An application of the augmented Lagrangian approach to the optimal control of a biharmonic system with state–space constraints

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AN APPLICATION OF THE AUGMENTED LAGRANGIAN APPROACH TO THE OPTIMAL CONTROL OF A BIHARMONIC SYSTEM WITH STATE-SPACE CONSTRAINTS

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For the solution of certain optimal control problems with elliptic systems and inequality state-space constraints the augmented Lagrangian approach is proposed. In order to ensure the existence of a solution of the appropriate dual problem, special class of augmented Lagrangians is constructed on such Sobolev spaces for which the continuous imbedding into C_0 is guaranteed (see e.g. [7]). This approach is demonstrated on the case of a system governed by a biharmonic equation.

0. Introduction

The application of the augmented Lagrangian approach, which is very well known and exploited for the finite-dimensional mathematical programming, meets various difficulties in the infinite-dimensional case. One of the main troubles is how to ensure the existence of a Kuhn–Tucker vector. Rockafellar in [15] discussed the infinite-dimensional programming problem, but the constraint space was supposed to be Euclidian. The first attempt to treat the case of the infinite-dimensional constraint space is [18], where the range space of the inequality constraints is Hilbert. By means of a generalized convexity concept, various augmented Lagrangians were obtained for constraints even in certain Banach spaces (reflexive or non-reflexive — cf. [2, 9]). Another class of augmented Lagrangians was applied to the optimal control of parabolic systems in [14].

In our paper we shall slightly modify the developed theory for the case of a general convex set of admissible observations. In Section 2 we apply it to a biharmonic system and we shall notice the numerical aspects of biharmonic problems, especially the numerical solution of the projection onto the considered convex set in the space $\mathring{H}^2(\Omega)$. The advantage of our method is that the existence of a Kuhn–Tucker vector is ensured due to the Slater condition, and because of using the space $\mathring{H}^2(\Omega)$, the numerical method is not so difficult as it would be the case for $H^2(\Omega)$.

1. Duality theory and numerical approximation

Let U and X be two reflexive Banach spaces, let H be a Hilbert space. Let us have a convex continuous functional J defined and lower bounded on $U \times X$. Let A be a continuous operator from $U \times X$ into X^* (the dual space of X). Let the derivatives A'_x, J'_x, A'_u, J'_u exist in the Fréchet sense, suppose J'_x, A'_x to be continuous as operators on $U \times X$. Let us denote by $\chi_{\mathfrak{D}}$ the indicatory function for an arbitrary convex set $\mathfrak{D}$, let $C \subset U$, $K \subset H$ be closed convex sets. Let us denote $\hat{J} = J + \chi_C$, assume $\hat{J}$ to be coercive with respect to u independently of $x \in X$ on $U \times X$. Let q be a continuous operator from X into H such that the functional $[p, u] \to \operatorname{dist}(p - q(\Pi(u + y_0)), K)$ is convex and continuous, where $\Pi(u + y_0)$ is the solution of the problem $A(u + y_0, x) = 0$ with prescribed u, y_0 is a fixed element of U and $p \in H$ (we suppose the existence and the uniqueness of that solution). The condition of the convexity and continuity of the function $[p, u] \to \operatorname{dist}(p - q(\Pi(u + y_0)), K)$ is fulfilled, e.g. if both q and Π are continuous and linear or if Π is continuous and linear, K is a closed convex cone with the vertex in the origin and q is K-convex.

Let us look for the solution of the following problem:

$$\operatorname{Inf}\{J(u, x);\, u \in C, q(x) \in K,\ A(u + y_0, x) = 0\}. \tag{P}$$

Firstly, we define the perturbed essential objective

$$F_r(u, p) = J(u, \Pi(u + y_0)) + \chi_C(u) + \chi_K(q(\Pi(u + y_0)) - p) + r\,\|p\|_H^2, \quad r > 0 \tag{1.1}$$

and we obtain by the usual way the augmented Lagrangian

$$L_r(u, p^*) = -(F_r)_u^*(p^*) = \hat{\hat{J}}(u) + \inf_{v \in H}[r\|q\|_H^2 - \langle p^*, q\rangle + \langle p^* - 2rq, v\rangle + r\|v\|_H^2 + \chi_K(v)],$$

where the convex function

$$\hat{\hat{J}}(u) = \hat{J}(u, \Pi(u + y_0)),$$

$(F_r)_u^*(p^*)$ is the conjugate function to F_r with respect to p (for fixed u), the argument of q is dropped and $q - v$ is substituted for p. Denoting $\alpha = q - \dfrac{p^*}{2r}$, we obtain

$$L_r(u, p^*) = \hat{\hat{J}}(u) + r\|q\|_H^2 - \langle p^*, q\rangle + 2r \inf_{v \in K}\left\langle -\alpha + \frac{v}{2}, v\right\rangle.$$

The infimum is attained for $w \in K$ iff $\langle -\alpha + w, v - w\rangle \geq 0$ for all $v \in K$, thus iff $w = \alpha^K$ (the projection of α onto K). So we have

$$L_r(u, p^*) = \hat{\hat{J}}(u) + r\left\|\left(q - \frac{p^*}{2r}\right) - \left(q - \frac{p^*}{2r}\right)^K\right\|_H^2 - r\left\|\frac{p^*}{2r}\right\|_H^2. \tag{1.2}$$

The dual objective function

$$g_r(p^*) = - F_r^*(0, p^*) = \inf_{u \in H} L_r(u, p^*) = - p_r^*(p^*),$$

where

$$p_r(p) = \inf\{\hat{\hat{J}}(u) + r\|p\|_H^2; q(\Pi(u + y_0)) \in p + K\} = p_0(p) + r\|p\|_H^2.$$

So we have

$$g_r(p^*) = \sup_{a^* \in H}\left(g_0(a^*) - \frac{1}{4r}\|p^* - a^*\|_H^2\right) =$$
$$= -\frac{1}{2r}\inf_{a^*}\left(\frac{1}{2}\|p^* - a^*\|_H^2 - 2rg_0(a^*)\right) = -\frac{1}{2r}\left(\frac{1}{2}\|p^* - \tilde{p}^*\|_H^2 - 2rg_0(\tilde{p}^*)\right), \tag{1.3}$$

where $\tilde{p}^* = \mathrm{prox}_{-2rg_0}(p^*)$. The dual problem has the form

$$\sup\{g_r(p^*); p^* \in H\}. \tag{D}$$

Lemma 1.1. Let the mentioned suppositions be satisfied, furthermore let $q \circ \Pi$ be a continuous mapping. Then the problem (P) is normal in the Ekeland and Temam sense (cf. [4]). Moreover, if K has a non-empty interior and there exists such $u \in C$ that $\Pi(u + y_0) \in \mathrm{Int}\, K$, then the problem (P) is stable (a Kuhn–Tucker vector exists) for every $r \geq 0$.

The proof is well known, the normality is a straightforward consequence of the lower weak semicontinuity and the coercivity of J, the convexity and closeness of K and the properties of Π and q. The stability is based on the obvious Slater condition.

Lemma 1.2. Let φ be a proper convex lower semicontinuous functional defined on some Hilbert space $\varkappa$. Let us denote $P_\varphi(x) = \mathrm{prox}_\varphi(x)$. Then for the mapping $\beta(x) = \frac{1}{2}\|x - P_\varphi(x)\|_\varkappa^2 + \varphi(P_\varphi(x))$ there exists the Fréchet derivative $\beta'(x) = x - P_\varphi(x)$ and it holds

$$0 \leq \beta(x + h) - \beta(x) - \beta'(x)h \leq \frac{1}{2}\|h\|_\varkappa^2. \tag{1.4}$$

The proof is well known. It is a straightforward consequence of the variational formulation of the definition of $P_\varphi(x)$ and the convexity of β (cf. also e.g. [11]).

Corollary. The mapping $\vartheta : w \to \frac{1}{2} ||w - w^{\mathfrak{D}}||^2_{\varkappa}$, where $w^{\mathfrak{D}}$ is the projection onto a closed convex set $\mathfrak{D} \subset \varkappa$, has the Fréchet derivative in the form

$$\vartheta'(w) = w - w^{\mathfrak{D}}. \tag{1.5}$$

Proof: Put the indicatory function $\chi_{\mathfrak{D}}$ as φ into lemma 1.2.

By the corollary of lemma 1.2 it holds

$$(L_r)'_{p^*}(u, p^*) = \left(q(\Pi(u + y_0)) - \frac{p^*}{2r}\right)^K - q(\Pi(u + y_0)). \tag{1.6}$$

Numerical approximation of the solution of the problem (P) is based on the following schema (cf. e.g. [15])

k-th step:
k.1. given $p_k^* \in H$, determine $u_k \in C$ such that $L_r(u_k, p_k^*) \leq \inf_{u \in C} L_r(u, p_k^*) + \alpha_k$;

k.2. set $p_{k+1}^* = p_k^* + 2r\left[\left(q(\Pi(u_k + y_0)) - \frac{p_k^*}{2r}\right)^K - q(\Pi(u_k + y_0))\right]$.

Under suppositions mentioned above, the schema has the following properties:

Lemma 1.3. Suppose $\sum_{k=1}^{+\infty} \sqrt{\alpha_K} < +\infty$ and let the Slater condition hold. Then there exists a Kuhn–Tucker vector p^* such that $p_k^* \to p^*$.

The proof of this lemma is well known and is based on the following propositions:

Proposition 1.3.1. It holds

$$r\,||(L_r)'_{p^*}(u_k, p_k^*) - g_r'(p_k^*)||^2 \leq \alpha_k. \tag{1.7}$$

The proof consists in using of the mentioned properties of the function g_r and lemma 1.2.

Proposition 1.3.2. $\{p_k^*\}$ is a bounded sequence.

This proposition is a straightforward consequence of the preceding proposition, if we consider that

$$p_{k+1}^* - p_k^* = 2r(L_r)'_{p^*}(u_k, p_k^*) \text{ and } p_k^* - \operatorname{prox}_{-2rg_0}(p_k^*) = -2rg_r'(p_k^*)$$

(cf. lemma 1.2).

All optimal solutions $\hat{p}^*$ of (D) (cf. lemma 1.1) are fixed points of the mapping prox_{-2rg_0} ~~(p_k^*) which~~ is non-expansive.

Proposition 1.3.3. $\{p_k^*\}$ is a maximizing sequence of the problem (D).

For the proof the inequality

$$g_r(p_{k+1}^*) \geq g_r(p_k^*) + r\,||g_r'(p_k^*)|| - 2c\sqrt{r\alpha_k} \tag{1.8}$$

is very essential, being a consequence of the properties of g_r (c is the local Lipschitz constant of g_r). Using the concavity of g_r and the fact $g_r'(p_k^*) \to 0$, we obtain the proved assertion.

Proposition 1.3.4. $\{p_k^*\}$ is convergent.

From propositions 1.3.3 and 1.3.4 we immediately obtain lemma 1.3. To prove proposition 1.3.4 we shall verify the Bolzano–Cauchy condition. From proposition 1.3.1 and the proof of proposition 1.3.2 and using (1.8) we obtain the following inequalities for $l < k$:

$$||p_k^* - p_l^*|| \leq \sum_{l=l}^{k-1} 2r\,||g_r'(p_l^*)|| + 2\sqrt{r\alpha_l} \leq 2(g_r(p_k^*) - g_r(p_l^*)) +$$
$$+ 2(1 + 2c)\sqrt{r}\sum_{l=l}^{k-1}\sqrt{\alpha_l}. \tag{1.9}$$

Using proposition 1.3.3 and the summability of $\sum_{l=1}^{+\infty}\sqrt{\alpha_l}$, we obtain the required result.

Lemma 1.4. $\{u_k\}$ is a bounded asymptotically minimizing sequence for (P). Every weak cluster point of $\{u_k\}$ is an optimal solution of (P).

Proof: The boundedness of $\{u_k\}$ is implied by the form of $L_r(u_k, p_k^*)$ and the coercivity of $\hat{J}$. Because of (1.8), (1.7) and (1.6) dist $(q(\Pi(u + y_0)), K) \to 0$. From this fact and lemma 1.3 we can see that $\{u_k\}$ is asymptotically minimizing for (P). Let u_0 be a weak cluster point of $\{u_k\}$. Due to weak lower semicontinuity of the functionals $u \to \hat{\hat{J}}(u)$ and $u \to \text{dist}(p - q(\Pi(u + y_0)), K)$ and the preceding assertion of our lemma, we can see that u_0 is an optimal solution of (P).

The solution of the intermediate problem

$$\min\,\{L_r(u, p_k^*);\, u \in C\} \tag{1.10}$$

(resp. of finding u_k such that the condition of step k.1 of our schema will be satisfied) is substantially simplified under the assumption that $J(\cdot,\cdot)$ is continuously Fréchet differentiable on $U \times X$. In such a case, lemma 1.5 below can be utilized for the computation of the Fréchet derivatives of the function $L_r(u, p_k^*) - \chi_c(u)$ with respect to control.

Lemma 1.5. Let X, U, A, Π be as above, let Π be locally Lipschitz in the neighborhood of a point $u \in U$. Let $\mathfrak{F}$ be a functional on $U \times X$, let $\mathfrak{F}'_x$ exist in the Fréchet sense and be continuous as the operator from $U \times X$ into X^*. Define $\mathfrak{L}(u, x, \varkappa) = \mathfrak{F}(u, x) + \langle \varkappa, A(u + y_0, x) \rangle$, where $\varkappa \in X$ is a solution of the problem

$$\mathfrak{F}'_x(u, x) + \varkappa \circ A'_x(u + y_0, x) = 0 \in X^*. \tag{1.11}$$

Let $x = \Pi(u + y_0)$, $x' = \Pi(v + y_0)$. Then it holds

$$\mathfrak{F}(v, x) - \mathfrak{F}(u, x) = \mathfrak{L}(v, x, \varkappa) - \mathfrak{L}(u, x, \varkappa) + o(||u - v||). \tag{1.12}$$

The proof based on the direct calculation and obvious manipulations using all suppositions is made in [10].

In such a way, the intermediate problem is in fact a constrained minimization on the space U which can be solved by various existing methods. For simple control constraints such as upper and lower bounds in some spaces we recommend to apply some variant of the conjugate gradient method (cf. [7]) and to treat these constraints by the "clipping off" technique described in [13].

2. Optimal control of a system described by a biharmonic equation with the state-space and control constraints

Now, we shall apply the general theory described in the previous section to the following problem: Let Ω be a bounded domain in R^2 with a sufficiently smooth boundary Γ (our method is suitable also for the case $\Omega \subset R^3$). Let us take $H \doteq X = \mathring{H}^2(\Omega)$, $U = L_2(\Omega)$, $q = Id_X$.

Let c_1, c_2, c_3, D be positive numbers. Let $\psi \in L_\infty(\Omega)$ fulfil the condition $\psi(x) \leq -c_3$ on some neighborhood of Γ. Let Ω' be a measurable subset of Ω. We suppose that $\varphi_0 \in L_2(\Omega)$ such that $\varphi_0(x) \in [c_1, c_2]$ a.e. on Ω. Let us assume

$$J(u) = \int_\Omega \varphi_0 u \, dy; \tag{2.1}$$

$$A(u, x) = D \Delta^2 x - u, \quad y_0 \equiv f \in L_2(\Omega), \tag{2.2}$$

$$K = \{x \in \mathring{H}^2(\Omega), x \geq \psi \text{ a.e. on } \Omega'\}, \tag{2.3}$$

$$C = \{u \in L_2(\Omega);\ 0 \leq u \leq \gamma \text{ a.e. on } \Omega\};\ \gamma \in L_\infty(\Omega). \tag{2.4}$$

On the space $\dot{H}^2(\Omega)$ we consider the norm $\|\cdot\|_2$ in the form

$$\|x\|_2 = \left(\int_\Omega \sum_{i,j=1}^{2} \left(\frac{\partial^2 x}{\partial y_i \partial y_j} \right)^2 dy \right)^{\frac{1}{2}} = \left(\int_\Omega (\Delta x)^2 dy \right)^{\frac{1}{2}}. \tag{2.5}$$

This system equation describes e.g. the deflection of the floor of a warehouse. This floor must not deflect from various reasons (safety, operation etc.), deeper than some previously stated limit. Function f represents the own weight of the floor, control u is the weight distribution of the stored material and the upper limit for u is given by the height of the ceiling and the density of the stored material. Function φ_0 appearing in the objective is the price for a unit of the storage area depending on the allocation in the warehouse. The number D (the modulus of the flexural rigidity) can be considered without any loss of generality to be equal to 1. Our task is to maximize the profit of the warehouse or, if $\varphi_0 \equiv$ const., the maximal capacity of the warehouse under preserving the given state-space constraint (feasible deflection of the floor).

We can see that the spaces, the operators and the set C satisfy the requirements of Section 1. Due to the declared properties of ψ and Sobolev's imbedding theorem for $\dot{H}^2(\Omega)$ dim $\Omega \leq 3$, we have Int $K \neq \emptyset$. If we suppose the existence of such $u \in C$ that the corresponding solution of the state equation belongs to Int K, we shall ensure the stability of (P). To solve the problem (P) described by means of (2.1)–(2.5), the method derived in Section 1 can be applied. For the determination of u_k in the step k.1 of the numerical schema we exploit the assertion of lemma 1.5 for $\mathcal{F}(u, x) = L_r(u, x, p^*) - \chi_c(u)$ for the fixed parameter p_k^*. From (1.11) we state that the corresponding form of the co-state equation is

$$\varkappa = p_k^* - 2rx + 2r \left(x - \frac{p_k^*}{2r} \right)^K. \tag{2.6}$$

From (2.6) and (1.12) we can see that

$$\mathcal{F}'_u(u, \Pi u) = \frac{\partial \mathfrak{L}}{\partial u} (u, x, \varkappa) = \mathcal{F}'_u(u, x) + \varkappa \circ A'_u(u + f, x) =$$

$$= \varphi_0 + 2rx - p_k^* - 2r \left(x - \frac{p_k^*}{2r} \right)^K. \tag{2.7}$$

The minimization of step k.1 can be numerically carried out e.g. by the variant of the conjugate gradient method described for the case $U = L_\infty(\Omega)$ in [18]. The control constraints are respected in such a way that we involve two disjoint subsets of Ω at the beginning of the iteration procedure as estimations of "active constraint sets"

$$\alpha_0 = \{y \in \Omega; u(y) = 0\}, \quad \beta_0 = \{y \in \Omega; u(y) = \gamma(y)\}.$$

They are currently updated in such a manner that they approach the real "active constraint sets" for u. The construction of a conjugate direction is correspondingly modified. Thus, we must be able to compute for every control $u \in C$ the corresponding value of the objective and the Fréchet derivative given by (2.7). For the evaluation of the cost value the system equation must be solved. If, moreover, the Fréchet derivative is to be evaluated, the co-state equation (2.6) must be solved. This is, in fact, still more complicated than the solution of the system equation, because the projection onto the set K is a complicated biharmonic problem with an obstacle on the domain Ω.

So, from the practical numerical point of view, we have two main auxiliary problems. The more difficult problem, namely to find the projection onto the set K (i.e. to find such $v' \in K$ that $\|v' - s\|_2^2 \leq \|w' - s\|_2^2$ for every $w' \in K$ for some previously given s), can be described (on setting v for $v' - s$ and $w = w' - s$) in the following form: We define $K_s = \{w \in \mathring{H}^2(\Omega); w \geq \psi - s$ a.e. on $\Omega\}$ and look for such $v \in K_s$ that for every $w \in K_s$ the following inequality holds

$$\int_\Omega \Delta v \Delta(w - v)\, dy \geq 0. \tag{2.8}$$

Among several possibilities of the numerical solution of the problem we look upon the following two methods as suitable:

1° The equivalent formulation of our problem is

$$\min \left\{\frac{1}{2} \int_\Omega |\Delta w|^2\, dy,\ w \in K_s\right\}. \tag{2.9}$$

This problem can be solved directly using the finite element method. For the sake of simplicity we consider rectangular Ω. Let $\{\mathcal{R}_h\}$, $h \to 0_+$, $h = (h_1, h_2)$ be a regular system of rectangulations of $\bar{\Omega}$. Let $\mathcal{N}_h$ be the set of all vertices of rectangles of $\mathcal{R}_h$. We suppose $\mathcal{N}_h \subset \mathcal{N}_{h'}$ for $h > h'$. On $\mathcal{R}_h$ we define the Ahlin elements

$$Q_3(R_k) = \left\{\mathfrak{a}(y_1, y_2);\ \mathfrak{a}(y_1, y_2) = \sum_{i,j=0}^{3} \alpha_{ij} y_1^i y_2^j,\ [y_1, y_2] \in R_k\right\},\quad R_k \subset \mathcal{R}_h.$$

By the usual way we define the appropriate approximative spaces $V_h \subset \mathring{H}^2(\Omega)$

and the corresponding approximative convex sets $K_s^h \subset V^h$. For problem (2.9) the finite-dimensional Ritz approximation has the form

$$\min \left\{ \frac{1}{2} \int_\Omega |\Delta w|^2 \, dy; \; w \in K_s^h \right\} \tag{2.9$_h$}$$

and it is a quadratic programming problem. Using the Sobolev imbedding theorem ($\dot{H}^2(\Omega) \hookrightarrow C_0(\bar{\Omega})$) and the coercivity of the functional in (2.9), we can prove the convergence of the method (see [6]). (2.9)$_h$ can be solved exploiting the well-known SOR method (for details see [5]). Similar results can be obtained by means of Ari–Aridini's elements.

2° We have $w - \psi + s \in L_\infty(\Omega)$. Let Λ be the dual positive cone in $(L_\infty(\Omega))^*$ and $\langle\langle \cdot, \cdot \rangle\rangle$ the duality pairing between $L_\infty(\Omega)$ and $(L_\infty(\Omega))^*$. Let us define the following Lagrangian on $\dot{H}^2(\Omega) \times (L_\infty(\Omega))^*$

$$l(w, \check{c}) = \frac{1}{2} \int_\Omega |\Delta w|^2 \, dy + \langle\langle \check{c}, \psi - s - w \rangle\rangle.$$

Since Int $K_s \neq \emptyset$, l has a saddle point (v, τ), where v is a solution of (2.9) (cf. [4]). To find some saddle point, we use the following schema:

$$\begin{aligned} &\lambda^0 \geq 0 \text{ arbitrary} && \Delta^2 w^n = \lambda^n \text{ in } \Omega \\ &w^n = \frac{\partial w^n}{\partial n} = 0 \text{ on } \Gamma, && \lambda^{n+1} = [\lambda^n + g_n(\psi - s - w^n)]_+, \quad g_n > 0, \end{aligned} \tag{2.10}$$

where $[\check{z}]_+ : y \to \max(\check{z}(y), 0)$ for a function $\check{z}$ on Ω. Then we solve the sequence of biharmonic equations. This direct way is advantageous especially for rectangular Ω; we can apply the same elements as in 1° and we use the standard procedure.

With help of a factorization we can convert the biharmonic equation into two 2$^{\text{nd}}$ order elliptic problems in $\dot{H}^1(\Omega)$ as follows:

$$\begin{aligned} &\lambda^0 \geq 0 \text{ arbitrary} && -\Delta m^n = \lambda^n \text{ in } \Omega, \qquad -\Delta w^n = m^n \text{ in } \Omega \\ &w^n = m^n = 0 \text{ on } \Gamma, && \lambda^{n+1} = [\lambda^n + \varrho_n(\psi - s - w^n)]_+, \; \varrho_n > 0. \end{aligned} \tag{2.11}$$

This procedure can be applied for polygons. We use the triangulation of Ω and the elements composed from linear or quadratic polynomials on triangles. If Ω is a general domain with the Lipschitz boundary, we can approach it by a certain polygon Ω^h from its interior and use the regular triangulation of Ω^h. The convergence is proved in [5]. The order of the convergence is independent of $S^h = \Omega \setminus \Omega^h$. For the problems with the operator Δ cf. also [1], where some combination of procedures of 1° and 2° is described, or [17], where the external approximations are used.

Of course, the second main numerical difficulty, namely the state equation, can be solved by means of the same finite elements as in 2°. For details confront [5].

3. Conclusion

Our paper has two purposes: firstly to modify the primal–dual method for solving non-linear programs of [15], [16] and [18], in order to extend its area of applicability. As the space of perturbations a certain type of Sobolev spaces was chosen in order to guarantee the existence of a solution of the appropriate dual problem. The second aim is to introduce a non-trivial application of this method; namely the optimal control problem with a biharmonic equation and state–space and control constraints having a straightforward practical interpretation. The resulting numerical problems, especially the corresponding projection, are also considered. However, these numerical operations seem to be a little bit time-consuming at present. For a further study, the analysis of different generalized Lagrangians in this context supplies an attractive research field. The numerical solution of concrete problems of this type will be the subject of an intended paper.

References

1. *Brezzi, F., Hager, W. W., Raviart, P. A.*, Error estimates for the finite element solution of variational inequalities. Num. Math., Vol. **28**, *4*, 1977, 431–444 (Part I); Vol. **31**, *1*, 1978, 1–16 (Part II).
2. *Dolecki, S., Kurcyusz, S.*, On Φ-convexity in extremal problems. SIAM J. Contr. and Optimiz., Vol. **16**, 1978, 277–300.
3. *Duvaut, G., Lions, J. L.*, Les Inéquations en Mécanique et en Physique. Dunod, Paris, 1972.
4. *Ekeland, I., Temam, R.*, Analyse Convexe et Problèmes Variationnels. Dunod, Paris, 1974.
5. *Glowinski, R., Lions, J. L., Trémolières, R.*, Analyse Numérique des Inéquations Variationelles. Dunod, Paris, 1976.
6. *Haslinger, J.*, On numerical solution of a variational inequality of the 4th order by finite element method, Apl. mat., Vol. **23**, *5*, 1978, 334–345.
7. *Koivo, H. N., Mayorga, R. V.*, A robust conjugate-gradient algorithm for optimal control problems. Proc. 7-th. Congress IFAC, Helsinki, 1978.
8. *Kufner, A., John, O., Fučík, S.*, Function Spaces. Academia, Praha, 1977.
9. *Kurcyusz, S.*, Some remarks on generalized Lagrangians. Proc. 7-th IFIP Conference on Optimization Techniques, Nice, 1975.
10. *Luenberger, D. E.*, Optimization by Vector Space Methods, J. Wiley and Sons, N. Y., 1968.
11. *Moreau, J. J.*, Proximité et dualité dans un espace Hilbertien. Bull. Soc. Math. France, **93**, 1965, 273–299.
12. *Outrata, J. V., Schindler, Z.*, An augmented Lagrangian method for a class of convex continuous optimal control problems. This journal (to appear).

13. *Pagurek, B., Woodside, C. M.*, The conjugate gradient method for optimal control problems with bounded control variables. Automatica, Vol. **4**, 1968, 337–349.
14. *Pillo, G. di, Grippo, L.*, The multiplier method for optimal control problems of parabolic systems. Rap. Istit. Automatica Univ. Roma e Centro studio sist. contr. e calcolo autom. C.N.R., nov. 1978.
15. *Rockafellar, R. T.*, A dual approach to solving nonlinear programming problems by unconstrained optimization. Math. Programming, Vol. **3**, 1973, 354–373.
16. *Rockafellar, R. T.*, The multiplier method of Hestenes and Powel applied to convex programming. J. Optimiz. Theory and Appl., Vol. **9**, 1973, 555–562.
17. *Trémolières, R.*, Inéquation variationneles; Existence, approximations, résolution. Thèse, Univ. de Paris VI, 1972.
18. *Wierzbicki, A. P., Kurcyusz, S.*, Projection on a cone, penalty functionals and duality theory for problems with inequality constraints in Hilbert space. SIAM J. Contr. and Optimiz., Vol. **15**, 1977, 25–56.

Применение обобщенных функций Лагранжа для оптимального управления бигармонической системой с ограничениями на состояние

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(Прага)

Для решения одного класса выпуклых задач оптимального управления с эллиптическими системами и ограничениями типа неравенств на фазовые координаты предлагается подход, основанный на применении обобщенных функций Лагранжа. Чтобы обеспечить существование решения двойственной задачи, используется специальный класс этих функций на пространствах Соболева, для которых гарантировано вложение в пространство непрерывных функций. Этот подход применен к решению задачи с системой, описанной бигармоническим уравнением.

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