Assignment

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Question : Suppose that (X, Y) has joint probability mass function

$$P(X = 0, Y = 0) = P(X = 1, Y = 1) = \theta, \quad (1)$$

$$P(X = 1, Y = 0) = P(X = 0, Y = 1) = \frac{1}{2} - \theta. \quad (2)$$

where $0 \leq \theta \leq \frac{1}{2}$ is an unknown parameter. Consider testing $H_0 : \theta = \frac{1}{4}$ against $H_1 : \theta = \frac{1}{3}$; based on a random sample $(X_1, Y_1), (X_2, Y_2), \dots (X_n, Y_n)$ from the above probability mass function. Let M be the cardinality of the set $\{i : X_i = Y_i, 1 \leq i \leq n\}$. If m is the observed value of M, then which one of the following statements is true?

A) The likelihood ratio test rejects H_0 if $m > c$ for some c.
B) The likelihood ratio test rejects H_0 if $m < c$ for some c.
C) The likelihood ratio test rejects H_0 if $c_1 < m < c_2$ for some c_1 and c_2.
D) The likelihood ratio test rejects H_0 if $m < c_1$ or $m > c_2$ for some c_1 and c_2.

Solution: Given that,

$$H_0 : \quad \theta = \theta_0 = \frac{1}{4}, \quad (3)$$

$$H_1 : \quad \theta = \theta_1 = \frac{1}{3}. \quad (4)$$

and the pmf is given by

$$p_{XY}(0,0) = p_{XY}(1,1) = \theta p_{XY}(0,1) = p_{XY}(1,0) = \frac{1}{2} - \theta \quad (5)$$

Then for the given random sample of data,

$$\Pr(X_i, Y_i) = \begin{cases} 2\theta & X_i = Y_i \\ 1 - 2\theta & X_i \neq Y_i \end{cases} \quad (6)$$

$$(7)$$

Then the likelihood of the data under H_0 is given by:

$$L(\theta_0 \mid data) = \prod_{i=1}^{n} \Pr(X_i, Y_i) \quad (8)$$

$$= (2\theta_0)^m (1 - 2\theta_0)^{n-m} \quad (9)$$

$$= \left(\frac{1}{2}\right)^m \left(\frac{1}{2}\right)^{n-m} \quad (10)$$

Then the likelihood of the data under H_1 is given by:

$$L(\theta_1 \mid data) = \prod_{i=1}^{n} \Pr(X_i, Y_i) \quad (11)$$

$$= (2\theta_1)^m (1 - 2\theta_1)^{n-m} \quad (12)$$

$$= \left(\frac{2}{3}\right)^m \left(\frac{1}{3}\right)^{n-m} \quad (13)$$

The likelyhood ratio will be

$$\lambda(data) = \frac{L(\theta_1 \mid x)}{L(\theta_0 \mid x)} \quad (14)$$

$$= \frac{\left(\frac{2}{3}\right)^m \left(\frac{1}{3}\right)^{n-m}}{\left(\frac{1}{2}\right)^m \left(\frac{1}{2}\right)^{n-m}} = (2)^m \left(\frac{2}{3}\right)^n \quad (15)$$

Let the critical value be denoted by c_1, then the likelihood ratio test rejects H_0 if

$$\implies \lambda(data) \underset{H_0}{\overset{H_1}{\gtrless}} c_1 \quad (16)$$

$$(17)$$

From (15),

$$\implies (2)^m \left(\frac{2}{3}\right)^n \underset{H_0}{\overset{H_1}{\gtrless}} c_1 \quad (18)$$

$$\implies (2)^m \underset{H_0}{\overset{H_1}{\gtrless}} c_1 \left(\frac{2}{3}\right)^n \quad (19)$$

$$\implies m \underset{H_0}{\overset{H_1}{\gtrless}} \log_2 \left(c_1 \left(\frac{2}{3}\right)\right)^n \quad (20)$$

$$\implies m \underset{H_0}{\overset{H_1}{\gtrless}} c \quad \exists c \in \mathbb{R} \quad (21)$$

$\therefore$ From (21), Option A is correct and Options B,C,D are incorrect