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Assignment 1

Problem

How to ensure continuity while merging cubic Bezier curves?

Bezier Curves

Explicit formulation

Let V_i's be the control points.

Let $\{V_0, \ldots, V_n\}$ be the control polygon.

An explicit formulation for Q(u):

In general,

$$Q(u) = \sum_{i=0}^{n} V_i \, {}^{n}C_i \, u^i \, (1-u)^{n-i}$$

Note that $Q(0) = V_0$ and $Q(1) = V_n$ from the above formulation

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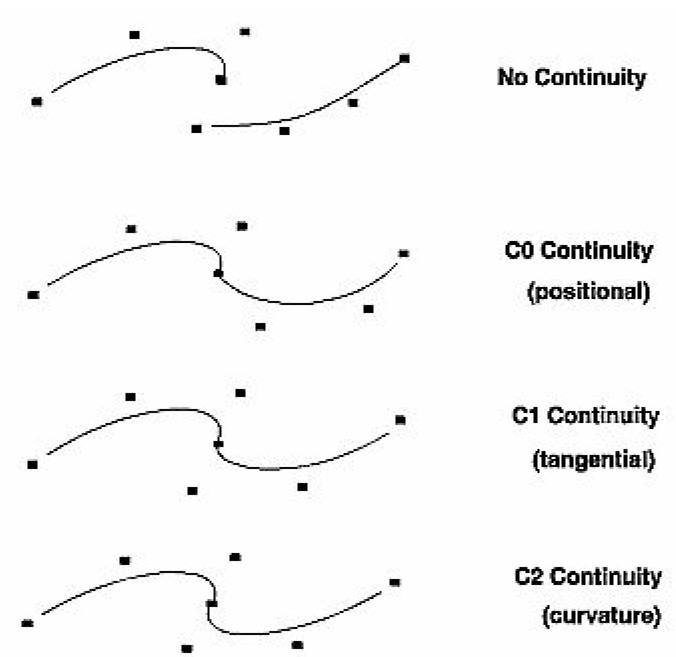

Ensuring Positional (C^0) continuity

Let there be two cubic curves B_1 and B_2.

B_1 has control points V_0, V_1, V_2 and V_3.

B_2 has control points W_0, W_1, W_2, W_3.

Then, $Q_1(u)$ is defined as: $Q_1(u) = (1 - u)^3 V_0 + 3u(1 - u)^2 V_1 + 3u^2(1 - u) V_2 + u^3 V_3$

Also, $Q_2(u)$ is defined as: $Q_2(u) = (1 - u)^3 W_0 + 3u(1 - u)^2 W_1 + 3u^2(1 - u) W_2 + u^3 W_3$

Hence, to ensure Positional (C^0) continuity, $Q_1(1)$ should be equal to $Q_2(0)$.

$Q_1(1) = V_3$

$Q_2(0) = W_0$

Therefore, $V_3 = W_0$

Ensuring Derivative (C^1) continuity

Operator Calculus

We need to use a tool known as operator calculus.

Define the operator D by:

$$D V_i \equiv V_{i+1}$$

In general, $V_i = D^i V_0$

Rewriting our explicit formulation in this notation gives:

$$\begin{aligned} Q(u) &= \sum_{i=0}^{n} {}^nC_i\, u^i (1 - u)^{n-i} V_i \\ &= \sum_{i=0}^{n} {}^nC_i\, u^i (1 - u)^{n-i} D^i V_0 \\ &= (uD + 1 - u)^n V_0 \end{aligned}$$

Taking the first derivative, we get:

$$Q'(u) = n(uD + 1 - u)^{n-1}(D - 1) V_0$$

Now, $(D - 1) V_0 = D V_0 - V_0 = V_1 - V_0$

Also, $\sum_{i=0}^{n-1} {}^{n-1}C_i (uD)^i (1 - u)^{n-1-i} = (uD + 1 - u)^{n-1}$

Hence,

$$Q'(u) = n \sum_{i=0}^{n-1} {}^{n-1}C_i (uD)^i (1 - u)^{n-1-i} (V_1 - V_0)$$

This gives us a general expression for the derivative Q'(u).

Now:

$Q'(0) = n\,(V_1 - V_0)$

$Q'(1) = n\,(V_n - V_{n-1})$

Derivative (C^1) continuity

For cubic curve (n = 3):

$$Q_1'(1) = 3\,(V_3 - V_2)$$

$$Q_2'(0) = 3\,(W_1 - W_0)$$

Hence, for C^1 continuity, we need to set

$$Q_1'(1) = Q_2'(0)$$

$$V_3 - V_2 = W_1 - V_3 \qquad \text{[Due to } C^0 \text{ continuity, } V_3 = W_0 \text{]}$$

$$V_3 = \frac{V2 + W1}{2} \quad \text{OR} \quad W_0 = \frac{V2 + W1}{2}$$

$$W_1 = 2\,V_3 - V_2$$

Ensuring Curvature (C^2) continuity

We already know that $Q'(u) = n\,(uD + 1 - u)^{n-1}\,(D - 1)\,V_0$

Differentiating it again, we get

$$Q''(u) = n\,(n - 1)\,(uD + 1 - u)^{n-2}\,(D - 1)^2\,V_0$$

Simplifying Further,

$$Q''(u) = n\,(n - 1)\,\sum_{i=0}^{n-2} {}^{n-2}C_i\,u^i\,(1 - u)^{n-2-i}\,D^i\,(V_2 - 2\,V_1 + V_0)$$

This gives us a general expression for the double derivative Q''(u).

$Q''(0) = n\,(n-1)\,(V_2 - 2\,V_1 + V_0)$

$Q''(1) = n\,(n-1)\,(V_n - 2\,V_{n-1} + V_{n-2})$

Curvature (C^2) continuity

For cubic curve (n = 3):

$$Q_1''(1) = 6\,(V_3 - 2\,V_2 + V_1)$$

$$Q_2''(0) = 6\,(W_2 - 2\,W_1 + W_0)$$

Hence, for C^2 continuity, we need to set

$$Q_1''(1) = Q_2''(0)$$

$$6\,(V_3 - 2\,V_2 + V_1) = 6\,(W_2 - 2\,W_1 + W_0)$$

$$(V_3 - V_2) - (V_2 - V_1) = (W_2 - W_1) - (W_1 - W_0)$$

Also, from C^0 and C^1 continuities, we have

$$V_3 = W_0 \qquad [\ C^0 \text{ continuity}\ ]$$

$$V_3 - V_2 = W_1 - W_0 \qquad [\ C^1 \text{ continuity}\ ]$$

Putting it together,

$$W_2 = 4\,(V_3 - V_2) + V_1$$