# Homology

## A Short Course in Computational Geometry and Topology

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# Triangulation

## Definition

A *triangulation* of a topological space $X$ is a simplicial complex $K$ and a homeomorphism $\Phi : |K| \to X$.

In the case of $2-$ *manifolds*,since the triangles are geometry and the condition of homeomorphism requires that any two triangles be

- disjoint
- share an edge
- share a vertex

# Triangulation of a Torus

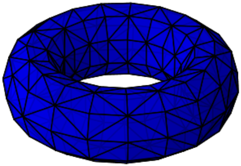

Figure 1: Triangulation of Torus

# Tracing

- The little creatures traverses the surface, providing detailed reports of their journeys.
- Among them, those returning to their starting point yield valuable closed paths, while others, mysteriously connecting left to right, are identified by consulting open paths initially deemed less informative.
- To streamline the collection, closed paths encompassing distinct real estate are grouped, bringing order to the diverse records.

# Example

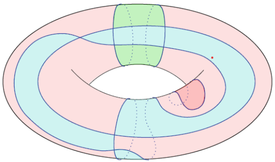

**Fig. 9.1** The torus embedded in $\mathbb{R}^3$. Of the six paths, one surrounds real estate by itself, two form a pair with a ring-shaped piece of real estate between them, and three collaborate to surround a complicated piece of real estate that reaches all the way around the torus

Figure 2: Paths traced around a Torus

## Free Abelian Group

### Definition

An Abelian group $(G)$ with a subset $S$ that generates the entire group. Then $G$ along with it's basis $S$ is considered as a free abelian group.

## Chain Group and Complex

- Let $K$ be a simplicial complex and $p$ a dimension. A $p$-chain is a formal sum of $p$-simplices in $K$.
- The standard notation for this is $c = \sum_i a_i \sigma_i$, where the $\sigma_i$ are the $p$-simplices and the $a_i$ are the coefficients.
- In computational topology, we mostly work with coefficients $a_i$ that are either 0 or 1, called modulo 2 coefficients $\mathbb{Z}_2$.

- Two $p$-chains are added componentwise, like polynomials. Specifically, if $c = \sum_i a_i \sigma_i$ and $c' = \sum_i b_i \sigma_i$ then $c + c' = \sum_i (a_i + b_i)\sigma_i$, where the coefficients satisfy $1 + 1 = 0$.
- In set notation, the sum of two $p$-chains is their symmetric difference. The $p$-chains together with the addition operation form the group of $p$-chains denoted as $(C_p, +)$.

# Homomorphism

## Definition

A homomorphism $\phi$ between two algebraic structures $(G, *)$ and $(H, \circ)$ is a mapping $\phi : G \rightarrow H$ that preserves the operation, i.e., for all $a, b \in G$, $\phi(a * b) = \phi(a) \circ \phi(b)$ and $\phi(e_G) = e_H$

## Boundary Homomorphism

For a $p$-chain, $c = \sum a_i \sigma_i$, the boundary is the sum of the boundaries of its $(p-1) - simplices$, $\partial_p c = \sum a_i \partial_p \sigma_i$.

$$\partial_p : C_p \to C_{p-1}$$

Taking the boundary commutes with addition. We will therefore refer to $\partial_p$ as the boundary homomorphism or, shorter, the boundary map for chains. The chain complex is the sequence of chain groups connected by boundary homomorphisms,

$$\ldots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \ldots$$

# Cycles

A $p$-cycle is a $p$-chain with empty boundary, $\partial c = 0$. Since $\partial$ commutes with addition, we have a group of $p$-cycles, denoted as $Z_p = Z_p(K)$, which is a subgroup of the group of $p$-chains. In other words, the group of $p$-cycles is the kernel of the $p$-th boundary homomorphism

$$Z_p = \ker \partial_p$$

## Boundaries

A $p$-boundary is a $p$-chain that is the boundary of a $(p+1)$-chain, $c = \partial d$ with $d \in C_{p+1}$. Since $\partial$ commutes with addition, we have a group of $p$-boundaries, denoted by $B_p = B_p(K)$, which is again a subgroup of the $p$-chains. In other words, the group of $p$-boundaries is the image of the $(p+1)$-st boundary homomorphism

$$B_p = \operatorname{im} \partial_{p+1}$$

# Fundamental Lemma of Homology

## Lemma

$\partial_p \partial_{p+1} \tau = 0$ for every integer $p$ and every $(p+1)$-chain $\tau$.

**Proof.**

- We just need to show that $\partial_p \partial_{p+1} \tau = 0$ for a $(p+1)$-simplex $\tau$. The boundary, $\partial_{p+1} \tau$, consists of all $p$-faces of $\tau$. Every $(p-1)$-face of $\tau$ belongs to exactly two $p$-faces, so $\partial_p(\partial_{p+1} \tau) = 0$.
- It follows that every $p$-boundary is also a $p$-cycle or, equivalently, that $B_p$ is a subgroup of $Z_p$.

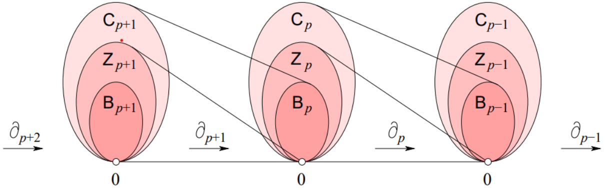


Figure 3: Chain Complex Homomorphisms

1. Background
2. Little Creatures
3. Chain Groups
4. Cycles and Boundaries
5. **Homology**
6. Euler-Poincaré Formula

# Homology

## Definition

The $p$-th homology group is the $p$-th cycle group modulo the $p$-th boundary group

$$H_p = Z_p / B_p$$

Since the boundaries form subgroups of the cycle groups, we can take quotients. In other words, we can partition each cycle group into classes of cycles that differ from each other by boundaries. This leads to the notion of homology groups and their ranks.

## Betti Number

- The rank of the $p$-th homology group is referred to as the $p$-th Betti number of $K$, denoted as $\beta_p = \text{rank} H_p$.
- Recalling that the elements in $H_p$ are classes of size $2^r$, where $r = \text{rank} B_p$, and since these classes partition $Z_p$, we have

$$2^{\text{rank} Z_p} = 2^{\text{rank} B_p} \cdot 2^{\text{rank} H_p}$$

$$\beta_p = \text{rank} Z_p - \text{rank} B_p$$

# Euler's Characteristic for simplicial complex

## Definition

$\chi = s_0 - s_1 + s_2 - \cdots \pm s_k$, where $k = \dim K$ and $s_i$ is the number of $i - simplicies$

- $s_p = \text{rank} C_p$. The boundary mapping $\partial_p$ divides the chain group into two groups.
- $rank(C_p) = rank(ker\partial_p) + rank(img\partial_p)$
- $rankC_p = rankZ_p + rankB_{p-1}$

$$\chi = \sum_{p=0}^{k} (-1)^p (\mathrm{rank} Z_p + \mathrm{rank} B_{p-1})$$

$$= \sum_{p=0}^{k} (-1)^p (\mathrm{rank} Z_p - \mathrm{rank} B_p)$$

$$= \sum_{p=0}^{k} (-1)^p \beta_p.$$

## Example

- Computing the betti numbers of a circle $S^1$.
- Computing the betti numbers of a hollow sphere $S^2$