Assignment 2

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Question 5(b)

Verify Rolle's theorem for the following function:
$f(x) = e^{-x} \sin x$ on $[0, \pi]$

Solution: Given, $f(x) = e^{-x} \sin x$ on $[0, \pi]$

For the function to satisfy the Rolle's theorem, the following conditions should be satisfied:

1) $f(x)$ should be continuous in $[0, \pi]$
2) $f(0) = f(\pi) = 0$
3) $f(x)$ should be differentiable in $(0, \pi)$

Here, e^{-x} is an exponential and continuous function and $\sin x$ is a trigonometric and continuous function $\therefore f(x) = e^{-x} \sin x$ is continuous on $[0, \pi]$.

Also,

$$f(0) = e^0 \sin 0 = 0 \tag{1}$$

$$f(\pi) = e^{\pi} \sin \pi = 0 \tag{2}$$

$$\therefore f(0) = f(\pi) = 0 \tag{3}$$

$f(x)$ is differentiable on $(0, \pi)$ if Left hand derivative(LHD) and right hand derivative(RHD) exists and equal for every $x \in (0, \pi)$ as we already know $f(x)$ is continuous.

If $c \in (0, \pi)$, then

$$\text{LHD} = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \tag{4}$$

$$= \lim_{h \to 0^-} \frac{e^{-(c+h)} \sin(c+h) - e^{-c} \sin c}{h} \tag{5}$$

$$= e^{-c} \lim_{h \to 0^-} \left(\frac{e^{-h} \sin(c+h) - \sin c}{h} \right) \tag{6}$$

Similarly,

$$\text{RHD} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \tag{7}$$

$$= \lim_{h \to 0^+} \frac{e^{-(c+h)} \sin(c+h) - e^{-c} \sin c}{h} \tag{8}$$

$$= e^{-c} \lim_{h \to 0^+} \left(\frac{e^{-h} \sin(c+h) - \sin c}{h} \right) \tag{9}$$

We know that,

$$\sin(x+h) = \sin x \cos h + \cos x \sin h \tag{10}$$

$$\text{Also, } \lim_{h \to 0^-} \frac{\sin h}{h} = \lim_{h \to 0^+} \frac{\sin h}{h} = 1, \tag{11}$$

$$\lim_{h \to 0^-} \cos h = \lim_{h \to 0^+} \cos h = 1, \tag{12}$$

$$\lim_{h \to 0^-} e^{-h} = \lim_{h \to 0^+} e^{-h} = 1 \tag{13}$$

From equations (10) (11) (12) (13), equation (6) becomes

$$\text{LHD} = e^{-c} \lim_{h \to 0^-} \left(e^{-h} \cos c + \frac{e^{-h} sinc - sinc}{h} \right) \tag{14}$$

$$= e^{-c} \left(\cos c + \sin c \lim_{h \to 0^-} \frac{e^{-h} - 1}{h} \right) \tag{15}$$

Similarly, from equations (10) (11) (12) (13), equation (9) becomes

$$\text{RHD} = e^{-c} \lim_{h \to 0^+} \left(e^{-h} \cos c + \frac{e^{-h} sinc - sinc}{h} \right) \tag{16}$$

$$= e^{-c} \left(\cos c + \sin c \lim_{h \to 0^+} \frac{e^{-h} - 1}{h} \right) \tag{17}$$

We know that, $\lim_{h \to 0^-} \frac{1 - e^{-h}}{h}$ and $\lim_{h \to 0^+} \frac{1 - e^{-h}}{h}$ exist.

$\implies$ both LHD and RHD exists.

$$\lim_{h \to 0^-} \frac{1 - e^{-h}}{h} = \lim_{h \to 0^+} \frac{1 - e^{-h}}{h} = 1, \tag{18}$$

$$\implies \text{LHD} = \text{RHD} = e^{-x} (\cos x - \sin x) \tag{19}$$

$\because$ LHD and RHD exists and equal, we can say that $f(x)$ is differentiable.

$\implies f(x)$ is differentiable in $(0, \pi)$.

Thus, all the conditions of the Rolle's theorem are satisfied.

$\therefore \exists c \in (0, \pi)$ such that $f'(c) = 0$.

By differentiating $f(x)$,

$$f'(x) = e^{-x} (\cos x - \sin x) \tag{20}$$

$\because f'(c) = 0$, substituing $f'(x)$ at $x = c$ from equation (20),

$$e^{-c}\left(\cos c - \sin c\right) = 0 \tag{21}$$

$$\implies \cos c - \sin c = 0 \tag{22}$$

$$\implies c = \frac{\pi}{4} \tag{23}$$

Clearly, $\frac{\pi}{4} \in (0, \pi)$
Hence, Rolle's theorem is verified.
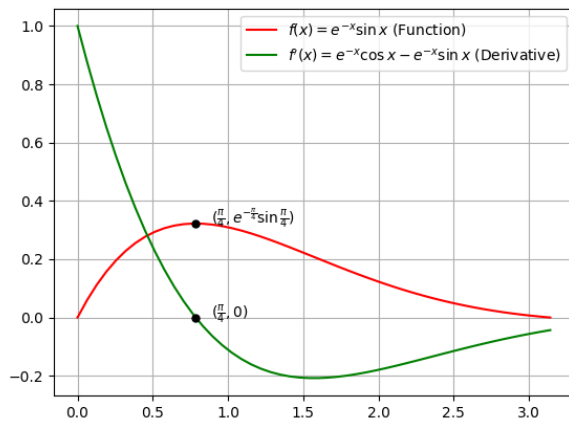Also, we can see from the graph clearly that all the necessary conditions for Rolle's theorem are satisfied and $f'(\frac{\pi}{4}) = 0$

Fig. 1. Graph of $f(x)$ and $f'(x)$ in $[0, \pi]$