Tutorial 2 - Problem 1

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Problem 1 :

Die Roll → $\underbrace{\{1,2,3,4\}}_{P_{D_1}}$ ⟶ Coin toss

Die Roll → $\underbrace{\{5,6\}}_{P_{D_2}}$ ⟶ Two coin tosses

let X be the random variable whose value is the number of observed heads

$\therefore$ $\text{Range}(X) \triangleq R_X = \{0,1,2\}$

let D be the random variable whose value is the observed value of the die roll

$\therefore$ $\text{Range}(D) \triangleq R_D = \{1,2,3,4,5,6\}$

$P(X=x, D=d) = P(D=d) \cdot P(X=x \mid D=d)$

X \ D	1	2	3	4	5	6
0	1/12	1/12	1/12	1/12	1/24	1/24
1	1/12	1/12	1/12	1/12	1/12	1/12
2	0	0	0	0	1/24	1/24

{ Joint Entropy of the outcome of the die throw and the number of heads obtained

$$H(X,D) = \sum_{x \in Supp(P_X)} \sum_{d \in Supp(P_D)} p(x,d) \log \frac{1}{p(x,d)}$$

$$= \cancel{10}^{5} \cdot \frac{1}{\cancel{12}\,6} \cdot \log 12 + \cancel{4} \cdot \frac{1}{\cancel{24}\,6} \log 24$$

$$= \frac{5}{6} \log 12 + \frac{1}{6}\left(\log 12 + \cancel{\log}\, 2\right)$$

$$= \log 12 + \frac{1}{6} = \boxed{\log 3 + \frac{13}{6}}$$

{ Average Uncertainty in the number of heads if the outcome of the die is known

→ $H(X \mid D)$

$H(X \mid D) = H(X,D) - H(D)$

$$H(D) = \sum_{d \in R_D} P(D=d) \log \frac{1}{P(D=d)}$$

$$= 6 \cdot \frac{1}{6} \cdot \log 6 = \log 6$$

$$= 1 + \log 3$$

$$\therefore\ H(X \mid D) = \left(\frac{13}{6} + \cancel{\log 3}\right) - \left(1 + \cancel{\log 3}\right)$$

$$= \boxed{\frac{7}{6}}$$

{ Average Uncertainty in the number of heads

→ $H(X)$

Using total probability theorem

$$P(X=x) = \sum_{d \in R_D} P(X=x, D=d)$$

$$P(X=0) = \frac{1}{12}+\frac{1}{12}+\frac{1}{12}+\frac{1}{12}+\frac{1}{24}+\frac{1}{24} = \frac{5}{12}$$

$$P(X=1) = \frac{1}{12}+\frac{1}{12}+\frac{1}{12}+\frac{1}{12}+\frac{1}{12}+\frac{1}{12} = \frac{1}{2}$$

$$P(X=2) = 0+0+0+0+\frac{1}{24}+\frac{1}{24} = \frac{1}{12}$$

$$H(X) = \sum_{x \in R_X} P(X=x) \log \frac{1}{P(X=x)}$$

$$= \frac{5}{12} \log \frac{12}{5} + \frac{1}{2} \log 2 + \frac{1}{12} \log 12$$

$$= \frac{5}{12}\left(\log 12 - \log 5\right) + \frac{1}{2} + \frac{1}{12} \log 12$$

$$= \frac{1 + \log 12}{2} - \frac{5}{12} \log 5$$

$$= \boxed{\frac{3 + \log 3}{2} - \frac{5}{12} \log 5}$$

$$\simeq 1.325$$

$$H(X \mid D) = \frac{7}{6} \simeq 1.17 < 1.325$$

$$\therefore\ \boxed{H(X \mid D) < H(X)}$$