Lecture - 22

Monday, July 12, 2021 10:12 AM

DFT

* properties of DFT

* circular convolution in matrix form (circulant matrices)

* linear convolution using circular convolution (by zero-padding)

(1) $y[n] = x[n] * h[n]$ (required)

$x[n], h[n]$ - length N signals

* linear convolution complexity : $(2N-1)N$ multiplications

Using circular convolution property

$$(2)\begin{cases} x[n] \longrightarrow \tilde{x}[n] \xleftrightarrow{DFT} \tilde{X}[k] \\ h[n] \longrightarrow \tilde{h}[n] \xleftrightarrow{DFT} \tilde{H}[k] \end{cases} \longrightarrow \tilde{X}[k]\tilde{H}[k] \xleftrightarrow{DFT} \tilde{x}[n] \circledast \tilde{h}[n] = y[n]$$

zero-padding

multiply

circ. conv. property

No direct implementation of circular convolution, implicit in property

claim : process (2) is faster than process (1) by efficient computation of DFT

* direct DFT computation

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \quad , \quad k = 0,1,\cdots N-1$$

* complex multiplications : $N(N) = N^2$
* complex additions : $N(N-1) = N^2 - N$

$\Big\}\ O(N^2)$

1 complex multiplication $\equiv$ 4 real multiplications (+ 2 real addition)

$(a+jb)(c+jd) = (ac-bd) + j(bc+ad)$

1 complex addition $\equiv$ 2 real additions

⊛ Fast Fourier transform (FFT)

★ collection of algorithms to efficiently compute DFT

★ $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$ ★ , $k = 0, 1, \cdots N-1$

$W_N = e^{-j\frac{2\pi}{N}}$ phase factor / twiddle factor

★ properties / observations about W_N

① $W_N^0 = 1$ ② $W_N^{(p+N)} = W_N^p W_N^N = W_N^p$

③ $W_N^{kN} = 1$ k, p – any integers.

N – even

④ $W_N^{2p} = e^{-j\frac{2\pi}{N}2p} = e^{-j\frac{2\pi}{(N/2)}p} = W_{N/2}^p$

⑤ $W_N^{(p+N/2)} = W_N^p W_N^{N/2} = W_N^p \left(e^{-j\frac{2\pi}{N}\cdot\frac{N}{2}}\right) = W_N^p \left(e^{-j\pi}\right) = -W_N^p$

⊛ Radix-2 FFT

assume $N = 2^m$

★ decimation in time algorithm ★

★ $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$, $k = 0, 1 \cdots N-1$

$$= \sum_{n\text{-even}} x[n] W_N^{kn} + \sum_{n\text{-odd}} x[n] W_N^{kn}$$

$n = 2r$ $n = 2r+1$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2r] W_N^{2kr} + \sum_{r=0}^{N/2-1} x[2r+1] \underbrace{W_N^{k(2r+1)}}_{\to\, W_N^k W_N^{2kr}}$$

$$= \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r] W_{N/2}^{kr}}_{\frac{N}{2}\text{ point DFT of } x[2r] \;\equiv\; G_1[k]} + W_N^k \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r+1] W_{N/2}^{kr}}_{\frac{N}{2}\text{ point DFT of } x[2r+1] \;\equiv\; G_2[k]}$$

$$X[k]\begin{cases} X[k] = G_1[k] + W_N^{k}\, G_2[k] & k = 0, 1, \cdots \frac{N}{2}-1 \\ X[k+\frac{N}{2}] = G_1[k+\frac{N}{2}] + W_N^{k+\frac{N}{2}}\, G_2[k+\frac{N}{2}] & k = 0, 1, \cdots \frac{N}{2}-1 \\ \quad = G_1[k] - W_N^{k}\, G_2[k] & k = 0, 1, \cdots \frac{N}{2}-1 \end{cases}$$

★ computations (complex multiplications)

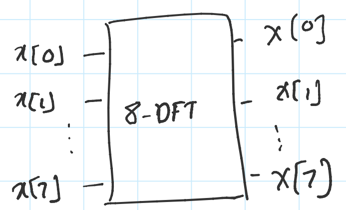
$$G_1 : \frac{N}{2}\left(\frac{N}{2}\right) = \frac{N^2}{4} \quad \& \quad G_2 : \frac{N^2}{4}$$

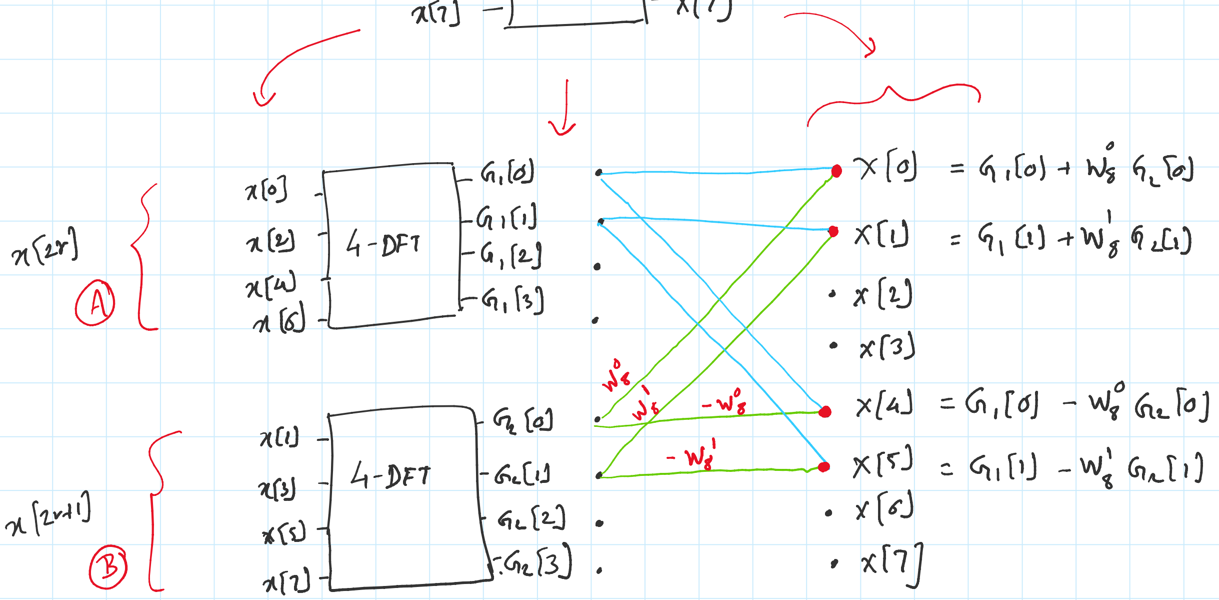
$$X[k] : \underbrace{\frac{N^2}{4}}_{G_1} + \underbrace{\frac{N^2}{4}}_{G_2} + \underbrace{\frac{N}{2}}_{\text{to combine } G_1 \& G_2} = \frac{N^2+N}{2} \quad \text{\# cmplx. multiplications.}$$

★ we further compute $\frac{N}{2}$ point DFT using decimation in time

to get $\frac{N}{4}$ point DFTs and so on ....

★ lets look at 8-point DFT example

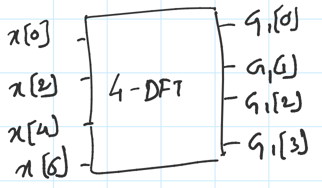




Butterfly diagram for FFT

* Block A above can be further expressed as follows :

A



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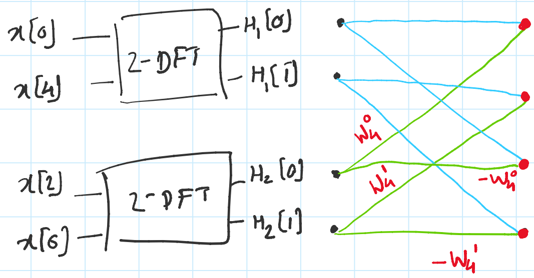


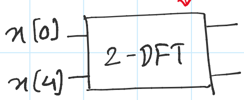
$G_1[0] = H_1[0] + W_4^0 H_2[0]$

$G_1[1] = H_1[1] + W_4^1 H_2[1]$

$G_1[2] = H_1[0] - W_4^0 H_2[0]$

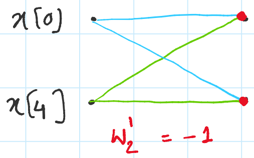
$G_1[3] = H_1[1] - W_4^1 H_2[1]$

2 point DFT block – very easy to implement



$H_1[0] = x[0] + x[4]$

$H_1[1] = x[0] - x[4]$



$H_1[0] = x[0] + x[4]$

$H_1[1] = x[0] - x[4]$