on what the statement is saying. □

Proof of theorem. Theorem 1.1 says that U_w can be decomposed into two parts, a trivial part $UU_\eta(\mathfrak{n})$ and the other UZ_V. The content lies in proving $U_w \subseteq UZ_V + UU_\eta(\mathfrak{n})$, but that by (1) is equivalent to proving "If $u \in U_w$ then $v \in \tilde{A} \otimes W_V(\bar{\mathfrak{b}})$ where $v = u^\eta$", the rest follows from the lemma. □

Remark 1.3

- Theorem 1.1 implies, along with remark 1.2, that a Whittaker module V is determined upto equivalence by the central ideal Z_V.
- Note that the proof depends on non-singularity of η. It relies on Theorem 2.5 in the paper, which in turn relies on 2.4.3 in Theorem 2.4.1 in the paper, the part which assumes non-singularity of η.

Read 2.4.1 in paper

Theorem 1.2 – Relation between Whittaker modules and ideals of the Center of U Let U, V, U_V, Z_V be as above. Then the correspondence

$$V \mapsto Z_V \tag{2}$$

is a bijection between the set of equivalence classes of Whittaker modules and the set of ideals of Z.

Proof. Injectivity is clear be Theorem 1.1. For surjectivity, let Z_* be an ideal of Z. Then we would like to construct a Whittaker module V such that $Z_V = Z_*$. But we already know that Z_V determines V upto equivalence. So let $L = UZ_* + UU_\eta(\mathfrak{n})$, then $V = U/L$ is a Whittaker module with $U_w = L$. □

Remark 1.4 This also gives us a correspondence between the set of ideals of center and the set of annihilators of Whittaker modules. $Z_* \mapsto U_w = UZ_* + UU_\eta(\mathfrak{n})$.

Fact 1.1 The subalgebra $ZU(\mathfrak{n})$ can be written as $Z \otimes U(\mathfrak{n})$. [a]

Then we can write Z/Z_* as a Z-module and thus, $(Z/Z_*)_\eta$ as a $Z \otimes U(\mathfrak{n})$-module, if $u \in Z, v \in U(\mathfrak{n})$ and $y \in (Z/Z_*)_\eta$, then $uvy = \eta(v)uy$.

[a]Reference: Section 3.3 in paper, Theorem 0.12 in *Lie Group Representations on Polynomial Rings by Kostant*

Theorem 1.3 – Every Whittaker module is an induced module Let V be any U-module then, V is a Whittaker module if and only if we have the isomorphism $V \cong U \otimes_{Z\otimes U(\mathfrak{n})} Z/Z_*$ of U-modules for some ideal Z_* of Z. And furthermore, in such a case Z_* is unique, and $Z_* = Z_V$.

Write proof, not hard, not interesting

Proof. □

Define Universal Whittaker Module

Remark 1.5 This resembles the construction using Verma modules.

Theorem 1.4 – Whittaker Vectors in a Whittaker Module Let V be any U-module with a cyclic Whittaker vector w. Then any $v \in V$ is a Whittaker vector if and only if v is form $v = uw$ where $u \in Z$.

Remark 1.6 This theorem neatly characterizes Whittaker vectors in a Whittaker module. And in other words it says that the space of Whittaker vectors is a Z-cyclic module isomorphic to $Z/Z_V w$ since we have a Z-module homomorphism $u \in Z \mapsto uw \in V$ with kernel $Z_V = Z_w$ and image being the space of Whittaker vectors.

For any U-module V, let $\operatorname{End}_U V$ be the algebra of operators on V commuting with U, and if $\pi_V : U \to \operatorname{End} V$ is a representation of U on V, then $\pi_V(Z) \subseteq \operatorname{End}_U V$ and $\pi_V(Z) \cong Z/Z_V$ as algebras.

Theorem 1.5 – $\operatorname{End}_U V$ in a Whittaker Module is a commutative algebra Let V be a Whittaker module, then $\operatorname{End}_U V = \pi_V(Z)$ and $\operatorname{End}_U V \cong Z/Z_V$ as algebras. And thus $\operatorname{End}_U V$ is a commutative algebra.

Proof. Let w be a cyclic Whittaker vector, and if $\alpha \in \operatorname{End}_U V$, then αw is also a Whittaker vector. But by the previous theorem, $\alpha w = uw$ for some $u \in Z$. Now for any $v \in U$, we have $\alpha v w = v \alpha w = v u w = u v w$ thus $\alpha = \pi_V(u)$. □

Read remark 3.5 in paper

Remark 1.7 This resembles the result from Dixmier's book on enveloping algebras, that the centralizer of a regular element is commutative. Probably why Prof. Michael was excited about that.

We will refer to a homomorphism $\zeta : Z \to \mathbb{C}$ as a central character. Let $Z_\zeta = \operatorname{Ker} \zeta$, since image of ζ is a field, Z_ζ is a maximal ideal of Z. Using 2.3.2 in paper, we can naturally parametrize the set of central characters by $\mathbb{C}^l$.

Read 2.3.2 in paper

If V is a U-module, we say V admits an infinitesimal character ζ if there exists a central character ζ if ζ is a central character such that $uv = \zeta(v)$ for all $u \in Z, v \in V$. By Dixmier's Theorem any irreducible U-module admits an infinitesimal character.

Find reference

For a central character ζ, let $\mathbb{C}_{\zeta,\eta}$ be the 1-dimensional $Z \otimes U(\mathfrak{n})$-module defined so that if $u \in Z, v \in U(\mathfrak{n})$ and $y \in \mathbb{C}_{\zeta,\eta}$, then $uvy = \zeta(u)\eta(v)y$. Also let $Y_{\zeta,\eta} = U \otimes_{Z\otimes U(\mathfrak{n})} \mathbb{C}_{\zeta,\eta}$.

Why is this?

Why? Verma modules are not irreducible. Harish-Chandra isomorphism?

Does this not mean that the map is just surjective?

$Y_{\zeta,\eta}$ admits an infinitesimal character and that is ζ.

Let η_0 be the trivial character, and let Y_{ζ,η_0} be defined the same way. And let $\lambda : \mathfrak{h} \to \mathbb{C}$ be a linear functional, then it defines a homomorphism $\lambda : U(\mathfrak{h}) \to \mathbb{C}$ and the corresponding Verma module is $M_\lambda = U \otimes_{U(\mathfrak{h})\otimes U(\mathfrak{n})} \mathbb{C}_\lambda$ where $U(\mathfrak{h})$ acts on $\mathbb{C}_\lambda$ via λ and $U(\mathfrak{n})$ acts trivially. Now one knows that M_λ admits an infinitesimal character $\zeta = \zeta(\lambda)$. And hence considering the cyclic generator, one has the exact sequence $Y_{\zeta,\eta_0} \to M_\lambda \to 0$ of U-modules. But M_λ is in general not irreducible, thus in particular Y_{ζ,η_0} is not in general irreducible. In contrast the next theorem asserts that $Y_{\zeta,\eta}$ is always irreducible when η is non-singular.

Theorem 1.6 – Big Theorem Let V be any Whittaker module for U, the universal enveloping algebra of a semisimple Lie algebra $\mathfrak{g}$, then the following are equivalent:

1. V is an irreducible U-module.
2. V admits an infinitesimal character.
3. The corresponding ideal Z_V of Z, given by Theorem 1.1 is a maximal ideal.
4. The space of Whittaker vectors in V is 1-dimensional.
5. All non-zero Whittaker vectors in V are cyclic.
6. The centralizer $\mathrm{End}_U V$ reduces to the constants.
7. V is isomorphic to $Y_{\zeta,\eta}$ for some central character ζ.

Theorem 1.7 Let V be any $U-$module which admits an infinitesimal character. Assume $w \in V$ is a Whittaker vector, then the submodule Uw is irreducible. If ζ is the infinitesimal character and w_ζ is the cyclic Whittaker generator of $Y_{\zeta,\eta}$, then $Uw \cong Y_{\zeta,\eta}$ as U-modules with $uw_\zeta \mapsto uw$. This isomorphism is unique up to a scalar multiple.

If V_1 and V_2 are irreducible U-modules with the same infinitesimal characters, and if they have Whittaker vectors, they are unique up to a scalar multiple and $V_1 \cong V_2$ as U-modules.

If V is a Whittaker module, Z_V its corresponding ideal, and if $V_* \subseteq V$ a submodule, then let $Z(V_*) = \{u \in Z \mid u(V) \subseteq V_*\}$ so that $Z(V_*)$ is an ideal in Z containing Z_V.

Remark 1.8 If $w \in V$ is a cyclic Whittaker vector, and if $u \in Z$, then $u \in Z(V_*)$ if and only if $uw \in V_*$.

Theorem 1.8 Let V be a Whittker module, then the map $V_* \to Z(V_*)$ is a bijection between the set of submodules of V and the set of ideals of Z containing Z_V. Furthermore, $Z(V_1+V_2) = Z(V_1) + Z(V_2)$ and $Z(V_1 \cap V_2) = Z(V_1) \cap Z(V_2)$ and the inverse to the correspondece is given by $Z_* \to Z_*V$ for any ideal Z_* of Z containing Z_V.

Remark 1.9 While we had the correspondece between Z_V an ideal in Z and V a Whittker module, this theorem tells us that ideals Z_* of Z containing Z_V correspond to $U-$submodules $V_* = Z_*V$ of V and the ideal corresponding to V_* is $Z(V_*)$. This correspondece is an order reversing lattice isomorphism similar to what we see in Galois theory.

Theorem 1.9 – Corollary Let Y_η be the induced $U-$module denoted by $Y_\eta = U \otimes_{U(\mathfrak{n})} \mathbb{C}_\eta$. The action of Z on Y_η induces an isomorphism $Z \to \mathrm{End}_U Y_\eta$. Furthermore if Z_* is any ideal in Z the map $Z_* \to Z_*Y_\eta$ sets up a bijection of the sets of all ideals in Z and the set of all $U-$submodules of Y_η.

Remark 1.10 This is just the earlier theorem applied to the universal Whittaker module.

Let $\lambda \in \mathfrak{h}'$ be a linear functional on $\mathfrak{h}$ and let M_λ be the corresponding Verma module. Let $C \subseteq \mathfrak{h}'$ be the $\mathbb{Z}$-cone spanned by the negative roots and zero and let $C_\lambda = \lambda + C$. If $M_\lambda(\mu) \subseteq M_\lambda$ is the weight space corresponding to the weight $\mu \in \mathfrak{h}'$, then one has the direct sum $M_\lambda = \bigoplus_{\mu \in C_\lambda} M_\lambda(\mu)$. And if $\phi \in \Delta$ one knows that $e_\phi M_\lambda(\mu) \subseteq M_\lambda(\mu + \phi)$.

Let $U(\mathfrak{n})$ be a $\mathfrak{b}$ module defines as $x \cdot u = xu$ for $x \in \mathfrak{n}$ and $u \in U(\mathfrak{n})$ and $x \cdot u = [xu]$ for $x \in \mathfrak{h}$ and $u \in U(\mathfrak{n})$. Let $U(\mathfrak{n})'$ be the dual module.

Definition 1.3 – Restricted dual to $U(\mathfrak{n})$ Let $U(\mathfrak{n})^*$ be the subspace of all $\beta \in U(\mathfrak{n})'$ such that β vanishes on a power of the augmentation ideal $\mathfrak{n}U(\mathfrak{n})$. Then $U(\mathfrak{n})^*$ is a $U(\mathfrak{b})$-module. We refer to $U(\mathfrak{n})^*$ as the restricted dual to $U(\mathfrak{n})$.

Let $0 \neq m_\lambda \in M_\lambda$ be a highest weight vector and l be a linear functional on M_λ such that $l(m_\lambda) = 1$ and l vanishes on $M_\lambda(\mu)$ for $\mu \neq \lambda$. For any $v \in M_\lambda$ let $\beta_v \in U(\mathfrak{n})'$ be defined by $\langle \beta_v, u \rangle = l(\check{u}v)$ for $u \in U(\mathfrak{n})$. Where $\check{u}$ denotes the image of the linear isomorphism defined by $\check{wu} = \check{u}\check{w}$ and $\check{x} = -x$ for $x \in \mathfrak{g}$.

Lemma 1.2 The correspondece $v \mapsto \beta_v$ defines a map $M_\lambda \to U(\mathfrak{n})^*$ which is a $\mathfrak{n}$-module homomorphism. And this is an isomorphism of $\mathfrak{n}-$modules iff the Verma module M_λ is irreducible.