Solutions to HW10

Note: Most of these solutions were generated by R. D. Yates and D. J. Goodman, the authors of our textbook. I have added comments in italics where I thought more detail was appropriate. Those solutions I have written myself are designated by my initials.

Problem 10.1.2 •

$X_1, \ldots, X_n$ are independent uniform random variables, all with expected value $\mu_X = 7$ and variance $\mathrm{Var}[X] = 3$.

(a) What is the PDF of X_1?

(b) What is $\mathrm{Var}[M_{16}(X)]$, the variance of the sample mean based on 16 trials?

(c) What is $P[X_1 > 9]$, the probability that one outcome exceeds 9?

(d) Would you expect $P[M_{16}(X) > 9]$ to be bigger or smaller than $P[X_1 > 9]$? To check your intuition, use the central limit theorem to estimate $P[M_{16}(X) > 9]$.

Problem 10.1.2 Solution

$X_1, X_2 \ldots X_n$ are independent uniform random variables with mean value $\mu_X = 7$ and $\sigma_X^2 = 3$

(a) Since X_1 is a uniform random variable, it must have a uniform PDF over an interval $[a, b]$. From Appendix A, we can look up that $\mu_X = (a + b)/2$ and that $\mathrm{Var}[X] = (b - a)^2/12$. Hence, given the mean and variance, we obtain the following equations for a and b.

$$(b-a)^2/12 = 3 \qquad (a+b)/2 = 7 \tag{1}$$

Solving these equations yields $a = 4$ and $b = 10$ from which we can state the distribution of X.

$$f_X(x) = \begin{cases} 1/6 & 4 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(b) From Theorem 7.1, we know that

$$\mathrm{Var}[M_{16}(X)] = \frac{\mathrm{Var}[X]}{16} = \frac{3}{16} \tag{3}$$

(c)

$$P[X_1 \geq 9] = \int_9^\infty f_{X_1}(x)\, dx = \int_9^{10} (1/6)\, dx = 1/6 \tag{4}$$

(d) The variance of $M_{16}(X)$ is much less than $\text{Var}[X_1]$. Hence, the PDF of $M_{16}(X)$ should be much more concentrated about $E[X]$ than the PDF of X_1. Thus we should expect $P[M_{16}(X) > 9]$ to be much less than $P[X_1 > 9]$.

$$P[M_{16}(X) > 9] = 1 - P[M_{16}(X) \le 9] = 1 - P[(X_1 + \cdots + X_{16}) \le 144] \tag{5}$$

By a Central Limit Theorem approximation,

$$P[M_{16}(X) > 9] \approx 1 - \Phi\left(\frac{144 - 16\mu_X}{\sqrt{16}\sigma_X}\right) = 1 - \Phi(2.66) = 0.0039 \tag{6}$$

As we predicted, $P[M_{16}(X) > 9] \ll P[X_1 > 9]$.

Problem 10.1.4 ■

Let $X_1, X_2, \ldots$ denote a sequence of independent samples of a random variable X with variance $\text{Var}[X]$. We define a new random sequence $Y_1, Y_2, \ldots$ as

$$Y_1 = X_1 - X_2$$

and

$$Y_n = X_{2n-1} - X_{2n}$$

(a) What is $E[Y_n]$?

(b) What is $\text{Var}[Y_n]$?

(c) What are the mean and variance of $M_n(Y)$?

Problem 10.1.4 Solution

(a) Since $Y_n = X_{2n-1} + (-X_{2n})$, ...the expected value of the difference is

$$E[Y] = E[X_{2n-1}] + E[-X_{2n}] = E[X] - E[X] = 0 \tag{1}$$

(b) [Because the random variables are independent, they are uncorrelated, so the variance of the sum is the sum of the variances. Thus] the variance of the difference between X_{2n-1} and X_{2n} is

$$\text{Var}[Y_n] = \text{Var}[X_{2n-1}] + \text{Var}[-X_{2n}] = 2\,\text{Var}[X] \tag{2}$$

(c) Each Y_n is the difference of two samples of X that are independent of the samples used by any other Y_m. Thus $Y_1, Y_2, \ldots$ is an iid random sequence. By Theorem 7.1, the mean and variance of $M_n(Y)$ are

$$E[M_n(Y)] = E[Y_n] = 0 \tag{3}$$

$$\text{Var}[M_n(Y)] = \frac{\text{Var}[Y_n]}{n} = \frac{2\,\text{Var}[X]}{n} \tag{4}$$

Problem 10.2.2 •

For an arbitrary random variable X, use the Chebyshev inequality to show that the probability that X is more than k standard deviations from its expected value $E[X]$ satisfies

$$P[|X - E[X]| \geq k\sigma] \leq \frac{1}{k^2}$$

For a Gaussian random variable Y, use the $\Phi(\cdot)$ function to calculate the probability that Y is more than k standard deviations from its expected value $E[Y]$. Compare the result to the upper bound based on the Chebyshev inequality.

Problem 10.2.2 Solution

We know from the Chebyshev inequality that

$$P\left[|X - E\left[X\right]| \geq c\right] \leq \frac{\sigma_X^2}{c^2} \tag{1}$$

Choosing $c = k\sigma_X$, we obtain

$$P\left[|X - E\left[X\right]| \geq k\sigma\right] \leq \frac{1}{k^2} \tag{2}$$

The actual probability the Gaussian random variable Y is more than k standard deviations from its expected value is

$$\begin{aligned} P\left[|Y - E\left[Y\right]| \geq k\sigma_Y\right] &= P\left[Y - E\left[Y\right] \leq -k\sigma_Y\right] + P\left[Y - E\left[Y\right] \geq k\sigma_Y\right] && (3)\\ &= 2P\left[\frac{Y - E\left[Y\right]}{\sigma_Y} \geq k\right] && (4)\\ &= 2Q(k) && (5) \end{aligned}$$

The following table compares the upper bound and the true probability:

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
Chebyshev bound	1	0.250	0.111	0.0625	0.040
$2Q(k)$	0.317	0.046	0.0027	6.33×10^{-5}	5.73×10^{-7}

(6)

The Chebyshev bound gets increasingly weak as k goes up. As an example, for $k = 4$, the bound exceeds the true probability by a factor of 1,000 while for $k = 5$ the bound exceeds the actual probability by a factor of nearly 100,000.

Problem 10.2.3 Solution (SK)

Elevators arrive randomly at the ground floor of an office building. Because of a large crowd a person will wait for a time W to board the third arriving elevator. Let X_1 denote the time (in seconds) before the first elevator arrives, X_2 be the interval between the arrivals of the second and third elevators and X_3 be the interval after between the arrival of the second and third elevators. Suppose that the X_i are independent uniform(0,30) random variables. Find upper bounds to the probability that W exceeds 75 seconds using the Markov, Chebyshev, and Chernoff bounds.

$$P(W) = X_1 + X_2 + X_3,$$

so by additivity of the expectation,

$$E(W) = \mathrm{E}[X_1] + \mathrm{E}[X_2] + \mathrm{E}[X_3].$$

For a continuous uniform random variable on the interval (a,b), the expected value is $(a+b)/2$ so $\mathrm{E}[X_i] = 15$. Thus $\mathrm{E}[W] = 45$. Because the three random variables X_i are independent,

$$\mathrm{Var}[W] = \mathrm{Var}[X_1] + \mathrm{Var}[X_2] + \mathrm{Var}[X_3].$$

For a continuous uniform random variable on an interval (a, b) the variance is $(b-a)^2/12$ so $\mathrm{Var}[X_i] = 900/12 = 300/4 = 75$. Thus $\mathrm{Var}[W] = 225$.

(a) the Markov inequality,

The Markov inequality applies because the wait time is non-negative, and gives

$$P[W > 75] \leq \mathrm{E}[W]/c^2 = 45/75 = 3/5.$$

(b) the Chebyshev inequality,

The Chebyshev inequality applies because $c = 75 - \mathrm{E}[W] = 30 > 0$, so applying

$$P[|W - \mathrm{E}[W]| \geq c] \leq \mathrm{Var}[W]/c^2$$

yields

$$P[W \geq 75] = P[|W - \mathrm{E}[W]| \geq 30] \leq 225/30^2 = 9/36 = 1/4.$$

(c) the Chernoff bound.

The Chernoff bound applies for any random variable X and constant c.

The moment generating function of a sum of independent random variables is the product of their moment generating functions so

$$\phi_W(s) = (\phi_X(s))^3 .$$

The moment generating function for a uniform(a,b) random variable is

$$\frac{e^{bs} - e^{as}}{s(b-a)}.$$

Thus

$$\begin{aligned} P[W \geq 75] &\leq \min_{s\geq 0} e^{-s75}\phi_W(s) & (7)\\ &= \min_{s\geq 0} e^{-s75}\frac{\left(e^{30s}-e^{0s}\right)^3}{s^3 30^3} & (8)\\ &= \frac{1}{2700}\min_{s\geq 0} e^{-s75}\left(e^{30s}-1\right)^3 s^{-3} & (9)\\ & & (10) \end{aligned}$$

Thus we want to minimize $h(s) = e^{-s75}(e^{30s}-1)^3 s^{-3}$.

Differentiating, we find

$$\begin{aligned} \frac{d}{ds}h(s) &= -75e^{-s75}\left(e^{30s}-1\right)^3 s^{-3} + e^{-s75}3\left(e^{30s}-1\right)^2 30e^{30s}s^{-3} - 3e^{-s75}\left(e^{30s}-1\right)^3 s^{-4}\\ &= e^{-s75}\left(e^{30s}-1\right)^2 s^{-3}\left(-75(e^{30s}-1)+30e^{30s}+3\left(e^{30s}-1\right)/s\right). \end{aligned}$$

Solving for s would be nontrivial, so I used Matlab to evaluate $h(s)$. After doing some preliminary plots to determine a range in which the minimum might lie, I evaluated $h(s)$ on the interval $[0.00001, 2]$. Using the Matlab commands

```
     >> for index = 0.00001:0.00001:2,
            hoi(round(10000*index)) = (exp(30*index)-1)^3/index^3/exp(75*index);
        end;

>>[val,loc]=min(hoi)

val =

  2.4912e+003


loc =

   197
```

identifies the minimum value 0.9227 as occurring at $s = 0.1968$. Thus in this case, the Chernoff bound does not give a better bound than the other two inequalities. This is a bit disappointing for all that work. However, as noted in the text is is not surprising since teh Chernoff bound is useful when c is large relative to $E[X]$. Apparently $75/45 = 5/3$ is not large.

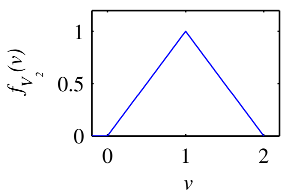
Problem 10.2.4 ■

Let X equal the arrival time of the third elevator in the previous problem. Find the exact value of $P[W \geq 75]$. Compare your answer to the upper bounds derived in the previous problem.

Problem 10.2.4 Solution

The hard part of this problem is to derive the PDF of the sum $W = X_1 + X_2 + X_3$ of iid uniform $(0, 30)$ random variables. In this case, we need to use the techniques of Chapter 6 to convolve the three PDFs. To simplify our calculations, we will instead find the PDF of $V = Y_1 + Y_2 + Y_3$ where the Y_i are iid uniform $(0, 1)$ random variables. By Theorem 3.20 to conclude that $W = 30V$ is the sum of three iid uniform $(0, 30)$ random variables.

To start, let $V_2 = Y_1 + Y_2$. Since each Y_1 has a PDF shaped like a unit area pulse, the PDF of V_2 is the triangular function



$$f_{V_2}(v) = \begin{cases} v & 0 \leq v \leq 1 \\ 2 - v & 1 < v \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The PDF of V is the convolution integral

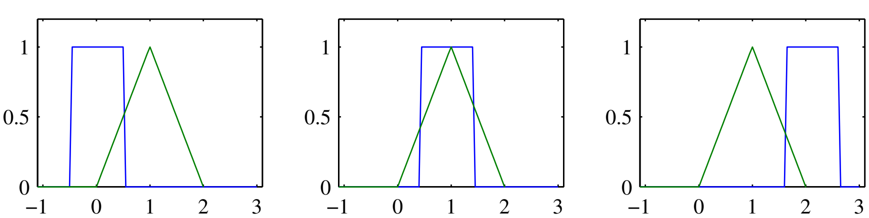
$$f_V(v) = \int_{-\infty}^{\infty} f_{V_2}(y) f_{Y_3}(v - y)\, dy \tag{2}$$

$$= \int_0^1 y f_{Y_3}(v - y)\, dy + \int_1^2 (2 - y) f_{Y_3}(v - y)\, dy. \tag{3}$$

Evaluation of these integrals depends on v through the function

$$f_{Y_3}(v - y) = \begin{cases} 1 & v - 1 < v < 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

To compute the convolution, it is helpful to depict the three distinct cases. In each case, the square "pulse" is $f_{Y_3}(v - y)$ and the triangular pulse is $f_{V_2}(y)$.



$0 \leq v < 1$ $\quad$ $1 \leq v < 2$ $\quad$ $2 \leq v < 3$

From the graphs, we can compute the convolution for each case:

$$0 \leq v < 1: \qquad f_{V_3}(v) = \int_0^v y\, dy = \frac{1}{2} v^2 \tag{5}$$

$$1 \leq v < 2: \qquad f_{V_3}(v) = \int_{v-1}^1 y\, dy + \int_1^v (2 - y)\, dy = -v^2 + 3v - \frac{3}{2} \tag{6}$$

$$2 \leq v < 3: \qquad f_{V_3}(v) = \int_{v-1}^2 (2 - y)\, dy = \frac{(3 - v)^2}{2} \tag{7}$$

To complete the problem, we use Theorem 3.20 to observe that $W = 30V_3$ is the sum of three iid uniform $(0, 30)$ random variables. From Theorem 3.19,

$$f_W(w) = \frac{1}{30} f_{V_3}(v_3)\, v/30 = \begin{cases} (w/30)^2/60 & 0 \le w < 30, \\ [-(w/30)^2 + 3(w/30) - 3/2]/30 & 30 \le w < 60, \\ [3 - (w/30)]^2/60 & 60 \le w < 90, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Finally, we can compute the exact probability

$$P[W \ge 75] = \frac{1}{60} \int_{75}^{9} 0[3 - (w/30)]^2\, dw = -\left.\frac{(3 - w/30)^3}{6}\right|_{75}^{90} = \frac{1}{48} \tag{9}$$

For comparison, the Markov inequality indicated that $P[W < 75] \le 3/5$ and the Chebyshev inequality showed that $P[W < 75] \le 1/4$. As we see, both inequalities are quite weak in this case.

Problem 10.2.8 Solution (SK)

Let K be a Poisson randiom variable with expected value α. Use the Chernoff bound to find an upper bound on $\mathrm{P}[K \ge c]$.

The Chernoff bound for an arbitrary random variable X and constant c is

$$P[X \ge c] \le \min_{s \ge 0} e^{-sc} \phi_X(s).$$

For a Poisson(α) random variable K,

$$\phi_K(s) = e^{\alpha(e^s - 1)},$$

so the Chernoff bound yields

$$P[X \ge c] \le \min_{s \ge 0} e^{-sc} e^{\alpha(e^s - 1)} = \min_{s \ge 0} e^{\alpha(e^s - 1) - sc}.$$

Let

$$h(s) = e^{\alpha(e^s - 1) - sc}.$$

Rather than taking the derivative of $h(s)$, we can use the fact that the natural log is increasing and instead find the mimimum of $\ln h(s)$. We find that

$$\ln h(s) = \alpha e^s - \alpha - sc$$

and

$$\frac{d}{ds} \ln h(s) = \alpha e^s - c.$$

Equating this to zero and solving for s we obtain $s = \ln c/\alpha$. If $c > \alpha$ then $\ln c/\alpha > 0$ and

$$\begin{aligned} P[X \ge c] &\le \min_{s \ge 0} e^{-cs} e^{\alpha(e^s - 1)} & (10) \\ &= e^{-c \ln c/\alpha + \alpha((c/\alpha) - 1)} & (11) \\ &= e^{c \ln \alpha/c + c - \alpha} & (12) \\ &= (\alpha/c)^c\, e^{c - \alpha}. & (13) \end{aligned}$$

If $c \leq \alpha$ then $\ln c/\alpha < 0$ and so we evaluate the bound at $s = 0$, in which case we obtain

$$P[X \geq c] \leq \min_{s \geq 0} e^{-cs} e^{\alpha(e^s - 1)} \tag{14}$$
$$= 1. \tag{15}$$

Finally

$$P[X \geq c] \leq \begin{cases} (\alpha/c)^c \, e^{c-\alpha} & c > \alpha \\ 1 & c \leq \alpha \end{cases}$$

Problem 10.3.4 ■

In communication systems, the error probability $P[E]$ may be difficult to calculate; however it may be easy to derive an upper bound of the form $P[E] \leq \epsilon$. In this case, we may still want to estimate $P[E]$ using the relative frequency $\hat{P}_n(E)$ of E in n trials. In this case, show that

$$P\left[\left|\hat{P}_n(E) - P[E]\right| \geq c\right] \leq \frac{\epsilon}{nc^2}.$$

Problem 10.3.4 Solution

Since the relative frequency of the error event E is $\hat{P}_n(E) = M_n(X_E)$ and $E[M_n(X_E)] = P[E]$, we can use Theorem 7.12(a) to write

$$P\left[\left|\hat{P}_n(A) - P[E]\right| \geq c\right] \leq \frac{\text{Var}[X_E]}{nc^2}. \tag{1}$$

Note that $\text{Var}[X_E] = P[E](1 - P[E])$ since X_E is a Bernoulli ($p = P[E]$) random variable. Using the additional fact that $P[E] \leq \epsilon$ and the fairly trivial fact that $1 - P[E] \leq 1$, we can conclude that

$$\text{Var}[X_E] = P[E](1 - P[E]) \leq P[E] \leq \epsilon. \tag{2}$$

Thus

$$P\left[\left|\hat{P}_n(A) - P[E]\right| \geq c\right] \leq \frac{\text{Var}[X_E]}{nc^2} \leq \frac{\epsilon}{nc^2}. \tag{3}$$

Problem 10.5.2 •

X is a Bernoulli random variable with unknown success probability p. Using 100 independent samples of X find a confidence interval estimate of p with confidence coefficient 0.99. If $M_{100}(X) = 0.06$, what is our interval estimate?

Problem 10.5.2 Solution

Since $E[X] = \mu_X = p$ and $\text{Var}[X] = p(1-p)$, we use [the Weak Law of Large Numbers] to write

$$P[|M_{100}(X) - p| < c] \geq 1 - \frac{p(1-p)}{100c^2} = 1 - \alpha. \tag{1}$$

For confidence coefficient 0.99, we require

$$\frac{p(1-p)}{100c^2} \leq 0.01 \quad \text{or} \quad c \geq \sqrt{p(1-p)}. \tag{2}$$

Since p is unknown, we must ensure that the constraint is met for every value of p. The worst case occurs at $p = 1/2$ which maximizes $p(1-p)$. In this case, $c = \sqrt{1/4} = 1/2$ is the smallest value of c for which we have confidence coefficient of at least 0.99.

If $M_{100}(X) = 0.06$, our interval estimate for p is

$$M_{100}(X) - c < p < M_{100}(X) + c. \tag{3}$$

Since $p \geq 0$, $M_{100}(X) = 0.06$ and $c = 0.5$ imply that our interval estimate is

$$0 \leq p < 0.56. \tag{4}$$

Our interval estimate is not very tight because because 100 samples is not very large for a confidence coefficient of 0.99.

Problem 10.5.4 ■

When we perform an experiment, event A occurs with probability $P[A] = 0.01$. In this problem, we estimate $P[A]$ using $\hat{P}_n(A)$, the relative frequency of A over n independent trials.

(a) How many trials n are needed so that the interval estimate

$$\hat{P}_n(A) - 0.001 < P[A] < \hat{P}_n(A) + 0.001$$

has confidence coefficient $1 - \alpha = 0.99$?

(b) How many trials n are needed so that the probability $\hat{P}_n(A)$ differs from $P[A]$ by more than 0.1% is less than 0.01?

Problem 10.5.4 Solution

Both questions can be answered using the following equation from Example 7.6:

$$P\left[\left|\hat{P}_n(A) - P[A]\right| \geq c\right] \leq \frac{P[A](1 - P[A])}{nc^2} \tag{1}$$

The unusual part of this problem is that we are given the true value of $P[A]$. Since $P[A] = 0.01$, we can write

$$P\left[\left|\hat{P}_n(A) - P[A]\right| \geq c\right] \leq \frac{0.0099}{nc^2} \tag{2}$$

(a) In this part, we meet the requirement by choosing $c = 0.001$ yielding

$$P\left[\left|\hat{P}_n(A) - P[A]\right| \geq 0.001\right] \leq \frac{9900}{n} \tag{3}$$

Thus to have confidence level 0.01, we require that $9900/n \leq 0.01$. This requires $n \geq 990{,}000$.

(b) In this case, we meet the requirement by choosing $c = 10^{-3}P[A] = 10^{-5}$. This implies

$$P\left[\left|\hat{P}_n(A) - P[A]\right| \geq c\right] \leq \frac{P[A](1 - P[A])}{nc^2} = \frac{0.0099}{n10^{-10}} = \frac{9.9 \times 10^7}{n} \tag{4}$$

The confidence level 0.01 is met if $9.9 \times 10^7/n = 0.01$ or $n = 9.9 \times 10^9$.