Assignment 2

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Question 5(b)
Verify Rolle's theorem for the following function: $f(x) = e^{-x}\sin x$ on $[0, \pi]$
Solution: Given, $f(x) = e^{-x}\sin x$ on $[0, \pi]$
For the function to satisfy the Rolle's theorem, the following conditions should be satisfied:

1) $f(x)$ should be continuous in $[0, \pi]$
2) $f(0) = f(\pi) = 0$
3) $f(x)$ should be differentiable in $(0, \pi)$

Here, e^{-x} is an exponential and continuous function and $\sin x$ is a trigonometric and continuous function
$\therefore f(x) = e^{-x}\sin x$ is continuous on $[0, \pi]$.
Also,

$$f(0) = e^{0}\sin 0 = 0 \tag{1}$$
$$f(\pi) = e^{\pi}\sin \pi = 0 \tag{2}$$
$$\therefore f(0) = f(\pi) = 0 \tag{3}$$

$f(x)$ is differentiable on $(0, \pi)$ if Left hand derivative(LHD) and right hand derivative(RHD) exists and equal for every $x \in (0, \pi)$ as we already know $f(x)$ is continuous.
If $c \in (0, \pi)$, then

$$\text{LHD} = \lim_{h\to 0^-} \frac{f(c+h) - f(c)}{h} \tag{4}$$
$$= \lim_{h\to 0^-} \frac{e^{-(c+h)}\sin(c+h) - e^{-c}\sin c}{h} \tag{5}$$
$$= e^{-c}\lim_{h\to 0^-}\left(\frac{e^{-h}\sin(c+h) - \sin c}{h}\right) \tag{6}$$

Similarly,

$$\text{RHD} = \lim_{h\to 0^+} \frac{f(c+h) - f(c)}{h} \tag{7}$$
$$= \lim_{h\to 0^+} \frac{e^{-(c+h)}\sin(c+h) - e^{-c}\sin c}{h} \tag{8}$$
$$= e^{-c}\lim_{h\to 0^+}\left(\frac{e^{-h}\sin(c+h) - \sin c}{h}\right) \tag{9}$$

We know that,

$$\sin(x+h) = \sin x\cos h + \cos x\sin h \tag{10}$$
$$\text{Also, } \lim_{h\to 0^-}\frac{\sin h}{h} = \lim_{h\to 0^+}\frac{\sin h}{h} = 1, \tag{11}$$
$$\lim_{h\to 0^-}\cos h = \lim_{h\to 0^+}\cos h = 1, \tag{12}$$
$$\lim_{h\to 0^-} e^{-h} = \lim_{h\to 0^+} e^{-h} = 1 \tag{13}$$

From equations (10) (11) (12) (13), equation (6) becomes

$$\text{LHD} = e^{-c}\lim_{h\to 0^-}\left(e^{-h}\cos c + \frac{e^{-h}sinc - sinc}{h}\right) \tag{14}$$
$$= e^{-c}\left(\cos c + \sin c\lim_{h\to 0^-}\frac{e^{-h} - 1}{h}\right) \tag{15}$$

Similarly, from equations (10) (11) (12) (13), equation (9) becomes

$$\text{RHD} = e^{-c}\lim_{h\to 0^+}\left(e^{-h}\cos c + \frac{e^{-h}sinc - sinc}{h}\right) \tag{16}$$
$$= e^{-c}\left(\cos c + \sin c\lim_{h\to 0^+}\frac{e^{-h} - 1}{h}\right) \tag{17}$$

We know that, $\lim_{h\to 0^-}\frac{1-e^{-h}}{h}$ and $\lim_{h\to 0^+}\frac{1-e^{-h}}{h}$ exist.
$\implies$ both LHD and RHD exists.

$$\lim_{h\to 0^-}\frac{1-e^{-h}}{h} = \lim_{h\to 0^+}\frac{1-e^{-h}}{h} = 1, \tag{18}$$
$$\implies \text{LHD} = \text{RHD} = e^{-c}(\cos c - \sin c) \tag{19}$$

$\because$ LHD and RHD exists and equal, we can say that $f(x)$ is differentiable.
$\implies f(x)$ is differentiable in $(0, \pi)$.
Thus, all the conditions of the Rolle's theorem are satisfied.
$\therefore \exists c \in (0, \pi)$ such that $f'(c) = 0$.
By differentiating $f(x)$,

$$f'(x) = e^{-x}(\cos x - \sin x) \tag{20}$$

$\because$ $f'(c) = 0$, substituing $f'(x)$ at $x = c$ from equation (20),

$$e^{-c}(\cos c - \sin c) = 0 \tag{21}$$
$$\implies \cos c - \sin c = 0 \tag{22}$$
$$\implies c = \frac{\pi}{4} \tag{23}$$

Clearly, $\frac{\pi}{4} \in (0, \pi)$
Hence, Rolle's theorem is verified.
Also, we can see from the graph clearly that all the necessary conditions for Rolle's theorem are satisfied and $f'(\frac{\pi}{4}) = 0$

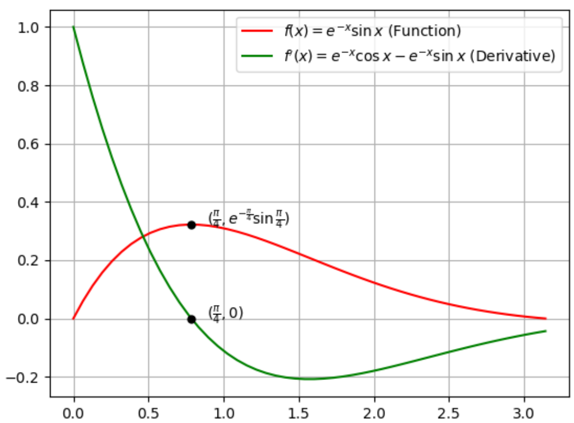

Fig. 1. Graph of $f(x)$ and $f'(x)$ in $[0, \pi]$