Assignment

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Question : Suppose that (X, Y) has joint probability mass function

$$P(X = 0, Y = 0) = P(X = 1, Y = 1) = \theta, \tag{1}$$

$$P(X = 1, Y = 0) = P(X = 0, Y = 1) = \frac{1}{2} - \theta. \tag{2}$$

where $0 \le \theta \le \frac{1}{2}$ is an unknown parameter. Consider testing $H_0 : \theta = \frac{1}{4}$ against $H_1 : \theta = \frac{1}{3}$; based on a random sample $(X_1, Y_1), (X_2, Y_2), \dots (X_n, Y_n)$ from the above probability mass function. Let M be the cardinality of the set $\{i : X_i = Y_i, 1 \le i \le n\}$. If m is the observed value of M, then which one of the following statements is true?

A) The likelihood ratio test rejects H_0 if $m > c$ for some c.
B) The likelihood ratio test rejects H_0 if $m < c$ for some c.
C) The likelihood ratio test rejects H_0 if $c_1 < m < c_2$ for some c_1 and c_2.
D) The likelihood ratio test rejects H_0 if $m < c_1$ or $m > c_2$ for some c_1 and c_2.

Solution: Given that,

$$H_0 : \quad \theta = \theta_0 = \frac{1}{4}, \tag{3}$$

$$H_1 : \quad \theta = \theta_1 = \frac{1}{3}. \tag{4}$$

Let x represent the random sample. Then the likelihood of the data under H_0 is given by:

$$L(\theta_0 \mid x) = \prod_{i=1}^{n} P(X_i, Y_i) \tag{5}$$

$$= \left(\frac{1}{4}\right)^m \left(\frac{3}{4}\right)^{n-m} \tag{6}$$

Then the likelihood of the data under H_1 is given by:

$$L(\theta_1 \mid x) = \prod_{i=1}^{n} P(X_i, Y_i) \tag{7}$$

$$= \left(\frac{1}{3}\right)^m \left(\frac{2}{3}\right)^{n-m} \tag{8}$$

The likelyhood ratio will be

$$\lambda(x) = \frac{L(\theta_1 \mid x)}{L(\theta_0 \mid x)} \tag{9}$$

$$= \frac{\left(\frac{1}{3}\right)^m \left(\frac{2}{3}\right)^{n-m}}{\left(\frac{1}{4}\right)^m \left(\frac{3}{4}\right)^{n-m}} = \left(\frac{3}{2}\right)^m \left(\frac{8}{9}\right)^n \tag{10}$$

Let the critical value be denoted by c_1, then the likelihood ratio test rejects H_0 if

$$\implies \lambda(x) > c_1 \tag{11}$$

$$\tag{12}$$

From (10),

$$\implies \left(\frac{3}{2}\right)^m \left(\frac{8}{9}\right)^n > c_1 \tag{13}$$

$$\implies \left(\frac{3}{2}\right)^m > c_1 \left(\frac{9}{8}\right)^n \tag{14}$$

$$\implies m > \log_{\frac{3}{2}} \left(c_1 \left(\frac{9}{8}\right)^n\right) \tag{15}$$

$$\implies m > c \quad \exists c \in \mathbb{R} \tag{16}$$

$\therefore$ From (16), Option A is correct and Options B,C,D are incorrect