Topics in Randomness

Karthik Ravishankar

December 12, 2022

Contents

1 Lecture 1

The main notions are Kolmogorav complexity and Martin Lof Randomness.
Kolmogorov complexity: This measures the information content or complexity of a finite binary string. $C(\sigma)$ is the legnth of the shortest binary description of σ.
Martin Lof Randomness: An infinite binary sequence $X \in 2^\omega$ is random if it is in no effective measure zero set (typical sequences are random).

How are these notions related? The idea is that random sequences should have incompressible initial segments.This is seen by the fact that For almost all $X \in 2^\omega$, $\exists c \exists^\infty n$ such that $C(X|_n) \geq n - c$ and this implies X is $ML-random$. (Miller 2005, NST 2006) An X as above $\iff$ X is $ML-random$ relative to $0'$ - ($2random$).
Levin(73) and Chaitin(75) used modified forms of Kolmogrov complexity to characterize randomness. Let $K : 2^{<\omega} \to \omega$ be prefix - free complexity, then by (Schnorr 75) : X is ML-random $\iff \exists c \forall n K(X|_n) \geq n - c$.
Schnorr(71) characterized ML-randomness in terms of certain (semi computable) betting games to formalize the fact that Random reals are 'unpredictable'.
Partial Randomness/Hausdorff dimension : Lutz(2000/2003) effectivized Hausdorff dimension. Now the effective Hausdorff dimension of a singleton need not be 0. Mayordomo (2002) shoed that $dim(X) = liminf_n K(X|_n)/n = liminf_n C(X|n)/n$. So $ML-random \implies dim(X) = 1$. In Lutz and Lutz 2018, they gave the point to set principle: $dim_H(E) = min_{Z \in 2^\omega} sup_{X \in E} dim^Z(X)$ for any $E \subset 2^\omega$. This has applications to geometric measure theory (Lutz, Lutz, Don Stall...).

2 Lecture 2

References: i) Computability and Randomness - Nies 2009
ii) Algorithmic Randomness and Complexity - Downey and Hirschfeldt 2010

2.1 Kolmogorov Complexity

Definition: Let $M : 2^{<\omega} \to 2^{<\omega}$ be any partial function. Then $C_M(\sigma) = min\{|\tau : M(\tau) = \sigma\}$ or ∞.
Let $\{M_k\}_{k\in\omega}$ be an effective listing of all partial computable functions $2^{<\omega} \to 2^{<\omega}$.
Then we define a partial computable $V : 2^{<\omega} \to 2^{<\omega}$ by $V(0^k 1\sigma) = M_k(\sigma)$. This is our universal machine and compresses as well as any other machine (upto a constant).
Definition: $C(\sigma) = C_V(\sigma)$ for V as above. This is called the Kolmogorov complexity of σ.
If $M : 2^{<\omega} \to 2^{<\omega}$ is any partial computable function, then $C(\sigma) \leq^+ C_M(\sigma)$ i.e. there is at most a constant blow up in complexity and this constant is independent of σ. If $\hat{V}$ is another universal machine then $C(\sigma) =^+ C_{\hat{V}}(\sigma)$. We also have $C(\sigma) \leq^+ |\sigma|$ as the identity function is partial computable.
If h is partial computable, then $C(h(\sigma)) \leq^+ C(\sigma)$. We always have incompressible strings :$\forall n \exists \sigma \in 2^n$ such that $C(\sigma) \geq n$.
$C(n) =^+ C(0^n) \leq^+ log_2(n+1)$. Here we are identifying $\sigma \in 2^{<\omega}$ with the natural number $n \in \omega$ if 1σ is the binary expansion of $n+1$.
C is not computable but there is a computable function approximating it from above (and so $0'$ computable).

Intuition: We want random sequences to be incompressible. But no sequence $X \in 2^\omega$ has the property that $C(X|_n) \geq^+ n$ as we see below.
Lemma: If $|\tau| = \sigma$ i.e σ is the string representing the number $|\tau|$ then $C(\sigma\tau) \leq^+ |\tau|$, here $|\tau|$ is a number..
Proof: Let M be the machine which takes in τ and outputs $\sigma\tau$ where $\sigma = |\tau|$. Then $C(\sigma\tau) \leq^+ C_M(\sigma\tau) = |\tau|$.
Theorem: If $X \in 2^\omega$ then $\exists^\infty n C(X|_n) \leq^+ n - log(n)$.
Proof: For $k \in \omega$, let $\sigma = X|_k$ and $n = k + \sigma$ (where we are treating σ as the number it codes) and $\sigma\tau = X|_n$. This means that $|\tau| = n - k = \sigma$. So $C(\sigma\tau) \leq^+ |\tau|$.
Now $k = log(\sigma)$ (treating σ as a number) and so $k =^+ log(n)$ and so we're done.

Another consequence is as follows:
Theorem: It is not always the case that $C(\sigma\tau) \leq^+ C(\sigma) + C(\tau)$.
Proof: Fix $k \in \omega$. Take a string μ such that $|\mu| \geq 2^{k+1} + k$ (μ is long enough) and $C(\mu) \geq |\mu|$. Let $\sigma = \mu|_{k+\mu|_k}$ and let τ be the rest of μ i.e. $\sigma\tau = \mu$.
Then $C(\sigma) \leq^+ |\sigma| - k$ (as in the previous theorem), and so $C(\sigma) + C(\tau) \leq^+ |\sigma| = k + |\tau| = |\mu| - k \leq C(\mu) - k = C(\sigma\tau) - k$.

The problem in all this is that we are using the length of an input to code extra bits of information. In other words program length can underestimate 'information content'.

3 Lecture 3

Last time we saw that for any sequence in 2^ω there are infinitely many initial segments which are compressible by upto a factor of $log(n)$.
Possible fixes to this: i) We could require monotonicity: Restrict to M such that $\sigma \prec \tau \implies M(\sigma) \prec M(\tau)$ if both halt. But this isn't a very useful direction to pursue. Levin defined a monotone complexity using a more permissive model.
ii) We will restrict to machines M which have prefix free domains, that is $\sigma, \tau \in dom(M) \implies \sigma \not\prec \tau$. We say $M, dom(M)$ are prefix -free.
Chaitin thought in terms of self delimiting Turing machines where $M(\tau) = \sigma$ if the machine has read exactly τ off the input tape before halting with output σ. This is clearly prefix free (by uniqueness of the run), and it is easy to see that any prefix - free M can be given by a self delimiting machine- the machine starts doing all possible computations and checks whether it agrees with τ.
Intuition behind prefix free machines: Length of the program (input) is intrinsic to the program and doesn't provide extra information.
We can effectively list the prefix free partial computable functions. Using this effective listing, we can define a universal machine as before: $U(0^k 1\sigma) = M_k(\sigma)$.
Note that this is prefix free- If two strings are comparable, they must be input to the same prefix free machine and so cannot be comparable!
Definition: The prefix-free complexity of σ denoted by $K(\sigma) = C_U(\sigma) = min\{|\tau| : U(\tau) = \sigma\}$.
We have $K(\sigma) \leq^+ |\sigma| + K(|\sigma|)$.
Proof: Define M to be $M(\eta\sigma) = \sigma$ if $n = |\sigma|$ and $U(\eta) = n$. This is prefix free and shows $K(\sigma) \leq^+ K_M(\sigma) = |\sigma| + K(|\sigma|)$.
We can give a weaker bound of $K(\sigma) \leq^+ 2|\sigma|$ (To get rid of the K in on the right hand side) above. To get this bound, just repeat digits and use 01 as a delimitter.
We can get $K(\sigma) \leq^+ |\sigma| + log|\sigma| + 2loglog|\sigma|...$
Now we finally have subadditivity:
$K(\sigma\tau) \leq^+ K(\sigma) + K(\tau)$.
Proof: Define M to be $M(\tau_0\tau_1) = U(\tau_0)U(\tau_1)$ if both converge. We are using prefix free ness of U here to make this well defined
K is not computable but is approximable from above. $0'$ computes K.
(Krafts Inequality): If $D \subset 2^{<\omega}$ is prefix-free, then $\sum_{\sigma \in D} 2^{-|\sigma|} \leq 1$.
Let $[D] = \cup_{\sigma \in D}[\sigma]$. Since D is prefix free $\sigma, \tau \in D$ with $\sigma \neq \tau \implies [\sigma] \cap [\tau] = \emptyset$.
Therefore $\mu([D]) = \mu(\cup[\sigma]) = \sum_\sigma \mu[\sigma] = \sum_{\sigma \in D} 2^{-|\sigma|}$ and we know $\mu[D] \leq 1$.
As a corollary we get $\sum_{\sigma \in 2^{<\omega}} 2^{-K(\sigma)} \leq 1$.

4 Lecture 4

Definition: $X \in 2^\omega$ is $1-$ random if $K(X|_n) \geq^+ n$.
In fact we later show that $lim\ K(X|_n) - n \to \infty$ when X is $1-$ random.
Proposition: Almost all $X \in 2^\omega$ are $1-$ random.
Proof: Let $S_c = \{\sigma \in 2^{<\omega} : K(\sigma) \leq |\sigma| - c\}$ and $U_c = [S_c] = \{X \in 2^\omega : \exists n K(X|_n) \leq n - c\}$. Note that X is not $1-$random $\iff X \in \cap_{c\in\omega} U_c$.
But $\mu(U_c) = \mu([S_c]) \leq \sum_{\sigma \in S_c} \mu([\sigma]) = \sum_{\sigma\in S_c} 2^{-|\sigma|} \leq \sum_{\sigma \in S_c} 2^{-K(\sigma)-c} \leq 2^{-c} \sum_{\sigma \in 2^{<\omega}} 2^{-K(\sigma)} \leq 2^{-c}$.. Hence $\mu(\cap U_c) = 0$.

Recall: If $S \subset 2^{<\omega}$ is a *c.e.* set then $[S]$ is a Σ_1^0 class and $2^\omega - [S]$ is a Π_1^0 class.
Definition: A Martin Lof test is an effective sequence of Σ_1^0 classes (effective open sets) $\{V_n\}_{n\in\omega}$ such that $\mu(V_n) \leq 2^{-n}$. We say $X \in 2^\omega$ passes $\{V_n\}_n$ if $X \notin \cap V_n$.
$X \in 2^\omega$ is Martin Lof random if it passes all $ML-$ tests.
$ML-$ random $\implies$ $1-$ random. Every $ML-$ test gives us a measure $0, G_\delta$ set of non $ML-$ randoms. There are only countably many $ML-$ tests. Almost every $X \in 2^\omega$ is $ML-$ random.
Aside: Every measure 0 set $E \subset 2^\omega$ is covered by open sets of arbitrarily small measure. So $E \subset \cap_{n\in\omega} V_n$ for a Martin - Lof test $\{V_n\}$ relative to some oracle Z (which basically codes the strings generating each V_n.)
$E \subset 2^\omega$ has non zero outer measure $\iff \forall Z$ there is a $Z-ML$ random $X \in E$.
Getting back to Prefix free complexity, the Kraft inequality has an effective converse.
Theorem: Let $\{d_i\}_{i\in\omega}$ be a sequence of natural numbers such that $\sum_{i\in\omega} 2^{-d_i} \leq 1$. Then there is a prefix free sequence $\{\sigma_i\}_{i\in\omega}$ such that $|\sigma_i| = d_i$. We can compute σ_i from $d_0, ..., d_i$.
Proof: At stage n we have determined $\sigma_0, ..., \sigma_{n-1}$. Let the terminating binary expansion of $1 - \sum_{i<n} 2^{-d_i} = x_0.x_1x_2...x_m$. Inductively we will have strings with $|\tau_j| = j$ for each $x_j \neq 0$ such that $\{\sigma_i\}_{i<n} \cup \{\tau_j\}_{x_j \neq 0}$ is prefix free.
Now if $x_{d_n} \neq 0$ let $\sigma_n = \tau_{d_n}$. Otherwise let $k < d_n$ be greatest such that $x_k \neq 0$. Such a k will always exist by the weight condition. Then let $\sigma_n = \tau_k \frown 0^{k-d_n}$, and we add $\tau_k 1, \tau_k 01, ..., \tau_k 0^{k-d_n-1}1$ for the next stage.
Corollary (Kraft Chaitin/Machine existence theorem) Given an effective list of requests $< d_i, \tau_i >$ with $\sum 2^{-d_i} \leq 1$ then there is a prefix free machine M such that $\forall i \exists \sigma_i$ such that $|\sigma_i| = d_i$ and $M(\sigma_i) = \tau_i$.
$K(\tau_i) \leq^+ d_i$. If we only have $\sum 2^{-d_i} < \infty$ then $K(\tau_i) \leq^+ d_i$. Such sets of requests are called bounded request sets.

5 Lecture 5

Theorem: X is $1-$ random $\iff$ x is $ML-$ random.
Proof: The backward direction is done- we constructed a ML test above which prevents compressibility of initial segments.
For the forward direction, assume that X is not ML random. There is an ML test $\{V_n\}_{n\in\omega}$ such that $X \in \cap V_n$. Let $\{S_n\}$ be an effective list of *c.e.* sets of strings such that $[S_n] = V_n$. WLOG we may assume that S_n is prefix free (Just don't put a prefix, instead put a subset of the prefix which covers the same set as the prefix would). We define the request set $W = \{< |\sigma| - n, \sigma >: \sigma \in S_{2n}\}$.
Then this is a bounded request set:
$\sum_{<d,\sigma>\in W} 2^{-d} \leq \sum_{n\in\omega} \sum_{\sigma\in S_{2n}} 2^{-|\sigma|+n} = \sum_{n\in\omega} 2^n \sum_{\sigma\in S_{2n}} 2^{-|\sigma|} = \sum_n 2^n \mu(V_{2n}) \leq \sum_n 2^n 2^{-2n} < \infty$ So $\sigma \in S_{2n} \implies K(\sigma) \leq^+ |\sigma| - n$. But $\forall n \exists k$ we have $X|_k \in S_{2n}$. So X is not $1-$ random.
Corollary: There is a universal $ML-$ test, that is a test $\{U_n\}_n$ such that $\cap_n U_n$ is eactly the non $ML-$ randoms.
Corollary: i) The set of $ML-$ randoms is Σ^0_2.
ii) $2^\omega - U_1$ is a nonempty Π^0_1 class containing only $ML-$ randoms.
iii) There is a (super) low $ML-$ random.
iv) The leftmost point in $2^\omega - U_1$ is left - c.e ML -random. (computably approximable from below).

Definition: If M is a prefix free machine taking binary strings to binary strings, then $\Omega_M = \mu([dom(M)]) = \sum_{\sigma\in dom(M)} 2^{-|\sigma|}$ is the halting probability of M.
Chaitin's Ω is $\Omega = \Omega_U$.
Theorem: Ω is a left c.e. ML random.
Proof: Let U_t be the stage t approximation to U. Assume U_t contains strings of length at most t. Let $\Omega_t = \mu([domU_t])$. Then $\{\Omega_t\}$ is a non decreasing computable sequence of rationals such that $\Omega = lim\Omega_t$.
Define a partial computable $g : 2^{<\omega} \to 2^{<\omega}$ as follows: On input x of length n wait for t such that $0.x \leq \Omega_t \leq 0.x + 2^{-n}$. Then output the least element y not in the range of U_t.
If $X = \Omega|_n$ then such a t exists. By stage t all $U-$ programs of length $\leq n$ have halted. So $K(y) > n$ where $y = g(\Omega|_n)$.
Therefore $K(\Sigma|_n) \geq^+ K(g(\Omega|_n)) > n$.
Other left c.e. $ML-$ randoms: $\sum_n 2^{-K(n)}$, $\mu(U_1)$. In general $\mu(V)$ where V is a Σ^0_1 class and $2^\omega - V$ is non empty and contains noly ML-randoms.
Theorem: K is the least (w.r.t $\leq^+$) function $D : 2^{<\omega} \to 2^{<\omega}$ computable from above and having $\sum 2^{-D(\sigma)} < \infty$.
Proof: $W = \{< D(\sigma) + k, \sigma >: \sigma \in 2^{<\omega}, k \in \omega\}$ is a bounded request set: $\sum_{<d,\sigma>\in W} 2^{-d} = \sum_{k\in\omega} \sum_{\sigma\in 2^{<\omega}} 2^{-D(\sigma)-k} = \sum_{k\in\omega} 2^{-k} \sum_{\sigma\in 2^{<\omega}} 2^{-D(\sigma)} = 2\sum_{\sigma\in 2^{<\omega}} 2^{-D(\sigma)}$. Therefore $K(\sigma) \leq^+ D(\sigma)$.

6 Lecture 6

For any prefix free machine M let $P_M(\sigma) = \mu[M^{-1}(\sigma)]$.
Coding Theorem: For any prefix free machine M, $P_M(\sigma) \leq^* 2^{-K(\sigma)}$.
$P_U(\sigma) =^* 2^{-K(\sigma)}$ where U is the universal machine and $\leq^*$ is '$\leq$ upto multiplicative constant'.
Corollary: $\exists c \forall \sigma$, σ has at most c shortest $U-$ descriptions.
Proof: Let $D(\sigma) = ciel(-logP_M(\sigma))$. Note that D is computable approximable from above and $D(\sigma) \geq -logP_M(\sigma) \leq D(\sigma) - 1$. So $2^{-D(\sigma)} \leq P_M(\sigma) \leq 2^{-D(\sigma)+1}$. So $\sum_\sigma$ of LHS $\leq \sum_\sigma P_M(\sigma) \leq 1$. Thus $2^{-K(\sigma)} \geq^* 2^{-D(\sigma)} \geq^* P_M(\sigma)$. Since $P_U(\sigma) \geq 2^{-K(\sigma)}$ the second statement follows.
Counting Theorem: There is a $c \in \omega$ such that:
i) $\forall d, n, |\{\sigma \in 2^n : K(\sigma) \leq n + K(n) - d\}| < 2^c 2^{n-d}$.
ii) $\forall b, n, |\{\sigma \in 2^n : K(\sigma) \leq K(n) + b\}| < 2^c 2^b$.

Remark: a) Most strings have complexity close$^+$ to the upper bound
b) $A \in 2^\omega$ is $K-$ trivial if $K(A|_n) \leq^+ K(n)$.
Solovay showed that $K-$ trivial $\not\Longrightarrow$ computable. But computable $\Longrightarrow$ $K-$ trivial..$C-$ trivial $\Longrightarrow$ computable. By $ii)$ above, at most $2^c 2^b$ $K-$ trivials with constant b. So only countable many $K-$ trivials. In fact all are Δ^0_2. ($0'$ computes it since $K \leq_T 0'$ and we have a $0'$ computable tree with only isolated paths - since tree is bounded width).
c)The counting theorem is not tight.

Proof (Counting Theorem): Let M be the prefix free machine $M(\tau) = |U(\tau)|$. By the coding theorem $\exists c, \forall n P_M(n) < 2^c 2^{-K(n)}$. Let $S_{n,d} = S = \{\sigma \in 2^n : K(\sigma) \leq |\sigma| + K(|\sigma|) - d\}$. We have $P_M(n) \geq |S| 2^{-n-K(n)+d}$. So $|S| 2^{-n-K(n)+d} < 2^c 2^{-K(n)}$ so $|S| < 2^c 2^{n-d}$.

Definition (Martingales) : Let $B(\sigma)$ be the capital remaining after betting on $|\sigma|$ bits and seeing σ. $B(\sigma) = \frac{B(\sigma 0)+B(\sigma 1)}{2}$.

7 Lecture 7

Betting on a binary string bit by bit. Start with $B(\lambda)$ 'dollars'. After betting along σ we have $B(\sigma)$. If we bet γ on 0, then $B(\sigma 0) = B(\sigma 1) = B(\sigma) + \lambda + B(\sigma) - \lambda = 2B(\sigma)$.

Definition(Martingales) $B : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ is a martingale if $\forall \sigma, B(\sigma) = \frac{B(\sigma 0)+B(\sigma 1)}{2}$

B succeeds on $X \in 2^\omega$ if $limsup B(X|_n) = \infty$. Requiring $liminf B(X|_n)$ gives the same notion although rate of convergence may change.

Example: $B_\tau(\sigma)$ bets $2^{|\sigma|}$ when $\sigma \prec \tau$ and $2^{|\tau|}$ if $\tau \prec \sigma$ and 0 otherwise. This martingale fails for all $X \in 2^\omega$.

Definition : A supermartingale is a generalization of a martingale where we replace the equality by the inequality $S(\sigma 0) + S(\sigma 1) \leq 2S(\sigma)$.

Proposition For each supermatingale S there is a martingale B with the same start capital, such that $\forall \sigma B(\sigma) \geq S(\sigma)$.

Proof: Send extra capital to the left.

Proposition: A weighted sum of (super)martingales is a (super) martingale as long as the start capital is finite.

(Kolmogrov's Inequality): If S is a supermartingale and $W \subset 2^{<\omega}$ is prefix free, then $\sum_{\sigma \in W} 2^{-|\sigma|} S(\sigma) \leq S(\lambda)$.

Proof: WLOG assume W is finite. We prove by induction on the length n of the longest string in W. Clearn for $n = 0$, let $W_0 = \{\sigma : 0\sigma \in W\}$ and $W_1 = \{\sigma : 1\sigma \in W\}$. Then $S(\lambda) \geq 1/2(S(0)+S(1)) \geq 1/2(\sum_{\sigma \in W_0} 2^{-|\sigma|} S(0\sigma) + \sum_{\sigma \in W_1} 2^{-|\sigma|} S(1\sigma)) = \sum_{\sigma \in W} 2^{-|\sigma|} S(\sigma)$.

Definition: A supermartingale S is (left) c.e. if $S(\sigma)$ is left c.e. (a limit of a computable non decreasing sequence of rationals) uniformly in σ. (also called c.e. or lower semicomputable).

Corollary: For a supermartingale S, $\mu\{Z \in 2^\omega : \exists n S(Z|_n) \geq b\} \leq S(\lambda)/b$.

Proof: Let W be the prefix free set of minimal strings σ such that $S(\sigma) \geq b$, then $\mu(W) = \sum_{\sigma \in W} 2^{-|\sigma|}$. By Kolmogrov $\sum_{\sigma \in W} 2^{-|\sigma|} b \leq \sum_{\sigma \in W} 2^{-|\sigma|} S(\sigma) \leq S(\lambda)$.

Proposition: The follow are equivalent for $A \in 2^\omega$.

i) No c.e. supermartingale succeeds on A

ii) No c.e. martingale succeeds on A

iii) $\sum 2^{n-K(A|_n)} < \infty$

iv) $lim K(A|_n) - n = \infty$.

v) $K(A|_n) \geq^+ n$

vi) A is ML random.

Proof: $vi) \implies i)$ - Given a supermartingale S with $S(\lambda) \leq 1$, define $V_n = \{Z \in 2^\omega : \exists m S(Z|_m) > 2^n\}$. Then $\{V_n\}_{n\in\omega}$ is a ML test. IF S succeeds on A, then $A \in \cap V_n$.

$ii) \implies iii)$ - Define $M(\sigma) = \sum_{\tau \prec \neq \sigma} 2^{|\tau| - K(\tau)} + \sum_{\sigma \prec \tau} 2^{|\sigma| - K(\tau)} = \sum_{\tau \in \sigma} 2^{-K(\tau)} B_\tau(\sigma)$. $M(\lambda) = \sum 2^{-K(\tau)} \leq 1$. By ii M does not succeed on A. So there is a b such that $M(A|_m) < b$ for all m, but $\sum_0^m 2^{n-K(A|_n)} \leq M(A|_m) < b \forall m$.

8 Lecture 8

As a corollary to the proof we get a universal *c.e.* martingale which succeeds on all sequences that some *c.e.* martingale succeeds on i.e. on all non ML randoms.

Note: i)Recall that we used $M(\sigma) = \sum_{\tau \in 2^{<\omega}} 2^{-K(\tau)} B_\tau(\sigma)$. $f(\sigma) = 2^{-K(\sigma)}$ is maximal with respect to $\geq^*$ among functions approximable from below with $\sum f(n) < \infty$, so its the best way to combine B_τ's in a weighted way.
ii) A is ML - random $\iff \sum 2^{n-K(A|_n)} < \infty$. Conversely if $\sum 2^{-f(n)} < \infty$, then there is an $ML-$ random $X \in 2^\omega$ such that $K(X|_n) \leq^+ n + f(n)$.
iii) A c.e. supermartingale is optimal if it is maximal with respect to $\geq^*$ among *c.e.* supermartingales.We can effectively list *c.e.* supermartingales $\{S_n\}_{n\in\omega}$ with $S(\lambda) \leq 1$. Just take $S(\sigma) = \sum 2^{-n-1} S_n(\sigma)$ is an optimal *c.e.* supermartingale. We can't effectively list the computable or *c.e.* martingales, in fact there is no optimal c.e. martingale, nor a universal computable martingale (given a computable martingale, you can compute a sequence it doesn't win against, and no computable sequence is computably random).
iv) **Def(A priori complexity)** :Let S be an optimal *c.e.* supermartingale and let $KM(\sigma) = |\sigma| - log_2 S(\sigma)$. This is approximable from above, $0 \leq KM(\sigma) \leq^+ |\sigma|$ (the upper bound follows from the lower bound on $S(\sigma)$ since we can have a martingale that never bets and S is an optimal martingale). We now get $X \in 2^\omega$ is ML random $\iff KM(X|_n) =^+ n$. $X \in 2^\omega$ is computable $\iff KM(X|_n) =^+ 0$. We also have monotonicity $KM(\sigma \frown i) \geq KM(\sigma)$ for $i \in \{0,1\}$.
$KM(\sigma) \leq^+ K(\sigma)$ since $S(\sigma) \geq^* M(\sigma) \geq 2^{|\sigma|-K(\sigma)}$, where M is our universal *c.e* martingale. So $KM(\sigma) = |\sigma| - log_2 S(\sigma) \leq^+ |\sigma| - log_2(2^{|\sigma|-K(\sigma)}) = |\sigma| - |\sigma| + K(\sigma)$. We also get $KM(\sigma) = K(\sigma) \pm O(log|\sigma|)$.
If τ is σ backwards, obviously $K(\sigma) =^+ K(\tau)$ but we don't always have $KM(\sigma) \neq^+ KM(\tau)$. (This is one advantage of K over KM since we want $\psi(\sigma)$ to be no more complex than σ to compress with ψ is partial computable).
Next time we will see the following theorem:
Theorem(Kucera,Gacs): Every set is (wtt) reducible to a ML random sequence.
Recall: A wtt reduction is just a Turing reduction with a computable bound on the use.

9 Lecture 9

Def: A supermartingale S strongly succeeds on $X \in 2^\omega$ if $limS(X|_n) = \infty$.
A *c.e.* supermartingale is strongly universal if it strongly succeeds on every non ML random.
Fact: $M(\sigma) = \sum_{\tau \in 2^{<\omega}} 2^{K(\tau)} B_\tau(\sigma)$. is strongly universal.
Proof: Let X be non ML random. Fix k and take $\tau \prec X$ such that $K(\tau) \leq |\tau| - k$. Then for any $n \geq |\tau|, M(X|_n) \geq 2^{-K(\tau)} B_\tau(X|_n) = 2^{-K(\tau)} 2^{|\tau|} \geq 2^k$. Since k was arbitrary M wins along X.
Corollary: An optimal c.e. supermartingale is strongly universal.
Theorem : (Kucera, Gacs) Every set is (wtt) reducible to ML random sequence.
Lemma: Given $\delta > 1$ and $k \in \omega$ we can compute a length $l = l(\delta, k)$ such that for any supermartingale S and any σ, $|\{\tau \in 2^l : S(\sigma\tau) \leq \delta S(\sigma)\}| \geq k$.
Proof: Let T = set of bad strings of length l. By Kolmogrov's inequality : $\sum_{\tau \in T} 2^{-|\tau|} S(\sigma\tau) \leq S(\tau)$. The left hand side is $\geq |T| 2^{-|\tau|} \delta S(\sigma)$. So $|T| \leq 2^l/\delta$. To find the l, pick l such that $2^l - |T| \geq (1 - 1/\delta)@^l \geq k$. We get $l \geq log_2 k + log_2 \delta - log_2(\delta - 1)$.
Proof of theorem: Fix a strongly universal *c.e.* supermartingale S with $S(\lambda) \leq 1$. Fix a computable sequence of rationals $\{\delta_s\}_{s \in \omega}$ such that $\delta_s > 1$ and $\prod \delta_s \leq 2$. Let $l_s = l(\delta_s, 2)$ for each s. Fix X. We build a sequence A that is ML random and $X \leq_{wtt} A$. The use $u(t) = \sum_{s \leq t} l_s$.
Say we have built $\sigma = A|u(t-1)$ and $S(\sigma) \leq \prod_{s<t} \delta_s$. There are at least 2 strings $\tau \in 2^{l_s}$ such that $S(\sigma\tau) \leq \prod_{s \leq t} \delta_s$. Let τ_0 be the left most and τ_1 the rightmost. Define $A|_{\mu(t)} = \sigma\tau_i$ where $i = X(t)$. Note, we can decode X from A! This reduction is uniform too!
Note: $A \geq_T 0' \oplus X$.
Corollary : Every Turing degree above $0'$ contains a ML random.

9.1 A little measure theory

Definition: Let μ be a measure. If $C \subset 2^\omega$ is a μ measurable and σ is a string satisfying $\mu[\sigma] \neq 0$, let $\mu(C|\sigma) = \mu(C \cap [\sigma])/\mu([\sigma])$.
Theorem: If $\mu(C) > 0$ then $\forall \delta < 1 \exists \sigma \in 2^{<\omega}$ such that $\mu(C|\sigma) \geq \delta$ (Weak form of Lebesgue density)
Proof: Let $\epsilon = (1/\delta - 1)\mu(C)$. There is an open set $A \supset C$, (assuming μ is regular) such that $\mu(A) - \mu(C) \leq \epsilon$. So $\delta\mu(A) \leq \mu(C)$. Let D be prefix free such that $A = [D]$. If no $\sigma \in D$ satisfies $\mu(C|\sigma) \geq \delta$, then $\mu(C) = \sum_{\sigma \in D} \mu(C \cap [\sigma]) < \delta \sum_{\sigma \in D} \mu([\sigma]) = \delta\mu(A) \leq \mu(C)$. But then $\mu(C) < \mu(C)$, a contradiction.

10 Lecture 10

We'll only use the weak form of Lebesgue density for Lebesgue measure μ, but the proof works for regular measures (outer) and its true for any Borel measure. Lebesgue density says if μ is a Borel measure, for any $\mu - a.e.X \in 2^\omega$, and C measurable we have: $lim_{\sigma \prec X} \mu(C|\sigma) = 0$ when $X \notin C$ and 1 otherwise.

Theorem: X is not computabe $\implies \mu\{A : A \geq_T X\} = 0$.
Proof: Assume $\mu(\{A : A \geq_T X\}) > 0$. Fix an index e such that $\mu\{A : \varphi_e^A = X\} > 0$. Let $C = \{A : \varphi_e^A = X\}$. Take σ such that $\mu\{A : \varphi_e^{\sigma A} = X\} > 2/3$. To compute $X(n)$ wait for either $\mu\{A : \varphi_e^{\sigma A}(n) = 0\} > 1/3$ or $\mu\{A : \varphi_e^A(n) = 1\} > 1/3$ whichever happens is correct.
Exercise: $i)\{X : X$ has PA degree $\} = \{X : X$ computes a $0-1$ valued DNC function $\}$. (Similar to previous theorem)
$ii)$ If X is ML random, then $f(e) = X|_e$ can only agree with $\varphi_e(e)$ finitely often, so $\mu\{X : X$ has DNC degree$\} = 1$.

10.1 Relativizing ML randomness

A ML test relative to A is an A computable of $\Sigma_1^0[A]$ classes $\{V_n^A\}_{n\in\omega}$ such that $\mu(V_n^A) \leq 2^{-n}$.
Definition: X is ML random relative to A if X passes every $A - ML$ test.
Example: X is $n-$ random if X is 0^{n-1} random. So $1-$ random $= ML$ random. We can relativize the other characterizations, and everything works.
Unsurprisingly $B \leq_T A$ and X is $A-$ random $\implies X$ is $B-$ random.
The construction of a universal $A - ML$ test is uniform in A. So there is a universal oracle $A - ML - test$: $\{U_n^\square\}_{n\in\omega}$ for every $A \in 2^\omega$, $\{U_n^A\}_{n\in\omega}$ is a universal $A - ML - test$.
Example: If X is not computable and Z is $X \oplus 0'$ random, then $Z \not\geq_T X$.
Proof: Fix e. Using $X \oplus 0'$ we can find a $\sigma_n \prec X$ such that $\mu\{Z : \varphi_e^Z \succ \sigma_n\} \leq 2^{-n}$. Hence $\{Z : \varphi_e^Z = X\}$ is covered by the $X \oplus 0'$ ML test above.
Fact: X is $K-$ trivial $(K(X|_n) \leq^+ K(n)) \iff$ there is an X random Z such that $Z \geq_T X$. Recall all $K-$ trivials are Δ_2^0.
Van Lambalgen's Theorem: $A \oplus B$ is $1-$ random $\iff B$ is $1-$ random and A is $B-$ random.
Corollary If $A, B \in 2^\omega$ are both $1-$ random, then A is B random $\iff B$ is $A-$ random.
Corollary If $A \oplus B$ is $1-$ random then $A|_T B$. A, B may not be a minimal pair but if $X \leq_T A, B$ with $A \oplus B$ $1-$ random, then X is $K-$ trival. Such X form a proper subclass of $K-$ trivials.
Note: If X is $1-$ random and Z is $X-$ random, then $Z \not\geq_T X$.
Proof: X is Z random by Van Lambalgen's.
$VanLambalgen's$ theorem has analogues: Its true for $1-$ generics. Product forcing (Set theory): (G, H) is generic for $\mathbb{P} \times \mathbb{Q}$ over $V \iff G$ is $\mathbb{P}$ generic over V and H is $\mathbb{Q}$ generic over $V[G]$. Fubini's theorem can also be seen as an analogue.

11 Lecture 11

Van Lambalgen's Theorem: $A \oplus B$ is $1-$ random $\iff$ B is $1-$ random and A is $B-$ random.
Proof: For the forward direction let $\{U_n^{\square}\}_{n\in\omega}$be a universal oracle test. Let $W_n = \{\sigma \oplus \tau : |\sigma| = |\tau| and [\sigma] \subset U_n^{\tau}\}$. So $[W_n] = \{X \oplus Y : X \in U_n^Y\}$. Note that $\mu[W_n] = \int_Y \mu\{X \in 2^\omega : X \in U_n^Y\} dY = \int_Y \mu(U_n^Y) dY \leq \int_Y 2^{-n} dY = 2^{-n}$. Hence $\{[W_n]\}_n$ is a ML test.
So $A \oplus B \notin \cap[W_n]$ which means for some n we have $A \oplus B \notin [W_n]$. So A is $B-$ random. Similarly B is $A-$ random and hence $1-$ random.
Lemma If $\{V_n\}_{n\in\omega}$ is an $ML-$ test and X is $1-$ random then $X \notin V_n$ for almost all n.
Proof: Let $V_n' = \cap_{m>n} V_m$ Then $\mu(V_n') \leq \sum_{m>n} 2^{-m} = 2^{-n}$ so $\{V_n'\}_{n\in\omega}$ is a ML test. But $X \notin V_n' \implies \forall m > n$ we have $X \notin V_m$.
For the backward direction of $Van\ Lambalgen's\ theorem$:
Let $W_n = \{\tau \in 2^{<\omega} : (\forall Y \prec \tau)\mu\{X \in 2^\omega : X \oplus Y \in U_{2n}\} > 2^{-n}\}$. This set is $c.e.$ by compactness.
Also $[W_n] = \{Y \in 2^\omega : \mu\{X \in 2^\omega : X \oplus Y \in U_{2n}\} > 2^{-n}\}$. If $\mu([W_n]) > 2^{-n}$ then $\mu(U_{2n}) \geq \int_{Y\in[W_n]} \mu\{X \in 2^\omega : X \oplus Y \in U_{2n}\} dY \geq \int_{Y\in[W_n]} 2^{-n} dY = 2^{-n}\mu([W_n]) > 2^{-2n}$, a contradiction. So $\{[W_n]\}_{n\in\omega}$ form a ML test. Since B is $1-$ random, $B \notin [W_n]$ for almost all n. Thus for sufficiently large n we have $\{X \in 2^\omega : X \oplus B \in U_{2n}\} \leq 2^{-n}$. let $V_n^B = \{X \in 2^\omega : X \oplus B \in U_{2n}\}$, this is a Σ_1^0 class relative to B. Thus $\{V_n^B\}_n$ is eventually a B ML test. So $A \notin V_n^B$ for some n, and so $A \oplus B \notin U_{2n}$ for this n. So $A \oplus B$ is ML random.
Theorem: If Y is $1-$ random, X is $n-$ random and $Y \leq_T X$, then Y is $n-$ random.
Proof: This is an application of Van Lambalgen's theorem. Assume $n > 1$. Let $Z \equiv_T 0^{n-1}$ be a $1-$ random (Kucera Gacs). So X is $n-$ random $\iff$ X is 0^{n-1} random $\iff$ X is $Z-$ random $\iff$ Z is $X-$ random $\implies$ Z is $Y-$ random $\iff$ Y is $Z-$ random $\iff$ Y is $n-$ random.
Fact: If Y is $1-$ random and X is $Z-$ random and $Y \leq_T X$ then Y is Z random for any Z.
Idea: Pull a $Z - ML$ test covering Y up using the reductions to a $Z-$ ML test covering X.
Definition: X is low for $\Omega = \sum_{\sigma\in domU} 2^{-|\sigma|}$ (Chaitin's Ω) if Ω is $X-$ random.
Note: If X is $1-$ random then X is low for Ω $\iff$ X is $\Omega-$ random $\iff$ X is $0'$ random $\iff$ $2-$ random.
Next time we'll see that all left c.e. 1 randoms are essentially the same. So low for Ω is well defined.

12 Lecture 12

12.1 Left c.e. ML randoms

We have seen examples $\Omega = \mu[domU]$ and $\sum_{\sigma\in 2^{<\omega}} 2^{-K(\sigma)}$ and if P is a non empty Π^0_1 class containing only randoms, take its left most element.
Theorem If α is a left c.e. ML random, then $\alpha \equiv_T 0'$ (even with respect to wtt)
Proof: $0'$ computes all left c.e. reals. For the other direction, α has c.e. DNC degree so $\alpha \geq_T 0'$ by Arslanov completness criteria.
Define $V_n = [\alpha_s|n]$ if n enters $0'$ at stage s and $\emptyset$ otherwise. So $\{V_n\}$ is a ML test, hence $\exists N \forall n \geq N \alpha \notin V_n$. But then $n \in 0' \iff n \in 0'_s$ where $\alpha_s|_n = \alpha|_n$. So $\alpha \geq_{wtt} 0'$.

12.2 Solavay Reducibility

Def: Let α, β be left- c.e. reals, then $\beta \leq_s \alpha$ if $\exists d \in \omega, \gamma$ left c.e. such that $2^{-d}\beta + \gamma = \alpha$.
$\leq_S$ is transitive and if $\beta \leq_S \alpha$ then from $\alpha|_n$ we can uniformly approximate β to within 2^{-n+d+1}. Therefore $\leq_S \implies \leq_T$ but not uniformly. $K(\beta|_n) \leq^+ K(\alpha|_n)$.
Theorem: α is left c.e. and ML random $\iff$ there is a universal prefix free R such that $\alpha = \Omega_R = \mu[domR] \iff \alpha$ is left c.e. and Solovay complete.
Corollary Any two left c.e. ML randoms are Solovay equivalent and so have the same initial segment complexity and are random relative to the same oracles. Therefore low for Ω is well defined.
Proof: Ω_R is left c.e. ML random follows from the same proof that Ω is.
Let α be left c.e ML random. If β is left c.e and $\{\alpha_s\}, \{\beta_s\}$ be left c.e. approximations to α, β and assume $\beta_s < \beta_{s+1}$ and $\beta_{-1} = 0$. Build a ML test $\{F_n\}$ on $[0,1]$ as follows: If $\alpha_s \in$ closure$(F_{n,s})$ do nothing, otherwise put $(\alpha_s, \alpha_s + 2^{-n}(\beta_{s+1} - \beta_{t_s}))$ into F_n where t_s is the last stage at which we put something into F_n. Note that $\mu(F_n) \leq 2^{-n}\beta \leq 2^{-n}$. If $\alpha \notin F_n$ then $\mu(F_n) = 2^{-n}\beta$. So $\{F_n\}$ is a ML test and so $\alpha \notin F_n$ for some n. Let $\gamma = \mu([0,\alpha] - F_n) = lim_s \mu([0,\alpha_s] - F_{n,s})$ which is non decreasing since we never add anything $F_{n,s+1}$ behind α_s. So γ is left c.e. and $2^{-n}\beta + \gamma = \alpha$ and so $\beta \leq_S \alpha$.
(Lecture 13) Let α be Solovay complete. Choose $d \in \omega, \gamma$ left c.e. such taht $2^{-d}\Omega_U + \gamma = \alpha$. We want $2^{-d} + \gamma \leq 1$ but we can get this. Fix $d' \geq d$ such that $2^{-d'} + \alpha \leq 1$. Then $\alpha = 2^{-d}\Omega + \gamma = 2^{-d'}\Omega + (2^{-d} - 2^{-d'})\Omega + \gamma = 2^{-d'}\Omega + \gamma'$. But now $2^{-d'} + \gamma' \leq 2^{-d'} + \alpha \leq 1$. WLOG let $d = d', \gamma = \gamma'$. Let $\{d_i\}_i$ be a computable sequence such that $\gamma = \sum 2^{-d_i}$, then $\{< d, 0 >, < d_i, i+1\}_i$ is a bounded request set bounded by 1 and so there is a prefix free machine M such that $\Omega_M = 2^{-d} + \gamma$, and $\exists \sigma \in 2^d$ with $M(\sigma) = 0$. Define R by $R(\sigma\tau) = U(\tau)$ and $R(\tau) = M(\tau)$ otherwise.Then R is universal and $\Omega_R = 2^{-d}\Omega_U + \gamma$.

Barmpalias and Lewis Pye: If α, β are left c.e. randoms and $\{\alpha_s\}, \{\beta_s\}$ are left c.e. approximations to α, β, then the limit $\alpha - \alpha_s / \beta - \beta_s$ exists and does not depend on the choice of approximations.

13 Lecture 13

Definition: (Solovay functions) A computable $g : 2^{<\omega} \to \omega$ is a Solovay function if
$i)K(\sigma) \leq^+ g(\sigma)$
$ii)\exists c, \exists^\infty \sigma$ such that $g(\sigma) \leq K(\sigma) + c$.
If g is right $c.e.$ and satisfies $i, ii)$ then call g a weak Solovay function. **Proposition** If $g : 2^{<\omega} \to \omega$ is right c.e., then $K(\sigma) \leq^+ g(\sigma) \iff \sum 2^{-g(\sigma)} < \infty$.
The backward direction is by optimality of K. The forward direction is since K satisfies the property that we require of g.
Proposition: There is a Solovay function.
Proof: Let $g(<\sigma, \tau, t>) = |\tau|$ if $U(\tau) = \sigma, t$ is the stage at which $U(\tau)\downarrow$ and $2| < \sigma, \tau, t > |$ (something big) otherwise.
Now $\sum_{\sigma \in 2^{<\omega}} 2^{-g(\sigma)} \leq \sum_{\sigma \in 2^{<\omega}} 2^{-2|\sigma|} + \sum_{\tau \in domU} 2^{-|\tau|} = 2 + \Omega < \infty$. So $K(\sigma) \leq^+ g(\sigma)$. Now let τ be a shortest U program for σ and $U(\tau)\downarrow$ at stage t. Then $g(<\sigma, \tau, t>) = |\tau| = K(\sigma) \leq^+ K(<\sigma, \tau, t>)$ for such τ, σ, t.

From now on we look at functions as being from $\omega \to \omega$ (instead of $2^{<\omega} \to \omega$).
Theorem Let $f : \omega \to \omega$ be right c.e.. Then f is a weak Solovay function $\iff \sum 2^{-f(n)}$ is finite and ML -random. (Note that the sum is left c.e.)
Proof: Big proof on next page
Facts: If f is any (weak) Solovay function then :
$a)A$ is K trivial $\iff K(A|_n) \leq^+ f(n)$.
$b)A$ is ML random $\iff C(A|_n) \geq^+ n - f(n)$.
Definition: A is low for Ω if Ω is $A-$ random.
A is weakly low for K if $\exists c \exists^\infty n$ such that $K(n) \leq K^A(n) + c$.
Theorem The following are equivalent for $A \in 2^\omega$.
i) A is low for Ω
ii) A is weakly low for K.
Proof: A is low for $\Omega \iff \Omega$ is $A-$ random $\iff \sum 2^{-K(n)}$ is A random $\iff \sum 2^{-K(n)}$ is finite and K is right c.e. (relative to A) $\iff K$ is a weak Solovay function relative to $A \iff A$ is weakly low for K.

Aside: For $A \in 2^\omega$ A is low for random (X ML random $\implies$ X is A random) $\iff A$ is low for K ($K(\sigma) \leq^+ K^A(\sigma)$) $\iff A$ is K trivial. Also all such A are Δ^0_2).

14 Lecture 14

Theorem: let $f : \omega \to \omega$ be right c.e. Then f is a weak Solovay function $\iff \sum 2^{-f(n)}$ is finite and ML random.
Proof: Assume $\sum 2^{-f(n)}$ is finite.
For the backward direction, assume that f doesn't satisfy $\exists c \exists^\infty n$ such that $f(n) \leq K(n) + c \iff \Omega_f = \sum 2^{-f(n)}$ is ML random.
Fix a compute approximation $\{f_s\}$ to f such taht $\forall n\ f_0(n) \leq f_1(n) \geq f_2(n)....$
We define a left c.e. approximation to Ω_f as follows: $a_0 = 0$ and $a_{i+1} = a_i + d_i$ where $\{d_i\}$ is defined as follows: To find d_i, search for the next pair of the form $< n, 0 >$ or $< n, s+1 >$ where $f_s(n) - f_{s+1}(n) > 0$. In the first case $d_i = 2^{-f_0(n)}$ and in the second case $d_i = 2^{-f_{s+1}(n)} - 2^{-f_s(n)}$.
We say d_i occurs due to n and let $b_i =$ sum of all d_j due to n for $j \leq i$. So $lim a_i = \Omega_f$.
We define a ML test $\{U_c\}_{c\in\omega}$ with the goal of covering Ω_f. Say that i is $c-$ matched if d_i occurs due to n and $2^{c+1}b_i \leq 2^{-K(n)}$. If we see that i is matched at stage s, add an interval of length $2d_i$ to U_c starting at $max(\{a_i, supU_{c,s}\})$.
Note $\sum\{d_i : d_i$ occurs for n and is i matched $\} \leq 2^{-K(n)}/2^{c+1}$. So $\mu(U_c) = 2\sum_i\{d_i : i$ is $c-$matched $\} \leq 2/2^{c+1} = 2^{-c}$.
So $\{U_c\}_{c\in\omega}$ is a ML test. $\Omega_f \in \cap U_c$. Since $lim_{n\to\infty} f(n) - K(n) = \infty$. Fix c. For n large enough, all corresponding i will be $c-$ matched. So there is an $i(c)$ such th at $i \geq i(c) \implies i$ is c matched. For these i's add intervals to U_c of length $\beta = 2(\Omega_f - a_{i(c)})$, all above $a_{i(c)}$. Since $\Omega_f - a_{i(c)} < \beta$ for $\Omega_f \notin U_c$ then it must be in one of the gaps. But each gap has supremum a_i for some i and $a_i \leq \Omega_f$ and $a_i \neq \Omega_f$. So Ω_f is not in any gap and the backward direction is done.
Forward direction in next lecture.
Theorem The following are equivalent for $A \in 2^\omega$.
i) A is 2 random
ii) A is 1 random and low for Ω
iii) A is $1-$ random and weakly low for K. iv) $\exists c, \exists^\infty n$ with $K(A|_n) \geq n + K(n) - c$
v) $\exists c \exists^\infty n$ such that $C(A|_n) \geq n - c$.

15 Lecture 15

We shall now show that if $f : \omega \to \omega$ is right c.e then f is a weak Solovay function $\implies \Omega_f = \sum 2^{-f(n)}$ is finite and ML random. So we want to show that if $\alpha = \Omega_f$ is not ML random then $f(n) - K(n) \to \infty$.
Suppose $\alpha = \Omega_f$ is not ML random. We build a bounded request set B as follows: If $U(\tau) = \sigma$ wait until $|\alpha_s - \sigma| < 2^{-|\sigma|}$ where $\alpha_s = \sum_{n \leq s} 2^{-f_s(n)}$. Then for all $n \geq s, m \geq 0$, put $< n, |\tau| - |sigma| + f(n) + 2 + m >$ into B upto a total weight of $2^{-|\tau|}$. So B is a bounded request set because the total weight of B is $\leq \sum_{\tau \in dom(U)} 2^{-|\tau|} \leq 1$.
If $\sigma = \alpha| - k$ then s exists and $|\alpha - \alpha_s| < 2^{-K+1}$ and so $\sum_{n>s} 2^{f(n)} \leq \alpha - \alpha_s < 2^{-k+1}$. So we are trying to add at most weight $\sum_{n>s} 2.2^{-|\tau|+k-f(n)-2} < 2.2^{-\tau|+k-(k+1)-2} = 2^{-|\tau|}$.
If $\sigma = \alpha|_k$ for $n > s$, we have $K(n) \leq^+ |\tau| - k + f(n)$, where the constant doesnt depend on τ, since for each c $\exists k$ with $K(\alpha|_k) \leq kc$ so for large n, $K(n) \leq^+ (k-c) - k + f(n) = f(n) - c$. So $K(n) - f(n) \to \infty$.
Theorem: The folowing are equivalent for $A \in 2^\omega$.
i) A is $2-$ random
ii)A is 1 random and low for Ω
iii) A is $1-$ random and WLK
iv) $\exists c, \exists^\infty n$ such that $K(A|_n) \geq n + K(n) - c$
v) $\exists c, \exists^\infty n$ such that $C(A|_n) \geq n - c$.
Proof: $i \iff ii$ is by VL theorem and $\Omega \equiv_T 0'$.
$ii \iff iii$ follows by low for $\Omega \iff WLK$.
$iii \iff iv$: If A is $1-$ random then $\sum 2^{n-K(A|_n)} < \infty - Ampleexcess$
Corollary: If A is $1-$ random then $K^A(n) \leq^+ K(A|_n) - n$ since $\{< K(A|_n) - n + m, n >_{m,n \in \omega}$ is a bounded request set which is $c.e.$ relative to A. So $K^A(n) \leq^+ K(A|_n) - n$.
Now since A is WLK, $K(A|_n) \geq^+ n + K^A(n) \geq^+ n + K(n)$ for infinitely many n.
$iv \implies v$: We have $K(\sigma) \leq^+ C(\sigma) + K(C(\sigma))$. Also $K(n) \leq^+ K(m - n) + K(m)$. So $K(\sigma) \leq^+ |\sigma| + K(|\sigma|) - (|\sigma| - C(\sigma) - K(|\sigma| - C(\sigma))$. Now let $d_n = n - C(A|_n)$ and assume $d_n \to \infty$. Then $n + K(n) - K(A|_n) \geq^+ d_n - K(d_n) \geq^+ d_n - 2log(d_n) \to \infty$.
$v \implies i$ K maximal $\implies$ C maximal. The reverse does not hold.

16 Lecture 16

v $\implies$ i (from previous lecture): Let U be a universal prefix free oracle machine. Suppose A is not 2− random. So $\forall c, \exists^\infty n K^{0'}(A|_n) < n - c$.
Let σ_c be a string witnessing this. Now define a plain machine M: On input γ let $t = |\gamma|$, then M looks for a $\sigma \prec \gamma$ such that $U_t^{0'_t}(\sigma)\downarrow$. (at most one such σ). If $\gamma = \sigma\tau$ then $M(\gamma) = U_t^{0'_t}(\sigma)\tau$. So we are using the extra bits τ to get a large enough t to get the computation correct. For large enough t, $U_t^{0'_t}(\sigma_c) = A|_n$. So $M(\sigma_c A|_{[n,t)}) = A|t$ and $C_M(A|_t) < t - c$.

16.1 A technical result

Recall: A *c.e.* set $A \subset \omega$ is simple if it is coinfinite and $\overline{A}$ has no infinite *c.e.* subset. Clearly simple $\implies$ not computable.
Lemma: If $H \subset 2^\omega$ is a null Σ_3^0 class. Then there is a simple set A satisfying $\forall Y \in H$ with Y ML random, $A \leq_T Y$.
Proof: First let H be Π_2^0. Then $H = \cap V_X$ for an effective sequence of Σ_1^0 classes. We may assume $V_{x+1} \subset V_x$ by taking intersections. Let $V_x = [D_x]$ where D_x is prefix free c.e. and let $V_{x,s} = [D_{x,s}]$. Want to satisfy
$R_e : |W_e| = \infty \implies W_e \cap A \neq \emptyset$.
Define $C(n,s) = \mu(V_{n,s})$, called the cost function. We know $\mu(V_n) \to 0$ as $n \to \infty$.
Put x into A at stage s for the sake of e if i) $W_{e,s} \cap A_s = \emptyset$. ii) $x \in W_{e,s}$ iii) $c(x,s) < 2^{-e}$ iv) $x \geq 2e$
We will meet all R_e so A is simple.
Define Γ^Y as follows: If σ goes into D_n at stage s let $\Gamma^\sigma(n) = A_s(n)$. This fails for any $Y \in V_{x,s}$ if x goes into A at stage s. But $\mu(V_{x,s}) < 2^{-e}$.
Let $U_n = \cup\{V_{x,s} : x$ goes into A at stage s for the sake of $e > n\}$. So $\mu(U_n) < 2^{-n}$ and $\{U_n\}_{n\in\omega}$ is a ML test.
If $Y \in H$ is ML random, so Γ^H will be total and $Y \notin U_n$ for some n so $\Gamma^Y =^* A$ (at most n mistakes). Therefore $Y \geq_T A$.
To extend this to a Σ_3^0 class $H = \cup_i \cap_x V_x^i$, take $c(x,s) = \sum_i 2^{-i}\mu(V_x^i, s)$. This works out (details left out)
Theorem (Kucera): If Y is a Δ_2^0 ML random then Y computes a simple set.
Proof: $\{Y\} = \{X : \forall n, t \exists s > t\ Y_s|_n \prec X\}$ is null Π_2^0.
Theorem (Friedberg Muchnik): There is a *c.e.* set strictly $\leq_T$ between 0 and $0'$.
Proof: There is a low ML random Y. (low basis theorem). Take $A \leq_T Y$ simple using previous theorem.

17 Lecture 17

Lemma If $H \subset 2^\omega$ is a null Σ^0_3 class then there is a simple set A such that $A \leq_T Y$ for each ML random $Y \in H$.
Proposition: If $\Omega = \Omega_0 \oplus \Omega_1$ then (Ω_0, Ω_1) do not form a minimal pair.
Proof: $\{\Omega_0, \Omega_1\}$ is a null Π^0_2 class.
Facts: i) $A \leq_T \Omega_0, \Omega_1$ must be $K-$ trivial (we will see this)
ii) These are not all K trivials (We will not see this)

17.1 Weak 2- randomness ('strong 1 random'- Joe)

Definition: $A \in 2^\omega$ is weak 2 random if A is not in any null Π^0_2 class.
So we have $2-random \implies$ weak $2-random \implies 1-random$: The second implication follows from the fact that $\{U_n\}$ being an ML test $\implies \cap U_n$ is a null Π^0_2 class. For the first implication given a null Π^0_2 class $\cap V_n$ with the $V_n's$ nested, then $0'$ can uniformly find indices m_n such that $\mu(V_{m_n}) \leq 2^{-n}$ and so we get a $0'$ ML test.
Ω is $1-$ random but not weak $2-$ random because $\{\Omega\}$ is a null Π^0_2 class.
We will see that every hyper immune free $1-$ random is weak $2-$ random but not $2-$ random.
Theorem: A is weakly $2-random \iff A$ forms a minimal pair with $9'$ (computes no noncomputable Δ^0_2 set) $\iff A$ does not compute a simple set.
Proof: If A is not weakly $2-$ random then it is in some null Π^0_2 class H but then there is a simple set $B \leq_T A$ by the lemma.
Clearly if $A, 0'$ are a minimal pair then A cannot compute a simple set.
If X is a non computable Δ^0_2 set then $\{Z : Z \geq_T X\}$ is null. Also $\{Z : Z \geq_T X\} = \cup_e\{Z : \forall n, t\exists s \geq t\varphi^Z_{e,s}(n) = X_s(n);\}$ and so is Σ^0_3 so $A \not\geq_T X$.

17.2 Hyperimmune degrees

Definition: Call $f : \omega \to \omega$ an escaping function if it is not dominated (equivalently majorized) by any computable function.
Definition: X has hyperimmune degree if it computes an escaping function, otherwise it has hyperimmune free degree.
Facts: Every nonempty Π^0_1 class contains a HIF member.
Every noncomputable Σ^0_2 set has hyperimmune degree.
Corollary: If A is HIF and $1-$ random then A is weakly $2-$ random.
Proof: If A is not weakly $2-$ random then $A \geq_T C$ where C is simple and so A computes an escaping function and so A would not be HIF.
Theorem: If X is $2-$ random then it has hyperimmune degree.
Lemma: Let P be a $\Pi^0_1[X]$ class of positive measure, then every $X-$ random has a tail in P.
Proof: Let V be $X-$ c.e, prefix free so that $P = [V]^c$. Let $V_1 = V$, and $V_{n+1} = \{\sigma\tau : \sigma \in V_n, \tau \in V\}$. $\mu[V_n] = (\mu[V_1])^n$. So a linear subsequence of $\{[V_n]\}_n$ is an $X-$ ML test. Let Z be $X-$ random and $n = \mu s Z \notin [V_n]$. So $\exists \sigma \in V_{n-1}\ \sigma \prec Z$ let $Z = \sigma Z_0$, so $Z_0 \in P$.

18 Lecture 18

Corollary: Let C be a degree invariant class and $P \subset C$ a positive measure Π^0_1 class. Then every X random is in C.
Lemma: There is a $\Pi^0_1[0']$ class P of positive measure all of whose elements compute an escaping function. (by a single ψ).
Proof: Let $R_e : \exists n\ \psi^A(n) \downarrow > \varphi_e(n)$ for all $A \in P$. We act independently for each R_e. At stage s if s is not currently a witness for some R_e ensure that $\psi^A(s) \downarrow \forall A$.
Action of the R_e strategy: Let $\sigma_0, ..., \sigma_{2^{e+2}}$ list string of length $e+2$ and set $i = 0$. Pick a fresh witness n and define $\psi^{\sigma_j}(n) = 0$ when $j \neq i$. Wait for $\varphi_e(n) \downarrow$. (measure risking strategy). Set $\psi^{\sigma_i}(n) = \varphi_e(n) + 1$. Increment i. If $i \leq 2^{e+2}, goto\ 2$.
Now if φ_e is total then R_e is satisfied eventually $\forall A$, otherwise we don't care. We might wait forever on some σ_i so $\psi^A(n) \uparrow \forall \sigma_i \prec A$. $0'$ can enumerate all strings on which some R_e waits forever. Let $P = \{A : \psi^A$ is total $\}$ so P is a $\Pi^0_1[0']$ class. ψ^A is escaping for $A \in P$ and $\mu(P) \geq 1 - \sum 2^{-e-2} = 1/2$.
Theorem: Every $2-$ random computse an escaping function i.e. has hyperimmune degree.
Corollary: There is a weak $2-$ random that is not $2-$ random.
Proof:Take any HIF $1-$ random, by HIF basis theorem.
Fact: Every $2-$ random computse a $1 - generic$.

18.1 Scott Sets

We will see an application of the above results.
Definition: i) B has PA degree relative to A if B computes an element of every nonempty $\Pi^0_1[A]$ class.
ii) $S \subset 2^\omega$ is a Turing ideal if $A \in S, B \leq_T A \implies B \in S$ and $A, B \in S \implies A \oplus B \in S$.
iii) $S \subset 2^\omega$ is a Scott set if it is a Turing ideal and $\forall A \in S \exists B \in S$ such that B is PA relative to A.
These arise in the study of models of PA. They are the $2nd$ order parts of ω models of WKL.
Question (Friedman, McAllister): If S is a Scott ideal and $A \in S$ is non computable is there a $B \in S$ such that $A|_T B$.
Answer (Kucera, Slaman): Yes! Broke down by cases: A is/is not K trivial.
Question: Can there be a maximal antichain of size 2.
Conidis:Generalized KS to ω models of $WWKL$ (weak weak Konig's lemma).
ω **model of** $WWKL$: Turing ideal S satisfying for every $A \in S, \exists B \in S$ that is random relative to A or equivalenetly $\forall A \in S$ every $\Pi^0_1[A]$ class of positive measure has an element in S.
Note : S a Scott set $\implies$ S is an $\omega-$ model of $WWKL$.

19 Lecture 19

S is a Scott set $\implies S$ is an ω model of $WWKL$.
Theorem(Westrick): If S is an $\omega-$ model of $WWKL$ then for every $A \in S$ there is a $B \in S$ that is either weak $2-$ random relative to A or $1-$ generic relative to A.
Proof: Let $B \in S$ be $1-$ random relative to A. If B is not weak $2-$ random relative to A then there is an $A - c.e.$ set $C \leq_T B \oplus A$ such that $C \not\leq_T A$.
Fact: Every non computable $c.e.$ set computes a $1-$ generic.
Relativizing if $C \not\leq_T A$ is $A-$ c.e. then there is $D \leq_T C \oplus A$ that is $1-$ generic relative to A. Note that $D \in S$.
Facts: Assume A is not computable. i) If B is $1-$ generic relative to A then $B|_T A$ (In fact they form a minimal pair)
ii) If B is weak 2 - random relative to A then $B|_T A$: Clearly B $A-$ random $\implies B \not\leq_T A$. for the other direction note that $\{z : z \geq_T A\} = \cup_e \{z : \varphi_e^Z = A\} = \cup_e \{z : \forall n \exists s\ \varphi_{e,s}^z(n) \downarrow = A(n)\}$ is a $\Sigma_3^0[A]$ class and so $B \not\geq_T A$.
Corollary (Westrick): If S is an $\omega-$ model of $WWKL$ and $A_1, ..., A_n \in S$ are noncomputable then there is a $B \in S$ such that $B|_T A_i \forall i$.
Proof: Let X be a $1-$ generic or weak 2 random relative to $A_1 \oplus ... \oplus A_n$ hence relative to each, hence T incomparable from each.

19.1 Lowness and K - triviality

The following are equivalent for $A \in 2^\omega$:
i) A is low for K - $K^A(\sigma) \geq^+ K(\sigma)$.
ii) A is low for random - $X 1-$ random $\implies X$ A random.
iii) A is a base for $1-$ randomness- There is an $A-$ random $X \geq_T A$
iv) A is $K-$ trivial: $K(A|_n) \leq^+ K(n)$.
iv) If $Z \not\geq_T 0'$ is $1-$ random then $Z \oplus A \not\geq_T 0'$.
v) A is computable from every low for Ω PA degree. (and a lot more)
Easy Implications: $1 \implies 2$: X is $1-$ random $\iff K(X|_n) \geq^+ n \iff K^A(X|_n) \geq^+ n$.
$2 \implies 3$ By Kucera Gacs, there is a $1-$ random $X \geq_T A$ but X is automatically $A-$ random.
$1 \implies 4$ $K(A|_n) \leq^+ K^A(A|_n) =^+ K^A(n) \leq^+ K(n)$.
(hard) $2 \implies 1$:
Definition: $A \leq_{LR} B$ if every $B-$ random is $A-$ random. $A \leq_{LK} B$ if $K^A(\sigma) \geq^+ K^B(\sigma)$. We want to show $A \leq_{LR} B \iff A \leq_{LK} B$. If $A \leq_{LK} B$ and X is $B-$ random then $K^B(X|_n) \geq^+ n$ so $K^A(X|_n) \geq^+ K^B(X|_n) \geq^+ n$ and hence X is $A-$ random.
Lemma: If there is a prefix free machine M such that $Z = z_0 z_1 z_2 z_3 ...$ where $\forall i K_M(z_i) \leq |z_i| - 1$ then Z is not ML random.
Proof Define a prefix free macine N such that on σ it searches for a $\gamma \prec \sigma$ such that $U(\gamma)\downarrow$, if $U(\gamma) = n$, then N searches for n $M-$ programs $y_0, ..., y_{n-1}$ such that $\sigma = \gamma y_0 ... y_{n-1}$. Then $N(\sigma) = M(y_0)...M(Y_{n-1})$. So $K(z_0...z_{n-1}) \leq^+ K_N(z_0...z_{n-1}) \leq K(n) + \sum_{i<n} K_M(z_i) \leq K(n) + |z_0...z_{n-1}| - n \leq^+ |z_0...z_{n-1}| -$

$(n - 2logn) \to \infty$ as $n \to \infty$. Hence Z is not $ML-$ random.
Lemma: If X is ML random and $X \in P$ where P is a Π^0_1 class then $\mu(P) > 0$.
Proof: Let $\{P_s\}_{s\in\omega}$ approximate P where each P_s is clopen. Let $V_n = P_{S_n}$ where $s_n = \mu s[\mu(P_s) \leq 2^{-n}]$. Then $\{V_n\}$ is a ML test covering P.
Definition: X is weakly $1-$ random (Kurtz random) if X avoids all null Π^0_1 classess weakly.
Note: Weakly $1-$ generic $\implies$ weakly $1-$ random.

20 Lecture 20

Proposition: The follow are equivalent for $A \in 2^\omega$.
i) $A \leq_{LR} B$
ii) There is a $\Sigma^0_1[B]$ class S such that $\mu(S) < 1$ and $\forall \sigma\ K^A(\sigma) \leq |\sigma| - 1 \implies [\sigma] \subset S$.
iii) For each prefix free oracle machine M and S with $\mu(S) < 1$, $\sigma\ K_{M^A}(\sigma) \leq |\sigma| - 1 \implies [\sigma] \subset S$.
Proof: $iii \implies ii$ done.
$ii \implies i$: By assumption S contains all non A randoms, so $P = 2^\omega - S$ is a positive measure $\Pi^0_1[B]$ class that only contains A randoms. If X is $B-$ random then X has a tail $X_0 \in P$ so X_0 is $A-$ random and so X is A random.
$i \implies iii$: Fix M such that iii fails. We build a set Z that is $B-$ random but not $A-$ random so $A \not\leq_{LR} B$. Let H be a $\Sigma^0_1[B] class$ such that $\mu(H) < 1$ and $2^\omega - H \subset MLR^B$. We build $Z = z_0 z_1 ...$ such that $\forall i : a) K_{M^A}(z_i) \leq |z_i| - 1$ i.e. Z is not A random. $b)[z_0 z_1 ... z_i] \not\subset H$ i.e. Z is B random.
Note that $[\varphi]$ satisfies $b)$. Inductively suppose we have $w = z_0 ... z_{n-1}$ such that $[w] \not\subset H$. Let $S = \{X : wX \in H\}$. So S is $\Sigma^0_1[B], S \neq 2^\omega$. Note that $2^\omega - S$ contains only B randoms, so it can't have measure 0. Hence $\mu(S) < 1$. Note that the conditino in $iii)$ fails for S so there is a z_n such that $K_{M^A}(z_n) \leq |z_n| - 1$ but $[z_n] \not\subset S$. In other words $[z_0 .... z_n] \not\subset H$.
Corollary: $A \leq_{LR} B \iff$ every $\Sigma^0_1[A]$ class G with $\mu(G) < 1$ is covered by a $\Sigma^0_1[B]$ class S with $\mu(S) < 1$.
Proof: Let $G = [\{\sigma : K^A(\sigma) \leq |\sigma| - 1\}]$. There is a $\Sigma^0_1[B]$ class S such that $\mu(S) < 1$ and $G \subset S$ so ii holds in the proposition above.This shows the backward direction.
For the forward direction apply Lebesgue desntiy to $2^\omega - G$ to get σ such that $\mu(H) \leq 1/2$ where $H = \{z : \sigma z \in G\}$. Let $H = [V]$ where V is $A-$ c.e. and prefix free. Then the request set relative to A : $\{< |\tau| - 1, \tau >: \tau \in V\}$ has weight $2 \times \mu(H) \leq 1$, so there is an oracle prefix -free machine M such that $V = \{\tau : K_{M^A}(\tau) \leq |\tau| - 1\}$. Hence there is a $\Sigma^0_1[B]$ class $\hat{S}$ such that $\mu(\hat{S}) < 1$ and $H = [V] \subset \hat{S}$. Let $S = \sigma\hat{S} \cup (2^\omega - [\sigma])$. Then $\mu(S) < 1$ and $G \subset S$.

Theorem: The following are equivalent:
$i) A \leq_{LR} B$
$ii)$ Let $f : \omega \to \omega$ be computable. $\forall I$ which is $A-$ c.e. such that $\sum_{i \in I} 2^{-f(i)} < \infty$ there is a $B-$ c.e. set $J \supset I$ such that $\sum_{i \in J} 2^{-f(i)} < \infty$.
$iii) A \leq_{LK} B$.
Proof: $ii) \implies iii)$: Let $f(< d, \tau >) = d$ and let $I = \{< |\sigma|, \tau >: U^A(\sigma) = \tau\}$. So I is $A-$ c.e. and $\sum_{<d,\tau> \in I} 2^{-f(<d,\tau>)} = \sum_{<d,\tau> \in I} 2^{-d} = \mu[dom U^A] \leq 1$. Fix $J \supset I$ such that J is $B - c.e.$ and $\sum_{<d,\tau> \in J} 2^{-f(<d,\tau>)} < \infty$, so this is a bounded request set relative to B and so $\forall < d, \tau > \in J$ we have $K^B(\tau) \leq^+ d$ but $\forall \tau$ the pair $< K^A(\tau), \tau > \in I \subset J$ so $K^B(\tau) \leq^+ K^A(\tau)$.
We already know $iii) \implies i)$.
Lemma: For a sequence $\{a_i\}$ with $a_i \in [0, 1)$, $\sum_i a_i < \infty \iff \prod(1 - a_i) > 0$.

21 Lecture 21

We showed $ii \implies iii \implies 1$ last time. Now for $i \implies 2$:
Without loss of generality assume $0 \notin range f$. To each $s \in \omega$ we associate a finite set V_s as follows. Assume that V_t is defined $\forall t < s$. Let $m = max\{|\sigma| : \sigma \in \cup_{t<s} V_t\}$. Set $V_s = \{\sigma 0^{f(s)} : \sigma \in 2^m\}$ This ensures that $k \subset \omega \implies \mu(\cap_{s \in K}[V_s]^c) = \prod_{s \in K}(1 - 2^{-f(s)})$ since each $[V_s]^c$ is independent from the others. I is $A - c.e.$ so $P = \cap_{s \in I}[V_s]^c$ is a $\Pi^0_1[A]$ class. By the lemma $\mu(P) = \prod_{s \in I}(1 - 2^{-f(s)}) > 0$. So there is a $\Pi^0_1[B]$ class $Q \subset P$ such that $\mu(Q) > 0$.
Let $J = \{t : [V_t] \cap Q = \emptyset\}$.Note that J is $B - c.e.$ and $J \supset I$. Also $\prod_{t \in J}(1 - 2^{-f(t)}) = \mu(\cap_{t \in J}[V_t]^c) \geq \mu(Q) > 0$ so $\sum_{t \in J} 2^{-f(t)} < \infty$ giving ii.
Corollary A is low for random ($A \leq_{LR} \emptyset$) $\iff$ A is low for K ($A \leq_{LK} \emptyset$).
We should see (but haven't yet): i) basis for $1-$ randomness $\implies$ low for K.
ii) There is a non computable low for random.

Lemma: If $A \leq_T B'$ and $A \leq_{LR} B$ then every $\Pi^0_1[A]$ class has a $\Sigma^0_2[B]$ subclass of the same measure.
Proof: Let $X \neq \emptyset$ be a $\Pi^0_1[A]$ class. Let $S^a \subset 2^{<\omega}$ be a prefix free $A - c.e.$ set of strings such that $X = 2^\omega - [S^A]$. Let $I = \{<\sigma, \tau>: \tau \in S^A$ with use $\sigma \prec A\}$. This is $A - c.e.$ and $\sum_{<\sigma,\tau> \in I} 2^{-|\tau|} < \infty$. So there is a $B - c.e.$ set $J \supset I$ such that $\sum_{<\sigma,\tau> \in J} 2^{-|\tau|} < \infty$. Let $\{A_s\}_{s \in \omega}$ be a B computable approximation to A and $T_s = \{<\sigma, \tau> \in J : \exists t \geq s\ \tau \in S^{A_t}_t$ with use $\sigma \prec A_t\}$. Let $U_s = \{\tau : \exists <\sigma, \tau> \in T_s\}$. So both U_s and T_s are uniformly $B - c.e.$ nested sequences of sets. If $Y = \cup_s [U_s]^c$ then Y is the desired $\Sigma^0_2[B]$ class. $S^A \subset U_s \forall s$ so $Y \subset X$. For each $<\sigma, \tau> \in T_0 - I$ there is a last stage t with $\sigma \prec A_t$ so $<\sigma, \tau> \notin T_s$ for any $s > t$. Fix $\epsilon > 0$, take n large enough to satisfy $\sum_{<\sigma,\tau> \in J, <\sigma,\tau> \geq n} 2^{-|\tau|} < \epsilon$. Take s large enough such that $<\sigma, \tau> \in T_0 - I$ and $<\sigma, \tau> < n \implies <\sigma, \tau> \notin T_s$.
Then $\mu(X - [U_s]^c) \leq \sum_{\tau \in U_s - S^A} 2^{-|\tau|} \leq \sum_{<\sigma,\tau> \in T_s - I} 2^{-|\tau|} \leq \sum_{<\sigma,\tau> \in J, <\sigma,\tau> \geq n} 2^{-|\tau|} < \epsilon$ so $\mu(X) = \mu(Y)$.

Theorem: The following are equivalent:
i) $A \leq_T B'$ and $A \leq_{LR} B$
ii) Every $\Pi^0_1[A]$ class has a $\Sigma^0_2[B]$ subclass of the same measure.
iii) Every $\Sigma^0_2[A]$ class has a $\Sigma^0_2[B]$ subclass of the same measure.
Proof: $ii \implies iii$: Let W be a $\Sigma^0_2[A]$ class where $W = \cup_{i \in \omega} X_i$ for $\Pi^0_1[A]$ classes X_i. Let $X = \{0^i 1 z : i \in \omega, z \in X_i\}$. So X is a $\Pi^0_1[A]$ class. Let $Y \subset X$ be a $\Sigma^0_2[B]$ class of the same measure. Let $Y_i = \{z \in 0^i 1 z \in Y\}$. Note $\mu(Y_i) = \mu(X_i)$ and $Y_i \subset X_i$ for each i. Let $Z = \cup Y_i$ so $Z \subset W$ and is a $\Sigma^0_2[B]$ class with $\mu(Z) = \mu(W)$.

22 Lecture 22

From the last lecture, we are yet to show $iii) \implies i)$. From $iii)$, every $\Pi^0_1[A]$ class of positive measure has a $\Pi^0_1[B]$ subclass of positive measure. So $A \leq_{LR} B$. Now consider the $\Sigma^0_1[A]$ hence $\Pi^0_2[A]$ class $U = \cup_{n\in A}[0^n1]$. By $iii)$ there is a $Q \supset U$ that is $\Pi^0_2[B]$ and has the same measure. Note that $n \in A \iff [0^n1] \subset Q \iff \forall k[0^n1] \in Q_k$ where $Q = \cap_k Q_k$ and Q_k is $\Sigma^0_1[B]$ uniformly. So A is $\Pi^0_2[B]$, using $\cup_{n\in A}[0^n1]$, $\overline{A}$ is $\Pi^0_2[B]$, so A is $\Delta^0_2[B]$ and so $A \leq_T B'$.
Recall: Low for random $\implies$ low for $K \implies K-$ trivial, and $\exists c \forall b, n |\{\sigma \in 2^n : K(\sigma) \leq K(n) + b\}| < 2^c 2^b$ (counting theorem).
Lemma: K trivial $\implies \Delta^0_2$.
Proof: Say $\forall n K(A|_n) \leq K(n) + b$. Then A is a path on $T = \{\sigma \in 2^{<\omega} : \forall t \leq |\sigma| K(\sigma|_t) \leq K(t) + b\}$. $T \leq_T 0'$ and T has at most $2^c 2^b$ infinite paths. So A is isolated in $[T]$ and so is $0'$ computable since $[T]$ is a $\Pi^0_1[0']$ class.
Proposition: If Z is Δ^0_2 and A random, then A is GL_1 i.e. $A' \leq_T A \oplus 0'$.
Proof: Let $\{Z_s\}_{s\in\omega}$ be a computable approximation to Z and $f(n) = \mu s\ \forall t \geq s Z_t|_n = Z|_n$. $f \leq_T 0'$. Define $G_e = [Z_s|e]$ if e enters A' at stage s and $\emptyset$ otherwise. So $\{G_e\}_{e\in\omega}$ is a ML test relative to A. Since Z is $A-$ random, for large enough $e, Z \notin G_e$ hence for large e, $e \in A' \iff e \in A'_{f(e)}$. So $A' \leq_T A \oplus f \leq_T A \oplus 0'$.
Corollary: Let $\Omega = \Omega_0 \oplus \Omega_1$. Then Ω_0, Ω_1 are both GL_1 and Δ^0_2 so both low.
Corollary: Low for random $\implies$ low.
Proof: A is low for random $\implies \Omega$ is A random so A is GL_1 but A is also Δ^0_2.
Theorem: The following are equivalent:
i) A is low for random
ii) Every weak $2-$ random is $A-$ random
iii) A is low for weak $2-$ random.
Proof: $iii) \implies ii)$ and $i) \implies ii)$ are clear.
for $i) \implies iii)$ let A be low for random. Since $A' \leq_T 0'$ and $A \leq_{LR} 0$, every $\Sigma^0_2[A]$ class has a Σ^0_2 subclass of the same measure. So every null $\Pi^0_2[A]$ class is contained in a null Π^0_2 class. If Z is not a weak $2-$ random for A then Z is not weak $2-$ random.
For $ii) \implies i)$ if A is not low for random, then there is a $\Sigma^0_1[A]$ class T of measure < 1 such that no Σ^0_1 class of measure < 1 covers T. Let $T = [W]$ where W is A c.e. and prefix free. Build $X = \sigma_0\sigma_1\sigma_2....$ such that each $\sigma_i \in W$. This ensures that X is not A random: Let $w_1 = w$. Let $w_{n+1} = \{\sigma\tau : \sigma \in W_n, \tau \in W\}$. Then $\mu([W_n]) = \mu([W])^n$. So some linear subsequence of $\{[W_n]\}$ is an $A-$ ML test covering X. We will construct a sequence $S_0 \subset S_1 \subset S_2...$ of Σ^0_1 classes and ensure that $[\sigma_0\sigma_1...\sigma_{n-1}] \not\subset S_n$. Let S_0 be a Σ^0_1 class containing all non ML randoms with $\mu(S_0) < 1$. Assume we have $w = \sigma_0...\sigma_{n-1}$ with $[w] \not\subset S_n$. Let $S = \{Z : wZ \in S_n\}$. Note that $2^\omega - S$ is non empty and contains only randoms, and is Π^0_1 (and so must have positive measure since it contains randoms) and so $\mu(S) < 1$. This means S does not cover T but $T = [w]$ so take $\sigma_n \in w$ such that $[\sigma_n] \not\subset S$ hence $[\sigma_0...\sigma_n] \not\subset S_n$. As before $[\sigma_0...\sigma_n] - S$ has positive measure, say at least $\epsilon > 0$. Let V be the n^{th} null Π^0_2 class, so V is contained in some Σ^0_1 class U of measure $< \epsilon$. Let $S_{n+1} = S_n \cup U$. Note $[\sigma_0...\sigma_n] \not\subset S_{n+1}$ and $X \notin V$.

23 Lecture 23

Definition: A is a base for $1-$ randomness if there is a $Z \geq_T A$ that is $A-$ random.
Theorem: A is a base for $1-$ randomness $\implies$ A is low for K.
Proof: Suppose $Z \geq_T A$ is A random. let $\varphi^Z = A$. We define an oracle ML test $\{[C_d^X]\}_{d\in\omega}$ and a uniformly $c.e.$ seuqence $\{L_d\}_{d\in\omega}$ of bounded request sets.
Idea: If $Z \notin [C_d^A]$ then L_d will ensure that A is low for K. When we see a computation $U^\eta(\sigma) = y$ with $\eta \prec A$ we should add $< |\sigma| + d + 1, y >$ to L_d. But we don't know A! To solve this, we build c.e. sets $C^\eta_{d,\sigma}$ such that if $\mu[C^\eta_{d,\sigma}]$ reaches $2^{-|\sigma|-d-1}$ we put $< |\sigma| + d + 1, y >$ into L_d.
We call $C^\eta_{d,\sigma}$ is a 'hungry set'. We 'feed' it with α such that $\eta \prec \varphi^\alpha_{|\alpha|}$ and we ensure that $[C^\eta_{d,\sigma}] \cap [C^{\hat\eta}_{d,\hat\sigma}] = \emptyset$ unless $\eta = \hat\eta$ and $\sigma = \hat\sigma$. Note that $\sum_{<k,y>\in L_d} 2^{-k} \leq 1$. Why do the right sets get fed?
Construction of the $C^\eta_{d,\sigma} \forall \eta, \sigma \in 2^{<\omega}$ and fixed d. Start with $C^\eta_{d,\sigma} = \emptyset$.
Stage s: In substages $t \in \{0, ..., 2^s - 1\}$ go through all $\alpha \in 2^s$. If α has been declared used (for d) goto the next substage. Otherwise, check if there are η, σ such that
i) $U^\eta_s(\sigma)\downarrow$ with η the exact use.
ii) $\eta \prec \varphi^\alpha_{\alpha|}$. iii) $\mu[C^\eta_{d,\sigma}] + 2^{-s} \leq 2^{-|\sigma|+d}$.
For least such σ put α into $C^\eta_{d,\sigma}$ and declare all $\rho \geq \alpha$ used for d.
Verification: For fixed d the sets $[C^\eta_{d,\sigma}]$ are disjoint so each L_d is a bounded request set. For $X \in 2^\omega$ let $C^X_d = \cup_{\eta\prec X, \sigma\in 2^{<\omega}} C^\eta_{d,\sigma}$. Note $\mu[C^X_d] \leq \sum_{\eta\prec X,\sigma\in 2^{<\omega}} \mu[C^\eta_{d,\sigma}] = \sum_{\sigma\in domU^X, use\ \eta} \mu[C^\eta_{d,\sigma}] \leq \sum_{\sigma\in domU^X} 2^{-|\sigma|-d} \leq 2^{-d}$.
Therefore $\{[C^X_d]\}_{d\in\omega}$ is an $X-$ ML test. Hence for some d, $Z \notin [C^A_d]$.
Claim: L_d works.
Proof: Suppose $U^A(\sigma) = \eta$ with use $\eta \prec A$. We claim that $\mu[C^\eta_{d,\sigma}] = 2^{-|\sigma|-a}$, so $< |\sigma| + d + 1, \eta > \in L_d$. If not consider s large enough that $U^\eta_s(\sigma) = y$, $\eta \prec \varphi_s^{Z|_s}$ and $\mu([C^n_{d,\sigma}]) + 2^{-s} \leq 2^{-|\sigma|-d}$. Then $\alpha = Z|_s$ enters $C^\eta_{d,\sigma}$ at stage s unless some $\beta \prec \alpha$ entered $C^{\hat\eta}_{d,\hat\sigma}$ already.
But $A = \varphi^Z$ so $\hat\eta \prec \varphi^\alpha$ must be a prefix of A. Either way, $Z \in [C^A_d]$, a contradiction!
Theorem: The following are equivalent:
i) A is low for K
ii)A is low for random
iii) A is low weak $2-$ random.
iv) A is a base for $1-$ randomness.
v) A is K trivial (we won't see v $\implies$ i since its too hard)
We have an alternate proof that ii $\implies$ i via iv.

24 Lecture 24

Fact: There is a noncomputable *c.e.* set low for K.
Proof: Let $\Omega = \Omega_0 \oplus \Omega_1$. Let $A \leq_T \Omega_0, \Omega_1$ be a simple set ($\{\Omega_0, \Omega_1\}$ is a null Π^0_1 class). Then Ω_0 is Ω_1 random, hence $A-$ random, So A is a base for $1-$ randomness.
Definition: A cost function is a computable map: $c : \omega \times \omega \to \mathbb{R}_{\geq 0}$. We say c satisfies the limit condition if $lim_{n\to\infty} sup_{s\geq n} c(n, s) = 0$.
We say $c(n, s)$ = cost of changing $A(n)$ at stage s.
Example: $c_K(n, s) = \sum_{n\leq m\leq s} 2^{-K_s(m)}$ satisfies the limit condition.
Definition: Let $\{A_s\}_s$ be a computable approximation to a Δ^0_2 set A. We say $\{A_s\}_s$ obeys c if $\sum\{c(x, s) : x \leq s, x \text{ is the least such that } A_{s-1}(x) \neq A_s(x)\} < \infty$ that is the total cost of the approximation is finite.
Proposition: If A has an approximation $\{A_s\}_{s\in\omega}$ that obeys c_K, then A is $k-$ trivial. It is also true that all $K-$ trivials have such an approximation (although this is hard to show and we won't see it here).
Proof: Consider the request set R where at stage s we put $< K_s(n), A_s|_n >$ into R if $n \leq s$ and i) $K_s(n) < K_{s-1}(n)$ or ii)$A_{s-1}|_n \neq A_s|_n$. If R is bounded then A is $K-$ trivial.
Requests enumerated for $i)$ weight at most $\Omega \leq 1$. Assume $< K_s(n), A_s|_n >$ is ennumerated for $ii)$ Let $x \leq n$ be leaset such that $A_{s-1}(x) \neq A_s(x)$. The cost $c(x, s) = \sum_{x\leq n\leq s} 2^{-K_s(n)}$ accounts for all $ii)$ requests at stage s. But A obeys c_K so R is a bounded request set.
Proposition: If c is a cost function satisfying the limit condition, then there is a promptly simple set A obeying C.
Proof: Let R_e be the requirement $|W_e| = \infty \implies W_e \cap A \neq \emptyset$. Let $A_0 = \emptyset$.
At stage $s > 0$ for each $e < s$ which hasn't been satisfied, if there is an $x > 2e$ with $x \in W_e$ and $c(x, s) \leq 2^{-e}$, then put the least such x into A_s.
Note: A is coinfinite and obeys c. If x is sufficiently large, then $c(x, s) \leq 2^{-e} \forall s \geq x$, so each R_e is satisfied.
Corollary (Solovay): There is a noncomputable (simple) $K-$ trivial.
Low for K construction.
Let U be a universal prefix free oracle machine. If A has Δ^0_2 approximation $\{A_s\}_s$ let $C_{U,A}(x, s) = \sum\{2^{-|\sigma|} : U^{A_{s-1}}_{s-1}(\sigma)\downarrow, x < use U^{A_{s-1}}_{s-1}(\sigma)\}$. $C_{U,A}$ depends on the approximation to A, and is called an adaptive cost function. Does it have the limit condition?-Maybe
Proposition: If A has an approximation $\{A_s\}_s$ that obeys $c_{U,A}$ then A is low for K.
Proof: Consider the request set R where at stage s we put $< |\sigma|, \eta >$ into R if $U^{A_s}_s(\sigma) \downarrow= \eta$ but $U^{A_{s-1}}_{s-1}$ does not converge to η. If R is a bounded request set , then A is low for K. Suppose a request $< |\sigma|, \eta >$ is put into R at stage s because $U^{A_s}_s(\sigma) = \eta$ with use w.
Case 1: $\forall t > s$ $A_s|_w = A_t|_w$. Then we can charge the request against $\Omega^A < 1$.
Case 2: $\exists t > s$ with $A_s|_w \neq A_t|_w$. Let t be the least. Let $x < w$ be least such that $A_{t-1}(x) \neq A_t(x)$. Then $2^{-|\sigma|}$ can be charged against $c_{U,A}(x, t)$.

25 Lecture 25

Recall: $c_{U,A}(x,s) = \sum\{2^{-|\sigma|} : U_{s-1}^{A_{s-1}}(\sigma)\downarrow$ and $x <$ use $U_{s-1}^{A_{s-1}}\}$.
Theorem: There is a simple set that is low for K.
Proof: We have the simplicity requirements $R_e : |W_e| = \infty \implies W_e \cap A \neq \emptyset$. Starting with $A_0 = \emptyset$, at stage s we wait to see if there is an $x \geq 2e, x \in W_e$ for each unsatisfied R_e with $e \leq s$ and $c_{U,A}(x,s) \leq 2^{-e}$. Then put the least such x into A_s. Note that A is coinfinite and obeys $C_{U,A}$.
We claim that for any large enough m, we have $sup_{s>m} c_{U,A}(m,s) \leq 2^{-e}$, so R_e is satisfied. This is because if $\sigma_0, ..., \sigma_{k-1} \in domU^A$ be such that if $\alpha = \sum_{i<k} 2^{-|\sigma_i|}$, then $\Omega^A - \alpha \leq 2^{-e-1}$. Choose $m \geq e+1$ such that all computations $U_m^{A_m}(\sigma_i)$ are stable and no R_j for $j \leq e+1$ acts at any stage $\geq m$.
At each stage $s > m$ we have $\Omega_{s-1}^{A_{s-1}} - \Omega^A \leq 2^{-e-1}$ since the total weight of all computations that can be injured at stages $\geq m$ is $\sum_{j>e+1} 2^{-j} = 2^{-e-1}$. But note that $C_{U,A}(m,s) \leq \Omega_{s-1}^{A_{s-1}} - \alpha \leq 2^{-e}$ for some $s > m$ which is what was needed.
Facts: i)Every low for K is low (we showed this), in fact super low: $A' \leq_{tt} 0'$.
ii) The low for K degrees form an ideal as they are closed under $\leq_T$ and join (by passing through K triviality).
Theorem: If A, B are K trivials then so is $A \oplus B$.
Proof: Let A, B be K trivial with constant b. We can then fix c such that $\forall n\ |\{\sigma \in 2^n : K(\sigma) \leq K(n) + b\}| < 2^{b+c}$. Consider a prefix free machine M such that $M(\tau\gamma_1\gamma_2)$ converges if $\tau \in dom(U)$ and $\gamma_1, \gamma_2 \in 2^{b+c}$. If $U(\tau) = n$, then M looks for the $\gamma_{1^{st}}$ and $\gamma_{2^{nd}}$ strings σ_1, σ_2 of length n with complexity $\leq |\tau| + b$. Then $M(\tau\gamma_1\gamma_2) = \sigma_1 \oplus \sigma_2$. So $K(A \oplus B|_{2n}) \leq^+ K_M(A \oplus B|_{2n}) = K(2n) + 2(b+c) \leq^+ K(2n)$. Odd lengths follow too.

Facts:iii) Every K trivial is $\leq_T$ a $c.e. K-$ trivial.
iv) Every $K-$ trivial (has an approximation that) satisfies C_K.
Theorem: Let A be c.e., X a Martin Lof random and $0' \not\leq_T A \oplus X$. Then X is Martin Lof random with respect to A. Proof: Let $\{U_n^Z\}$ be a universal oracle ML test. Let $U_n^A[s] = U_{n,s}^{A_s}$. Observe that $\mu(U_n^A[s]) \leq 2^{-n}$. If X is not ML random in A, then $X \in U_n^A \forall n$. Let $f(n) = \mu s[X \in U_n^A[s]$ with use $\sigma \prec A]$. Note $X \in U_n^A[t] \forall t \geq f(n)$ and $f \leq_T A \oplus X$. Since $f \not\geq_T 0'$ there are infinitely many n such that $n \in 0' - 0'_{f(n)}$. So we take $S_n = U_n^A[t]$ if n enters $0'$ at stage t and $S_n = \emptyset$ otherwise. $\mu(S_n) \leq 2^{-n}$ and $\exists^\infty n\ X \in S_n$ so X is not ML random.
Corollary: Any $c.e.$ set A that is computable from a $ML-$ random $X \not\geq_T 0'$ is low for K.
Proof: $X \oplus A \equiv_T X \not\geq_T 0'$ so X is $ML-$ random with respect to A, so A is a base for $1-$ randomness.
Fact: v) There is an $ML-$ random $X <_T 0'$ that computes all low for K sets.
vi) A is low for $K \iff \forall X\ X\ ML-$ random and $X \not\geq_T 0' \implies A \oplus X \not\geq_T 0'$.

26 Lecture 26- Hausdorff Dimension

Not all measure zero sets are created equal. Idea: In dimension n, a ball of radius r has volume about r^n.
Definition: A set D is an $n-$ cover if $\sigma \in D \implies |\sigma| \geq n$. D covers $R \subset 2^\omega$ if $R \subset [D]$. Let $H_n^S(R) = inf\{\sum_{\sigma \in D} 2^{-s|\sigma|} : D$ is an $n-$ cover of $R\}$ - The sum in the inf called $wt_s(D)$ is the dimension s weight of D.
Now the s dimensional (outer) Hausdorff measure of R is $H^S(R) = lim_{n\to\infty} H_n^S(R)$.
Note that for $s < 1$ we have $wt_s(\{\sigma 0, \sigma 1\}) > wt_s(\{\sigma\})$.
Proposition: For all $R \subset 2^\omega$, there is a $s \in [0, 1]$ such that :
i) $H^t(R) = 0$ for all $t > s$ ii) $H^t(R) = \infty$ for all $t \in [0, s)$.
Proof: All we need to show is that if $H^S(R) < \infty$ and $d > 0$ then $H^{s+d}(R) = 0$.
But for all n, $H_n^{std}(R) = inf\{\sum_{\sigma \in D} 2^{-(s+d)|\sigma|} : D$ is an n cover of $R\} \leq inf\{\sum_{\sigma \in D} 2^{-d|\sigma|} 2^{-s|\sigma|} : D$ is an n cover of $R\} = 2^{-dn} H_n^S(R)$.
So $H^{s+d}(R) = lim H_n^{s+d}(R) \leq lim 2^{-dn} H_n^s(R) = 0$.

Definition: The Hausdorff dimension of R is $dim_H(R) = sup\{s : H^s(R) = \infty\} = inf\{s : H^s(R) = 0\}$.
Examples: i) $R = \{X \oplus \emptyset : X \in 2^\omega\}$ has $dim_H(R) = 1/2$ because you can find a $2n-$ cover D of R with 2^n strings of length $2n$. $wt_{1/2}(D) = 2^n . 2^{-1/2 \times 2n} = 1$.
ii) Say R is made up by restricting on odd blocks to be 000...0 and anything goes on even blocks. If the growth rate of length of blocks is fast enough, then $dim_H(R) = 0$.
Facts: i) $H^1(X)$ is the outer Lebesgue measure
ii) If $\mu(X) \neq 0$ then $dim(X) = 1$.
iii) If $X \subset Y$, then $di(X) \leq dim(Y)$.
iv) dim$(\cup_i Y_i) = sup(dim(Y_i))$.
Proposition: $H^s(R) = 0 \iff \forall \epsilon > 0$ there is a cover D of R such that $wt_s(D) < \epsilon$.

26.1 $s-$ Randomness

Let s be a computable real between $0, 1$.
Definition: i) An $s - ML$ test is a computable sequence $\{V_k\}_{k\in\omega}$ of c.e. sets of strings such that $wt_s(V_k) \leq 2^{-k}$.
ii) We call $X \in 2^\omega$ $(ML)s-$ random if $X \notin \cap V_k$ for all $s-$ ML tests.
Fact: There is a universal $s-$ ML test.
Definition: The effective Hausdorff dimension of $X \in 2^\omega$, $dim(X) = sup\{s : X$ is $s-$ random $\}$.
Proposition-The point to set principle: Let $R \subset 2^\omega$.
$dim_H(R) = min_{z\in 2^\omega} sup_{X\in R} dim^Z(X)$.

27 Lecture 27

Proof of the point to set principle: For $\leq$, let $dim^Z(X) < s$ for some rational s and all $X \in R$. Then the universal $s - ML$ test relative to Z covers R, so $dim_H(R) \leq sup_{X \in R} dim^Z(X)$.
For the other direction - $\geq$: Take Z to code for every rational $s > dim_H(R)$ a sequence $\{V_k^s\}_{k \in \omega}$ of covers of R such that $\forall n \; wt_s(V_n^s) \leq 2^{-n}$. So $sup_{X \in R} dim^Z(X) \leq dim_H(R)$.
Goals:
Theorem: $\exists X \in 2^\omega \; dim(X) = 1/2$ and if $Y \leq_T X$ then $dim(Y) \leq 1/2$.
Theorem (P Lutz, Joe) Under CH there is an $E \subset 2^\omega$ such that
i) $dim_H(E) = 1/2$, in fact E is not σ finite in dim 1/2.
ii) $f : \omega \to \omega$ is a continuous function, then $dimH(f(E)) \leq 1/2$.
There is an analogy between effective (dim) and classical (dim_H)here. (add oracles). $f : 2^\omega \to 2^\omega$ computable to f continuous.
$X \leq_{tt} Y$ is defined as there being a total computable f such that $f(Y) = X$.
$X \leq_T Y \iff$ there is a partial computable $f(Y) = X$.

Theorem: X is $s-$ random $\iff K(X|_n) \geq^+ sn$.
Proof: (forward direction) Let $V_k = \{\sigma : K(\sigma) \leq s|\sigma| - k\}$. So $wt_s(V_k) = \sum_{\sigma \in V_k} 2^{-s|\sigma|} \leq \sum_{\sigma \in V_k} 2^{-K(\sigma)-k} \leq \sum_{\sigma \in 2^{<\omega}} 2^{-K(\sigma)-k} = 2^{-k}\Omega \leq 2^{-k}$. So $\{V_k\}_{k \in \omega}$ is an $s-$ ML test hence $X \notin [V_k]$ for some k so $\forall n K(X|_n) > sn - k$.
(backward direction): Suppose X is not s randomd. Let $\{V_k\}_{k \in \omega}$ be an $s-$ ML test covering X. Build a bounded request set L by putting $< s|\sigma| - k, \sigma >$ into L $\forall k$ and $\sigma \in V_{2k}$. $\sum_{<d,\sigma> \in L} 2^{-d} = \sum_k \sum_{\sigma \in V_{2k}} 2^{-s|\sigma|+k} = \sum_k wt_s(V_{2k}) 2^k \leq \sum_k 2^{-2k} 2^k = 2 < \infty$. Hence $\forall k \exists n K(X|_n) \leq^+ sn - k$ where the $\leq^+$ comes from the coding constant for L.
Corollary: $dim(X) = liminf_n K(X|_n)/n$.
Proof: ($\leq$) If the lim inf $< s$ for s rational then X is not $s-$ random by the theorem above. So $dim(X) \leq s$.
($\geq$) If $liminf > t$ for t rational. Then $K(X|_n) \geq^+ tn$, so X is $t-$ random. Thus $dim(X) \geq t$.
Recall: $K(\sigma) = C(\sigma) + O(log|\sigma|) = KM(\sigma) + O(log|\sigma|)$, so $dim(X)$ could be defined as $liminf$ of C or KM too.

Example: For $R, S \subset 2^\omega$, $dim_H(R) + dim_H(S) \leq dim_H(R \times S)$, where $R \times S \subset 2^\omega \times 2^\omega$. Let $\sigma, \tau \in 2^n$ let $wt_s(\{(\sigma, \tau)\}) = 2^{-sn}$. Define the Hausdorff measure/dimension in $2^\omega \times 2^\omega$ accordingly. Note that $wt_{s/2}(\sigma \oplus \tau) = 2^{s/2(2n)} = 2^{-sn}$. So the map $\oplus : 2^\omega \to 2^\omega \to 2^\omega$ given by $(X, Y) \to X \oplus Y$ exactly halves dimension.
So we can restate the example as for $R, S \subset 2^\omega$, $dim_H(R) + dim_H(S) \leq 2dim_H(R \oplus S)$.

28 Lecture 28

28.1 Conditional Complexity

Definition: A partial function $M : 2^{<\omega} \times 2^{<\omega} \to 2^{<\omega}$ is prefix free if $\forall y\{x : M(x,y)\downarrow\}$ is prefix free.
There is an effective list $\{M_n^2\}_{n\in\omega}$ of prefix free machines of the form above.
Definition: Let $U^2(0^n 1\tau, y) = M_n^2(\tau, y)$. This is universal. Let $K(\sigma|y) = min\{|\tau| : U^2(\tau, y) = \sigma\}$ which is the complexity of σ given y.
Facts: i) $K(\sigma 1\emptyset) =^+ K(\sigma)$. In fact for any y we have $K(\sigma|y) =^+ K(\sigma)$ but the constant depends on y.
ii) $\exists c\forall\sigma, y K(\sigma|y) \leq K(\sigma) + c$.
iii) $K(x|y) \leq^+ K(x|z) + K(z|y)$.
Definition: If $t = \mu s[K_s(\sigma) = K(\sigma)]$, let σ^* be the left most string such that $|\sigma^*| = K(\sigma)$ and $U_t(\sigma^*)\downarrow = \sigma$ - first discovered minimal program for σ.
Facts: i) If we fix a partial computable $N : 2^{<\omega} \to 2^{<\omega}$ then $\forall y, z$ we have $N(z)\downarrow \implies K(y|z) \leq^+ K(y|N(z))$. The constant here depends on N.
ii) $K(y|x^*) = K(y| < x, K(x) >)$ since we can compute $< x, K(x) >$ from x and vice versa.
Theorem: $K(x,y) := K(< x, y >) =^+ K(x) + K(y|x^*)$
For $\leq^+$ consider the prefix free machine M such that $M(\sigma\tau)\downarrow \iff \sigma \in dom(U)$ and $\tau \in domU^2(.|\sigma)$. Let $M(\sigma\tau) =< U(\sigma), U^2(\tau,\sigma) >$. Let τ be a minimal $U^2(., x^*)$ program for y. Then $M(x^*\tau) =< x, y >$ and $|x^*\tau| = K(x) + K(y|x^*)$.
For $\geq^+$ we prove $K(y|x^*) \leq^+ K(x,y) - K(x)$. Define $G(\sigma) = x$ whenever $U(\sigma) =< x, y >$ for some y. Recall that $f(x) = 2^{-K(x)}$ is maximal upto $\geq^*$ among left c.e. functions $f : 2^{<\omega} \to \mathbb{R}_{\geq 0}$ such that $\sum f(\sigma) < \infty$.
So there is a c such that $\forall x$ we have $2^c 2^{-K(x)} > \mathbb{P}_G(x)$ - probability that G outputs $x = \sum_{G(\sigma)=x} 2^{-|\sigma|}$.
Define the bounded rqeuest set L_σ as : If $U(\sigma)\downarrow = x$ and $U(\rho)\downarrow =< x, y >$ then put $< |\rho| - |\sigma| + c, y >$ into L_σ but only up to total weight 1 for L_σ. L_σ has weight at most 1.
Let M_σ be the prefix free machine for L_σ. Let $N(\gamma,\sigma) = M_\sigma(\gamma)$. Note N is prefix free.
If $\sigma = x^*$ then $2^c 2^{-|\sigma|} > \mathbb{P}_G(x) = \sum\{2^{-|\rho|} : \exists y U(\rho) =< x, y >\}$. But the total weight we want to put into L_σ is $P_G(x)2^{|\sigma|-c} < 1$. So L_σ is unconstrainted and $K(y|x^*) \leq^+ K_N(y|x^*) = K_N(y|\sigma) \leq K(x,y) - K(x) + c$.

Theorem: $R, S \subset 2^\omega$ we have$dim_H(R) + dim_H(S) \leq 2dim_H(R \oplus S)$. Proof: Assume $dim_H(R) = r, dim_H(S) = s$. Fix Z and let $\epsilon > 0$. We want to prove that there is an $X \oplus Y \in R \oplus S$ such that $dim^Z(X \oplus Y) \geq 1/2(r+s) - \epsilon$.
Take $x \in R$ such that $dim^Z(X) \geq r - \epsilon$. Let $Y \in S$ be such that $dim^{Z\oplus X}(Y) \geq s - \epsilon$. We want $K^Z(X \oplus Y|_{2n}) \geq^+ K^Z(X|_n) + K^{Z\oplus X}(Y|_n) - O(log(n))$.

29 Lecture 29

Last time we were showing $R, S \subset 2^\omega \implies dim_H(R) + dim_H(S) \leq 2dim_H(R \oplus S)$.
Proof: Assume $dim_H(R) = r$ and $dim_H(S) = s$. Fix Z and let $\epsilon > 0$. We want to prove that there is an $X \oplus Y \in R \oplus S$ such that $dim^Z(X \oplus Y) \geq r + s/2 - \epsilon$. Take $X \in R$ such that $dim^Z(X) \geq r - \epsilon$ and $Y \in S$ with $dim^{Z\oplus X}(Y) \geq s - \epsilon$. Note that $K^{Z\oplus X}(Y|_n) \leq^+ K^Z(n) + K^Z(K^Z(X|_n)) + X^Z(Y|_n \ |(X|_n)^*) \leq^+ O(logn) + K^Z(Y|_n|(X|_n)^*)$. So $K^Z(X \oplus Y|_{2n}) =^+ K^Z(X|_n, Y|_n) =^+ K^Z(X|_n) + K^Z(Y|_n|(X|_n)^*) \geq^+ K^Z(X|_n) + K^{Z\oplus X}(Y|_n) - O(log\ n)$.
Hence $dim^Z(X \oplus Y) = liminf K^Z(X \oplus Y|_{2n})/2n \geq 1/2 liminf(K^Z(X|_n) + K^{Z\oplus X}(Y|_n))/n \geq 1/2((r - \epsilon) + (s - \epsilon)) = 1/2(r + s) - \epsilon$.

Let $I_0 = [0, 1) \cap \omega$ and $I_n = [n!, (n + 1)!) \cap \omega$ for $n > 0$. Note that $|I_n| = n.n! = n. \sum_{i<n} |I_i|$. Let $R = \{X \in 2^\omega : \forall n\ X$ is zero on even intervals $I_{2n}\}$ and $S = \{X \in 2^\omega : \forall n\ X$ is all zero on odd intervals $I_{2n+1}\}$. If $X \in R$, then $K(X|_{(2n+1)!})/(2n + 1)! < (2n)!/(2n + 1)! = 1/(2n + 1)$ where the $<$ is ignoring $O(log(n))$.
So $dim(X) = 0$ and $dim_H(R) = 0$. Similarly $dim_H(S) = 0$.
Claim: $dim_H(R \oplus S) = 1/2$.
Easy to see that $dim_H(R \oplus S) \leq 1/2$. Now fix Z and let X be ML random with respect to Z. Define X_0, X_1 by $X_0(m) = X(m)$ if $m \in I_n$ where n is odd and 0 otherwise and $X_1(m) = X(m)$ if $m \in I_n$ where n is even and 0 otherwise. We can uniformly compute $X|_n$ from $X_0 \oplus X_1|_{2n}$ and vice versa.
So $K^Z(X_0 \oplus X_1|_{2n}) =^+ K^Z(X|_n) \geq^+ n$. So $dim^Z(X_0 \oplus X_1) \geq 1/2$ and so Z was arbitrary. So $dim_H(R \oplus S) \geq 1/2$.

29.1 Strong $s-$ randomness and supermartingales

Definition: If $V \subset 2^{<\omega}$ then $pfwt_s(V) = sup\{wt_s(S) : S \subset V$ prefix free $\}$. $pfwt-$ stands for prefix free s weight of V.
Note that $pfwt_s(V) \leq wt_s(V)$.
A strong $s - ML$ test is a computable sequence $\{V_k\}_{k\in\omega}$ of *c.e.* sets of strings such that $\forall k pfwt_s(V_k) \leq 2^{-k}$.
X is strong $s-$ random if $X \notin \cap[V_k]$ for every strong $s-$ ML test $\{V_k\}$. Note: strong $s-$ random $\implies$ $s-$ random.
Definition: We say that a (*super*) martingale d $s-$ succeeds on $X \in 2^\omega$ if $limsup_{n\to\infty} d(X|_n)/2^{(1-s)n} = \infty$.
Theorem: For s computable with $0 < s < 1$ then X is strongly $s-$ random $\iff$ no *c.e.* martingale $s-$ succeeds on X.

Let d be an optimal supermartingale. The a priori complexity $KM(\sigma) = n - log_2(d(\sigma))$.
Proposition: X is strongly $s-$ random $\iff KM(X|_n) \geq^+ sn$.

30 Lecture 30

Example: If $dim_H(R \times S) \geq 1+s$ then $dim_H(R), dim_H(S) \geq s$ and $2dim_H(R \oplus S) \geq 1+s$.
Proof: Assume $dim_H(R) < s$ say $dim_H(R) < s-\epsilon$. Fix Z such that $\forall X \in R\ dim^Z(X) < s-\epsilon$.
So $\forall X \in R \exists^\infty n K^Z(X|_n) < (s-\epsilon)n$. Fix $X \oplus Y \in R \oplus S$. Then $\exists^\infty n K^Z(X \oplus Y|_{2n}) =^+ K^Z(X|_n, Y|_n) \leq^+ K^Z(X|_n) + K^Z(Y|_n) \leq^+ (s-\epsilon)n + n + O(logn)$.
So $dim^Z(X \oplus Y) \leq (1+s-\epsilon)/2$. So $dim_H(R \oplus S) \leq (1+s-\epsilon)/2$.
Last time we defined strong $s-$ random and the notion of a supermartingale $s-$ succeeding on a sequence $X \in 2^\omega$. We will now prove the theorem stated in last class.
Theorem: For s computable with $0 < s < 1$ then X is strongly $s-$ random $\iff$ no *c.e.* (super)martingale $s-$ succeeds on X.
Proof: Suppose d is an $s-$ successful c.e supermartingale. Assume $d(\phi) \leq 1$. Let $V_k = \{\sigma \in 2^{<\omega} : d(\sigma) > 2^k 2^{(1-s)|\sigma|}\}$. So $X \in \cap[V_k]$. Let $S \subset V_k$ be prefix free. Then $1 \geq d(\phi) \geq \sum_{\sigma \in S} 2^{-|\sigma|} d(\sigma) \geq \sum_{\sigma \in S} 2^k 2^{s|\sigma|} = 2^k wt_s(S)$. So $wt_s(S) \leq 2^{-k}$ and so $pfwt_s(V_k) \leq 2^{-k}$ and $\{V_k\}_k$ is a strong $s-$ ML test. For the other direction let $\{V_k\}_{k \in \omega}$ be a strong $s-$ ML test covering X. Let $d(\sigma) = 2^{|\sigma|} \sum_{k \in \omega} k.pfwt_s(V_k \cap [\sigma]^{\prec})$ where $[\sigma]^{\prec}$ is the set of finite extensions of σ. Note $d(\phi) = \sum_{k \in \omega} k 2^{-k} < \infty$ and d has the supermartingale property because for any $V \subset 2^{<\omega}$ and any σ the $pfwt_s(V \cap [\sigma]^{\prec}) \geq pfwt_s(V \cap [\sigma 0]^{\prec}) + pfwt_s(V \cap [\sigma 1]^{\prec})$. If $\sigma \in V_k$ then $d(\sigma) \geq k 2^{|\sigma|} 2^{-s|\sigma|} = k 2^{(1-s)|\sigma|}$ so d $s-$ succeeds on X.

Recall: The a priori complexity $KM(\sigma) = |\sigma| - log_2(d(\sigma))$ where d is an optimal supermartingale.
Proposition: X is strongly $s-$ random $\iff KM(X|_n) \geq^+ sn$.
Proof: We just unwind definitions:
$KM(X|_n) \geq^+ sn \iff -log_2(d(X|_n)) \geq^+ sn - n \iff d(\sigma) \leq^* 2^{(1-s)n} \iff limsup\ d(X|_n)/2^{(1-s)n} < \infty$.
Theorem: If $s < t$ and $X \in 2^\omega$ is $t-$ random then it is strongly $s-$ random.
Corollary: $dim(X) = sup\{s : X$ is $s-$ random $\} = sup\{s : X$ is strongly $s-$ random $\} = inf\{s : X$ is not strongly $s-$ random $\}$.
Proof of theorem: For a given s, t there is a c such that if $A \subset 2^{<\omega}$ then $wt_t(A) \leq c \times pfwt_s(A)$ since $wt_t(A) = \sum_{\sigma \in A} 2^{-t|\sigma|} = \sum_{n \in \omega} \sum_{\sigma \in A \cap 2^n} 2^{-tn} = \sum_{n \in \omega} 2^{-(t-s)n} \sum_{\sigma \in A \cap 2^n} 2^{-sn} \leq \sum_{n \in \omega} 2^{-(t-s)n} pfwt_s(A) = cpfwt_s(A)$ where $c = \sum_{n \in \omega} 2^{-(t-s)n} < \infty$ since its a geometric series. So an appropriate tail of any strong $s-$ ML test is a $t-$ ML test.

Fact: X strong $s-$ random $\implies K(X|_n) - sn \to \infty \implies K(X|_n) \geq^+ sn$ but the converses are not true.
Definition: optimal $s-$ weight $opwt_s(S) = inf\{wt_s(V) : [S] \subset [V]\}$.
We get the same randomness notion for this as we did for $pfwt_s$.

31 Lecture 31

Definition: $opwt_s(S) = inf\{wt_s(V) : [S] \subset [V]\}$.
Note that $opwt_s(S) \leq 1$, essentially this is the cost of compressing some initial segment of every $X \in [S]$ to $s.length$.
Fix $s < 1$, assume $S \subset 2^{<\omega}$ is finite.
Claim: If $[S] = [V]$ and $\tau \in V$ is incomparable with every $\sigma \in S$ or properly extends some $\sigma \in S$ then we can do better than this V, i.e. there is a $\hat{V}$ with no such τ and $[S] \subset [\hat{V}]$ and $wt_s(\hat{V}) < wt_s(V)$. So for finite S we only need to consider finitely many V in the definition of $opwt_s(S)$.
Definition: The optimal cover of $S \subset 2^{<\omega}$ is a set $S^{OC} \subset 2^{<\omega}$, and $[S] \subset [S^{OC}]$ and $wt_s(S^{OC}) = opwt_s(S)$ and $[S^{OC}]$ has minimal measure among candidates.
Facts: S^{OC} if they exist must be prefix free.
For finite S, S^{OC} exists and is unique. (induction over length of longest string in S). S^{OC} and $opwt_s(S)$ are computable from S.
Example: $s = 1/2$ and $S = \{00, 11\}$, then $wt_s(S) = 1$ and $S^{OC} = S$ and not $\{\emptyset\}$ but for $S = \{0, 11\}$ we get $S^{OC} = \{\emptyset\}$.
What if S is infinite? Let $S = \cup S_t$ where $S_0 \subset S_1 ...$ are all finite. If $\sigma \in S_t^{OC}$ then either $\sigma \in S_{t+1}^{OC}$ or some $\tau \prec \sigma$ is in S_{t+1}^{OC}. Therefore $[S_t^{OC}] \subset [S_{t+1}^{OC}]$ and $lim S_t^{OC} = \hat{S}$ exists (pointwise limit) and $S^{OC} = \hat{S}$ and is unique.
Proposition: If S is c.e. then S^{OC} is Δ_2^0 (dce), $[S^{OC}]$ is a Σ_1^0 class and $opwt_s(S)$ is left *c.e.*.
Definition: (Kjor-Hanssen) X is vehemently $s-$ random if for every computable sequence of *c.e.* sets $\{V_k\}_{k\in\omega}$ with $opwt_s(V_k) \leq 2^{-k} \forall k$, $X \notin \cap [V_k]$.
Observe that $opwt_s(S) \leq pfwt_s(S)$ so every strong s test is also a vehement $s-$ test and $Vehement \implies strong$.
Lemma: For any *c.e.* set $S \subset 2^{<\omega}$ we can effectively find a *c.e.* $V \subset 2^{<\omega}$ such that $[V] = [S^{OC}]$ and if $P \subset V$ is prefix free, $wt_s(P) \leq opwt_s(S)$.
Corollary: X is strong $s-$ random $\iff$ X is vehemently $s-$ random.
Proof of lemma: Let $\{S_t\}_{t\in\omega}$ be an enumeratino of S. Let $V = \cup S_t^{OC}$. Note that V is c.e. and $[V] = [S^{OC}]$. If there is a prefix free $P \subset V$ with $wt_s(P) > opwt_s(S)$ then this would be true of a finite $P' \subset P$.
Assume $P \subset V$ is finite and prefix free. If $\tau \in V$ then we will show that $wt_s(P \cap [\tau]^{\prec}) \leq wt_s(\{\tau\})$. We prove this by induction on the distance k from τ to its longest extension in P. Its trivial if $P \cap [\tau]^{\prec} = \emptyset$. $k = 0$ is immediate. If $k > 0$ there is a unique t such that $\tau \in S_{t+1}^{OC} - S_t^{OC}$. Note that $wt_s(S_t^{OC}) \cap [\tau]^{\prec}) \leq wt_s(\{\tau\})$ or else we would have $\tau \in S_t^{OC}$. Also every element of $P \cap [\tau]^{\prec}$ must have entered V by stage t. Moreover every element of $P \cap [\tau]^{\prec}$ extends an element of $S_t^{OC} \cap [\tau]^{\prec}$. Apply the inductive hypothesis to every element of $S_t^{OC} \cap [\tau]^{\prec}$. So $wt_s(P \cap [\tau]^{\prec}) \leq wt_s(S_t^{OC} \cap [\tau]^{\prec}))$. Again every element of P extends some element of S^{OC} so $wt_s(P) \leq wt_s(S^{OC}) = opwt_s(S)$.

32 Lecture 32

Fact: $opwt_s(S) \leq wt_s($ minimal strings in $S) \leq pfwt_s(S)$.
Fact: Forall $S \subset 2^{<\omega}, S^{OC}$ is unique

32.1 Solovay $s-$ randomness

Definition: $V \subset 2^{<\omega}$, V c.e. is a Solovay $s-$ test if $wt_s(V) < \infty$. $X \in 2^\omega$ fails V if $\exists^\infty n X|_n \in V$.
X is Solovay $s-$ random if it passes all Solovay $s-$ tests.
Theorem: X is Solovay s random $\iff limK(X|_n) - sn = \infty$.
Proof: Let $V_b = \{\sigma \in 2^{<\omega} : K(\sigma) < s|\sigma| + b\}$. Then $wt_s(V_b) = \sum_{\sigma \in V_b} 2^{-s|\sigma|} < \sum_{\sigma \in V_b} 2^{-K(\sigma)+b} \leq 2^b$. If $liminf K(X|_n) - sn \leq b$ for some b, then X fails V_b and so is not Solovay s random.
For the other direction let V be a Solovay s test with weight $wt_s(V) < \infty$. Then $\{< s|\sigma|, \sigma >: \sigma \in V\}$ is a bounded request set. So if X is covered by V then $liminf K(X|_n) - sn < \infty$.

Corollary: Solovay $s-$ random $\implies$ $s-$ random.
Theorem: Strong $s-$ random $\implies$ Solovay $s-$ random.
Proof: Take $V \subset 2^{<\omega}$ such that $wt_s(V) \leq 2^b$. Let $U_n = \{\sigma \in V : \sigma$ has atleast $n-1$ prefixes in $V\}$ and $V_n = \{\sigma \in V : \sigma$ has exactly $n-1$ prefixes in $V\}$. U_n is always *c.e.* while V_n need not be. Notice that $[V_1], [V_2], ..., [V_n]$ all cover U_n and $V_1, V_2 ... V_n$ are all disjoint. Therefore $opwt_s(U_n) \leq wt_s(V)/n \leq 2^b/n$. So $\{U_{2^{n+b}}\}_{n\in\omega}$ is a vehement $s-$ test covering all X that fail V.

Fix $s = 1/2$.
Theorem: There is an $A \leq_T 0'$ such that $dim(A) = 1/2$ and if $Y \leq_T A$ then $dim(Y) \leq 1/2$.
Proof: We build A as the limit of a sequence of approximations. A condition is a pair $< \sigma, s >$ such that $\sigma \in 2^{<\omega}$ and $S \subset [\sigma]^{\prec}$ is *c.e.* and $\sigma \notin S^{OC}$.
Let $P_{<\sigma,s>} = [\sigma] - [S^{OC}]$. This is a non empty Π^0_1 class. $< \tau, T > \prec < \sigma, S >$ if $P_{<\tau,T>} \subset P_{<\sigma,S>}$.
Reference: Extracting information is hard: Dictionary: $W \to opwt, DW \to wt, PW \to pfwt$.
Lemma: Let $S = \{\sigma \in 2^{<\omega} : K(\sigma) \leq |\sigma|/2\}$. Then $< \emptyset, S >$ is a valid condition.
Proof: $wt(S) = \sum_{\sigma \in S} 2^{-|\sigma|/2} \leq \sum_{\sigma \in S} 2^{-K(\sigma)} < \sum_{\sigma \in 2^{<\omega}} 2^{-K(\sigma)} < 1$. Therefore $opwt(S) \leq wt(S) < 1$. So $\emptyset \notin S^{OC}$.
Note that every $A \in P_{<\emptyset,S>}$ has effective dim $dim(A) \geq 1/2$ since they are all $1/2-$ random.
Lemma: Let $\sigma \in 2^{<\omega}$ and $S \subset [\sigma]^{\prec}$. If $[\sigma] - [S^{OC}]$ is non empty then it has positive measure $\geq 2^{-|\sigma|/\sqrt{2}}$.

33 Lecture 33

Lemma: Let $< \sigma, S >$ be a condition. Then $dim(\mu(P_{<\sigma,S>})) \leq 1/2$ where $P_{<\sigma,s>} = [\sigma] - [S^{OC}]$.

Proof: We prove that $dim(\mu[S^{OC}]) \leq 1/2$ that is $\mu(_{<\sigma,s>}) = 2^{-|\sigma|} - \mu[S^{OC}]$. If S^{OC} is finite, then $\mu[S^{OC}]$ is rational. So assume its infinite. Let $w = opwt(S)$. Let $V = \cup S_t^{OC}$ where S_t are the finite approximations of S. So V is c.e. and $[V] = [S^{OC}]$ and $wt(P) \leq w$ whenever $P \subset V$ is prefix free. Note that $V \supset S^{OC}$ is infinite.
Let $\{V_t\}_{t\in\omega}$ be an enumeration of V with $V_0 = \emptyset$. Fix $s > 1/2$. We will produce a Solovay $s-$ test T covering $\mu[V] = \mu[S^{OC}]$. T will consist of rational intervals in $[0,1]$. It can be pulled back to a Solovay s test in $2^{<\omega}$ covering the binary expansion of $\mu[V]$. The pullback will have s weight $3 \times wt_s(T)$.
$T = T_0 \cup T_1$ where if $\tau \in V_{t+1} - V_t$ put $[\mu[V_{t+1}], \mu[V_{t+1}] + 2^{-|\tau|}]$ into T_0.
If $\mu[V_t \cap 2^{>n}] \leq k2^{-n}$ and $\mu[V_{t+1} \cap 2^{>n}] > k2^{-n}$ for some $k, n \in \omega$ then put $[\mu[V_{t+1}], \mu[V_{t+1}] + 2^{-n}]$ into T_1. So T is a *c.e.* set of rational intervals. Note that this T does not depend on s.
Since $V \cap 2^n$ is prefix free:
$wt_s(T_0) = \sum_{I\in T_0} |I|^s = \sum_{\tau\in V} 2^{-s|\tau|} = \sum_{n\in\omega} 2^{-sn}|V \cap 2^n| = \sum_{n\in\omega} 2^{(1/2-s)n}2^{-n/2}|V \cap 2^n| = \sum_{n\in\omega} 2^{(1/2-s)n} wt_{1/2}(V \cap 2^n) \leq \sum_{n\in\omega} 2^{(1/2-s)n} w < \infty$.
Now fix $n \in \omega$ and let k be the number of intervals of length 2^{-n} added to T_1. By construction $2^{-n}k < \mu[V \cap 2^{>n}]$. Let $P \subset V \cap 2^{>n}$ be prefix free such that $[P] = [V \cap 2^{>n}]$. Then $\mu[P] = \sum_{\sigma\in P} 2^{-|\sigma|} \leq \sum_{\sigma\in p} 2^{-n/2}2^{-|\sigma|/2} = 2^{-n/2}wt_{1/2}(P) \leq 2^{-n/2}w$. So $2^{-n}k < 2^{-n/2}w$. So $k < 2^{n/2}w$.
$wt_s(T_1) = \sum_{I\in T_1} |I|^s < \sum_{n\in\omega} 2^{n/2}w(2^{-n})^s = \sum_{n\in\omega} 2^{(1/2-s)n}w < \infty$. So T is a Solovay $s-$ test for any $s > 1/2$.
Next we prove that T covers $\mu[V]$. Call $\tau \in V_{t+1} - V_t$ timely if $V_{t+1} \cap 2^{\leq n} = V \cap 2^{\leq n}$ where $n = |\tau|$. Since V is infinitely there are ∞ many timely $\tau \in V$. Fix $\tau \in V_{t+1} - V_t$ timely and let $n = |\tau|$.
Claim: There is an interval length 2^{-n} in T that contains $\mu[V]$.
Note that if $u > t$ then $\mu[V] - \mu[V_u] = \mu[V \cap 2^{>n}] - \mu[V_u \cap 2^{>n}]$ because nothing shorter gets added. When τ enters V we put $[\mu[V_{t+1}], \mu[V_{t+1}] + 2^{-n}]$ into $T_0 \subset T$. Let $I = [\mu[V_u], \mu[V_u] + 2^{-n}]$ be the last interval of length 2^{-n} added to T. If $\mu[V] \notin I$ then $\mu[V] > \mu[V_u] + 2^{-n}$. But then $\mu[V \cap 2^{>n}] > \mu[V_u \cap 2^{>n}] + 2^{-n}$ so after stage u another interval of length 2^{-n} is added to T_1 a contradiction.
So $\mu[V] \in I$ and there are infintely many timely $\tau \in V$ so $\mu[V]$ fails T. Since $[V] = [S^{OC}]$ we have $dim(\mu[S^{OC}]) \leq 1/2$.

34 Lecture 34

Conditions: $<\sigma, S>$ such that $\sigma \in 2^{<\omega}$ with $S \subset [\sigma]^{\prec}$ is *c.e.* and $\sigma \notin S^{OC}$
Let $P_{<\sigma,S>} = [\sigma] - S^{OC}$. This is a nonempty Π^0_1 class and $<\tau, T>\prec<\sigma, S>$ if $P_{<\tau,T>} \subset P_{<\sigma,S>}$.
Lemma: Let $S = \{\sigma : K(\sigma) \leq |\sigma|/2\}$. Then $<\emptyset, S>$ is a condition.
Note: Every $A \in P_{<\emptyset,S>}$ has $dim(A) \geq 1/2$.
Lemma: If $<\sigma, S>$ is a condition then $dim(\mu(P_{<\sigma,S>}) \leq 1/2$.
Lemma: Let $\sigma \in 2^{<\omega}$ and $S \subset [\sigma]^{\prec}$ If $[\sigma] - [S^{OC}]$ is nonempty then it has positive measure $\geq (1 - \frac{1}{\sqrt{2}})2^{-|\sigma|}$.
Proof: Let $n = |\sigma|$. Since $[S] \subset [\sigma]$ we have $op_{wt}(S) \leq 2^{-n/2}$. Since $[\sigma]-[S^{OC}] \neq \emptyset$ we have $\sigma \notin S^{OC}$ and so $\tau \in S^{OC} \implies |\tau| \geq n + 1$.
So $\mu[S^{OC}] = \sum_{\tau \in S^{OC}} 2^{-|\tau|} \leq \sum_{\tau \in S^{OC}} 2^{-(n+1)/2}2^{-|\tau|/2} = 2^{-(n+1)/2} \sum_{\tau \in S^{OC}} 2^{-|\tau|/2} \leq 2^{(n+1)/2}2^{-n/2} = 2^{-n}/\sqrt{2}$ where we have used $\sum_{\tau \in S^{OC}} 2^{-|\tau|/2} = wt(S^{OC}) = opwt(S)$.
So $\mu([\sigma] - [S^{OC}]) \geq 2^{-n} - 2^{-n}/\sqrt{2} > 0$. So if $<\sigma, S>$ is a condition then $\mu(P_{<\sigma,S>}) > 0$.

Lemma: Let $\sigma \in 2^{<\omega}$ and $S \subset [\sigma]^{\prec}$ be such that $\mu([S]) < 2^{-|\sigma|}$ and $wt(S) < \infty$. Then for any $\epsilon > 0$ there is a τ extending σ with $[\tau] \not\subset [S]$ and $wt(S \cap [\tau]^{\prec}) < \epsilon 2^{-|\tau|/2}$.
Proof: Take $b \in \omega$ such that $\mu([\sigma] - [S]) \geq 2^{-b}$. For each $m \geq b$ define $D_m = \{\tau : \sigma \prec \tau, |\tau| = m, \tau' \prec \tau \implies \tau' \notin S\}$. Now $\mu([\sigma] - [S]) \leq |D_m|2^{-m}$ because if $\tau \in 2^m$ is not in D_m then $[\tau] \subset [S]$. Hence $|D_m| \geq 2^{m-b}$.
Let $S_\tau = S \cap [\tau]^{\prec}$. If $\exists \tau \in D_m$ we have $wt(S_\tau) < \epsilon 2^{-m/2}$ then we win. Otherwise $wt(S) \geq \sum_{\tau \in D_m} wt(S_\tau) \geq \sum_{\tau \in D_m} \epsilon 2^{-m/2} \geq 2^{m-b}\epsilon 2^{-m/2} = \epsilon 2^{m/2-b}$. For 'large' m this is false.
Lemma: Assume that $<\sigma_0, S_0>, <\sigma_1, S_1> \ldots <\sigma_n, S_n>$ is a sequence of conditions such that $\cap_i P_{<\sigma_i,S_i>}$ has positive measure. Then there is a condition $<\tau, T>$ such that $<\tau, T>\prec<\sigma_i, S_i>$ for all i.
Proof: The σ_i must be comparable. So if $\sigma = \cup\sigma_i$, by assumption let $S = [\sigma]^{\prec} \cap (\cup_i S_i^{OC})$. Note that no prefix of σ is in $\cup S_i^{OC}$.
$wt(S) \leq (n + 1) < \infty$. By assumption $[\sigma] - [S] = [\sigma] - [\cup S_i^{OC}] = \cap_i P_{<\sigma_i,S_i>}$ has positive measure. Apply the lemma with $\epsilon = 1$ to get τ extending σ with $wt(S_\tau) < 2^{-|\tau|}$ and $[\tau] \not\subset [S]$.
Let $T = S_\tau = S \cap [\tau]^{\prec}$. So $<\tau, T>$ is a condition because $opwt(T) \leq wt(T) < 2^{-|\tau|}$ so $\tau \notin T^{OC}$. Also $P_{<\tau,T>} = [\tau] \cap ([\sigma] - [S]) \subset [\sigma] - [S] \subset [\sigma] - [S_i^{OC}] \subset [\sigma_i] - [S_i^{OC}] = P_{<\sigma_i,S_i>} \forall i$.

Theorem: There is an $A \leq_T 0'$ with $dim(A) = 1/2$ and $B \leq_T A \implies dim(B) \leq 1/2$.
Proof: Using $0'$ we build a sequence of conditions $<\sigma_0, S_0>$ extended by $<\sigma_1, S_1>$ extended by ..., and take $A = \cup\sigma_t =$ unique element in $\cap P_{<\sigma_t,S_t>}$. Since $0'$ runs the construction we have $A \leq_T 0'$. Let $<\sigma_0, S_0>$ be such that $A \in P_{<\sigma_0,S_0>} \implies dim(A) \geq 1/2$. So $dim(A) \geq 1/2$. Proof in next lecture!

35 Lecture 35

Proof Continued: Let $\{\varphi_e\}$ ennumerate partial computable functionals. We will meet the requirements
$R_{<e,n>} : \varphi_e^A$ total $\implies \exists l \geq n K(\varphi_e^A|_l) \leq (1/2 + 2^{-n})l$.
So $B \leq_T A \implies dim(B) \leq 1/2$ (and so $dim(A) \leq 1/2$).
Stage $t+1 =< e, n >$: Take b such that $2^{-b} < \mu(P_{<\sigma_t, s_t})$ (note that $b = |\sigma_t|+1$ works).
We define a prefix free machine M such that $M(p)$ acts as follows: Wait for $U(p) \downarrow$ (so $dom(M) \subset dom(U)$). Let $\sigma = U(p)$ and $m = |\sigma|$. We are only interested in the case where σ is an initial segment of $\mu(P_{<\sigma_t, S_t>})$'s binary expansion.
For each $\tau \in 2^{<\omega}$ let $T_\tau = \{v \succ \sigma_t : \tau \prec \varphi_e^v\}$. Then T_τ is c.e. Now Search for $\tau \in 2^{m-b}$ such that $\mu(P_{<\sigma_t, S_t \cup T_\tau>}) < .\sigma$ where $.\sigma$ is the rational corresponding to σ and let $M(p) = \tau$.
We can effectively find a $c \in \omega$ such that $\forall \tau K(\tau) \leq K_M(\tau) + c$. Using $0'$ find σ that is an initial segment of $\mu(P_{<\sigma_t, S_t>})$ of length $m > n + b$ and $K(\sigma) + c \leq (1/2 + 2^{-n})(m - b)$. Let $p = \sigma^*$ (minimal U program for σ).
Case 1: $M(p) \downarrow = \tau$.
We have $\mu(P_{<\sigma_t, S_t>}) \geq .\sigma$ and $\mu(P_{<\sigma_t, S_t \cup T_\tau>}) < .\sigma$
Hence there is a $\sigma_{t+1} \in T_\tau$ such that $[\sigma_{t+1}] \not\subset [S_t^{OC}]$. We may also assume $\sigma_{t+1} \succ \sigma_t$ and $0'$ can find σ_{t+1}. Let $S_{t+1} = S_t \cap [\sigma_{t+1}]^{\prec}$. No prefix of σ_{t+1} is in S_t^{OC} so $S_{t+1}^{OC} = S_t^{OC} \cap [\sigma_{t+1}]^{\prec}$. So $< \sigma_{t+1}, S_{t+1} >$ is a valid condition that extends $< \sigma_t, S_t >$.
Why is $R_{<e,n>}$ satisfied? Take $A \in P_{<\sigma_{t+1}, S_{t+1}>}$, then $\sigma_{t+1} \prec A$ and $\sigma_{t+1} \in T_\tau$ and so $\varphi_e^A \succ \tau$. Let $k = |\tau| = m - b$ which is $> n$. $K(\varphi_e^A|_l) = K(\tau) \leq K_M(\tau) + c \leq |\rho| + c = K(\sigma) + c \leq (1/2 + 2^{-n})(m - b) = (1/2 + 2^{-n})l$.
Case 2: In this case $\mu(P_{<\sigma_t, S_t \cup T_\tau}) \geq .\sigma$ for all $\tau \in 2^{m-b}$. Therefore $< \sigma_t, S_t \cup T_\tau >$ is a condition and it extends $< \sigma_t, S_t >$ and furthermore $\mu(P_{<\sigma_t, S_t>} - P_{<\sigma_t, S_t \cup T_\tau}) \leq 2^{-m}$.
Note that $\mu(\cap_{\tau \in 2^{m-b}} P_{<\sigma_t, S_t \cup T_\tau>}) \geq \mu(P_{<\sigma_t, S_t>}) - 2^{m-b}2^{-m} > 2^{-b} - 2^{-b} > 0$.
Therefore there is a condition $< \sigma_{t+1}, S_{t+1} >$ extending $< \sigma_t, S_t \cup T_\tau >$ for all $\tau \in 2^{m-b}$ (Can take $\sigma_{t+1} \succ \sigma_t$). $0'$ can find this.

Why is $R_{<e,n>}$ satisfied? If φ_e^A is total, let $\tau = \varphi_e^A|_{m-b}$ Since $\sigma_t \prec A$ some $\gamma \prec A$ is in T_τ. But then $A \notin P_{<\sigma_t, S_t \cup T_\tau>} \geq P_{<\sigma_{t+1}, S_{t+1}>}$ and so φ_e^A is partial. □

36 Lecture 36

Lemma: Fix an oracle Z and a countable collection of functions f_i such that $f_i : 2^\omega \to 2^\omega$. is continuous. There is an $A \in 2^\omega$ such that $dim^Z(A) = 1/2$ and $dim^{f_i}(f_i(A)) \leq 1/2$.

Partial relativization - No access to Z when compressing $f_i(A)$. Total 'reductions' f_i.
Theorem: Assuming Continuum Hypothesis there is an $E \subset 2^\omega$ with $dim_H(E) = 1/2$ and for all continuous $f : 2^\omega \to 2^\omega$ we have $dim(f(E)) \leq 1/2$.
Proof: Let $\{Z_\lambda : \lambda < \omega_1 \text{ list } 2^\omega\}$ and $\{f_\lambda : \lambda < \omega_1\}$ list all continuous functions $2^\omega \to 2^\omega$. Let A_λ be given by the lemma applied to Z_λ and the countable collection $\{f_\gamma : \gamma \leq \lambda\}$. Finally let $E = \{A_\lambda : \lambda < \omega_1\}$. For any Z there is an $A \in E$ such that $dim^Z(A) = 1/2$. By point to set $dim_H(E) \geq 1/2$.
Now let $f : 2^\omega \to 2^\omega$ be a continuous function. Say $f = f_\lambda$. But $E = \{A_\gamma : \gamma < \lambda\} \cup \{A_\gamma : \gamma \geq \lambda\}$. Call the first set in the union E_0 and the second set E_1. Since E_0 is countable, $dim_H(f(E_0)) = 0$.On the other hand $\forall A \in E_1$ we have $dim^{f_\lambda}(f_\lambda(A)) \leq 1/2$ so by point to set $dim_H(f_\lambda(E_1)) \leq 1/2$. Hence $dim_H(f_\lambda(E)) \leq 1/2$.

Lemma: If $0 \leq s < t \leq 1$ and $\delta > 0$ then $\exists n \in \omega \forall N$ and $\forall f : 2^N \to 2^n$ there is an $A \subset 2^n$ such that
$i) wt_t(A) \leq \delta$
$ii) opwt_s(f^{-1}(A)) = 1$.
Lemma 1: Fix $s \in [0,1]$ and let $V \subset 2^{<\omega}$. If $opwt_s(V) < 1$ then $\mu[V^{OC(s)}] < 2^{s-1}$ (same as proof for s=1/2)
Lemma: If $a, b \geq 0$ and $r \in (0,1]$ then $(a+b)^r \leq a^r + b^r$ i.e. $x \to x^r$ is subadditive.
Lemma 2: Let $s \in (0,1]$. If $V \subset 2^{<\omega}$ then $wt_s(V) \geq (\mu[V])^s$.
Example: Let $V = \{\sigma\}, |\sigma| = n$, then $wt_s(V) = 2^{-sn}$ and $(\mu[V])^s = 2^{-ns}$.
Proof: Sufficient to prove for finite V because the inequality is preserved under a limit. $V = \emptyset$ is clear. Otherwise we use induction on the length n of the longest string in V. If $n = 0$ or even when $\emptyset \in V$, then $wt_s(V) \geq 1 = (\mu[V])^s$. Assume $n > 1$ and $\emptyset \notin V$.
For $i \in \{0,1\}$ let $V_i = \{\sigma \in 2^{<\omega} : i\sigma \in V\}$. By induction, $wt_s(V) = 2^{-s}(wt_s(V_0)+wt_s(V_1)) \geq 2^{-s}(\mu[V_0]^s+\mu[V_1]^s) \geq 2^{-s}(\mu[V_0]+\mu[V_1])^s = 2^{-s}.(2\mu[V])^s = \mu[V]^s$.
Lemma 3: $D \subset E \implies [D^{OC}] \subset [E^{OC}]$.
Lemma 4: If $B_0 \subset B_1 ... \subset B_i$, and $C = \cup_{j \leq i} B_j^{OC(s)}$ then $pfwt_s(C) \leq opwt_s(B_i)$.

37 Lecture 37

We'll now prove the lemma from last time.
Lemma: Let $0 \le s < t \le 1$ and $\delta > 0$. Then there is an $n \in \omega$ such that $\forall N \forall f : 2^N \to 2^n$ there is an $A \subset 2^n$ such that
i) $wt_t(A) \le \delta$
ii) $opwt_s(f^{-1}(A)) = 1$
Proof: All optimal covers are for dim s. Let $\gamma = (t+s)/2$ and take n large. We will define a sequence $\sigma_0, \sigma_1, ...$ of strings in 2^n. Let $B_i = f^{-1}(\{\sigma_0, ..., \sigma_i\})$. We want to show that we can pick the $\sigma_i's$ so that $opwt_s(B_m) = 1$ for a fairly small M.
We will let $A = \{\sigma_0, ..., \sigma_m\}$ and let $C_i = \cup_{i \le j} B_j^{OC}$. Note that $[C_i] = [B_i^{OC}]$. So $opwt_s(C_i) = opwt_s(B_i)$. By lemma 4, $pfwt_s(C_i) \le opwt_s(B_i) \le 1$.
Now $wt_{s+\gamma}(C_i) = \sum_l \sum_{\sigma \in C_i \cap 2^l} 2^{(-s+\gamma)l} = \sum_l 2^{-\gamma l} \sum_{\sigma \in C_i \cap 2^l} 2^{-sl} \le \sum_l 2^{-\gamma l} pfwt_s(C_i) \le 1/(1-2^{-\gamma}) = d_0$- Does not depend on n.
We will now argue that if $opwt_s(B_i) < 1$ then we can choose σ_{i+1} to push up $wt_{s+\gamma}(c_i)$ by a non trivlal amount.
By lemma 1 if $opwt_s(B_i) < 1$ then $\mu(2^\omega - [B_i^{OC}]) \ge 1 - 2^{s-1} > 0$. This means that there is a $\sigma_{i+1} \in 2^n$ such that $[f^{-1}(\sigma_{i+1}] - [B_i^{OC}]$ has measure at least $d_1 2^{-n}$ where $d_1 = 1 - 2^{s-1}$. Let $D = C_{i+1} - C_i$. Note that $[D] \supset [C_{i+1}] - [C_i] = [B_{i+1}^{OC}] - [B_i^{OC}] \supset [f^{-1}(\sigma_{i+1}] - [B_i^{OC}]$. Therefore $\mu[D] \ge d_1 2^{-n}$. By $lemma2$, $wt_{s+\gamma}(D) \ge d_1^{s+\gamma} 2^{-(s+\gamma)n}$.
So the choice of σ_{i+1} added at least $d_1^{s+\gamma} 2^{-(s+\gamma)n}$ to $wt_{s+\gamma}(C_{i+1})$. We can do this at most $d_0 d_1^{-(s+\gamma)} 2^{(s+\gamma)n}$ times. So $opwt_s(B_m) = 1$ for some M less than this.
For sufficiently large n, we have $wt_t(\{\sigma_0, ..., \sigma_M\}) \le d_0 d_1^{-s-\gamma} 2^{(s+\gamma)n} 2^{-tn} = d_0 d_1^{-s-\gamma} 2^{-\gamma n} \le \delta$.

Lemma: Fix Z and a countable collection $f_0, f_1, ...$ of continuous functions $f_i : 2^\omega \to 2^\omega$. There is an $A \in 2^\omega$ such that $dim^Z(A) \ge 1/2$ and $\forall i dim^{f_i}(f_i(A)) \le 1/2$.
Proof: We build a sequence of conditions $< \sigma_0, S_0 > \succ < \sigma_1, S_1 > \succ$ ... and take $A = \cup \sigma_i =$ the unique element of $\cap P_{< \sigma_t, S_t >}$. Let $< \sigma_0, S_0 > = < \emptyset, \{\tau \in 2^{<\omega} : K^Z(\tau) \le |\tau|/2\} >$. We will meet the requirements
$R_{i,m} : (\exists n \ge m) K^{f_i}(f_i(A)|_n) \le (1/2 + 2^{-m})n$.
Stage $t = < i, m > +1$: We try to satisfy $R_{< i,m >}$. Define a prefix free f_i machine M^{f_i} as follows: By the recursion theorem, we may assume that we know a constant c such that $\forall \tau K^{f_i}(\tau) \le K_{M^{f_i}}(\tau) + c$.
Proof finished next time.

38 Lecture 38

We now prove the lemma from last class: **Lemma:** Fix Z and a countable collection $f_0, f_1, ...$ of continuous functions $f_i : 2^\omega \to 2^\omega$. There is an $A \in 2^\omega$ such that $dim^Z(A) \geq 1/2$ and $\forall i \; dim^{f_i}(f_i(A)) \leq 1/2$.
Proof: We build a sequence of conditions $< \sigma_i, S_i >$ monotonically getting stronger such that $A = \cup \sigma_i =$ the unique element in $\cap P_{<\sigma_i, S_i>}$.
Let $< \sigma_0, S_0 >=< \emptyset, \{\tau : K^Z(\tau) \leq |\tau|/2\} >$ so that $dim(A) \geq 1/2$.
Let $R_{i,m} : \exists n \geq m \; K^{f_i}(f_i(A)|_n) \leq (1/2 + 2^{-m})n$.
Stage $v =< i, m > +1$ (Satisfy $R_{i,m}$): Define a prefix free machine M^{f_i} as follows: By the recursion theorem we may assume we know a constant c such that $\forall \tau K^{f_i}(\tau) \leq K_{M^{f_i}}(\tau) + c$. Let $s = 1/2$ and $t = 1/2 + 2^{-m}$. Let $\delta = 2^{-c-1}$. Fix n from the lemma and let $\hat{f} : 2^\omega \to 2^\omega$ be defined by $\hat{f}(B) = f_i(\sigma_v B)$.
By compactness there is a N such that $B|_N$ determines $\hat{f}(B)|_n$.
So let $f : 2^N \to 2^n$ be the resulting function.
So let $f : 2^N \to 2^n$ be the resulting function and apply the lemma to get $C \subset 2^n$ such that $wt_t(C) \leq \delta$ and $opwt_s(f^{-1}(C)) = 1$. Now M^{f_i} gives each $\tau \in C$ a program of length $\leq t|\tau| - c$. Note that $(\sigma_v \frown f^{-1}(C))^{OC} = \{\sigma_v\}$.
So $[\sigma_v \frown f^{-1}(c)] \not\subset [S_v^{OC}]$. Take $\sigma_{v+1} \in \sigma_v \frown f^{-1}(C)$ such that no prefix of σ_{v+1} is in S_v^{OC}.
Then $< \sigma_{v+1}, S_v \cap [\sigma_{v+1}]^{\prec} >$ is a condition that extends $< \sigma_v, S_v >$. Note that if $A \in P_{<\sigma_{v+1}, s_{v+1}>}$ then $f_i(A)|_n \in C$ so $K^{f_i}(f_i(A)|_n) \leq K_{M^{f_i}}(f_i(A)|_n) + c \leq tn - c + c = tn = (1/2 + 2^{-m})n$.

Theorem: Assume CH. Then $\forall s \in [0,1) \exists E \subset 2^\omega$ such that $dim_H(E) = s$ and if $f : 2^\omega \to 2^\omega$ is continuous then $dim_H(f(E)) \leq s$.
What about $\mathbb{R}^k$?
Let $p : 2^\omega \to [0,1]$ be the continuous surjection given by (binary expansion)$^{-1}$. Note that p preserves measure, $H-$ dim,....
Let $p_K : 2^\omega \to [0,1]^k$ by $P_k(A_0 \oplus ... \oplus A_{k-1}) = (p(A_0), p(A_1), ...)$. This preserves relative dimension: $dim_H(p_k(E)) = k \times dim_H(E)$.
Issue: A continuous function $\hat{f} : 2^\omega \to [0,1]^k$ cannot be approximated by a function $f : 2^N \to (2^n)^k$: Suppose we are mapping from $2^\omega \to$ the unit square. We can take N large enough to determine the image of $A|_N$ under the function up to n bits to within 2^k possibilities. Pick the 'leftmost' possibility, then the combinatorial lemma provides a $C \subset (2^n)^k$ which we expand to one of 'weight' $\leq 2^k \times \delta$.
Theorem: Assuming CH. $\forall s \in [0,1) \exists E \subset 2^\omega$ such that $dim_H(E) = s$ and for any continuous $f : 2^\omega \to 2^\omega$ we have $dim_H(f(E)) \leq s$ and for any continuous $f : 2^\omega \to \mathbb{R}^k$ we have $dim_H(f(E)) \leq sk$.
Take E from this theorem and let $\hat{E} = p_2(E)$ so $\hat{E} \subset \mathbb{R}^2$.
Corollary: There is an $\hat{E} \subset [0,1]^2$ such that $dim_H(\hat{E}) = 1$ and if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous then $dim_H(f(E)) \leq 1$.
Theorem: There is a $E \subset [0,1]^2$ such that $dim_H(E) = 1$ and if $f : [0,1]^2 \to \mathbb{R}$ is continuous then $dim_H(f(E)) = 0$.

39 Lecture 39

Note all sets of $s - dim_H$ are the same:
Definition: i)Given $E \subset 2^\omega$ we say E has essential dimension s if $dim_H(E) = s$ and E is not a countable union of sets of lower dimension.
ii)We say that E is $\sigma-$ finite in dimension s if it is a countable union of sets each of whose $s-$ dimension is finite.
Theorem: The theorem from last class is true if we instead take E to be not σ finite in dimension s.
Idea: Every $F \subset 2^\omega$ of finite H_s measure is covered by a $V_n \subset 2^{<\omega}$ of fixed finite weight with least string of length $\geq n$ and we can avoid countably many such families.

Theorem: Assuming CH for every $t \in (0,1]$ there is an $E \subset 2^\omega$ that is essentially dimension t and if $f : 2^\omega \to 2^\omega$ is continuous then $H_t(f(E)) = 0$.
Idea: Use $s < t$ lemma with t fixed and $s \to t$.
Fact: $E \subset 2^\omega$ has nonzero H_t measure $\iff$ $\forall Z \exists A \in E$ such that A is (strongly) t- random relative to Z.
Proposition: If $E \subset 2^\omega$ is $\sigma-$ finite in dimension s and $H_s(E) > 0$ then there is a continuous $f : 2^\omega \to 2^\omega$ such that $f(E)$ has non zero outer Lebesgue measure. In particular $dim(F(E)) = 1$.

39.1 (Uniform) Almost Everywhere Domination

Recall that we say f majorizes g if $\forall n f(n) \geq g(n)$ and f dominates g if $\exists N \forall n > N f(n) \geq g(n)$.
Definition(Dobrinen, Simpson): B is almost everywhere dominating (aed) if for almost all X every X computable function is dominated by a B computable function.
B is uniformly aed if there is an $f \leq_T B$ such that for almost every X every X computable function is dominated by f.
We will see that B is *aed* $\iff$ B is *uaed* $\iff$ $0' \leq_{LR} B$ (i.e. every B random is $2-$ random).
Facts: i) *uaed* $\implies$ high. ii)*uaed* $\implies$ *aed*. iii) 0 is not *aed* since every $2-$ random has hyperimmune degree. iv) $0'$ is uaed.
Theorem: B is *aed* $\iff$ for every Π^0_2 class Q and $\epsilon > 0$ there is a $\Pi^0_1[B]$ class $F \subset Q$ with $\mu(F) \geq \mu(Q) - \epsilon$.
Proof: Assume B is *aed*. Then given a Π^0_2 set $Q \subset 2^\omega$ let ψ be a partial computable function such that $Q = \{X \in 2^\omega : (\forall n)\psi^X(n) \downarrow\}$. Let $g^X(n) =$ least s such that $\psi^X_s(n) \downarrow$. For almost all $X \in Q$ there is an $f \leq_T B$ that majorizes g^X. Thus for $\epsilon > 0$ there is an $f \leq_T B$ such that $\mu(\{X \in Q : f$ majorizes $g^X\}) \geq \mu(Q) - \epsilon$.
So $F = \{X \in 2^\omega : \forall n \psi^X_{f(n)}(n) \downarrow\}$ is a $\Pi^0_1[B]$ subset of Q such that $\mu(F) \geq \mu(Q) - \epsilon$.

40 Lecture 40

Theorem: B is almost everywhere dominating $\iff$ for every Π^0_2 class Q and $\epsilon > 0$ there is a $\Pi^0_1[B]$ class $F \subset Q$ such that $\mu(F) \geq \mu(Q) - \epsilon$.
Proof: The forward direction was done last time.
For the backward direction given e, ϵ let $Q_e = \{X \in 2^\omega : \forall n \varphi^X_e(n) \downarrow\}$. This is Π^0_2 so let $F \subset Q$ be $\Pi^0_1[B]$ such that $\mu(F) \geq \mu(Q) - \epsilon$.
F is closed and $\forall X \in F \varphi^X_e$ is total. So by compactness we can B compute a function f majorizing every φ^X_e for $X \in F$.
Letting $\epsilon \to 0$ for almost every $X \in Q_e$ thre is an $f \leq_T B$ majorizing φ^X_e. This holds for all e so B is almost everywhere dominating.
Recall that $A \leq_{LR} B$ if and only if every $\Pi^0_1[A]$ class of positive measure has a $\Pi^0_1[B]$ subclass of positive measure.
Corollary: B is almost everywhere dominating $\implies 0' \leq_{LR} B$.
Fact: Every $\Pi^0_1[0']$ class is a Π^0_2 class.
Proof: Let $Q = \{X \in 2^\omega : \forall n X|_n \in T\}$ where $T \leq_T 0'$. T is Δ^0_2 and in particular Π^0_2. There is a computable R such that $\sigma \in T \iff \forall u \exists v R(u, v, \sigma)$. So $Q = \{X \in 2^\omega : \forall n \forall u \exists v R(u, v, X|_n)\}$ is Π^0_2.
Proof of corollary: Let Q be a $\Pi^0_1[0']$ class and hence a Π^0_2 class. Applying the theorem to get a positive measure $\Pi^0_1[B]$ subclass of Q.
Theorem: B is uniform almost everywhere dominating $\iff$ for every Π^0_2 class there is a $\Sigma^0_2[B]$ subclass $S \subset Q$ such that $\mu(S) = \mu(Q)$.
Proof: Let B be uniformly almost everywhere dominating as witnessed by $f \leq_T B$. Let Q be a Π^0_2 class and take ψ partial computable such that $Q = \{X \in 2^\omega : \forall n \psi^X(n) \downarrow\}$. Define $g^X(n) =$ least s such that $\psi^X_s(n) \downarrow$. Let $S = \{X \in 2^\omega : \exists k \forall n \psi^X_{f(n)+k}(n) \downarrow\}$. So $S \subset Q$ is $\Sigma^0_2[B]$ and $\mu(S) = \mu(Q)$.
For the backward direction let $Q = \{0^l 1 X : e \in \omega, X \in 2^\omega, \varphi^X_e \text{ total }\}$. So Q is Π^0_2. Take $S \subset Q$ a $\Sigma^0_2[B]$ class with $\mu(S) = \mu(Q)$. Write $S = \cup_{i \in \omega} P_i$ where each P_i is a $\Pi^0_1[B]$ class uniformly. Let $P_{i,e} = \{X : 0^e 1 X \in P_i\}$. So $X \in \cup_i P_{i,e} \implies \varphi^X_e$ total$\}$ and $\mu(\cup_i P_{i,e}) = \mu\{X : \varphi^X_e \text{ total }\}$.
Let $f_{i,e}(n) = max\{\varphi^X_e(n) : X \in P_{i,e}\}$ such that $f_{i,e} \leq_T B$ uniformly in i, e.
Let $f(n) = max\{f_{i,e}(n) : i, e \leq n\}$. Then f witnesses that B is uniformly almost everywhere dominating. $\square$

Claim: $0'$ is uniformly almost everywhere dominating
or equivalently every Π^0_2 class has a $\Sigma^0_2[0']$ subclass of same measure. We'll see the proof next time