Assignment - 2

I affirm that I have neither given nor received help or used any means which would make this assignment unfair.

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Q 4.36

$x(t) = [e^{-t} + e^{-3t}]\, u(t)$

$\Rightarrow X(\omega) = \dfrac{1}{1+j\omega} + \dfrac{1}{3+j\omega} = \dfrac{4+2j\omega}{(1+j\omega)(3+j\omega)}$

$y(t) = 2[e^{-t} - \cancel{x}e^{-4t}]\, u(t)$

$\Rightarrow Y(\omega) = 2\left(\dfrac{1}{1+j\omega} - \dfrac{1}{4+j\omega}\right) = \dfrac{2\times 3}{(1+j\omega)(4+j\omega)}$

(a) Frequency response of the system is $H(\omega)$. As it is an LTI system, we can say that

$Y(\omega) = H(\omega) \cdot X(\omega)$

$\therefore H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = \dfrac{\cancel{6}^{3}}{\cancel{(1+j\omega)}(4+j\omega)} \times \dfrac{\cancel{(1+j\omega)}(3+j\omega)}{\cancel{2}(2+j\omega)} = \boxed{\dfrac{9+3j\omega}{(4+j\omega)(2+j\omega)}}$

(b) The impulse response $h(t)$ can be found by using partial fractions.

$H(\omega) = \dfrac{9+3j\omega}{(4+j\omega)(2+j\omega)} = \dfrac{A}{(4+j\omega)} + \dfrac{B}{(2+j\omega)}$

$= \dfrac{2A + Aj\omega + 4B + Bj\omega}{(4+j\omega)(2+j\omega)}$

From this, $2A + 4B = 9$
$A + B = 3$ $\Rightarrow$ $\underline{A = B = \dfrac{3}{2}}$

So, using standard transforms, we can say that

$$\frac{3}{2}\left(\frac{1}{(2+j\omega)}+\frac{1}{(4+j\omega)}\right) \longleftrightarrow \left(\frac{3}{2}\left(e^{-2t}+e^{-4t}\right)\right)u(t)$$

$$\therefore\ h(t)=\frac{3}{2}\left(e^{-2t}+e^{-4t}\right)u(t)$$

(c) As we know that $\frac{dy(t)}{dt} = j\omega Y(\omega)$, from (a), we can say that,

$$\frac{Y(\omega)}{X(\omega)} = \frac{(9+3j\omega)}{(4+j\omega)(2+j\omega)}$$

$$\Rightarrow\ 8Y(\omega) + 6j\omega Y(\omega) + (j\omega)^2 Y(\omega) = 9X(\omega) + 3j\omega X(\omega)$$

On transforming this equation back to time domain, we get,

$$8y(t) + 6\frac{dy(t)}{dt} + \frac{d^2y(t)}{dt^2} = 9x(t) + 3\frac{dx(t)}{dt}$$

4.44: $$\frac{dy(t)}{dt} + 10y(t) = \int_{-\infty}^{\infty} x(\tau)\, z(t-\tau)\, d\tau - x(t)$$

$$z(t) = e^{-t}u(t) + 3\delta(t) \Rightarrow Z(\omega) = \frac{1}{1+j\omega} + 3 = \frac{4+3j\omega}{(1+j\omega)}$$

(a) For frequency response, we can convert to frequency domain and find $Y(\omega)/X(\omega)$:

PTO

So, we get,

$$j\omega\, Y(\omega) + 10\, Y(\omega) = \underbrace{X(\omega).\, Z(\omega)}_{\text{Because convolution in time domain}} - X(\omega)$$

$$\therefore \frac{Y(\omega)}{X(\omega)} = \frac{\frac{(4+3j\omega)}{(1+j\omega)} - 1}{10+j\omega} = \frac{3+2j\omega}{(1+j\omega)(10+j\omega)} = H(\omega)$$

(b) We can find impulse response using partial fractions.

$$\frac{3+2j\omega}{(1+j\omega)(10+j\omega)} = \frac{A}{(1+j\omega)} + \frac{B}{(10+j\omega)}$$

$$= \frac{10A + Aj\omega + B + Bj\omega}{(1+j\omega)(10+j\omega)}$$

$$\therefore 10A + B = 3 \; ; \quad A + B = 2$$

$$\Rightarrow A = \frac{1}{9} \quad \Rightarrow B = \frac{17}{9}$$

$$\therefore H(\omega) = \frac{1}{9(1+j\omega)} + \frac{17}{9(10+j\omega)}$$

$$\therefore h(t) = \frac{1}{9} e^{-t} u(t) + \frac{17}{9} e^{-10t} u(t)$$

6.33

(a) $H(j\omega)$ is a lowpass filter with cutoff $= \omega_{lp}$

PTO

So, only the low frequency part of $x(t)$ passes.

$H(j\omega) \cdot X(j\omega) \rightarrow X_l(\omega) \leftarrow$ only low frequency part.
($\because$ convolution in time domain is multiplication in frequency domain).
If $X_h(\omega)$ is the high frequency part of the signal, we can write:

$$X(\omega) = X_H(\omega) + X_l(\omega)$$

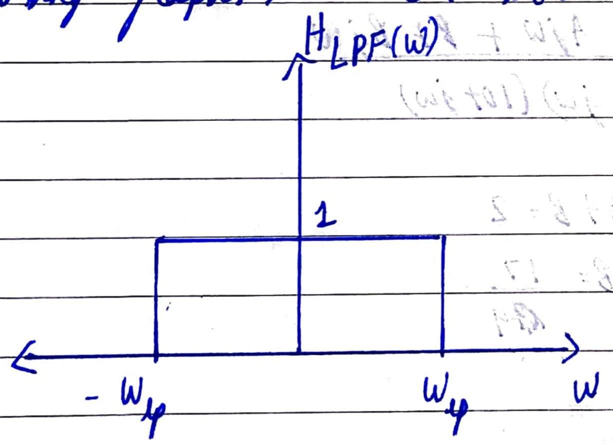
$$\therefore X(\omega) - X_H(\omega) = X_l(\omega)$$

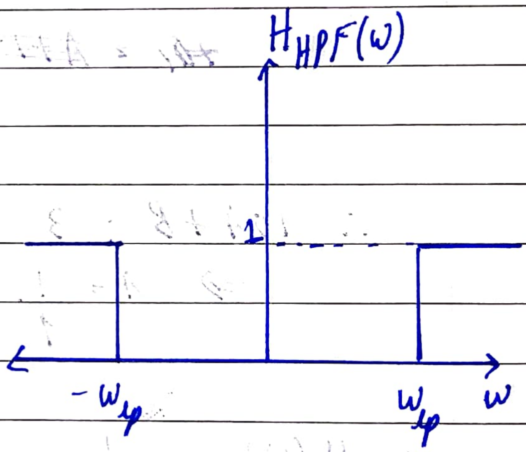
$$\Updownarrow \quad\quad \Updownarrow \quad\quad \Updownarrow \text{IFT}$$

$$x(t) - x_h(t) = x_l(t)$$

$$\therefore \underline{x(t) - x_l(t) = x_h(t)} \quad \therefore \text{High pass filter.}$$

or,
Using graphs, we can show that.





As we can see from the graphs,

$$1 - H_{LPF}(\omega) = H_{HPF}(\omega) \quad \text{—— (1)}$$

Multiplying by $X(\omega)$, $\leftarrow$ Highpass.

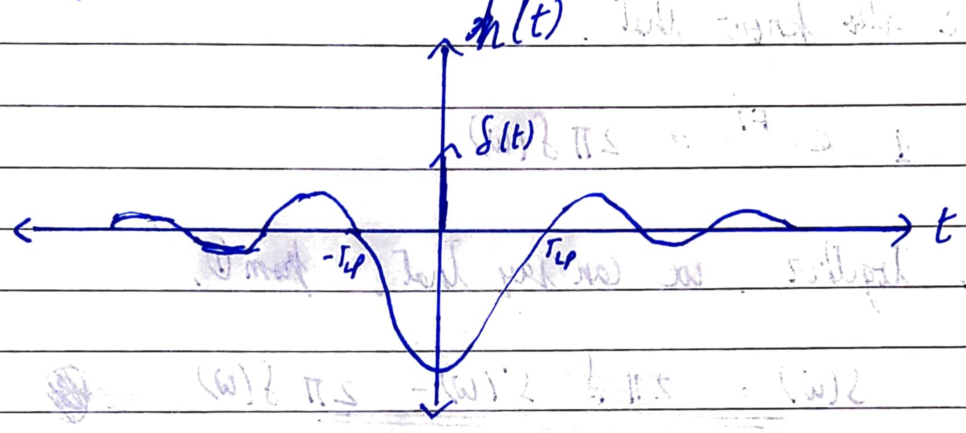
$$X(\omega) - X(\omega) H_{LPF}(\omega) = X(\omega) H_{HPF}(\omega)$$

$\therefore$ It acts as a high pass filter. Cutoff $\rightarrow W_{lp}$.

For the impulse response, from eqn ①,

$$1 - H_{LPF}(\omega) = H_{HPF}(\omega)$$
$$\updownarrow \quad \updownarrow \quad \updownarrow$$
$$\delta(t) - \frac{\sin(\omega_{lp} t)}{\pi t} = h_{HPF}(t)$$

Plotting the graph.



(b) Again, from equation ①, we can say that

$$1 - H_{HPF}(\omega) = H_{LPF}(\omega).$$

Multiplying by $X(\omega)$,

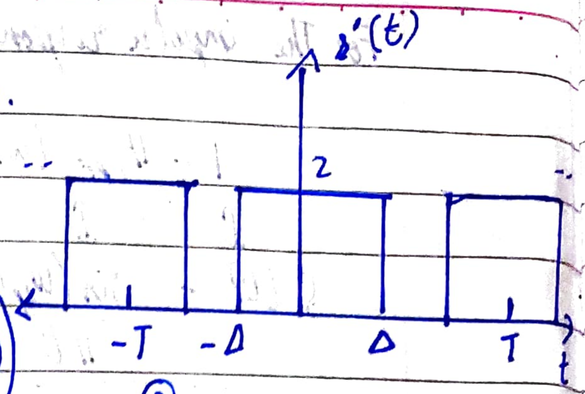
$$\underbrace{X(\omega) - X(\omega) H_{HPF}(\omega)}_{\text{what the whole system does}} = X(\omega) H_{LPF}(\omega) \quad \leftarrow \text{Lowpass.}$$

So, again, this acts as a lowpass filter with a cutoff frequency of ω_{lp}.

7.24 As it is easier to work with multiple rectangular pulses, we can shift $s(t)$ up by 1, and write is as,

$$s(t) = s'(t) - 1 \quad — ①$$
$$w(t) = s(t) \cdot x(t) \Rightarrow W(\omega) = S(\omega) * X(\omega) \frac{1}{2\pi} \quad — ②$$

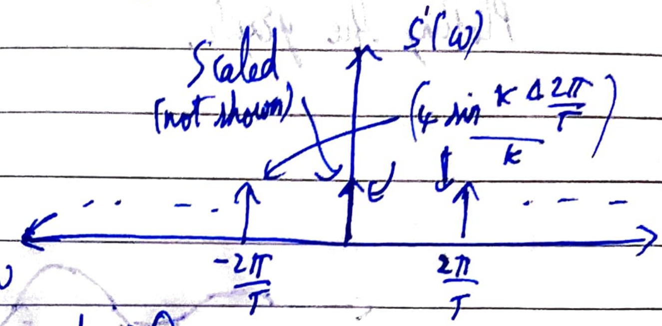
Now, it is easier to find the Fourier transform. By standard Fourier transforms, we can say that,



$$s'(\omega) = 2.\left(\sum_{k=-\infty}^{\infty} \frac{2\sin(k\Delta\, 2\pi/T)}{k}\,\delta\left(\omega - k\frac{2\pi}{T}\right)\right) \quad \text{—— ②}$$

We also know that,

$$1 \xleftrightarrow{FT} 2\pi\,\delta(\omega)$$



So, together, we can say that, from ①,

$$S(\omega) = \cancel{2\pi\delta}\; S'(\omega) - 2\pi\,\delta(\omega) \quad \text{—— ④}$$

So, we can conclude that $S(\omega)$ has scaled impulses every $2\pi/T$. Now,

(a) $\Delta = T/3$.

So, $$S(\omega) = \sum_{k=-\infty}^{\infty} \frac{4\sin k2\pi/3}{k}\,\delta\left(\omega - k\frac{2\pi}{T}\right) - 2\pi\,\delta(\omega)$$

Now, multiplying $x(t)$, we get $W(\omega)$ by convolving,

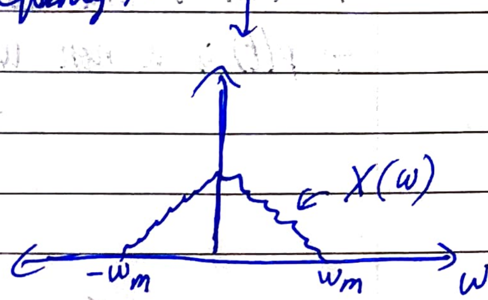
$$W(\omega) = \frac{1}{2\pi}\left(X(\omega) * S(\omega)\right) \quad (\because \text{eqn②})$$

$$= \frac{1}{2\pi}\left(\sum_{k=-\infty}^{\infty}\left(\frac{4\sin k2\pi/3}{k}\, X\left(\omega - \frac{k2\pi}{T}\right)\right) - 2\pi X(\omega)\right)$$

∴ This will form scaled replicas of $X(\omega)$ at $\frac{2\pi}{T}$ intervals.

So, as each copy is at $2\pi/T$ distance from the other, at minimum, $2W_m \leq \frac{2\pi}{T}$ (For no aliasing, assuming W_m is max frequency).

$\therefore T_{max} = \frac{\pi}{W_m}$



(b) $\Delta = T/4$.

So, $S(\omega) = \sum 4 \frac{\sin K\pi/2}{K} \delta\left(\omega - \frac{K2\pi}{T}\right) - 2\pi\delta(\omega)$

Here, as we can see, the terms with an even value of K become zero. So, we have impulses spaced $\frac{4\pi}{T}$ apart from each other in $S(\omega)$.

Now, $W(\omega)$ is,

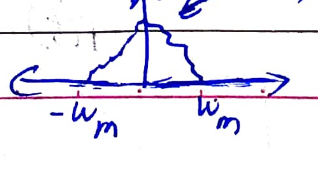
$$W(\omega) = \frac{1}{2\pi}\left(X(\omega) * S(\omega)\right) \quad \text{(eqn 2)}.$$

$$= \frac{1}{2\pi}\left(\sum_{K=-\infty}^{\infty}\left(4\frac{\sin K\pi/2}{K} X\left(\omega - \frac{K2\pi}{T}\right)\right) - 2\pi X(\omega)\right)$$

So this gives us scaled copies of $X(\omega)$ spaced $4\pi/T$ from each other. So, we can say that the minimum requirement for no aliasing will be,

$$2W_m \leq \frac{4\pi}{T} \quad \text{(Assuming } W_m \text{ is highest frequency).}$$

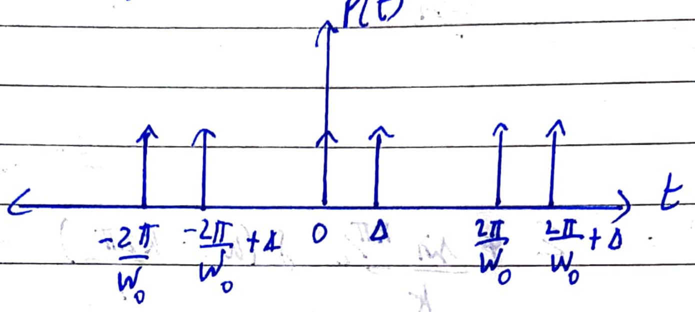
$$\rightarrow \boxed{T \leq \frac{2\pi}{W_m}}$$



7.37 Given in the question,

→ $x(t)$ is band limited, i.e. $X(\omega) = 0$ for $|\omega| > W_0$.

→ $p(t)$ is a non uniformly spaced pulse train.



→ $f(t)$ is periodic with $T = 2\pi/W_0$, where $f(0) = a$ and $f(\Delta) = b$.

→ $H_1(\omega)$ is a phase shifter.

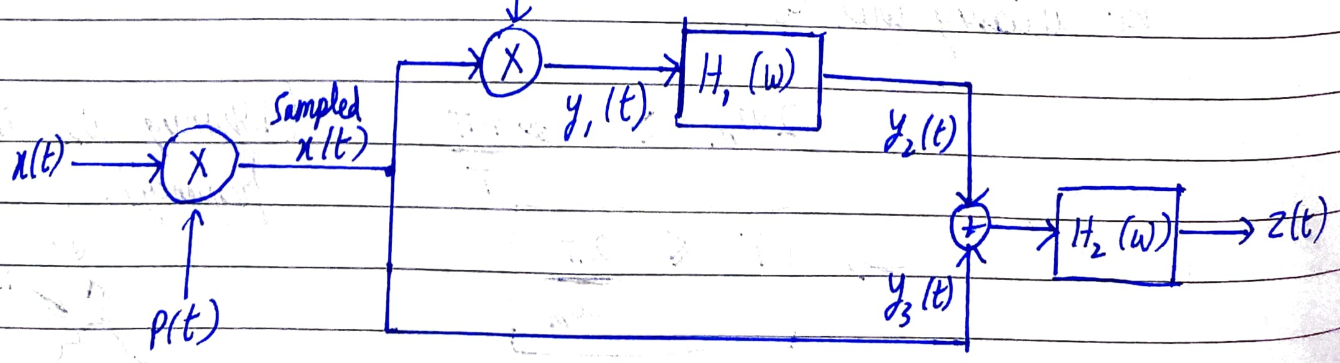
$$H_1(\omega) = \begin{cases} j & ; \quad \omega > 0 \\ -j & ; \quad \omega < 0 \end{cases}$$

For ease of checking, $W = W_0$

→ $H_2(\omega)$ is an ideal lowpass filter.

$$H_2(\omega) = \begin{cases} k & 0 < \omega < W_0 \\ k^* & -W_0 < \omega < 0 \\ 0 & |\omega| > W_0 \end{cases}$$

System



(a) $p(t)$ can be represented as the sum of ~~two~~ two impulse trains.

$$p(t) = \sum_{k=-\infty}^{\infty} \delta\left(t - k\frac{2\pi}{W_0}\right) + \delta\left(t - \frac{k2\pi}{W_0} - \Delta\right)$$

$$\therefore \text{FT of } \sum \delta\left(t - \frac{k2\pi}{W_0}\right) = \frac{2\pi}{2\pi/W_0} \sum \delta\left(W - \frac{k2\pi}{2\pi/W_0}\right)$$

$$= W_0 \sum_{k=-\infty}^{\infty} \delta(W - kW_0)$$

Similarly,

$$\text{FT of } \sum \delta\left(t - \frac{k2\pi}{W_0} - \Delta\right) = W_0 \sum_{k=-\infty}^{\infty} e^{-jkW\Delta} \delta(W - kW_0)$$

So,

$$\text{FT of } p(t) = W_0 \sum_{k=-\infty}^{\infty} \left(1 + e^{-jkW\Delta}\right) \delta(W - kW_0)$$

Now,

$$y_1(t) = x(t) \cdot p(t) \cdot g(t) = x(t) \cdot m(t).$$

First, $\underbrace{p(t) \cdot g(t)}_{m(t)} =$

$$\sum_{k=-\infty}^{\infty} a\,\delta\left(t - k\frac{2\pi}{W_0}\right) + b\,\delta\left(t - k\frac{2\pi}{W_0} - \Delta\right)$$

$$\Updownarrow \text{FT}$$

$$M(W) = W_0 \left(\sum_{k=-\infty}^{\infty} \left(a + b e^{-jkW\Delta}\right) \delta(\omega - kW_0)\right)$$

So,

$$y_1(t) = x(t) \cdot m(t) \Rightarrow Y_1(\omega) = \frac{1}{2\pi}\left(X(\omega) * M(\omega)\right)$$

PTO

$$\Rightarrow Y_1(w) = \frac{W_0}{2\pi}\left(\sum_{k=-\infty}^{\infty}\left(a+be^{-jkW_0\Delta}\right)X(w-kW_0)\right)$$

Now, $y_2(t) = y_1(t) * h_1(t)$

$$\Rightarrow Y_2(w) = Y_1(w)\,.\,H_1(w)$$

$$= \begin{cases} \dfrac{jw}{2\pi}\left(\displaystyle\sum_{k=-\infty}^{\infty}\left(a+be^{-jkW_0\Delta}\right)X(w-kW_0)\right) ; & w>0 \\ \dfrac{-jw}{2\pi}\left(\displaystyle\sum_{k=-\infty}^{\infty}\left(a+be^{-jkW_0\Delta}\right)X(w-kW_0)\right) ; & w<0 \end{cases}$$

$$y_3(t) = x(t).\,p(t)$$

$$= \left(X(w) * P(w)\right)\frac{1}{2\pi}$$

$$= \underline{\underline{\frac{W_0}{2\pi}\left(\sum_{k=-\infty}^{\infty}\left(1+e^{-jkW_0\Delta}\right)X(w-kW_0)\right)}}$$

(b) $z(t) = \left(y_2(t)+y_3(t)\right) * h_2(t)$

$$\Rightarrow Z(w) = Y_2(w).\,H_2(w) + Y_3(w).\,H_2(w)$$

$$\therefore Z(w) = \begin{cases} \dfrac{kjW_0}{2\pi}\left((a+b)X(w) + \left(a+be^{-jW_0\Delta}\right)X(w-W_0)\right) \\ \quad + \dfrac{kW_0}{2\pi}\left(2X(w) + \left(1+e^{-jW_0\Delta}\right)X(w-W_0)\right) & \text{for } 0<w<W \\ & \text{(opened the summation)} \\ \dfrac{-k^* jW_0}{2\pi}\left((a+b)X(w) + \left(a+be^{jW_0\Delta}\right)X(w+W_0)\right) \\ \quad + \dfrac{k^* W_0}{2\pi}\left(2X(w) + \left(1+e^{jW_0\Delta}\right)X(w+W_0)\right) & \text{for } -W<w<0. \end{cases}$$

On simplifying, we get.

$$z(w) = \begin{cases} \dfrac{kW_0}{2\pi}\left[(2+aj+bj)X(w) + \left(1+e^{-jw_0\Delta}+ja+be^{-jw_0\Delta}j\right)X(w-W_0)\right] & \text{for } 0<w<W_0 \\[2ex] \dfrac{k^* W_0}{2\pi}\left[(2-aj-bj)X(w) + \left(1+e^{jw_0\Delta}-aj-bje^{jw_0\Delta}\right)X(w+W_0)\right] & \text{for } -W_0<w<0 \end{cases}$$

Now, we need $z(w) = X(w)$.

For $0 < w < W_0$.

$$\frac{kW_0}{2\pi}(2+aj+bj) = 1 \quad \text{(and)} \quad 1+e^{-jW_0\Delta}+ja+jbe^{-jW_0\Delta} = 0$$

$$\Downarrow$$

$$1+\cos(W_0\Delta) - j\sin(W_0\Delta) + ja + jb\cos(W_0\Delta) + b\sin(W_0\Delta) = 0$$

$$k = \frac{1}{(2+aj+bj)} \times \frac{2\pi}{W_0}$$

$\therefore$ Making real part zero and imaginary part 0, we get,

$$b = \frac{-1-\cos(W_0\Delta)}{\sin(W_0\Delta)}$$

$$a = \sin(W_0\Delta) - b\cos(W_0\Delta)$$

For $-W_0 < w < 0$,

$$\frac{k^* W_0}{2\pi}(2-aj-bj) = 1 \quad \text{(and)} \quad 1+e^{jW_0\Delta}-aj-bje^{jW_0\Delta} = 0$$

$$\Downarrow$$

$$k^* = \frac{1}{(2-aj-bj)} \cdot \frac{2\pi}{W_0}$$

PTO

$$1 + e^{jw_0\Delta} - aj - bje^{jw_0\Delta} = 0$$

$$\Rightarrow 1 + \underset{\cos(w_0\Delta) + j\sin(w_0\Delta)}{\underbrace{e^{jw_0\Delta}}} - aj - bj\left(\cos(w_0\Delta) + j\sin(w_0\Delta)\right) = 0$$

$$= 1 + \cos(w_0\Delta) + j\sin(w_0\Delta) - aj - bj\cos(w_0\Delta) + b\sin(w_0\Delta) = 0$$

Making real and imaginary parts zero, we get

$$b = \frac{-1 - \cos(w_0\Delta)}{\sin(w_0\Delta)} \quad ; \quad a = -b\cos(w_0\Delta) + \sin(w_0\Delta)$$

$$= \sin(w_0\Delta) + \frac{(1 + \cos w_0\Delta)}{\tan(w_0\Delta)}$$

$\therefore$ For $z(t) = x(t)$,

$$K = \frac{2\pi}{(2 + aj + bj)w_0} \quad ; \quad a = \sin(w_0\Delta) + \frac{(1 + \cos w_0\Delta)}{\tan(w_0\Delta)} \quad ; \quad b = \frac{-1 - \cos(w_0\Delta)}{\sin(w_0\Delta)}$$

7.39.

$$x(t) = \cos\left(\frac{w_s}{2}t + \phi\right)$$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT) \quad ; \quad T = 2\pi/w_s$$

(a) $$x(t) = \left(\cos\left(\frac{w_s t}{2} + \phi\right)\right)$$

$$= \cos\frac{w_s t}{2}\cos\phi - \underbrace{\sin\frac{w_s t}{2}\sin\phi}_{g(t)}$$

So, $g(t) = -\sin \frac{\omega_s t}{2} \sin\phi$.

(b) $g(nT) = 0$. for $n \in \mathbb{Z}$.

$$g(nT) = -\sin \frac{\omega_s nT}{2} \sin\phi$$

$$= -\sin \frac{n}{2} \omega_s \cdot \frac{2\pi}{\omega_s} \sin\phi$$

$$= -\underbrace{\sin n\pi}_{0} \sin\phi$$

$$= \underline{\underline{0}}$$

(c) Lowpass filter with $\omega_c = \omega_s/2$.

$$x(t) = \frac{e^{j\left(\frac{\omega_s t}{2}+\phi\right)} + e^{-j\left(\frac{\omega_s t}{2}+\phi\right)}}{2} + g(t)$$ ← will become zero.

$$\Big\updownarrow FT$$

$$X(\omega) = \frac{e^{j\phi}}{2}\left(2\pi\delta\left(\omega - \omega_s/2\right)\right) + \frac{e^{-j\phi}}{2}\left(2\pi\delta\left(\omega + \omega_s/2\right)\right) + \cancel{G(\omega)}^{\,0}.$$

as for all nT that we require it will be zero.

$$\therefore X_p(\omega) = \frac{1}{2\pi}\left(X(\omega) * \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)\right)$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

As we have a low pass filter, all values in the summation except $n=0$ get cut.

$$\therefore X_{pf}(\omega) = \frac{1}{T} X(\omega) + \frac{1}{T} X(\omega - \omega_s) + \frac{1}{T} X(\omega + \omega_s)$$
(unscaled)

$$= \frac{1}{T}\left(2\pi \frac{\left(e^{j\phi}\delta(\omega - \omega_s/2) + e^{-j\phi}\delta(\omega + \omega_s/2)\right)}{2}\right)$$

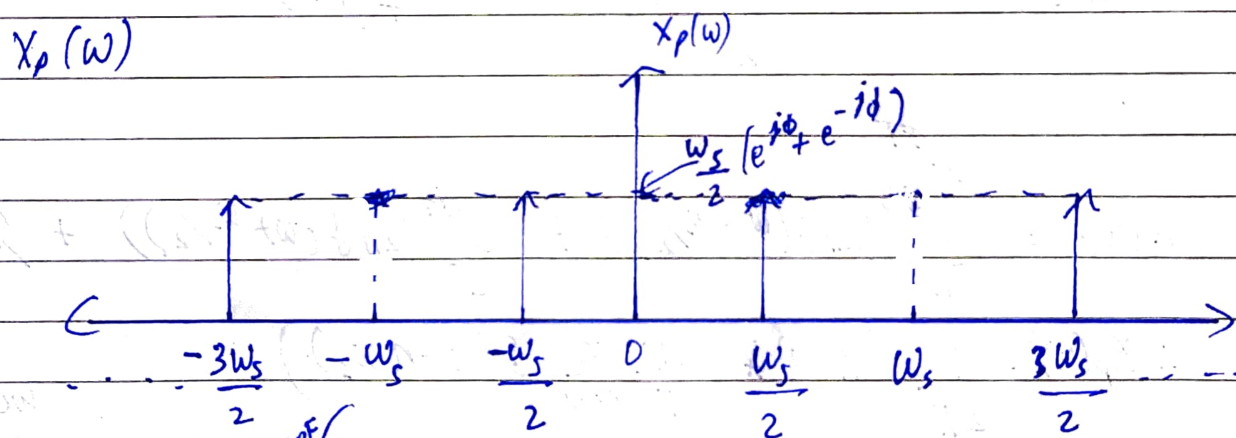
$$+ \frac{1}{T}\left(\cancel{2\pi e^{j\phi}}\, 2\pi \frac{\left(0 + e^{-j\phi}\delta(\omega - \omega_s/2)\right)}{2}\right)$$

$$+ \frac{1}{T}\left(2\pi \frac{\left(e^{j\phi}\delta(\omega + \omega_s/2) + 0\right)}{2}\right)$$

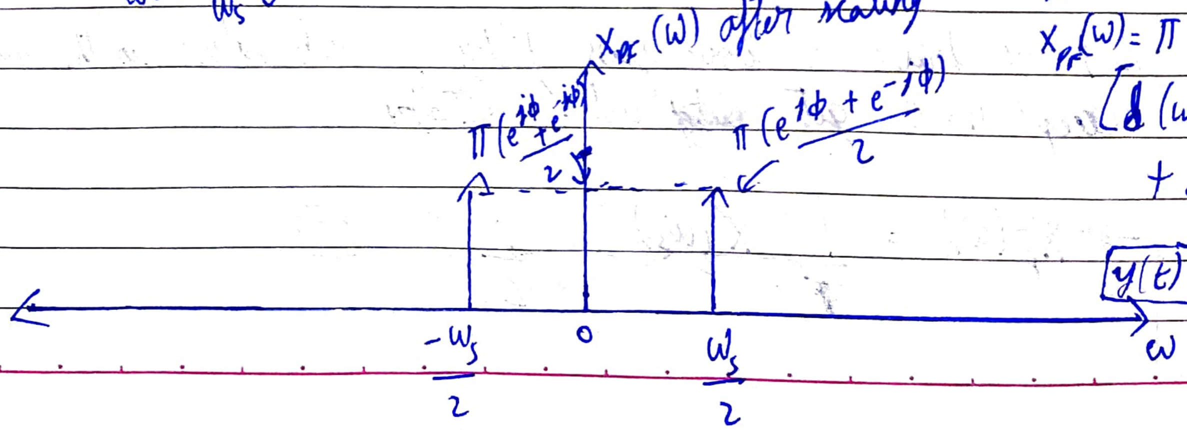
$$= \boxed{\frac{2\pi}{T}\left(\frac{e^{j\phi} + e^{-j\phi}}{2}\left(\delta(\omega - \omega_s/2) + \delta(\omega + \omega_s/2)\right)\right)}$$

Now this on ~~putting~~ scaling will give us the required output with inverse FT $= \cos\phi \cdot \cos\frac{\omega_s t}{2}$

Graphically,



Ideal LPF with $\frac{\pi}{\omega_s}$



$X_{pf}(\omega) = \pi\cos\phi\left[\delta(\omega - \omega_s/2) + \delta(\omega + \omega_s/2)\right]$

$\boxed{y(t) = \cos\phi \cdot \cos\frac{\omega_s t}{2}}$