CS461 – Assignment 3

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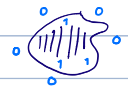
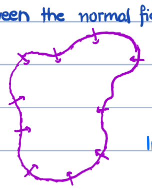
Problem Statement : Can you define an implicit function given a set of points ?

We show that surface reconstruction from oriented points can be cast as a spatial Poisson problem. This Poisson formulation considers all points at once.
We have to define a function with positive values inside the model and negative values outside and we have to extract the zero set.

$$\chi_M(p) = \begin{cases} 1 & \text{if } p \in M \\ 0 & \text{otherwise} \end{cases}$$



In practice, we define the indicator function to be −1/2 outside the shape and 1/2 inside, so that the surface is the zero level set. We also smooth the function a little, so that the zero set is well defined.
There is a relationship between the normal field at the shape boundary, and the gradient of the smoothed indicator function.



Indicator Gradient

$\nabla \chi_M$

Represent the point normals by a vector field V
Find the function χ whose gradient best approximates V

$$\min_{\chi} \| \nabla \chi - V \|^2$$

Now we try to convert this into a poisson problem by applying the divergence operator

$$\nabla \cdot (\nabla \chi) = \nabla \cdot V \iff \underbrace{\Delta \chi}_{\text{Laplacian}} = \nabla \cdot V$$

Euler Lagrange Formulation

Let $f : \mathbb{R} \to \mathbb{R}$, Imagine we didn't know f, but we did know g

$g = df/dx$, $f = \int g \, dx$

But what if g is not analytically integrable ?
Then we can look for approximate solutions, drawn from some parametrized family of candidate functions

Assume we have a family of functions F
Let's minimize the mean squared approximation error over some interval Ω and functions $f \in F$

minimize $\displaystyle\int_{\Omega} \left| \frac{df}{dx} - g \right|^2 dx$

$\displaystyle\int_{\Omega} L(x, f(x), f'(x))\, dx$ are obtained as solutions f to PDE $\dfrac{\partial L}{\partial f} - \dfrac{d}{dx}\dfrac{\partial L}{\partial f'} = 0$

In our case $L = (f'(x) - g(x))^2$ so

$$\frac{\partial L}{\partial f} = 0$$

$$\frac{\partial L}{\partial f'} = 2\left(f'(x) - g(x)\right)$$

$$\frac{d}{dx}\frac{\partial L}{\partial f'} = 2\left(f''(x) - g'(x)\right) \Rightarrow \boxed{f'' = g'}$$

Let's try to discretize this:

★ Sample n consecutive points $\{x_i\}$ from Ω

For simplicity, they're evenly spaced $\because x_{i+1} - x_i = h$

★ We want to minimize $\sum_i (f'(x_i) - g(x_i))^2$

Approximately: $f'(x_i) \approx \dfrac{f_{i+1} - f_i}{h} = \dfrac{1}{h}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} f_i \\ f_{i+1} \end{bmatrix}$

All derivatives can be listed in one big matrix multiplication

$Af = g$ where

$$A = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \\ 0 & \cdots & & & 1 \\ & & & & -1 \end{bmatrix} \quad f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{bmatrix} \quad g = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_n \end{bmatrix}$$

f and g are discrete approximations of continuous functions f and g

Now coming back to our problem,

We need to solve set of equations $Af = g$ in a least squares sense

minimize $\|r\|^2 = \|g - Af\|^2$

The minimum is achieved when all directional derivatives are zero

giving the normal equations:

$$\boxed{A^T A f = A^T g}$$

The discrete operator A we constructed is full rank (invertible)

and gives a unique solution $A^{-1}g$ for f

In higher dimensions

We have a function $f : \mathbb{R}^p \to \mathbb{R}^q$

We can discretize the domain as before and obtain discrete analogues of the gradient $\nabla(A)$, divergence $\nabla \cdot (-A^T)$ and Laplacian $\Delta = (\nabla \cdot)\nabla\,(-A^T A)$

Takeaway

So we have achieved a continuous variation problem into a discrete one by the following mappings

Cont. function $\rightarrow$ Discrete vector of values

Cont. operator $\rightarrow$ Discrete Matrix

Function Composition $\rightarrow$ Matrix multiplication

Euler - Lagrange $\rightarrow$ Linear Least Squares

Conclusion :

So to solve $\Delta x = \nabla \cdot v$, which is our original problem
We can discretize the system by representing the
functions as vectors of values at sample points

Solving the resulting linear system gives a least
squares fit at sample position.