HOMEWORK-2

NUMERICAL ALGORITHMS

1 ANSWER:

(a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

We want, $A = U \Sigma V^*$

$\Rightarrow A^*A = V \Sigma^* \Sigma V^*$

$A^*A = V \Sigma' V^*$ $\quad (\Sigma' = \Sigma^* \Sigma)$

$\Rightarrow$ This is same as the eigen decomposition,

→ Then, finding eigen values of A^*A:

$(A^*A - \lambda I) = 0$

$\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$

$\begin{bmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} = 0$

$(2-\lambda)^2 - 4 = 0 \Rightarrow (2-\lambda) = \pm 2$

$\Rightarrow \lambda = 0, 4$

$\boxed{\Rightarrow \sigma = 0, 2} \Rightarrow (\Sigma) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

→ Eigen vectors of A^*A:

(i) $(A^*A - 4I)x = 0$

$\Rightarrow \begin{bmatrix} 2-4 & 2 \\ 2 & 2-4 \end{bmatrix} x = 0$

$\Rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} x_1 = 0 \Rightarrow \boxed{x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$\Rightarrow \left.\begin{matrix} -2x_{11} + 2x_{12} = 0 \\ +2x_{11} + (-2)x_{12} = 0 \end{matrix}\right\}$ Solving these gives $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(ii) $(A^*A - 0I)x_2 = 0$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} x_2 = 0$$

$\Rightarrow x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (Solving $2x_{21} + 2x_{22} = 0$ and $2x_{21} + 2x_{22} = 0$)

Let $x_{21} = 1$, $\Rightarrow x_{22} = -1$.

$\Rightarrow V = [x_1 \ x_2]$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$\because$ V needs to be orthonormal, $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

and $A = U'\Sigma' V^{*'}$

$\Rightarrow A'V'\Sigma^{-1} = U\Sigma$

$\Rightarrow U\Sigma' = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cancel{\begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix}}$

$\Rightarrow U'\Sigma' = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \cancel{\begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} \neq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$$U' \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \quad \text{—— (1)}$$

Then we can write U as

$U' = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\because U\Sigma' = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$

and colums of U are still orthonormal

$\Rightarrow A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$ ($\because$ Taking $\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$ inside Σ').

$$\boxed{\Rightarrow A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}$$

where $U = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$V^* = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

① (b) ANSWER : Given $A = U\Sigma V^*$

$$\begin{bmatrix} A & A \\ A & A \end{bmatrix} = \begin{bmatrix} U\Sigma V^* & U\Sigma V^* \\ U\Sigma V^* & U\Sigma V^* \end{bmatrix}$$

$$= \begin{bmatrix} U\Sigma V^* & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & U\Sigma V^* \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ U\Sigma V^* & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & U\Sigma V^* \end{bmatrix}$$

$$= \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & \Sigma \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & V^* \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \Sigma & 0 \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \Sigma \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & V^* \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix}}_{} + \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} 0 & \Sigma \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix}$$

This doesn't affect the result of multiplication

$$+ \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \Sigma & 0 \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix} + \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \Sigma \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix}$$

$$= \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}\left\{ \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \Sigma \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \Sigma & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \Sigma \end{bmatrix} \right\}\begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix}$$

$$= \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} \Sigma & \Sigma \\ \Sigma & \Sigma \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix} \quad \text{—— ①}$$

$\Rightarrow$ we can write $\begin{bmatrix} \Sigma & \Sigma \\ \Sigma & \Sigma \end{bmatrix}$ as $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} \Sigma & \Sigma \\ \Sigma & \Sigma \end{bmatrix}$$

Putting in eq^n ①

$$\Rightarrow \begin{bmatrix} A & A \\ A & A \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix}$$

$$\begin{bmatrix} A & A \\ A & A \end{bmatrix} = \begin{bmatrix} U & U \\ U & -U \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V^* & V^* \\ V^* & -V^* \end{bmatrix} \quad \text{—— ②}$$

$\Rightarrow SVD\left(\begin{bmatrix} A & A \\ A & A \end{bmatrix}\right) \Rightarrow U'\Sigma'V'^{*}$ (Given $A = U\Sigma V^{*}$)

where $U' = \begin{bmatrix} U & U \\ U & -U \end{bmatrix}$ $\quad V'^{*} = \begin{bmatrix} V^{*} & V^{*} \\ V^{*} & -V^{*} \end{bmatrix}$

$\Sigma' = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$ $\quad V = \begin{bmatrix} V & V \\ V & -V \end{bmatrix}$

$\Rightarrow$ In part (a):

consider $[A] = [1]$

$SVD(A) = U\Sigma V^{*}$

$\Rightarrow U = [1] \quad \Sigma = [1] \quad V^{*} = [1]$

Then $SVD\left(\begin{bmatrix} A & A \\ A & A \end{bmatrix}\right) = \begin{bmatrix} U & U \\ U & -U \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V^{*} & V^{*} \\ V^{*} & -V^{*} \end{bmatrix}$

$SVD\left(\begin{bmatrix} A & A \\ A & A \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

which is the same result as when

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\Rightarrow$ It agrees with the result in part (a).

②

(a) : ANSWER:

→ For (AB^*) to be a ~~proper~~ projector:

$$(AB^*)^2 = AB^*$$

$$\Rightarrow (AB^*)(AB^*)$$

$$\Rightarrow A(B^*A)B^* = AB^*$$

$$\Rightarrow AB^* \text{ only if } \underline{\underline{B^*A = I}}.$$

→ for (BA^*) to be a projector

$$(BA^*)^2 = BA^*$$

$$\Rightarrow (BA^*)(BA^*)$$

$$= BA^*BA^* = \underline{\underline{BA^*}} \text{ only if } \underline{\underline{A^*B = I}}$$

$\Rightarrow$ ~~BA~~

Now, consider the converse:

→ if $A^*B = I$

Then $A^*B = I$

$BA^*B = B$ $\quad (I \cdot B = B)$

$BA^*BA^* = BA^*$

$\boxed{\Rightarrow (BA^*)^2 = BA^*}$ $\quad [P^2 = P]$

$\Rightarrow BA^*$ is a projector

→ if $B^*A = I$

Then $B^*A = I$

$AB^*A = A$

$AB^*AB^* = AB^*$

$\boxed{(AB^*)^2 = AB^*}$

$\Rightarrow AB^*$ is a projector $\underline{(AB^*)^2 = AB^*}$.

$\Rightarrow$ AB^* and BA^* are projectors if and only if $A^*B = B^*A = I$.

② (b) ANSWER:

$\Rightarrow$ Consider an orthogonal project P, then P follows that:

(i) $P = P^*$

(ii) $P^2 = P$

$\Rightarrow (AB^*)$ must also follow these conditions,

$$\Rightarrow (AB^*) = (AB^*)^*$$

$$\Rightarrow AB^* = BA^*$$

Let SVD of $A = U_1 \Sigma_1 V_1^*$
$B = U_2 \Sigma_2 V_2^*$

$$\Rightarrow U_1 \Sigma_1 V_1^* (U_2 \Sigma_2 V_2^*)^* = U_2 \Sigma_2 V_2^* (U_1 \Sigma_1 V_1^*)^*$$

$$\Rightarrow U_1 \Sigma_1 V_1^* V_2 \Sigma_2^* U_2^* = U_2 \Sigma_2 V_2^* V_1 \Sigma_1^* U_1^*$$

($\because \Sigma$ = diagonal matrix $\Sigma^* = \Sigma$)

$$\Rightarrow U_1 \Sigma_1 V_1^* V_2 \Sigma_2 U_2^* = U_2 \Sigma_2 V_2^* V_1 \Sigma_1 U_1^*$$

($\because \Sigma$ has non negative real entries)

Let V_1 and V_2 be ~~orthogonal~~ parallel subspaces.

Then $V_1^* V_2 = I$ (since V_1, V_2 are orthonormal)

$\Rightarrow V_1, V_2$ differ only in sign, or:

$$\Rightarrow U_1 \Sigma_1 \Sigma_2 U_2^* = U_2 \Sigma_2 \Sigma_1 U_1^*$$

And since Σ_1, Σ_2 are both diagonal matrices then

$\Sigma_1 \Sigma_2 = \Sigma_2 \Sigma_1 = \Sigma$

$\Rightarrow \; U_1 \Sigma U_2^* = U_2 \Sigma U_1^*$

and let $U_1 = U_2$ (V_1 is also parallel to U_2)

Then $U_1 \Sigma U_1^* = U_1 \Sigma U_1^*$ (Thus both become equal)

$\Rightarrow$ But their singular values might be different.

$\rightarrow$ Then the assumptions included here are:

(i) V_1, V_2 are equal or differ in sign and thus are parallel subspaces.

(ii) U_1, U_2 are also equal or differ in sign.

But their singular values might differ and thus the matrices A and B are not necessarily equal.

$\Rightarrow$ Since if $A=B$, then AA^* is always an orthogonal projector.

③ QR Decomposition:

(a) $C \in \mathbb{C}^{p\times m}$, $p<m$

Since, C is full rank

$\Rightarrow$ rank (C) = rank $(CC^*) = p$

Let the full QR decomposition of $C^* = QR$

→ And since C^* has rank (p), then it has p orthogonal vectors that can be used to generate the orthonormal basis.

$\Rightarrow$ Q has dimensions $= m\times m$

where $Q = [\underbrace{Q_1}_{p} \; \underbrace{Q_2}_{(m-p)}] \}m$

and $R = \begin{bmatrix} R \\ 0 \end{bmatrix} \begin{matrix}]\, p\times p \\ ]\, m-p\times p \end{matrix} \Big\} m\times p$ (for full QR decomposition)

$\Rightarrow C^* = QR$

$C^* = [Q_1 \; Q_2] \begin{bmatrix} R \\ 0 \end{bmatrix}$

$\begin{bmatrix} R \\ 0 \end{bmatrix} = \begin{bmatrix} Q_1^* \\ Q_2^* \end{bmatrix} C^*$

$\Rightarrow Q_2^* C^* = 0$

$\boxed{\Rightarrow \; CQ_2^* = 0}$

$\Rightarrow$ null $(C) = Q_2^*$ having dimensions ~~$(m-p)\times$~~ $m\times(m-p)$

$\Rightarrow$ null space of $C = Q_2^*$ where Q_2 is obtained by taking full QR decomposition of C^*.

③

ⓑ null(C) $\cap$ null(D), $C \in \mathbb{C}^{p_1 \times m}$, $C \in \mathbb{C}^{p_2 \times m}$, $p_1 p_2 <$ ~~p_1~~$+p_2 < m$

Let QR decomposition of $C^* = QR_1$

$$C^* = [\underbrace{Q_{11}}_{p_1} \;\; \underbrace{Q_{12}}_{(m-p_1)}] \begin{bmatrix} R_{11} \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} R_{11} \\ 0 \end{bmatrix} = \begin{bmatrix} Q_{11}^* \\ Q_{12}^* \end{bmatrix} C^*$$

$$\Rightarrow Q_{12}^* C^* = 0$$

$$\Rightarrow C Q_{12} = 0$$

$\Rightarrow$ null (C) $= Q_{12}$. , where $Q_{12} \in \mathbb{C}^{\cancel{(m-p_1)\times} m \times (m-p_1)}$

Similarly, null (D) $= Q_{22}$.
where let QR decomposition of $D^* = Q_2 R_2$

$$D^* = [\underset{p_2}{Q_{21}} \;\; \underset{(m-p_2)}{Q_{22}}] \begin{bmatrix} R_{21} \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} Q_{21}^* \\ Q_{22}^* \end{bmatrix} D^* = \begin{bmatrix} R_{21} \\ 0 \end{bmatrix}$$

$$\Rightarrow Q_{22}^* D^* = 0$$

$$\Rightarrow D Q_{22} = 0$$

$\Rightarrow$ null (D) $= Q_{22}$ where $Q_{22} \in \mathbb{C}^{m \times (m-p_2)}$

$\Rightarrow$ null (C) $\cap$ null (D) $= Q_{12} \cap Q_{22}$

where $Q_{12} \in \mathbb{C}^{m \times (m-p_1)}$
$Q_{22} \in \mathbb{C}^{m \times (m-p_2)}$

and Q_{12} is obtained by taking QR decomposition of C^*
Q_{22} is obtained by taking QR decomposition of D^*

3

ⓒ ANSWER:

Given $A \in \mathbb{C}^{m\times n}$, $n<m$

$\Rightarrow$ Let QR decomposition of $A = Q_1 R_1$, $Q \in \mathbb{C}^{m\times m}$, $R \in \mathbb{C}^{m\times n}$

$$A = \begin{bmatrix} \underbrace{Q_{11}}_{n} & \underbrace{Q_{12}}_{m-n} \end{bmatrix} \}m \begin{bmatrix} R_{11} \\ 0 \end{bmatrix} \begin{matrix} n\times n \\ (m-n)\times n \end{matrix}$$

~~A = [~~ $\Rightarrow$ $A [R_{11}^* \ \ 0] = [Q_{11} \ \ Q_{12}]$

$\Rightarrow$ $AR_{11}^* = Q_{11}$

$\Rightarrow \text{range}(A) = Q_{11}$

$\because$ Q_{11} is orthogonal basis.

$C \in \mathbb{C}^{p\times m}$ & $p<m$

$\Rightarrow$ QR decomposition of $C^* = Q_2 R_2$

$\Rightarrow$ $C^* = \begin{bmatrix} \underset{(p)}{Q_{21}} & \underset{(m-p)}{Q_{22}} \end{bmatrix} \begin{bmatrix} R_{21} \\ 0 \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} Q_{21}^* \\ Q_{22}^* \end{bmatrix} C^* = \begin{bmatrix} R_{21} \\ 0 \end{bmatrix}$

$\Rightarrow$ $CQ_{22} = 0$

$\Rightarrow$ Q_{22} = basis of orthonormal basis.

$\Rightarrow$ range(A) $\cap$ null(C)

$= \underline{Q_{11} \cap Q_{22}}$ where $Q_{11} \in \mathbb{C}^{m\times n}$
$Q_{22} \in \mathbb{C}^{m\times (m-p)}$

③

(d) ANSWER:

$A \in \mathbb{C}^{m \times n_1}$, $n_1 < m$

$\Rightarrow$ QR decomposition of $A = Q_1 R_1$

$$A = \begin{bmatrix} Q_{11} & Q_{12} \end{bmatrix} \begin{bmatrix} R_{11} \\ 0 \end{bmatrix}$$

$\Rightarrow$ range $(A) = Q_{11}$ [basis for range(A)]

$\because$ $A \, [R_{11}^{*} \; 0] = [Q_{11} \;\; Q_{12}]$

$AR_{11}^{*} = Q_{11}$ where $R_{11} \in \mathbb{C}^{n \times n}$

and let QR decomposition of $B = Q_2 R_2$

$$B = \begin{bmatrix} Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} R_{21} \\ 0 \end{bmatrix}.$$

$B \, [R_{21}^{*} \; 0] = [Q_{21} \;\; Q_{22}]$

$\Rightarrow$ $BR_{21}^{*} = Q_{21}$

$\Rightarrow$ Basis for range $(B) = Q_{21}$, $Q_{21} \in \mathbb{C}^{m \times n_2}$.

$\Rightarrow$ range$(A) \cap$ range (B)

$= \underline{\underline{Q_{11} \cap Q_{21}}}$. where $Q_{11} \in \mathbb{C}^{m \times n_1}$
$Q_{21} \in \mathbb{C}^{m \times n_2}$.

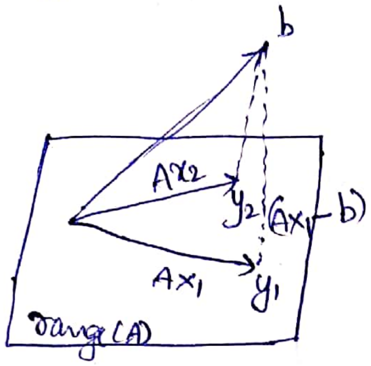
④

(a) ANSWER:

Given weighted inner product $(u,v)_W = u^* W v$

$\|u\|_W = \sqrt{(u,u)_W}$

Let range(A) be shown as:



And the point denoted by $Ax = y$

Then $(y, a)_W = 0$ for all $a \in$ range(A), then it is unique minimizer.

⇒ Let there be two points y_1, y_2 on range(A) such that they both minimize $(Ax_1 - b)$ and $(Ax_2 - b)$

Then for the inner product

→ $(y_1, a)_W = 0$ and $(y_2, a)_W = 0$

⇒ The inner product will be zero, only if we project b orthogon-ally on range of A.

And b will come to a single (image) point y' on range of A after the projection.

⇒ This point y' will give the inner product of all $a \in$ Range(A) as 0.

⇒ $y' = y_1 = y_2$

$\Rightarrow$ Both the points y_1, y_2 are the same.

Thus, there exists a unique minimizer.

\# If, it is a minimizer, then

$(Ax-b, q)_W = 0$

~~(Ax-b)~~, Then $Ax-b$ is orthogonal to A.

$\Rightarrow (A^*, (Ax-b))_W = 0$

$\Rightarrow A^* W (Ax-b) = 0$

$\Rightarrow A^* W A x - A^* W b = 0$

$$\boxed{\Rightarrow A^* W A x = A^* W b}$$

$\Rightarrow$ Also, since W is a positive Hermitial matrix and u, v are orthogonal, then even after multiplying the weights the product will still be 0 since W is symmeteric and has positive real values.

So, we are applying weights symmetrically on both u and v.

Scanned by CamScanner

(4)
(b) ANSWER :

→ Using Modified Gram Scmidt.

we will follow the same procedure but instead of using 2norm and inner product, we will use weighted norm and weighted inner product.

using the same algorithm, we have :

for $j = 1 \dots n$
  $v_j = a_j$
for $i = 1 \dots n$
  $r_{ii} = \|\vec{v}_i\|_W$ → $\|v_i\|_W = \sqrt{v_i^* W v_i}$
  $z_{ii} = v_i / r_{ii}$
  for $j = i+1 \dots n$
    $r_{ij} = (\vec{z}_i, v_j)_W$ → where $(z_i, v_j)_W = z_i^* W v_j$
    $v_j = v_j - r_{ij} z_i$ → subtracting component of z_i from v_j

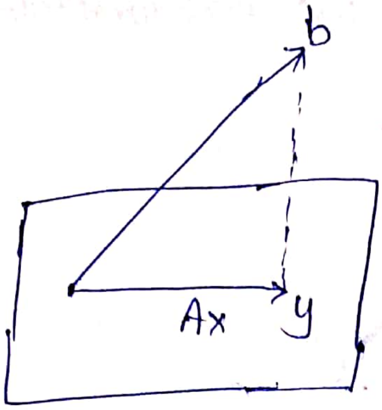
⇒ It will generate, $A = ZR$ such that $(z_i, z_i)_W = 1$ and $(z_i, z_j)_W = 0$ for $i \neq j$.

⇒ When tested on the given example, the algorithm produced $A = ZR$, when $A = I_n$, $W = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}_n$ with errors in $\approx O(10^{-20})$ when tested on large values of n.

④

(c)

Consider the least squares problem:



→ $y =$ orthogonal projector for b

and if $A = ZR$, Z represents the orthogonal basis (weighted)

Then orthogonal projector of $b = ZZ^*$

$\Rightarrow \quad y = ZZ^* b$

and $\quad Ax = y$

$ZRx = y$

$ZRx = ZZ^* b$

$Rx = (Z^* Z) Z^* b$

$Rx = I Z^* b \qquad [\because (z_i, z_i)_w = 1$ otherwise $(z_i, z_j)_w = 0]$

$\boxed{Rx = Z^* b} \Rightarrow \boxed{Rx = (z_i^* b)_w}$

⇒ we can calculate $z^* b$ in $O(m^2)$ as each value calculation takes (m), for m rows, it takes $O(m^2)$.

Then $Rx = z^* b$, to calculate x use backsubstitution, since R is an upper triangular matrix.

which will also take $O(m^2)$.

⇒ Thus, we take an overall $\underline{O(m^2)}$..

(5)

(a) ANSWER :

→ consider $v = \frac{x + sign(x_1)\|x\|_2 e_1}{\|x + sign(x_1)\|x_2\| e_1\|}$, let $x' = sign(x_1)\|x\|_2 e_1$

$$\Rightarrow v = \frac{x + x'}{\|x + x'\|}$$

There is some instability due to floating point operations,

$$\Rightarrow x' = sign(x_1)\|x\|_2 \circledast e_1 = sign(x_1)\|x\|(1+\epsilon_1) \circledast e_1$$

$x' = sign(x_1)\|x\|_2 \circledast e_1 (1+\epsilon_2)$ [Assuming that norms, vector multiplication is backward stable]

$= sign(x_1)\|x\|_2 e_1 (1+\epsilon_2)(1+\epsilon_1)$

$$v = [x \oplus x'] \oslash \|x + x'\|$$

$$= (x+x')(1+\epsilon_3) \oslash (\|x+x'\|)(1+\epsilon_4)$$

$$= \frac{(x+x')(1+\epsilon_3)(1+\epsilon_5)}{\|x+x''\|_2 (1+\epsilon_4)}$$

$$= \frac{(x+x')(1+\epsilon_3)(1+O(\epsilon_m))}{\|x+x'\|_2 (1+O(\epsilon_m))}$$

$$= \frac{[x + x'(1+\epsilon_2)(1+\epsilon_1)](1+\epsilon_3)}{\|x+x'\|_2}$$

$$= \frac{[x + x'(1+\epsilon_1+\epsilon_2+\epsilon_1\epsilon_2)](1+\epsilon)}{\|x+x'\|_2}$$

$$= \frac{[x + x'(1+O(\epsilon))][1+O(\epsilon)]}{\|x+x'\|_2}$$

$$= \frac{x + x' + x'O(\epsilon) + \cancel{x} + x'O(\epsilon) + xO(\epsilon) + x'O(\epsilon^2)}{\|x+x'\|_2}$$

$$= \frac{(x+x') + (x+x')\,O(\epsilon)}{\|x+x'\|_2}$$

$$\hat{v} = \frac{(x+x')(1+O(\epsilon))}{\|x+x'\|_2}$$

$$\Rightarrow \|\hat{v}-v\| = \frac{(x+x')}{\|x+x'\|_2}(1+O(\epsilon)) - \frac{(x+x')}{\|x+x'\|_2}$$

$$\Rightarrow \|\hat{v}-v\| = \underline{\underline{O(\epsilon)}}$$

$$\because \quad \hat{v}-v = \frac{(x+x')\,O(\epsilon)}{\|x+x'\|_2}$$

$$\|\hat{v}-v\|_2 = \left\| \frac{(x+x')O(\epsilon)}{\|x+x'\|_2} \right\|_2$$

$\because$ if we take the norm again, same value will be returned

$$= \frac{\|x+x'\|_2\,O(\epsilon)}{\|x+x'\|_2}$$

$$\boxed{\Rightarrow \|\hat{v}-v\|_2 = O(\epsilon)}$$

Scanned by CamScanner

(5)

(b) ANSWER :

→ From the previous part we know that $\|\check{v} - v\|_2 = O(\epsilon_m)$

→ To compute $b = y - 2(v^* y)v$ we are using $\check{v}$ and then incur floating point error, for some $\|\check{y} - y\|_2 = O(\|y\|_2 \epsilon_m)$

let $\check{v} = v + \delta v$ (some perturbation added to v)

Then

$$\tilde{b} = y \ominus 2(\check{v} \circledast y)\,\check{v} \quad \text{with floating point arithmetic error}$$

$$= y \ominus 2(\check{v}^* y)\check{v}(1+\epsilon_1)$$

$$= [y - 2(\check{v}^* y)\check{v}(1+\epsilon_1)](1+\epsilon_2)$$

$$= y(1+\epsilon_2) - 2(\check{v}^* y)\check{v}(1+\epsilon_1)(1+\epsilon_2)$$

$$\approx y(1+\epsilon_2) - 2(\check{v}^* y)\check{v}(1+\epsilon_3) \quad \text{[ignoring higher order terms in } \epsilon]$$

$$\tilde{b} = [y - 2(\check{v}^* y)\check{v}](1 + O(\epsilon_m)). \text{ — (1)}$$

consider $b' = y - 2(\check{v}^* y)\check{v}$

$$= y - 2((v+\delta v)^* y)(v + \delta v)$$

$$= y - 2(v^* y + \delta v^* y)v - 2(v^* y + \delta v^* y)\delta v$$

$$= y \underbrace{- 2(v^* y + \delta v^* y)\delta v}_{= \delta y} - 2(v^* y + \underbrace{\delta v^* y}_{= v \delta y})v \text{ — (2)}$$

$$\rightarrow \frac{\|2(v^* y + \delta v^* y)\delta v\|}{\|y\|} \approx O(\epsilon_m) \text{ [for it to be equal to } \delta y]$$

$$\Rightarrow \frac{\|2(v^* y)\delta v\|}{\|y\|} + \frac{\|2(\delta v^* y)\delta v\|}{\|y\|}$$

$$\leq \frac{2\,\|v^* y\|\cdot\|\delta v\|}{\|y\|} + \frac{2\,\|\delta v^* y\|\,\|\delta v\|}{\|y\|}$$

$\because$ v is a unit vector $\Rightarrow \|v\|_2 = 1$

$$\leq \frac{2\,\cancel{\|y\|}\,\|\delta v\|}{\cancel{\|y\|}} + \frac{2\,\|\delta v\|\|y\|\,\|\delta v\|}{\|y\|} \quad \text{and} \quad \delta v = O(\epsilon_m)$$

$$\leq 2\epsilon_m + 2\epsilon_m$$

$$\leq O(\epsilon_m)$$

and also

$$\# \frac{\|\delta v^* y\|}{\|v^* \delta y\|} = O(\epsilon_m)$$

$$\leq \frac{\|\delta v\|\,\|y\|}{\|y\|} \quad (\because \text{ v is a unit vector})$$

$$\leq \|\delta v\|$$

$$\leq O(\epsilon_m) \quad \text{since } \delta v = O(\epsilon_m)$$

$$\Rightarrow b' = y - O(\epsilon_m) - 2(v^* y + O(\epsilon_m))\,v$$

$$b' = \hat{y} - 2(v^* \hat{y})v$$

Putting this in eq^n ①

$$\Rightarrow \hat{b} = (\hat{y} - 2(v^* \hat{y})v)(1+\epsilon_m)$$

$$\hat{b} = [\hat{y} - 2(v^* \hat{y})v] + \underbrace{\epsilon_m(\hat{y} - 2(v^* \hat{y})v)}_{}$$

This is also equal to $O(\epsilon_m)$ with respect to $\hat{y}$

$$\because \frac{\|\epsilon_m(\check{y} - 2(v^* \check{y})v)\|}{\|\check{y}\|} \leq \epsilon_m \cdot \left[\frac{\|\check{y}\|}{\|\check{y}\|} - \frac{2\|v\| \, \|\check{y}\| \, \|v\|}{\|\check{y}\|}\right]$$

$$\leq \epsilon_m [1-2]$$

$$\leq O(\epsilon_m) \quad (\because \|v\|_2 = 1 \text{, } v \text{ is a unit vector})$$

$$\boxed{\Rightarrow \check{b} = \check{y} - 2(v^* \check{y})v}$$

$$\because \check{b} = \check{y} - 2(v^* \check{y})v + O(\epsilon_m)$$

$$= \check{y} + O(\epsilon_m) - 2(v^* \check{y})v$$

$$\boxed{\check{b} = \check{y} - 2(v^* \check{y})v}$$

⑥ ANSWER :

ⓐ given ellipse of the form, $f(x,y) = (1+a)x^2 + (1-a)y^2 + 2bxy + cx + dy + e = 0$

$\Rightarrow (1+a)x^2 + (1-a)y^2 + 2bxy + cx + dy + e = 0$

$\Rightarrow x^2 + ax^2 + y^2 - ay^2 + 2bxy + cx + dy + e = 0$

$\Rightarrow (x^2 - y^2)a + 2bxy + cx + dy + e = -(x^2 + y^2)$

and $\theta = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$ having dimensions (5×1)

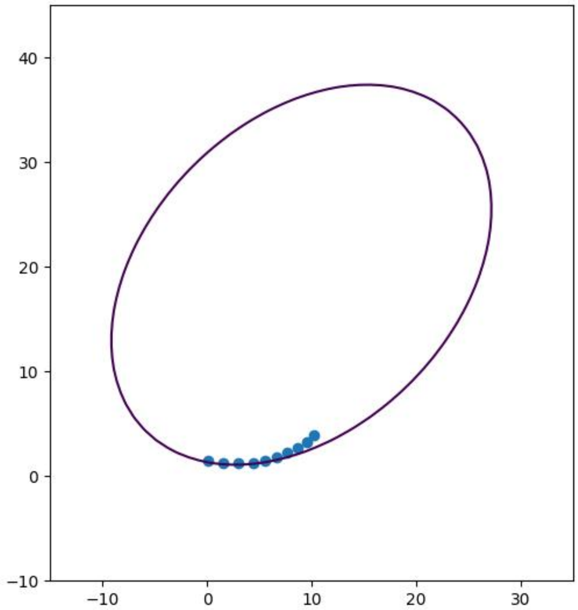
Then we have $A = \begin{bmatrix} x_1^2 - y_1^2 & 2x_1y_1 & x_1 & y_1 & 1 \\ x_2^2 - y_2^2 & 2x_2y_2 & x_2 & y_2 & 1 \\ & \vdots & & & \end{bmatrix}$ (having dimensions $n \times 5$) where n is the number of available samples

and $b = \begin{bmatrix} -(x_1^2 + y_1^2) \\ -(x_2^2 + y_2^2) \\ \vdots \end{bmatrix}$ where b has dimensions $n \times 1$

and we need to minimize $\|A\theta - b\|_2^2$

\# And using the least squares minimization, we get

$\theta = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} -0.57205 \\ -0.14871 \\ -1.99915 \\ -13.78811 \\ 17.58059 \end{bmatrix}$ (using qr decomposition)



40
30
20
10
0
−10
−10
0
10
20
30

⑥

(b)

using the formulas for the above given values: (x_i, y_i)

$\rightarrow K_{b\to\theta} = \dfrac{K(A)}{\eta \cos\theta'}$ [calculating values using numpy] (for the given set of points)

and $K(A) = \|A\|_2 \|A^+\|_2 = 4823.4753$

$\eta = \dfrac{\|A\|_2 \|\theta\|_2}{\|A\theta\|_2} = 22.511325$

and $\cos\theta' = \dfrac{\|y\|}{\|b\|}$ where $y = A\theta$

$= \dfrac{\|A\theta\|_2}{\|b\|_2} = 0.9999$

$\boxed{\Rightarrow K_{b\to\theta} = 214.26977}$ (using the above formula)

$\rightarrow K_{A\to\theta} = K(A) + \dfrac{K(A)^2 \tan\theta}{\eta}$

and $\tan\theta = \dfrac{\|y-b\|_2}{\|y\|_2} = \dfrac{\|A\theta - b\|_2}{\|A\theta\|_2} = 0.0029785$

then $\boxed{K_{A\to\theta} = 7901.8472}$ (using above formula)

$\Rightarrow$ These values are calculated using linalg module in numpy. and since these are very big values then we can say that the problem is ill-conditioned.

6 (c)

→ To find condition number of θ with respect to vector v containing original data.

we have:

$$K = \frac{\|\delta\theta\|/\|\theta\|}{\|\delta v\|/\|v\|} \quad \text{(using } \infty \text{ norm for the calculations)}$$

$$= \frac{\|\delta\theta\|/\|\theta\|}{\|\delta b\|/\|b\|} \cdot \frac{\|\delta b\|/\|b\|}{\|\delta v\|/\|v\|}$$

$$K = K_{b\to\theta} \cdot \frac{\|\delta b\|/\|b\|}{\|\delta v\|/\|v\|} \quad \text{— (1)}$$

where v is the vector containing $[x_i, y_i]^T$

and b is a vectors that contains $[-(x_i^2+y_i^2)]^T$

⇒ Finding $\frac{\|\delta b\|/\|b\|}{\|\delta v\|/\|v\|}$ using ∞ norm

consider we changed the values x, y by δ

$$\Rightarrow \frac{\| -[(x+\delta)^2 + (y+\delta)^2] + (x^2+y^2) \|}{\|\delta v\|} \cdot \frac{\|v\|}{\|b\|}$$

$$\Rightarrow \frac{\| -(\cancel{x^2} + \delta^2 + 2x\delta + \cancel{y^2} + \delta^2 + 2y\delta - \cancel{x^2} - \cancel{y^2}) \|}{\delta} \cdot \frac{\|v\|}{\|b\|}$$ (since $\delta v = \delta$ max change in any value $= \delta$)

$$= \frac{|-2\delta(x+y)|}{\delta} \cdot \frac{\|v\|}{\|b\|}$$

(ignoring higher order δ)

$$\Rightarrow \frac{2\cancel{8}\,|x+y|^{\cancel{2}}}{\cancel{8}} \cdot \frac{\|v\|}{\|b\|}$$

$$\Rightarrow 2|x+y| \cdot \frac{\max(x_i, y_i)}{|x^2+y^2|} \quad \forall x_i, y_i \in V$$

$$= 2\frac{|x+y|}{|x^2+y^2|}\max(x_i, y_i) = \frac{2\max(|x_i+y_i|)}{\max(|x_i^2+y_i^2|)} \cdot \max(x_i, y_i)$$

$$\Rightarrow \boxed{K = K_{b\to\theta}\frac{2\max(|x_i+y_i|)}{\max(|x_i^2+y_i^2|)}\max(x_i, y_i)} \quad \forall x_i, y_i \in V$$

$$\Rightarrow \boxed{K_{v\to\theta} = K_{b\to\theta} \cdot 2\frac{\max(|x_i+y_i|)}{\max(|x_i^2+y_i^2|)}\max(x_i, y_i)} \quad \forall x_i, y_i \in V,$$

and the value of

$\boxed{K_{v\to\theta} = 516.8348}$ for the set of data points given.