# Tutorial 2 - Problem 2

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Problem -2

1(a). $H(Y,Z|X) \le H(Y|X) + H(Z|X)$

with equality iff $\underbrace{p(y_j, z_k|x_i) = p(y_j|x_i)\,p(z_k|x_i)}_{\text{Conditional independence}}$ $\forall i,j,k$

$$H(Y,Z|X) = H(Y,Z,X) - H(X)$$

$$H(Y|X) = H(Y,X) - H(X)$$

$$H(Z|X) = H(Z,X) - H(X)$$

**Toolbox of Inequalities (Bounds)**

$$0 \le H(X) \le \log_2 |R_X|$$

$$0 \le H(X|Y) \le H(X)$$

$$D(p(x)\|p(y)) \ge 0$$

Consider

$$H(Y|X) + H(Z|X) - H(Y,Z|X)$$

$$= H(Y,X) - H(X) + H(Z,X) - H(\cancel{X}) - H(Y,Z,X) + H(\cancel{X})$$

$$= H(Z,X) - H(X) - \left(H(Y,Z,X) - H(Y,X)\right)$$

$$= H(Z|X) - H(Z|Y,X)$$ part (c)

part (b)

$$H(A|B) = \sum_b p(b)\,H(A|B=b)$$

$$= \sum_b p(b) \sum_a p(a|b) \log \frac{1}{p(a|b)}$$

$$= \sum_{a,b} p(a,b) \log \frac{1}{p(a|b)}$$

Similarly,

$$H(A|B,C) = \sum_{b,c} p(b,c)\,H(A|B=b,C=c)$$

$$= \sum_{a,b,c} p(a,b,c) \log \frac{1}{p(a|b,c)}$$

$$= \sum_{z,x} p(z,x) \log \frac{1}{p(z|x)} - \sum_{z,x,y} p(z,x,y) \log \frac{1}{p(z|x,y)}$$

$$= \sum_{z,x,y} p(z,x,y) \log \frac{1}{p(z|x)} - \sum_{z,x,y} p(z,x,y) \log \frac{1}{p(z|x,y)}$$

$$= \sum_{z,x,y} p(z,x,y) \log \left(\frac{p(z|x,y)}{p(z|x)}\right)$$

$$= \sum_{z,x,y} p(z,x,y) \log \left(\frac{p(z,y|x)\cdot p(x)}{p(x,y)\cdot p(z|x)}\right)$$

$$= \sum_{z,x,y} p(z,x,y) \log \left(\frac{p(z,y|x)}{p(y|x)\cdot p(z|x)}\right)$$

$q_1(x,y,z)$, $q_2(x,y,z)$ Are valid dist on $X, Y, Z$

$$= D\left(q_1(x,y,z) \,\|\, q_2(x,y,z)\right) \ge 0$$

Therefore,

$$H(Y|X) + H(Z|X) \ge H(Y,Z|X)$$

For equality,

$$\boxed{p(y,z|x) = p(z|x)\cdot p(y|x)}$$

$D(p(x)\|q(x)) = 0$ iff. $p(x) = q(x)$

$\forall\, y \in R_Y,\ z \in R_Z,\ x \in R_X$

part b. & c.

covered within the solution of part a.

Generalisation:

Show that

$$H(A_1, A_2, \dots A_n|X) \le H(A_1|X) + H(A_2|X) + \dots + H(A_n|X)$$

Try proof by induction

$H(A_n, X)$

$$H(A_1|X) + \dots + H(A_n|X) - H(A_1, A_2, \dots A_n) + H(X)$$

$$= H(A_1|X) + H(A_2|X) + \dots H(A_{n-1}|X) - H(A_1, A_2, \dots A_{n-1}|A_n, X)$$

$$\ge H(\underbrace{A_1, A_2, \dots A_{n-1}}_{Z}|X) - H(\underbrace{A_1, A_2, \dots A_{n-1}}_{Z}|\underbrace{A_n}_{Y}, X) \ge 0$$

Assuming holds for n-1

Base case