Assignment

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Question : Consider communication over a memoryless binary symmetric channel using a (7, 4) Hamming code. Each transmitted bit is received correctly with probability$(1-\epsilon)$, and flipped with probability ϵ. For each codeword transmission, the receiver performs minimum Hamming distance decoding, and correctly decodes the message bits if and only if the channel introduces at most one bit error.
For $\epsilon = 0.1$, the probability that a transmitted codeword is decoded correctly is _______(rounded off to two decimal places). (rounded off to two decimal places).
Solution: Given that, Let X be a random variable defined in the Table I;

RV	Value	Description
n (or) p	7	The total number of bits
ϵ	0.1	Probability of error in transimmited bit
X	$0 \leq X \leq 7$	The number of bit errors in transmission

TABLE I
RANDOM VARIABLE X DECLARATION

Then, $X \sim Bin(n, p)$ where

$$n = 7 \quad p = \epsilon = 0.1 \tag{1}$$

the pmf of X is given by

$$p_X(k) = {}^7C_k(\epsilon)^k(1-\epsilon)^{7-k} \tag{2}$$

the cdf of X is given by

$$F_X(k) = \sum_{i=0}^{k} {}^7C_i(\epsilon)^i(1-\epsilon)^{7-i} \tag{3}$$

From equation (3), the probability of getting one or less error is given by

$$F_X(1) = \sum_{i=0}^{1} {}^7C_i(\epsilon)^i(1-\epsilon)^{7-i} \tag{4}$$
$$= {}^7C_0(\epsilon)^0(1-\epsilon)^7 + {}^7C_1(\epsilon)^1(1-\epsilon)^6 \tag{5}$$
$$= (1-\epsilon)^7 + 7(\epsilon)^1(1-\epsilon)^6 \tag{6}$$

From (1) and (6),

$$F_X(1) = (1-0.1)^7 + 7(0.1)^1(1-0.1)^6 \tag{7}$$
$$= 0.85 \tag{8}$$

$\therefore$ the probability that a transmitted codeword is decoded correctly is 0.85.

Gaussian
Let parameters be defined in the Table II;

RV	Value	Description
$\mu = np$	0.7	Mean of Binomial distribution
$\sigma^2 = npq$	0.63	Varience of Binomial distribution

TABLE II
PARAMETERS

Let Y is the Gaussian obtained by approximating binomial with parameters n,p then By Central limit theroem,

$$X \approx Y \sim \mathcal{N}(np, npq) \tag{9}$$

We need to find

$$\Pr(Y \leq 1) = F_Y(1) \tag{10}$$

After corrections to make the vlaues more accurate, we need to find

$$\Pr(Y \leq 1.33) = F_Y(1.33) \tag{11}$$

then CDF of Y is:

$$F_Y(x) = \Pr(Y \leq x) \tag{12}$$
$$= \Pr(Y - \mu \leq x - \mu) \tag{13}$$
$$= \Pr\left(\frac{Y-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) \tag{14}$$

Since,

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{15}$$

Q function is defined

$$Q(x) = \Pr(Y > x) \; \forall x \in Y \sim \mathcal{N}(0,1) \tag{16}$$

From (14) and (16),

$$F_Y(x) = 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \tag{17}$$
$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right), & x > \mu \\ Q\left(\frac{\mu-x}{\sigma}\right), & x < \mu \end{cases} \tag{18}$$

From (18) and Table II,

$$F_Y(1.33) = 1 - Q\left(\frac{1.33-0.7}{0.63}\right) \tag{19}$$
$$= 1 - Q(1) \tag{20}$$
$$= 0.8413 \tag{21}$$

$\therefore$ the probability that a transmitted codeword is decoded correctly is 0.8413.
The Binomial CDF vs. Guassian CDF plot is given in fig1

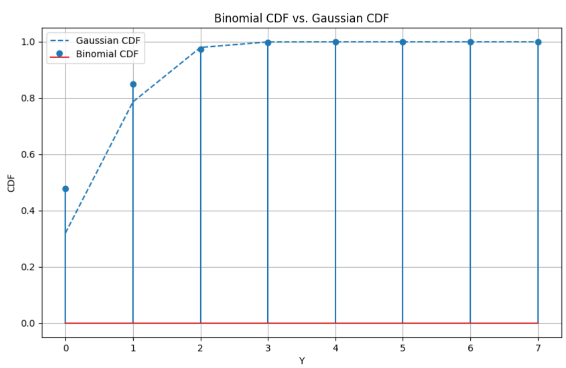

Fig. 1. Binomial CDF vs. Guassian CDF