Machine Learning with Python

MTH786U/P 2020/21

Estimating the height of a person

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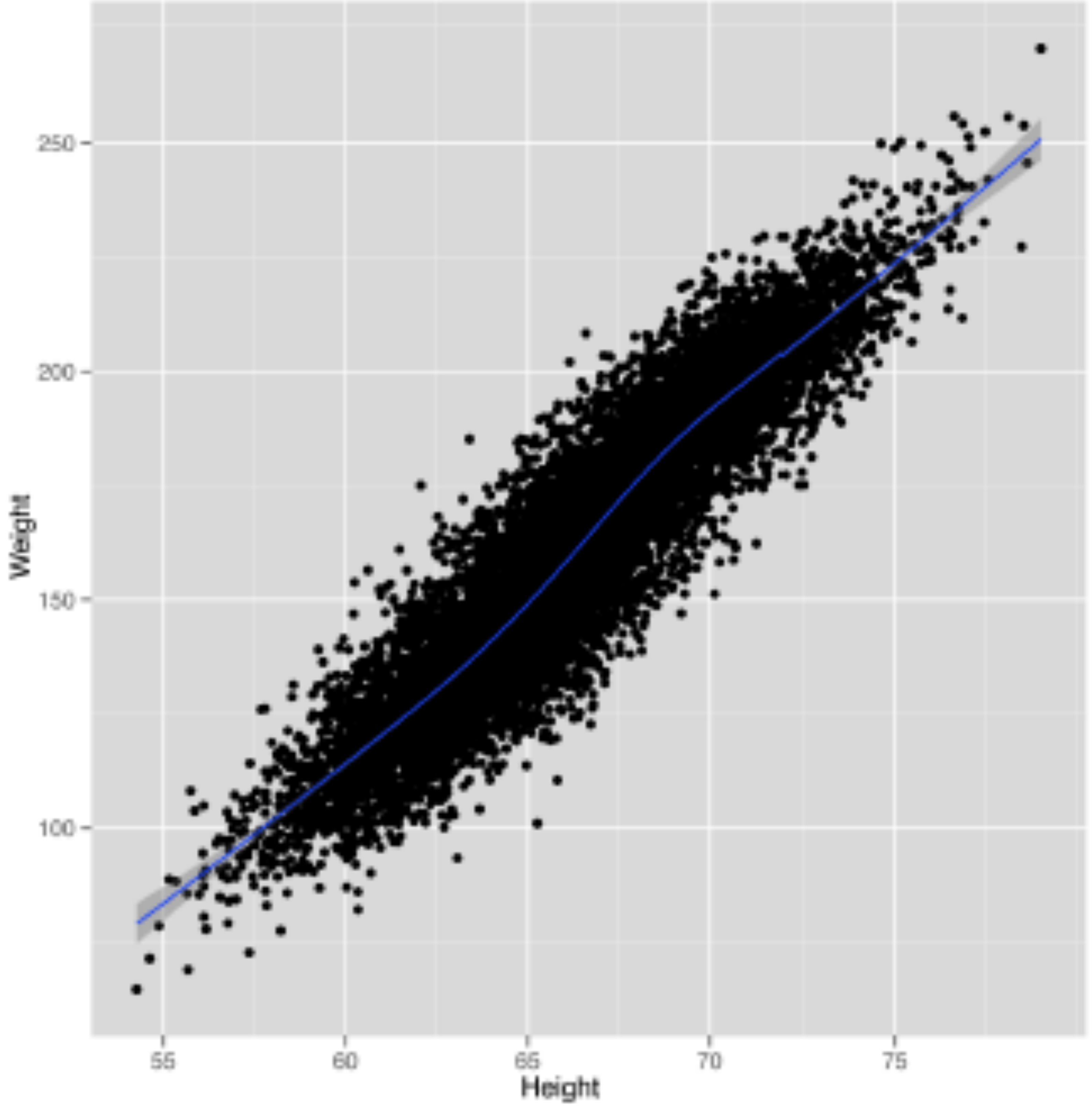
What is regression?

Examples:



From “Machine Learning for Hackers” by Conway & White

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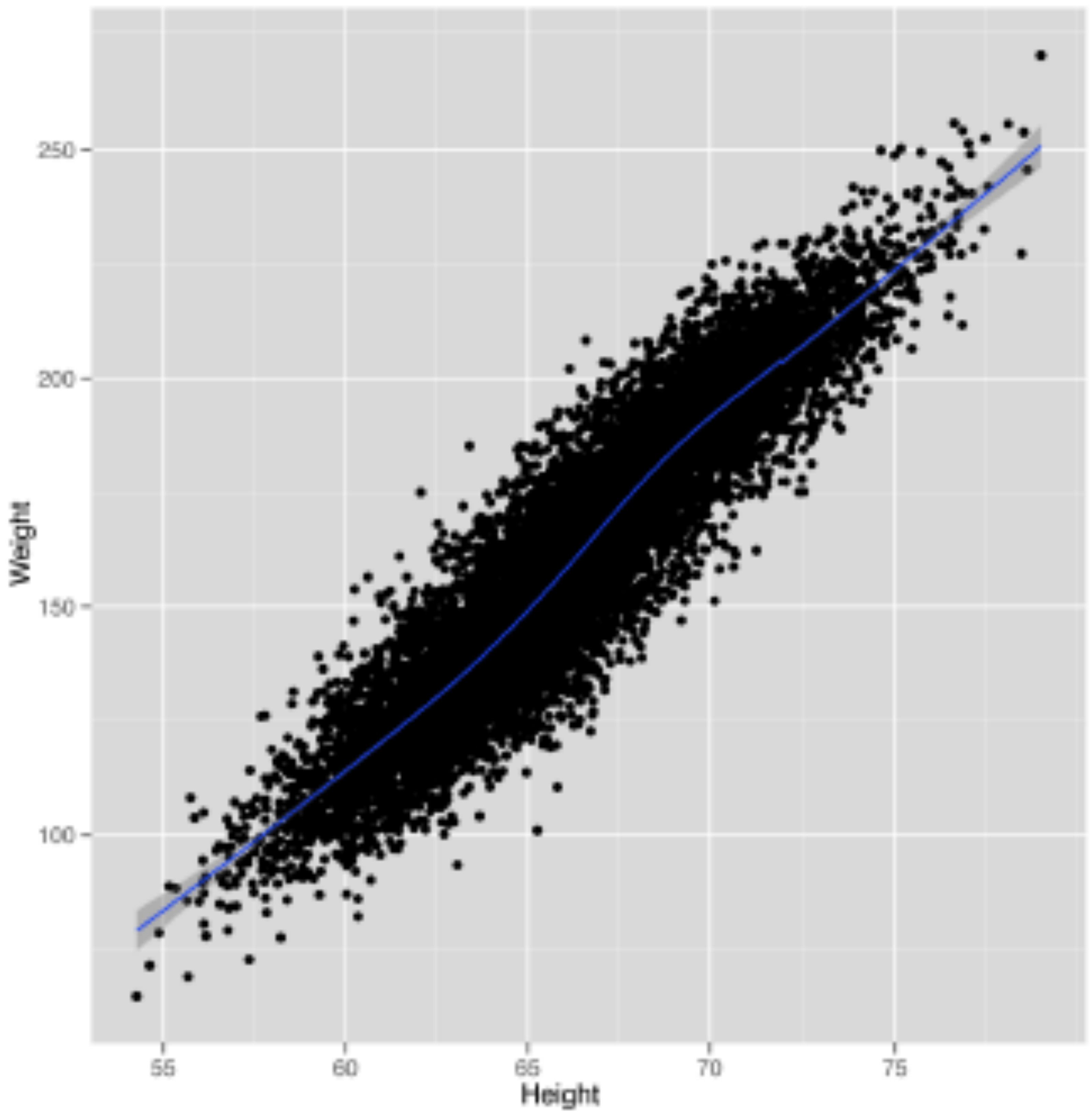

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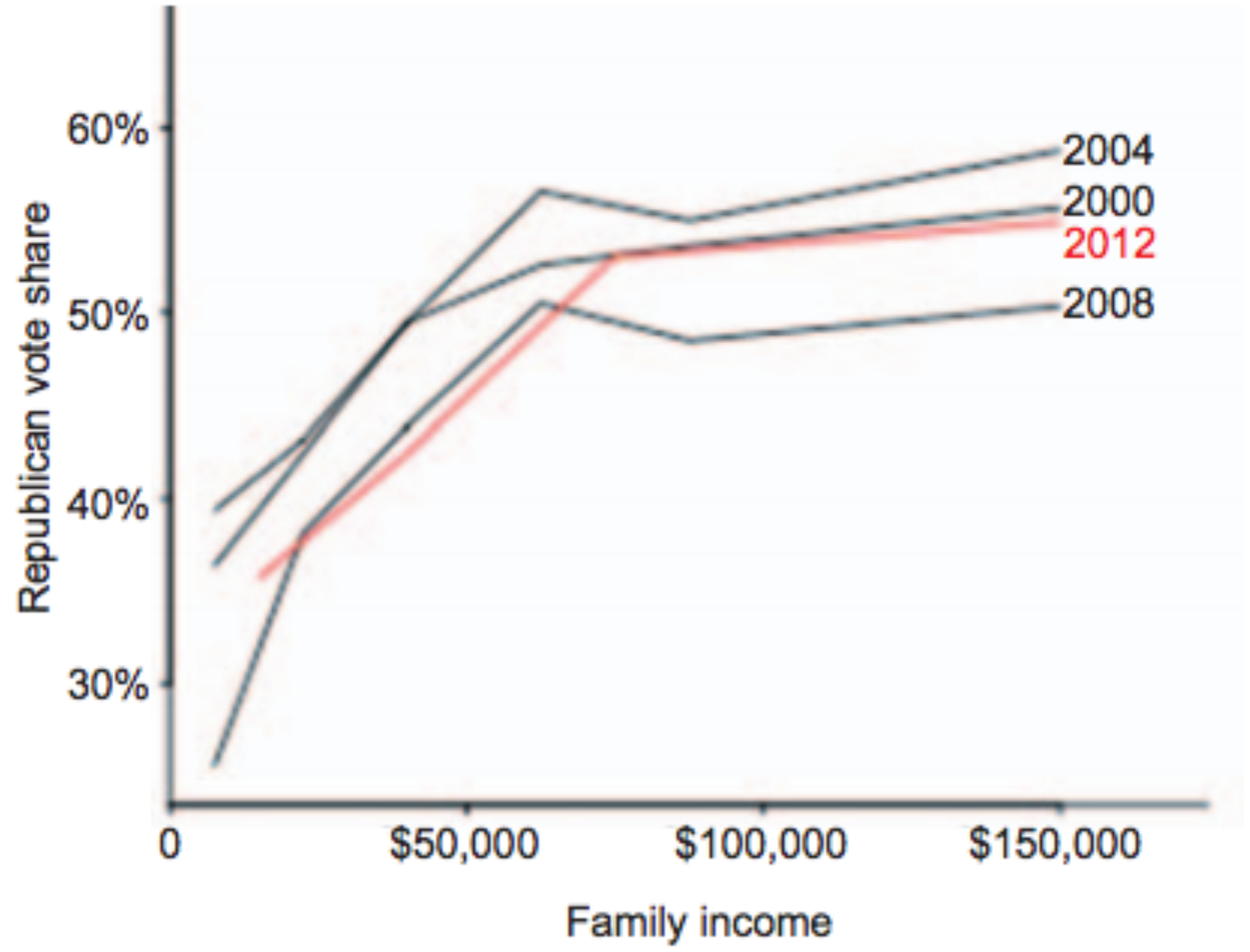

From Avi Feller et al. 2013

What is regression?

Mathematical formulation:

Given input/output pairs $\{(x_i, y_i)\}_{i=1}^{s}$ find function f with

$$y_i \approx f(x_i) \qquad \forall i \in \{1, \ldots, s\}$$

Example: linear regression

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Affine linear transformation of vector $x = (x_1, \ldots, x_d)$ with weights $w \in \mathbb{R}^{d+1}$

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Note that $i \neq j$:

$$f(x_i) = w_0 + \sum_{j=1}^{d} w_j \, x_{ij}$$

Cost function

Notation: $f(x) = w_0 + \sum_{j=1}^{d} w_j x_j = \langle w, x \rangle$ with $x := \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} \in \mathbb{R}^{d+1}$

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Imagine $s = 3$ and $d = 2$:

$$w_0 + x_{11}w_1 + x_{12}w_2 = y_1$$
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This system of linear equations has a unique solution if…?

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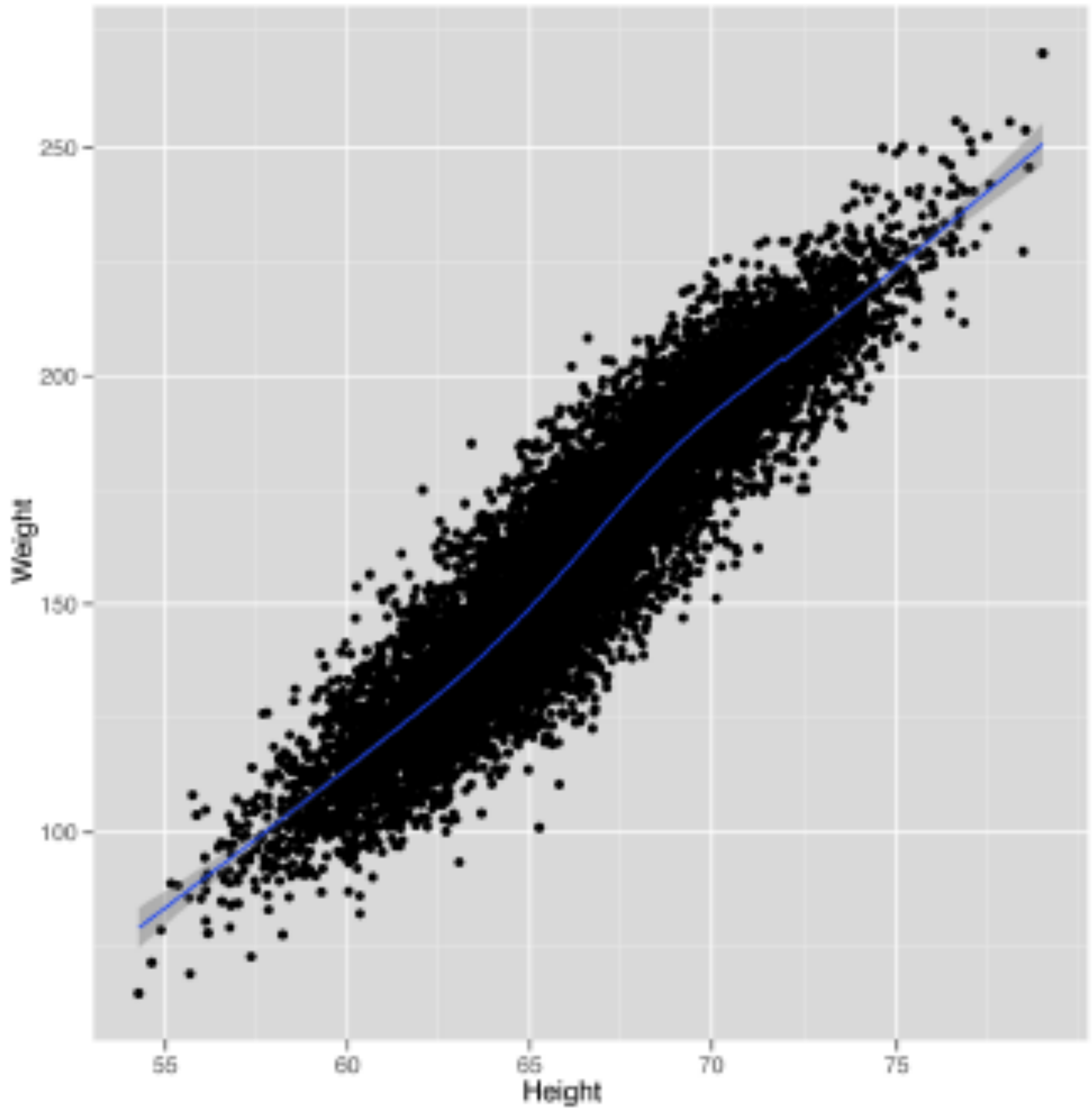

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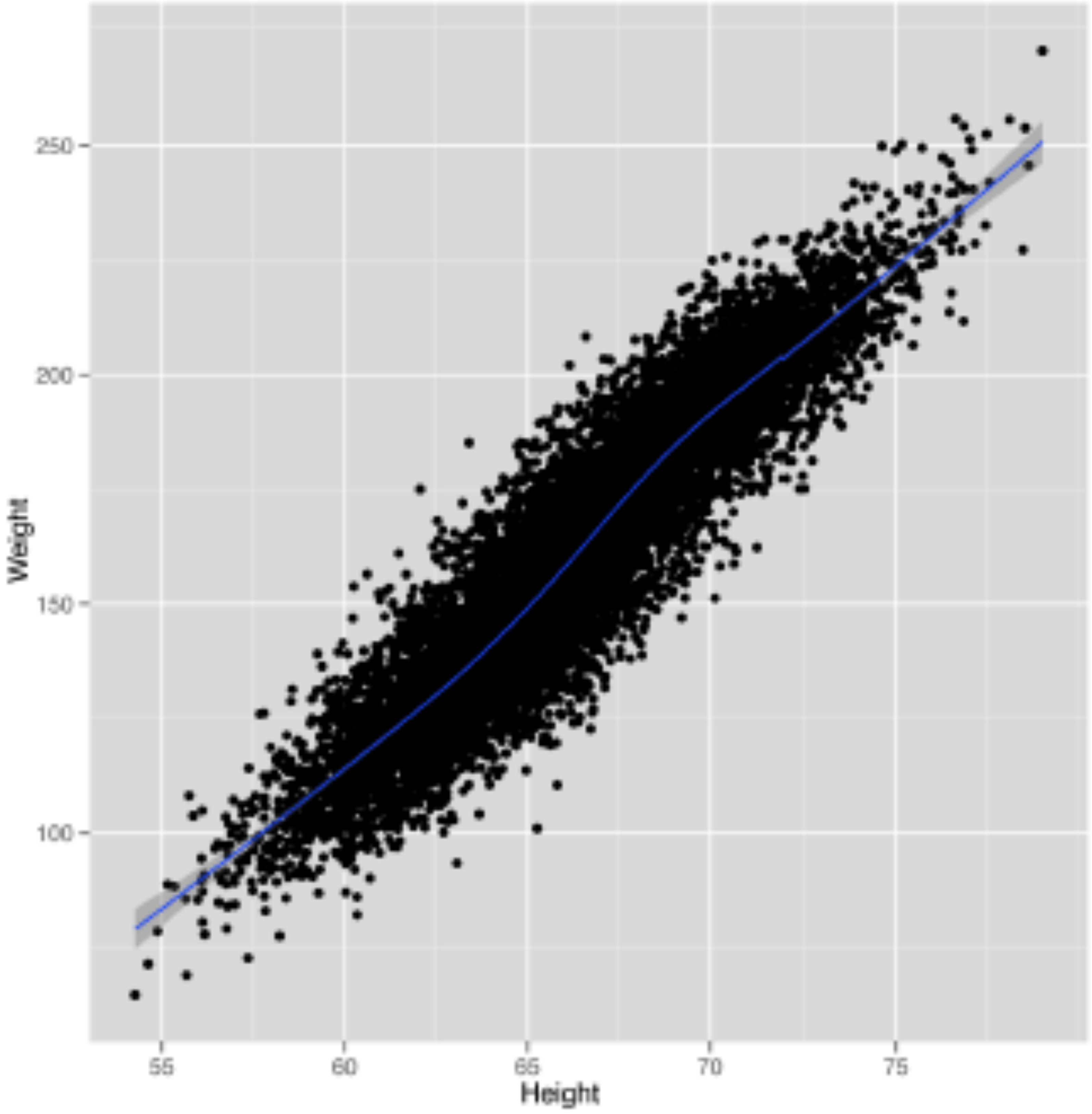

$$s \gg d + 1 = 2$$

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Instead we need to find an approximation that is optimal in some sense

Example: Mean-Square Error (MSE)

$$\mathrm{MSE}(w) := \frac{1}{2s}\sum_{i=1}^{s} |f(x_i) - y_i|^2$$

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How can we do this?

How do we compute $\hat{w}$?

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We do what we did in school: we compute the derivative and set it to zero:

$$\nabla \mathrm{MSE}(\hat{w}_0) = \mathrm{MSE}'(\hat{w}_0) = \frac{1}{s}\sum_{i=1}^{s}(\hat{w}_0 - y_i) \overset{!}{=} 0$$

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$$\Rightarrow \qquad \hat{w}_0 = \frac{1}{s} \sum_{i=1}^{s} y_i$$

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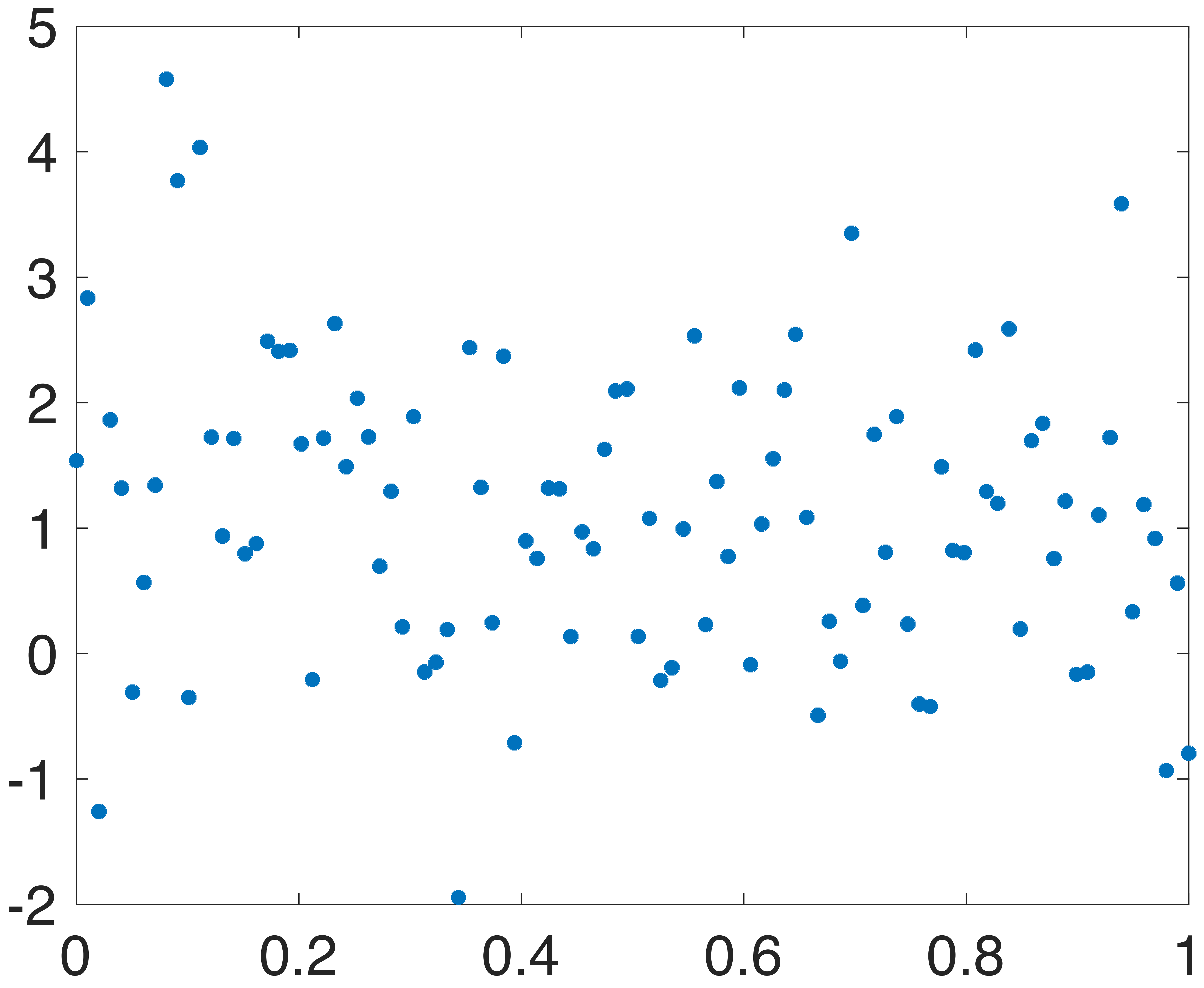

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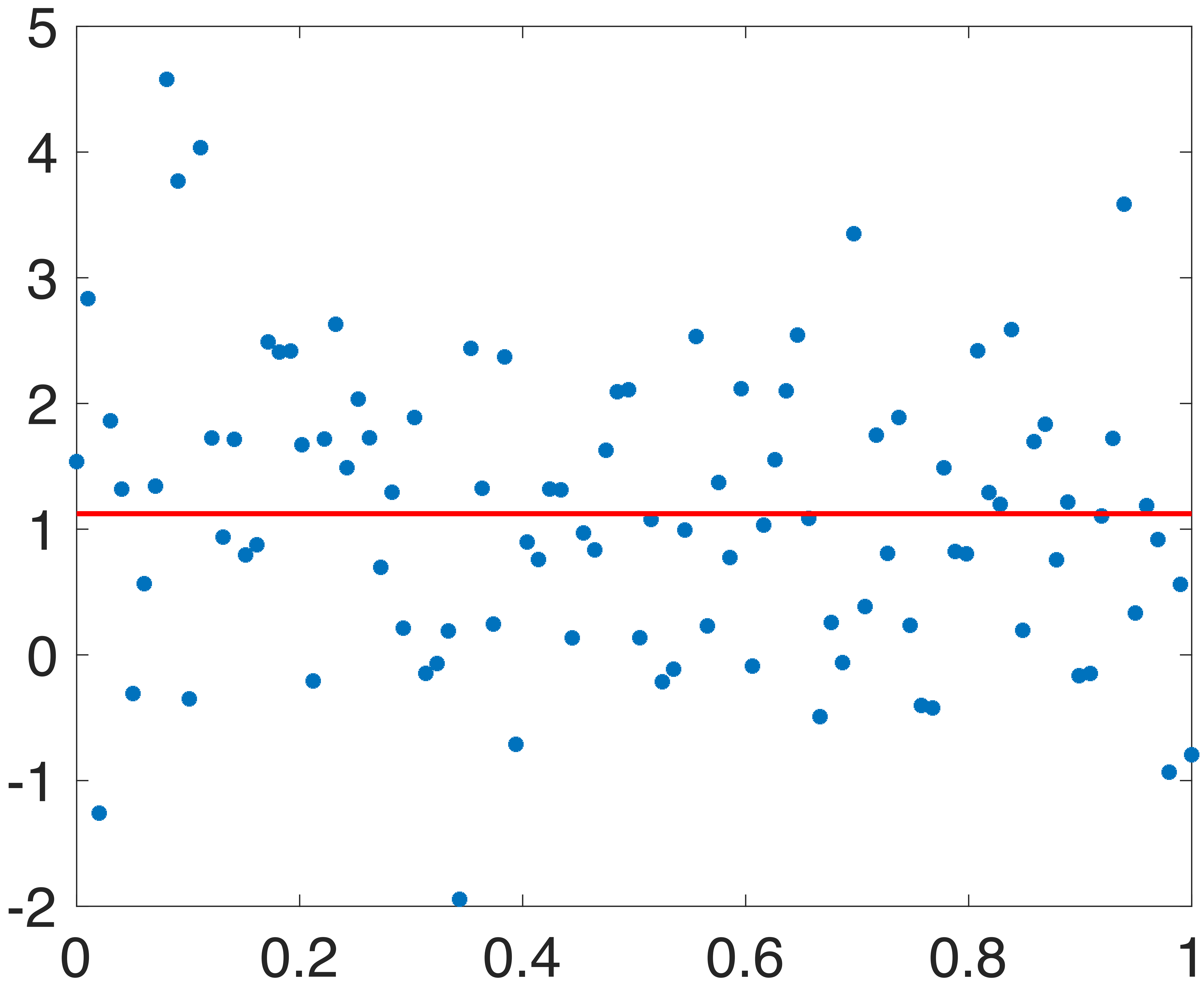

$\hat{w}_0 \approx 1.1231$

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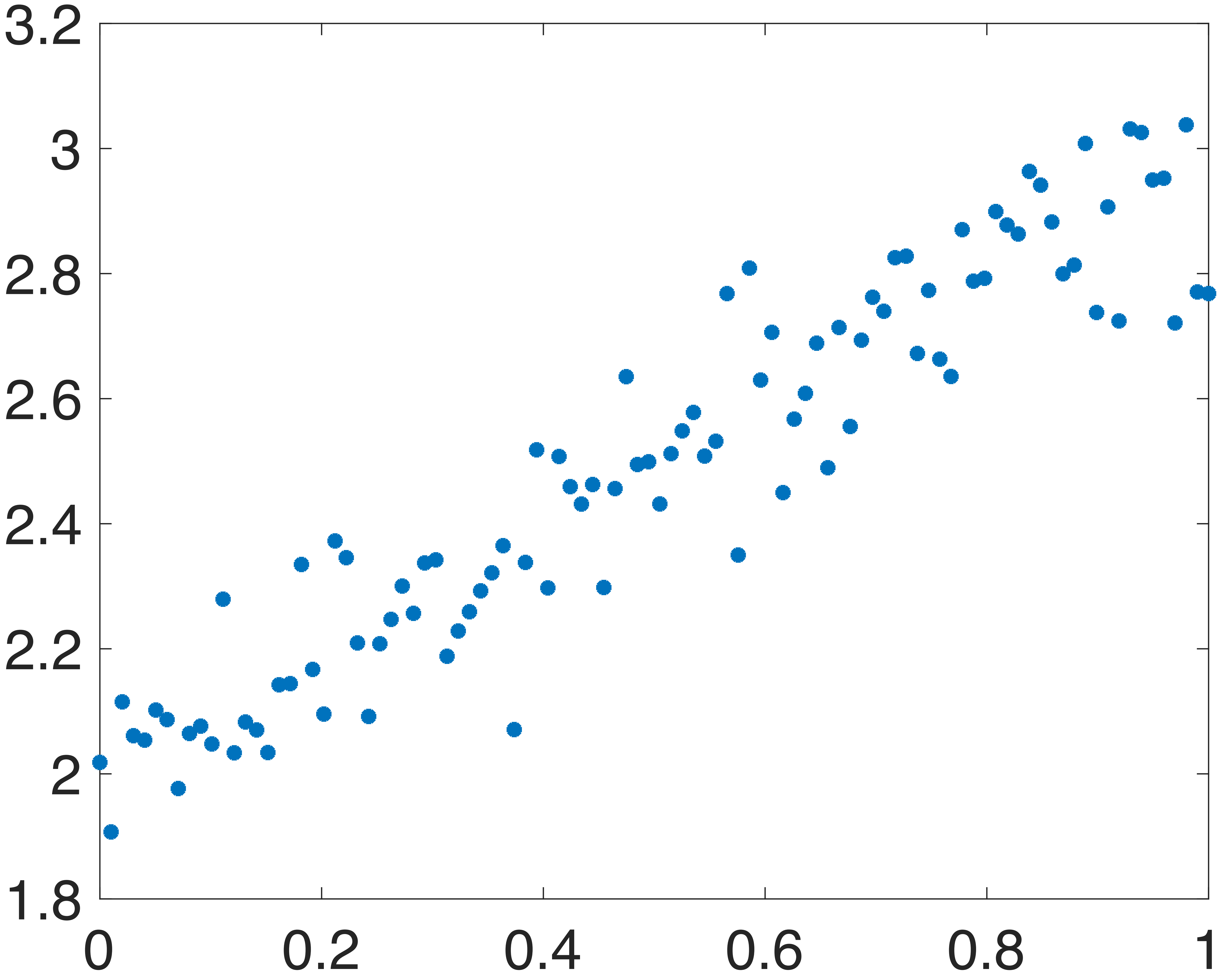

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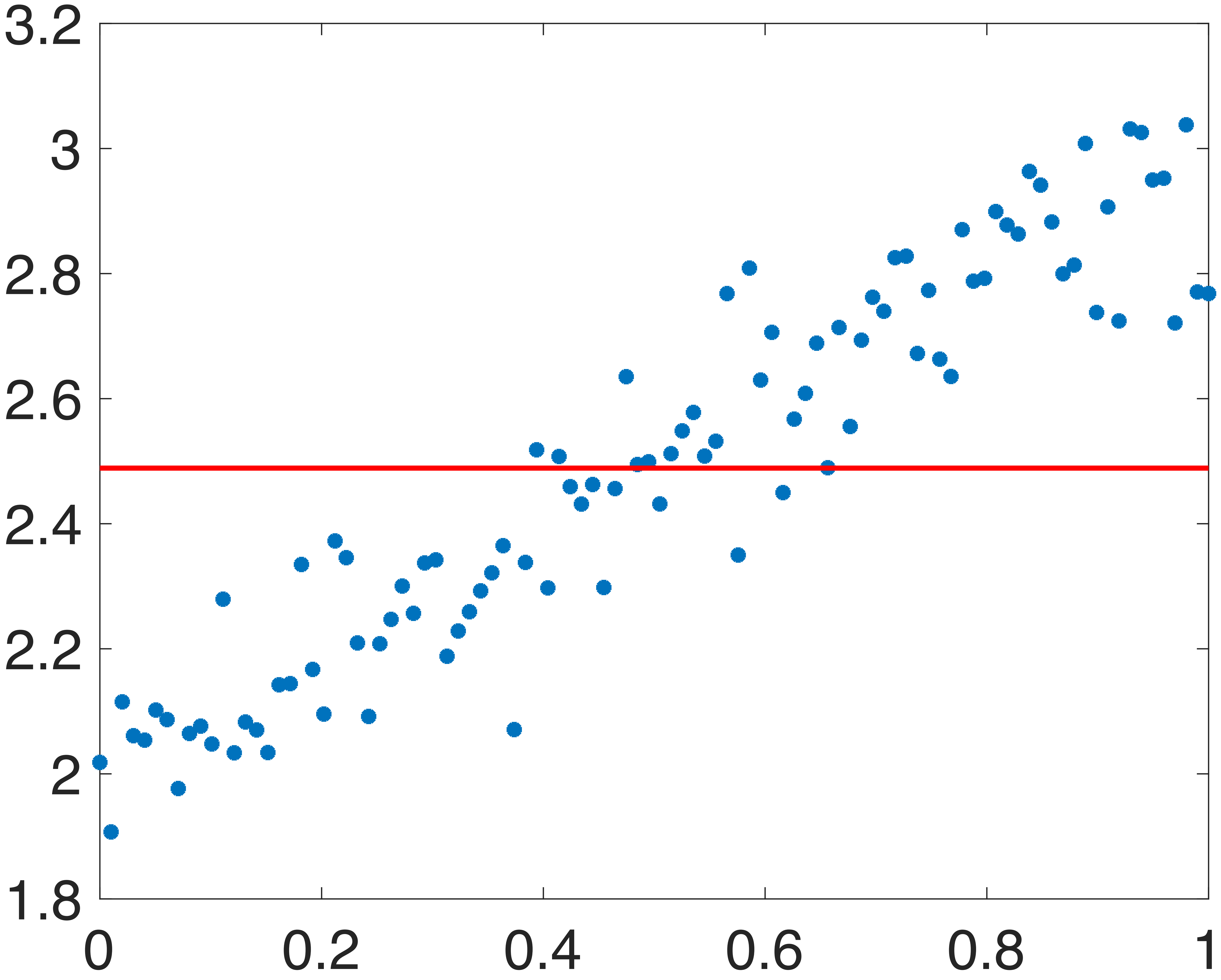

$\hat{w}_0 \approx 2.4889$

We will discuss how to compute a better approximation now