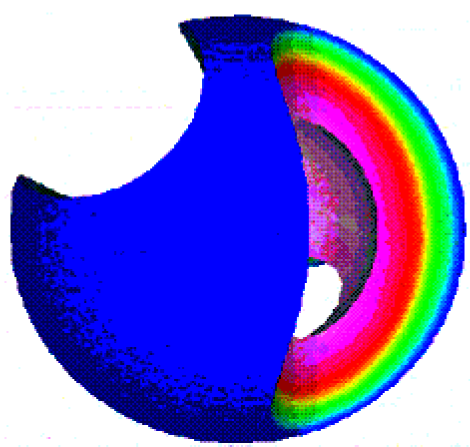
Homework 12

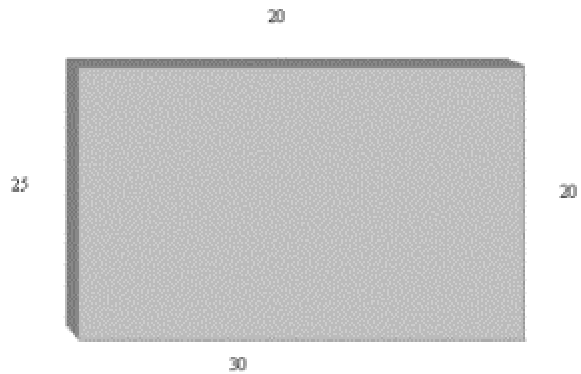
Read the application to temperature distribution below. Create a 3 x 3 grid on the dam and determine the 9 equations that will solve for the temperature distribution. Your homework should have the following.

1. Layout of the grid
2. Equations in the form of the average for each point
3. Equations in the form for **A**x = **B**
4. The matrices **A**, x and **B**.
5. The solution for the 9 temperatures from octave.

Application to Temperature Distribution



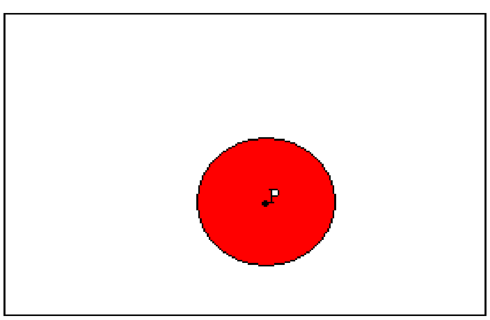
Introduction Consider the cross section of a long rectangular dam on a river. As you can imagine, the boundaries of the dam are subject to three factors: the temperature of the air, the temperature of the water, and the temperature of the ground at its base. The following diagram represents this situation:



where the number represent the temperatures (in degree Celsius) of the boundaries. Engineers are interested in knowing the temperature distribution inside the dam in a specific period of time so they can determine the thermal stress to which the dam is subjected. Assuming the boundary temperatures are held constant during that specific period of time, the temperature inside the dam will reach certain equilibrium after some time has passed. Finding this equilibrium temperature distribution at different points on the plate (the dam) is desirable, but extremely difficult. However, one can consider a few points on the plate and **approximate** the temperature of these points. This approximation is based on a very important physical property called the **Mean-Value Property:**

If a plate has reached a thermal equilibrium, and P is a point on the plate, C is a circle centered at P and fully contained in the plate, then the temperature at P is the average value of the temperature function over C.

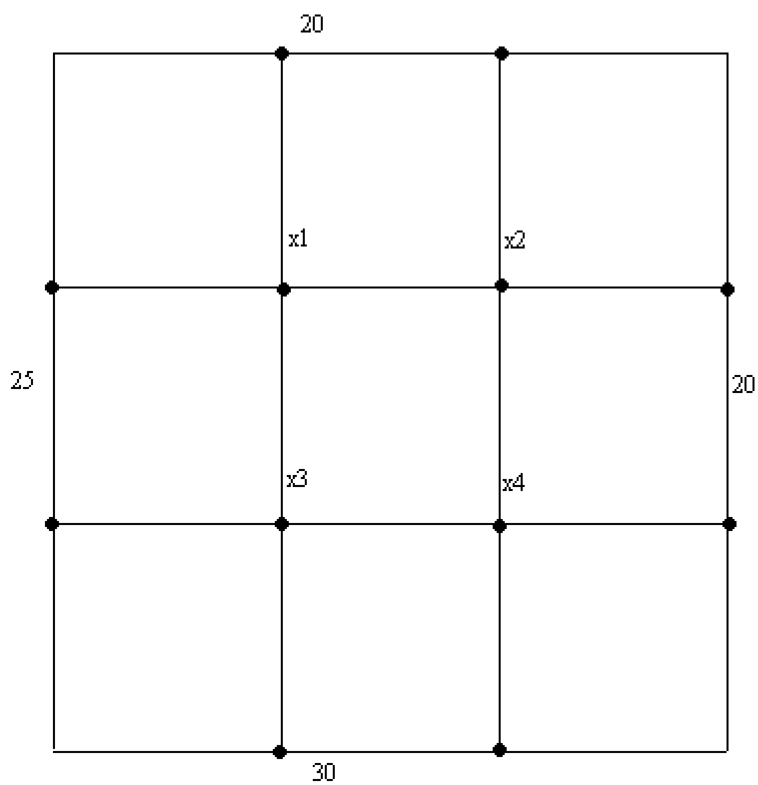
The following diagram illustrates this property:



To see how the property works, place a grid over the plate (the cross section of the bridge in our case) and consider the points with the lines of the grid meet. We will be interested in the temperatures at these points only in the plate. Design the grid in such a way that some of the points considered lie on the boundary of the plate. Studying the temperature at these grid points requires the following practical version of the Mean-Value Property:

If a plate has reached a thermal equilibrium and P is a grid point not on the boundary of the plate, then the temperature at P is the average of the temperatures of the four closest grid points to P.

Let us start with a grid with four interior points, and let x_1, x_2, x_3, x_4 be the temperatures at these four points. The situation is illustrated in the following diagram:



By the second version of the Mean-Value Property, we have the following system of linear equations:

$$\begin{aligned}
x_1 &= \frac{20+25+x_2+x_3}{4} \\
x_2 &= \frac{20+20+x_1+x_4}{4} \\
x_3 &= \frac{25+30+x_1+x_4}{4} \\
x_4 &= \frac{20+30+x_2+x_3}{4}
\end{aligned}$$

which after simplification gives:

$$\begin{aligned}
4x_1 - x_2 - x_3 &= 45 \\
-x_1 + 4x_2 - x_4 &= 40 \\
-x_1 + 4x_3 - x_4 &= 55 \\
-x_2 - x_3 + 4x_4 &= 50
\end{aligned}$$

The matrix form of the system is *AX=b*, where

$$A = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad b = \begin{bmatrix} 45 \\ 40 \\ 55 \\ 50 \end{bmatrix}$$

X is called the **vector of equilibrium temperatures.** The solution for the above system is then

$$X = A^{-1}b$$

provided, of course, that *A* is invertible. Using the methods you learned in your first linear algebra course, you can compute the inverse of a square matrix. In our case, we find:

$$A^{-1} = \begin{bmatrix} 7/24 & 1/12 & 1/12 & 1/24 \\ 1/12 & 7/24 & 1/24 & 1/12 \\ 1/12 & 1/24 & 7/24 & 1/12 \\ 1/24 & 1/12 & 1/12 & 7/24 \end{bmatrix}$$

and the vector of equilibrium temperatures is

$$X = A^{-1}b = \begin{bmatrix} 23.125 \\ 21.875 \\ 25.625 \\ 24.375 \end{bmatrix}$$

These answers are in degrees Celsius.