Assignment 10

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Outline

Question

Probability, Random Variables and Stochastic Processes Chapter 2, Problem 2-25

A train and a bus arrive at the station at random between 9 A.M. and 10 A.M. The train stops for 10 minutes and the bus for x minutes. Find x so that the probability that the bus and the train will meet equals 0.5

Solution

Let's denote the random variable X_1 map to the set $\{0, 1\}$ where $X_1 = 0$ denote that bus and train don't meet and $X_1 = 1$ denote that they meet.

Let's denote the random variable X_2 map to the set $\{0, 1\}$ where $X_2 = 0$ denote that bus arrives first and $X_2 = 1$ denote that train arrives first.

Method-I

Given, train stops for 10 mins and bus stops for x minutes.
Let's draw a graph with Arrival time of bus in mins on X-axis and Arrival time of train in mins on Y-axis.
For the region in which bus and train meet(in blue color),
$Y < X + x$(train should arrive within x minutes after the bus) and
$X < Y + 10$(bus should arrive within 10 minutes after the train)

Graph

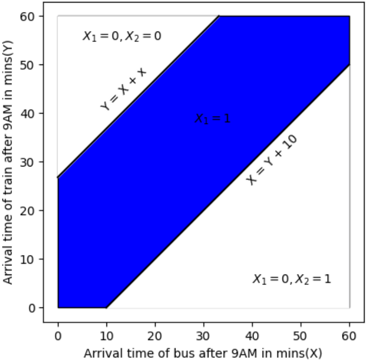

Figure 1: Arrival times of Bus and train

Finding the Value of x

$$\text{Given, } \Pr\left(X_1 = 1\right) = 0.5 \tag{1}$$
$$\implies \Pr\left(X_1 = 0\right) = 0.5 \tag{2}$$

Substituting the values from the Figure (1) in equation (2),

$$\frac{\frac{1}{2}\left[\left(60 - x\right)^2 + 50 \times 50\right]}{60 \times 60} = 0.5 \tag{3}$$
$$\implies \left(60 - x\right)^2 + 50 \times 50 = 60 \times 60 \tag{4}$$
$$\implies \left(60 - x\right)^2 = 1100 \tag{5}$$
$$\implies x = 60 - 10\sqrt{11} \approx 26.83 \text{ mins} \tag{6}$$

Method - II

Let the bus arrive at time t_B and train at t_T.
Let's take the area $1min^2$ as 1 unit

$$\implies n\left(\Sigma_{i=0}^{1} X_1 = i\right) = \int_0^{60} \left(\int_0^{60} dt_T\right) dt_B \tag{7}$$

$$= 3600 \tag{8}$$

If train arrives first, it can arrive in first 50 mins or last 10 mins.
If train arrives in first 50 mins, for each value of t_T, bus should arrive within $t_T + 10$ mins.
It train arrives in 50 to 60 mins, for each value of t_T, bus should arrive within 60 mins.

$$\implies n\left(X_1 = 1, X_2 = 1\right) = \int_0^{50} \left(\int_{t_T}^{t_T+10} dt_B\right) dt_T + \int_{50}^{60} \left(\int_{t_T}^{60} dt_B\right) dt_T \tag{9}$$

If Bus arrives first

If bus arrives first, it can arrive in 60 – x mins or last x mins.
If bus arrives in first 60 – x mins, for each value of t_B, train should arrive within $t_B + 10$ mins.
It bus arrives in 60 – x to 60 mins, for each value of t_T, train should arrive within 60 mins.

$$\implies n\left(X_1 = 1, X_2 = 0\right) = \int_0^{60-x} \left(\int_{t_B}^{t_B+x} dt_T\right) dt_B + \int_{60-x}^{60} \left(\int_{t_B}^{60} dt_T\right) dt_B \tag{10}$$

$$\text{Also, } n(X_1 = 1) = n\left(X_1 = 1, X_2 = 0\right) + n\left(X_1 = 1, X_2 = 1\right) \tag{11}$$

If Bus and train meet

$$n(X_1 = 1) = 500 + x\,(60 - x) + \int_{50}^{60} (60 - t_T)\,dt_T + \int_{60-x}^{60} (60 - t_B)\,dt_B \tag{12}$$

$$= 1100 + 120x - x^2 - \int_{50}^{60} t_T\,dt_T - \int_{60-x}^{60} t_B\,dt_B \tag{13}$$

$$= 1100 + 120x - x^2 - \left(\frac{60^2 - 50^2}{2}\right) - \left(\frac{60^2 - (60 - x)^2}{2}\right) \tag{14}$$

$$= \frac{1100 + 120x - x^2}{2} \tag{15}$$

$$\implies |n\,(X_1 = 1)\,| = \frac{(60 - x)^2 + 2500}{2} \tag{16}$$

Finding the Value of x

We know that,

$$\Pr\left(X_1 = 1\right) = \frac{n\left(X_1 = 1\right)}{n\left(\Sigma_{i=0}^{1} X_1 = i\right)} = 0.5 \tag{17}$$

$$\frac{\frac{1}{2}\left[\left(60 - x\right)^2 + 50 \times 50\right]}{60 \times 60} = 0.5 \tag{18}$$

$$\implies \left(60 - x\right)^2 + 50 \times 50 = 60 \times 60 \tag{19}$$

$$\implies \left(60 - x\right)^2 = 1100 \tag{20}$$

$$\implies x = 60 - 10\sqrt{11} \approx 26.83 \text{ mins} \tag{21}$$