DATA STRUCTURES AND ALGORITHMS

Hashing

Introduction of hash table

- Data structure that offers very fast insertion and searching, almost O(1).
- Relatively easy to program as compared to trees.
- Based on arrays, hence difficult to expand.
- No convenient way to visit the items in a hash table in any kind of order.

Hashing

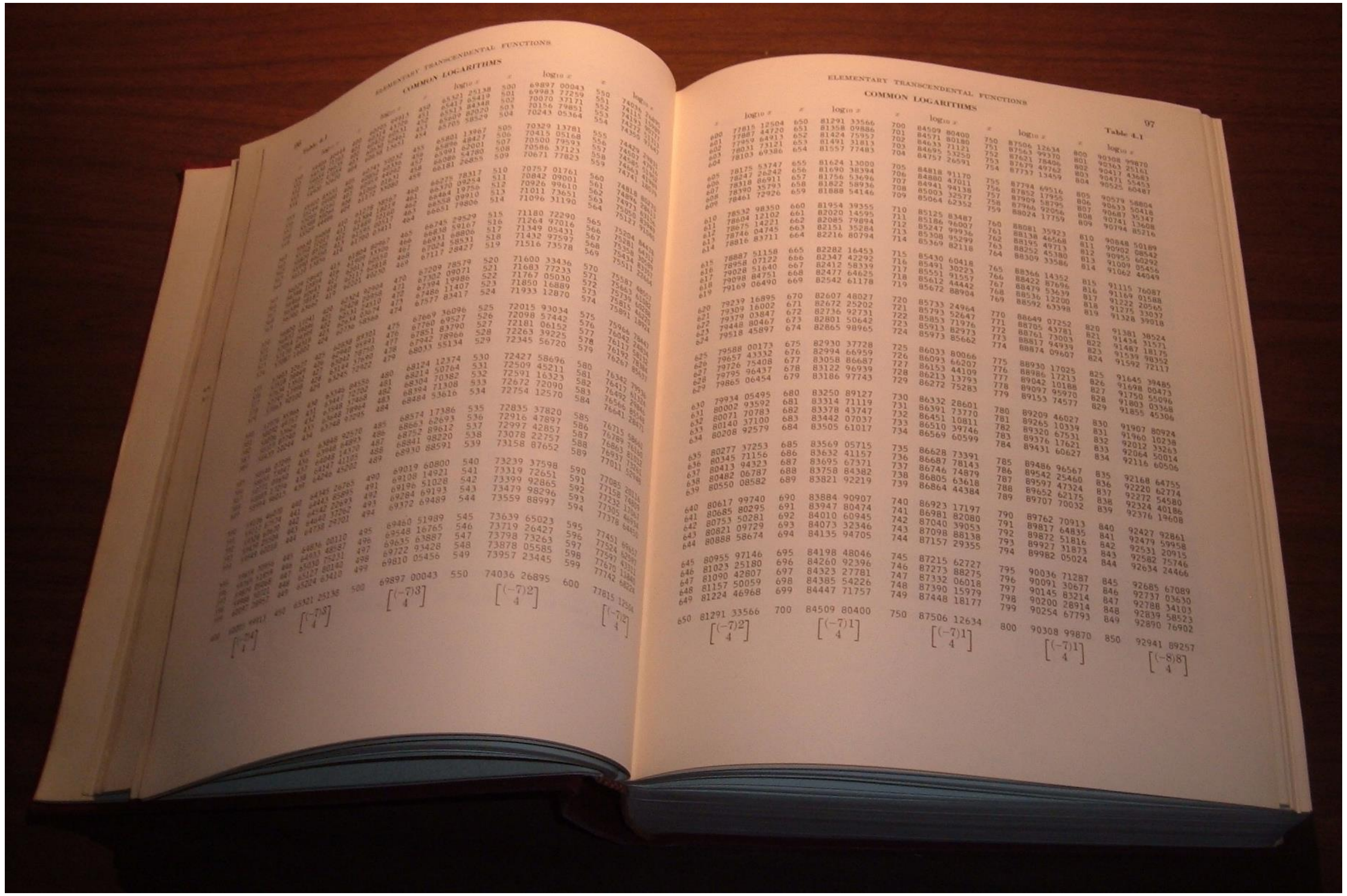
- A range of key values can be transformed into a range of array index values.
- A simple array can be used where each record occupies one cell of the array and the index number of the cell is the key value for that record.
- But keys may not be well arranged.
- In such a situation hash tables can be used.

Concept of Hashing

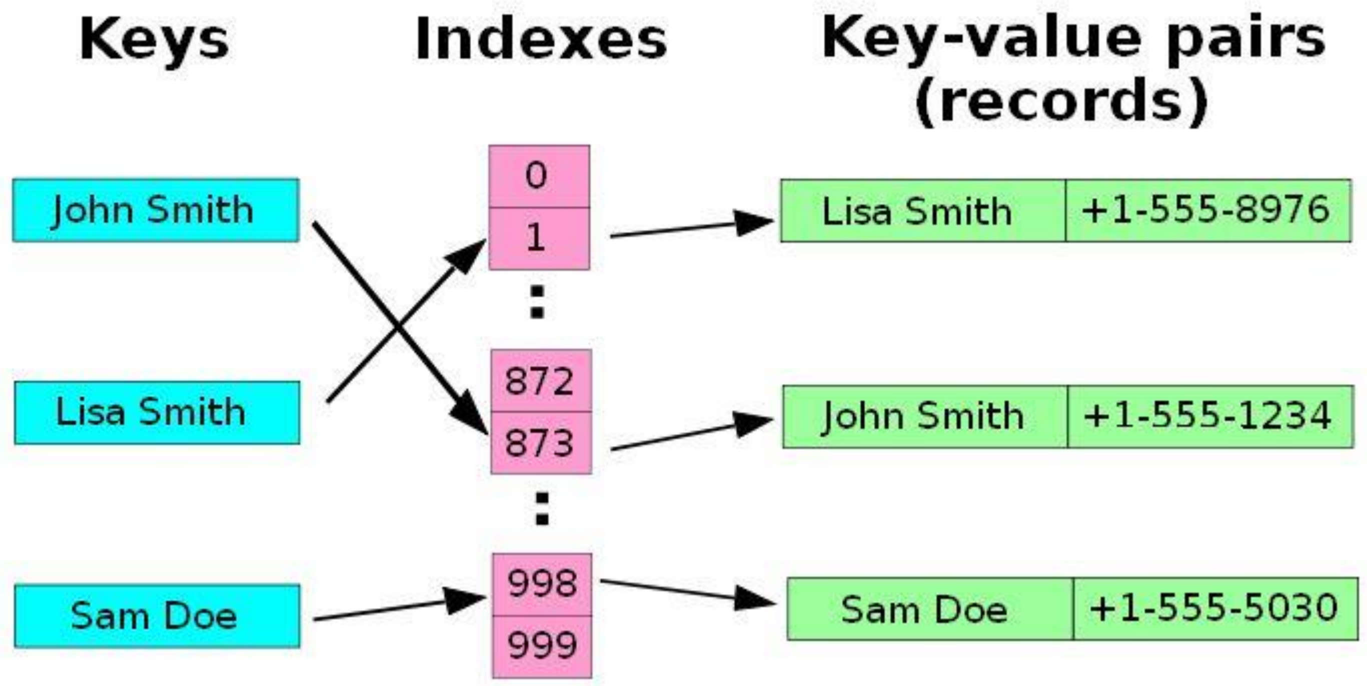
In CS, a **hash table**, or a **hash map**, is a data structure that associates keys (names) with values (attributes).

- ✓ Look-Up Table
- ✓ Dictionary
- ✓ Cache
- ✓ Extended Array

Tables of logarithms



Example



A small phone book as a hash table.

(Figure is from Wikipedia)

Dictionaries

Collection of pairs.

- (key, value)
- Each pair has a unique key.

Operations.

- Get(theKey)
- Delete(theKey)
- Insert(theKey, theValue)

Just An Idea

Hash table :

Collection of pairs,

Lookup function (Hash function)

Hash tables are often used to implement associative arrays,

Worst-case time for Get, Insert, and Delete is O(size).

Expected time is O(1).

Origins of the Term

The term "hash" comes by way of analogy with its standard meaning in the physical world, to "chop and mix." **D. Knuth** notes that **Hans Peter Luhn** of IBM appears to have been the first to use the concept, in a memo dated January 1953; the term hash came into use some ten years later.

Search vs. Hashing

Search tree methods: key comparisons

- Time complexity: O(size) or O(log n)

Hashing methods: hash functions

- Expected time: O(1)

Types

- Static hashing (section 8.2)
- Dynamic hashing (section 8.3)

Static Hashing

Key-value pairs are stored in a fixed size table called a *hash table*.

- A hash table is partitioned into many ***buckets***.
- Each bucket has many ***slots***.
- Each slot holds one record.
- A hash function f(x) transforms the identifier (key) into an address in the hash table

Hash table

s slots

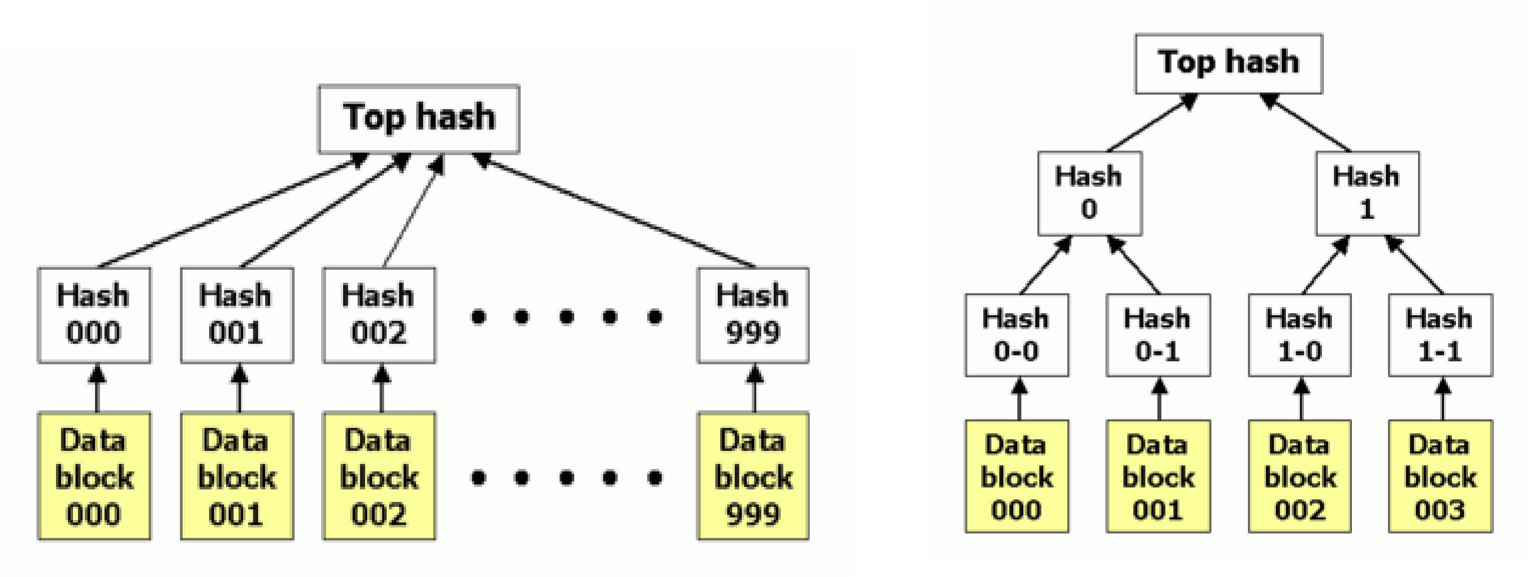
b buckets

	0	1		s-1
0			. . .	
1				
	. . .	. . .		. . .
b-1			. . .	

Data Structure for Hash Table

```
#define MAX_CHAR  10
#define TABLE_SIZE  13
typedef struct {
   char key[MAX_CHAR];
   /* other fields */
} element;
element hash_table[TABLE_SIZE];
```

Other Extensions



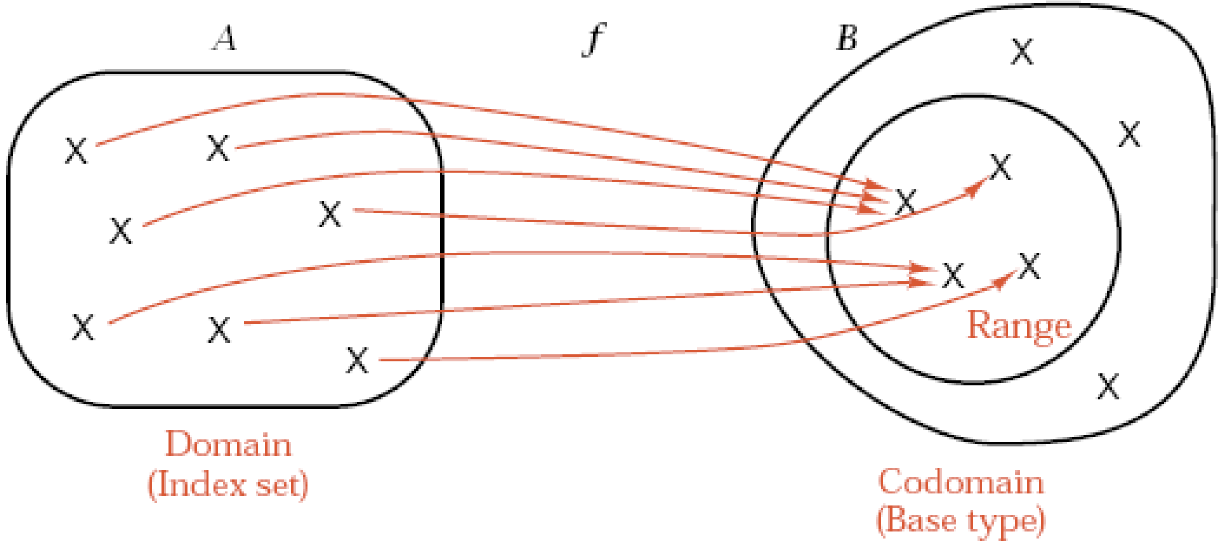
Hash List and Hash Tree

(Figure is from Wikipedia)

Formal Definition

Hash Function

In addition, one-to-one / onto



The Scheme

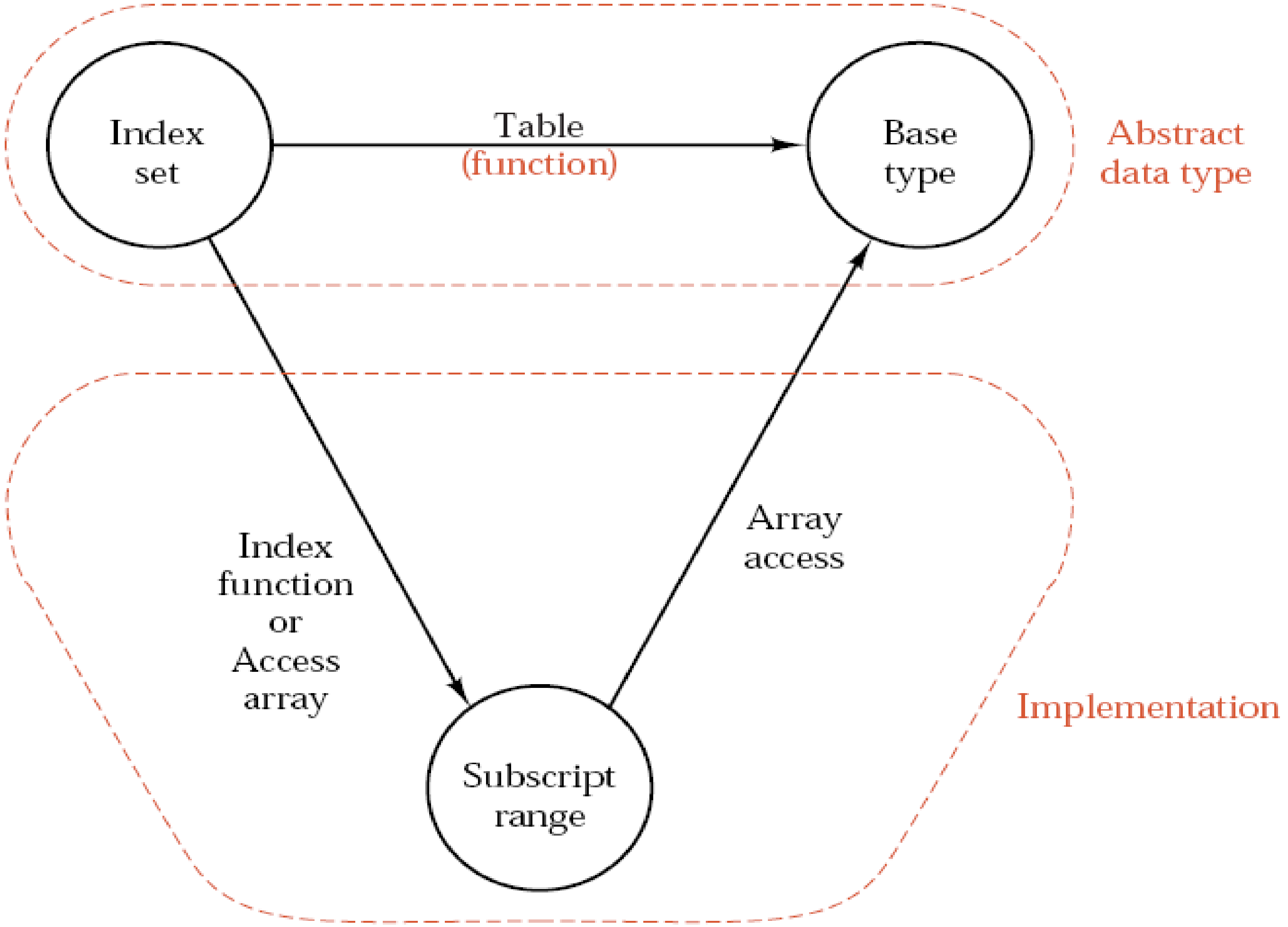


Figure is from Data Structures and Program Design In C++, 1999
Prentice-Hall, Inc.

Ideal Hashing

- Uses an array table[0:b-1].
  - ✓ Each position of this array is a bucket.
  - ✓ A bucket can normally hold only one dictionary pair.

- Uses a hash function f that converts each key k into an index in the range [0, b-1].

- Every dictionary pair (key, element) is stored in its home bucket table[f[key]].

Example

Pairs are: (22,a), (33,c), (3,d), (73,e), (85,f).

Hash table is table[0:7], b = 8.

Hash function is key (mod 11).

(3,d)		(22,a)	(33,c)			(73,e)	(85,f)
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]

What Can Go Wrong?

(3,d)		(22,a)	(33,c)			(73,e)	(85,f)
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]

Where does (26,g) go?

Keys that have the same home bucket are synonyms.

22 and 26 are synonyms with respect to the hash function that is in use.

The bucket for (26,g) is already occupied.

Some Issues

Choice of hash function.

Really tricky!

To avoid collision (two different pairs are in the same the same bucket.)

Size (number of buckets) of hash table.

Overflow handling method.

Overflow: there is no space in the bucket for the new pair.

Example (fig 8.1)

synonyms:
char, ceil,
clock, ctime

↑

overflow

	Slot 0	Slot 1
0	acos	atan
1		
2	char	ceil
3	define	
4	exp	
5	float	floor
6		
…		
25		

synonyms

synonyms

Choice of Hash Function

- Requirements
  - ✓ easy to compute
  - ✓ minimal number of collisions

- If a hashing function groups key values together, this is called clustering of the keys.

- A good hashing function distributes the key values uniformly throughout the range.

Some hash functions

- Middle of square

  H(x):= return middle digits of x^2

- Division

  H(x):= return x % k

- **Multiplicative:**

  H(x):= return the first few digits of the fractional part of x*k, where k is a fraction.

  advocated by D. Knuth in TAOCP vol. III.

Some hash functions II

Folding:

Partition the identifier x into several parts, and add the parts together to obtain the hash address

e.g. x=12320324111220; partition x into 123,203,241,112,20; then return the address 123+203+241+112+20=699

Shift folding vs. folding at the boundaries

Digit analysis:

If all the keys have been known in advance, then we could delete the digits of keys having the most skewed distributions, and use the rest digits as hash address.

Hashing By Division

- Domain is all integers.
- For a hash table of size *b*, the number of integers that get hashed into bucket *i* is approximately **$2^{32}/b$**.
- The division method results in a uniform hash function that maps approximately the same number of keys into each bucket.

Hashing By Division II

In practice, keys tend to be correlated.

If divisor is an even number, odd integers hash into odd home buckets and even integers into even home buckets.

20%14 = 6, 30%14 = 2, 8%14 = 8

15%14 = 1, 3%14 = 3, 23%14 = 9

divisor is an odd number, odd (even) integers may hash into any home.

20%15 = 5, 30%15 = 0, 8%15 = 8

15%15 = 0, 3%15 = 3, 23%15 = 8

Hashing By Division III

- Similar biased distribution of home buckets is seen in practice, when the divisor is a multiple of prime numbers such as 3, 5, 7, …
- The effect of each prime divisor p of b decreases as p gets larger.
- Ideally, choose large prime number b.
- Alternatively, choose b so that it has no prime factors smaller than 20.

Hash Algorithm via Division

```
void init_table(element ht[])
{
  int i;
  for (i=0; i<TABLE_SIZE; i++)
    ht[i].key[0]=NULL;
}

int transform(char *key)
{
  int number=0;
  while (*key) number += *key++;
  return number;
}
```

```
int hash(char *key)
{
  return (transform(key)
        % TABLE_SIZE);
}
```

Criterion of Hash Table

- The key density (or identifier density) of a hash table is the ratio n/T

  n is the number of keys in the table

  T is the number of distinct possible keys

- The loading density or loading factor of a hash table is ⍰ = n/(sb)

  s is the number of slots

  b is the number of buckets

Example

synonyms:
char, ceil,
clock, ctime

↑

overflow

	Slot 0	Slot 1
0	acos	atan
1		
2	char	ceil
3	define	
4	exp	
5	float	floor
6		
…		
25		

synonyms

synonyms

b=26, s=2, n=10, α=10/52=0.19, f(x)=the first char of x

Overflow Handling

An overflow occurs when the home bucket for a new pair (key, element) is full.

We may handle overflows by:

- Search the hash table in some systematic fashion for a bucket that is not full.
  - Linear probing (linear open addressing).
  - Quadratic probing.
  - Random probing.

- Eliminate overflows by permitting each bucket to keep a list of all pairs for which it is the home bucket.
  - Array linear list.
  - Chain.

Linear probing (linear open addressing)

Open addressing ensures that all elements are stored directly into the hash table, thus it attempts to resolve collisions using various methods.

Linear Probing resolves collisions by placing the data into the next open slot in the table.

Linear Probing – Get And Insert

divisor = b (number of buckets) = 17.
Home bucket = key % 17.

0				4				8				12				16
34	0	45				6	23	7			28	12	29	11	30	33

- Insert pairs whose keys are 6, 12, 34, 29, 28, 11, 23, 7, 0, 33, 30, 45

Linear Probing – Delete

0				4				8				12				16
34	0	45				6	23	7			28	12	29	11	30	33

Delete(0)

0				4				8				12				16
34	🟨	45				6	23	7			28	12	29	11	30	33

- Search cluster for pair (if any) to fill vacated bucket.

0				4				8				12				16
34	45	🟨				6	23	7			28	12	29	11	30	33

Linear Probing – Delete(34)

0				4				8				12				16
34	0	45				6	23	7			28	12	29	11	30	33

0				4				8				12				16
🟨	0	45				6	23	7			28	12	29	11	30	33

Search cluster for pair (if any) to fill vacated bucket.

0				4				8				12				16
0	🟨	45				6	23	7			28	12	29	11	30	33

0				4				8				12				16
0	45	🟨				6	23	7			28	12	29	11	30	33

Linear Probing – Delete(29)

0				4				8				12				16
34	0	45				6	23	7			28	12	29	11	30	33

0				4				8				12				16
34	0	45				6	23	7			28	12	🟨	11	30	33

Search cluster for pair (if any) to fill vacated bucket.

0				4				8				12				16
34	0	45				6	23	7			28	12	11	🟨	30	33

0				4				8				12				16
34	0	45				6	23	7			28	12	11	30	🟨	33

0				4				8				12				16
34	0	🟨				6	23	7			28	12	11	30	45	33

Performance Of Linear Probing

0				4				8				12				16
34	0	45				6	23	7			28	12	29	11	30	33

Worst-case find/insert/erase time is □(n), where n is the number of pairs in the table. This happens when all pairs are in the same cluster.

Expected Performance

0				4				8				12				16
34	0	45				6	23	7			28	12	29	11	30	33

α = loading density = (number of pairs)/b.

α = 12/17.

S_n = expected number of buckets examined in a successful search when n is large

U_n = expected number of buckets examined in a unsuccessful search when n is large

Time to put and remove is governed by U_n.

Expected Performance

alpha	S_n	U_n
0.50	1.5	2.5
0.75	2.5	8.5
0.90	5.5	50.5

S_n ~ ½(1 + 1/(1 – ⍰))

U_n ~ ½(1 + 1/(1 – ⍰)^2)

Note that 0 <= ⍰ <= 1.

The proof refers to D. Knuth’s TAOCP vol. III

α <= 0.75 is recommended.

Linear Probing (program 8.3)

```
void linear_insert(element item, element ht[]){
 int i, hash_value;
 i = hash_value = hash(item.key);
 while(strlen(ht[i].key)) {
 if (!strcmp(ht[i].key, item.key)) {
                fprintf(stderr, “Duplicate entry\n”);  exit(1);
          }
     i = (i+1)%TABLE_SIZE;
     if (i == hash_value) {
          fprintf(stderr, “The table is full\n”); exit(1);
     } }
 ht[i] = item;
}
```

Problem of Linear Probing

- Identifiers tend to cluster together
- Adjacent cluster tend to coalesce
- Increase the search time

Coalesce Phenomenon

bucket	x	bucket searched	bucket	x	bucket searched
0	acos	1	1	atoi	2
2	char	1	3	define	1
4	exp	1	5	ceil	4
6	cos	5	7	float	3
8	atol	9	9	floor	5
10	ctime	9	…		
…			25		

Average number of buckets examined is 41/11=3.73

Quadratic Probing

- Linear probing searches buckets (H(x)+i2)%b
- Quadratic probing uses a quadratic function of i as the increment
- Examine buckets H(x), (H(x)+i2)%b, (H(x)-i2)%b, for 1<=i<=(b-1)/2
- b is a prime number of the form 4j+3, j is an integer

Random Probing

- Random Probing works incorporating with random numbers.

  $H(x) := (H'(x) + S[i]) \% b$

  S[i] is a table with size b-1

  S[i] is a random permuation of integers [1,b-1].

Rehashing

Rehashing: Try H_1, H_2, …, H_m in sequence if collision occurs. Here H_i is a hash function.

Double hashing is one of the best methods for dealing with collisions.

If the slot is full, then a second hash function is calculated and combined with the first hash function.

$H(k, i) = (H_1(k) + i\, H_2(k)) \% m$

Summary: Hash Table Design

Performance requirements are given, determine maximum permissible loading density. Hash functions must usually be custom-designed for the kind of keys used for accessing the hash table.

We want a successful search to make no more than 10 comparisons (expected).

$S_n \sim \frac{1}{2}(1 + 1/(1 - \alpha))$

$\alpha <= 18/19$

Summary:
Hash Table Design II

We want an unsuccessful search to make no more than 13 comparisons (expected).

Un ~ ½(1 + 1/(1 – ⍰)2)

⍰ <= 4/5

So ⍰ <= min{18/19, 4/5} = 4/5.

Summary:
Hash Table Design III

Dynamic resizing of table.

Whenever loading density exceeds threshold (4/5 in our example), rehash into a table of approximately twice the current size.’

Fixed table size.

Loading density <= 4/5 => b >= 5/4*1000 = 1250.

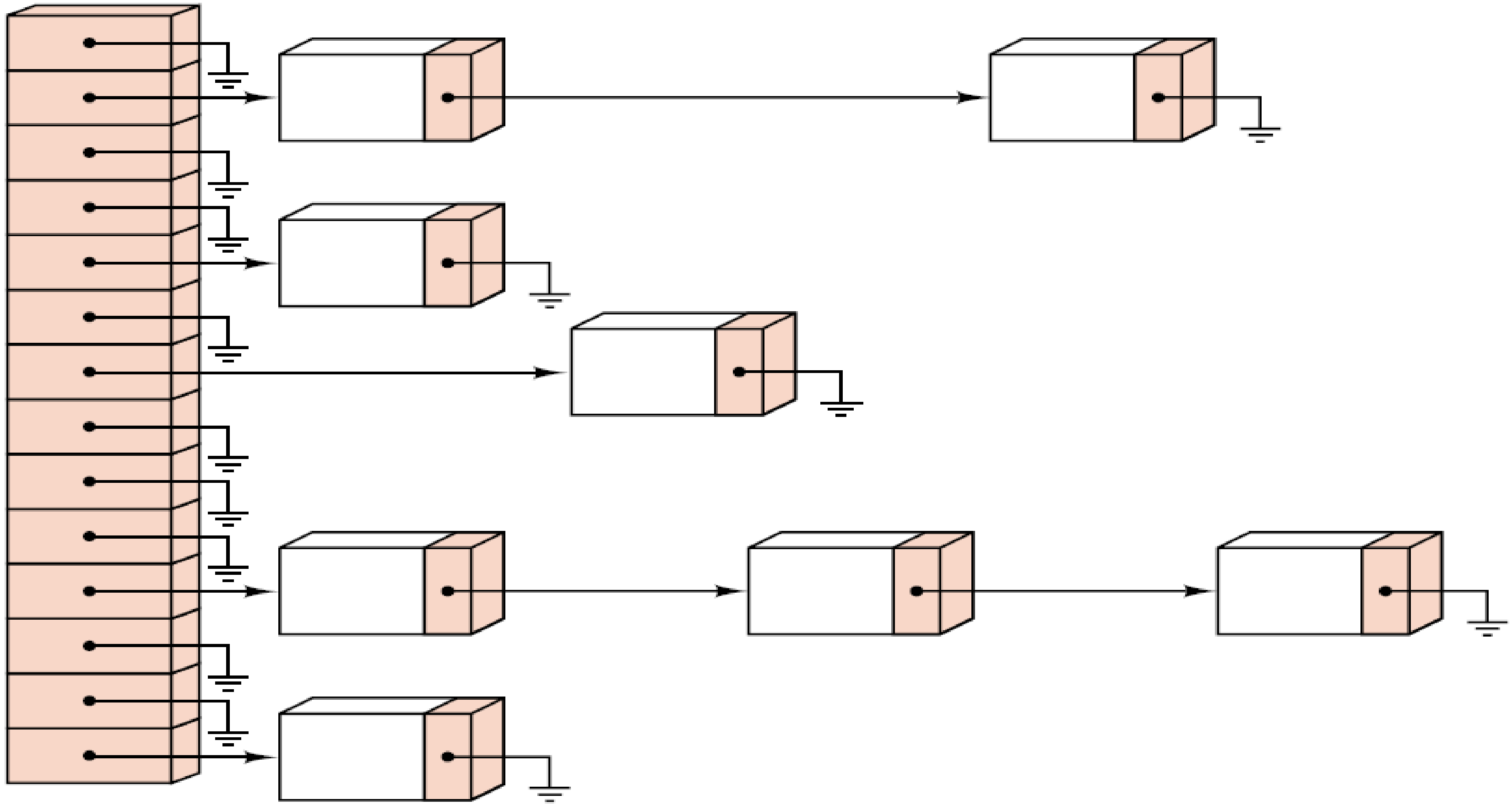
Pick b (equal to divisor) to be a prime number or an odd number with no prime divisors smaller than 20.

Data Structure for Chaining

```
#define MAX_CHAR 10
#define TABLE_SIZE 13
#define IS_FULL(ptr) (!(ptr))
typedef struct {
   char key[MAX_CHAR];
   /* other fields */
} element;
typedef struct list *list_pointer;
typedef struct list {
   element item;
   list_pointer link;
};
list_pointer hash_table[TABLE_SIZE];
```

The idea of **Chaining** is to combine the linked list and hash table to solve the overflow problem.

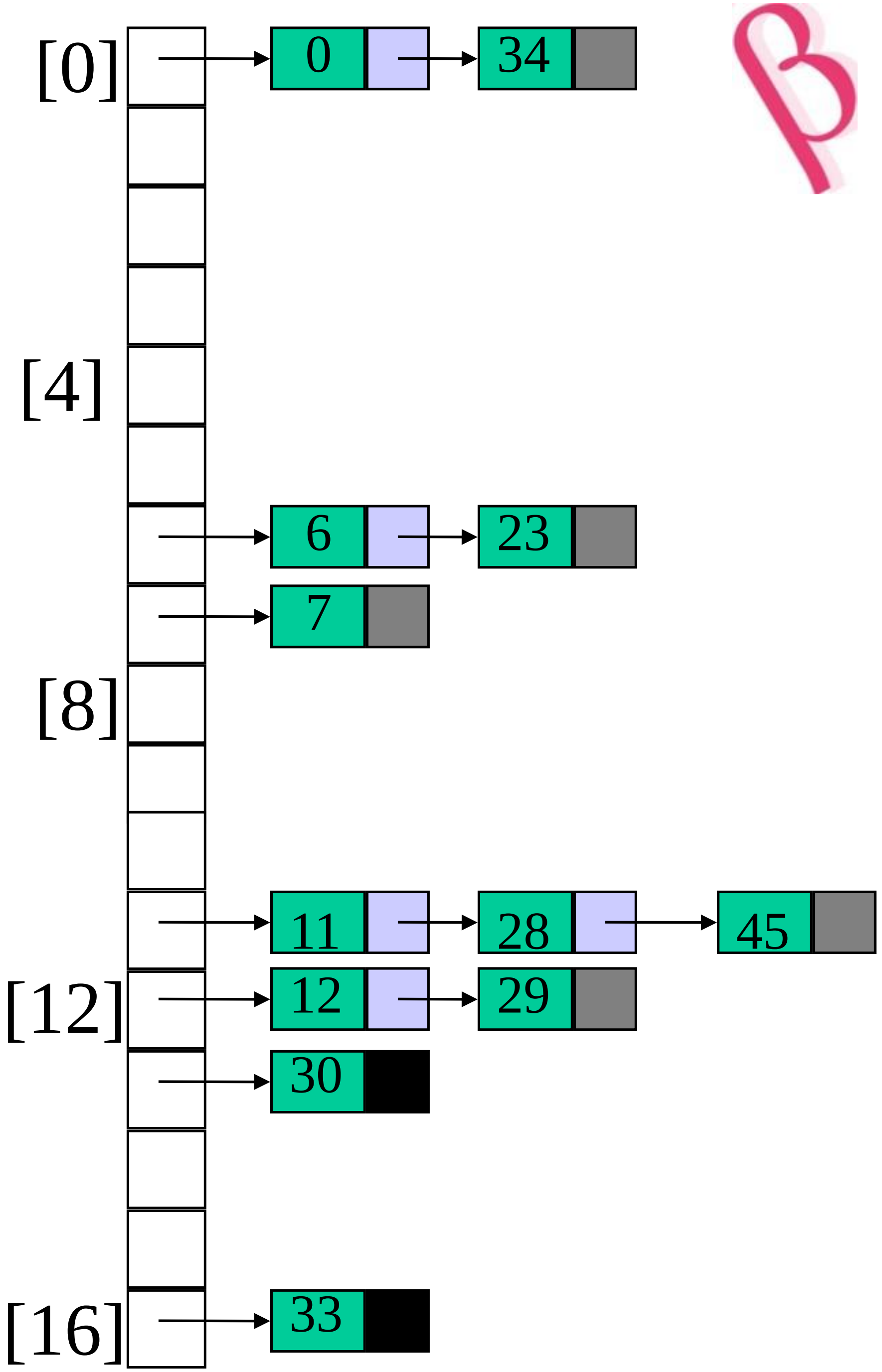
Figure of Chaining



Sorted Chains

- Put in pairs whose keys are 6, 12, 34, 29, 28, 11, 23, 7, 0, 33, 30, 45

- Bucket = key % 17.



Expected Performance

Note that ⍰ >= 0.

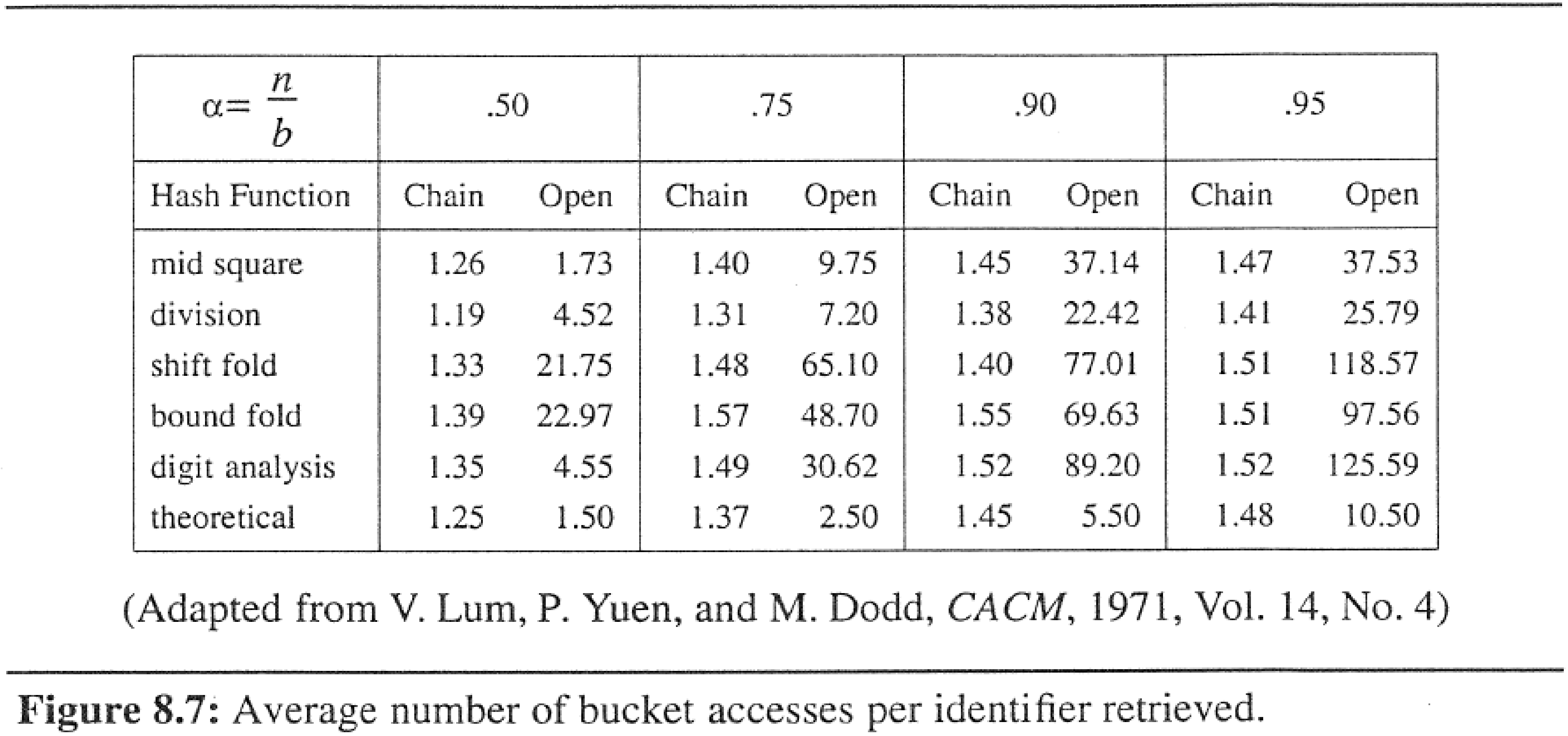
Expected chain length is ⍰.

S_n ~ 1 + ⍰/2.

U_n ~ ⍰

Refer to the theorem 8.1 of textbook, and refer to D. Knuth’s TAOCP vol. III for the proof.

Comparison



$\alpha = \frac{n}{b}$	.50		.75		.90		.95	
Hash Function	Chain	Open	Chain	Open	Chain	Open	Chain	Open
mid square	1.26	1.73	1.40	9.75	1.45	37.14	1.47	37.53
division	1.19	4.52	1.31	7.20	1.38	22.42	1.41	25.79
shift fold	1.33	21.75	1.48	65.10	1.40	77.01	1.51	118.57
bound fold	1.39	22.97	1.57	48.70	1.55	69.63	1.51	97.56
digit analysis	1.35	4.55	1.49	30.62	1.52	89.20	1.52	125.59
theoretical	1.25	1.50	1.37	2.50	1.45	5.50	1.48	10.50

(Adapted from V. Lum, P. Yuen, and M. Dodd, *CACM*, 1971, Vol. 14, No. 4)

Figure 8.7: Average number of bucket accesses per identifier retrieved.

Comparison : Load Factor

If **open addressing** is used, then each table slot holds at most one element, therefore, the loading factor can **never** be greater than 1.

If **external chaining** is used, then each table slot can hold many elements, therefore, the loading factor **may be** greater than 1.

Conclusion

The main **tradeoffs** between these methods are that **linear probing** has the best cache performance but is most sensitive to clustering, while **double hashing** has poorer cache performance but exhibits virtually no clustering; **quadratic probing** falls in between the previous two methods.

Dynamic Hashing (extensible hashing)

- In this hashing scheme the set of keys can be varied, and the address space is allocated dynamically
  - File **F**: a collection of records
  - Record **R**: a key + data, stored in pages (buckets)
  - space utilization

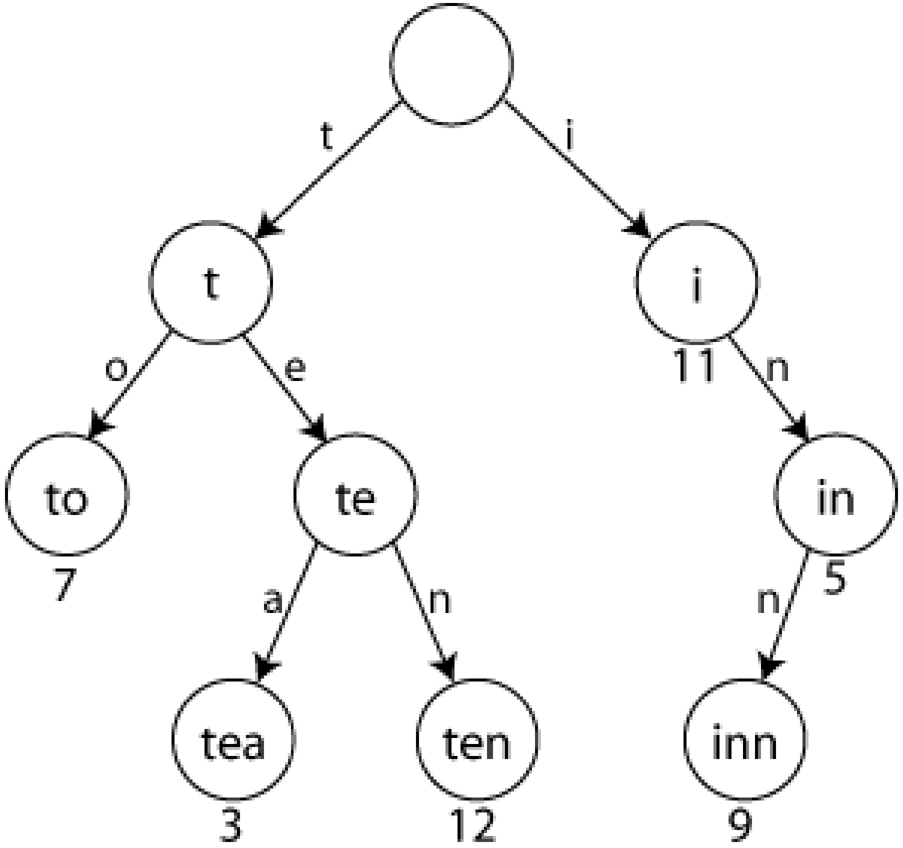
$$\frac{NumberOfRecord}{NumberOfPages * PageCapacity}$$

Trie

Trie: a binary tree in which an identifier is located by its bit sequence

Key lookup is faster. Looking up a key of length *m* takes worst case O(*m*) time.

Refer to textbook section 10.9 for more details.



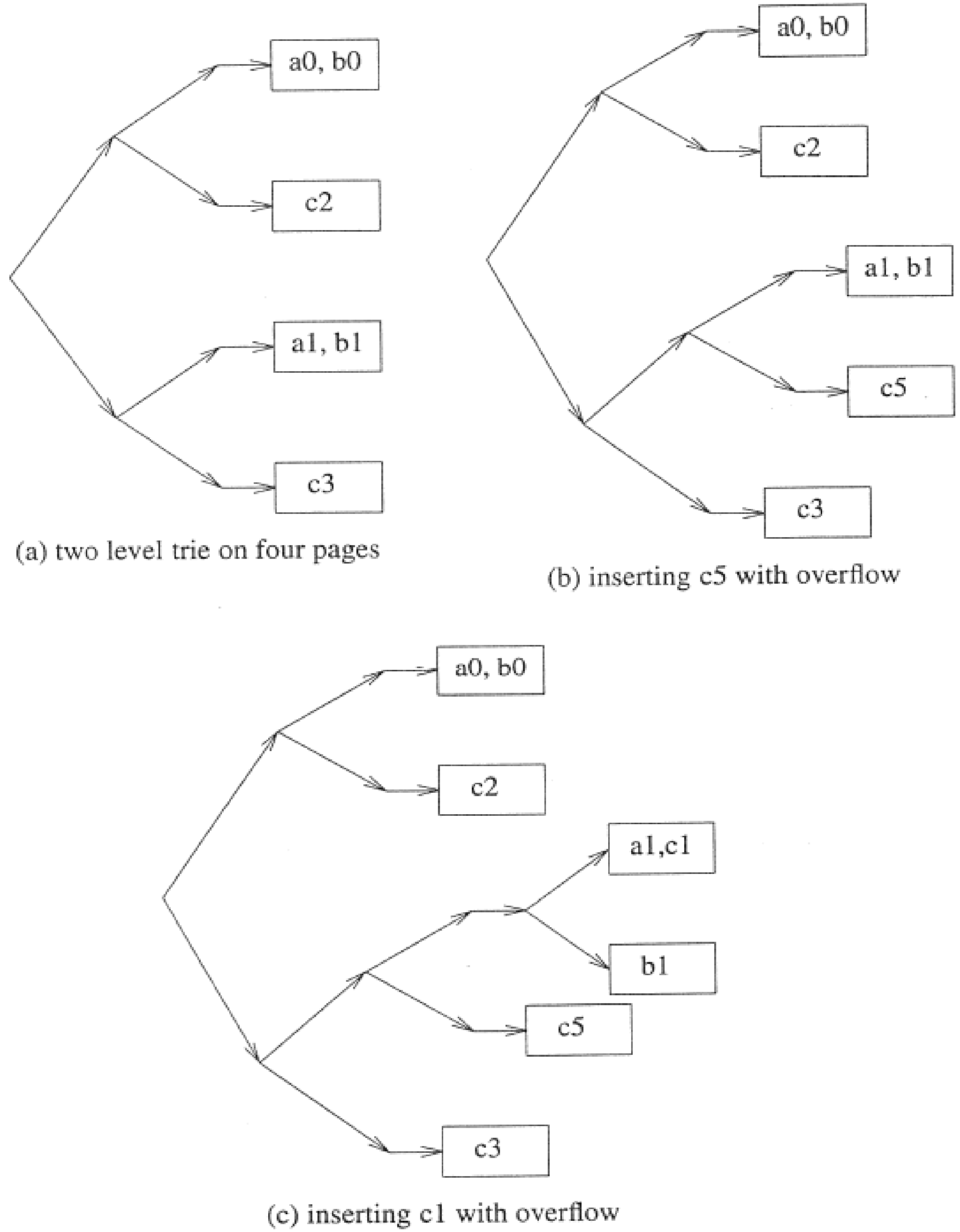
Dynamic Hashing Using Directories

Identifiers	Binary representaiton
a0	100 000
a1	100 001
b0	101 000
b1	101 001
c0	110 000
c1	110 001
c2	110 010
c3	110 011

Example:
M (# of pages)=4,
P (page capacity)=2

Allocation: lower order two bits

Figure 8.8: Some identifiers requiring 3 bits per character(p.414)



(a) two level trie on four pages

(b) inserting c5 with overflow

(c) inserting c1 with overflow

Figure 8.9: A trie to hold identifiers

Figure 8.9:

A trie to hole identifiers (p.415)

Read it in reverse order.

c5: 110 101

c1: 110 001

Dynamic Hashing Using Directories II

We need to consider some issues!

Skewed Tree,

Access time increased.

Fagin et. al. proposed extendible hashing to solve above problems.

Ronald Fagin, **Jürg Nievergelt**, **Nicholas Pippenger**, and **H. Raymond Strong**, *Extendible Hashing - A Fast Access Method for Dynamic Files*, ACM Transactions on Database Systems, 4(3):315-344, 1979.

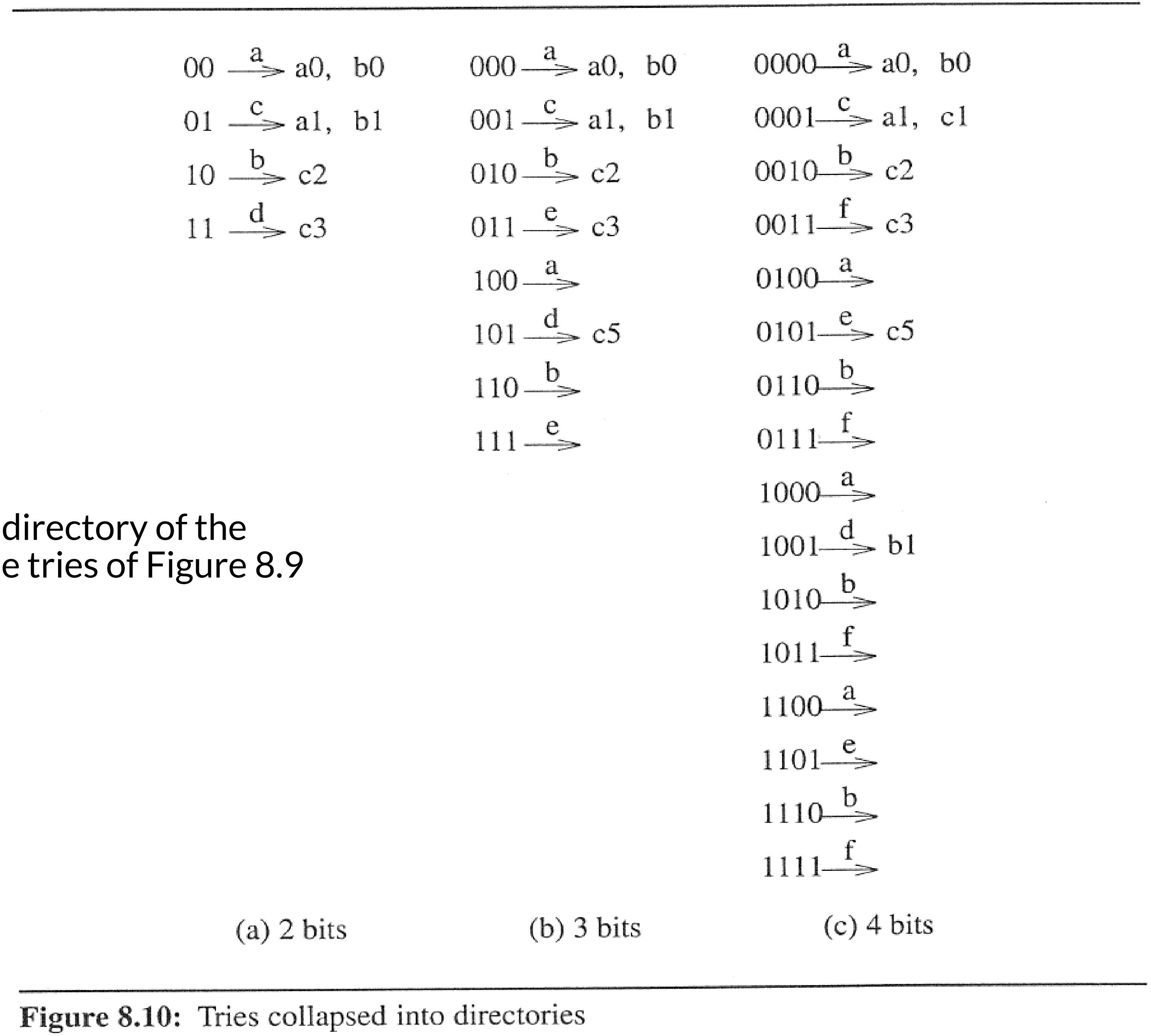
Dynamic Hashing Using Directories III

A directories is a table of pointer of pages.

The directory has k bits to index 2^k entries.

We could use a hash function to get the address of entry of directory, and find the page contents at the page.

The directory of the
three tries of Figure 8.9



$00 \xrightarrow{a} a0,\ \ b0$

$01 \xrightarrow{c} a1,\ \ b1$

$10 \xrightarrow{b} c2$

$11 \xrightarrow{d} c3$

(a) 2 bits

$000 \xrightarrow{a} a0,\ \ b0$

$001 \xrightarrow{c} a1,\ \ b1$

$010 \xrightarrow{b} c2$

$011 \xrightarrow{e} c3$

$100 \xrightarrow{a}$

$101 \xrightarrow{d} c5$

$110 \xrightarrow{b}$

$111 \xrightarrow{e}$

(b) 3 bits

$0000 \xrightarrow{a} a0,\ \ b0$

$0001 \xrightarrow{c} a1,\ \ c1$

$0010 \xrightarrow{b} c2$

$0011 \xrightarrow{f} c3$

$0100 \xrightarrow{a}$

$0101 \xrightarrow{e} c5$

$0110 \xrightarrow{b}$

$0111 \xrightarrow{f}$

$1000 \xrightarrow{a}$

$1001 \xrightarrow{d} b1$

$1010 \xrightarrow{b}$

$1011 \xrightarrow{f}$

$1100 \xrightarrow{a}$

$1101 \xrightarrow{e}$

$1110 \xrightarrow{b}$

$1111 \xrightarrow{f}$

(c) 4 bits

Figure 8.10: Tries collapsed into directories

Dynamic Hashing Using Directories IV

It is obvious that the directories will grow very large if the hash function is clustering.

Therefore, we need to adopt the uniform hash function to translate the bits sequence of keys to the random bits sequence.

Moreover, we need a family of uniform hash functions, since the directory will grow.

Dynamic Hashing Using Directories IV

a family of uniform hash functions:

$$hash_i : key \rightarrow \{0 \ldots 2^{i-1}\},\ 1 \leq i \leq d$$

If the page overflows, then we use hashi to rehash the original page into two pages, and we coalesce two pages into one in reverse case.

Thus we hope the family holds some properties like hierarchy.

Analysis

1. Only two disk accesses.
2. Space utilization

~ 69 %

If there are k records and the page size p is smaller than k, then we need to distribute the k records into left page and right page. It should be a symmetric binomial distribution.

Analysis II

If there are j records in the left page, then there are k-j records in the right page. The probability is:

$$\binom{k}{j} \ (1/2)^k$$

$$L(k) = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \{L(j) + L(k-j)\}$$

$$= 2^{1-k} \sum_{j=0}^{k} \binom{k}{j} L(j)$$

Analysis III

$$L(k) \sim \frac{k}{p \ln 2}$$

Thus the space utilization is

$$\frac{k}{pL(k)} \sim \ln 2 \sim 0.69$$

Overflow pages

To avoid doubling the size of directory, we introduce the idea of overflow pages, i.e.,

If overflow occurs, than we allocate a new (overflow) page instead of doubling the directory.

Put the new record into the overflow page, and put the pointer of the overflow page to the original page. (like chaining.)

Overflow pages II

Obviously, it will improve the storage utilization, but increases the retrieval time.

Larson et. al. concluded that the size of overflow page is from p to p/2 if 80% utilization is enough. (p is the size of page.)

Overflow pages III

- For better space utilization, we could monitor
  - Access time
  - Insert time
  - Total space utilization
- Fagin et al. conclude that it performed at least as well or better than B-tree, by simulation.

Extendible Hashing: Bibl.

- Fagin, R., Nievergelt, J., Pippenger, N., and Strong, H. R. Extendible Hashing - A Fast Access Method for Dynamic Files. ACM Trans. Database System 4, 3(Sept. 1979), 315-344.
- Tamminen, M. Extendible Hashing with Overflow. Information Processing Lett. 15, 5(Dec. 1982), 227-232.
- Mendelson, H. Analysis of Extendible Hashing. IEEE Trans. on Software Engineering, SE-8, 6(Nov. 1982), 611-619.
- Yao, A. C. A Note on the Analysis of Extendible Hashing. Information Processing Letter 11, 2(1980), 84-86.

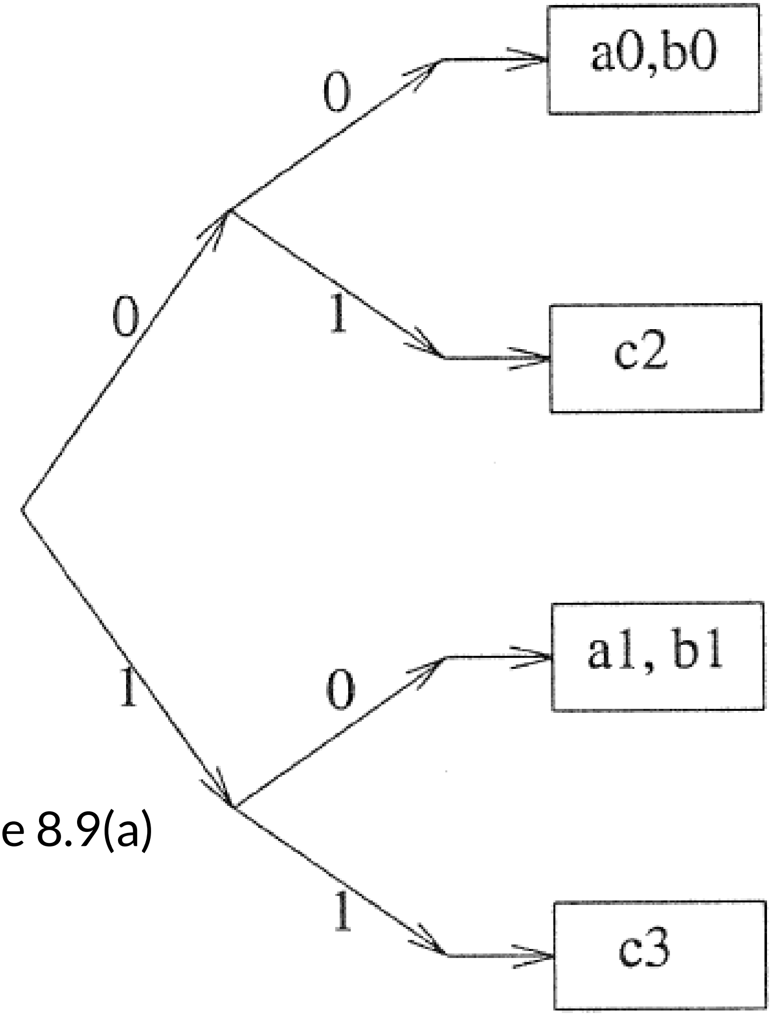
Directoryless Dynamic Hashing(Linear Hashing)

Ref. "Linear Hashing: A new tool for file and database addressing", VLDB 1980. by W. Litwin.

Ref. Larson, “Dynamic Hash Tables,” Communications of the ACM, pages 446–457, April 1988, Volume 31, Number 4.

If we have a contiguous space that is large enough to hold all the records, we could estimate the directory and leave the memory management mechanism to OS, e.g., paging.

Figure 8.12



Trie in figure 8.9(a)

Map a trie to the contiguous space without directory.

00	a0
	b0
01	c2
	-
10	a1
	b1
11	c3
	-

Linear Hashing II.

- **Drawback** of previous mapping: It wastes space, since we need to double the contiguous space if page overflow occurs.

- How to **improve**: Intuitively, add only one page, and rehash this space!

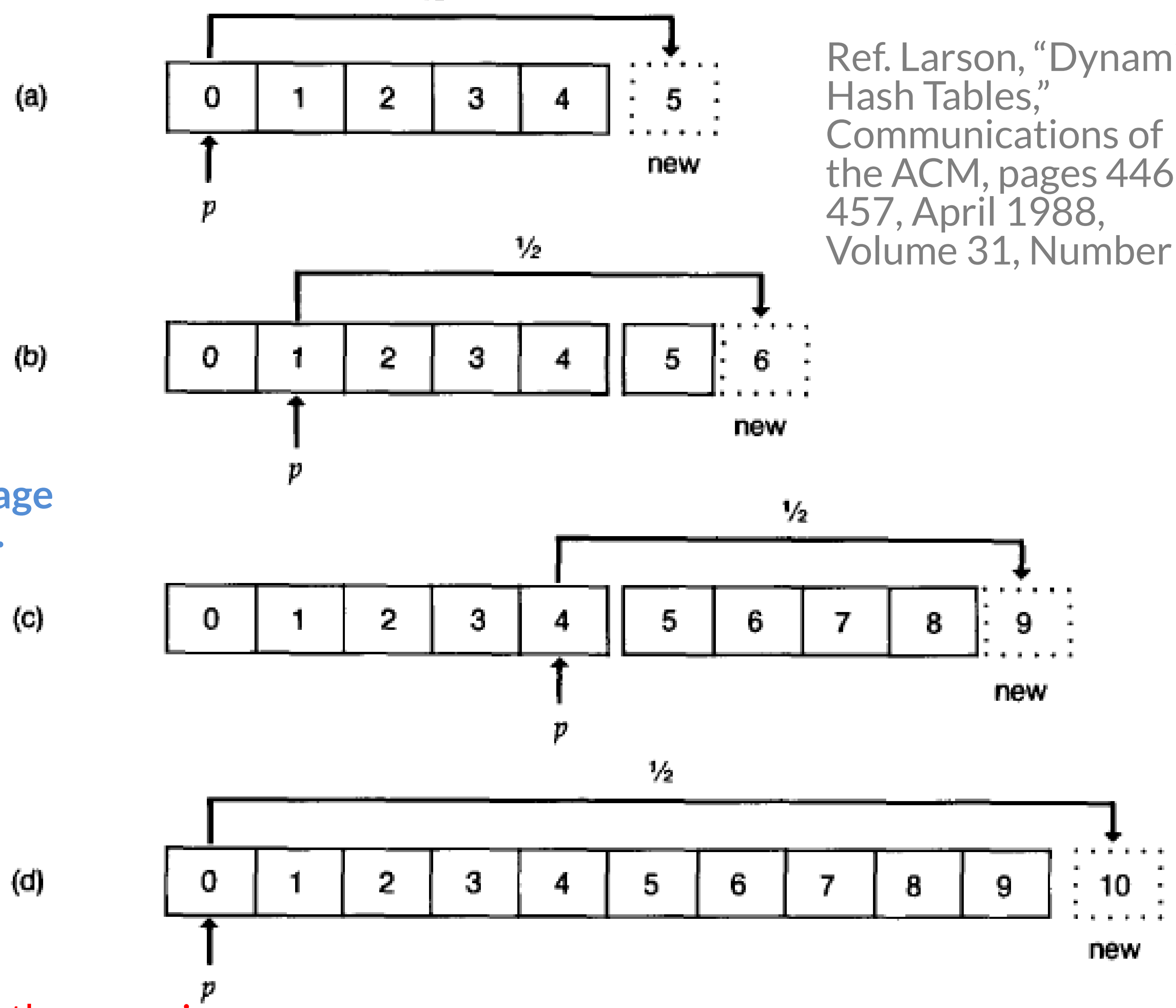


FIGURE 1. Illustration of the Expansion Process of Linear Hashing

Ref. Larson, "Dynamic Hash Tables," Communications of the ACM, pages 446–457, April 1988, Volume 31, Number 4.

Add new page one by one.

Eventually, the space is doubled. Begin new phase!

Linear Hashing II.

The suitable family of hashing functions:

$$h_j(K) = g(K) \bmod (N \times 2^j), \quad j = 0, 1, \ldots$$

$$g(K) = (cK) \bmod M,$$

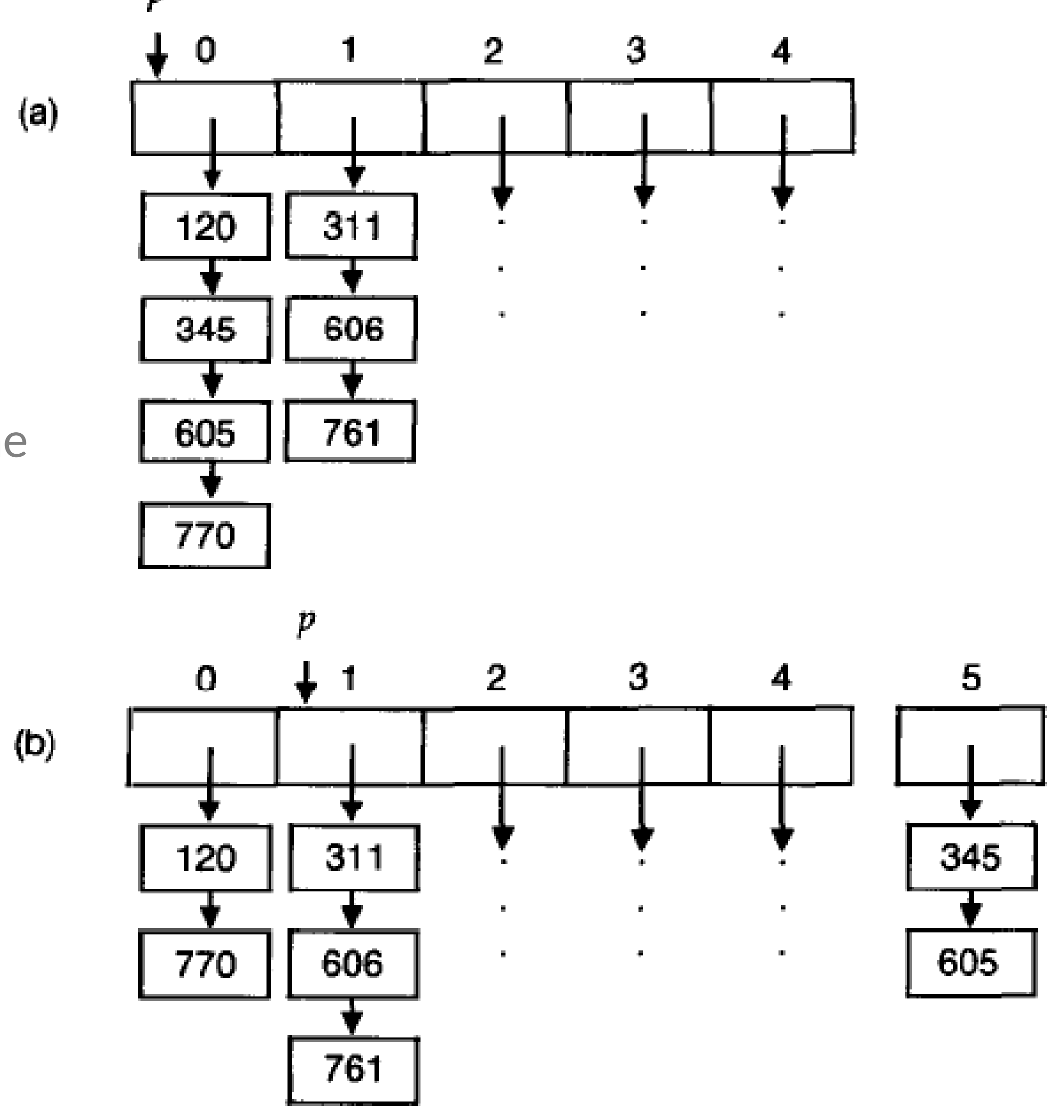
Where N is the minimum size of hash table, c is a constant, and M is a large prime.

This family of hash functions is given by Larson, “Dynamic Hash Tables,” Communications of the ACM, pages 446–457, April 1988, Volume 31, Number 4.

Example

Ref. Larson, "Dynamic Hash Tables," Communications of the ACM, pages 446–457, April 1988, Volume 31, Number 4.

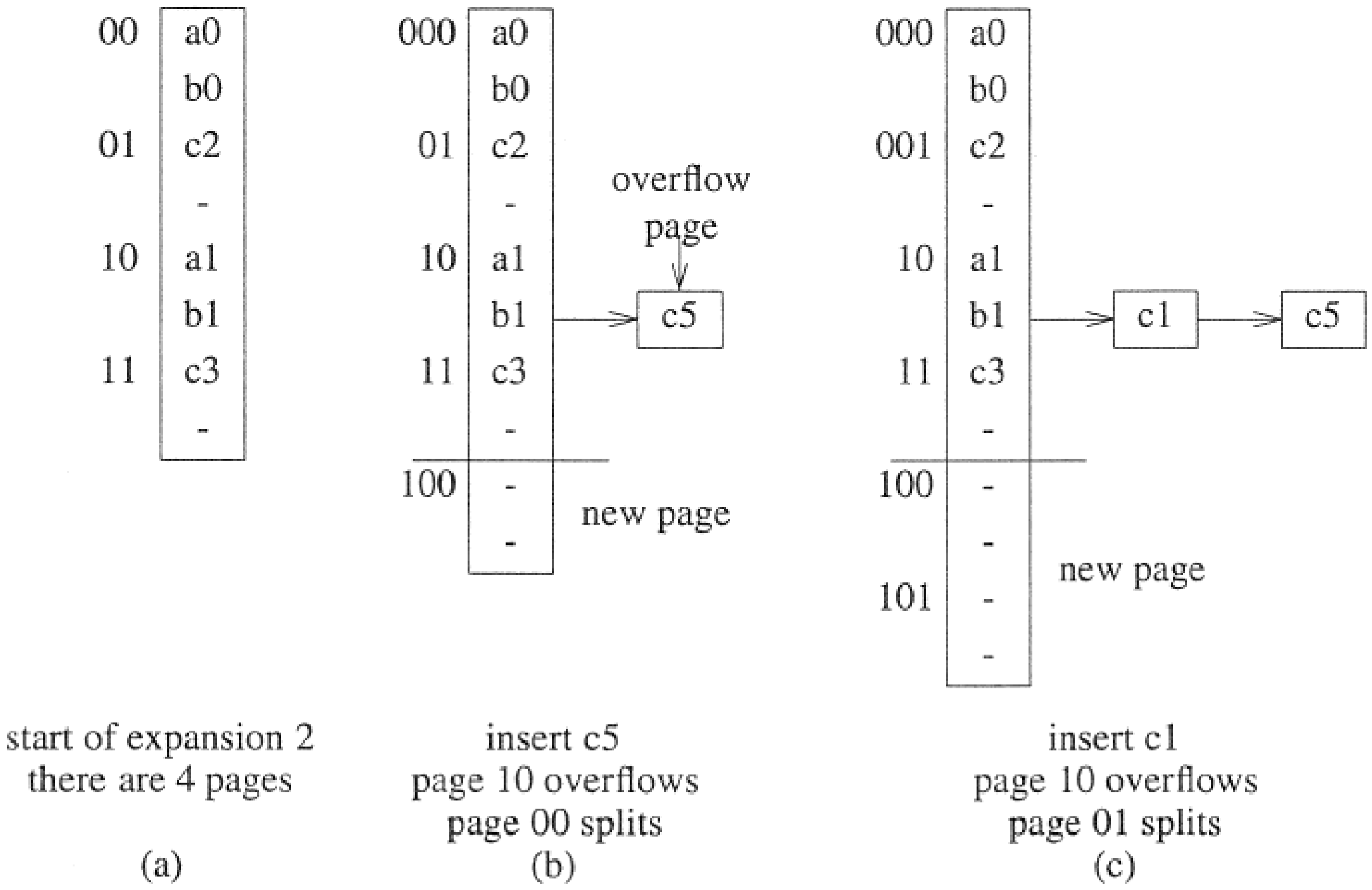
The case that keys is rehashed into new page.



Hash functions: $h_0(K) = K \bmod 5$
$h_1(K) = K \bmod 10$

FIGURE 2. An Example of Splitting a Bucket

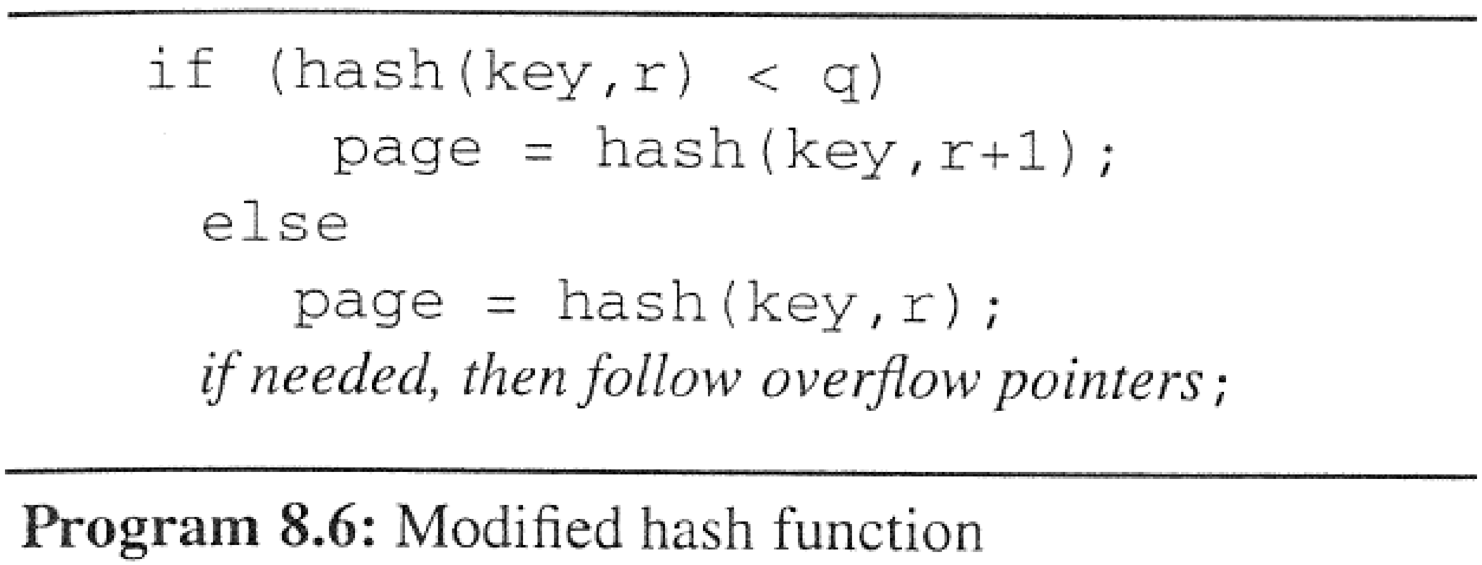
Figure 8.13



Recall Overflow pages: If overflow occurs, than we allocate a new (overflow) page instead of doubling the directory

No new keys be rehashed into new pages

Linear Hashing III.



```
if (hash(key,r) < q)
        page = hash(key,r+1);
  else
       page = hash(key,r);
  if needed, then follow overflow pointers;
```

Program 8.6: Modified hash function

The family of hash function in the textbook, hash(key, r) := key (mod 2^{r-1})

Analyzsis

Space utilization is not good!

[Litwin] ~ 60%

Litwin suggested to keep overflows until the space utilization exceeds the predefined amount.

It can also be solved by open addressing, etc.

Thank You!