Computing the Non-Linear term:

We need to compute this Non-Linear term.

$$\phi(\alpha^i) = \int \varphi_k(x) \frac{exp(\sum_{j=1}^{N} \alpha_j^i \varphi_j(x))}{\int exp(\sum_{j=1}^{N} \alpha_j^i \varphi_j(z))dz} dx$$

We can write the above in as follows:

$\phi(\alpha^i) = \int \varphi_k(x)p(x)dx$

Further

$\phi(\alpha^i) = \int f(x)p(x)dx$

Where $p(x) = \frac{exp(\sum_{j=1}^{N} \alpha_j^i \varphi_j(x))}{\int exp(\sum_{j=1}^{N} \alpha_j^i \varphi_j(z))dz}$

$p(x)$ gives the probability with which we need sample the input values 'x' for the function f(x).

We can compute $\phi(\alpha^i)$ in two ways:

1. We can compute the denominator of $p(x)$ which is $\int exp(\sum_{j=1}^{N} \alpha_j^i \varphi_j(z))dz$ using Monte Carlo Integration and use the value in computing $\phi(\alpha^i)$ which is also computed with Monte Carlo Integration. This method resulted in large error when used for evaluating the integral.

2. Since we do not have any proper method to compute $\int exp(\sum_{j=1}^{N} \alpha_j^i \varphi_j(z))dz$ we can compute $p(x)$ using Markov Chain Monte Carlo (MCMC) by creating a model based on the Sparse Grid structure and assuming particular distribution (preferable Uniform) for the random variables. Once we sample the values for the random variables (x) using MCMC, we can evaluate the function f(x) on these values and compute the mean which will be the expected value of the function f(x).

$E[f(x)] = \int f(x)p(x)dx$

For computing the expected value of the function iteratively based on the updated model, we can use the sample values obtained in the previous sampling run. We can either store the sampled values of the random variables and use them to initiate the sampling process which should result in faster convergence of the sampling run, or we can sample few new points and calculate the expected value by reusing the output of the function values.