Assignment - 5

I affirm that I have neither given nor received help or used any means which would make this assignment unfair.

- L. Lakshmanan
(2020112024)

Q1. $H(z) = \dfrac{1}{1 - 2r(\cos\theta) z^{-1} + r^2 z^{-2}}$ $\quad |z| > r > 0$

Substituting $z^{-1} = p$ to get a quadratic equation.

$$H(p) = \frac{1}{1 - 2r(\cos\theta) p + r^2 p^2}$$

Roots of the denominator $= \dfrac{2r\cos\theta \pm \sqrt{4r^2\cos^2\theta - 4r^2}}{2r^2}$

$$= \frac{\cancel{2}\cancel{r}\cos\theta \pm \cancel{2}\cancel{r} j\sin\theta}{\cancel{2}r\cancel{^2}}$$

$$= r^{-1}e^{j\theta},\ r^{-1}e^{-j\theta}$$

$$\therefore H(p) = \frac{1}{r^2(p - r^{-1}e^{j\theta})(p - r^{-1}e^{-j\theta})}$$

$$\Rightarrow H(z) = \frac{1}{r^2(z^{-1} - r^{-1}e^{j\theta})(z^{-1} - r^{-1}e^{-j\theta})}$$

Using partial fractions,

$$\frac{A}{r^2(z^{-1} - r^{-1}e^{j\theta})} + \frac{B}{r^2(z^{-1} - r^{-1}e^{-j\theta})}$$

$$\frac{Az^{-1} - Ar^{-1}e^{-j\theta} + Bz^{-1} - Br^{-1}e^{j\theta}}{r^2(z^{-1}-r^{-1}e^{j\theta})(z^{-1}-r^{-1}e^{-j\theta})} = \frac{1}{r^2(z^{-1}-r^{-1}e^{j\theta})(z^{-1}-r^{-1}e^{-j\theta})}$$

$\Rightarrow A + B = 0$ —— 1 ($\because z^{-1}$ term has cofficient 0).

$$-Ar^{-1}e^{-j\theta} - Br^{-1}e^{j\theta} = 1$$

$$\Rightarrow Ar^{-1}e^{-j\theta} + Br^{-1}e^{j\theta} = -1$$

$$\Rightarrow Ae^{-j\theta} + Be^{j\theta} = -r$$

$$\Rightarrow \cancelto{0}{(A+B)\cos\theta} + j(-A+B)\sin\theta = -r \quad (\because 1)$$

$$\Rightarrow B - A = \frac{-r}{j\sin\theta} \quad \text{—— } 2$$

Now, doing 1 + 2, we get

$$2B = \frac{-r}{j\sin\theta} \Rightarrow B = \frac{-r}{2j\sin\theta}$$

$$\Rightarrow A = \frac{r}{2j\sin\theta} \quad (\because 1), \quad B = \frac{-r}{2j\sin\theta}$$

$\therefore$ Now we have,

$$H(z) = \frac{Az}{r^2(1-r^{-1}ze^{j\theta})} + \frac{Bz}{(1-r^{-1}ze^{-j\theta})r^2}$$

$$= \left(\left(\frac{-1}{r^{-1}e^{j\theta}}\right)\frac{Az}{\left(z-\frac{1}{r^{-1}e^{j\theta}}\right)} + \left(\frac{-1}{r^{-1}e^{-j\theta}}\right)\frac{Bz}{\left(z-\frac{1}{r^{-1}e^{-j\theta}}\right)}\right)\frac{1}{r^2}$$

$$= \left(\frac{-A}{r^{-1}e^{j\theta}}\frac{z}{z-re^{-j\theta}} - \frac{B}{r^{-1}e^{-j\theta}}\frac{z}{z-re^{j\theta}}\right)\frac{1}{r^2}$$

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$$= \frac{B r e^{-j\theta}}{r^2}\left(\frac{z}{z - r e^{-j\theta}}\right) + \frac{A r e^{j\theta}}{r^2}\left(\frac{z}{z - r e^{j\theta}}\right) \quad (\because A = -B)$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$h[n] = \frac{B r e^{-j\theta}}{r^2}(r e^{-j\theta})^n u[n] + \frac{A r e^{j\theta}}{r^2}(r e^{j\theta})^n u[n]$$

$$= \frac{-r^{-1}}{2j\sin\theta}(r e^{-j\theta})^{n+1} u[n] + \frac{r^{-1}}{2j\sin\theta}(r e^{j\theta})^{n+1} u[n]$$

$$= \frac{r^n\left(\cos(n+1)\theta + j\sin(n+1)\theta\right) - r^n\left(\cos(n+1)\theta - j\sin(n+1)\theta\right)}{2j\sin\theta} u[n]$$

$$= \frac{2\cancel{j} r^n \sin(n+1)\theta \; u[n]}{2\cancel{j}\sin\theta}$$

$$= \frac{r^n \sin(n+1)\theta \; u[n]}{\sin\theta} \qquad \text{Proved.}$$

2.

$$H(z) = \frac{1 - z^{-2}}{1 - (1+\alpha)\cos\omega_c z^{-1} + \alpha z^{-2}}$$

$$\Rightarrow H(e^{j\omega}) = \frac{1 - e^{-2j\omega}}{1 - (1+\alpha)\cos\omega_c e^{-j\omega} + \alpha e^{-2j\omega}}$$

$$= \frac{e^{2j\omega} - 1}{e^{2j\omega} - (1+\alpha)\cos\omega_c e^{j\omega} + \alpha}$$

$$= \frac{\cancel{e^{j\omega}}\left(e^{j\omega} - e^{-j\omega}\right)}{\cancel{e^{j\omega}}\left(e^{j\omega} - (1+\alpha)\cos\omega_c + \alpha e^{-j\omega}\right)}$$

$$= \frac{2j \sin w}{\cos w + j\sin w - (1+\alpha)\cos w_c + \alpha \cos w - \alpha j \sin w}$$

$$= \frac{2j \sin w}{(1+\alpha)(\cos w - \cos w_c) + j \sin w (1-\alpha)}$$

$$= \frac{2j}{(1+\alpha)\left(\cot w - \frac{\cos w_c}{\sin w}\right) + j(1-\alpha)}$$

Now, magnitude of this is

$$|H(e^{jw})| = \frac{2}{\sqrt{(1+\alpha)^2\left(\cot w - \frac{\cos w_c}{\sin w}\right)^2 + (1-\alpha)^2}}$$

To maximise this value, we must minimise the denominator. As we can clearly see, at $w = w_c$, the first term becomes zero. The rest cannot be altered as they are constants.

$$\cot w - \frac{\cos w_c}{\sin w} = 0$$

$$\Rightarrow \cot w \cdot \sin w = \cos w_c$$

$$\cos w = \cos w_c$$

$\Rightarrow w = w_c$ is where we get the maximum value.

Putting $w = w_c$

$$|H(e^{jw})| = \frac{2}{\sqrt{(1+\alpha)^2 \cdot 0 + (1-\alpha)^2}} = \underline{\frac{2}{1-\alpha}}$$

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3.

$y[n] = 2(-0.2)^n u[n] \longleftrightarrow Y(z) = \frac{2z}{z+0.2}$

$x[n] = 3(0.5)^n u[n] \longleftrightarrow X(z) = \frac{3z}{z-0.5}$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2\cancel{z}}{z+0.2} \times \frac{z-0.5}{3\cancel{z}}$$

$$= \frac{2z-1}{3z+0.6}$$

$$= \frac{2}{3}\left(\frac{z}{z+0.2} - \frac{0.5}{z+0.2}\right)$$

Using standard inverse z Transform formulae, we get,

$$\frac{2}{3}\left(\frac{z}{z+0.2} - \frac{0.5}{z+0.2}\right) \longleftrightarrow \frac{2}{3}\left((-0.2)^n u[n] - 0.5(-0.2)^{n-1} u[n-1]\right)$$

$$\left(\because \frac{1}{z-a} \longleftrightarrow (a)^{n-1} u[n-1]\right)$$

$$\therefore h[n] = \frac{2}{3}\left((-0.2)^n u[n] - 0.5(-0.2)^{n-1} u[n-1]\right)$$

4.

$$y[n] = 0.4\,y[n-1] + 0.05\,y[n-2] + 3x[n]$$

(a) Taking z Transform of the given equation

$$Y(z) = 0.4\,Y(z)z^{-1} + 0.05\,Y(z)z^{-2} + 3X(z) \quad \text{(shift property)}$$

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$$Y(z)\left(1-0.4z^{-1}-0.05z^{-2}\right) = 3X(z)$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{3}{1-0.4z^{-1}-0.05z^{-2}} = \underline{\underline{H(z)}}$$

$$\therefore H(z) = \underline{\underline{\frac{3}{1-0.4z^{-1}-0.05z^{-2}}}}$$

(b) As $H(z) = \dfrac{3}{1-0.4z^{-1}-0.05z^{-2}}$

Substituting $p = z^{-1}$ to get a quadratic equation in the denominator

$$H(p) = \frac{3}{1-0.4p-0.05p^2} = \frac{60}{20-8p-p^2} = \frac{-60}{p^2+8p-20}$$

Roots of the denominator

$$p = \frac{-8 \pm \sqrt{64+80}}{2}$$

$$= \frac{-8 \pm 12}{2} = \underline{-10, 2}$$

$$\therefore H(p^{-1}) = \frac{-60}{(p+10)(p-2)}$$

Using partial fractions,

$$\frac{A}{(p+10)} + \frac{B}{(p-2)} = \frac{-60}{(p+10)(p-2)}$$

$$\frac{Ap-2A+Bp+10B}{(p+10)(p-2)} = \frac{-60}{(p+10)(p-2)}$$

$A+B=0$ —① ($\because$ coefficient of p is 0)

$-2A+10B=-60$ —②

If we do $2\times①+②$.

$$12B=-60$$
$$\Rightarrow B=\underline{\underline{-5}}$$
$$\Rightarrow A=\underline{\underline{5}} \quad (\because ①)$$

$$\therefore H(p^{-1}) = \frac{5}{(p+10)} - \frac{5}{(p-2)}$$

Substituting $z^{-1}=p$,

$$H(z) = \frac{5}{z^{-1}+10} - \frac{5}{z^{-1}-2}$$
$$= \frac{5z}{1+10z} - \frac{5z}{1-2z}$$
$$= \frac{1}{10}\left(\frac{5z}{z+\frac{1}{10}}\right) + \frac{1}{2}\left(\frac{5z}{z-\frac{1}{2}}\right)$$
$$= \frac{1}{2}\left(\frac{z}{z+0.1}\right) + \frac{5}{2}\left(\frac{z}{z-0.5}\right)$$

Taking inverse Z transform by using standard formulae,

$$h[n] = \frac{1}{2}(-0.1)^n u[n] + \frac{5}{2}(0.5)^n u[n]$$

PTO

(c) Step response $s[n]$

$$u[n] = \begin{cases} 1 & ; \quad 0 \leq n < \infty \\ 0 & ; \quad n < 0 \end{cases}$$

Taking Z Transform of $u[n]$

$$U(z) = \sum_{-\infty}^{\infty} x[n] \, z^{-n}$$

$$= \sum_{0}^{\infty} z^{-n}$$

This is the sum of an infinite GP. This can be simplified as,

$$U(z) = \frac{1}{1 - z^{-1}} \qquad (\because \text{Sum of infinite GP}).$$

$$|z^{-1}| < 1$$

$$\Rightarrow |z| > 1$$

$$= \frac{z}{z-1}$$

$\therefore$ For step response, we multiply $U(z)$ with $H(z)$ and take inverse Z Transform.

$$U(z)\,H(z) = \frac{-60}{(1-z^{-1})(z^{-1}+10)(z^{-1}-2)}$$

Substituting $z^{-1} = p$.

$$U(z)\,H(z) = \frac{60}{(p-1)(p+10)(p-2)}$$

Using partial fractions,

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$$\frac{A}{(p-1)} + \frac{B}{(p+10)} + \frac{C}{(p-2)} = \frac{60}{(p-1)(p+10)(p-2)}$$

$$= \frac{Ap + 10A + Bp - B}{(p-1)(p+10)} + \frac{C}{(p-2)}$$

$$= \frac{Ap^2 + 10Ap + Bp^2 - Bp - 2Ap - 20A - 2Bp + 2B + Cp^2 + 9Cp - 10C}{(p-1)(p+10)(p-2)} = \frac{60}{(p-1)(p+10)(p-2)}$$

$$\Rightarrow (A+B+C)p^2 + (8A - 3B + 9C)p - 20A + 2B - 10C = 60$$

($\because$ Equating Numerators).

So, we get,

$A+B+C = 0$ — ①

$8A - 3B + 9C = 0$ — ②

$-20A + 2B - 10C = 60$ — ③

Doing $-2 \times$ ① $+$ ③, $3 \times$ ① $+$ ②,

$-22A - 12C = 60$ — ④

$11A + 12C = 0$ — ⑤

Doing ④ + ⑤,

$-11A = 60$

$\Rightarrow A = \underline{\underline{\frac{-60}{11}}}$

Using this in ⑤,

$\Rightarrow 12C = 60$

$C = \underline{\underline{5}}$

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Using these values in ①,

$$\frac{-60}{11} + B + 5 = 0$$

$$\Rightarrow B = \frac{+5}{11}$$

$$\therefore U(z) H(z) = \frac{-60/11}{(p-1)} + \frac{5/11}{(p+10)} + \frac{5}{(p-2)}$$

Substituting $p = z^{-1}$,

$$U(z) H(z) = \frac{-60/11}{(z^{-1}-1)} + \frac{5/11}{(z^{-1}+10)} + \frac{5}{(z^{-1}-2)} = S(z)$$

$$= \frac{-60z}{11(1-z)} + \frac{5z}{11(1+10z)} + \frac{5z}{(1-2z)}$$

$$= \frac{60z}{11(z-1)} + \frac{1}{\cancel{10}_{2}}\frac{\cancel{5}z}{11(z+\frac{1}{10})} - \frac{5z}{2(z-1/2)}$$

$$= \frac{60}{11}\frac{z}{(z-1)} + \frac{1}{22}\frac{z}{z+1/10} - \frac{5}{2}\frac{z}{z-1/2}$$

Using standard z transform formulae, we get,

$$S[n] = \frac{60}{11}[1]^n u[n] + \frac{1}{22}[0.1]^n u[n] - \frac{5}{2}[0.5]^n u[n]$$

PTO

5.

$y[n] = x[n] + \alpha x[n-M]$

(a) Taking z Transform of the given equation,

$Y(z) = X(z) + \alpha X(z) z^{-M}$

$\therefore \dfrac{Y(z)}{X(z)} = 1 + \dfrac{\alpha}{z^{+m}} = 1 + \alpha z^{-M} = H(z)$

As we know that the z transform of $\delta[n]$ is 1, we can say that z^{-M} is the shifted version of it.
So, Taking inverse z transform, we get,

$h[n] = \underline{\delta[n] + \alpha \delta[n-M]}$

(b) The causal inverse system of $H(z)$ is $\dfrac{1}{H(z)}$ $(\because H(z) \cdot G(z) = 1)$.
(By definition)

$\therefore G(z) = \dfrac{1}{H(z)} = \dfrac{1}{1 + \alpha z^{-M}}$

This can be expanded as the sum of an infinite G.P.

$G(z) = 1 + (-\alpha z^{-M}) + (\alpha z^{-M})^2 \ldots\ldots$

As the inverse z transform of z^{-M} is $\delta(n-M)$ as shown before, we have,

$g[n] = \delta[n] - \alpha \delta[n-M] + \alpha^2 \delta[n-2M] \ldots$

As it is causal,
$(|\alpha z^{-M}| < 1$
$\Rightarrow |z^{-M}| < \dfrac{1}{|\alpha|}$
$\Rightarrow |z^{M}| > |\alpha|)$
$\Rightarrow$ Causal $\Rightarrow$ Sum of GP.

$$\Rightarrow g[n] = \sum_{k=0}^{\infty} (-\alpha)^k \delta[n - kM]$$

(c) $G(z) = \dfrac{1}{1+\alpha z^{-M}}$ $\Rightarrow$ Pole $= 1 + \alpha z^{-M} = 0$

$$\Rightarrow z^{-M} = \frac{-1}{\alpha}$$

$$\Rightarrow z^{M} = -\alpha$$

$$\Rightarrow z = \sqrt[M]{-\alpha}$$

$\therefore$ If $\left|\sqrt[M]{-\alpha}\right| < 1$, all poles lie within the unit circle and then the system is stable ($\because$ unit circle is in ROC then).