Tutorial - 3

Information & Communication

PROBLEM - I (Chain Rule and Mutual Information)

The conditional mutual information of Random variables X and Y given Z is defined by

$$I(X;Y|Z) = H(X|Z) - H(X|Y,Z)$$

(a) Prove that (Chain Rule for Mutual Information)

$$I(X_1, X_2, \ldots X_n; Y) = \sum_{i=1}^{n} I(X_i ; Y | X_{i-1}, X_{i-2}, \ldots, X_1)$$

(b) Prove that

$$I(X;Y|Z) \geq 0$$

with equality if and only if

$$p(x|z) \cdot p(y|z) = p(x,y|z)$$

(c) Suppose if

$$p(x,y,z) = p(x) \cdot p(y|x) \cdot p(z|y)$$

then prove that

$$I(X;Y) \geq I(X;Z)$$

Problem - II (Functions of Random Variables)

(a) Let X and Y be two random variables on the set of non-negative integers. Show that if $Y = 2X$ then

$$H(X|Y) = H(Y|X) = 0$$

(b) Let $Y = g(X)$ for some function g. Show that $H(Y|X) = 0$. Under what conditions on g is $H(X|Y)$ also $= 0$? Analyze

(c) Let X and Y be two random variables. Let E be a random variable defined as

$$E = \begin{cases} 0 & \text{if } Y = X \\ 1 & \text{if } Y \neq X \end{cases}$$

Prove that

$$P(E | X, Y) = 0$$

Problem - 3 (Chain Rule)

Let $\{X_i, 1 \le i \le 3\}$ be random variables over $\{0,1\}$. Let Y be a random variable over $\{0,1\}$.

X_1, X_2 / X_3, Y	$X_1 = 0$, $X_2 = 0$	$X_1 = 0$, $X_2 = 1$	$X_1 = 1$, $X_2 = 0$	$X_1 = 1$, $X_2 = 1$
$X_3 = 0$, $Y = 0$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{32}$	$\frac{1}{64}$
$X_3 = 0$, $Y = 1$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
$X_3 = 1$, $Y = 0$	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$
$X_3 = 1$, $Y = 1$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$

Check if

(a) $H(X_1, X_2, X_3) = H(X_1) + H(X_2 | X_1) + H(X_3 | X_2, X_1)$

(b) $H(X_1, X_2, X_3 | Y) = H(X_1 | Y) + H(X_2 | X_1, Y) + H(X_3 | X_2, X_1, Y)$