Lecture 9 notes

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Key Ideas

1. Lemmas for the CDF
2. Types of random variables

Lemma : For any $x \in R$

$P(X = x) = P(X \leq x) - P(X < x)$

Then, by continuity of probability,

$P(X = x) = F_X(x) - \lim_{\epsilon \to 0} F_X(x - \epsilon)$

Corollary : $F_X()$ is left continuous iff $P(X = x) = 0 \;\; \forall \;\; x \in R$

This implies $F_X()$ is continuous iff $P(X = x) = 0 \;\; \forall \;\; x \in R$

Types of random variables

- Continuous Random Variable
- Discrete Random Variable
- Mixed Random Variable

  (Not done in class) There is a fourth type of random variable called a SINGULAR random variable, that is a fundamentally different random variable. It is mostly only of academic interest as it does not have many applications. Any probability measure on the Real numbers can be broken down into these three fundamental components, i.e., Discrete, Continuous and singular.

Continuous Random variables A random variable X with CDF $F_X()$ is said to be continuous if $F_X()$ is continuous. In the context of continuous RVs, probabilities of intervals give useful info, as probabilities of points are always 0.

If $F_X()$ is a differentiable function, then we can find another function for the continuous RV, which is defined as,

$$f_X(x) = \frac{dF_X(x)}{dx}$$

which is called the **probability density function**. We can integrate this function from $-\infty$ to whatever value we want to get the CDF.

So we can say that $P(a \leq x \leq b) = \displaystyle\int_a^b f_X(x) \cdot dx$

And from this, if we find $P(x \leq X \leq x + \Delta x)$ and make $\Delta x \to 0$, we can approximate it to a rectangle with area $f_X(x) \cdot \Delta x$.

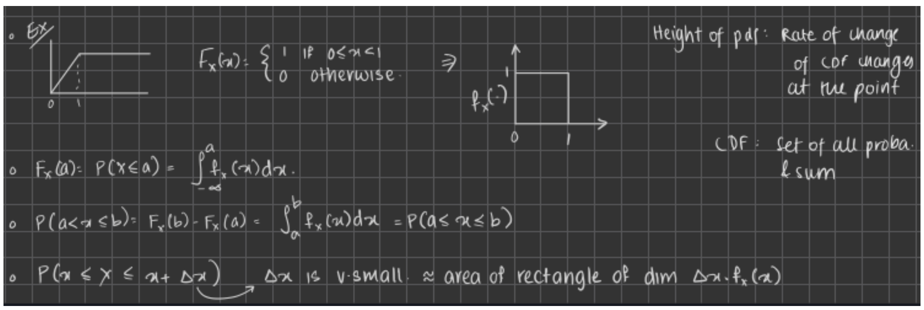

Figure 1: Example

Properties of a PDF

- $f_X(x) \geq 0$ (Because CDF is monotonically non decreasing)
- $\displaystyle\int_{-\infty}^{\infty} f_X(x) \cdot dx = 1$

PDF by itself does not indicate any probability.

Radon Nikodyn theorem (Not done in class) Let X be a continuous RV. There exists a non negative measurable function $f_x() : R \to [0, \infty)$ such that for any B in the Borel sigma algebra, we have

$$P_X(B) = \int_B f_x \cdot d\lambda$$

This basically means for any Borel set, we can represent its probability law as the integral of a non negative function. As we have not done measure theory yet, we can understand it as an integration over the real line for our purposes. This is the basis for the equation that relates the CDF of continuous random variables to the PDF, i.e.,

$$P_X((-\infty, x]) = F_X(x) = \int_{-\infty}^{x} f_X(x)dx$$

Discrete Random Variables X is said to be a discrete random variable if the range of X is either finite or countably infinite in R.

The CDF of a discrete random variable will look like a staircase, constant everywhere, with jumps at certain points.

Here, $P(X = x_i) > 0$, not necessarily, but possible. CDF for this would be

$$F_X(a) = \sum_{x_i \leq a} P(X = x_i)$$

Now, $P(X = x_i) = P_X(x_i)$ for x_i in the range of X.

$P_X()$ is called the **probability mass function**.

Properties of PMF:

- $P_X(x_i) \geq 0$

For S = range of X = $\{x_1, x_2, ...\}$

- $\sum_{x_i \in S} P_X(x_i) = 1$
