HW 1

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Problem 1

The Bayesian Update rule in terms of densities is given by

$$\pi_1(\omega) = \pi_0(\omega)\frac{\mathbb{P}(X|\omega)}{\int_\Omega \mathbb{P}(X|\omega)d\pi_0(\omega)}$$

alternatively written as

$$p(\omega|X) = p(\omega)\frac{\mathbb{P}(X|\omega)}{\int_\Omega \mathbb{P}(X|\omega)dp(\omega)}$$

where $\pi_0 = p$ is the density of the prior distribution, $\pi_1(\omega) := p(\omega|X)\forall\omega$ is density of the posterior distribution, and X is the evidence/observations on which this update is based on.

Subpart a

The pdf of a Beta prior for fixed parameters α, β is given by

$$p(\omega) = c_{\alpha,\beta}\omega^{\alpha-1}(1-\omega)^{\beta-1}$$

where $\omega \in [0,1]$. If the Beta prior describes the parameter p of a Geometric random variable with mean $1/p$, then

$$\mathbb{P}(X|\omega) = (1-\omega)^{X-1}\omega$$

thus the posterior becomes

$$c'\omega^{\alpha+1-1}(1-\omega)^{\beta+X-1-1} = c_{\alpha+1,\beta+X-1}\omega^{\alpha+1-1}(1-\omega)^{\beta+X-1-1}$$

(last equality due to the fact that it is a density) which is the density of the Beta distribution with parameters $\alpha+1, \beta+X-1$ thus the posterior is also Beta for any observation X distributed as a Geometric. This proves Beta is conjugate prior to Geometric.

Subpart b

The pdf of a Gamma prior for fixed parameters k, θ is given by

$$p(\omega) = c_{k,\theta}\omega^{k-1}e^{-\omega/\theta}$$

If the Gamma prior describes the parameter λ of a Poisson random variable with mean λ, then

$$\mathbb{P}(X|\omega) = \frac{\omega^X e^{-\omega}}{X!}$$

thus the posterior becomes

$$c'\omega^{k+X-1}e^{-\omega(1+\theta^{-1})} = c_{k+X,(1+\theta^{-1})^{-1}}\omega^{k+X-1}e^{-\omega(1+\theta^{-1})}$$

(last equality due to the fact that it is a density) which is the density of the Gamma distribution with parameters $k+X,(1+\theta^{-1})^{-1}$ thus the posterior is also Gamma for any observation X distributed as a Poisson. This proves Gamma is conjugate to prior to Poisson.

Subpart c

The pdf of a Normal prior for fixed parameters μ, σ^2 is given by

$$p(\omega) = c_\sigma \exp\left(-\frac{(\omega-\mu)^2}{2\sigma^2}\right)$$

If the Normal prior describes the parameter μ of a Normal random variable X with mean μ and variance 1, then

$$\mathbb{P}(X|\omega) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(X-\omega)^2}{2}\right)$$

thus the posterior becomes

$$c'\exp\left(-\frac{(\omega-\mu)^2}{2\sigma^2} - \frac{(X-\mu)^2}{2}\right) = c_{\sigma'}\exp\left(-\frac{1}{2}\left[\left(\frac{1}{\sigma^2}+1\right)\omega^2 - 2\left(\frac{\mu}{\sigma^2}+X\right)\omega\right]\right)$$

which is the kernel of a Normal distribution. Completing the square, we find that the posterior is Normal with parameters

$$\mu' = \frac{\frac{\omega}{\sigma^2}+X}{\frac{1}{\sigma^2}+1}$$

and

$$\sigma'^2 = \frac{1}{\frac{1}{\sigma^2}+1}$$

thus the posterior is also Normal for any observation X distributed as a Normal. This proves Normal is conjugate prior to Normal (with variance 1).

Problem 2

Subpart a

Let $f_t(x,y)$ be the probability mass function for the 2-dimensional random variable $(\hat{\mu}_t(1), \hat{\mu}_t(2))$, where $(x,y) \in \Omega_t := \{\frac{i}{j} : i \text{ and } j \neq 0 \in \{0,1,\ldots,t\}\}^2$ We know

$$\begin{aligned}
\mathrm{Reg}(n) &= \sum_{t=0}^{n} \Delta\mathbb{P}(I_t = 1) \\
&\geq \sum_{t=3}^{n} \Delta \sum_{\Omega_t} \mathbb{P}(I_t = 1, (\hat{\mu}_t(1), \hat{\mu}_t(2)) = (x,y)) f_t(x,y) \\
&= \sum_{t=3}^{n} \Delta \sum_{\Omega_t} \frac{e^{\beta\hat{\mu}_t(1)}}{e^{\beta\hat{\mu}_t(1)} + e^{\beta\hat{\mu}_t(2)}} f_t(x,y) \\
&= \sum_{t=3}^{n} \Delta \sum_{\Omega_t} \frac{1}{1+e^{\beta(\hat{\mu}_t(1)-\hat{\mu}_t(2))}} f_t(x,y)
\end{aligned}$$

Now

$$\min_{x,y\in\Omega_t} \frac{1}{1+e^{\beta(\hat{\mu}_t(1)-\hat{\mu}_t(2))}} = \frac{1}{1+e^{\beta\frac{t}{t}}} = \frac{1}{1+e^{\beta}}$$

. Thus, we have

$$\begin{aligned}
\mathrm{Reg}(n) &\geq \sum_{t=3}^{n} \Delta \frac{1}{1+e^{\beta}} \sum_{\Omega_t} f_t(x,y) \\
&= \sum_{t=3}^{n} \Delta \frac{1}{1+e^{\beta}} \\
&= \frac{n-2}{1+e^{\beta}}
\end{aligned}$$

Thus the regret is lower bounded by $\frac{n-2}{1+e^{\beta}}$ and the Boltzmann Exploration Algorithm cannot achieve sublinear regret.

Subpart b

If the rewards are deterministically equal to μ_1 and μ_2, then the sample means $\hat{\mu}_t(1)$ and $\hat{\mu}_t(2)$ are equal to μ_1 and μ_2 respectively. Then

$$\begin{aligned}
\mathbb{E}[N_n(1)] &= \sum_{t=1}^{n} \mathbb{P}(I_t = 1) \\
&= \sum_{t=1}^{n} \frac{e^{\beta_t \mu_1}}{e^{\beta_t \mu_1} + e^{\beta_t \mu_2}} \\
&= \sum_{t=1}^{n} \frac{1}{1+e^{\beta_t(\mu_2-\mu_1)}} \\
&= \sum_{t=1}^{n} \frac{1}{1+e^{\beta_t \Delta}}
\end{aligned}$$

We want this summation to asymptotically grow as $\frac{\log n}{\Delta^2}$. Recall that the summation $\sum_{i=1}^{n} \frac{1}{i}$ asymptotically grows as $\log n$. Thus if we set $\beta_t = \log(t\Delta^2)/\Delta$, we get

$$\mathbb{E}[N_n(1)] = \sum_{t=1}^{n} \frac{1}{1+e^{\beta_t \Delta}} = \sum_{t=1}^{n} \frac{1}{1+e^{\log(t\Delta^2)}} = \sum_{t=1}^{n} \frac{1}{1+t\Delta^2}$$

which for large n grows as

$$\sum_{t=1}^{n} \frac{1}{t\Delta^2} = \frac{\log n}{\Delta^2}$$

Problem 3

Subpart a

An upperbound of the regret of the Explore then Commit algorithm is given by

$$\mathrm{Reg}(ETC, n) \leq O(m\Delta + n\Delta \exp \frac{-m\Delta^2}{2})$$

but if we knew the time horizion and the order of Δ, then we can set

$$m \approx \frac{2}{\Delta^2}\log(n\Delta)$$

so that the upperbound becomes

$$\text{Reg}(ETC, n) \leq O(\frac{2}{\Delta^2}\log(n))$$

Part b

If m is chosen to be larger than $n^{2/3}$, i.e. $m = n^{2/3+\delta}$ where $\delta > 0$. Then

$$\text{Reg}(ETC, n) \leq O(n^{2/3+\delta}\Delta + n\Delta \exp{(\frac{-n^{2/3+\delta}\Delta^2}{2})})$$

Choose Δ so that the equation

$$\exp{(-n^{2/3+\delta}\Delta^2/2)} = n^{-1/3+\delta}(\Delta^{-1} - 1)$$

is satisfied. Then the regret becomes

$$\text{Reg}(ETC, n) \leq O(n^{2/3+\delta}\Delta + n\Delta n^{-1/3+\delta}(\Delta^{-1} - 1)) = O(n^{2/3+\delta})$$

The implicit equation has a solution because when $\Delta = 1$, the RHS is 0 and LHS $= \exp{(-n^{2/3+\delta}/2)} > 0$, i.e. LHS $>$ RHS. When $\Delta > 0$ but for values arbitrarily close to zero, RHS diverges to ∞ while LHS converges to 1, i.e. LHS $<$ RHS.

Since both LHS and RHS are continuous functions of Δ, the equation has a solution $\Delta_n \in (0, 1)$ for each n.

Part c

If $m = n^{2/3-\delta}$, then

$$\text{Reg}(ETC, n) \leq O(n^{2/3-\delta}\Delta + n\Delta \exp{(\frac{-n^{2/3-\delta}\Delta^2}{2})})$$

Choose Δ so that the equation

$$\exp{(-n^{2/3+\delta}\Delta^2/2)} = n^{-1/3}(\Delta^{-1} - n^{-\delta})$$

is satisfied. Then the regret becomes

$$\text{Reg}(ETC, n) \leq O(n^{2/3-\delta}\Delta + n\Delta n^{-1/3}(\Delta^{-1} - n^{-\delta})) = O(n^{2/3})$$

The implicit equation has a solution because when $\Delta = n^{-\delta} < 1$, the RHS is 0 and LHS $= \exp{(-n^{2/3-\delta}/2)} > 0$, i.e. LHS $>$ RHS. When $\Delta > 0$ but for values arbitrarily close to zero, RHS diverges to ∞ while LHS converges to 1, i.e. LHS $<$ RHS.

Since both LHS and RHS are continuous functions of Δ, the equation has a solution $\Delta_n \in (0, n^{-\delta}) \subset (0, 1)$ for each n.

Part d

$$R(n,m,\Delta) = m\Delta + n\Delta \exp \frac{-m\Delta^2}{2}$$

$$\begin{aligned}
\min_{1\le m\le n} \max_{0\le\Delta\le 1} R(n,m,\Delta) = \min(& \\
& \min_{1\le m\le n^{2/3-\delta}} \max_{0\le\Delta\le 1} R(n,m,\Delta), \\
& \min_{n^{2/3-\delta}\le m\le n^{2/3+\delta}} \max_{0\le\Delta\le 1} R(n,m,\Delta), \\
& \min_{n^{2/3+\delta}\le m\le n} \max_{0\le\Delta\le 1} R(n,m,\Delta) \\
&)
\end{aligned}$$

But from above

$$\begin{aligned}
\min_{1\le m\le n} \max_{0\le\Delta\le 1} R(n,m,\Delta) = \min(& \\
& O(n^{2/3+\delta}), \\
& \min_{n^{2/3-\delta}\le m\le n^{2/3+\delta}} \max_{0\le\Delta\le 1} R(n,m,\Delta), \\
& O(n^{2/3}) \\
&)
\end{aligned}$$

But this is true for all $\delta > 0$. So,

$$\begin{aligned}
\min_{1\le m\le n} \max_{0\le\Delta\le 1} R(n,m,\Delta) = \min(& \\
& O(n^{2/3+0}), \\
& O(n^{2/3}) \\
&)
\end{aligned}$$

That is,

$$\min_{1\le m\le n} \max_{0\le\Delta\le 1} R(n,m,\Delta) = O(n^{2/3})$$

Problem 4

INCOMPLETE; WILL UPDATE SOON