Camlin Page
Date / /

Assignment - 4

I affirm that I have neither given nor received help or used any means which would make this assignment unfair

- L. Lakshmanan
(2020112024)

Q 4.18

(a) $x_1(n) = \{1, 1, \underset{\uparrow}{1}, 1, 1\}$

$$X_1(\omega) = \sum_{-\infty}^{\infty} x_1(n) e^{-j\omega n}$$
$$= e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega}$$
$$= 2\cos 2\omega + 2\cos\omega + 1$$

(b) $x_2(n) = \{1, 0, 1, 0, \underset{\uparrow}{1}, 0, 1, 0, 1\}$

$$X_2(\omega) = \sum_{-\infty}^{\infty} x_2(n) e^{-j\omega n}$$

$$= e^{j4\omega} + e^{j2\omega} + 1 + e^{-j2\omega} + e^{-j4\omega}$$
$$= 2\cos 4\omega + 2\cos 2\omega + 1$$

(c) $x_3(n) = \{1, 0, 0, 1, 0, 0, \underset{\uparrow}{1}, 0, 0, 1, 0, 0, 1\}$

$$\Rightarrow X_3(\omega) = \sum_{-\infty}^{\infty} x_3(n) e^{-j\omega n}$$

$$= e^{j6\omega} + e^{j3\omega} + 1 + e^{-j3\omega} + e^{-j6\omega}$$
$$= 2\cos 6\omega + 2\cos 3\omega + 1$$

(d) As seen, $X_2(\omega) = X_1(2\omega)$ and $X_3(\omega) = X_1(3\omega)$. The structure of these graphs are the same.

(e) $x_k(n) = \begin{cases} x(n/k) & ; \; n/k \in Z \\ 0 & ; \; \text{otherwise} \end{cases}$

$$X_k(\omega) = \sum_n x_k(n)\, e^{-j\omega n}$$

$$= \sum_{n;\frac{n}{k}\in Z} x_k(n)\, e^{-j\omega n}$$

$$= \sum_{n;\frac{n}{k}\in Z} x(n)\, e^{-j\omega k n} \qquad \left(\because x_k(n) = x(n/k)\right) \rightarrow x_k(kn) = x(n)$$

$$= \underline{X(k\omega)}$$

Q 7.13

$$x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$$

$$x_p(n) \xleftrightarrow[3N]{DFT} X_3(k)$$

(a) $X_1(k) = \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi nk}{N}}$

$$X_3(k) = \sum_{n=0}^{3N-1} x_p(n)\, e^{-j\frac{2\pi kn}{3N}}$$

$$= \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi kn}{3N}} + \sum_{n=N}^{2N-1} x_p(n)\, e^{-j\frac{2\pi kn}{3N}} + \sum_{n=2N}^{3N-1} x_p(n)\, e^{-j\frac{2\pi kn}{3N}}$$

$$= X_1(k/3) + e^{-j\frac{2\pi k}{3}} X_1(k/3) + e^{-j\frac{2\times 2\pi k}{3}} X_1(k/3)$$

$$= \underline{X_1(k/3)\left(1 + e^{-j2\pi k/3} + e^{-j4\pi k/3}\right)}$$

(b) $x_p(n) = \{\ldots, 1, \underset{\uparrow}{2}, 1, 2, \ldots\}$

$N=2$, ($\because$ Sequence is periodic with $N=2$).

$$\therefore X_1(k) = \sum_{n=0}^{1} x_p(n)\, e^{\frac{-2\pi j n k}{N}}$$

$$= 2 + e^{-j\pi k}$$

$$X_2(k) = \sum_{n=0}^{5} x_p(n)\, e^{\frac{-2\pi j n k}{N}}$$

$$= 2 + e^{-j\frac{2\pi k}{6}} + 2e^{-j\frac{4\pi k}{6}} + e^{-j\pi k} + 2e^{-j\frac{8\pi k}{6}} + e^{-j\frac{10\pi k}{6}}$$

$$= \left(2 + (e^{-j\pi})^{k/3}\right) + e^{-j\frac{4\pi k}{6}}\left(2 + (e^{-j\pi})^{k/3}\right) + e^{-j\frac{8\pi k}{6}}\left(2 + (e^{-j\pi})^{k/3}\right)$$

$$= \left(2 + e^{-j\pi k/3}\right)\left(1 + e^{-j\frac{2\pi k}{3}} + e^{-j\frac{4\pi k}{3}}\right)$$

$$= \underline{X_1(k/3)}\left(1 + e^{-j2\pi k/3} + e^{-j4\pi k/3}\right)$$

$\therefore$ Proved.

7.16

$$h(n) = \delta(n) - \frac{1}{4}\delta(n-k_0)$$

$N = 4k_0$

$$G(k) = \frac{1}{H(k)}$$

Now, $H(k)$ is the N point DFT of $h(n)$, which is

$$H(k) = \sum_{n=0}^{N-1} h(n)\, e^{-j\frac{2\pi k n}{N}}$$

$$= 1 + \left(-\frac{1}{4}\right) e^{-j\frac{2\pi k k_0}{4k_0}}$$

$$= 1 - \frac{1}{4} e^{-j\frac{\pi k}{2}}$$

$$G(k) = \frac{1}{H(k)}$$

$$= \frac{1}{1 - \frac{1}{4} e^{-j\frac{\pi}{2}k}}$$

This can be represented as the sum of an infinite G.P.

$$G(k) = 1 + \frac{1}{4} e^{-j\frac{\pi}{2}k} + \left(\frac{1}{4} e^{-j\frac{\pi}{2}k}\right)^2 - - - -$$

Now, the sequence for $G(k)$ is,

$$G(k) = \left\{ \frac{1}{1 - \frac{1}{4} e^{-j\frac{\pi}{2}k}} \right\}$$

$$= \underbrace{\left\{ \frac{4}{3}, \frac{16-4j}{17}, \frac{4}{5}, \frac{16+6j}{17} \ldots \right\}}_{N \text{ length}}$$

$$g(n) = \frac{1}{4K_0} \sum_{k=0}^{4K_0 - 1} G(k)\, e^{j\left(\frac{2\pi k n}{4K_0}\right)}$$

$$= \frac{1}{4K_0} \left(\frac{4}{3} \sum_{k|k=4p} e^{j\left(\frac{\pi k n}{2K_0}\right)} + \frac{16-4j}{17} \sum_{k|k=4p+1} e^{j\frac{\pi k n}{2K_0}} + \frac{4}{5} \sum_{k|k=4p+2} e^{\frac{j\pi k n}{2K_0}} + \frac{16+4j}{17} \sum_{k|k=4p+3} e^{\frac{j\pi k n}{2K_0}} \right)$$

Now, the summation can be ~~[illegible]~~ written as S for simplicity,

$$= \frac{1}{4K_0} \left(\frac{4}{3} S + \frac{16-4j}{17} e^{j\frac{\pi}{2K_0} n} \cdot S + \frac{4}{5} e^{j\pi \frac{n}{K_0}} + \frac{16+4j}{17} e^{j\frac{3\pi}{2K_0} n} \cdot S \right)$$

with $S = \sum_{p=0}^{K_0 - 1} e^{j\frac{2\pi n p}{K_0}}$

Now, S can be simplified as the sum of a GP.

$$S = \frac{\left(e^{\frac{2\pi n}{K_0}}\right)^{K_0} - 1}{e^{\frac{2\pi n}{K_0}} - 1}$$

As we can clearly see, this expression is K_0 for all $n = mK_0$. In our case, as n is in $[0, 4K_0)$, we have 0, K_0, $2K_0$ and $3K_0$ to satisfy this. $S = 0$ everywhere else.

$$\therefore g(n) = 0 \quad \forall n \neq mK_0$$

$$g(0) = \frac{1}{4}\left(\frac{4}{3} + \frac{16-4j}{17} + \frac{4}{5} + \frac{16+4j}{17}\right)$$

$$= \frac{256}{255}$$

$$g(K_0) = \frac{1}{4}\left(\frac{4}{3} + j\left(\frac{16-4j}{17}\right) - \frac{4}{5} - j\left(\frac{16+4j}{17}\right)\right)$$

$$= \frac{64}{255}$$

$$g(2K_0) = \frac{1}{4}\left(\frac{4}{3} - \left(\frac{16-4j}{17}\right) + \frac{4}{5} - \left(\frac{16+4j}{17}\right)\right)$$

$$= \frac{16}{255}$$

$$g(3K_0) = \frac{1}{4}\left(\frac{4}{3} - j\left(\frac{16-4j}{17}\right) - \frac{4}{5} + j\left(\frac{16+4j}{17}\right)\right)$$

$$= \frac{4}{255}$$

Camlin Page
Date / /

Now, if we convolve them,

$$y(n) = h(n) * g(n)$$
$$= \left(\delta(n) - \frac{1}{4}\delta(n-k_0)\right) * g(n)$$

$$= g(n) - \frac{1}{4} g(n-k_0)$$

So, this will only have non zero values at $n = 0, k_0, 2k_0$ and $3k_0$ (Circular convolution, $N = 4k_0$).

$$y(0) = g(0) - \frac{1}{4} g(-k_0) = y(4k_0) \quad \text{(circular)}$$

$$= g(0) - \frac{1}{4} g(3k_0) = \frac{256}{255} - \frac{1}{4} \times \frac{4}{255} = \underline{1}$$

$$y(k_0) = g(k_0) - \frac{1}{4} g(0) = \frac{64}{255} - \frac{1}{4} \times \frac{256}{255} = \underline{0}$$

$$y(2k_0) = g(2k_0) - \frac{1}{4} g(k_0) = \frac{16}{255} - \frac{1}{4} \times \frac{64}{255} = \underline{0}$$

$$y(3k_0) = g(3k_0) - \frac{1}{4} g(2k_0) = \frac{4}{255} - \frac{1}{4} \times \frac{16}{255} = \underline{0}$$

So, it is not a perfect inverse, as the IDFT has impulses after every $4k_0$ gap.
~~(This y(k) is also periodic, so this pattern repeats)~~
If we consider only the interval from 0 to $4k_0$, it is an inverse.

7.26

$$\underbrace{\sum_{l=-\infty}^{\infty} \delta(n + lN)}_{x(n)} = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi kn}{N}} \quad \text{(To prove)}$$

As given in the hint, if we take DFT of the LHS, we get

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk} \quad \text{(N point DFT)}$$

$$= 1 + 0 = \underline{1} \quad \text{(Impulse train with period N)}$$

$\therefore X(k) = 1$ ($\because$ only term at 0 will be one as period = N).

Now, if we try finding the IDFT of X(k),

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}nk}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}nk}$$

$\therefore$ As $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}nk}$, we have proved the statement.

$$\therefore \sum_{l=-\infty}^{\infty} \delta(n+lN) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}nk}$$

7.30

$$x(n) = \cos 2\pi f_1 n + \cos 2\pi f_2 n$$

$$f_1 = \frac{1}{18};\quad f_2 = \frac{5}{128};\quad f_c = \frac{50}{128}$$

$$x_c(n) = \cos 2\pi f_c n$$
$$x_{am}(n) = x(n) \cos 2\pi f_c n$$

(a), (b), (c) and (d) are plots.

(e) As $x_{am}(n) = x(n)\cos 2\pi f_c n$, its DFT can be calculated as,

$$X_{am}(k) = \sum_{n=0}^{N-1} x(n) \cos(2\pi f_c n)\, e^{-\frac{2\pi k}{N} n}$$

PTO

Changing $k/N = f$; $2\pi f = \omega$;

$$\Rightarrow \cancel{X_{am}(k)}$$

$$X_{am}(\omega) = \sum_{n=0}^{N-1} \frac{x(n)}{2} \left(e^{-j2\pi(f-f_c)n} + e^{-j2\pi(f+f_c)n}\right)$$

$$\left[\because \cos(2\pi f_c n) = \frac{e^{-j2\pi f_c n} + e^{j2\pi f_c n}}{2}\right)$$

$$\Rightarrow X_{am}(\omega) = \frac{1}{2}\left(\sum_{n=0}^{N-1} x(n) e^{-j(\omega-\omega_c)n} + \sum_{n=0}^{N-1} x(n) e^{-j(\omega+\omega_c)n}\right)$$

$$= \frac{1}{2}\left(X(\omega-\omega_c) + X(\omega+\omega_c)\right)$$

8.18

$$X'(k) = \begin{cases} X(k) & 0 \le k \le k_0 - 1 \\ 0 & k_0 \le k \le LN - k_0 \\ X(k+N-LN) & LN-k_0+1 \le k \le LN-1 \end{cases}$$

$$x'(n) \xleftrightarrow[LN]{DFT} X'(k)$$

$$x(n) \xleftrightarrow{DFT} X(k)$$

To prove :- $\cancel{Lx'(Ln)}$ $Lx'(Ln) = x(n)$; $0 \le n \le N-1$

Now, $x'(n)$ is,

$$x'(n) = \frac{1}{LN} \sum_{k=0}^{LN-1} X'(k)\, e^{\frac{j2\pi kn}{LN}}$$

$$= \frac{1}{LN}\left(\sum_{k=0}^{k_0-1} X(k)\, e^{\frac{j2\pi kn}{LN}} + 0 + \sum_{k=LN-k_0+1}^{LN-1} X(k+N-LN)\, e^{\frac{j2\pi kn}{LN}}\right)$$

This can be rewritten as,

PTO

$$x'(n) = \frac{1}{LN}\left(\sum_{k=0}^{K_0-1} X(k)\, e^{\frac{j2\pi kn}{LN}} + \sum_{k=N-K_0+1}^{N-1} X(k)\, e^{\frac{j2\pi(LN-N+k)n}{LN}}\right)$$

$$= \frac{1}{LN}\left(\sum_{k=0}^{K_0-1} X(k)\, e^{\frac{j2\pi kn}{LN}} + \sum_{k=N-K_0+1}^{N-1} X(k)\, \underbrace{\cancel{e^{j2\pi n}}}_{1} \cdot e^{\frac{j2\pi(k-N)n}{LN}}\right)$$

Now,

$$x'(Ln) = \frac{1}{LN}\left(\sum_{k=0}^{K_0-1} X(k)\, e^{\frac{j2\pi kn}{N}} + \sum_{k=N-K_0+1}^{N-1} X(k)\, e^{\frac{j2\pi(k-N)n}{N}}\right)$$

$$= \frac{1}{LN}\left(\sum_{k=0}^{K_0-1} X(k)\, e^{\frac{j2\pi kn}{N}} + \sum_{k=N-K_0+1}^{N-1} X(k)\, \underbrace{\cancel{e^{-j2\pi n}}}_{1}\, e^{\frac{j2\pi kn}{N}}\right)$$

$$= \frac{1}{LN}\left(\sum_{k=0}^{K_0-1} X(k)\, e^{\frac{j2\pi kn}{N}} + \sum_{k=N-K_0+1}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}}\right)$$

$$\therefore L\, x'(Ln) = \frac{1}{N}\left(\sum_{k=0}^{K_0-1} X(k)\, e^{\frac{j2\pi kn}{N}} + \sum_{k=N-K_0+1}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}}\right)$$

$$= \underline{\underline{x(n)}}$$ (The middle Terms are 0, so resulting sum is the same).

Now, when $N=4$, $L=1$, $X(k)=\{1,0,0,1\}$, no zeroes are inserted.
So, $x'(n) = x(n)$.

$\therefore$ This gives us $x(n) = x'(n) = \frac{1}{4}\left(\sum_{k=0}^{3} X(k)\, e^{\frac{j2\pi kn}{4}}\right)$

$$= \frac{1}{4}\left(1 + e^{j\frac{3\pi}{2}n}\right)$$

This gives us,

$$x(n) = x'(n) = \left\{\frac{1}{2},\ \frac{1}{4}(1-j),\ 0,\ \frac{1}{4}(1+j)\right\}$$

This padding of zeroes helps us get closer to the shape of the continuous time domain signal.