AE322 Assignment-1:

1. a) $Y(t) = 6\sin(5x(t)) + 8\cos(10x(t))$

Linearity $\rightarrow$ Satisfy Superpositn $\rightarrow$ additivity, $\rightarrow$ Homogenity

$Y(t) \rightarrow$ does not Satisfy Superpositn & on replacing/shifting $t \rightarrow t-t_0$ $Y(t)$ does not shifts.

$\Rightarrow$ System is non-linear & time-Variant

b) $Y(t) = t\sin(Kt) + x(t)$

$y_1(t) + y_2(t) = 2t\sin(Kt) + x_1(t) + x_2(t) \Rightarrow y_3(t) = 2t\sin(Kt) + x_3(t)$

$\Rightarrow$ non linear, time-Variant

fail Superpositn principle

$\because t \rightarrow t-t_0 \Rightarrow t\sin(Kt)$ term does not Shift Same way.

c) $y(t) = ax(t) + b$

$$F(\alpha x_1 + \beta x_2) = a(\alpha x_1 + \beta x_2) + b \neq \alpha y_1 + \beta y_2$$

$\Rightarrow x(t) \rightarrow x(t-t_0)$, $y(t)$ shifts the same way.

So

$\Rightarrow$ non linear, time invariant.

d) $y(t) = Kx(t-c)$ $c \in \mathbb{R}$

$$F(\alpha x_1 + \beta x_2) = K(\alpha x_1(t-c) + \beta x_2(t-c)) = \alpha y_1 + \beta y_2$$

$t \rightarrow (t-t_0)$ $y(t) = Kx(t-c-t_0) \rightarrow$ still maintain shift.

$\Rightarrow$ Linear & time Invariant.

e) $y(t) = t^2 x(t)$

$$F(\alpha x_1 + \beta x_2) = t^2(\alpha x_1 + \beta x_2) = \alpha t^2 x_1 + \beta t^2 x_2$$
$$= \alpha y_1 + \beta y_2$$

$t \to t - t_0 \Rightarrow y(t) = (t-t_0)^2 x(t-t_0)$ not same as original eq^n.

$\Rightarrow$ Linear, Time-Variant.

2). Transfer fn^c

$$H(s) = \frac{Y(s)}{X(s)}$$ where $Y(s)$ & $X(s)$ are Laplace transforms of output & input signals

$\downarrow$

only exists for LTI (Linear time invariant systems)

so a, b, c & e $\Rightarrow$ T.F does not exist.

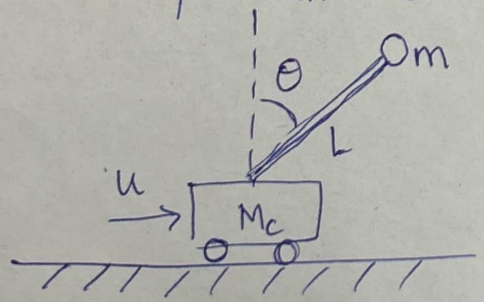
d) $y(t) = k x(t-c)$ Ass^n Zero initial Cond^n

Taking Laplace transform

$$Y(s) = k e^{-cs} X(s)$$

Transfer function. $$H(s) = \frac{Y(s)}{X(s)} = k \cdot e^{-cs}$$

3. Inverted pendulum - Cart System



$m = 0.1\,kg$ $M_c = 1\,kg$

$L = 0.5\,m$ $g = 9.81\ m/s^2$

Eq^n of motion:

$$\ddot{\theta} = F(\theta, \dot{\theta}) + G(\theta)u$$

$$F(\theta, \dot{\theta}) = \frac{\sin\theta}{L(M_c + m\sin^2\theta)} \left\{ -mL\dot{\theta}^2 \cos\theta + (M_c + m)g \right\}$$

$$G(\theta) = \frac{-\cos\theta}{L(M_c + m\sin^2\theta)}$$

a) $$\ddot{\theta} = F(\theta,\dot{\theta}) + G(\theta)u$$

for equilibrium points, Setting $\dot{\theta}=0$ & $\ddot{\theta}=0$

$$\Rightarrow \quad 0 = \frac{\sin\theta}{L(M_c+m\sin^2\theta)}(M_c+m)g - \frac{u\cos\theta}{L(M_c+m\sin^2\theta)}$$

$$\Rightarrow \quad u = (M_c+m)\,g\tan\theta$$

equilibrium without any input $\Rightarrow u=0$

Eq llbm Points: $\theta = 0$ & π are equilibrium points.

$(\theta,\dot{\theta}) = (0,0) \rightarrow$ unstable $\because$ inverted Pendulum, at $\theta=0$ for small pertubation, due to gravity it does not return to its initial state.

$(\theta,\dot{\theta}) = (\pi,0) \rightarrow$ stable.

b) Linearisation & State space eqn's

Lets define State variables $x_1=\theta$ & $x_2=\dot{\theta}$

System Equations $\Rightarrow$ $\dot{x}_1 = x_2 \Rightarrow f_1$

$$\ddot{\theta} = \dot{x}_2 = F(x_1,x_2) + G(x_1)u \Rightarrow f_2$$

at $u=0$

Linearisation around $\theta=0$:

$$A \rightarrow \text{Jacobian matrix} \Rightarrow \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ \frac{(M_c+m)g}{LM_c} & 0 \end{bmatrix}$$

$$f_2 = \frac{-m(\sin 2x_1)x_2^2}{2(M_c+m\sin^2 x_1)} + \frac{(M_c+m)g\sin x_1}{L(M_c+m\sin^2 x_1)} \qquad B = \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ \frac{-1}{LM_c} \end{bmatrix}$$

$$\left.\frac{\partial f_2}{\partial x_1}\right|_{x_1=0} = \frac{(M_c+m)g}{LM_c} \qquad \left.\frac{\partial f_2}{\partial x_2}\right|_{x_1=0} = 0.$$

State Space representation $AX + Bu$

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{(M_c+m)g}{LM_c} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-1}{LM_c} \end{bmatrix} u$$

c) Transfer function

$$\ddot{\theta} = \frac{(M_c+m)g}{LM_c}\theta - \frac{1}{LM_c}u$$

Taking Laplace transform, Considering $U(s)$ as input

$$s^2\theta(s) = \frac{(M_c+m)g}{LM_c}\theta(s) - \frac{1}{LM_c}U(s)$$

$$G(s) = \frac{\theta(s)}{U(s)} = \frac{-1}{LM_c\left(s^2 - \frac{(M_c+m)g}{LM_c}\right)}$$

$L = 0.5$ $M_c = 1$
$m = 0.1$ $g = 9.81$

$$G(s) = \frac{-1}{\frac{1}{2}(s^2 - 21.58)} \Rightarrow G_p(s) = \frac{-2}{s^2 - 21.582}$$

d) Characteristic equation $\Rightarrow s^2 - 21.58 = 0$

Roots $s^2 = 21.58$

$s = \pm\sqrt{21.58}$

$\because$ One root is +ve, $(\theta = 0) \Rightarrow$ unstable.

Routh array

s^2 1 -21.58

s 0 0

Considering $A(s) = s^2 - 21.58$

$$\frac{dA(s)}{ds} = 2s$$

2 & 0 → replace in s' row

s^2 1 -21.58

s 2 0

s^0 -21.58 → $-$ve $\Rightarrow$ it has +ve real part

$\therefore \theta = 0 \Rightarrow$ unstable eqllbm point.

for $\theta = \pi$:

Similarly we obtain state space representation as

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-(M_c+m)g}{LM_c} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LM_c} \end{bmatrix} u$$

Transfer function: $\ddot{\theta} = \frac{-(M_c+m)g}{LM_c}\theta + \frac{1}{LM_c}u$

Laplace transform $\Rightarrow$ $s^2\theta(s) = \frac{-(M_c+m)g}{LM_c}\theta(s) + \frac{1}{LM_c}u(s)$

$$G(s) = \frac{\theta(s)}{u(s)} = \frac{1}{LM_c\left(s^2 + \frac{(M_c+m)g}{LM_c}\right)} = \frac{1}{\frac{1}{2}(s^2+21.58)}$$

characteristic eq^n $\boxed{s^2+21.58=0}$

$s = \pm i\sqrt{21.58}$ $\Rightarrow$ both roots are purely imaginary

$\therefore$ $\theta = \pi$ is neutrally stable eqlb^m pt.

Routh array

s^2	1	21.58
s^1	$\cancel{0}$ 2	0
s^0	21.58	

all are +ve

$A(s) = s^2+21.58$

$\frac{dA(s)}{ds} = 2s \rightarrow [2 \quad 0]$

$\Rightarrow$ $\theta = \pi$ is stable eqlb^m pt.

$$G_p(s) = \frac{2}{s^2+21.58}$$

e) Initial cond^n $\rightarrow$ eqllb^m points $\Rightarrow$ $\theta = \pi$, $\dot{\theta} = 0$ $\quad 0 < t \leq 20\,sec$

Unit step input $\Rightarrow$ $u = 1$

abt^w $\theta = 0$ $\Rightarrow$ $\ddot{\theta} = 21.582\,\theta - 2u$

for $\theta = \pi$ $\Rightarrow$ $\ddot{\theta} = -21.582\,\theta + 2u$

f). Initial Condⁿ → $\theta = 10^\circ$ $\dot{\theta} = 0$ deg/s $u(t) \equiv 0$

abt eqlbᵐpt $(\theta, \dot{\theta}) = (0,0)$ for $u = 0$ ⇒ $\ddot{\theta} = 21.582\theta$

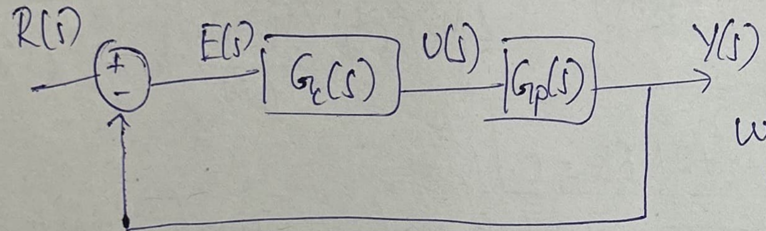
g). PD Controller design $(\theta, \dot{\theta}) = (0,0)$

$t_s = 1$ sec $M_p = < 10\%$ Initial Condⁿ $\theta = 10^\circ$ $\dot{\theta} = 0$ rad/s

for $M_p = 10\% = e^{\frac{-\pi \varepsilon}{\sqrt{1-\varepsilon^2}}} \times 100\%$ ⇒ $\frac{\varepsilon}{\sqrt{1-\varepsilon^2}} = 0.7333$

Damping ratio ⇒ $\varepsilon \approx 0.5913$

$t_s \approx \frac{4.4}{\varepsilon \omega_n}$ ⇒ $\omega_n = \frac{4.4}{0.5913 \times 1} = 7.4412$ rad/s



we know
$G_p(s) = \frac{-2}{s^2 - 21.582}$

for PD Control $U(t) = K_p e(t) + K_d \dot{e}(t)$

$$\Rightarrow \frac{U(s)}{E(s)} = K_p + K_d s \rightarrow G_c(s)$$

$$\frac{Y(s)}{R(s)} = \frac{G_p(s) \cdot G_c(s)}{1 + G_p(s) G_c(s)} = \frac{\left(\frac{-2}{s^2 - 21.582}\right)(K_p + K_d s)}{1 - \frac{2(K_p + K_d s)}{(s^2 - 21.582)}}$$

$$\Rightarrow \frac{Y(s)}{R(s)} = \frac{-2K_p - 2K_d s}{s^2 - 2K_d s - (2K_p + 21.582)}$$

we know Characteristic eqⁿ ⇒ $s^2 - 2K_d s - (2K_p + 21.582) = 0$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

From Coeff Comp ⇒

$$K_d = -\xi\omega_n = \underline{\underline{-4.4}}.$$

$\omega_n^2 = -(2K_p + 21.582)$

$K_p = \dfrac{-\omega_n^2 - 21.582}{2} = -38.476$

h) for max overshoot 15% $= e^{-\frac{\pi \xi}{\sqrt{1-\xi^2}}} \times 100$ $\quad T_s = 0.5$ sec

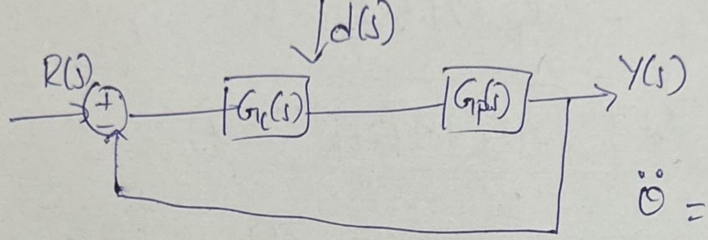
$\downarrow$

$\dfrac{\xi}{\sqrt{1-\xi^2}} = 0.60417 \Rightarrow \xi = 0.5170$ $\quad \Rightarrow \xi\omega_n \approx 8.8$

$\omega_n \approx 17.0212$

$K_d = -\xi\omega_n = -8.8$, $\quad K_p = \dfrac{-\omega_n^2 - 21.582}{2} = -155.6516$

i)



$\ddot{\theta} = 21.582\theta - 2u$

$u = -K_p\theta - K_d\dot{\theta} + 1$

for case of 15% M_p & 0.5 sec T_s $\Rightarrow K_p = -155.65$ & $K_d = -8.8$

$\Rightarrow u = 155.65\theta + 8.8\dot{\theta} + 1$

we know $E(s) = \dfrac{1}{1 + G_p(s)G_c(s)} R(s) - \dfrac{G_p(s)}{1 + G_p(s)G_c(s)} d(s)$ $\quad$ & $d(s) = \frac{1}{s}$ ($\because$ unit step dist)

for steady state error

$$e = \lim_{s \to 0} s.E(s)$$

$$= \lim_{s\to 0} \frac{-\left(\frac{-2}{s^2 - 21.582}\right)}{1 + \left(\frac{-2}{s^2 - 21.582}\right)(K_p + K_d s)} = \frac{\frac{-2}{21.582}}{1 + \frac{2K_p}{21.582}}$$

$$= \frac{-2}{2K_p + 21.582}$$

for $K_p = -155.65$

$e \approx 0.0069$ rad . (or) $0.39°$.

j. for $e < 0.5^\circ \Rightarrow e < 0.5 \times \frac{\pi}{180}$

$$\frac{-2}{2K_p + 21.582} < 8.72 \times 10^{-3}$$

$$\Rightarrow K_p < -125.7$$

So Choosing $K_p = -128$

$$\Rightarrow \text{Error} = 0.4883^\circ < 0.5^\circ.$$

K. for marginal stability

Charact eq^n $\quad s^2 - 2K_d s - (2K_p + 21.582) = 0$

$$s = \frac{2K_d \pm \sqrt{4K_d^2 + 4(2K_p + 21.582)}}{2}$$

$$= K_d \pm \sqrt{K_d^2 + 2K_p + 21.582}$$

for poles at zero → Marginal stability Cond^n

$$-2K_d = 0 \Rightarrow K_d = 0$$

$$\Rightarrow s = \pm\sqrt{2K_p + 21.582}$$

for $s = 0 \Rightarrow K_p = -10.79$ & $K_d = 0 \Rightarrow$ Marginally stable

$K_d = 0$ & $K_p > -10.79$ (unstable)

$K_d = 0$ & $K_p < -10.79$ (stable)