Camlin Page
Date / /

Assignment - 1 (Signal Processing)

I affirm that I have neither given nor received help or used any means which would make this assignment unfair.

– L. Lakshmanan
(2020 11 2026)

3.34

$$h(t) = e^{-4|t|}$$

$$\therefore H(\omega) = \int_{-\infty}^{\infty} e^{-4|t|} e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-4t} e^{-j\omega t}\, dt + \int_{-\infty}^{0} e^{4t} e^{-j\omega t}\, dt$$

$$= \left[\frac{e^{(-4-j\omega)t}}{-4-j\omega}\right]_0^{\infty} + \left[\frac{e^{(4-j\omega)t}}{4-j\omega}\right]_{-\infty}^{0}$$

$$= \left[0 + \frac{1}{4+j\omega}\right] + \left[\frac{1}{4-j\omega} + 0\right]$$

$$= \underline{\underline{\frac{1}{4+j\omega} + \frac{1}{4-j\omega}}} \longrightarrow \text{Fourier transform of } h(t) \text{ for convolution case.}$$

(a) $x(t) = \sum_{n=-\infty}^{\infty} \delta(t-n)$

This is an impulse train with $T = 1$. $\therefore \omega_0 = 2\pi$. Now, to find Fourier series coefficients, Taking $-\frac{1}{2}$ to $\frac{1}{2}$

$$a_k = \frac{1}{T}\int_{-1/2}^{1/2} \delta(t)\, e^{-jk\omega_0 t} = \underline{\underline{1}} \quad (\because e^0 \; (\because \text{Property of } \delta(t))$$

$\therefore$ All $a_k = 1$.

So, now to get Fourier coefficients of $y(t)$, we can just multiply the Fourier Transform of $h(t)$ with the respective a_k. This is because of convolution property.

$$\therefore b_k = H(k\omega_0) \cdot a_k$$

$$= \frac{1}{4 + j\omega_0 k} + \frac{1}{4 - j\omega_0 k} = \frac{1}{4 + 2\pi j k} + \frac{1}{4 - 2\pi j k}$$

(b) $x(t) = \sum_{n=-\infty}^{\infty} (-1)^n \delta(t-n)$

This is an alternating pulse train with $T = 2$. $\therefore \omega_0 = \frac{2\pi}{2} = \pi$.

$$\therefore a_k = \frac{1}{T}\int_{-1/2}^{3/2} \left(\delta(t) - \delta(t-1)\right) e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-1/2}^{3/2} e^{-jk\omega_0 t}\, \delta(t) - \int_{-1/2}^{3/2} \delta(t-1)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{2}\left(1 - e^{-jk\omega_0}\right) \qquad (\because \text{Property of delta function})$$

$$= \frac{1 - e^{-jk\pi}}{2}$$

Now, since $a_k = \frac{1 - e^{-jk\pi}}{2}$, we can take 2 cases.

k = odd number

$$\Rightarrow \frac{1 - e^{-jk\pi}}{2} = \frac{1 - \left(\cos(k\pi) - j\overbrace{\sin(k\pi)}^{0}\right)}{2}$$

$$= \frac{1 - (-1)}{2} = 1 \qquad (\because k \text{ is odd})$$

k is even.

$$\Rightarrow \frac{1 - e^{-jk\pi}}{2} = \frac{1 - \left(\cos(k\pi) - j\overbrace{\sin k\pi}^{0}\right)}{2} = \frac{1-1}{2} = 0$$

Camlin Page
Date / /

$$\therefore a_k = \begin{cases} 1 & ; \ k \text{ is odd.} \\ 0 & ; \ k \text{ is even.} \end{cases}$$

Now, to find Fourier series coefficients of $y(t)$, we can use convolution property like in (a).

$$b_k = a_k \cdot H(k\omega_0)$$

$$= \frac{1}{4+jk\omega_0} + \frac{1}{4-jk\omega_0} \longrightarrow a_k = 1 \ (k \text{ is odd})$$

(or)

$$= 0 \longrightarrow a_k = 0 \ (k \text{ is even}).$$

$$\therefore b_k = \begin{cases} \frac{1}{4+jk\pi} + \frac{1}{4-jk\pi} & ; \ k \text{ is odd} \\ 0 & ; \ k \text{ is even} \end{cases}$$

3.44

→ $x(t)$ is a real signal.
From this, we can deduce that,

$$a_k = a^*_{-k}$$

But, as $x(t) = x^*(t)$ ($\because x(t)$ is real).

$$a_{-k} = a^*_{k}$$

$$\Rightarrow \underline{\underline{|a_k| = |a_{-k}|}}$$

As a_1 is a positive real number, we can say $a_1 = a_{-1}$.

→ $T = 6 \Rightarrow \omega_0 = \frac{2\pi}{6} = \underline{\underline{\frac{\pi}{3}}}$

→ $a_k = 0$ for $k = 0$ and $k > 2$.
This means only a_1, a_{-1}, a_2, a_{-2} are non zero.

Camlin Page
Date / /

So, now we can write the function as,

$$x(t) = a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_2 e^{2j\omega_0 t} + a_{-2} e^{-2j\omega_0 t}$$

As $x(t)$ is real, it will only consist of cos components.
So,

$$x(t) = A\cos\omega_0 t + B\cos(2\omega_0 t + \theta)$$

(θ as we don't know if a_2 is $= a_{-2}$)

Now, $x(t-3)$,

$$x(t-3) = A\cos(\omega_0 t - 3\omega_0) + B\cos(2\omega_0 t - 6\omega_0 + \theta)$$

(Phase shift)

$$\therefore -x(t-3) = -A\cos(\omega_0 t - 3\omega_0) - B\cos(2\omega_0 t - 6\omega_0 + \theta)$$

Now, for them to be equal to $x(t)$,

$$\cos(\omega_0 t - 3\omega_0) = -\cos(\omega_0 t)$$

and

$$\cos(2\omega_0 t - 6\omega_0 + \theta) = -\cos(2\omega_0 t + \theta)$$

This is only possible with an odd multiple of π.
As $6\omega_0$ can never be an odd multiple, we can conclude that the term is zero.

$$\therefore \text{Now, } x(t) = \underline{\underline{A\cos(\omega_0 t)}}$$

PTO

Date / /

$\rightarrow \frac{1}{6}\int_{-3}^{3} |x(t)|^2 dt = \frac{1}{2}$

By Parseval's relation for periodic signals,

$\frac{1}{6}\int_{-3}^{3} |x(t)|^2 = \sum_{k=-\infty}^{\infty} |a_k|^2$

As only two coefficients exist here (proven before), we can say.

$|a_1|^2 + |a_{-1}|^2 = \frac{1}{2}$

As $a_1 = a_{-1}$,

$2p^2 = \frac{1}{2} \rightarrow (p = a_1 = a_{-1})$

$\therefore p = \sqrt{\frac{1}{4}} = \frac{1}{2}$ ($\because a_1$ is positive).

So, we can write $x(t)$ as,

$x(t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$

$= \frac{1}{2}(\cos(\omega_0 t) + j\sin(\omega_0 t) + \cos(\omega_0 t) - j\sin(\omega_0 t))$

$= \cos \omega_0 t$

$= \boxed{\cos(\pi/3\, t)}$

$\therefore$ According to the q format the answers are,

$A = 1$; $B = \pi/3$; $C = 0$

$x(t) = \cos(\pi/3\, t)$

Date / /

3.61

(a) $h(t) = \delta(t)$.

$\Rightarrow H(\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t} = \underline{\underline{1}}$

$\therefore$ All functions are eigenfunctions to this LTI system, and all of them have an eigenvalue of 1.

(b) $h(t) = \delta(t - T)$.

According to the hint given, if we take an impulse train,

$x(t) = \sum_{k=-\infty}^{\infty} \delta(t + kT)$ (Impulse train).

Now, if we do $h(t) * x(t)$.

$$y(t) = \int_{-\infty}^{\infty} \sum \delta(\tau + kT) \cdot \delta(t - \tau - T)\, d\tau$$

$$= \sum_{k} \int_{-\infty}^{\infty} \delta(\tau + kT) \cdot \delta(t - T - \tau)\, d\tau$$

$$= \sum_{k} \delta(t - T + kT)$$

$$= \sum_{k=-\infty}^{\infty} \delta(t + (k-1)T)$$ which is again an impulse train.

$\therefore \sum_{k=-\infty}^{\infty} \delta(t + kT)$ is a function with eigenvalue 1.

PTO.

As we can see, the impulse train has been shifted to the right. So, now if we had an impulse train of the form,

$$\sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k \delta(t+kT)$$

The coefficient will remain the same, but the impulse will be shifted, hence resulting in the (k-1)th pulse having $(1/2)^k$ as coefficient. This is essentially multiplying 1/2. In equations,

$$x(t) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k \delta(t+kT)$$

$$\therefore \; y(t) = \int_{-\infty}^{\infty} \sum_k \left(\frac{1}{2}\right)^k \delta(\tau+kT) \cdot \delta(t-T-\tau) \, d\tau$$

$$= \sum_k \left(\frac{1}{2}\right)^k \delta(t+(k-1)T) = \underline{\frac{1}{2}} \sum_k \left(\frac{1}{2}\right)^{k-1} \delta(t+(k-1)T)$$

$\therefore$ The train has been multiplied by 1/2. So, it has an eigenvalue of 1/2.

~~A~~ The same logic can be applied to 2.

So, a function with eigenvalue 1/2 is

$$\underline{\sum_{k=-\infty}^{\infty} \frac{1}{2^k} \delta(t+kT)}$$

and eigenvalue 2 is

$$\underline{\sum_{k=-\infty}^{\infty} 2^k \delta(t+kT)}$$

PTO

Camlin Page
Date / /

(c) We can express cosine and sine as the linear combination of $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$. So, if we prove that $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have the same eigenvalues, we can prove that cos t and sin t are eigenfunctions.

$$e^{j\omega_0 t} \longrightarrow \boxed{h(t)}$$ ($h(t)$ = real and even). (ω_0 is frequency of the cosine and sine waves here).

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega_0(t-\tau)} . d\tau$$

$$= \underline{\underline{e^{j\omega_0 t} H(\omega_0)}}$$

$$e^{-j\omega_0 t} \longrightarrow \boxed{h(t)}$$

ω_0 here is same as ω in question

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau) . d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega_0(t-\tau)} . d\tau$$

$$= e^{-j\omega_0 t} H(-\omega_0)$$

$$= e^{-j\omega_0 t} H(\omega_0)$$ (As $h(t)$ is real and even, $H(\omega)$ is also even).

Now, as both are scaled by $H(\omega_0)$, we can say that cos and sin are eigenfunctions.

$$\cos \omega t = \left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right) \qquad \sin \omega t = \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)$$

$\therefore$ As they are linear combinations, and both $e^{j\omega t}$ and $e^{-j\omega t}$ are scaled by the same factor, we can say that they are eigenfunctions. (As LTI systems are linear).

4.13 $x(t) \xleftrightarrow{FT} X(\omega) = \delta(\omega) + \delta(\omega - \pi) + \delta(\omega - 5)$

$h(t) = u(t) - u(t-2)$.

(a) $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} \, d\omega$

$= \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t} \, d\omega + \int_{-\infty}^{\infty} \delta(\omega - \pi)\, e^{j\omega t} \, d\omega + \int_{-\infty}^{\infty} \delta(\omega - 5)\, e^{j\omega t}\, d\omega \right)$

$= \frac{1}{2\pi} \left(1 + e^{j\pi t} + e^{j5t} \right)$

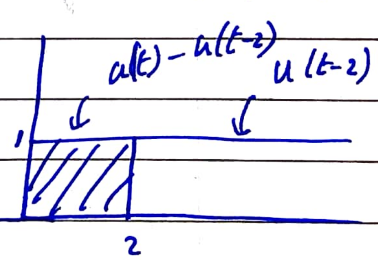
As the frequencies of these exponentials are not multiples of the other, they are not related and hence, do not form a periodic signal.

$\therefore$ $x(t)$ is NOT periodic.

(b) $x(t) * h(t)$

$\Rightarrow H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t} \, dt$ (To use convolution property).

$= \int_{-\infty}^{\infty} (u(t) - u(t-2))\, e^{-j\omega t} \, dt$



$= \int_{0}^{2} e^{-j\omega t} \, dt$

$= \left[\frac{e^{-j\omega t}}{-j\omega} \right]_0^2$

$= \frac{e^{-2j\omega} - 1}{-j\omega} = \boxed{\frac{j\,(e^{-2j\omega} - 1)}{\omega}}$

PTO

Camlin Page
Date / /

$$\therefore Y(\omega) = X(\omega) . H(\omega)$$

$$= \left(\delta(\omega) + \delta(\omega - \pi) + \delta(\omega - 5)\right) \frac{j\left(e^{-2j\omega} - 1\right)}{\omega}$$

$$= \delta(\omega) \underbrace{\frac{j\left(e^{-2j\omega} - 1\right)}{\omega}}_{\text{Limit}} + 0 + \delta(\omega - 5)\, j\left(\frac{e^{-10j} - 1}{5}\right)$$

$$\lim_{\omega \to 0} \left(\frac{e^{-2j\omega} - 1}{\omega}\right) = \frac{e^{-2j\omega}(-2j)}{1} = -2j\, e^{-2j\omega}$$

$$\Rightarrow Y(\omega) = \delta(\omega).(2e^{0}) + \delta(\omega - 5)\, j\left(\frac{e^{-10j} - 1}{5}\right)$$

$$= \boxed{2\delta(\omega) + \delta(\omega - 5)\, j\left(\frac{e^{-10j} - 1}{5}\right)}$$

If we consider the constants to be C_1 and C_2, we have,

$$Y(\omega) = C_1 \delta(\omega) + C_2 \delta(\omega - 5).$$

$$\therefore y(t) = \frac{1}{2\pi}\left(C_1 \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} . d\omega + C_2 \int_{-\infty}^{\infty} \delta(\omega - 5) e^{j\omega t} . d\omega\right)$$

$$= \frac{C_1}{2\pi} + \boxed{\frac{C_2\, e^{j5t}}{2\pi}}$$

Periodic because complex exponential

Adding a constant or multiplying a constant doesn't change the periodicity of the signal here, so $y(t) = x(t) * h(t)$ is periodic.

(c) The above example convolved two aperiodic signals to get a periodic signal. Hence, it is possible.

4.21

(e) $x(t) = [t e^{-2t} \sin 4t]\, u(t)$

$$= t e^{-2t} u(t) \left[\frac{1}{2j} \left(e^{j4t} - e^{-j4t} \right) \right]$$

$$= \frac{t e^{-2t}}{2j} e^{j4t} u(t) - \frac{t e^{-2t}}{2j} e^{-j4t} u(t)$$

$$\therefore X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \frac{1}{2j} \left(\int_0^{\infty} t e^{(-2+4j-j\omega)t} dt - \int_0^{\infty} t e^{(-2-4j-j\omega)t} dt \right)$$

Integral of $t e^{kt}$

$$\int t e^{kt} dt$$

$$= t \int e^{kt} - \iint e^{kt}$$

$$= \frac{t e^{kt}}{k} - \frac{e^{kt}}{k^2}$$

Substituting where required,

$$X(\omega) = \frac{1}{2j} \left(\left[\frac{t e^{(-2+4j-j\omega)t}}{(-2+4j-j\omega)} - \frac{e^{(-2+4j-j\omega)t}}{(-2+4j-j\omega)^2} \right]_0^{\infty} - \left[\frac{t e^{(-2-4j-j\omega)t}}{(-2-4j-j\omega)} - \frac{e^{(-2-4j-j\omega)t}}{(-2-4j-j\omega)^2} \right]_0^{\infty} \right)$$

On substituting limits, we get,

$$X(\omega) = \frac{1}{2j} \left(\left[0 - \left(0 - \frac{1}{(-2+4j-j\omega)^2} \right) \right] - \left[0 - \left[0 - \frac{1}{(-2-4j-j\omega)^2} \right] \right] \right)$$

Camlin Page
Date / /

$$\therefore X(\omega) = \frac{1}{2j}\left(\frac{1}{(-2+4j-j\omega)^2} - \frac{1}{(-2-4j-j\omega)^2}\right)$$

(f) $x(t) = \left[\frac{\sin \pi t}{\pi t}\right]\left[\frac{\sin 2\pi\,(t-1)}{\pi(t-1)}\right]$

If we divide this function into 2 parts in multiplication, we can then convolve their Fourier transforms to get our result.

Part-1

FT of $\frac{\sin(\pi t)}{\pi t} = \begin{cases} 1 & ; |\omega| < \pi \\ 0 & ; |\omega| \geq \pi. \end{cases}$

Part-2

FT of $\frac{\sin(2\pi(t-1))}{\pi(t-1)}$

Now, if we start from $\frac{\sin \pi t}{\pi t} = f(t)$,

$$f(t) \longleftrightarrow F(\omega)$$

Scaling time by 2, we get,

$$f(2t) \longleftrightarrow \frac{1}{|2|} F\left(\frac{\omega}{2}\right)$$

But since we only have πt in the denominator, we must multiply by 2.

$$\therefore \frac{\sin(2\pi t)}{\pi t} \longleftrightarrow F\left(\frac{\omega}{2}\right)$$

Camlin Page
Date / /

Now, with a time shift,

$$\frac{\sin(2\pi(t-1))}{\pi(t-1)} \longleftrightarrow F\left(\frac{w}{2}\right) e^{-jw}$$

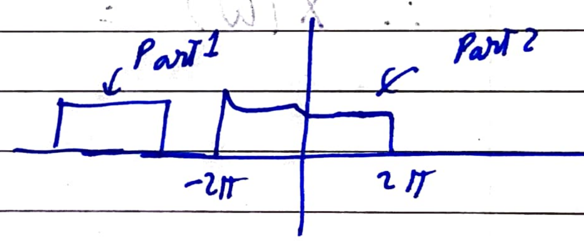
∴ FT of part 2

$$\frac{\sin 2\pi(t-1)}{\pi(t-1)} \longleftrightarrow \begin{cases} e^{-jw} & |w| < 2\pi \\ 0 & |w| \geq 2\pi \end{cases}$$

Now, we must convolve these two transforms. So, we take cases.

→ $\underline{(w+\pi) < -2\pi}$

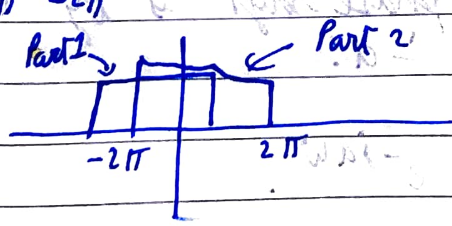
$$\frac{1}{2\pi}\int 0 \cdot X_2(w) = 0$$



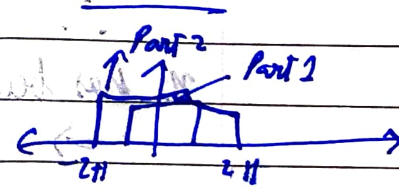
→ $(w+\pi) > -2\pi$ and $w-\pi < -2\pi$

$$\frac{1}{2\pi}\int_{-2\pi}^{w+\pi} 1 \cdot e^{-j\tau}\, d\tau = \frac{1}{2\pi}\left[\frac{e^{-j\tau}}{-j}\right]_{-2\pi}^{w+\pi} = \frac{-e^{-j(w+\pi)} + e^{2\pi j}}{2\pi j}$$

$$= \frac{1 - e^{-j(w+\pi)}}{2\pi j}$$



→ $w-\pi > -2\pi$ and $w+\pi < 2\pi$ ⟶

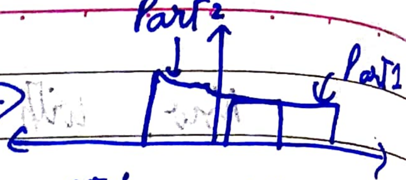


$$\frac{1}{2\pi}\int_{w-\pi}^{w+\pi} 1 \cdot e^{-j\tau}\, dt = \frac{1}{2\pi}\left[\frac{e^{-j\tau}}{-j}\right]_{w-\pi}^{w+\pi} = \frac{-e^{-j(w+\pi)} + e^{-j(w-\pi)}}{2\pi j}$$

$$= 0$$

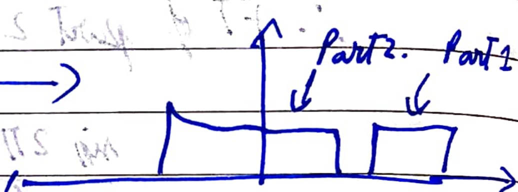
Camlin Page
Date / /

→ $w - \pi < 2\pi$ and $w + \pi > 2\pi$ ⟶



$$\frac{1}{2\pi}\int_{w-\pi}^{2\pi} 1 \cdot e^{-j\tau}\, d\tau = \frac{1}{2\pi}\left[\frac{e^{-j\tau}}{-j}\right]_{w-\pi}^{2\pi} = \frac{-e^{-2\pi j} + e^{-j(w-\pi)}}{2\pi j}$$

→ $w - \pi > 2\pi$ ⟶



$$\frac{1}{2\pi}\int 0 \cdot e^{-jt}\, dt = \underline{\underline{0}}$$

∴ The entire function can be defined as.

$$\therefore X(w) = \begin{cases} 0 & ; \quad w < -3\pi \\ (1 + e^{-jw})/2\pi j & ; \quad w > -3\pi \text{ and } w < -\pi \\ 0 & ; \quad -\pi < w < \pi \\ (-1 - e^{-jw})/2\pi j & ; \quad w < 3\pi \text{ and } w > \pi \\ 0 & ; \quad w > 3\pi \end{cases}$$

4.29. For $X_a(jw)$, we can clearly see that it is a phase shifted version of $X(jw)$. The phase shift is by e^{-jaw}, because of the slope now being '$-a$'.

$$\therefore \quad X_a(jw) = X(jw)\, e^{-jaw}.$$

x has been Time shifted then. So,

$$\Rightarrow \quad x(t-a) \longleftrightarrow X_a(jw).$$

$$\therefore \boxed{x_a(t) = x(t-a)}.$$

PTO

Camlin Page
Date / /

For $X_b(j\omega)$, it has been phase shifted to have a slope of b. So, the phase shift can be shown by $e^{jb\omega}$.

$$\therefore \; X_b(j\omega) = X(j\omega)\, e^{jb\omega}. \qquad (\because \text{Positive slope}).$$

This phase shift corresponds to a time shift in the time ~~domain~~ domain. So, we get

$$\boxed{x_b(t) = x(t+b)}$$

For $X_c(j\omega)$, we can see a ~~1st~~ phase ~~shift~~ reversal. This is basically taking the conjugate of the signal. So,

$$X_c(j\omega) = X^*(j\omega)$$

Conjugation in frequency domain is equivalent to conjugation AND time ~~rota~~ reversal in time domain. So,

$$x_c(t) = x^*(-t)$$

But $x(t)$ is real valued, so $x^*(-t) = x(-t)$.

$$\therefore \; \boxed{x_c(t) = x(-t)}$$

For $X_d(j\omega)$ it is conjugation and phase shift of the original signal. First, we conjugate the signal, and then phase shift it.

$$\therefore \; X_d(j\omega) = X^*(j\omega) \cdot e^{jd\omega}.$$

So, now we convert it to time domain, which is conjugation, time shift and reversal. So, we get,

$$x_d(t) = x^*(-t-d) = x(-t-d) \qquad (\because x(t) \text{ is real}).$$

$$\therefore \; \boxed{x_d(t) = x(-t-d)}$$