Assignment - Random Numbers

Kotikalapudi Karthik (CS21BTECH11030)

Contents

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files

```
wget https://github.com/Karthik−Kotikalapudi
    /AI1110/blob/main/Assignment−
    RandomNum/Codes/functions.h
wget https://github.com/Karthik−Kotikalapudi
    /AI1110/blob/main/Assignment−
    RandomNum/Codes/1.1.c
```

then compile and execute the C program with

```
gcc 1.1.c
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The following code plots Fig. (1.2)

```
wget https://github.com/Karthik−Kotikalapudi
    /AI1110/blob/main/Assignment−
    RandomNum/Codes/1.2.py
```

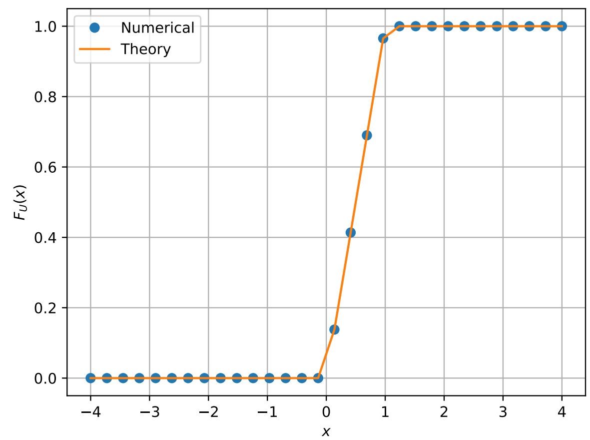

Fig. 1.2: The CDF of U

Run this code using

```
$ python3 1.2.py
```

1.3 Find a theoretical expression for $F_U(x)$
Solution: Given, U is a uniform random variable. Let

$$p_U(x) = 1 \text{ if } x \in [0,1] \tag{1.2}$$

$$F_U(x) = \Pr(U \le x) \tag{1.3}$$

$$= \int_{-\infty}^{x} p_U(x)dx \tag{1.4}$$

$$= \begin{cases} 0, & x \in (-\infty, 0) \\ \int_0^x dx, & x \in [0,1] \\ 1, & x \in (1,\infty) \end{cases} \tag{1.5}$$

$$\therefore F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in [0,1] \\ 1, & x \in (1,\infty) \end{cases} \tag{1.6}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.7}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.8}$$

Write a C program to find the mean and variance of U.
Solution: Mean: 0.50007 Variance: 0.083301
Download the following files

```
wget https://github.com/Karthik-Kotikalapudi
    /AI1110/blob/main/Assignment-
    RandomNum/Codes/1.4.c
```

then compile and execute the C program using

```
gcc 1.4.c -lm
./a.out
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.9}$$

Solution: Substituting $k = 1$ in the equation (1.9), and $F_U(x)$ from equation (1.6)

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.10}$$

$$E[U] = \int_0^1 x dx \tag{1.11}$$

$$\implies E[U] = \frac{1}{2} \tag{1.12}$$

$$E\left[U^2\right] = \int_0^1 x^2 dx \tag{1.13}$$

$$\implies E\left[U^2\right] = \frac{1}{3} \tag{1.14}$$

$$Var[U] = E\left[U^2\right] - (E[U])^2 \tag{1.15}$$

$$\implies Var[U] = \frac{1}{6} \tag{1.16}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following file

```
wget https://github.com/Karthik-Kotikalapudi
    /AI1110/blob/main/Assignment-
    RandomNum/Codes/2.1.c
```

```
gcc 2.1.c
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: We know that,

$$Q(x) = \Pr(X > x) \tag{2.2}$$

$$F_X(x) = 1 - Q(x) \tag{2.3}$$

The CDF of X is plotted in Fig. 2.2

```
wget https://github.com/Karthik-Kotikalapudi
    /AI1110/blob/main/Assignment-
    RandomNum/Codes/2.2.py
```

```
$ python3 2.2.py
```

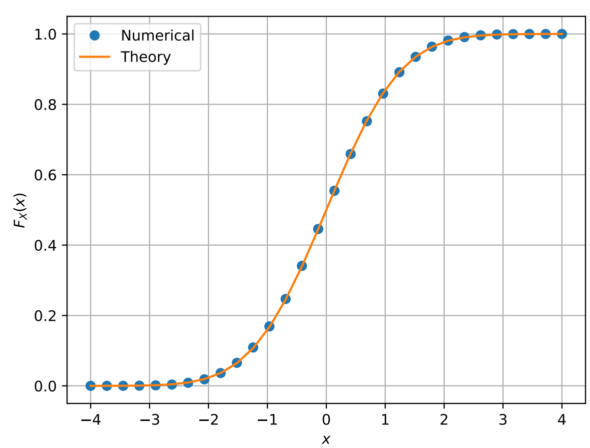

Fig. 2.2: The CDF of X

The properties of CDF are:

a) Monotonic Increasing function.
b)

$$\lim_{x \to -\infty} F(x) = 0 \tag{2.4}$$

$$\lim_{x \to \infty} F(x) = 1 \tag{2.5}$$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.6}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/Karthik-Kotikalapudi
    /AI1110/blob/main/Assignment-
    RandomNum/Codes/2.3.py
```

```
$ python3 2.2.py
```

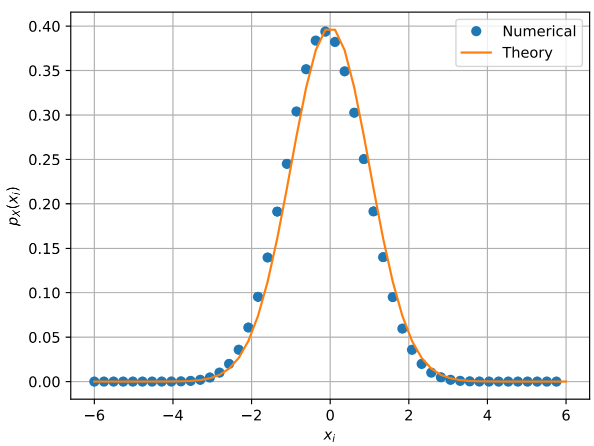

Fig. 2.3: The PDF of X

The properties of PDF are:

a)
$$\int_{-\infty}^{\infty} p(x)dx = 1 \tag{2.7}$$

b) $\forall x \in \mathbb{R} p(x) \geq 0$

2.4 Find the mean and variance of X by writing a C program.
Solution: Mean: 0.000294, variance: 0.999561
Download codes from

```
wget https://github.com/Karthik-Kotikalapudi
    /AI1110/blob/main/Assignment-
    RandomNum/Codes/2.4.c
```

then compile and execute using

```
gcc 2.4.c
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.8}$$

repeat the above exercise theoretically. **Solution:** From equation (1.9),

$$E[X] = \int_{-\infty}^{\infty} x p_x(x)\, dx \tag{2.9}$$

$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{\frac{-x^2}{2}} dx \tag{2.10}$$

Here $xe^{\frac{-x^2}{2}}$ is an odd function. Therefore,

$$E[X] = 0 \tag{2.11}$$

$$Var[X] = \int_{-\infty}^{\infty} x^2 p_x(x)\, dx \tag{2.12}$$

$$Var[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{\frac{-x^2}{2}} dx \tag{2.13}$$

$$\int x^2 e^{\frac{-x^2}{2}} = -xe^{\frac{-x^2}{2}} + \int e^{\frac{-x^2}{2}} dx \tag{2.14}$$

$$\implies Var[x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx \tag{2.15}$$

$$\implies Var[x] = \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \tag{2.16}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution: Download the files from

```
wget https://github.com/Karthik-Kotikalapudi
    /AI1110/blob/main/Assignment-
    RandomNum/Codes/3.1.c
wget https://github.com/Karthik-Kotikalapudi
    /AI1110/blob/main/Assignment-
    RandomNum/Codes/3.1.py
```

then compile and execute the C program then run the python program using

```
gcc 3.1.c
./a.out
python3 3.1.py
```

The CDF of V is plotted in Fig. 3.1

3.2 Find a theoretical expression for $F_V(x)$.
Solution: We know that

$$F_V(v) = \Pr(V \leq v) \tag{3.2}$$

$$V = -2\ln(1-U) \tag{3.3}$$

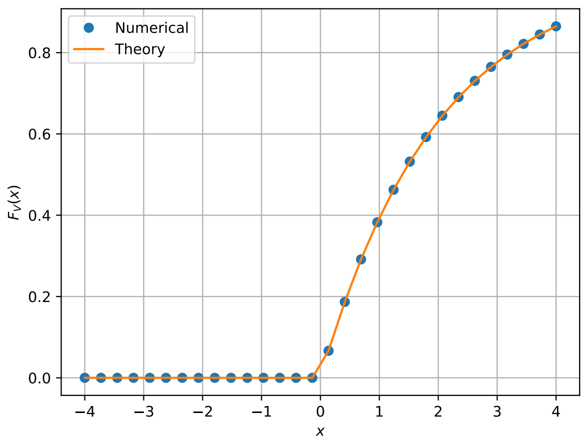

Fig. 3.1: The CDF of V

Substituting equation (3.3) in equation (3.2),

$$F_V(x) = \Pr(-2\ln(1-U) \le x) \tag{3.4}$$

$$= \Pr\left(U \le 1 - e^{-\frac{x}{2}}\right) \tag{3.5}$$

$$= F_U\left(1 - e^{-\frac{x}{2}}\right) \tag{3.6}$$

From equation (1.6),

$$F_V(x) = 1 - e^{-\frac{x}{2}} \tag{3.7}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the files from

```
wget https://github.com/Karthik-Kotikalapudi
    /AI1110/blob/main/Assignment-
    RandomNum/Codes/4.1.c
```

then compile and execute the C program using

```
gcc 4.1.c
./a.out
```

4.2 Find the CDF of T.
Solution: The CDF of T is plotted in the Fig. 4.2
Download the files from

```
wget https://github.com/Karthik-Kotikalapudi
    /AI1110/blob/main/Assignment-
    RandomNum/Codes/4.2.py
```

then run the python program using

```
python3 4.2.py
```

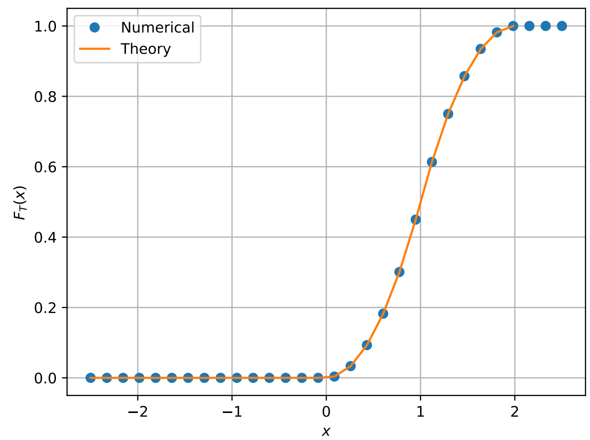

Fig. 4.2: The CDF of T

4.3 Find the PDF of T. **Solution:** The CDF of T is plotted in the Fig. 4.3
Download the files from

```
wget https://github.com/Karthik-Kotikalapudi
    /AI1110/blob/main/Assignment-
    RandomNum/Codes/4.3.py
```

then run the python program using

```
python3 4.3.py
```

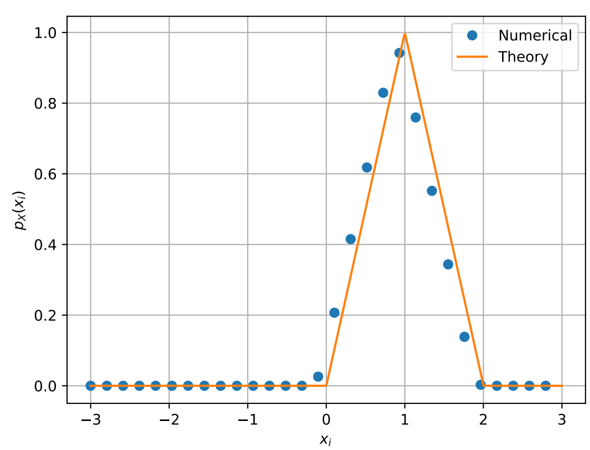

Fig. 4.3: The PDF of T

4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution: For calculating PDF, we know that

$$T = U_1 + U_2 \tag{4.2}$$

$$f_T(t) = \int_{-\infty}^{\infty} f_{U_1}(u) f_{U_2}(t-u) dt \tag{4.3}$$

$$f_T(t) = \int_{0}^{2} f_{U_1}(u) f_{U_2}(t-u) dt \tag{4.4}$$

If $0 < t < 1$,

$$f_T(t) = \int_{0}^{1} f_{U_2}(t-u) dt \tag{4.5}$$

$$f_T(t) = \int_{0}^{t} f_{U_2}(t-u) dt \tag{4.6}$$

$$f_T(t) = t \tag{4.7}$$

If $1 < t < 2$,

$$f_T(t) = \int_{1}^{2} f_{U_2}(t-u) dt \tag{4.8}$$

$$f_T(t) = \int_{1}^{t} f_{U_2}(t-u) dt \tag{4.9}$$

$$f_T(t) = 2 - t \tag{4.10}$$

For CDF,

$$F_T(x) = \int_{-\infty}^{x} p_T(t) dt \tag{4.11}$$

$$F_T(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ \frac{x^2}{2}, & x \in (0,1) \\ -\frac{x^2}{2} + 2x - 1, & x \in (1,2) \\ 1, & x \in (2, \infty) \end{cases} \tag{4.12}$$

4.5 Verify your results through a plot.
Solution: The theoritical CDF and PDF of T are plotted in the Figures, Fig. 4.2, Fig. 4.3 respectively.

5 Maximul Likelihood

5.1 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, $X \in \{1, -1\}$, is Bernoulli and $N \sim \mathcal{N}(0,1)$.

5.2 Plot Y.

5.3 Guess how to estimate X from Y.

5.4 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.3}$$

5.5 Find P_e.

5.6 Verify by plotting the theoretical P_e.

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.

6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.3}$$