Lecture - 21

Friday, July 9, 2021 10:17 AM

Discrete Fourier Transform (DFT)

- zero-padding : adding zeros to sequence $\longrightarrow$ dense sampling of DTFT

  useful for numerical computation / visualization of DTFT via DFT

- circular time-shift of sequence : $x[n-n_0] \equiv x[\langle n-n_0 \rangle_N]$

  using modulo-N indexing

- time-reversed sequence : $x[-n] \equiv x[N-n] = \{x[0], x[N-1], x[N-2], \cdots x[1]\}$

  ... (using periodicity)

Similarly for DFT sequence, $X[-k] = X[N-k]$

e.g. $e^{j\omega n}$ $\quad \omega \in [0, 2\pi)$ $\quad$ period $= 5$

for periodicity $\omega = 2\pi \frac{k}{N}$ $\quad$ (period $= N$)

$N=5$, $\omega \in \left\{0, \frac{2\pi}{5}, 2\pi\frac{2}{5}, 2\pi\frac{3}{5}, 2\pi\frac{4}{5}\right\}$

- Circular convolution : convolution of two N-length sequences to give another N-length sequence.

$$x_3[n] = x_1[n] \circledast x_2[n] = \sum_{m=0}^{N-1} x_1[m]\, x_2[n-m]$$

circular shift

- circular convolution in matrix form :

Ex. $N=4$ $\quad y[n] = x[n] \circledast h[n] = \sum_{m=0}^{3} x[m]\, h[n-m]$

$n=0$ $\quad y[0] = x[0]h[0] + x[1]h[3] + x[2]h[2] + x[3]h[1]$

$n=1$ $\quad y[1] = x[0]h[1] + x[1]h[0] + x[2]h[3] + x[3]h[2]$

$n=2$ $\quad y[2] = x[0]h[2] + x[1]h[1] + x[2]h[0] + x[3]h[3]$

$n=3$ $\quad y[3] = x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0]$

matrix form $\rightarrow$

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix} = \begin{bmatrix} h[0] & h[3] & h[2] & h[1] \\ h[1] & h[0] & h[3] & h[2] \\ h[2] & h[1] & h[0] & h[3] \\ h[3] & h[2] & h[1] & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

Circulant matrix

* any circular convolution can be expressed in this way

Linear convolution using circular convolution

* we are most often interested in linear convolution
* linear convolution arises naturally in LTI systems
* write linear convolution using matrix product

Ex. $y[n] = x[n] * h[n]$ ... (linear convolution)

$x[n] = \{x[0], x[1], x[2]\}$ length - 3

$h[n] = \{h[0], h[1], h[2], h[3]\}$ length - 4

$\Rightarrow$ $y[n]$ will have length - 6

$$y[n] = \sum_{m=0}^{2} x[m]\,h[n-m]$$

$n=0$ $\quad y[0] = x[0]h[0]$, $n=1$ $\quad y[1] = x[0]h[1] + x[1]h[0]$

$n=2$ $\quad y[2] = x[0]h[2] + x[1]h[1] + x[2]h[0]$

Similarly $y[3]$, $y[4]$ & $y[5]$

matrix form arrangement

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ \vdots \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 \\ h[1] & h[0] & 0 \\ h[2] & h[1] & h[0] \\ h[3] & h[2] & h[1] \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix}$$

linear convolution.

$$\begin{bmatrix} y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix} = \begin{bmatrix} h[2] & h[1] & h[0] \\ h[3] & h[2] & h[1] \\ 0 & h[3] & h[2] \\ 0 & 0 & h[3] \end{bmatrix} \begin{bmatrix} x[2] \end{bmatrix}$$

make circulant by adding columns

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 & h[3] & h[2] & h[1] \\ h[1] & h[0] & 0 & 0 & h[3] & h[2] \\ h[2] & h[1] & h[0] & 0 & 0 & h[3] \\ h[3] & h[2] & h[1] & h[0] & 0 & 0 \\ 0 & h[3] & h[2] & h[1] & h[0] & 0 \\ 0 & 0 & h[3] & h[2] & h[1] & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

circulant matrix   zero-padding

$$y[n] = \tilde{x}[n] \circledast \tilde{h}[n]$$

$$\tilde{x}[n] = \{ x[0], x[1], x[2], 0, 0, 0 \} \qquad \text{length}-6$$

$$\tilde{h}[n] = \{ h[0], h[1], h[2], h[3], 0, 0 \} \qquad \text{length}-6$$

$$\underbrace{y[n]}_{\text{length}-(L+M-1)} = \underbrace{x[n]}_{\text{length}-L} * \underbrace{h[n]}_{\text{length}-M} \equiv \underbrace{\tilde{x}[n] \circledast \tilde{h}[n]}_{\text{length } (L+M-1)}$$

* $\tilde{x}$ & $\tilde{h}$ are obtained by zero padding of x & h.

* any linear convolution can be done using circular convolution

Circular convolution property of DFT

$$x_1[n] \xleftrightarrow{\text{DFT}} X_1[k] \qquad x_2[n] \xleftrightarrow{\text{DFT}} X_2[k]$$

$$x_1[n] \circledast x_2[n] \xleftrightarrow{\text{DFT}} X_1[k]\, X_2[k]$$ ... (proof HW)

* motivation : $x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] = x[n] * h[n]$ — ①

(for doing linear conv. using circular conv.)

x & h – length N $\rightarrow$ y – length $(2N-1)$

multiplications $= (2N-1)N \sim O(N^2)$

convolution using DFT

$x[n] \longleftrightarrow X[k]$

$h[n] \longleftrightarrow H[k]$

$\Big\}$ computing DFT can be fast

$$X[k]\,H[k] \xleftrightarrow{\text{DFT}} x[n] \circledast h[n] \qquad \text{— ②}$$

by zero padding $\quad y[n] = \tilde{x}[n] \circledast \tilde{h}[n] \xleftrightarrow{\text{DFT}} \tilde{X}[k]\,\tilde{H}[k]$

$\downarrow$ I DFT (fast)

$y[n]$

⊛ Properties of DFT

① Linearity : $\alpha x_1[n] + \beta x_2[n] \xleftrightarrow{\text{DFT}} \alpha X_1[k] + \beta X_2[k]$

② Circular time-shift : $x[n] \longleftrightarrow X[k]$

$$x[n-n_0] \longleftrightarrow X[k]\, e^{-j\frac{2\pi}{N} n_0 k}$$

③ Frequency shift : $e^{j\frac{2\pi}{N}k_0 n}\, x[n] \xleftrightarrow{\text{DFT}} X[k-k_0]$

④ time-reversal : $x[n] \longleftrightarrow X[k]$

$$x[-n] \longleftrightarrow X[-k]$$

⑤ complex conjugate : $x^*[n] \longleftrightarrow X^*[-k]$

⑥ Parseval's Theorem :

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

7) circular convolution (discussed above)

8) multiplication :

$$x_1[n]\, x_2[n] \xleftrightarrow{\text{DFT}} \frac{1}{N} X_1[k] \circledast X_2[k]$$