Lecture 13 Notes

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Key Ideas

1. Variance
2. Examples and Properties of Discrete Random Variables
3. Examples and Properties of Continuous Random Variables

Variance of a Random Variable

Variance can be understood as a quantity that represents the avergae variation of a R.V about the mean. Given a R.V X, its variance is given by

$Var(X) = E[(X - E(X))^2] = E(X^2) - (E(X))^2$

If X is discrete, then

$E(X) = \mu$

$Var(X) = \sum_{x_i} (x_i - \mu)^2 P_X(x_i)$

If X is continuous, then

$E(X) = \mu$

$Var(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f_X(x) dx$

Properties of Variance

Variance is always ≥ 0, with equality iff the random variable is constant.

Examples of Discrete Random Variables

1. Bernoulli Random Variables

- Defined by

$$P_X(x) = \begin{cases} p & for\ x = 1 \\ 1-p & for\ x = 0 \\ 0 & otherwise \end{cases}$$

where $0 < p < 1$.

- $E(X) = 0(1-p) + 1(p) = p$
- $Var(X) = (0-p)^2(1-p) + (1-p)^2(p) = p(1-p)$

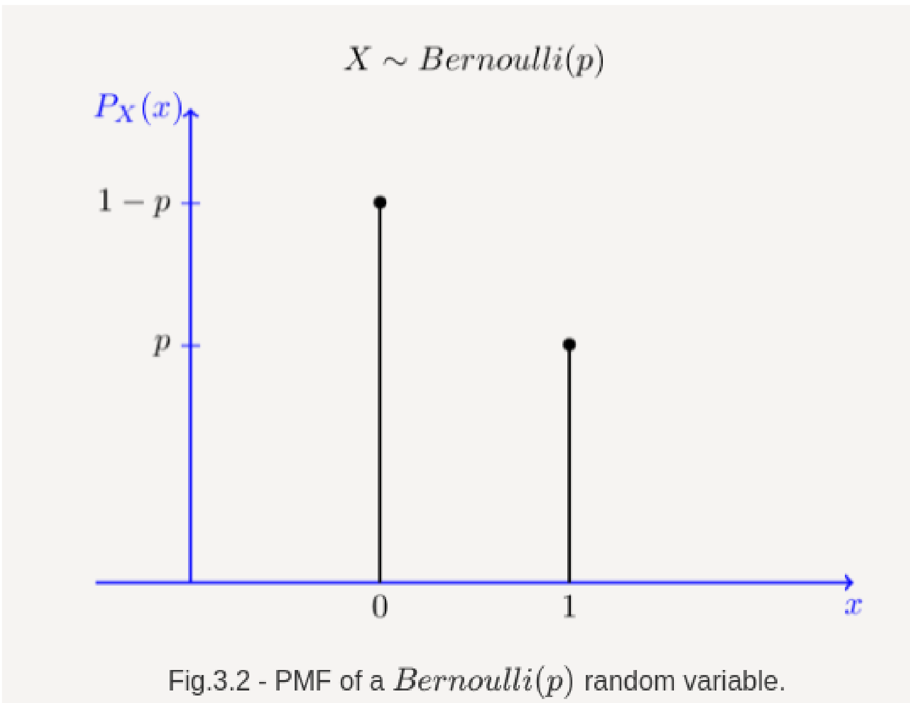

Figure 1: PMF of a Bernoulli random variable

- This random variable is mainly used to model random experiments that have two possible outcomes, which are sometimes referred to as "success" and "failure".
- The Bernoulli random variable is also called the **Indicator random variable**, which tells us if a particular event has occured or not.

2. Binomial Random Variables

- Experiment: Biased coin tossed n times with probability of heads being p; outcomes mapped to k where k is the number of heads obtained.

- This variable can be represented as the sum of n independent Bernoulli R.Vs.
- $E(X) = \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i} = np$
- $Var(X) =$ Sum of variances of n Bernoulli variables $= nVar(X) = np(1-p)$

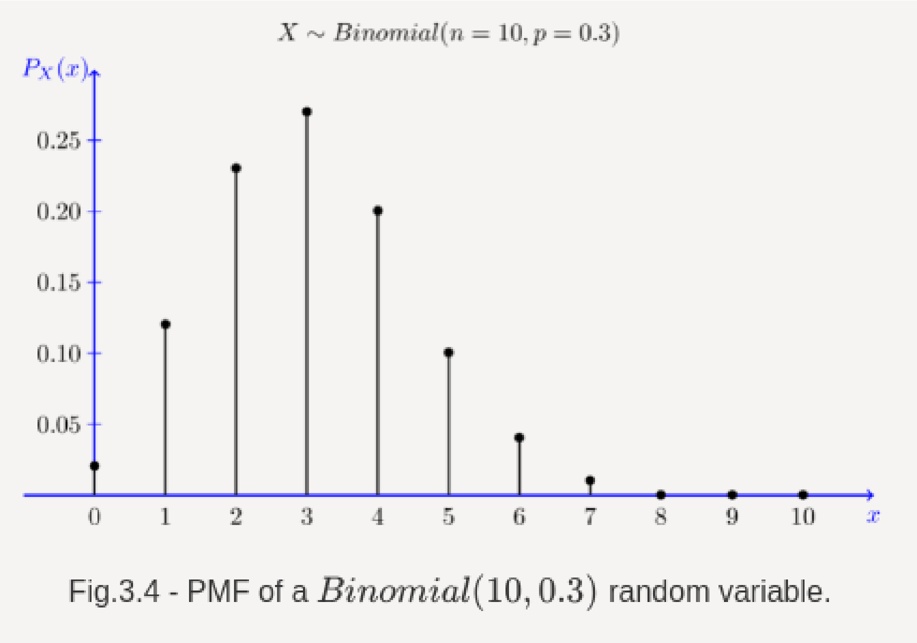

Figure 2: PMF of a binomial random variable

3. Geometric Random Variables

- Experiment: Toss a biased coin till a head is obtained. X is the position of the head.
- $P_X(k) = (1-p)^{k-1}p$
- $E(X) = \sum_{i=0}^{n} i(1-p)^{i-1}p = 1/p$
- $Var(X) = E(X^2) - (E(X))^2 = E(X^2) - 1/p^2$

  $E(X^2) = \sum_{k=1}^{\infty} k^2 p(1-p)^{k-1} = (2-p)/p^2$
- $\therefore Var(X) = (2-p-1)/p^2 = (1-p)/p^2$

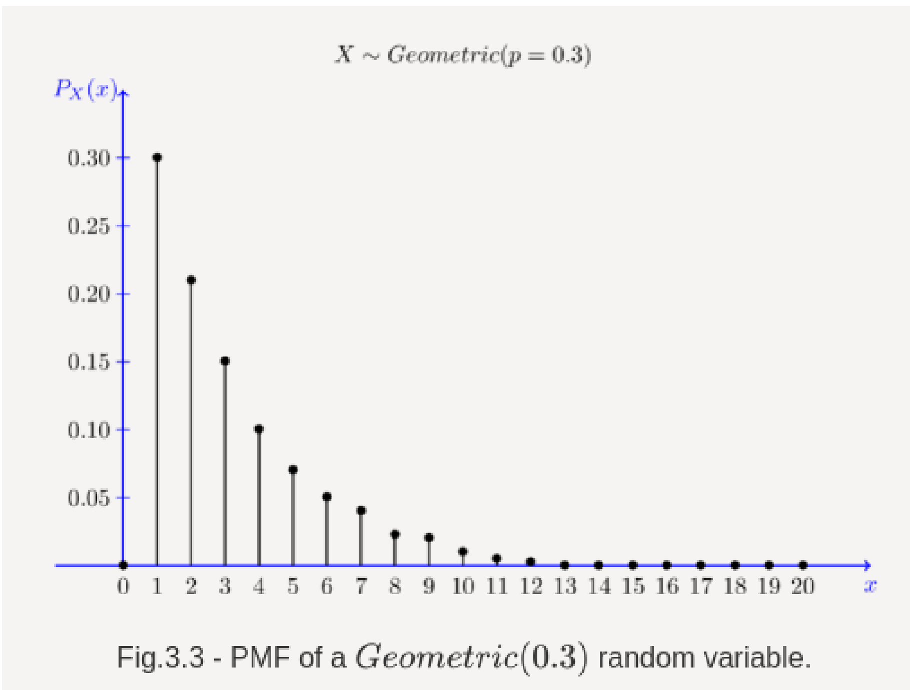

Figure 3: PMF of a geometric random variable

4. **Poisson Random Variables**

- Used to model rare events
- $P_X(k) = \lambda^k e^{-\lambda}/k!$
- $E(X) = \lambda$
- $Var(X) = \lambda$

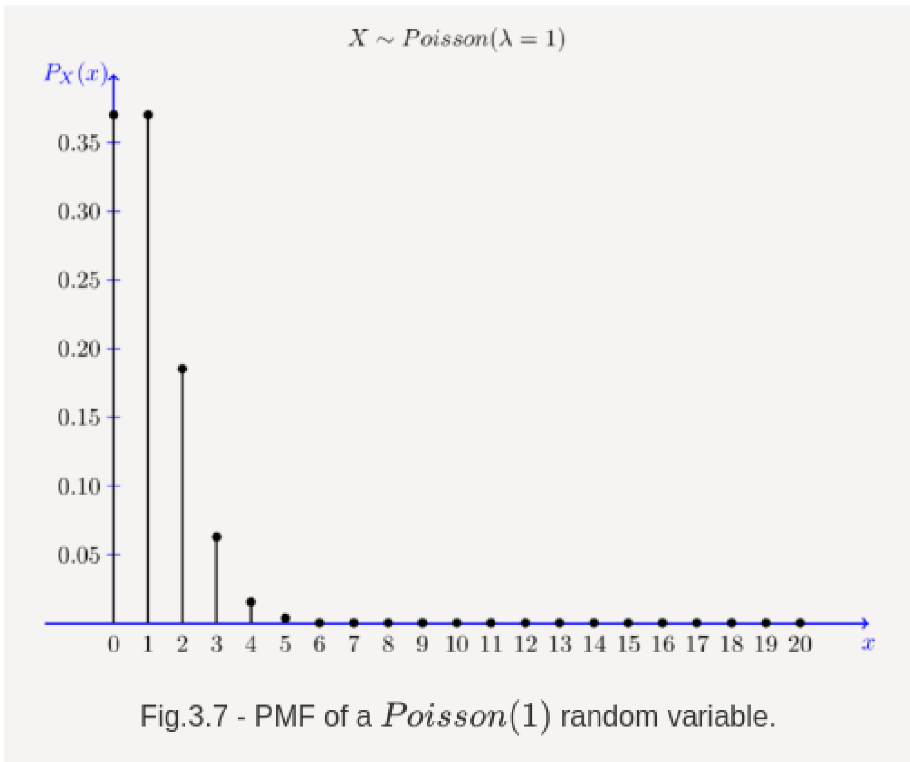

Figure 4: PMF of a poisson random variable

5. **Pascal Random Variables**

- This R.V can be understood as a generalisation of the geometric distribution.
- Experiment: Biased coin with probability of heads being p tossed repeatedly until m heads are observed, $m \in \mathcal{N}$.
- As we can deduce from the above definition, a geometric variable is simply the case when $m = 1$ for a Pascal random variable.

- $P_X(x) = \begin{cases} \binom{k-1}{m-1} p^m (1-p)^{k-m} & for\ k = m, m+1, m+2.... \\ 0 & otherwise \end{cases}$
- $E(X) = m/p$, can be derived by expressing it as the sum of multiple independent geometric random variables.
- $Var(X) = m \cdot (1-p)/p^2$

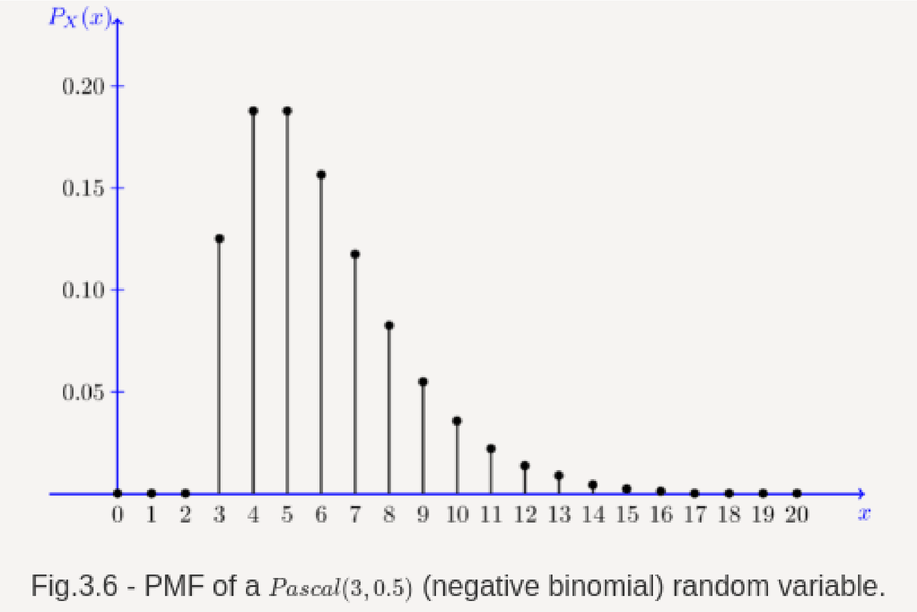

Figure 5: PMF of a Pascal random variable

Examples of Continuous Random Variables

1. Uniform Random Variables

- $f_X(x) = \begin{cases} 1/(b-a) & for\ a \leq x \leq b \\ 0 & otherwise \end{cases}$
- $E(X) = (a+b)/2$
- $Var(X) = (b-a)^2/12$

2. Exponential Random Variables

- Used to model completion time

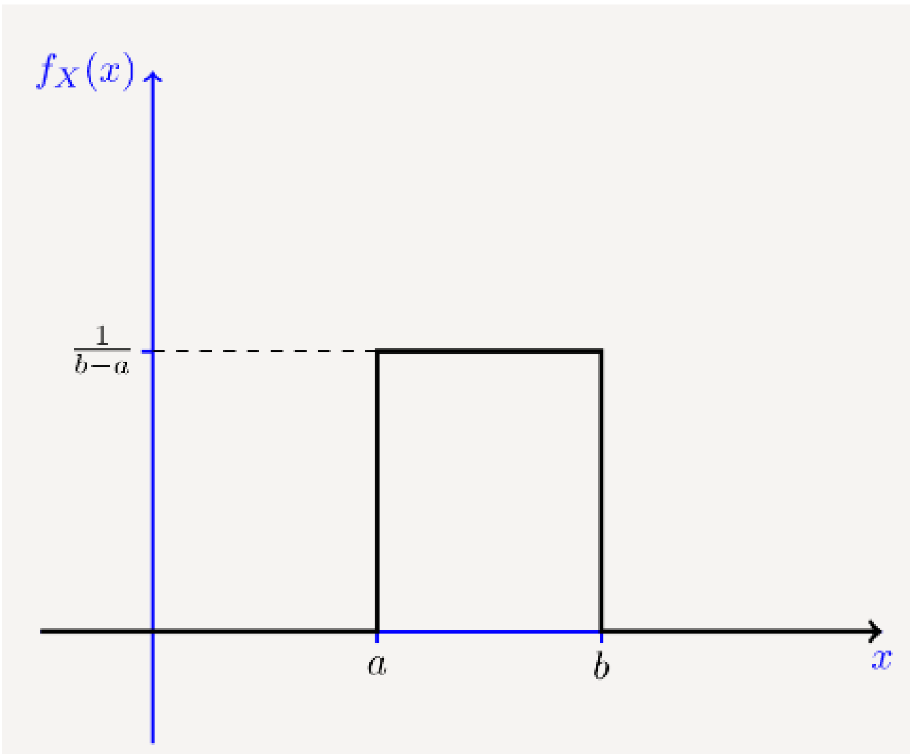

Figure 6: PDF of a uniform random variable

- $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & for x \geq 0 \\ 0 & otherwise \end{cases}$
- $E(X) = \int_0^\infty x\lambda e^{-\lambda x} dx = 1/\lambda$
- $Var(X) = 1/\lambda^2$

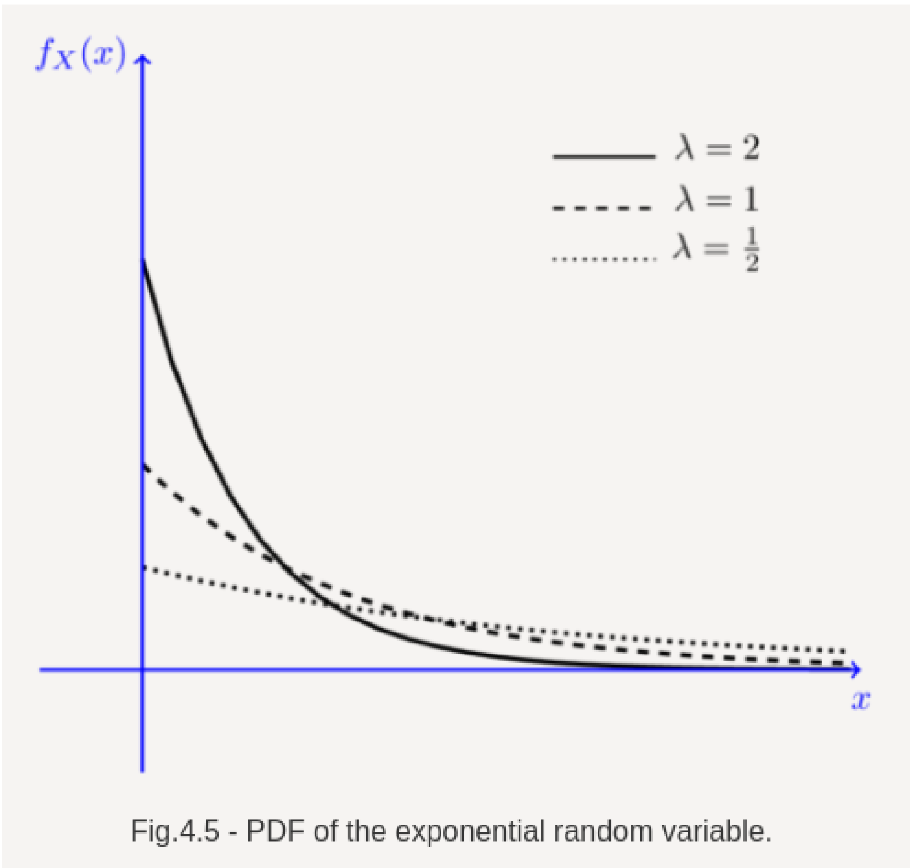

Figure 7: PDF of an exponential random variable

3. Gaussian Random Variables

- Used to model noise. It is a sum of many independent R.V.s
- $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$
- $E(X) = \mu$
- $Var(X) = \sigma^2$

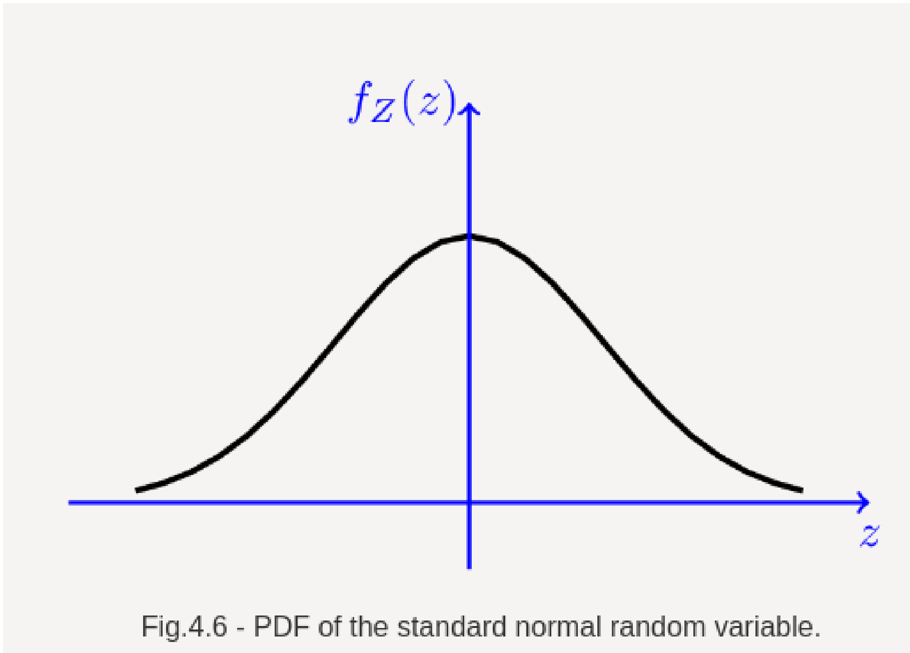

Figure 8: PDF of a standard normal random variable