ASSIGNMENT-II

INFORMATION & COMMUNICATION

1\.

1.14 Show that the Hamming distance satisfies the triangle inequality. That is, for three binary vectors **x**, **y**, and **z** of length n, show that

$$d_H(\mathbf{x}, \mathbf{z}) \leq d_H(\mathbf{x}, \mathbf{y}) + d_H(\mathbf{y}, \mathbf{z}).$$

$$d_H(\underline{X}, \underline{Y}) = \sum_{i=1}^{n} \mathbb{I}(x_i \neq y_i)$$

Indicator function

$\underline{X} = x_1 x_2 \ldots x_n$

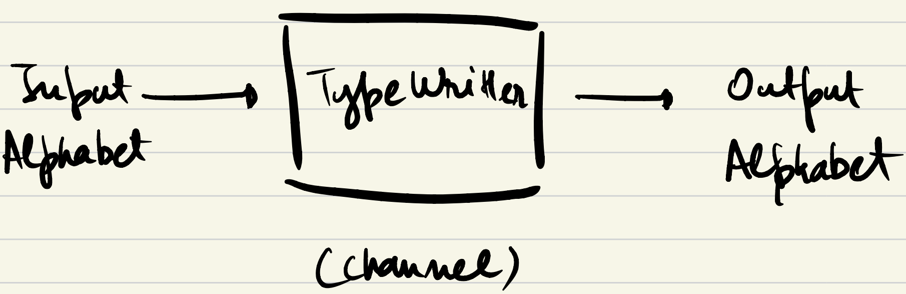
$\underline{Y} = y_1 y_2 \ldots y_n$

2\.

7.6 *Noisy typewriter.* Consider a 26-key typewriter.

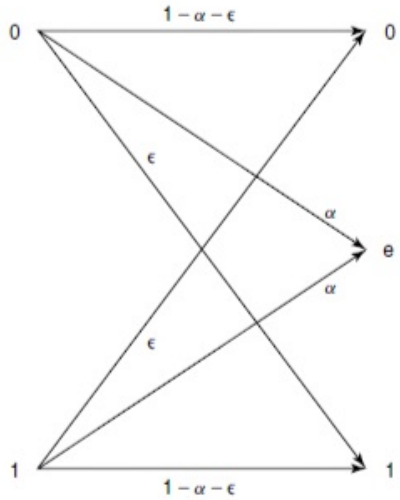
(a) If pushing a key results in printing the associated letter, what is the capacity C in bits?

(b) Now suppose that pushing a key results in printing that letter or the next (with equal probability). Thus, $A \to A$ or $B, \ldots, Z \to Z$ or A. What is the capacity?



3.

7.13 *Erasures and errors in a binary channel.* Consider a channel with binary inputs that has both erasures and errors. Let the probability of error be ϵ and the probability of erasure be α, so the channel is follows:



(a) Find the capacity of this channel.

(b) Specialize to the case of the binary symmetric channel ($\alpha = 0$).

(c) Specialize to the case of the binary erasure channel ($\epsilon = 0$).

4. **7.19** *Capacity of the carrier pigeon channel.* Consider a commander of an army besieged in a fort for whom the only means of communication to his allies is a set of carrier pigeons. Assume that each carrier pigeon can carry one letter (8 bits), that pigeons are released once every 5 minutes, and that each pigeon takes exactly 3 minutes to reach its destination.

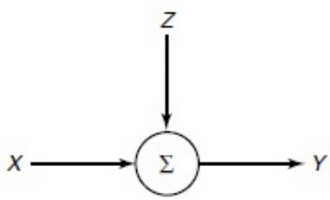
(a) Assuming that all the pigeons reach safely, what is the capacity of this link in bits/hour?

(b) Now assume that the enemies try to shoot down the pigeons and that they manage to hit a fraction α of them. Since the pigeons are sent at a constant rate, the receiver knows when the pigeons are missing. What is the capacity of this link?

(c) Now assume that the enemy is more cunning and that every time they shoot down a pigeon, they send out a dummy pigeon carrying a random letter (chosen uniformly from all 8-bit letters). What is the capacity of this link in bits/hour?

Set up an appropriate model for the channel in each of the above cases, and indicate how to go about finding the capacity.

5. **7.24** *Noise alphabets.* Consider the channel



$\mathcal{X} = \{0, 1, 2, 3\}$, where $Y = X + Z$, and Z is uniformly distributed over three distinct integer values $\mathcal{Z} = \{z_1, z_2, z_3\}$.

(a) What is the maximum capacity over all choices of the $\mathcal{Z}$ alphabet? Give distinct integer values z_1, z_2, z_3 and a distribution on $\mathcal{X}$ achieving this.

(b) What is the minimum capacity over all choices for the $\mathcal{Z}$ alphabet? Give distinct integer values z_1, z_2, z_3 and a distribution on $\mathcal{X}$ achieving this.

6.

7.14 *Channels with dependence between the letters.* Consider the following channel over a binary alphabet that takes in 2-bit symbols and produces a 2-bit output, as determined by the following mapping: $00 \to 01$, $01 \to 10$, $10 \to 11$, and $11 \to 00$. Thus, if the 2-bit sequence 01 is the input to the channel, the output is 10 with probability 1. Let X_1, X_2 denote the two input symbols and Y_1, Y_2 denote the corresponding output symbols.

(a) Calculate the mutual information $I(X_1, X_2; Y_1, Y_2)$ as a function of the input distribution on the four possible pairs of inputs.

(b) Show that the capacity of a pair of transmissions on this channel is 2 bits.

(c) Show that under the maximizing input distribution, $I(X_1; Y_1) = 0$. Thus, the distribution on the input sequences that achieves capacity does not necessarily maximize the mutual information between individual symbols and their corresponding outputs.

7.

Exercise 10.18: Alice wants to send Bob the result of a fair coin flip over a binary symmetric channel that flips each bit with probability $p < 1/2$. To avoid errors in transmission, she encodes heads as a sequence of $2k + 1$ zeroes and tails as a sequence of $2k + 1$ ones.

(a) Consider the case where $k = 1$, so heads is encoded as 000 and tails as 111. For each of the eight possible sequences of 3 bits that can be received, determine the probability that Alice flipped a heads conditioned on Bob receiving that sequence.

(b) Bob decodes by examining the 3 bits. If two or three of the bits are 0, then Bob decides the corresponding coin flip was a heads. Prove that this rule minimizes the probability of error for each flip.

(c) Argue that, for general k, Bob minimized the probability of error by deciding the flip was heads if at least $k + 1$ of the bits are 0.

(d) Give a formula for the probability that Bob makes an error that holds for general k. Evaluate the formula for $p = 0.1$ and k ranging from 1 to 6.

8. **5.42** *Ternary codes*. Which of the following codeword lengths can be the word lengths of a 3-ary Huffman code, and which cannot?

(a) (1, 2, 2, 2, 2)

(b) (2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3)

9. **5.43** *Piecewise Huffman*. Suppose the codeword that we use to describe a random variable $X \sim p(x)$ always starts with a symbol chosen from the set $\{A, B, C\}$, followed by binary digits $\{0, 1\}$. Thus, we have a ternary code for the first symbol and binary thereafter. Give the optimal uniquely decodable code (minimum expected number of symbols) for the probability distribution

$$p = \left(\frac{16}{69}, \frac{15}{69}, \frac{12}{69}, \frac{10}{69}, \frac{8}{69}, \frac{8}{69}\right). \qquad (5.160)$$