Probability Assignment 1

EE22BTECH11026 - KARTHIKEYA HANU PRAKASH KANITHI

Question : Suppose X is a binomial distribution $B\left(6, \frac{1}{2}\right)$. Show that $X = 3$ is the most likely outcome. (Hint : $P(X = 3)$ is the maximum among all $P(x_i), x_i = 0, 1, 2, 3, 4, 5, 6$)

Solution: Given that, X is a binomial distribution with parameters

$$n = 6 \qquad p = \frac{1}{2} \tag{1}$$

the probability of getting exactly k successes in n trials is given by

$$p_X(k) = {}^nC_k p^k (1-p)^{n-k} \tag{2}$$

From equations in (1) , The pmf simplifies as,

$$p_X(k) = {}^nC_k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \tag{3}$$
$$= {}^nC_k \left(\frac{1}{2}\right)^6 \tag{4}$$

We know that ${}^nC_k$ can be written as,

$$ {}^nC_k = \frac{n!}{(n-k)!k!} \tag{5}$$

If pmf is the greatest, then ${}^nC_k$ is the maximum for $k \in [0, n]$, Therefore It can be said that,

$$ {}^nC_k \geq {}^nC_{k-1} \quad and \tag{6}$$
$$ {}^nC_k \geq {}^nC_{k+1} \tag{7}$$

From (5) and (6), we can state that

$$\frac{n!}{(n-k)!k!} \geq \frac{n!}{(n-k+1)!(k-1)!} \tag{8}$$
$$\implies \frac{n!}{(n-k)!k!} \geq \frac{n!}{(n-k)!k!} \frac{k}{n-k+1} \tag{9}$$
$$\implies 1 \geq \frac{k}{n-k+1} \tag{10}$$
$$\therefore k \leq \frac{n+1}{2} \tag{11}$$

From (5) and (7), we can state that

$$\frac{n!}{(n-k)!k!} \geq \frac{n!}{(n-k-1)!(k+1)!} \tag{12}$$
$$\implies \frac{n!}{(n-k)!k!} \geq \frac{n!}{(n-k)!k!} \frac{n-k}{k+1} \tag{13}$$
$$\implies 1 \geq \frac{n-k}{k+1} \tag{14}$$
$$\therefore k \geq \frac{n-1}{2} \tag{15}$$

From (11) and (15), we can state that

$$\frac{n-1}{2} \leq k \leq \frac{n+1}{2} \tag{16}$$

We know that, $k \in \mathbb{W}$ and $k \in [0, n]$ and from (16),

$$\text{If } n \text{ is even:} \quad k = \frac{n}{2} \tag{17}$$
$$\text{If } n \text{ is odd:} \quad k = \frac{n+1}{2} \quad \text{or} \quad \frac{n-1}{2} \tag{18}$$

As,

$$n = 6 \tag{19}$$

We can confirm from (17) that

$$k = \frac{n}{2} \tag{20}$$
$$= 3 \tag{21}$$

Hence proved that,

$$X = 3 \tag{22}$$

is the most likely outcome and $p_X(3)$ is

$$p_X(3) = \binom{6}{3}\left(\frac{1}{2}\right)^6 \tag{23}$$
$$= \frac{5}{16} \tag{24}$$

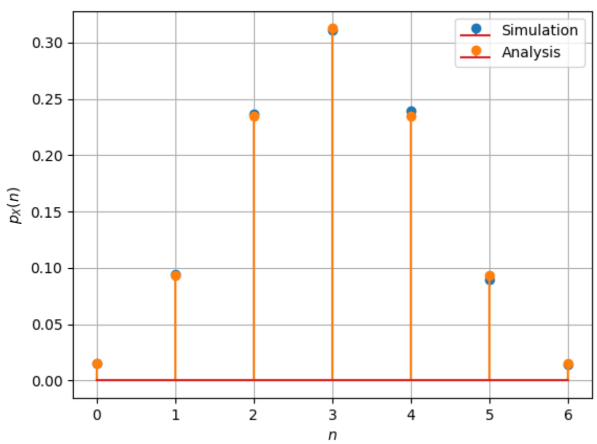

Fig. 0. Figure compares the therotical and simulation output