Tutorial 2 - Problem 3

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Let X be a random variable

$$R_X = \{x_1, x_2, x_3, \dots x_M\}$$

According to the initial distribution,

$$P(X = x_1) = p_1$$

$$P(X = x_2) = p_2 \qquad \text{Suppose } p_1 > p_2$$

$$\vdots$$

$$P(X = x_M) = p_M$$

$$H(X) = f(p_1, p_2, \dots p_M) = \sum_i p_i \log \frac{1}{p_i}$$

According to the final distribution,

$$P(X = x_1) = p_1' = p_1 - \Delta p \qquad \Delta p > 0$$

$$P(X = x_2) = p_2' = p_2 + \Delta p \qquad p_1' > p_2'$$

$$P(X = x_3) = p_3' = p_3$$

$$\vdots$$

$$P(X = x_M) = p_M' = p_M$$

$$H(X) = f(p_1', p_2', \dots p_M') = \sum_i p_i' \log \frac{1}{p_i'}$$

Required to show:

$$f(p_1, p_2, \dots p_M) < f(p_1', p_2', \dots p_M')$$

$$D\big(\underbrace{p(x)}_{\text{Initial distribution}} \,\|\, \underbrace{q(x)}_{\text{Final distribution}}\big) \geq 0$$

$$\Rightarrow \sum_i p_i \log \frac{p_i}{p_i'} \geq 0$$

$$\Rightarrow p_1 \log \frac{p_1}{p_1 - \Delta p} + p_2 \log \frac{p_2}{p_2 + \Delta p} + 0 + \dots + 0 \geq 0 \qquad -(i)$$

Consider

$$f(p_1', p_2', \dots, p_M') - f(p_1, p_2, \dots p_M)$$

$$= \sum_i p_i' \log \frac{1}{p_i'} - \sum p_i \log \frac{1}{p_i}$$

$$= (p_1 - \Delta p) \log \frac{1}{p_1 - \Delta p} - p_1 \log \frac{1}{p_1}$$

$$+ (p_2 + \Delta p) \log \frac{1}{p_2 + \Delta p} - p_2 \log \frac{1}{p_2}$$

$$= \underbrace{p_1 \log \frac{p_1}{p_1 - \Delta p} + p_2 \log \frac{p_2}{p_2 + \Delta p}}_{\geq 0 \ (\text{Ineq } (i))} + \underbrace{\Delta p \cdot \log \underbrace{\frac{p_1 - \Delta p}{p_2 + \Delta p}}_{>1}}_{> 0}$$

$$\because p_1 - \Delta p > p_2 + \Delta p$$

$$\therefore f(p_1', p_2', \dots p_M') > f(p_1, p_2, \dots p_M)$$