Assignment

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Question : Suppose that (X, Y) has joint probability mass function

$$P(X=0, Y=0) = P(X=1, Y=1) = \theta, \tag{1}$$

$$P(X=1, Y=0) = P(X=0, Y=1) = \frac{1}{2} - \theta. \tag{2}$$

where $0 \le \theta \le \frac{1}{2}$ is an unknown parameter. Consider testing $H_0 : \theta = \frac{1}{4}$ against $H_1 : \theta = \frac{1}{3}$; based on a random sample $(X_1, Y_1), (X_2, Y_2), \dots (X_n, Y_n)$ from the above probability mass function. Let M be the cardinality of the set $\{i : X_i = Y_i, 1 \le i \le n\}$. If m is the observed value of M, then which one of the following statements is true?

A) The likelihood ratio test rejects H_0 if $m > c$ for some c.
B) The likelihood ratio test rejects H_0 if $m < c$ for some c.
C) The likelihood ratio test rejects H_0 if $c_1 < m < c_2$ for some c_1 and c_2.
D) The likelihood ratio test rejects H_0 if $m < c_1$ or $m > c_2$ for some c_1 and c_2.

Solution: Given that,

$$H_0 : \quad \theta = \theta_0 = \frac{1}{4}, \tag{3}$$

$$H_1 : \quad \theta = \theta_1 = \frac{1}{3}. \tag{4}$$

and the pmf is given by

$$p_{XY}(0,0) = p_{XY}(1,1) = \theta \tag{5}$$

$$p_{XY}(0,1) = p_{XY}(1,0) = \frac{1}{2} - \theta \tag{6}$$

Then for the given random sample of data,

$$\Pr(X_i, Y_i) = \begin{cases} 2\theta & X_i = Y_i \\ 1 - 2\theta & X_i \ne Y_i \end{cases} \tag{7}$$

$$\tag{8}$$

Then the likelihood of the data under H_0 is given by:

$$L(\theta_0 \mid data) = \prod_{i=1}^{n} \Pr(X_i, Y_i) \tag{9}$$

$$= (2\theta_0)^m (1 - 2\theta_0)^{n-m} \tag{10}$$

$$= \left(\frac{1}{2}\right)^m \left(\frac{1}{2}\right)^{n-m} \tag{11}$$

Then the likelihood of the data under H_1 is given by:

$$L(\theta_1 \mid data) = \prod_{i=1}^{n} \Pr(X_i, Y_i) \tag{12}$$

$$= (2\theta_1)^m (1 - 2\theta_1)^{n-m} \tag{13}$$

$$= \left(\frac{2}{3}\right)^m \left(\frac{1}{3}\right)^{n-m} \tag{14}$$

The likelyhood ratio will be

$$\lambda(data) = \frac{L(\theta_1 \mid x)}{L(\theta_0 \mid x)} \tag{15}$$

$$= \frac{\left(\frac{2}{3}\right)^m \left(\frac{1}{3}\right)^{n-m}}{\left(\frac{1}{2}\right)^m \left(\frac{1}{2}\right)^{n-m}} = (2)^m \left(\frac{2}{3}\right)^n \tag{16}$$

Let the critical value be denoted by c_1, then the likelihood ratio test rejects H_0 if

$$\implies \lambda(data) \underset{H_0}{\overset{H_1}{\gtrless}} c_1 \tag{17}$$

$$\tag{18}$$

From (16),

$$\implies (2)^m \left(\frac{2}{3}\right)^n \underset{H_0}{\overset{H_1}{\gtrless}} c_1 \tag{19}$$

$$\implies (2)^m \underset{H_0}{\overset{H_1}{\gtrless}} c_1 \left(\frac{2}{3}\right)^n \tag{20}$$

$$\implies m \underset{H_0}{\overset{H_1}{\gtrless}} \log_2 \left(c_1 \left(\frac{2}{3}\right)\right)^n \tag{21}$$

$$\implies m \underset{H_0}{\overset{H_1}{\gtrless}} c \quad \exists c \in \mathbb{R} \tag{22}$$

$\therefore$ From (22), Option A is correct and Options B,C,D are incorrect