MODULE-1

Unit-1

THE SAMPLING PROCESS

The Process of converting an analog signal to a digital signal involves sampling the signal, holding it for conversion and converting it to the corresponding digital value. The sampling frequency must be high enough so as to avoid aliasing. Aliasing is a phenomenon due to which a high frequency signal when sampled using a low sampling rate becomes a low frequency signal that may interfere with the signal of interest. To avoid aliasing the sampling theorem states that the following requirement must be satisfied

$$f_s = 1/T_s > 2f_{max}$$

where, f_s is the sampling frequency in Hz

T_s is the sampling intervals in seconds &

f_{max} is the highest frequency contents of the analog signal.

For example, if we sample a signal with its highest frequency content as 10KHz it must be sampled using a sampling rate of more than 20KHz

The Minimum sampling rate needs to be twice the highest frequency content of the signal. This frequency is called Nyquist limit. ie $f_{nyquist} = 2 * f_{max}$

Sampling in the Frequency domain

In the digital signal processing, all the signals are discrete. when the analog signals such as speech, music etc are to be Processed by the DSP system. then such signals are converted to discrete form by Analog to Digital converters. After Processing, the signals are converted to analog form by Digital to Analog Converters. Many Applications demand the Processing of signals in frequency domain. For example frequency content, periodicity, energy and Power spectrums etc can be better analysed in frequency domain. Hence the Signals are transformed from time domain to frequency domain. Such transformations Can be obtained with the help of fourier transform (FT) and discrete Fourier Transform. once the required analysis and Processing is Performed in frequency domain, the signals are transformed back in time domain by Inverse Discrete Fourier Transform (IDFT).

we now that the Fourier Transform of a discrete time signal is called as Frequency Response and it is given by

$$X(W) = X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn} \longrightarrow (1)$$

eventhough $x(n)$ is discrete time signal, $X(w)$ is Continuous function from 0 to 2π. This continuous ~~spectrum~~

Spectrum $X(\omega)$ cannot be directly Processed by digital Signal Processor due to this the continuous spectrum $X(\omega)$ is Sampled by using frequency domain Sampling Process.

We know that the continuous spectrum, $X(\omega)$ is Periodic with the Period of 2π. Hence the samples taken from period 0 to 2π are only important in Signal Processing.

Let us Assume that there are 'N' number of Samples are taken from 0 to 2π. then the spacing between successive samples will be $\frac{2\pi}{N}$. The spacing between sucessive Samples called Frequency Resolution therefore $\frac{2\pi}{N}$ is called digital frequency resolution

Therefore to spectrum $X(\omega)$ is Sampled uniformly

Put $\omega = \frac{2\pi}{N}k$ in the equation (1) we get.

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi}{N}kn} \quad \text{for } k = 0, 1, 2 \ldots (N-1) \rightarrow (2)$$

For Simplicity $X\left(\frac{2\pi}{N}k\right)$ can be written alway as $X(k)$.

$$\therefore X(k) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi}{N}kn} \quad 0 \le k \le N-1$$

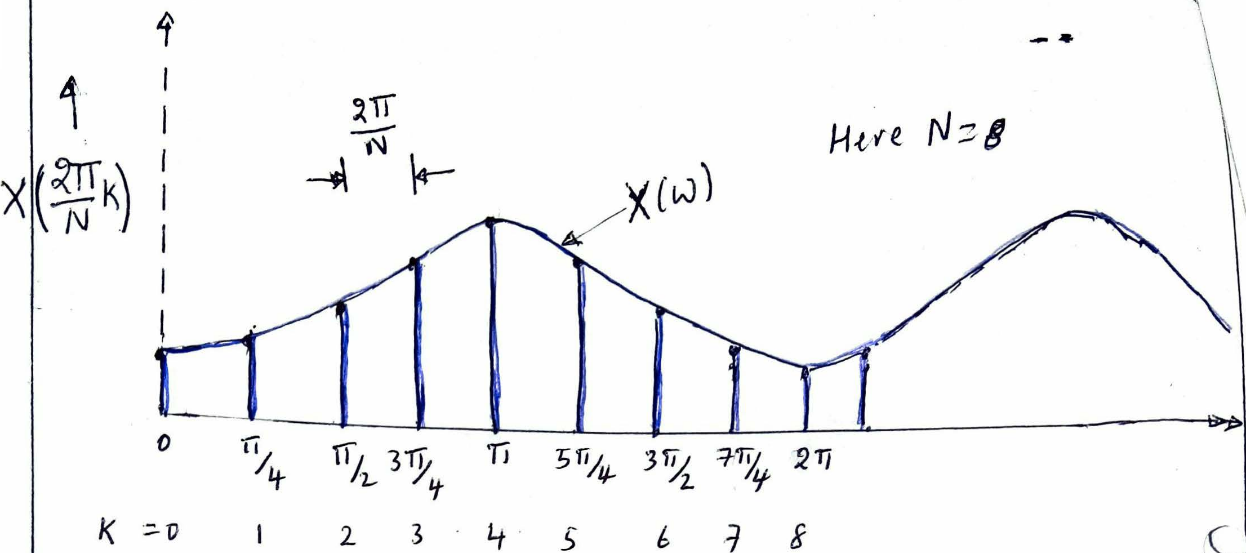
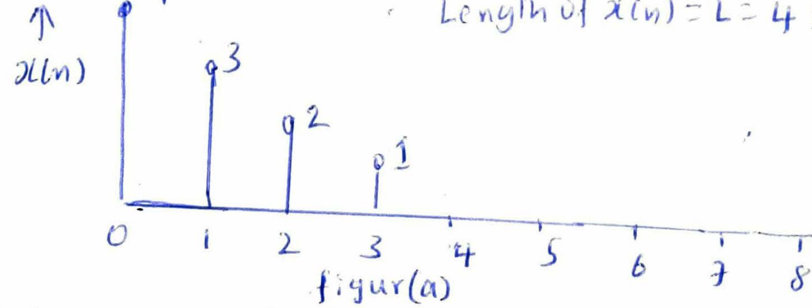


Figure Frequency domain Sampling of the spectrum X(w)

Thus 'k' is an index for frequency domain samples. and X(w) is calculated only at discrete values. therefore Output of frequency domain Sampling is the Discrete Fourier Transform (DFT)

To Determine Minimum value of N ie minimum number of samples in frequency domain. Let us consider a non-periodic signal x(n). It contains 'L' samples it is shown in figure below.



Length of x(n) = L = 4 samples

figur(a)

Let us assume that Period N greater than number of samples in time domain 'L' for Ex N=6. in figure(b)

observe that the samples at $n=4$ and $n=5$ are zero.
This is because the signal repeats at $n=6, 12, \ldots$ etc
Here $N > L$ hence there is no aliasing.

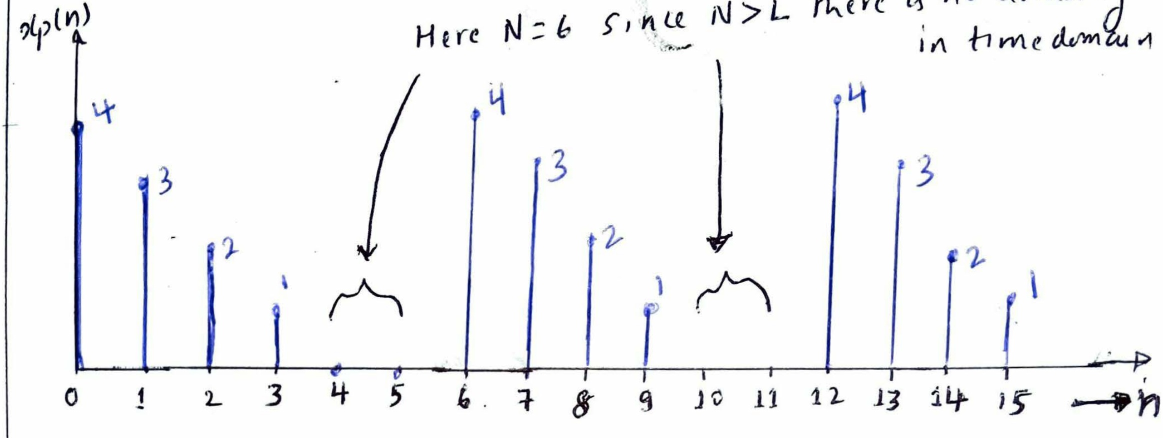
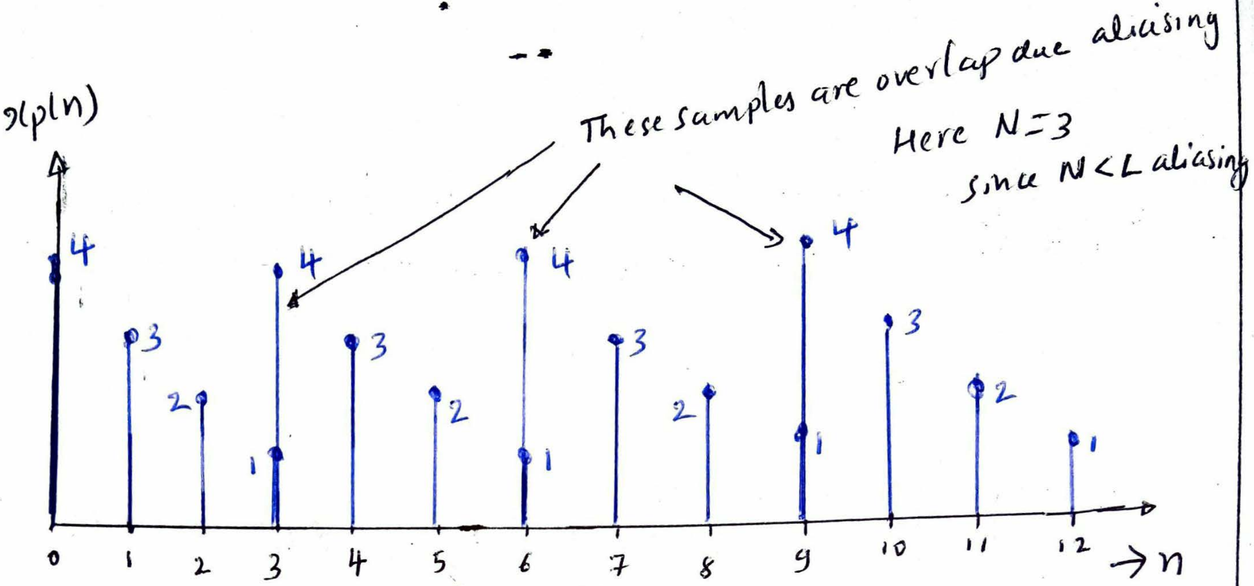


Figure (b)

Consider the waveform of periodic sequence $x_p(n)$ which is obtained by taking inverse Discrete Fourier transform for the frequency domain samples $X(K)$. In this waveform observe that $N=3$. Since $N<L$. the two samples are overlapping at $n=3, 6 \ldots$ etc. This is called aliasing due to less number of samples 'N' considering in the frequency domain sampling process. Hence it is not possible to obtain sequence $x(n)$ from $x_p(n)$. Thus

To avoid aliasing in time domain, the number of samples in the frequency spectrum must be greater than number of samples in the time domain sequence i.e $N \geq L$

above relation shows that we should take at-least 'L' samples over the period 0 to 2π. This process is illustrated in figure (c) below.



Figure(c)

– Above Frequency domain sampling Process shows that when we take N-point DFT there are N-frequency components between 0 to f_s. hence the frequency separation or The Frequency Resolution. $\Delta f = f_s/N$. This f_s/N is called Analog frequency Resolution. it is denoted by Δf or F.

<u>Example 1</u> A Signal $x(t)$ is bandlimitted to 10 KHz is sampled at 20 KHz. DFT of 1000 Samples computed a) what is the spacing between spectral Samples

b) what are the analog frequency corresponding to K=200, K=350

<u>Soln</u> Given $f_s = 20$ KHz DFT points N=1000

a) The Frequency spacing, $\Delta f = f_s/N = \frac{20\,KHz}{1000} = 20\,Hz$

b) The Analog frequency corresponding to index K=200

$f_K = K \times \Delta f$ for K=200 $f_{200} = 200 \times 20 = 4000\,Hz$ &

for K=350 $f_{350} = 350 \times 20 = 7000\,Hz$.

Example 2 Consider a signal with 0.5 sec duration with a bandwidth 5KHz. The resolution of the spectrum is 50Hz or less. Calculate the value of N

Sol^n Given $f_m = 5\,KHz$ & frequency Resolution $\Delta f = 50\,Hz$

By Sampling Theorem $f_s \geq 2 f_m$

$$f_s = 2 \times 5\,KHz$$

$$f_s = 10\,KHz$$

we know that Frequency Resolution, $\Delta f = f_s/N$

$$\therefore N = f_s/\Delta f = \frac{10\,KHz}{50\,Hz}$$

$$N = 200$$

Choose the next largest power of 2

we get $N = 256$

Example 3 A signal with a bandwidth of 2KHz is sampled with a sampling rate of 4KHz. If 512 samples of this signal are used to compute the Fourier transform of X(K). determine the frequency spacing between adjacent X(K) elements. what is the analog frequency corresponding to K=64 & 128.

Solution:- Given $f_s = 4\,KHz$ & $f_{max} = 2\,KHz$ & $N = 512$

we know that Frequency spacing $\Delta f = f_s/N = \frac{4000}{512}$

$$\Delta f = 7.8125\,Hz$$

The Analog frequency corresponding to index $K=64$ is $f_K = K \times \Delta f$

$K=64$, $f_{64} = 64 \times \Delta f = 64 \times 7.8125Hz = 500Hz$

$K=128$ $f_{128} = 128 \times \Delta f = 128 \times 7.8125Hz = 1000Hz$

Example 4

Given the following $x(n)$: $x(n) = \delta(n) + \delta(n-1) + \delta(n-2)$

i) Find the Fourier transform $X(e^{jw})$ & plot the $|X(e^{jw})|$

ii) Get the magnitude of the 4-point DFT of the first four samples of $x(n)$

iii) Get the magnitude of the 8-point DFT of first eight samples of $x(n)$

— Dec/Jan-2014, 10marks

Solution:- Given $x(n) = \delta(n) + \delta(n-1) + \delta(n-2)$

we that Fourier transform of $x(n)$ is called frequency response & it is given by $X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jwn}$

$$X(e^{jw}) = \sum_{n=0}^{2} [\delta(n) + \delta(n-1) + \delta(n-2)] e^{-jwn}$$

$$X(e^{jwn}) = \sum_{n=0}^{2} \delta(n) e^{-jwn} + \sum_{n=0}^{2} \delta(n-1) e^{-jwn} + \sum_{n=0}^{2} \delta(n-2) e^{-jwn}$$

$$X(e^{j\omega}) = \sum \{ \delta(0) e^{-j\omega(0)} + \delta(1) e^{-j\omega(1)} + \delta(2) e^{-j\omega(2)} \}$$
$$+ \{ \delta(0-1) e^{-j\omega(0)} + \delta(1-1) e^{-j\omega(1)} + \delta(2-1) e^{-j2(2)} \}$$
$$+ \{ \delta(0-2) e^{-j\omega(0)} + \delta(1-2) e^{-j\omega(1)} + \delta(2-2) e^{-j\omega(2)} \}$$

$$= \{1+0+0\} + \{0 + 1\cdot e^{-j\omega} + 0\} + \{0+0+1\cdot e^{-j2\omega}\}$$

$$X(e^{j\omega}) = 1 + e^{-j\omega} + e^{-j2\omega}$$

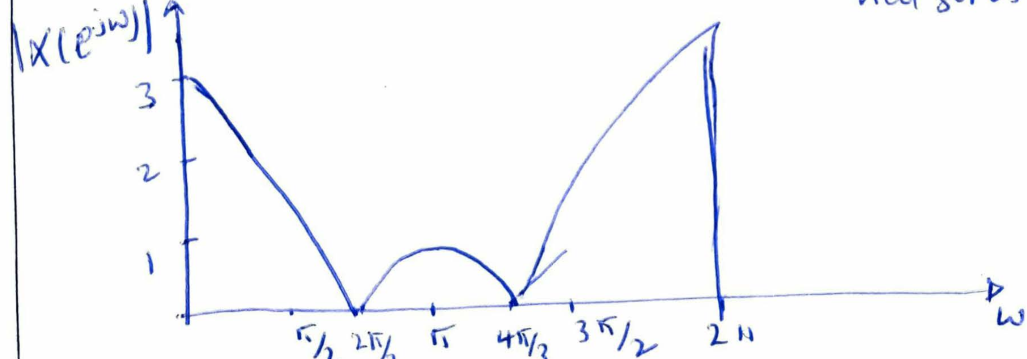
$$X(e^{j\omega}) = 1 + e^{-j\omega} + e^{-j\omega}\cdot e^{-j\omega} = e^{-j\omega}\left\{2\times\frac{e^{j\omega}+e^{-j\omega}}{2} + 1\right\}$$

$$= e^{-j\omega}\left[2\,\frac{e^{j\omega}+e^{-j\omega}}{2} + 1\right]$$

$$X(e^{j\omega}) = e^{-j\omega}[2\cos\omega + 1] = (1 + 2\cos\omega)e^{-j\omega}$$

Magnitude of $X(e^{j\omega})$, $|X e^{j\omega}| = |(1 + 2\cos\omega)|$

has zeros at $\omega = 2\pi/3$ & $4\pi/3$



magnitude $X(e^{j\omega})$

ii) 4-point DFT gives sampled version of $X(e^{j\omega})$

$$|X(K)| = |X(e^{j\omega})|_{\omega = \frac{2\pi}{4}K}$$

$$= |X(e^{j\frac{\pi}{2}K})|, \quad K = 0, 1, 2, 3$$

$$|X(K)| = |1 + 2\cos\frac{2\pi}{4}K| \quad \text{for } K = 0, 1, 2, 3$$

$$= |1 + 2\cos(\tfrac{\pi}{2} \times K)| \quad \text{for } K = 0, 1, 3$$

when $K=0$, $|X(0)| = |1 + 2\cos 0| = 3$

$K=1$, $|X(1)| = |1 + 2\cos(\tfrac{\pi}{2} \times 1)| = 1$

$K=2$, $|X(2)| = |1 + 2\cos(\tfrac{\pi}{2} \times 2)| = 1$

$K=3$, $|X(3)| = |1 + 2\cos(\tfrac{\pi}{2} \times 3)| = 1$

iii) Similarly Eight point DFTs are obtained by sampled $|X(e^{j\omega})|$ uniformly at $\omega = \frac{2\pi}{N}K$ for $N=8$

$$|X(K)| = |X(e^{j\omega})|_{\omega = 2\pi/8 \, K}$$

$$= |X(e^{-j\frac{2\pi}{8}K})| \quad \text{for } K = 0, 1, 2, 3, 4, 5, 6, 7$$

RELATIONSHIP OF THE DFT TO OTHER TRANSFORMS

1. RELATIONSHIP BETWEEN FOURIER TRANSFORM (FT) AND DFT

The Fourier transform, $X(e^{jw})$ of a finite duration sequence, $x(n)$ having length N is given by

$$X(e^{jw}) = \sum_{n=0}^{N-1} x(n) e^{-jwn} \longrightarrow (1)$$

where $X(e^{jw})$ is a continuous function of w.

The Discrete Fourier transform of $x(n)$ is given by

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}Kn} \quad \text{for } K=0,1,2\ldots(N-1) \rightarrow (2)$$

By comparing equation (1) and (2) we find that the DFT of $x(n)$ is a sampled version of Fourier transform of the sequence and is given by

$$X(K) = X(e^{jw})\Big|_{w=\frac{2\pi}{N}K} \quad \text{for } K=0,1,2\ldots(N-1)$$

2. Relationship Between DFT and Z-Transform

we know that Z-transform of finite duration sequence $x(n)$ is given as

$$X(Z) = \sum_{n=0}^{N-1} x(n) Z^{-n} \longrightarrow (1)$$

The discrete fourier transform of $x(n)$ is given by $X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}Kn}$ for $K=0,1,2\ldots(N-1) \rightarrow (2)$

By comparing equation ① and ② we find that the

$$X(K) = X(z)\Big|_{z=e^{j\frac{2\pi}{N}K}} \text{ for } K=0,1,2\ldots(N-1)$$

This means if z-transform is evaluated on unit circle at evenly spaced points then it becomes DFT.

And also we know that By Definition of IDFT

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)e^{j\frac{2\pi}{N}Kn} \rightarrow ③$$

By substituting equation ③ in equation ① we get

$$X(z) = \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{K=0}^{N-1} X(K)e^{j\frac{2\pi}{N}Kn}\right) z^{-n}$$

$$X(z) = \frac{1}{N}\sum_{K=0}^{N-1} X(K) \sum_{n=0}^{N-1}\left(e^{j\frac{2\pi}{N}K} z^{-1}\right)^{n}$$

By using geometric finite series sum formula

$$\sum_{n=0}^{N-1} a^{n} = \frac{1-a^N}{1-a} \text{ for } a\neq 1$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K) \times \frac{1-\left(e^{j\frac{2\pi}{N}K} z^{-1}\right)^{N}}{1-e^{j\frac{2\pi}{N}K} z^{-1}}$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K) \frac{1-e^{j\frac{2\pi K}{\cancel{N}}\times \cancel{N}}\cdot z^{-N}}{1-e^{j\frac{2\pi}{N}K} z^{-1}} \quad (\because e^{j2\pi K}=1)$$

$$\boxed{X(z) = \frac{1-z^{-N}}{N}\sum_{K=0}^{N-1}\frac{X(K)}{1-e^{j\frac{2\pi}{N}K} z^{-1}}}$$ is the another Relation b/w DFT & Z-T

3. RELATIONSHIP BETWEEN DFT and DFS (Discrete Fourier Series)

we know that $x_p(n)$ is periodic with period N. it can be expressed by discrete Fourier series as

$$x_p(n) = \sum_{K=0}^{N-1} C(K) e^{j\frac{2\pi}{N}Kn} \quad \text{for } n=0,1,2 \ldots (N-1) \rightarrow (1)$$

where $C(K)$ = fourier coeffients are given as

$$C(K) = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi}{N}Kn} \quad \text{for } K=0,1,2 \ldots (N-1) \rightarrow (2)$$

By the Definition of DFT

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}Kn} \quad \text{for } K=0,1,2 \ldots (N-1) \rightarrow (3)$$

By comparing equation (2) & (3) we can write

$$N \cdot C(K) = X(K)$$

$$\frac{1}{N} \cdot X(K) = C(K) \quad \text{for } K=0,1,2 \ldots (N-1)$$

This equation gives relationship between DFT & DFS coefficients. If we know the DFS coefficents then DFT can be obtained by above equations.

1) Find the 4-point DFT of the sequence $x(n) = \cos\left(\frac{n\pi}{4}\right)$

solution:- Given $x(n) = \cos(n\pi/4)$

we need to find 4-point DFT. so the first four samples of $x(n)$ can be obtained by putting $n=0,1,2,3$

when $n=0$, $x(0) = \cos\left(\frac{n\pi}{4}\right) = \cos(0\times\pi/4)$

$x(0) = 1$

when $n=1$, $x(1) = \cos(1\times\pi/4) = 1/\sqrt{2} = 0.707$

When $n=2$, $x(2) = \cos(2\times\pi/4) = \cos(\pi/2) = 0$

When $n=3$, $x(3) = \cos(3\times\pi/4) = \cos(3\pi/4) = -1/\sqrt{2} = -0.707$

By using matrix Relation for computing DFT

$\bar{X}_N = [W_N]\, x_N$

For $N=4$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -j \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0.707 \\ 0 \\ -0.707 \end{bmatrix}$$

$X(K) = [1,\ 1-j1.414,\ 1,\ 1+j1.414]$

(2) Find the DFT of the sequence $x(n) = 0.5^n u(n)$; $0 \le n \le 3$

Solution :- Given $x(n) = 0.5^n u(n)$

By the Definition of DFT

$$X(K) = N\text{-point DFT}\{x(n)\} = \sum_{n=0}^{N-1} x(n) W_N^{Kn} \quad 0 \le K \le N-1$$

$$X(K) = \sum_{n=0}^{3} 0.5^n W_N^{Kn}$$

$(\because N=4$ ie $n=0$ to $3)$

$$= \sum_{n=0}^{3} (0.5 W_N^K)^n = \frac{1-(0.5W_N^K)^N}{1-0.5W_N^K} = \frac{1-(0.5)^4}{1-0.5e^{-j\frac{2\pi}{4}K}}$$

$$X(K) = \frac{0.9375}{1-0.5e^{-j\frac{\pi}{2}K}}$$