Signal Processing Course

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- Intro stuff (course overview)
- What are signals? Examples? (Same definition as IC, only asked for intuition)

Signals studied will majorly include be functions of time, but time dependent signals are **NOT** the only kind, as noted in IC course notes today.

Classification of signals

- Discrete time signals & Continuous time signals. -> Discrete time is represented as x[n], continuous time as x(t).
- Periodic and Aperiodic signals
- Even and Odd signals
- Analog and Digital signals
- Deterministic and Random signals

Discrete time and continuous time signals are differentiated according to the division of time, while analog signals and digital signals are differentiated according to the range of the values that the signal can attain. DO NOT confuse them.

Signal Processing: Signals are processed to extract information, represent them in different forms (transforms) and to modify signals when necessary. A system can be used to perform the required processing on a signal.

Classification of systems

- Causal and non causal
- Linear and Non linear

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Fourier series

- Periodic signals -> Represented as a weighted sum of sinusoids.
- Fourier series can be applied to both real and complex signals.

Trigonometric Fourier Series representation

Synthesis equation:

$$\rightarrow x(t) = c_0 + \sum_{n=1}^{\infty}(c_n \cdot cos(n\omega_0 t) + b_n \cdot sin(n\omega_0 t))$$

where $\omega_0 = \frac{2\pi}{T}$.

Each of the terms in the infinite sum are called the harmonics, and c_0 is the constant/DC part of the function.
n = 1 IS CALLED THE FUNDAMENTAL COMPONENT AND NOT THE FIRST HARMONIC

Called the synthesis equation as we can use it to *reconstruct* a signal with frequency f_0. If we only use a finite number of terms in the sum, then we get a function $x'(t)$ which will be an approximation of the original function.

Reconstruction error -> $e = x'(t) - x(t) \neq 0$ When $e = 0$, we say that it is a *perfect reconstruction.*

- Determining coefficients for a given signal is *analysis.*
- Generating a periodic signal of frequency f_0 by progressively adding weighted harmonics to a sinusoid of frequency f_0 (Fundamental component) is *synthesis.*

Exponential Fourier Series representation

Synthesis equation:

$$\rightarrow x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t}$$

To convert from exponential form, use $e^{j\theta} = \cos\theta + j\sin\theta$

Analysis equation:

$\rightarrow a_0 = c_0$ (from trigonometric series)

$$\rightarrow a_k = \frac{1}{T}\int_{<T>} x(t) \cdot e^{-jk\omega_0 t} dt$$

Dot product of functions is the integral over their period in this case.

Analysis equation for a_k actually comes from the logic behind projections of vectors (from NeSS), i.e.

$a_k = \frac{<x(t), e^{-jk\omega_0 t}>}{<e^{-jk\omega_0 t}, e^{-jk\omega_0 t}>}$

WORKS BECAUSE $\sin(n\theta)$ ***is orthogonal to*** $\sin(m\theta)$***. Same for cosine. So every term that isn't having the same coefficient becomes 0 in the dot product, hence giving us the required component, which is the Fourier coefficient here.***

a_k are the Fourier series coefficients. They are also called the spectral coefficients.

- Signal representation by orthogonal signal set (Comparison with basis decomposition of vectors, similar idea here, where sinusoids are the basis functions).

We represent $x(t)$ using orthogonal signals (Can be compared to basis vectors, sine and cosine functions here as mentioned above).

- Solving examples of Fourier series questions

Fourier series is RESTRICTED TO periodic signals.

Fourier Transform

Consider an aperiodic signal $x(t)$. Now consider a periodic extension $x_p(t)$ with period $T > 2T_1$ ($2T_1$ is time range of original signal)

That is,

$$x_p(t) = x(t) + \sum_{n=1}^{\infty}(x(t-nT) + x(t+nT))$$

Now, we can find Fourier series coefficients of $x_p(t)$, as it is periodic. For this we can use the analysis equations.

Now, if we integrate from $\frac{-T}{2}$ to $\frac{T}{2}$, the only integrable portion is from $-T_1$ to T_1, i.e.,

$$a_k = \frac{1}{T}\left(\int_{-\frac{T}{2}}^{\frac{T}{2}} x_p(t) \cdot e^{-jk\omega_0 t}\, dt\right)$$

Here, as the function is aperiodic and the only integral part is from $-T_1$ to T_1, we can change the limits to $-\infty$ and ∞ to get a_k and change x_p to x as well.

If we take $k\omega_0 = \omega$, we can define X(ω) as,

$$X(\omega) = \left(\int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt\right)$$

where ω is a real continuous variable. This is the ***FOURIER TRANSFORM*** of $x(t)$. So according to this equation, we can say,

$$X(\omega) = a_k \cdot T$$

So, $X(\omega)$ is like the envelope of a_k.

Now, for the synthesis equations,

$$\rightarrow x_p(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t}$$

Here, we can substitute a_k with $\frac{X(k\omega_0)}{T}$. We can then substitute that $\frac{1}{T}$ with $\frac{\omega_0}{2\pi}$, hence resulting in the equation

$$\rightarrow x_p(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\omega_0) \cdot e^{jk\omega_0 t} \cdot \omega_0$$

If $T \rightarrow \infty$, $\omega_0 \rightarrow 0$. So, we can now make the summation an integration, that integration being (using $\omega = k\omega_0$),

$$\rightarrow x(t) = \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} d\omega \right)$$

To summarise,

- Analysis equation (Fourier Transform)

  $$X(\omega) = \left(\int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\ dt \right)$$

- Synthesis equation (Inverse Fourier Transform)

  $$x(t) = \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} d\omega \right)$$

- Fourier transform applies to general aperiodic signals.

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Fourier Transform, continued

- Both time and frequency are continuous variables here, whereas frequency was discrete in Fourier series.
- Fourier transform is basically derived from a limiting case of Fourier series.
- Fourier Transform is a weighted linear combination (integration, instead of summation like in Fourier series) of complex sinusoids.
- In general, all frequencies are present in the transform, i.e., integration goes from $-\infty$ to ∞.
- $X(\omega)$ quantifies the contribution of $e^{j\omega t}$, similar to a_k in Fourier Series.
- $X(\omega)$ is complex valued in general.
- Notation : $X(\omega) \longleftrightarrow x(t)$

What the Inverse Fourier Transform basically does is:

- Multiplies the given frequency domain signal by sinusoids($-\infty \rightarrow \infty$) according to their frequency.
- Adds them all up to give us our signal in time domain.

Dirac delta function ($\delta(t)$) or unit impulse signal:

- $\delta(t) = 0 \ when \ t \neq 0$
- $\left(\int_{-\infty}^{\infty} \delta(t) \ dt\right) = 1$

Main use of $\delta(t)$ is the sifting property, which is,

- $x(t) \cdot \delta(t - t_0) = x(t_0) \cdot \delta(t - t_0)$
- $\int_{-\infty}^{\infty} \delta(t - t_0) \cdot x(t) \ dt = x(t_0)$

Gibbs' Phenomenon : This phenomenon is the existence of an overshoot in the Fourier sum of a signal when there is a jump discontinuity. This error does not go away as more errors are added to the sum. However, the error does decrease in width and energy on taking more terms into consideration.

- A square wave in time has a sinc ($\frac{2\sin(\omega t)}{\omega}$) function as its Fourier transform. Graph of a sinc function is like a *single slit diffraction pattern* (Middle band is twice as big and stuff).
- Did example qs for Fourier Transform.

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Note: All double sided arrows imply FT here, or FS in a few places.

The Fourier Transform can be mapped to the Fourier series too.

Q. Consider the following Fourier Transform,

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

find $x(t)$.

Using the Inverse FT formula,

$\rightarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} d\omega$

Q. consider the following Fourier Transform :

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, a_k\, \delta(\omega - k\omega_0) \quad ; \text{ find the signal } x(t).$$

$$\longrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{ 2\pi \sum_{-\infty}^{\infty} a_k\, \delta(\omega - k\omega_0) \right\} e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \cdot 2\pi \sum_{-\infty}^{\infty} a_k \int_{-\infty}^{\infty} \delta(\omega - k\omega_0)\, e^{j\omega t}\, d\omega = \sum_{-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$$

$$x(t) = \sum_{-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$$

... FS representation of $x(t)$

... a_k are FS coefficients of $x(t)$

$$2\pi \sum_{-\infty}^{\infty} a_k\, \delta(\omega - k\omega_0) \xleftrightarrow{FT} x(t) \xleftrightarrow{FS} \{a_k\}$$

The foruier series only exists for periodic $x(t)$.

Properties of Fourier Transform

- Linearity: $x(t) \longleftrightarrow X(\omega)$; $y(t) \longleftrightarrow Y(\omega)$

$\Longrightarrow \alpha x(t) + \beta y(t) \longleftrightarrow \alpha X(\omega) + \beta Y(\omega)$

- Time shift: $x(t) \longleftrightarrow X(\omega)$

$\Longrightarrow x(t - t_0) \longleftrightarrow X(\omega) \cdot e^{-j\omega t_0}$

Basically a phase shift.

- Time and Frequency Scaling: $x(t) \longleftrightarrow X(\omega)$ $x(at) \longleftrightarrow \frac{1}{|a|}X(\omega/a)$

Makes sense intuitively, as an increase in time should result in a decrease in frequency, and vice versa.

Then a discussion about playback speeds on YT, so higher speeds imply the same voice needs to be fit into a shorter span of time, hence frequency needs to be increased. So, we get higher pitched voices and stuff (As I've noticed). Similarly, slowing down creates a lower frequency, which adds bass, hence the weird droning and hilarious playbacks at 0.25x. As for a physical representation of the constant a, the energy of a signal changes when it is compressed???? Should it not increase in frequency if I compress it in time? Will need to clarify.

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2 June 2021

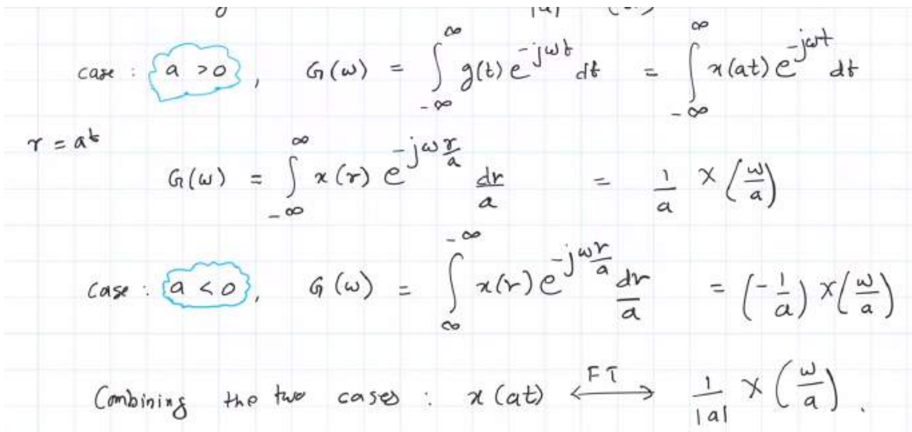

Figure 1: Proof for scaling

Note: Fourier Transform is a special case of Laplace transform.

Laplace transform	Fourier Transform
$x(t) \longleftrightarrow X(s)$	$x(t) \longleftrightarrow X(\omega)$
s is a complex variable.	ω is a real variable.
$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt$	$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-\omega t} dt$
$s = \sigma + j\omega$	When $\sigma = 0$ in Laplace transform
Region of Convergence (ROC):	Imaginary axis in s plane
Region of s plane where that integral converges.	

Trying to solve a known transform in reverse...

$\delta(t) \longleftrightarrow 1$. Starting from $X(\omega)$, find $x(t)$. Using synthesis formula, we get,

$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega$ Here we are at a fix, as we cannot simplify frther, as it would give undefined values. So signals may not have a converging integral at all times. We must take this into consideration too.

Now, if we take the Fourier transform of $cos(\omega_0 t)$, we still get $\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$. But here if we try to find the ROC, we find that the imaginary axis is not a part of it. So it does not converge. The transform will be 2 poles in the S plane. So when we try going forwards from $cos(\omega_0 t)$, we get stuck. As I understand it, any function that has a transform that contains a δ function will not cnoverge as it is not well defined.

Continuing with properties of the Fourier Transform...

- Parseval's theorem: Energy is same in time and frequency domains upto a scaling factor. i.e. $\int_{-\infty}^{\infty} |x(t)|^2 dt \underset{7}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$
- Symmetry and Conjugacy properties: $x(t) \longleftrightarrow X(\omega)$ $x(-t) \longleftrightarrow X(-\omega)$ (Can be derived from scaling property too) $x^*(t) \longleftrightarrow X^*(-\omega)$ From these, we can derive,
  - If $x(t)$ is real, $x(t) = x^*(t) \implies X(\omega) = X^*(-\omega)$ from first and third points. This is called conjugate symmetry.

- If $x(t)$ is even,
  $x(t) = x(-t) \implies X(\omega) = X(-\omega)$ from first and second points.
- If $x(t)$ is odd,
  $x(-t) = -x(t) \implies X(-\omega) = -X(\omega)$ from first and second points.
- Differentiation in time:
  $\frac{d(x(t))}{dt} \longleftrightarrow j\omega X(\omega)$

- Differentiation in frequency: $tx(t) \longleftrightarrow \frac{d(X(\omega))}{d\omega}$
- Frequency shift (Analogous to time shift) $x(t)e^{j\omega_0 t} \longleftrightarrow X(\omega - \omega_0)$

Linear Time Invariant (LTI) Systems

Many systems can be represented as LTI systems, like RLC circuits are LTI systems. Any system with a differential equation representation is an LTI system. Examples of non LTI systems:

- Transistors
- Op amps
- Diodes
- Any digital system

Linearity: $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$

$\implies \alpha x_1(t) + \beta x_2(t) \to \alpha y_1(t) + \beta y_2(t)$

Time invariance: $x(t) \to y(t)$

$\implies x(t - t_0) \to y(t - t_0)$

Any LTI system can be uniquely and fully characterised by its impulse response, i.e., the response it gives when the input is an impulse.

$\delta(t) \to h(t)$, where $h(t)$ is the impulse response of the system.

Given an arbitrary input $x(t)$ to an LTI system with impulse response $h(t)$, the output of the system will be an CONVOLUTION,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

where $x(t) * h(t)$ is $x(t)$ CONVOLVED with $h(t)$.

Intuition for convolution:

$-\infty$

$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$ ... weighted linear combination of shifted impulses

$\Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$ i.e. convolution.

The first $x(t)$ expression in the above image is a weighted linear representation of the given signal using shifted impulses.

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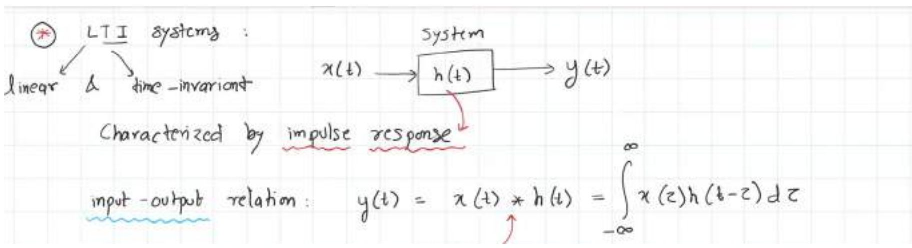


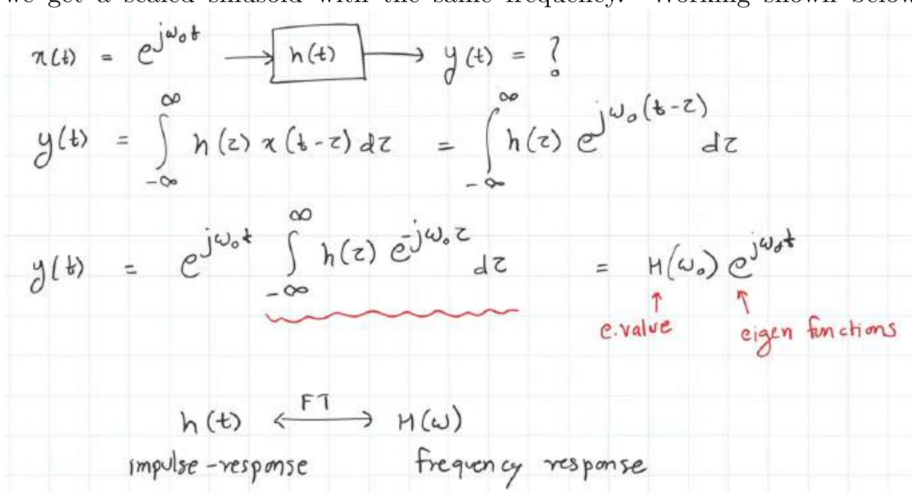
This is what we learned about LTI systems in the previous class.

Properties of convolution

- Commutativity $x_1(t) * x_2(t) = x_2(t) * x_1(t)$
- Distributivity $h(t) * [x_1(t) + x_2(t)] = h(t) * x_1(t) + h(t) * x_2(t)$
- Associativity $x_1(t) * [x_2(t) + x_3(t)] = [x_1(t) * x_2(t)] + x_3(t)$

Frequency analysis of LTI systems:

If we give a complex sinusoid as input to an LTI system, i.e., $x(t) = e^{j\omega_0 t}$, we get a scaled sinusoid with the same frequency. Working shown below.



So the complex sinusoid is an EIGENFUNCTION, and the Fourier transform $H(w_0)$ is the EIGENVALUE.

If we give a periodic signal as input to an LTI system, the same happens, as we can represent all periodic signals as a sum of complex sinusoids by their Fourier Series representation. If $x(t)$ is a periodic function,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$

On convolving with impulse response $h(t)$, we get

$$y(t) = \sum_{k=-\infty}^{\infty} a_k \cdot H(k\omega_0) e^{j\omega_0 kt}$$

Cosine can be represented as the sum of complex sinusoids too, in the same way of course. Working shown below

(c) consider $\cos(\omega_0 t)$ as input.

$\cos(\omega_0 t) = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] \Rightarrow y(t) = \frac{1}{2}\left[H(\omega_0) e^{j\omega_0 t} + H(-\omega_0) e^{-j\omega_0 t}\right]$

consider system with real $h(t)$

$\Rightarrow H^*(-\omega) = H(\omega) \Rightarrow H(-\omega) = H^*(\omega)$

$y(t) = \frac{1}{2}\left[H(\omega_0) e^{j\omega_0 t} + H^*(\omega_0) e^{-j\omega_0 t}\right]$

$H(\omega_0) = r e^{j\theta}$

$H^*(\omega_0) = r e^{-j\theta}$

$= \frac{1}{2} |H(\omega_0)| \left[e^{j\theta} e^{j\omega_0 t} + e^{-j\theta} e^{-j\omega_0 t}\right]$

$y(t) = |H(\omega_0)| \cos(\omega_0 t + \theta)$

As seen, the final function is just scaled and phase shifted.

Note, $|H(\omega)| \to$ Magnitude response $\angle H(\omega) \to$ Phase response (θ in the above image)

An LTI system NEVER adds frequencies. Only propagates or removes existing frequencies. This can be used as a criteria to find LTI systems. Proof is given a few lines from now.

If we give a general aperiodic signal as input, the FT is found by...

- Convolution property Fourier transform:

  If $x(t) * h(t) = y(t)$, then $X(\omega) \cdot H(\omega) = Y(\omega)$.

  Here, a convolution is converted into a multiplication by FT. This eases calculation by a lot. Now we can apply IFT to $Y(\omega)$ and get $y(t)$ which is what we need. Proof is pretty straightforward. $H(\omega)$ is the frequency response of the LTI system.

  From here we can say that if $X(\omega)$ is zero at any frequency, $Y(\omega)$ is NECESSARILY zero there too, so an LTI system cannot add frequencies to a given signal.

Did problems on convolution.

Ex 1: A delay system, no need for properties

Ex. ① $x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = ?$ given: $h(t)$

ⓐ direct convolution

$$y(t) = x(t) * \delta(t - t_0)$$

$$y(t) = x(t - t_0)$$

Ex 2: A derivative system

② derivative System : $x(t) \rightarrow \boxed{LTI} \rightarrow y(t) = \frac{dx}{dt}$

find $H(\omega)$.

$$\Rightarrow \quad Y(\omega) = \underbrace{j\omega}_{H(\omega)} X(\omega)$$

This system ~~boosts~~ SCALES the weights of the frequencies.

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LTI systems can be interpreted as ***Frequency selective filters***. Mainly because $H(\omega)$ can nullify any frequencies by having that component with zero wieght, hence excluding that frequency from the final output. Example q to demonstrate $\rightarrow$

Convolution property:

③ $x(t) = e^{-bt}u(t)$ & $h(t) = e^{-at}u(t)$ & $a, b > 0$, $a \neq b$.

using FT, find $y(t) = x(t) * h(t)$.

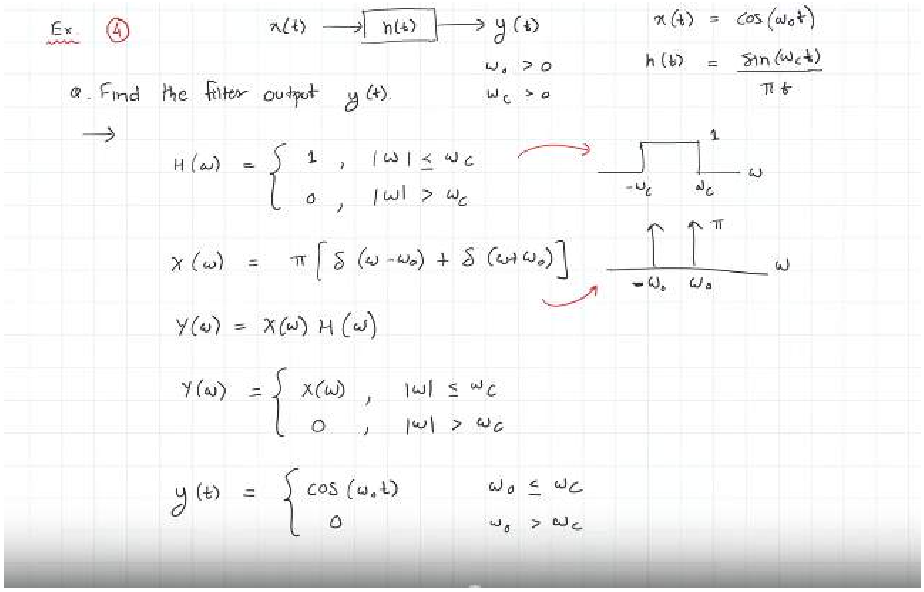
$$\Rightarrow \quad Y(\omega) = X(\omega) \cdot H(\omega)$$

$$Y(\omega) = \frac{1}{b + j\omega} \cdot \frac{1}{a + j\omega}$$ $\Rightarrow$ apply partial fraction analysis.

$$\Rightarrow \quad Y(\omega) = \frac{A}{a + j\omega} + \frac{B}{b + j\omega}$$ where, $A = \frac{1}{b - a}$, $B = \frac{-1}{b - a}$

$$\Rightarrow \quad y(t) = A e^{-at} u(t) + B e^{-bt} u(t)$$

An example q that shows how a sinc function acts as a filter.



Pretty straightforward. This is an ideal low pass filter.

A high pass filter is just that graph reversed.

A band pass filter is like a collection of bands that are allowed, and the rest which are blocked.

We cannot design an ideal low pass filter, as a sinc function is infinte, i.e., it's unbounded and extends for infinite time, and hence requires an infinite convolution, which we don't have the resources for. It's also unstable? Idk what that really means, but apparently it's unstable, i.e, a bounded input may give an unbounded output, and so we don't do it. We can design filters that can give something close to an ideal filter, and deal with it.

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9 June 2021

Analysis of a series RC circuit (using Fourier transform)

The differential eqn for an RC system is

$$V_{in}(t) - RC\frac{dV_{out}}{dt} = V_{out}(t)$$

Taking Fourier Transform of this,

$$V_{in}(\omega) - RCj\omega V_{out}(\omega) = V_{out}(\omega)$$
$$\implies V_{out}(\omega) = \frac{1}{1 + j\omega RC} V_{in}(\omega)$$

As seen, this will attenuate the higher frquencies, as it is inversely proportional to ω.

So,

$$H(\omega) = \frac{\frac{1}{RC}}{\frac{1}{RC} + j\omega}$$

So, using standard fourier transform formulae, we can get

$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$

The plots for those functions are as given below.

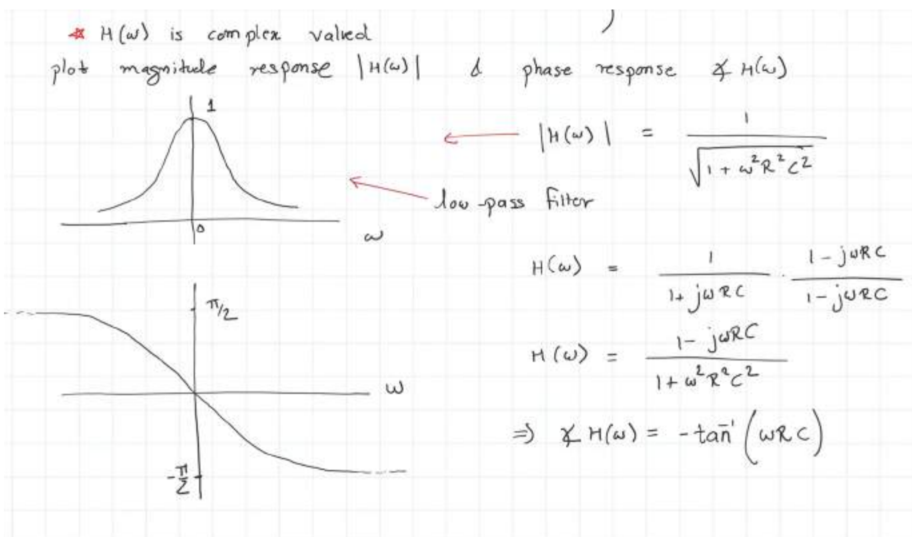


Figure 2: plot1

Multiplication property:

$$x(t) \leftrightarrow X(\omega) \; and \; y(t) \leftrightarrow Y(\omega)$$

$\implies x(t)y(t) \longleftrightarrow \frac{1}{2\pi}[X(\omega) * Y(\omega)]$

This is basically the reverse of the convolution property. Also called modulation property. This finds use because while multiplication with impulses is difficult, convolution is very easy.

Example question

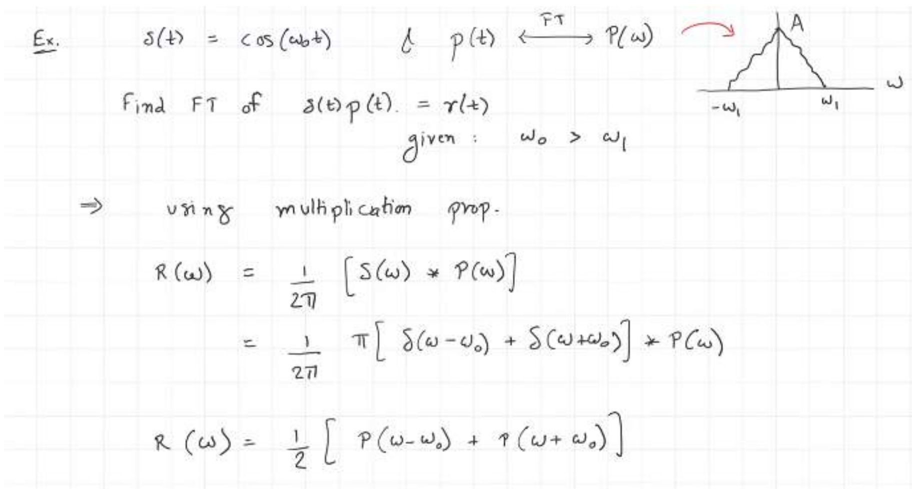
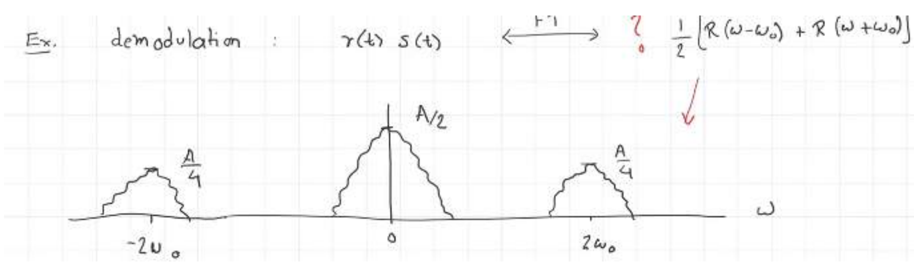


Figure 3: exq

This is an example of amplitude modulation. As we can see, the amplitude has decreased. In the above case, cosine is the carrier signal, and the weird thing is the message signal.

For demodulation, we just multiply the received signal with the same carrier wave.



We'll need to scale it as required. Low pass filter to get that middle copy.

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11 June 2021

Sampling Theorem

It's a link between continuous time signals and discrete time signals. Matlab plots graphs by discretising the input, as continuous time computation is not possible, hence the time_grid thing.

We can take infinite samples of a signal, but then this is practically not useful, so we need a limit. We need to take a certain amount of samples so that we have enough to reconstruct the signal when necessary. We do this because we can use it for:

- Compression
- Storage or transmission
- Computations

Fourier series is a discrete representation of periodic signals.

For a general aperiodic signal though, we sample its value at cerain instants of time and "join the dots". For this, we need to have enough samples, or we cannot have a definitive reconstruction of the signal.

We can reconstruct a band limited signal perfectly.

Band limited singal: A signal $x(t)$ is band limited if there is a frequency ω_m such that the FT $X(\omega)$ is zero for $|\omega| > \omega_m$. Ex: speech signals only extend upto a certain frequency.

Sampling theorem

A Band limited signal with some maximum frequency ω_m can be perfectly recovered/reconstructed from its samples if the sampling frequency ω_s satisfies

$$\omega_s > 2\omega_m$$

We also call $2\omega_m$ as the NYQUIST RATE of the signal.

Sampling interval is defined as

$T_s = \frac{2\pi}{\omega_s}$

Intuitively speaking, this is because when a signal has higer frequency, it varies quickly, which means we need more samples in any region when compared to a signal with a lesser frequency.

If the signal is not band limited, there is no limit on how high its frequency can go, hence, we can never PERFECTLY reconstruct it, but we can get a good approximation in a few cases (Idk if it's a few or many, but you can).

Proof: We use an impulse train to do the sampling.

$p(t) = \displaystyle\sum_{k=-\infty}^{\infty} \delta(t - kT_s)$

Now, we sample $x(t)$.

$$x(t) \longleftrightarrow x[k] = x(kT_s)$$

We use impulse train only for analytical proofs.

To sample, we multiply,

$x_p(t) = x(t) \cdot p(t)$

which gives us

$x_p(t) = \sum_{-\infty}^{\infty} x(kT_s)\delta(t - kT_s)$

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14 June 2021

Time limited signals may or may not be recoverable from sampling, they may not be band limited.

Periodic signals may not be recoverable from their samples, as periodic signals can potentially have infinite harmonics, which we cannot sample.

Continuing from last class...

If we use multiplication property on $x_p(t)$ after converting it to frequency domain, we get

$$X_p(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)$$

which gives us

$$X_p(\omega) = \frac{1}{2\pi} X(\omega) * \frac{2\pi}{T_s} \sum_{-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{T_s})$$

which simplifies to,

$$X_p(\omega) = \frac{1}{T_s} \sum_{-\infty}^{\infty} X(\omega - k\frac{2\pi}{T_s})$$

which is an infinite chain of **SCALED COPIES** of $X(\omega)$.

Diagrammatically,

As we can see from the figure, a lot of info seems to be lost in the time domain, but an exact scaled copy of the original signal is still there in the frequency domain, we can now use a suitable low pass filter to obtain our signal and then we can scale it as required.

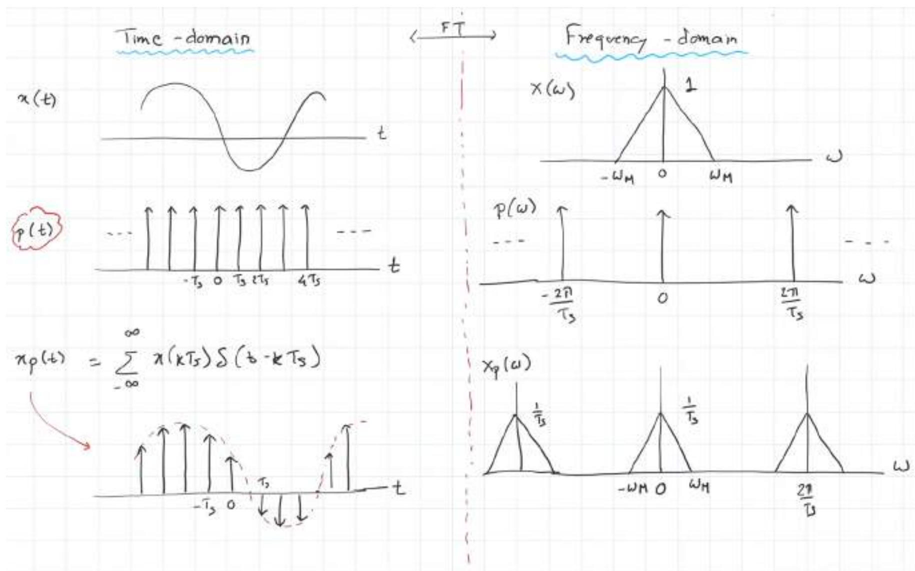

Figure 4: diag

The system that does the convolution is not LTI. A simple way of confirming that it is not LTI is as it creates multiple copies of the same signal at different frequencies, but we know that an LTI system cannot add new frequencies to a given signal.

The Nyquist rate is basically defined to prevent any overlaps between those copies. Overalapping causes aliasing. As seen from the picture, we get the inequality,

$$\frac{2\pi}{T_s} - \omega_m > \omega_m$$

to prevent overlaps. This directly gives us the Nyquist rate condition. Hence proved.

Now this reconstruction is in frequency domain. We want reconstruction in time domain too.

Low pass filtering in frequency domain is basically multiplication. So this is equivalent to convolution in time domain.

So,

$$x_r(t) = x_p(t) * h_{LPF}(t)$$

Ideal Reconstruction

If we take an ideal LPF, we get $h_{LPF}(t)$ as the sinc function.

$h_{LPF}(t) = T_s \frac{sin(\omega_c t)}{\pi t}$

Now, convolving,

$$x_r(t) = x_p(t) * h_{LPF}(t)$$

$$= \sum_{-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s) * T_s.\,\frac{\sin(\omega_c t)}{\pi t}$$

$$x_r(t) = T_s \sum_{-\infty}^{\infty} x(nT_s)\,\frac{\sin(\omega_c(t - nT_s))}{\pi(t - nT_s)} = x(t) \;\cdots \text{with appropriate } T_s \;\&\; \omega_c.$$

Figure 5: con

ω_c here is the cutoff frequency of the filter. This ideal reconstruction is basically a combination of weighted and shifted sinc shapes. An ideal reconstruction uses infinite sinc shapes.

THIS IS CALLED SINC INTERPOLATION.

Non ideal reconstructions

- Zero order hold reconstruction This basically takes the sample and holds its value till the next sample, and so on, so the graph looks a little like steps if the samples are spaced out.
- Linear interpolation (Connect the dots with lines) Just connect the obtained samples with straight lines.

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16 June 2021

For real signals, we can ignore the negative frequency spectrum and modify the graph to make it smaller and stuff.

- Zero Order Hold reconstruction (Piecewise constant approximation) h(t) is just a rectangular pulse from 0 to Ts. On convolving, it will hold the value till Ts, and then take the next sample's value and so on. This is basically using a non ideal low pass filter instead of an ideal low pass filter.

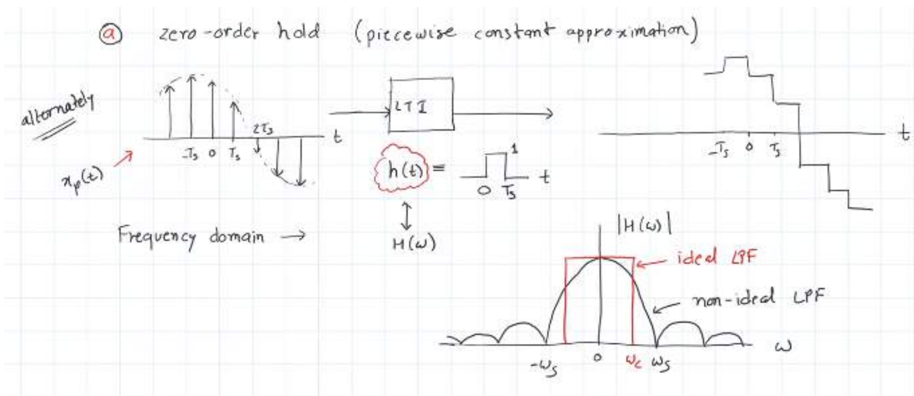

Figure 6: CLICK

- Linear interpolation (First order hold) Take a triangular pulse instead of a rectangular pulse here, from -Ts to Ts. The fourier transform of a triangular pulse is a squared sinc function, as it is formed by the convolution of two rectangular pulses. This is a better non ideal filter.

Aliasing (Spectral folding)

When the copies overlap, i.e. when we take a sampling frequency lower than the Nyquist rate. We ideally don't want aliasing to happen. High frequencies will appear at lower frequencies, basically they occur a little sooner than they are supposed to. If the signal is not band limited, no matter how high our sampling rate is, we will not be able to reconstruct it perfectly because of aliasing.

Solution is to use an anti aliasing filter. This just means pass the input signal into a low pass filter before sampling, where the filter has a cutoff frequency of $\omega_s/2$. This is basically forcing the signal to be band limited. This of course, will not be the same signal, but it will not undergo aliasing. This is what happens in those videos of cars and stuff, where the wheels seem to slow down after a while even though the car speeds up, because we aren't sampling quick enough. At exactly double the frequency, i.e. when the car has an rpm exactly double of our sampling rate, the wheels will appear stationary. This is an example of aliasing.

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Recap

Analog to digital conversion is actually more than one step. One of them is sampling, the sampled values are then **quantised**. Quantisation is basically

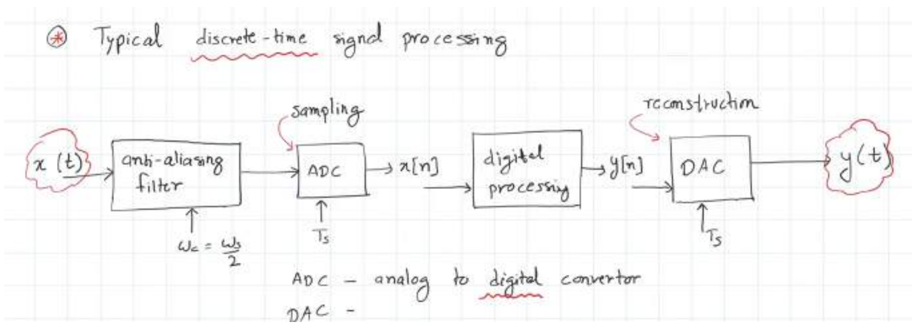

Figure 7: img

the discretization of the sample values, along with encoding, i.e. mapping them to bit sequences. ADC takes samples and converts them to binary values and outputs them. More bits implies better resolution. There is an error as we are mapping a certain continuous range to a single bit sequence. DAC takes the proccessed input and using the quantised values, it gives us analog signals.

Bitrate can be defined as,

$$b_s = B \cdot f_s$$

Where B is the number of bits used in the quantisaton of samples, and f_s is the sampling rate in Hz. We won't worry about quantisation here after this point, use it as a black box ig.

Discrete time signals

In x[n], n is necessarily an integer.

Examples:

- Unit impulse: $\delta[n]$, defined as 1 at n = 0, and 0 for all other points.
- Unit step: $u[n]$ is 1 $\forall n \geq 0$ and 0 everywhere else.
- Sinusoids: $x[n] = sin(\omega_0 n)$ Similar for cosine.
- Exponentials: $x[n] = a^n, a \in R$

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21 June 2021

Complex exponentials and sinusoids revision

Energy $\rightarrow \sum_{-\infty}^{\infty} |x[n]|^2$ If energy of a signal is finite, the signal is called an energy signal.

Power $\rightarrow \lim_{N\rightarrow\infty} \frac{1}{2N+1} \sum_{-N}^{N} |x[n]|^2$ If power of a signal is finite,the signal is called a power signal.

Even signals: $x[-n] = x[n]\ \forall n$ Odd sgnals: $x[-n] = -x[n]\ \forall n$ This implies $x[0] = 0$ for an odd signal.

Periodic signals: A signal is periodic if

$$x[n+N] = x[n]\ \forall\ n\ and\ N > 0$$

Smallest period of the signal $\rightarrow N$, called fundamental period.

Ex: $sin(5\pi n) \rightarrow sin(5\pi(n+N)) = sin(5\pi n) \implies 5\pi N = \pi k, k \in Z \implies N = \frac{k}{5}$ $\implies N = 1$ is the smallest period.

Frequency $= \frac{1}{N}$

Ex2: $cos(5\pi n) \rightarrow cos(5\pi(n+N)) = cos(5\pi n) \implies 5\pi N = 2\pi k, k \in Z$ $\implies N = \frac{2k}{5} \implies N = 2$ is the smallest period.

In general, for signals like $sin(\omega_0 n), cos(\omega_0 n), e^{j\omega_0 n}$ we can say that

$sin(\omega_0 n) \rightarrow sin(\omega_0(n+N)) = sin(\omega_0 n) \implies \omega_0 N = 2\pi k, k \in Z \implies \omega_0 = \frac{2\pi k}{N}$ for periodicity.

For example, if we take $cos(5n)$, as 5 does not satisfy that condition for any integers k and N, we can say that it is not periodic in discrete time.

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$e^{j\omega n} \rightarrow$ Highest frequency $= \pm\pi$, Lowest frequency $= 0$.

Discrete time LTI systems (Also called LSI, S for shift)

Systems that follow are linear and time/shift invariant, just like in continuous time.

- Linearity: $x_1 \rightarrow y_1, x_2 \rightarrow y_2 \implies \alpha x_1 + \beta x_2 \rightarrow \alpha y_1 + \beta y_2$
- Time invariance: $x[n] \rightarrow y[n] \implies x[n-n_0] \rightarrow y[n-n_0]$

Examples:

- $y[n] = x[n-n_0]$, is LTI.

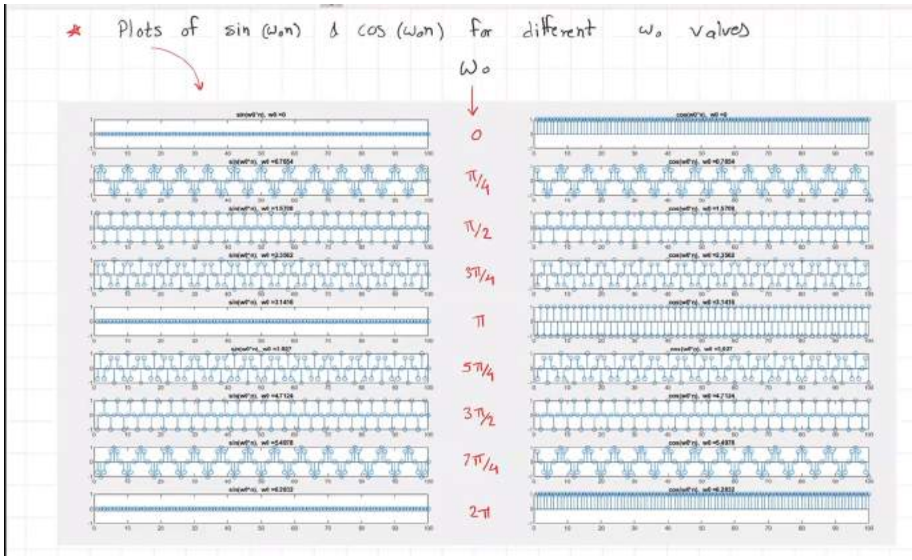

Figure 8: sinf

- $y[n] = x^2[n]$, is time invariant only.
- $y[n] = x[n] + x[n-1]$, is LTI.
- $y[n] = nx[n]$, is only linear.

Representing signals using impulses $x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[-1]\delta[n+1]$......

Impulses in discrete time are well behaved, and are just equal to one, not tending to infinity or something.

Impulse response of LTI system $\delta[n] \to h[n]$ in LTI system.

So, from this we can say that on passing a function into an LTI system, we get

$$y[n] = \sum_{-\infty}^{\infty} x[k]h[n-k]$$

$$y[n] = x[n] * h[n]$$

That summation is the convolution sum.

Properties of convolution $\to$ commutative, associative, distributive.

Unit impulse (discrete time) properties

- $x[n]\delta[n-k] = x[k]$
- $x[n] * \delta[n-k] = x[n-k]$

- $x[n] * \delta[n] = x[n]$

Discrete time Fourier Transform (DTFT)

Analogous to continuous time Fourier transform.

We define the DTFT of a function $x[n]$ as

$$g(\omega) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$$

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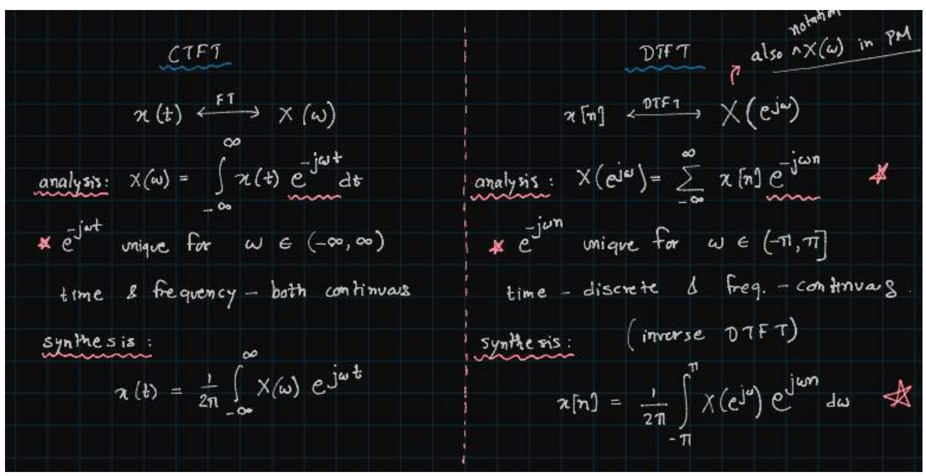
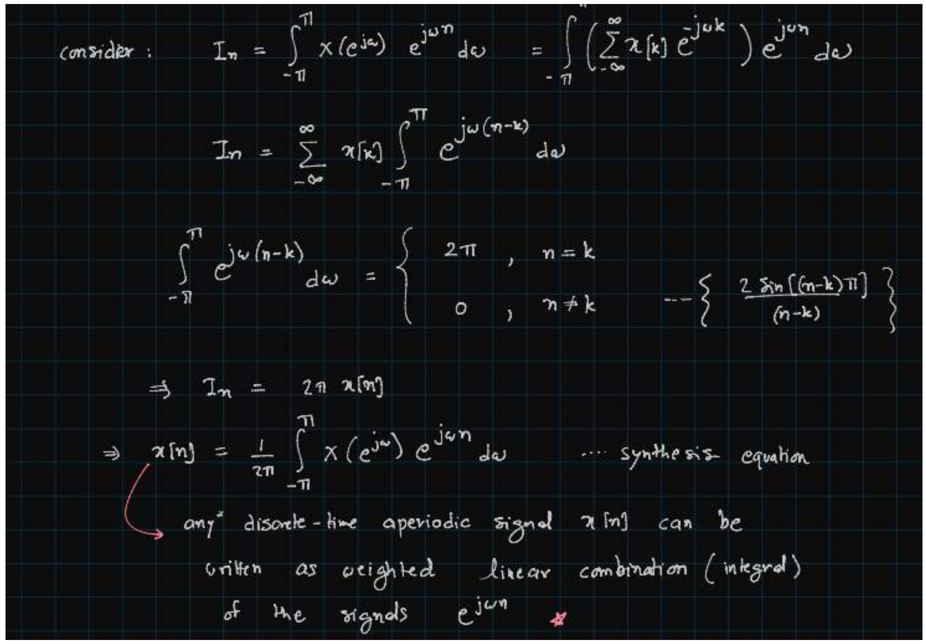
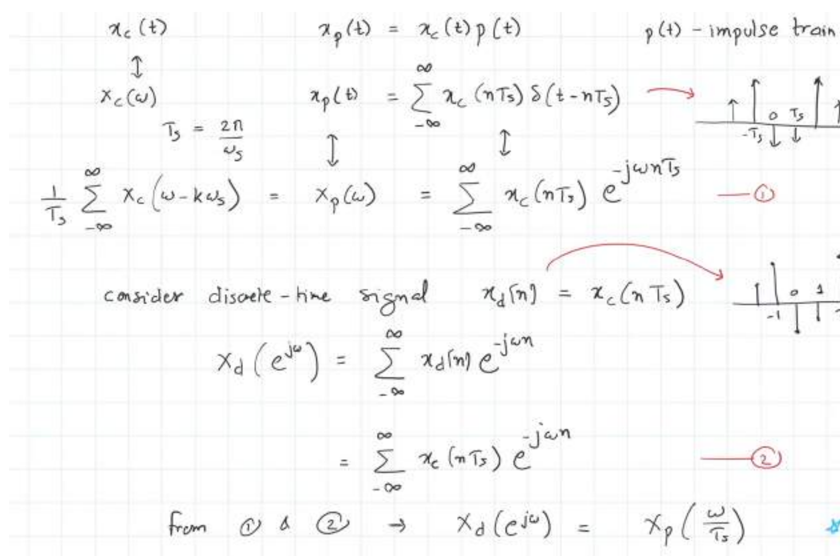
Notice that the DTFT for any signal is periodic with period 2π. As shown below,

$g(\omega + 2\pi) = \sum_{-\infty}^{\infty} x[n] \cdot e^{-j(\omega+2\pi)n} = \sum_{-\infty}^{\infty} x[n] \cdot e^{-j\omega n} \cdot e^{-j2\pi n} = \sum_{-\infty}^{\infty} x[n] \cdot e^{-j\omega n} = g(\omega)$

Nice table: 

Proof for synthesis: 

This can be compared to impulse train sampling, as impulse train sampling also made copies of the signal.

Ultra cool map that's kinda confusing: 

DTFT can also be compared to Fourier Series. It's like the reverse, because here, frequency is continuous, and we have a perioic signal in frequency domain. In FS, we get a periodic and continuous signal in time domain. We also have discrete time in DTFT, while we have discrete frequencies in FS.

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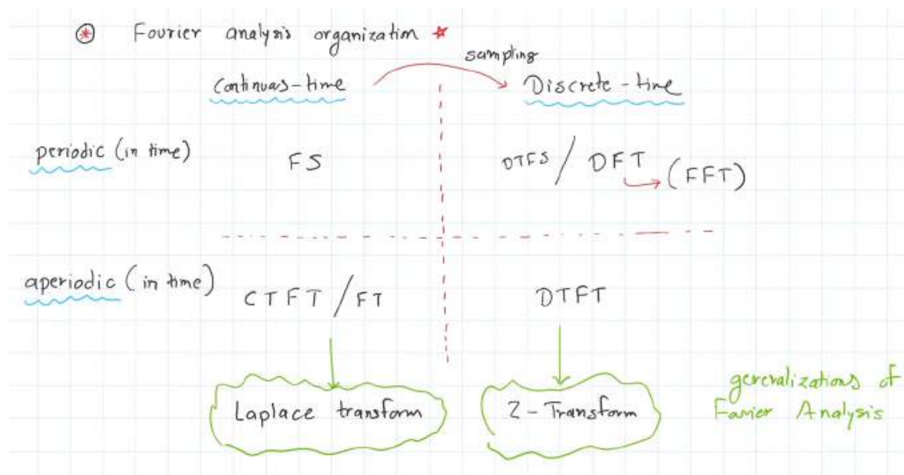


Organization of Fourier Analysis:

Standard signals and their DTFT:

- $\delta[n] \longleftrightarrow 1$
- $\delta[n-n_0] \longleftrightarrow e^{-j\omega n_0}$
- Unit impulse in frequency (Impulse train, because it has to be periodic)

  $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$

  So, $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega}d\omega$

  $\implies x[n] = \frac{1}{2\pi}\forall n$
- Shifted impulse in frequency Like before (still impulse train, but we find for one time period, also like before)

  $\delta(\omega-\omega_0) \longleftrightarrow \frac{1}{2\pi}e^{j\omega_0 n}$
- One sided exponential

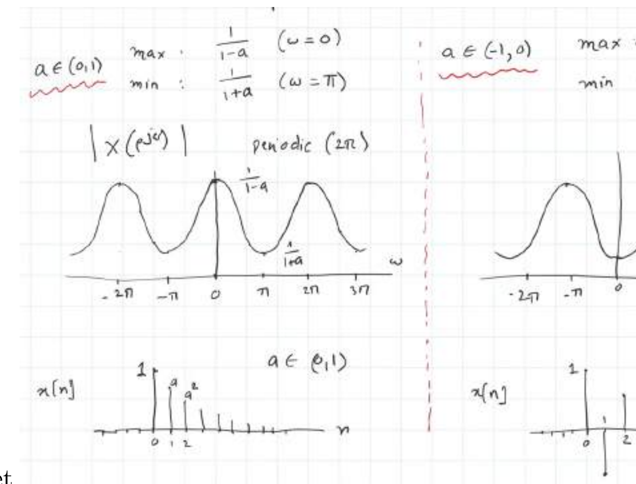
  $x[n] = a^n u[n], |a| < 1$

  Its DTFT is $x[n] \longleftrightarrow \frac{1}{1-ae^{-j\omega}}$

  From this we get

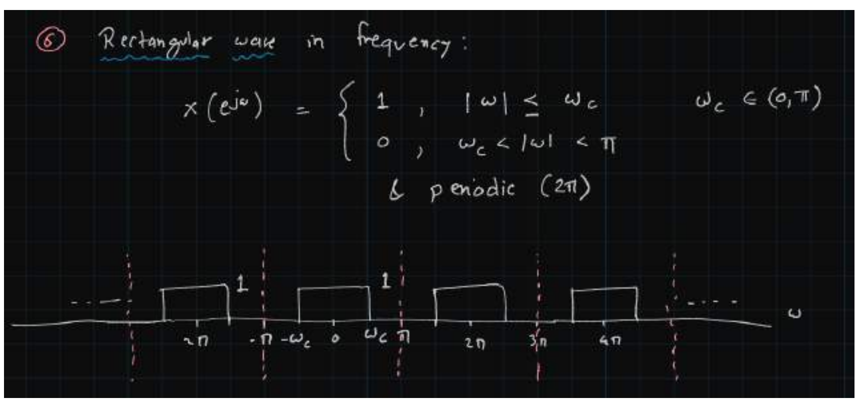
  $|X(e^{j\omega})| = 1/\sqrt{1+a^2-2acos(\omega)}$

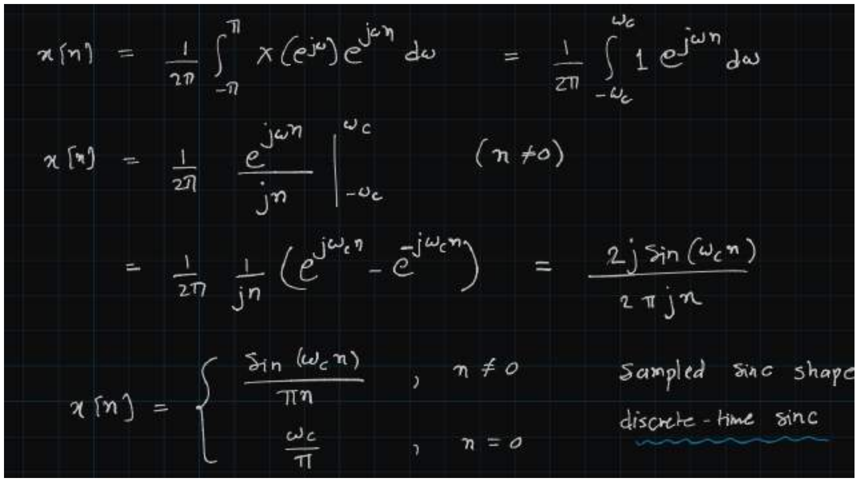
On plotting this for positive a and negative a, we get



As we can see, we can use the negatove a graph as a non ideal high pass filter, and the positive a graph as a non ideal low pass filter, as done in lab.

- Rectangular pulse in frequency, given below





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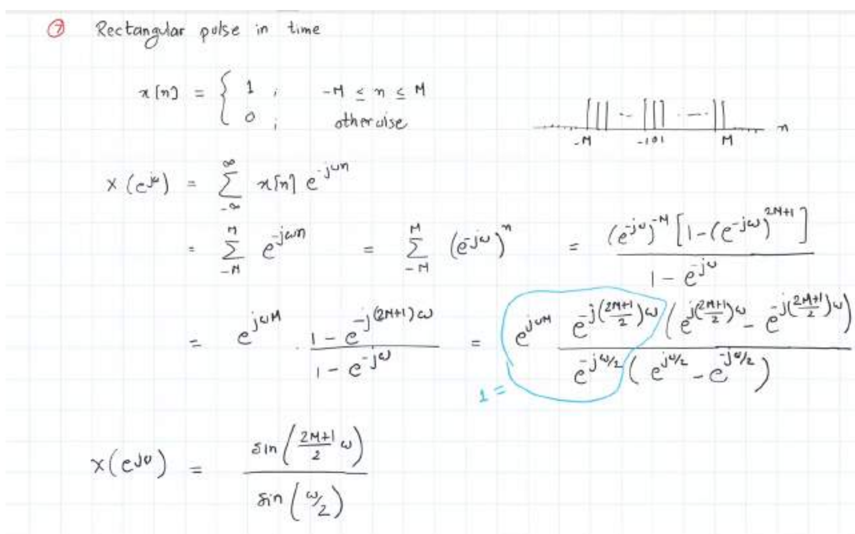
30 June 2021

Continuing from last class…

- Rectangular pulse in time



Derived below.
This is like a periodic sinc, with period 2π, and is real valued.

Discrete time LTI systems analysis using DTFT

When $x[n] = e^{j\omega n}$,

$$y[n] = x[n] * h[n] = \sum_{-\infty}^{\infty} x[k]h[n-k]$$

Substituting x[n],

$$y[n] = \sum_{-\infty}^{\infty} e^{j\omega k} h[n-k]$$

$$\implies y[n] = \sum_{-\infty}^{\infty} e^{j\omega(n-m)} h[m]$$

$$\implies y[n] = e^{j\omega n} \sum_{-\infty}^{\infty} e^{-j\omega m} h[m]$$

$$\implies y[n] = H(e^{j\omega})e^{j\omega n}$$

Thus, complex exponents are eigenfunctions.

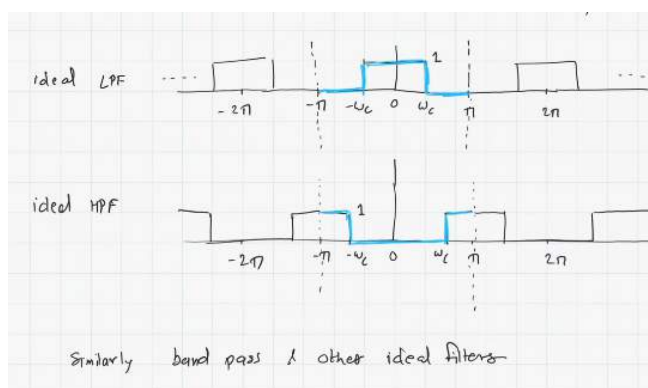
- Convolution property

  $x[n] * h[n] = y[n]$

  $Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$

The frequency selectivity of an LTI system is determined by its frequency response $H(e^{j\omega})$. We usually pay attention to the magnitude response, which is $|H(e^{j\omega})|$.

Filters:



- Ideal LPF and HPF: Graph of Magnitude response

  Their graphs will be sampled sincs in time domain.

- All pass filter: $h[n] = \delta[n - n0]$, magnitude response is just a constant.

- Non ideal filters Did these filters in lab.

  – Moving average filter (Non ideal low pass filter)
  – Digital differentiator filter (Non ideal High pass filter)

The plot for the magnitude responses of these filters allows us to find the nature of their filtering.

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Properties of DTFT

- Convolution
- Linearity
- Time shift
- Frequency shift
- Symmetry
- Differentiation in Frequency
- Parsevals relation
  $$\sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$
- Multiplication property
  $y[n] = x_1[n] \cdot x_2[n]$
  $\implies Y(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\theta) \cdot X_2(\omega - \theta) d\theta$
  Regular convolution will not work, as the integral may tend to infinity.

Discrete Fourier Transform (DFT)

DFT is very machine friendly. Consider a finite sequence $x[n], n \in \{0, 1, 2..., N - 1\}$.

We can sample the DTFT of the sequence at $\omega_k = 2\pi k/N, k = 0, 1, ..., N - 1$. This gives us

$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\omega_k n}$ (Because the rest are 0).

This is the DFT of the N length sequence x[n].

Inverse DFT is getting x[n] from X[k]. Consider

$S_m = \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}$. On substituting the formula for X[k], we get

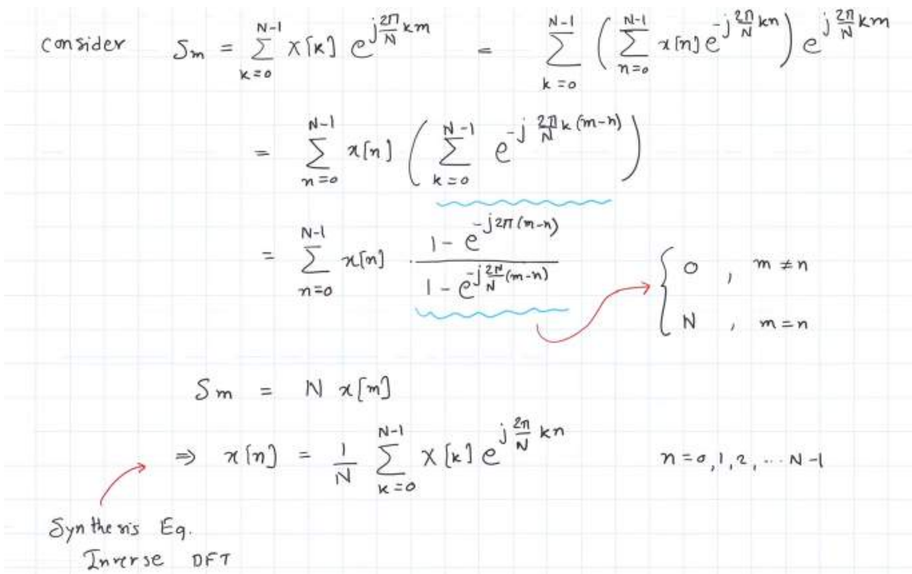
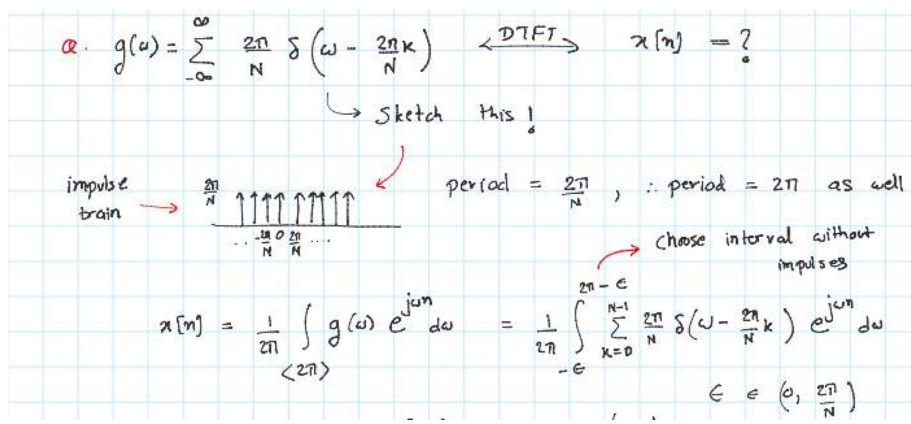


Figure 9: IDFTd

DFT is an orthogonal transformation.

Derivation for impulse train:



$$= \frac{1}{2\pi} \cdot \frac{2\pi}{N} \sum_{k=0}^{N-1} \int_{-\epsilon}^{2\pi-\epsilon} \delta\left(\omega - \frac{2\pi}{N}k\right) e^{j\left(\frac{2\pi}{N}k\right)n} \, d\omega$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}kn} \underbrace{\int_{-\epsilon}^{2\pi-\epsilon} \delta\left(\omega - \frac{2\pi}{N}k\right) d\omega}_{=1}$$

$$\Rightarrow x[n] = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}kn} = \begin{cases} \frac{1}{N} \frac{1-e^{j2\pi n}}{1-e^{j\frac{2\pi}{N}n}} & n \neq lN \\ \frac{1}{N} \cdot N & n = lN \end{cases}$$

$$\Rightarrow x[n] = \begin{cases} 0, & n \neq lN \\ 1, & n = lN \end{cases}$$

$$\Rightarrow x[n] = \sum_{l=-\infty}^{\infty} \delta[n - lN] \quad *$$

$\Rightarrow$ impulse train in frequency $\longleftrightarrow$ impulse train in time

spacing $= \frac{2\pi}{N}$ spacing $= N$

---

5 July 2021

Recap:

⊛ Ex. impulse-train in frequency ⟷ impulse-train in time

spacing $\frac{2\pi}{N}$ — spacing N

* $X(e^{j\omega}) = \frac{2\pi}{N}\sum_{-\infty}^{\infty}\delta(\omega - \frac{2\pi}{N}k) \xleftrightarrow{DTFT} \sum_{l=-\infty}^{\infty}\delta[n-lN] = x[n]$ — ①

Dirac delta — Kronecker delta

⊛ Discrete Fourier Transform (DFT)

* finite length sequence $x[n]$, $n=0,1,\cdots N-1$ (N-length seq.)
  * samples of $X(e^{j\omega})$ at $\omega_k = \frac{2\pi}{N}k$, $k=0,1,\cdots N-1$

* N-point DFT : $X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$, $k = 0,1 \cdots N-1$ Analysis

* N-point IDFT : $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}$, $n = 0,1,\cdots N-1$ Synthesis

DFT as linear combination of signals

A DFT can be understood as a linear combination of complex sinusoids of the form $e^{j2\pi kn/N}$, with n as the time variable.

Time period for all these signals is N (not necessarily the fundamental period). Here, we only have a finite set of signals to linearly combine,as it is discrete.

Periodicity of DFT

$X[k+N] = X[k]$

$x[n+N] = x[n]$ This works if we want to extend the intial n length sequence periodically. Bascially when we sample the DTFT, we get a periodic discrete signal in frequency domain. This sampling is done by multiplying the DTFT by an impulse train.

So, we can use convolution property here, which says that this will be equivalent to convolution in time domain, with another impulse train (as DFT of impulse train is impulse train). This gives us our periodic time domain signal, which is basically copies of our original n point time domain signal.

We also NEED a minimum of n points in our DFT, anything less than that would cause aliasing in time domain, but anything more is good. More would give us spaces in between the copies. Shown below:

DFT as an orthogonal linear transformation

$X = F_N \cdot x$, where F_N is a matrix.

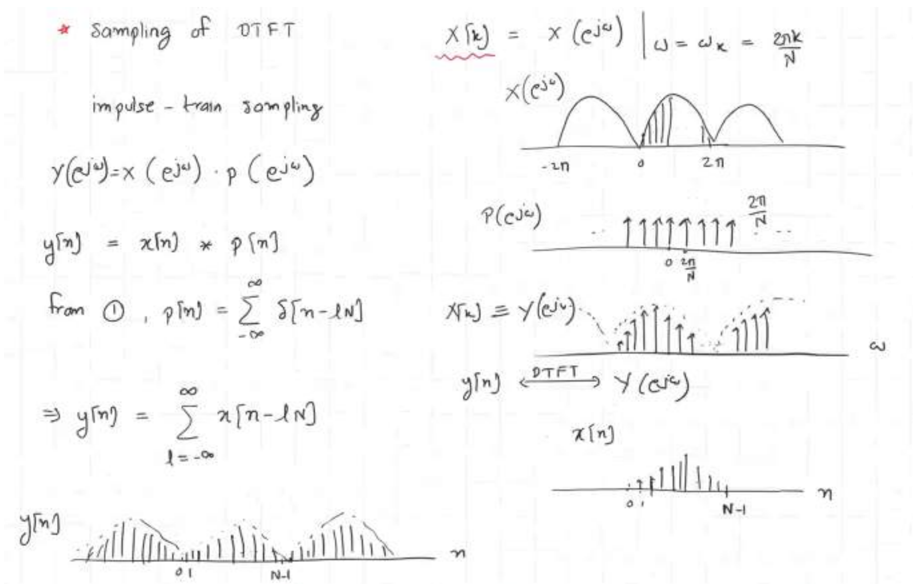


Figure 10: Periodicity of DFT

Let $W_N = e^{-j2\pi/N}$. With this, we can derive F_N.

Clearly, $(F_N)_{k,n} = W_N^{kn}$. Using this, if we fill in the matrix of F_4 for example, we get

The W_Ns used are called **twiddle factors**. As we can see from the image above, F_2 has a very simple structure, and we exploit this structure in the implementation of an efficient algorithm for calculating the DFT of a sequence called the Fast Fourier Transform, or the FFT.

The inverse DFT can be calculated as shown above in the image.

Ex :

Ex 2:

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7 July 2021

Recap of DFT till now:

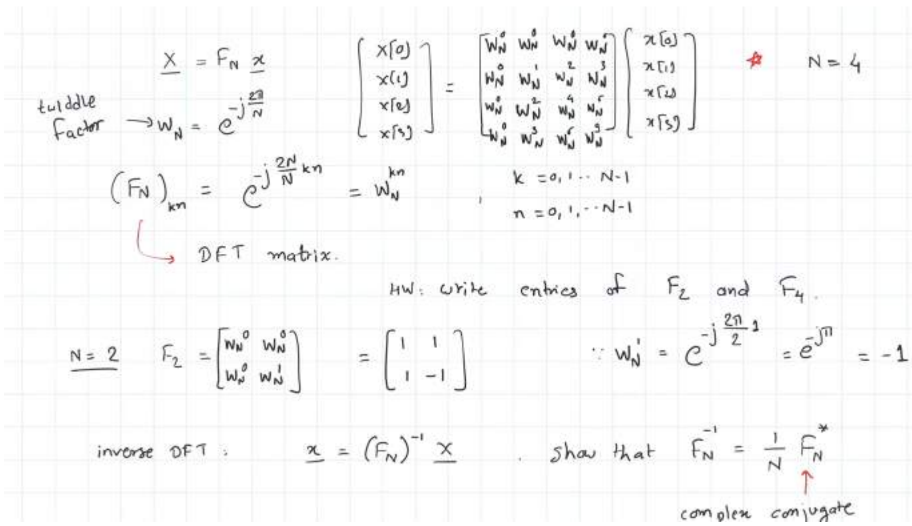

Figure 11: F_4 example

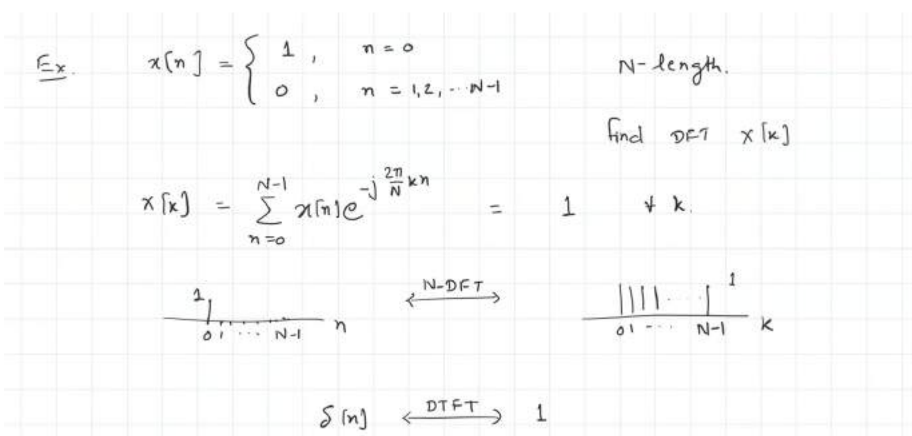

Figure 12: N point DFT example

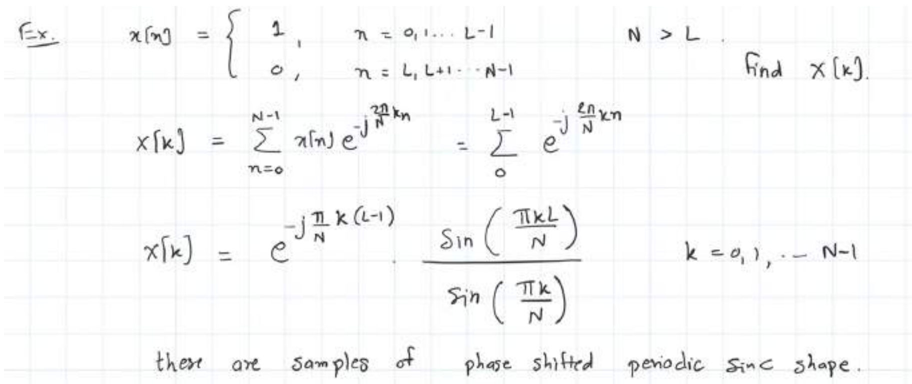

Figure 13: N point DFT of a rectangular pulse type discrete wave

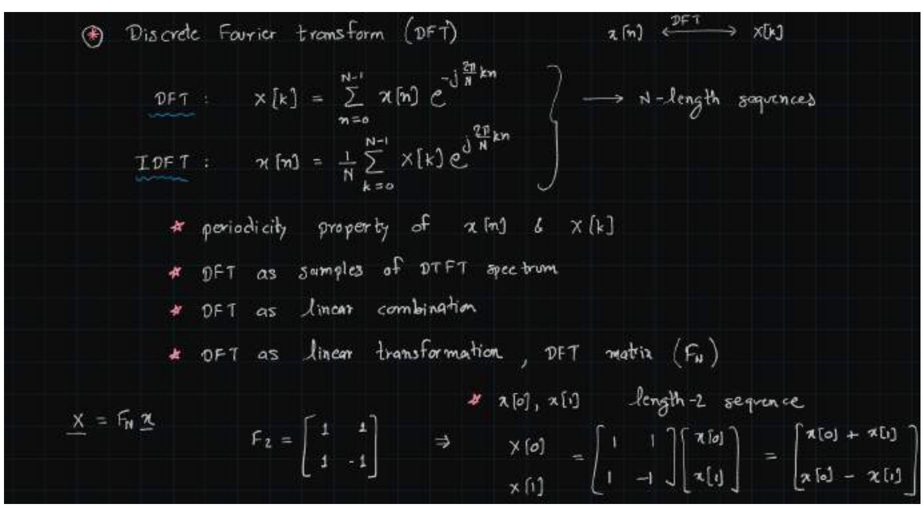

Figure 14: Recap

14 July 2021

- Direct DFT computation $\rightarrow O(N^2)$, multiplication and addition operations.
- Radix 2 FFT (Decimation in time algorithm)

  Assume $N = 2^m$

  By decimation in time, we can represent the DFT as a sum of two DFTs.

  $X[k] = G_1[k] + W_N^k \cdot G_2[k],\ \forall k = 0, 1, ..., N/2 - 1$ $X[k + N/2] = G_1[k] - W_N^k \cdot G_2[k],\ \forall k = 0, 1, ..., N/2 - 1$

  $G_1[k]$ and $G_2[k]$ are N/2 point DFTs. Diagrammatically, we get,

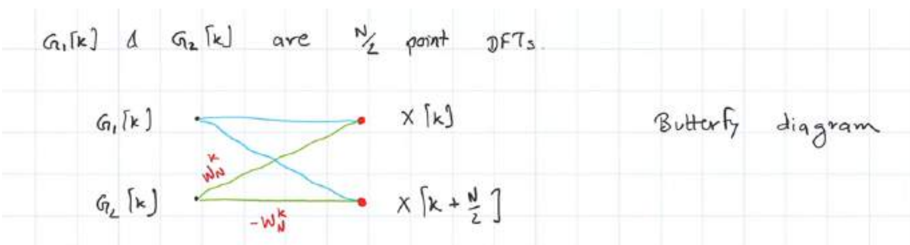

Figure 15: Structure of the above equations, Butterfly diagram

  We can recursively perform decimation in time to achieve a lower complexity. Each butterfly diagram has 1 complex multiplication and 2 complex additions. One N point can be split to 2 N/2 point, that can be further split into four N/4 point DFTs, and so on until we get 2 point DFTs. 2 points give one butterfly, so N points will have N/2 butterflies. So for an 8 point DFT, we will have 4 multiplications and 8 additions per stage. A stage here refers to a recursive step.

  So, in general, for a $N = 2^m$ point DFT, we have

  - Total multiplications $= N/2 + N/2 + ...$ (m times) $= N/2log_2(N)$
  - Total additions $= N + N + ...$ (m times) $= Nlog_2(N)$

  This is $O(Nlog_2(N))$ complexity, which is a very sizable decrease from $O(N^2)$.

  If we take N = 2^{10}, direct computation gives us approximately 2^{20} operations, while FFT gives us $2^9 \cdot 10$ computations, which is a huge improvement.

  Few remarks on the FFT

  - Radix need not only be 2, can be other numbers as well, usually powers of 2.

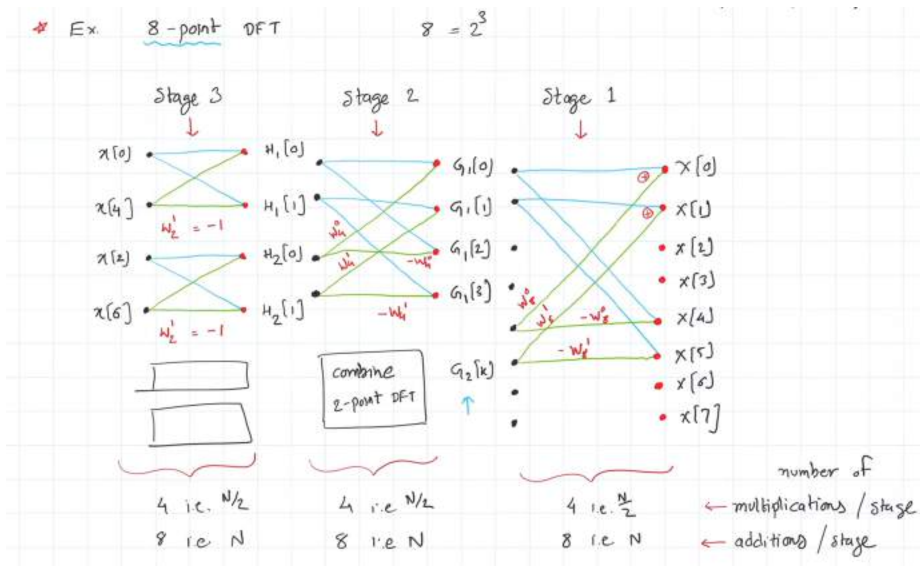

Figure 16: Example simplification of Radix 2 FFT for N = 8

- The Cooley Tukey algorithm uses a composite number (not a power of 2) N to implement a divide and conquer algorithm.
- Zero padding always helps us reach the required N.
- Inverse DFT also has a decimation in time implementation of a similar kind because it is very similar to DFT.
- DFTs are used so often that we get ICs specifically designed just for that purpose.
- Radix 2 FFT (decimation in frequency) is another method, but it does not give us any benefits in terms of computation when compared to decimation in time.

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16 July 2021

Z transform

Z transform is a generalisation of DTFT, like Laplace transform is a generalisation of Fourier Transform.

$e^{-j\omega t}$ $(FT) \rightarrow e^{-\delta t}$ $(\delta = \sigma + j\omega)$ $(Laplace\ Transform)$

$e^{-j\omega n}$ $(DTFT) \rightarrow z^{-n}$ $(z = re^{j\omega})$ $(Z - Transform)$

The Z transform of a signal $x[n]$ is a complex valued function. The expression is

$$X(z) = \sum_{-\infty}^{\infty} x[n] \cdot z^{-n}$$

As we can clearly see, the DTFT is a special case of the Z transform, as it is the same as what we get when we subsitute $r = 1$ for z in the above equation.

On the Z plane, this special case is represented by the unit circle.

Convergence of Z Transform Region of convergence (ROC) $\rightarrow$ region in z plane where $X(z)$ converges.

If unit circle does not exist in the ROC, then the DTFT for that signal does not exist.

Examples:

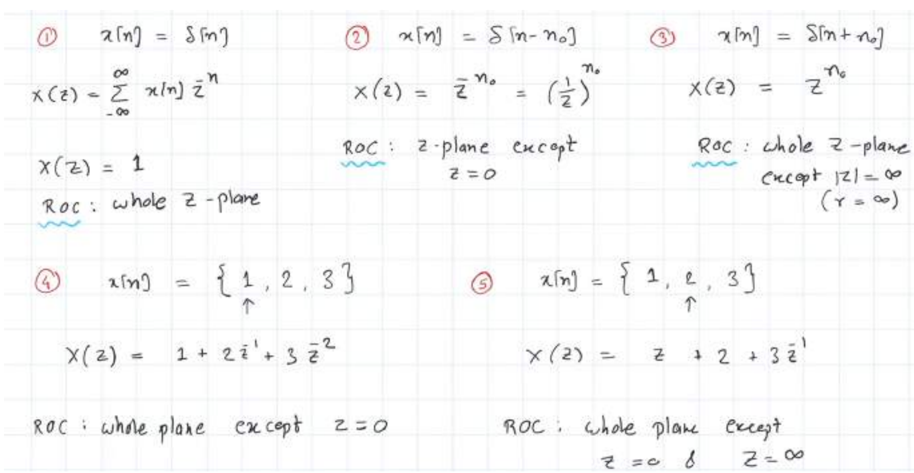

Figure 17: Examples of solved z transforms

We are mainly interested in Z transforms of the form

$$X(z) = \frac{N(z)}{D(z)}$$

i.e., ratio of polynomials in z. Here, when $N(z) = 0$, we have **zeroes** of $X(z)$, and when $D(z) = 0$, we have **poles** of $X(z)$. Poles are points where $X(z)$ is not finite, while zeroes are quite obviously points where $X(z)$ is zero.

Without specifying an ROC the expression for a Z transform is incomplete. As seen in a few examples below, the same $X(z)$ could correspond to different ROCs and hence, different signals. Examples:

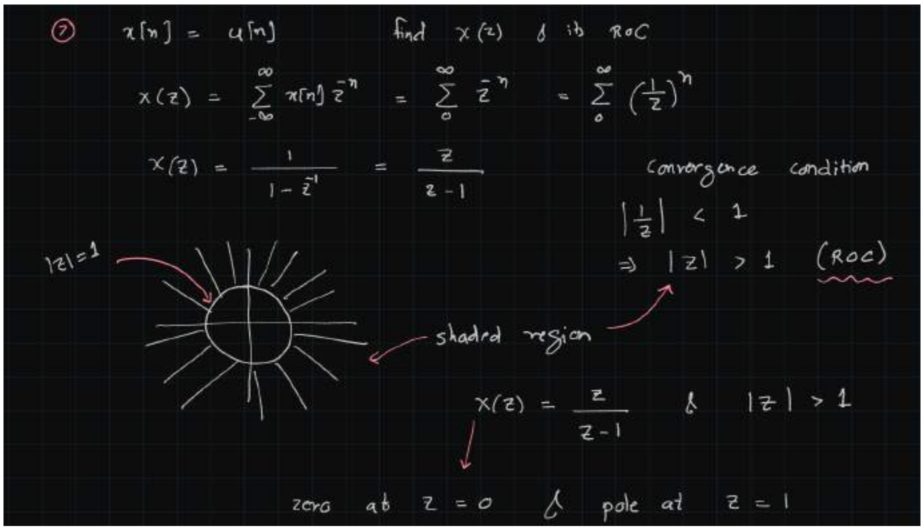

Figure 18: Example 1

As seen in the above two examples, the expression we get for $X(z)$ is the same, but the ROC is different, which leads to different signals. We must also note that the ROC always has circular symmetry. Can be of three types,

- $|z| > r_0$
- $|z| < r_0$
- $r_1 < |z| < r_2$

Generalised example would be

The first example here can be called a causal signal ($x[n] = 0 \; for \; n < 0$), and the second would be called an anti causal signal.

The above example can hence, correspond to 3 signals, depending on which ROC you take.

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19 July 2021

Example of a two-sided signal (If a signal is not causal or anti causal, it's this)

Another example q:

Table with signals and ROCs

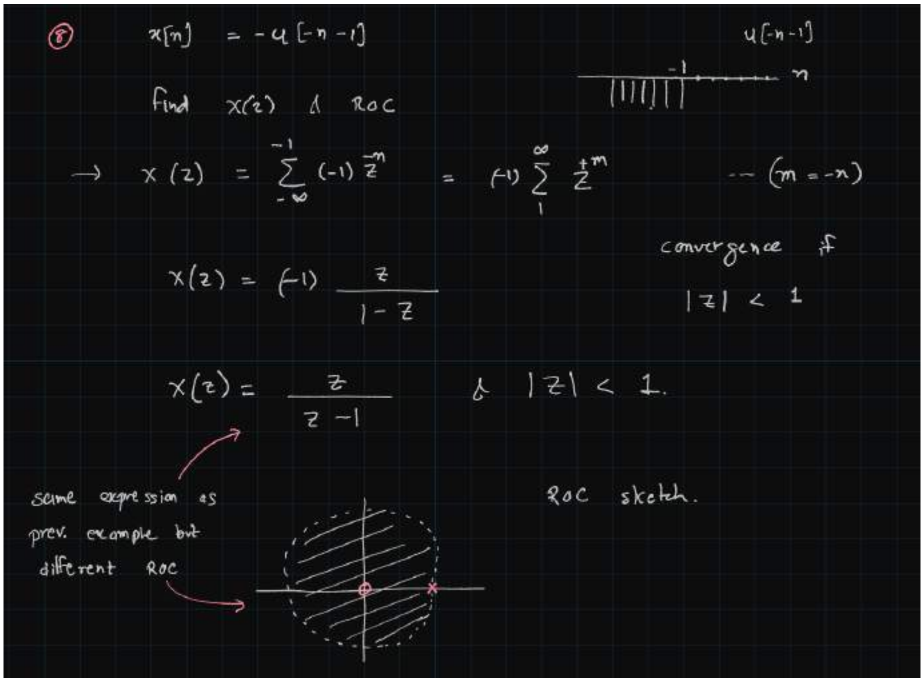

Figure 19: Example 2

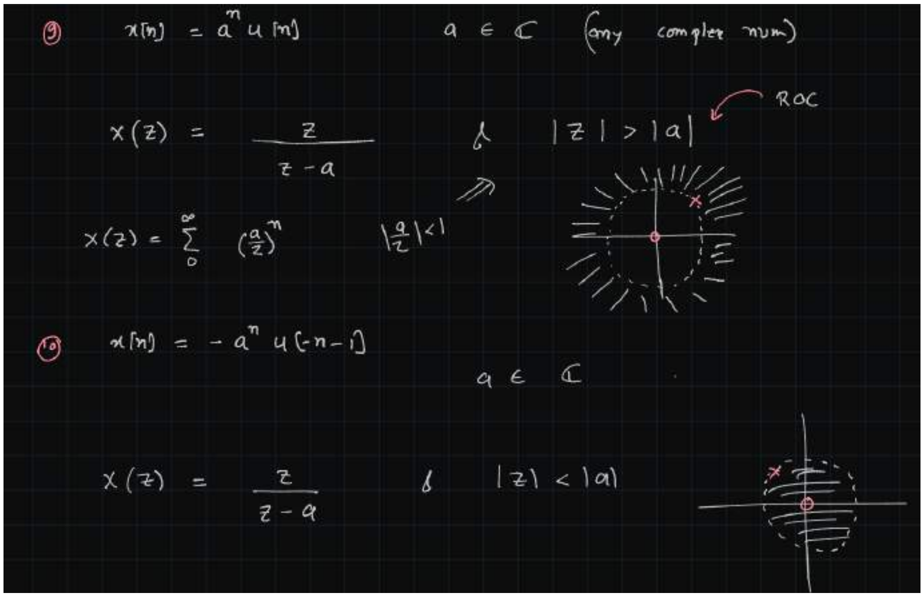

Figure 20: General examples of example 1 and 2

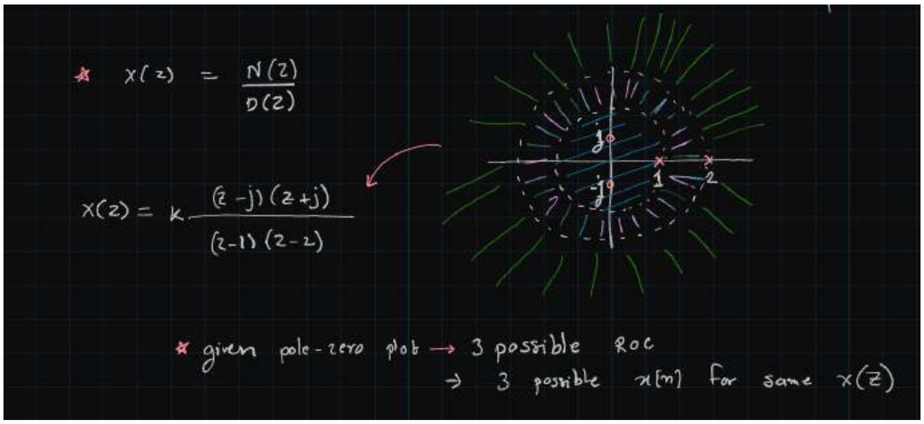

Figure 21: One transform with 3 possible ROCs

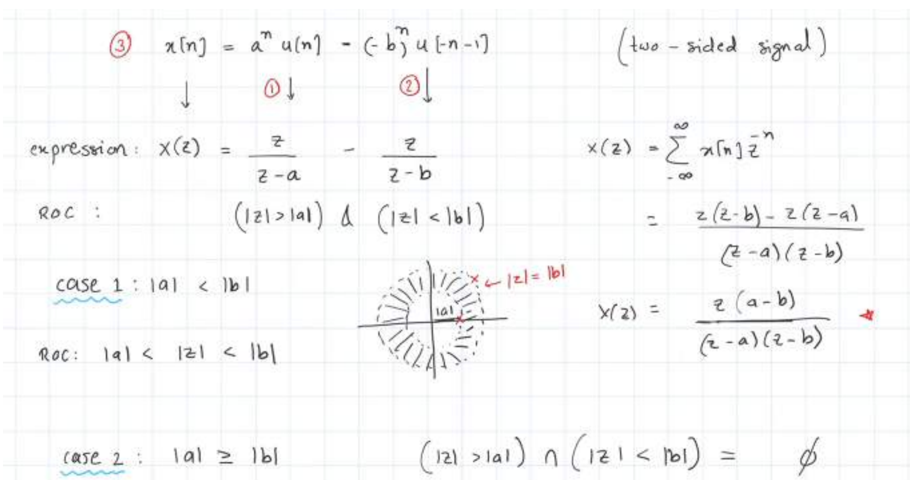

Figure 22: A two sided signal example

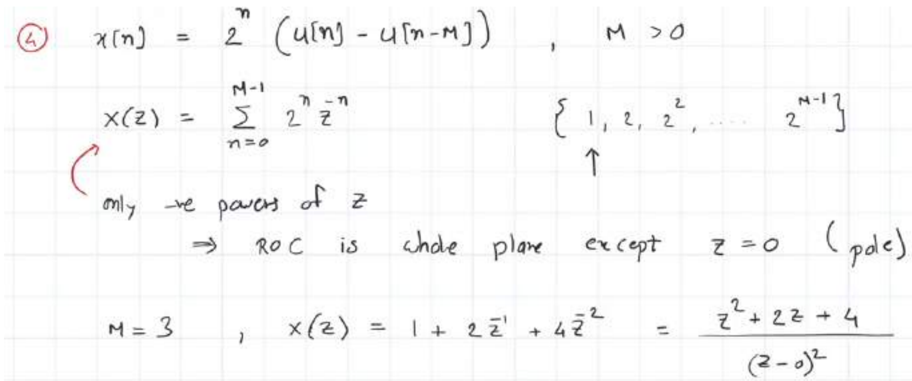

Figure 23: Example for finite length sequence Z transform

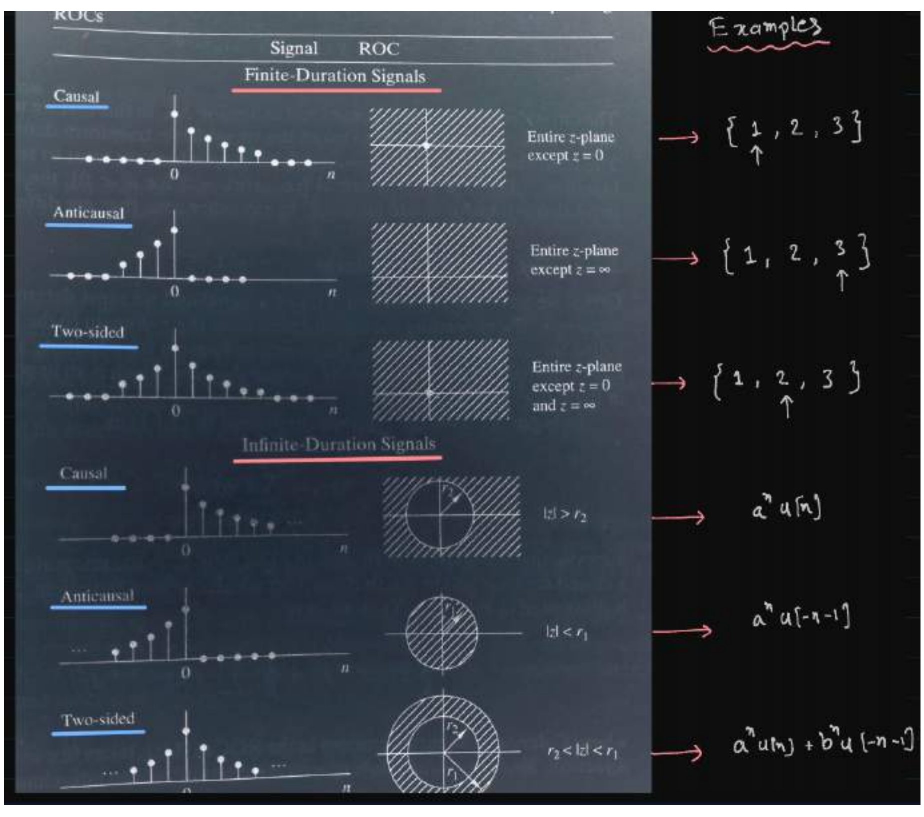

Figure 24: Nice table

Did more examples, a little confused about why poles at infinity are weird.

Poles for a **real signal** always exist in conjugate pairs. Zeroes exist as complex conjugate pairs.

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Ex:

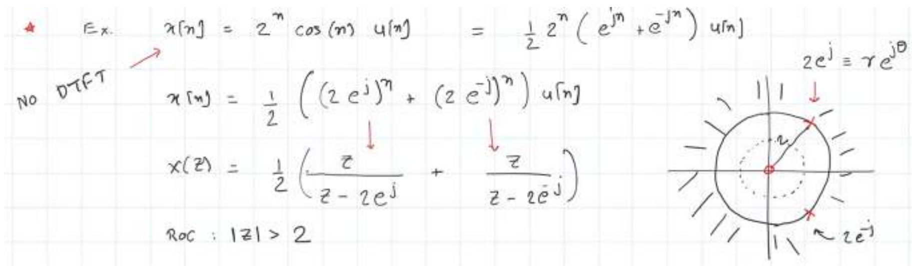

Figure 25: Another example

The above example has no DTFT as the unit circle in not part of its ROC.

Inverse Z transform

This refers to finding $x[n]$ given $X(z)$ and the ROC.

The general method for finding the inverse z transform of a given function is by using complex integrals, closed contour integration in the ROC using Cauchy's integral theorem (RA stuff). Won't be doing all that here.

We are mostly interested in only a few cases, so we'll learn those.

- Case $\rightarrow$ Finite polynomial terms (in z and z^{-1})

  Ex:

  $X(z) = 6z^{-2} + z^{-1} + z + 4z^3$

  ROC $\rightarrow$ C This is already in the form of a summation of coefficents with powers of z. So we get the sequence $x[n] = \{4, 0, 1, 0, 1, 6\}$ where we take the zero power coefficient as the zero indexed term.

- Case $\rightarrow$ when X(z) is a ratio of polynomials

  $X(z) = N(z)/D(z)$

  ROC $\rightarrow |z| > 1$

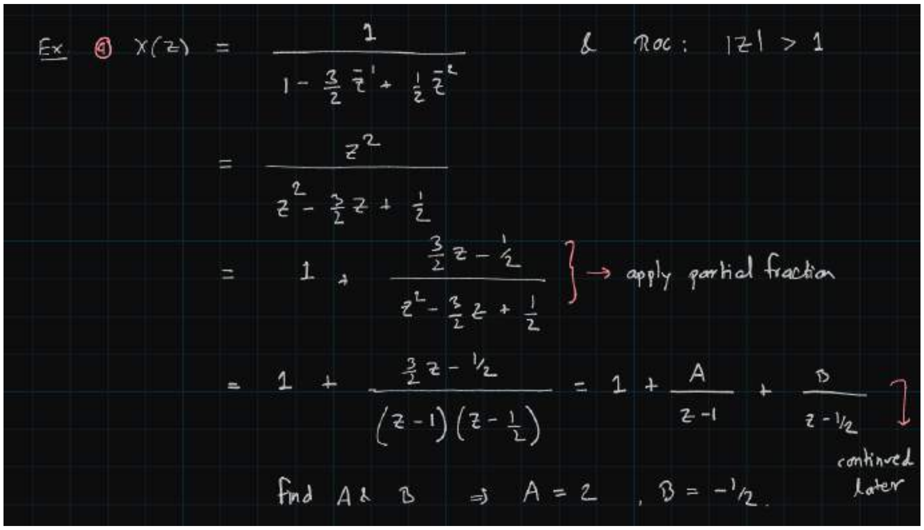

Figure 26: Example of using partial fractions (part 1)

Here, we will use partial fractions.

The z transform of this format has not been done yet, so this will be continued a little later in the lecture.

Properties of Z transform

- Linearity

  $x_1[n] \longleftrightarrow X_1(z)$, ROC : R_1
  $x_2[n] \longleftrightarrow X_2(z)$, ROC : R_2
  $\alpha x_1[n] + \beta x_2[n] \longleftrightarrow \alpha X_1(z) + \beta X_2(z)$, ROC contains $R_1 \cap R_2$, may be bigger.

- Time-shift

  $x[n] \longleftrightarrow X(z)$, ROC : R
  $x[n - n_0] \longleftrightarrow z^{-n_0} \cdot X(z)$, ROC : $R \pm \{0, \infty\}$, as it can change the poles and zeroes at these places.

  Ex:

  $x[n] = a^n \cdot u[n]$, find Z transform of $x[n-1]$ and $x[n+1]$.

  $a^n \cdot u[n] \longleftrightarrow z/(z-a)$, $|z| > a$

  $\implies a^{n-1} \cdot u[n-1] \longleftrightarrow z^{-1} \cdot z/(z-a) = 1/(z-a)$, $|z| > a$

  $\implies a^{n+1} \cdot u[n+1] \longleftrightarrow z \cdot z/(z-a) = z^2/(z-a)$, $(|z| > a) - \{\infty\}$ as infinity becomes a pole in this case.

Continuing that partial fraction example from before

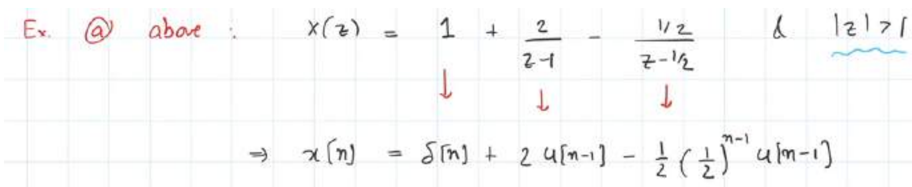

Figure 27: Example of using partial fractions (part 2)

- Time reversal

  $x[n] \longleftrightarrow X(z)$, ROC : R
  $x[-n] \longleftrightarrow X(1/z),\ i.e.\ X(z^{-1})$

- Convolution

  $x_1 \longleftrightarrow X_1$, ROC : R_1
  $x_2 \longleftrightarrow X_2$, ROC : R_2

  $x_1[n] * x_2[n] \longleftrightarrow X_1(z) \cdot X_2(z)$, ROC contains $R_1 \cap R_2$, may be bigger.

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Z Transform for LTI system / Filter

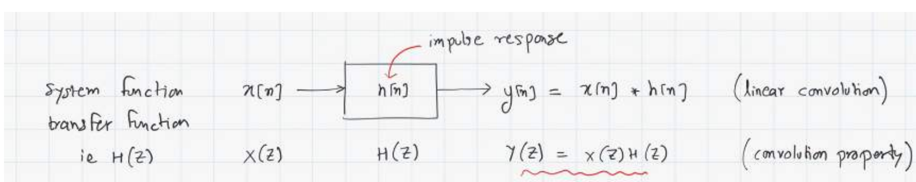

Figure 28: Schematic of an LTI system

$H(z)$ is called the system function, or the transfer function.

Input output pairs for this system

- $\delta(n) \to h[n]$
- $e^{j\omega n} \to H(e^{j\omega}) \cdot e^{j\omega n}$
- $z^n \to H(z) \cdot z^n$

z^n are EIGENFUNCTIONS of the LTI system, as they are general complex exponentials.

Equivalent representations of an LTI system

$$H(e^{j\omega})\ (DTFT) \longleftrightarrow h[n] \longleftrightarrow H(z)\ (+\ ROC)$$

Ex:

$h[n] = a^n \cdot u[n] \longleftrightarrow H(e^{j\omega}) = 1/(1-ae^{-j\omega}) \longleftrightarrow H(z) = z/(z-a) = 1/(1-az^{-1})$

As noticed before, DTFT may not exist, but z transform may exist (ROC does not contain unit circle).

System analysis using Z transform

- When is a system causal?
  - The system output should not depend on future inputs.
    $y[n] = \sum_{m=-\infty}^{\infty} h[m] \cdot x[n-m]$
    Here, in this convolution sum, $x[n-m]$ for m < 0 should not appear. So $h[m] = 0 \forall m < 0$.
  - In terms of the z transform, we say that a system is causal if
    $H(z) = \sum_{m=0}^{\infty} h[n] \cdot z^{-n}$
    and ROC is of the form $|z| > r$. ROC is basically outside the outermost pole in z plane. This directly implies that it cannot have a pole at infinity.
- When is a system stable?
  - Bounded input gives us a bounded output (BIBO stability). The condition on h[n] is
    $\sum_{-\infty}^{\infty} |h[n]| < \infty$
    So, we get
    We can infer that unit circle is part of the ROC of h[n]. So this is our condition for stability.

Examples

- When is a system causal and stable?
  - When it is causal, and stable =P.
  - Combining the above two conditions, we get the condition "A causal LTI system is stable iff all the poles are inside the unit circle."

LTI systems are characterised by **linear constant coefficient difference equations**.

Examples of linear constant coefficient difference equations:

$$H(z) = \sum_{-\infty}^{\infty} h[n]\, z^{-n}$$

$$|H(z)| = \left| \sum_{-\infty}^{\infty} h[n]\, z^{-n} \right|$$

$$|H(z)| \leq \sum_{-\infty}^{\infty} \left| h[n]\, z^{-n} \right|$$

$$= \sum_{-\infty}^{\infty} |h[n]|\, |z|^{-n}$$

let $|z| = 1$ i.e. unit circle

$$\Rightarrow |H(z)| \leq \sum_{-\infty}^{\infty} |h[n]| \quad \text{for } |z| = 1$$

if system is BIBO stable $\Rightarrow |H(z)| < \infty$ on $|z| = 1$

Bounded input bounded output

condition on $h[n]$

$$\sum_{-\infty}^{\infty} |h[n]| < \infty$$

$x[n]$ bounded $\Rightarrow y[n]$ bounded

$|x[n]| < C \quad \forall n$

C is finite.

Figure 29: BIBO stability

Ex.	$h[n]$	$H(z)$	ROC	Causal	Stable
	$\delta[n-1]$	z^{-1}	$\lvert z\rvert > 0$	Y	Y
	$\delta[n+1]$	z	$\lvert z\rvert < \infty$	N	Y
	$2^n u[n]$	$\frac{z}{z-2}$	$\lvert z\rvert > 2$	Y	N
	$(\frac{1}{2})^n u[n]$	$\frac{z}{z-\frac{1}{2}}$	$\lvert z\rvert > \frac{1}{2}$	Y	Y

Figure 30: Examples for checking causality and stability

- $y[n] = x[n] - x[n-1]$ For $y[n]$ to be this with $x[n]$ as input to an LTI system, we can deduce that the impulse response is

  $h[n] = \delta[n] - \delta[n-1]$

  $H(e^{j\omega}) = 1 - e^{-j\omega}$

  $H(z) = 1 - z^{-1},\ |z| > 0$

- $y[n] = x[n] + x[n-1]/3 + y[n-1]/2$

  This is a little more complex than the previous one as it has something like a feedback. Here, finding $h[n]$ is not easy, but H(z) is.

  Taking z transform of y,

  $Y(z) = X(z) + X(z) \cdot z^{-1}/3 + Y(z) \cdot z^{-1}/2$

  So,

  $Y(z) \cdot [1 - z^{-1}/2] = X(z)[1 + z^{-1}/3]$

  Which gives us,

  $Y(z)/X(z) = H(z) = [1 + z^{-1}/3]/[1 - z^{-1}/2]$

  $H(z) = (z + 1/3)/(z - 1/2)$

  This gives us 2 possible systems which can be found using the inverse Z transform.

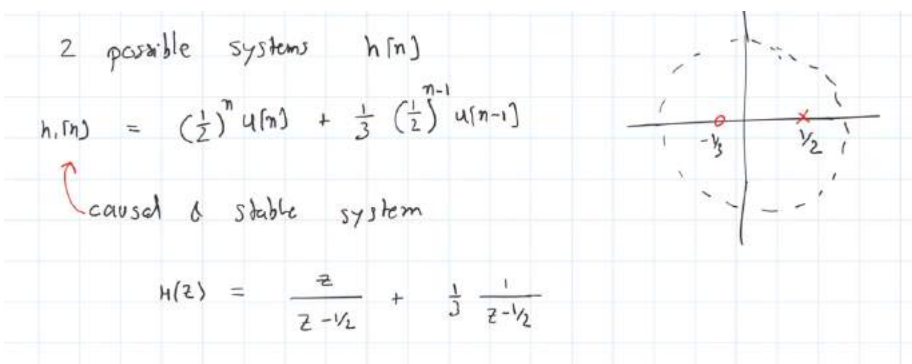

Figure 31: Inverse FT computation

Both above examples can be easily implemented. The first one is called FIR filter(Finite impulse response filter) while the second is called IIR filter (infinite impulse response).

In general:

$$\sum_{k=0}^{N} a_k \cdot y[n-k] = \sum_{l=0}^{M} b_l \cdot x[n-l]$$

Taking z transform

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{l=0}^{M} b_l \cdot z^{-l}}{\sum_{k=0}^{N} a_k \cdot z^{-k}}$$

Difference equations are easy to implement in a computer.

Inverse system

A system that nullifies the effect of another system. If $H_1(z)$ and $H_2(z)$ are the responses of the system and its inverse, then

$$H_1(z) \cdot H_2(z) = 1$$

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26 July 2021

Digital Filter Design

Equivalent system representations

- $h[n]$
- $H(e^{j\omega})$
- $H(z)$ + ROC

Remarks on filter design We design frequency selective filters. This means we specify what the filter does on the basis of the frequencies it allows and cancels. Ex: Ideal LPF, etc. It basically uses convolution sum to do this, and a small error in a process can carry over, causing a large error in the end.

We will focus more on causal and stable filters. If the filter is unstable, for a bounded input, we could get an arbitrarily large output. We prefer avoiding this.

Ideal filters like the ideal LPF for example, are not practical because their impulse response like in the case of sinc is infinite in extent and is also not causal.

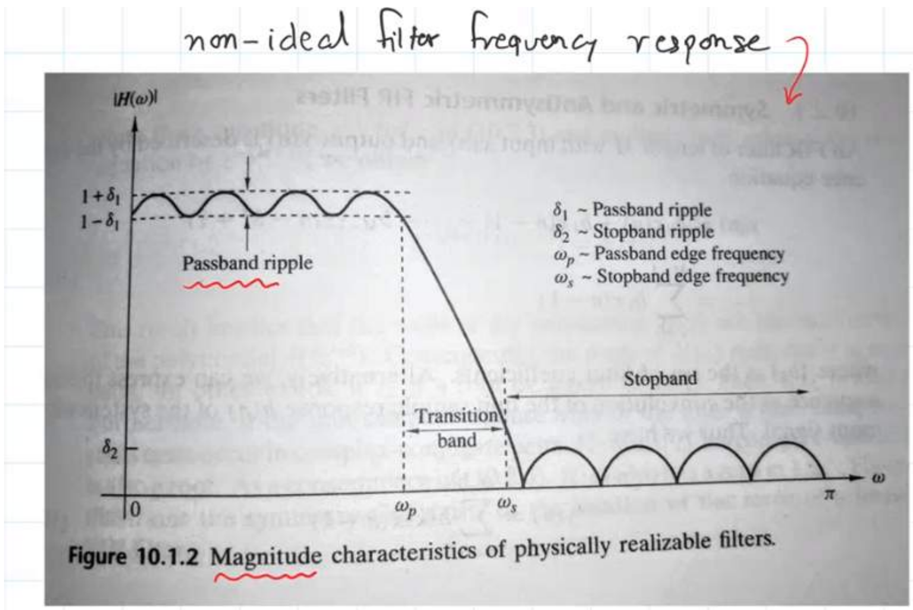

Figure 32: Frequency response of a practically realisable filter

We cannot have a perfectly flat response with a non ideal filter as seen in the passband above, and we cannot have a sharp cutoff frequency limit either, it has to span a band as shown in the figure. The figure shows a general filter, and these non idealities have to be kept in mind.

Software and tools for filter design Specificatons are given in terms of:

- Pass-band, stop-band
- Ripples in the pass-band and stop-band
- Filter order, filter type, etc.

The software then designs a filter accordingly. Nice. We don't have to specify all, we only need to give a few specifications, and it will decide the rest.

FIR vs IIR filters We will mostly discuss cases where $H(z)$ is of rational form.

$$\frac{N(z)}{D(z)} = H(z) = \frac{\displaystyle\sum_{l=0}^{M} b_l \cdot z^{-l}}{\displaystyle\sum_{k=0}^{N} a_k \cdot z^{-k}}$$

i.e.,

$$?\sum_{k=1}^{N} a_k y[n-k] = \sum_{l=0}^{M} b_l x[n-l]$$

- FIR filters

  In an FIR filter, $a_k = 0$. This implies $h[n] = b_n$. FIR filters are used mainly when linear phase is required. This definition of $a_k = 0$ restricts the signal to only look at past inputs, hence enforcing the system to be causal.

  A filter with linear phase means the output signal is just a delayed version of the input signal. If the phase is not linear, then the signal may be distorted.

  Ex:

  With linear phase (FIR), (input $\rightarrow$ output format below)

  $sin(\omega_1 n) + sin(\omega_2 n) + sin(\omega_3 n) \rightarrow sin(\omega_1(n-n_0)) + sin(\omega_2(n-n_0))$

  Assuming, ω_3 is nullified by the filter.

  With non linear phase, we get something like

  $sin(\omega_1 n) + sin(\omega_2 n) + sin(\omega_3 n) \rightarrow sin(\omega_1(n-n_1)) + sin(\omega_2(n-n_2))$

  As the components are shifted by different ammounts, the signal is distorted.

- IIR filters

  At least one of the a_ks is non zero. Implementation using difference equation still has finite computations, which is nice.

  They cannot have linear phase. For a given filter specification, the benefit that IIR filters have is that they need a lesser number of coefficients than an FIR filter. This helps us reduce the number of computations required I guess. So unless linear pase is an absolute requirement, we always use IIR filters, as they are faster and require less computations.

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- FIR filters : Symmetric and anti symmetric filters for linear phase.
- Method of windows and frequency sampling used in FIR filter design.

IIR Filter design (Causal and stable)

- No linear phase here.
- Benefit over FIR $\rightarrow$ fewer parameters for a similar application.
- System function

$$H(z) = \frac{\displaystyle\sum_{k=0}^{M} b_k \cdot z^{-k}}{1 + \displaystyle\sum_{l=1}^{N} a_l \cdot z^{-l}}$$

- Common practice $\rightarrow$ deisgn digital IIR filters starting from analog IIR filters. Many such well studied analog filters already exist. This is because they were studied a lot previously. Ex: Butterworth filters, Chebyshev filters, Elliptic filters, etc.

An analog filter is a continuous time LTI system. Assuming that the system has an impulse response $h_a(t)$, we have

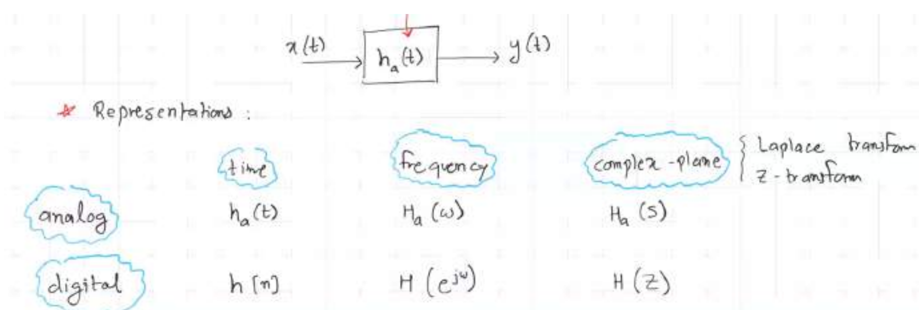

Figure 33: Representations of the system

IIR Filter design by impulse invariance Sample the analog IIR impulse response to get the discrete time domain response.

Time domain: $h[n] = h_a(nT_s)$

Frequency domain: $H(e^{j\omega} = 1/T_s \sum_{-\infty}^{\infty} H_a((\omega - 2\pi k)/T_s)$

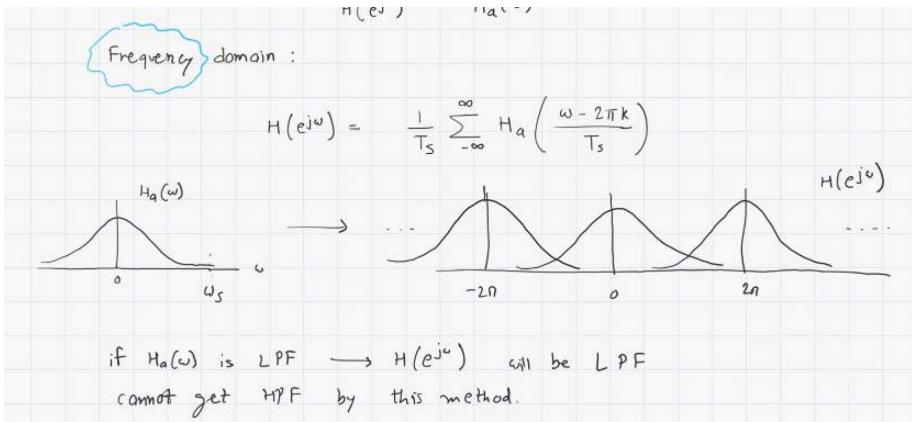

Figure 34: Sampling effects

We cannot get HPF with this method as it would cause a VERY significant amount of aliasing when sampling (Not even close to bandlimited). LPF can be done this way if the aliasing due to sampling is found to be negligible.

Complex domain : Assume $H_a(s)$ has a partial fraction expansion. So it will be of the form

$$H_a(s) = \sum k = 1^N c_k/(s - P_k)$$

, distinct poles at P_k.

$$\frac{1}{s-a} \longleftrightarrow e^{at}u(t)$$

In analog systems, all poles must lie in the left half plane of the s plane for a system to be causal and stable. On sampling

$$h[n] = h_a(nT_s) = \sum_{k=1}^{N} c_k e^{P_k n T_s} u(nT_s)$$

This implies that

$$H(z) = \sum_{k=1}^{N} c_k z/(z - e^{P_k T_s}, \quad |z| > |e^{P_k T_s}|$$

Analog pole at $P_k \to$ digital pole at $e^{P_k T_s}$. Now, if analog system is causal and stable, i.e. $real(P_k) < 0$, ten $|e^{P_k T_s}| < 1$. This implies that the digital system is ALSO stable and causal.

So now we have the poles and zeroes of the filter, which will let us construct a filter with finite computations.

IIR filter design by Bilinear transofrmation This method transforms $H_a(s) \longrightarrow H(z)$ directly.

Using $s = \frac{2}{T}(1 - z^{-1})/(1 + z^{-1})$. T is a parameter that helps us adjust things.

This maps left half plane of s plane inside the unit circle in z plane and maps the $j\omega$ axis of s plane to the unit circle only once (only one rotation = complete imaginary axis). This fixes the drawback of aliasing that we faced in the previous method.

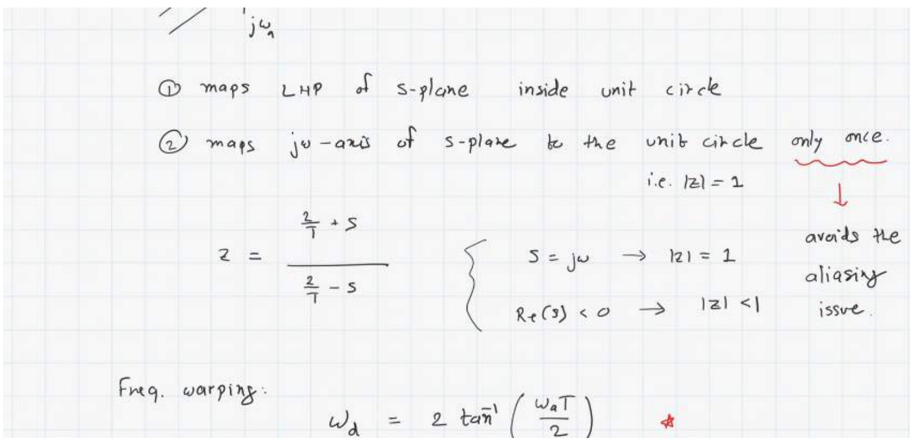

Figure 35: Weird stuff, but this maps ω_d from $-\pi$ to π for all the values of w_a, which is the whole imaginary axis

And that's about it for the course =P Take all courses that Santosh sir is responsible for whenever possible :gtodiayuh:
