Assignment - 3

I affirm that I have neither given nor received help or used any means which would make this assignment unfair.

- L.Lakshmanan
(2020112024)

OWN

5.31

(a)

$$\cos \omega_0 n \xrightarrow{LTI} \omega_0 \cos \omega_0 n \quad ; \quad -\pi \leq \omega_0 \leq \pi$$

Impulse response $\longrightarrow h[n]$
Frequency response $\longrightarrow H(e^{j\omega})$

As seen, on passing through the LTI system, we get a scaled version of the same cos function. By taking DTFT of both sides, we can see that,

$$\pi \sum_{k=-\infty}^{\infty} [\delta(\omega-\omega_0-2\pi k)+\delta(\omega+\omega_0-2\pi k)] \xrightarrow{LTI} \omega_0 \pi \sum_{k=-\infty}^{\infty} [\delta(\omega-\omega_0-2\pi k)+\delta(\omega+\omega_0-2\pi k)]$$

By convolution property, we can say that.

$$X(e^{j\omega}) \xrightarrow{LTI} H(e^{j\omega})\, X(e^{j\omega})$$

So, on comparison, we can say that (from $-\pi$ to π), $H(e^{j\omega})$ is $|\omega|$, as it scales both the impulses in that range by ω_0, regardless of the sign.

$$\therefore H(e^{j\omega}) = |\omega|$$

(b) To obtain impulse response, we can take IDTFT of the frequency response.

Date / /

$$\therefore h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{0} -\omega\, e^{j\omega n} d\omega + \int_{0}^{\pi} \omega\, e^{j\omega n}\, d\omega \right]$$

$$= \frac{1}{2\pi} \left[\int_{0}^{\pi} \omega e^{-j\omega n} d\omega + \int_{0}^{\pi} \omega\, e^{j\omega n} d\omega \right]$$

$$= \frac{1}{2\pi} \left[\int_{0}^{\pi} 2\omega \cos(\omega n)\, d\omega \right]$$

$$= \frac{1}{\pi} \left[\omega \int \cos \omega n\, d\omega - \int\int \cos \omega n\, d\omega \right] \quad \text{(By parts)}$$

$$= \frac{1}{\pi} \left[\frac{\omega \sin \omega n}{n} + \frac{\cos \omega n}{n^2} \right]_{0}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{\cos(\pi n)}{n^2} - \frac{1}{n^2} \right]$$

5.40

$$y[n] = x[n] * h[n]$$

Using convolution sum,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$\rightarrow y[0] = \sum_{k=-\infty}^{\infty} x[k]\, h[-k] \quad \text{—— (1)}$$

Using convolution property,

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$$

Camlin Page
Date / /

$$\Rightarrow y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) H(e^{j\omega}) e^{j\omega n} d\omega$$

$$\Rightarrow y[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) H(e^{j\omega}) d\omega \quad \text{—— (2)}$$

Now, to ~~prove~~ derive Parseval's relation,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega.$$

As we can see, we need the modulus of the function squared. This points us towards using the conjugate.

As we need $H(e^{j\omega}) = X^*(e^{j\omega})$, we can take $h[n] = x^*[-n]$, because of the properties of FT.

Using this in (1) and (2), and equating, we get,

$$\sum_{k=-\infty}^{\infty} x[k] x^*[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) X^*(e^{j\omega}) d\omega$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} |x[k]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2$$

Now, if $h[n] = z^*[-n]$, we get $H(e^{j\omega}) = Z^*(e^{j\omega})$, which gives us,

$$\sum_{k=-\infty}^{\infty} x[k] z^*[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) Z^*(e^{j\omega}) d\omega$$

$\therefore$ Derived.

5.43 $x[n] \longleftrightarrow X(e^{j\omega})$

$g[n] = x[2n] \longleftrightarrow G(e^{j\omega})$

(a)

$$v[n] = \left(\left(e^{-j\pi n}x[n]\right) + x[n]\right)/2$$
$$= \frac{x[n] + (-1)^n x[n]}{2}$$

$$\therefore v[n] = \begin{cases} x[n] & ; n \text{ is even} \\ 0 & ; n \text{ is odd} \end{cases}$$ (Interpretation)

As DTFT is linear,

$$\underline{V(e^{j\omega}) = \frac{X(e^{j\omega}) + X(e^{j(\omega+\pi)})}{2}}$$

Because of frequency shift property.

$$\left(\because e^{+j\omega_0 n}x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})\right)$$

(b)

$v[2n] = y[n]$

$$\therefore Y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} v[2k]e^{-j\omega k}$$

(odd samples are zero anyway).

Since this gives us even samples of v, on changing $2k$ to p, we get,

$$Y(e^{j\omega}) = \sum_{p=-\infty}^{\infty} v[p]e^{-j\omega p/2}$$
$$= \underline{\underline{V(e^{j\omega/2})}}$$

$\therefore$ Proved.

(c) $x[2n] \rightarrow$ only even samples of $x[n]$.
$v[n] \rightarrow$ odd samples $= 0$
even samples $= x[n]$.

$\therefore v[2n]$ will also be only even samples of $x[n]$, as $2n$ is always even.

As seen in (b), $v[2n] \longleftrightarrow V(e^{j\omega/2})$

$$= \frac{X(e^{j\omega/2}) + X(e^{j(\omega/2+\pi)})}{2}$$

(From (a))

$\therefore$ As $v[2n] = x[2n]$,

$g[n] = v[2n]$

$$\Rightarrow g[n] \longleftrightarrow G(e^{j\omega}) = V(e^{j\omega/2})$$

$$= \frac{X(e^{j\omega/2}) + X(e^{j(\omega/2+\pi)})}{2}$$

PM

2.25 $x(n) = a^n u(n)$; $0 < a < 1$

(a) $$x(n) = \sum_{k=-\infty}^{\infty} C_k \, x(n-k)$$

$$= \sum_{k=-\infty}^{n} C_k \, a^{n-k} \qquad (\because u(n) = 0 \ \forall \ n < 0).$$

Now, as we need to find C_k, we can do it by subtracting all the other terms, that is,

$$x(k) = \sum_{j=-\infty}^{k} C_j \, a^{k-j} \quad \text{——— (1)}$$

Now, if we take $x(k-1)$,

$$x(k-1) = \sum_{j=-\infty}^{k-1} c_j \, a^{k-j-1}$$

Now, we need to scale $x(k-1)$ by a, as all c_j for $j<k$ have a^{k-j-1}, which will not make them zero on subtraction.

$$a\,x(k-1) = \sum_{j=-\infty}^{k-1} c_j \, a^{k-j} \quad \text{——} \quad (2)$$

Now, if we do (1) − (2), we get,

$$x(k) - a\,x(k-1) = \sum_{j=-\infty}^{k} c_j \, a^{k-j} - \sum_{j=-\infty}^{k-1} c_j \, a^{k-j}$$

$$= c_k \, a^0$$

$$= \boxed{c_k}$$

∴ As c_k exists, we can represent $x(n)$ as that summation, and c_k will be defined as,

$$\underline{c_k = x(k) - a\,x(k-1)}$$

(b) $y(n) = T[x(n)]$; $g(n) = T[\gamma(n)]$

$$= T\left[\sum_{k=-\infty}^{\infty} c_k \, \gamma(n-k)\right]$$

Using linearity,

$$y(n) = \sum_{k=-\infty}^{\infty} c_k \, T[\gamma(n-k)]$$

Using time invariance, we get,

$$y(n) = \sum_{k=-\infty}^{\infty} C_k \, g(n-k)$$

(c) $\delta(n)$, $h(n) = T[\delta(n)]$

As $x(n) = \sum_{k=-\infty}^{n} C_k \, \gamma(n-k)$, we can construct a delta function using $x(n)$.

As $\delta(n)$ in discrete time is defined as 1 at 0 and zero every-where else, we can obtain this by making finding C_k.
To find coefficients we use

$$C_k = x(k) - a\,x(k-1) \qquad (\text{from (a)}).$$

Here, as we need $\delta(n)$,

$$C_k = \delta(k) - a\,\delta(k-1)$$

Now substituting this into the y(n) expression we got for $T[x(n)]$,

$$y(n) = \sum_{k=-\infty}^{\infty} C_k \, g(n-k)$$

$$\Rightarrow h(n) = \sum_{k=-\infty}^{\infty} \big(\delta(k) - a\,\delta(k-1)\big)\, g(n-k)$$

PTO

4.23.

(a) $y_1(n) = \begin{cases} x(n) & ; n \text{ even} \\ 0 & ; n \text{ odd.} \end{cases}$

$\therefore$ $y_1(n)$ can be interpreted as all even samples of $x(n)$, and so, we can represent it as,

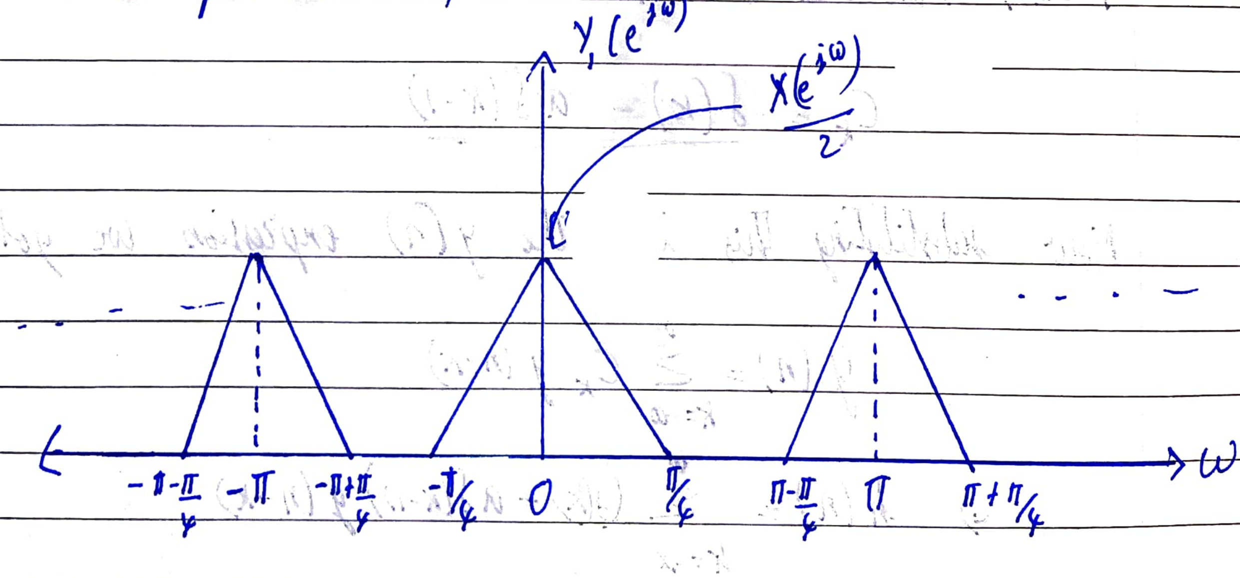
$$y_1(n) = \frac{x(n) + (-1)^n x(n)}{2}$$

$$= \frac{x(n) + e^{j\pi n} x(n)}{2}$$

If we take DTFT, we can use linearity and frequency shift properties to get,

$$Y_1(e^{j\omega}) = \frac{X(e^{j\omega}) + X(e^{j(\omega-\pi)})}{2}$$

$\therefore$ We can plot this as,



(b) $y_2(n) = x(2n)$

This will also only have even samples of $x(n)$.

PTO

Taking DTFT, we get,

$$Y_2(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[2k]\, e^{-j\omega k}$$

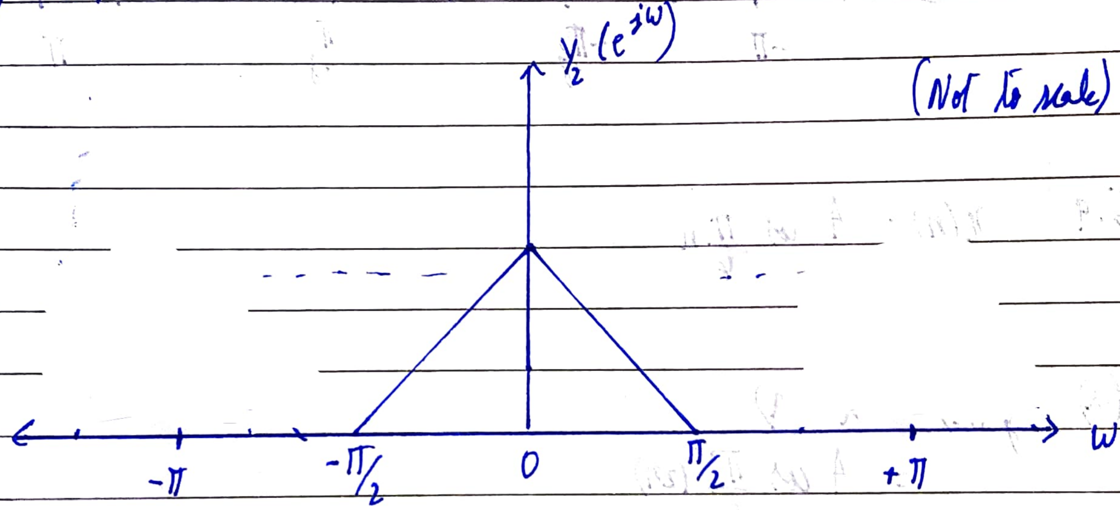
(assuming $x[P] = 0$ for odd P)

Substituting $2k = P$, we get,

$$Y_2(e^{j\omega}) = \sum_{P=-\infty}^{\infty} x[P]\, e^{-j\omega \frac{P}{2}}$$

$$= \underline{\underline{X(e^{j\frac{\omega}{2}})}}$$

Plotting it,



(c) $y_3(n) = \begin{cases} x(n/2) & ; n \text{ is even} \\ 0 & ; n \text{ odd.} \end{cases}$

As we can see, this definition of $y_3(n)$ will give us ALL samples of $x(n)$.

Taking DTFT,

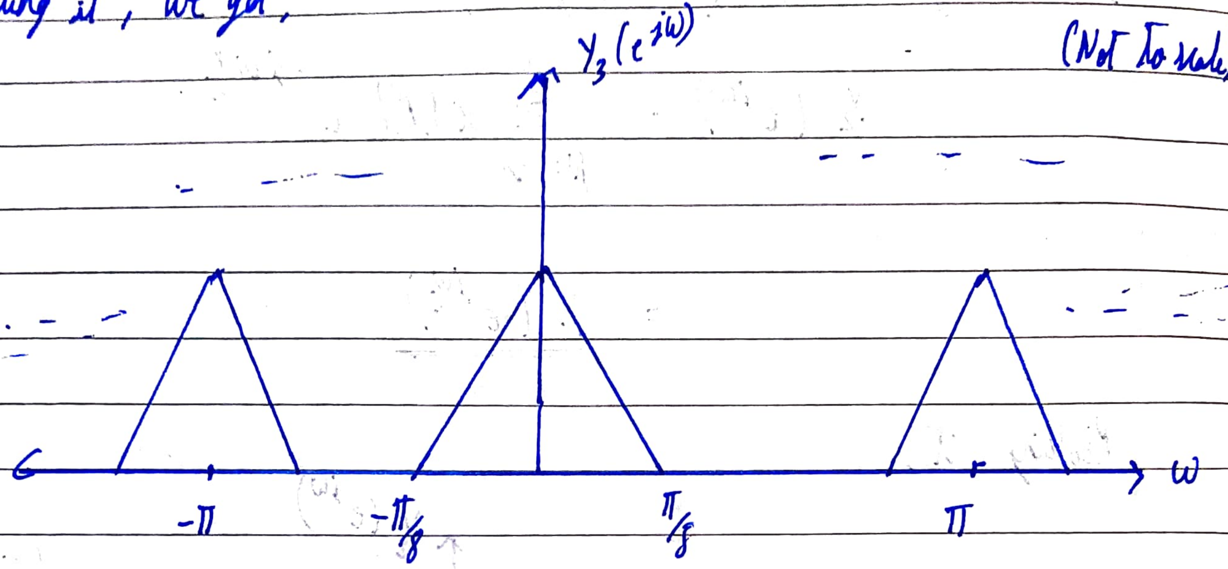
$$Y_3(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x(k/2)\, e^{-j\omega k} \quad ; k = \text{even}$$

Replacing ~~k/2 with P~~ $k/2 = P$, (because it still has all samples).

$$Y_3(e^{j\omega}) = \sum_{p=-\infty}^{\infty} x(p)\, e^{-2j\omega p}$$

$$= \underline{\underline{X(e^{2j\omega})}}$$

Plotting it, we get,



5.9 $\quad x(n) = A \cos \frac{\pi}{4} n$

(a) $\quad y(n) = x(2n)$

$$= A \cos \frac{\pi}{4}(2n)$$

$$= \underline{\underline{A \cos \frac{\pi}{2} n}}$$

To find frequency content, we take DTFT.

$$Y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} A \cos\left(\frac{\pi}{2} k\right) e^{j\omega k}$$

$$= A \sum_{k=-\infty}^{\infty} \cos \frac{\pi}{2} k \;\, e^{j\omega k}$$

DTFT of $\cos \omega_0 n$ formula.

$$= A \left(\pi \sum_{l=-\infty}^{\infty} \left[\delta(\omega - \pi/2 - 2\pi l) + \delta(\omega + \pi/2 - 2\pi l) \right] \right)$$

(b) $y(n) = x^2(n)$

$$= A^2 \cos^2 \frac{\pi}{4} n$$

$$= A^2 \frac{\left(1 + \cos \frac{\pi}{2} n\right)}{2}$$

$$= \frac{A^2}{2} + \frac{A^2}{2} \cos \frac{\pi}{2} n$$

DTFT

$$\underbrace{\frac{A^2}{2}\left[2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l)\right]}_{\text{DTFT of } x[n]=1 \text{ formula.}} + \underbrace{\frac{A^2}{2}\left[\pi \sum_{l=-\infty}^{\infty}\left[\delta(\omega - \pi/2 - 2\pi l) + \delta(\omega + \pi/2 - 2\pi l)\right]\right]}_{\text{DTFT of } \cos \omega_0 n \text{ formula.}}$$

$$\therefore Y(e^{j\omega}) = \frac{A^2 \pi}{2}\left(2 \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l) + \sum_{l=-\infty}^{\infty}\left[\delta(\omega - \pi/2 - 2\pi l) + \delta(\omega + \pi/2 - 2\pi l)\right]\right)$$

(c) $y(n) = \cos(\pi n)\, x(n)$

$$= A \cos(\pi n) \cos\left(\frac{\pi}{4} n\right)$$

$$= \frac{A}{2}\left(2 \cos(\pi n) \cos\left(\frac{\pi}{4} n\right)\right)$$

$\left(\because \cos C + \cos D = 2 \cos\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)\right)$

$$= \frac{A}{2}\left(\cos\left(\left(\pi + \frac{\pi}{4}\right) n\right) + \cos\left(\left(\pi - \frac{\pi}{4}\right) n\right)\right)$$

DTFT

DTFT of $\cos \omega_0 n$.

$$= \frac{A}{2}\left(\left(\pi \sum_{l=-\infty}^{\infty} \delta\left(\omega - \pi - \frac{\pi}{4} - 2\pi l\right) + \delta\left(\omega + \pi + \frac{\pi}{4} - 2\pi l\right)\right) + \left(\pi \sum_{l=-\infty}^{\infty} \delta\left(\omega - \pi + \frac{\pi}{4} - 2\pi l\right) + \delta\left(\omega + \pi - \frac{\pi}{4} - 2\pi l\right)\right)\right)$$

Camlin Page

Date / /

$\therefore Y(e^{j\omega}) =$

$$\frac{A}{2}\pi\left(\left(\sum_{l=-\infty}^{\infty}\delta\left(\omega-\pi-\frac{\pi}{4}-2\pi l\right)+\delta\left(\omega+\pi+\frac{\pi}{4}-2\pi l\right)\right)+\left(\sum_{l=-\infty}^{\infty}\delta\left(\omega-\pi+\frac{\pi}{4}-2\pi l\right)+\delta\left(\omega+\pi-\frac{\pi}{4}-2\pi l\right)\right)\right)$$

As clearly seen, none of these systems are LTI systems, as they are adding frequencies to the input.