COL 726 Homework 2

27 January–10 February, 2020

1. (a) Find the singular value decomposition of the 2×2 matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

   (b) Given a matrix $\mathbf{A} \in \mathbb{C}^{m\times m}$ with singular value decomposition $\mathbf{A} = \mathbf{U\Sigma V}^*$, find a singular value decomposition of $\begin{bmatrix} \mathbf{A} & \mathbf{A} \\ \mathbf{A} & \mathbf{A} \end{bmatrix} \in \mathbb{C}^{2m\times 2m}$. Does your answer agree with with that of part (a) when applied to $\mathbf{A} = [1]$?

2. (a) Consider two full-rank matrices $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{m\times n}$. Prove that $\mathbf{AB}^*$ and $\mathbf{BA}^*$ are projectors if and only if $\mathbf{A}^*\mathbf{B} = \mathbf{B}^*\mathbf{A} = \mathbf{I}$. You may use the fact that $\mathbf{A}^*\mathbf{A}$ and $\mathbf{B}^*\mathbf{B}$ are invertible.

   (b) Under what additional conditions is $\mathbf{AB}^*$ an orthogonal projector, other than the trivial case $\mathbf{A} = \mathbf{B}$? Give an answer in terms of the singular value decompositions of $\mathbf{A}$ and $\mathbf{B}$.

3. Show how to use the QR decomposition to find bases for the following subspaces of $\mathbb{C}^m$. You may assume all matrices are full rank.

   (a) An orthonormal basis for null($\mathbf{C}$), where $\mathbf{C} \in \mathbb{C}^{p\times m}$ with $p < m$.
   Hint: Consider the full QR decomposition of $\mathbf{C}^*$.

   (b) An orthonormal basis for null($\mathbf{C}$) $\cap$ null($\mathbf{D}$), where $\mathbf{C} \in \mathbb{C}^{p_1\times m}, \mathbf{D} \in \mathbb{C}^{p_2\times m}$ with $p_1, p_2 < p_1 + p_2 < m$.

   (c) A basis for range($\mathbf{A}$) $\cap$ null($\mathbf{C}$), where $\mathbf{A} \in \mathbb{C}^{m\times n}, \mathbf{C} \in \mathbb{C}^{p\times m}$ with $p < n < m$.

   (d) A basis for range($\mathbf{A}$) $\cap$ range($\mathbf{B}$), where $\mathbf{A} \in \mathbb{C}^{m\times n_1}, \mathbf{B} \in \mathbb{C}^{m\times n_2}$ with $n_1, n_2 < m < n_1 + n_2$.

4. Just as we have many different norms on $\mathbb{C}^m$, we can have different inner products other than the standard inner product $\mathbf{u}^*\mathbf{v}$. In particular, for any Hermitian positive definite matrix $\mathbf{W} \in \mathbb{C}^{m\times m}$, we can define a weighted inner product $(\mathbf{u}, \mathbf{v})_\mathbf{W} := \mathbf{u}^*\mathbf{W}\mathbf{v}$ and weighted norm $\|\mathbf{u}\|_\mathbf{W} = \sqrt{(\mathbf{u}, \mathbf{u})_\mathbf{W}}$.

   (a) A weighted least-squares problem is the problem of minimizing $\|\mathbf{Ax} - \mathbf{b}\|_\mathbf{W}$ for given $\mathbf{A} \in \mathbb{C}^{m\times n}$ and $\mathbf{b} \in \mathbb{C}^m$. Show that if $\mathbf{x}$ satisfies $(\mathbf{Ax} - \mathbf{b}, \mathbf{a})_\mathbf{W} = 0$ for all $\mathbf{a} \in$ range($\mathbf{A}$), then it is the unique minimizer. Conclude that $\mathbf{A}^*\mathbf{WAx} = \mathbf{A}^*\mathbf{Wb}$.

   (b) The analogue of the QR decomposition is a factorization $\mathbf{A} = \mathbf{ZR}$ (or $\hat{\mathbf{Z}}\hat{\mathbf{R}}$) where the columns of $\mathbf{Z}$ satisfy $(\mathbf{z}_i, \mathbf{z}_i)_\mathbf{W} = 1$, $(\mathbf{z}_i, \mathbf{z}_j)_\mathbf{W} = 0$ for $i \neq j$. Design and implement either a modified Gram-Schmidt or Householder algorithm for computing such a factorization.

Test your algorithm on $\mathbf{A} = \mathbf{I}$ and $\mathbf{W} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$.

(c) How would you use such a decomposition to solve the weighted least-squares problem in part (a) in $O(m^2)$ time?

5. In this exercise, you will prove the backward stability of one step in the Householder algorithm. You may assume that vector addition, scalar multiplication, norms, and inner products have been shown to be backward stable. Let ϵ_{m} denote the machine precision.

   (a) Let $\mathbf{x} = [x_1, \ldots, x_m]^T$ be a vector with $x_1 \neq 0$, and consider the corresponding Householder reflection, i.e. let $\mathbf{v}$ be the unit vector along $\mathbf{x} + \mathrm{sign}(x_1)\|\mathbf{x}\|_2 \mathbf{e}_1$. Suppose the vector computed with floating-point arithmetic is $\tilde{\mathbf{v}}$ instead. Show that $\|\tilde{\mathbf{v}} - \mathbf{v}\|_2 = O(\epsilon_{\mathrm{m}})$.

   (b) Consider a vector $\mathbf{y} \in \mathbb{C}^m$. We would like to compute $\mathbf{b} = \mathbf{y} - 2(\mathbf{v}^*\mathbf{y})\mathbf{y}$, but we use $\tilde{\mathbf{v}}$ instead of $\mathbf{v}$, then incur further floating-point error, and end up with a vector $\tilde{\mathbf{b}}$ instead. Prove that $\tilde{\mathbf{b}} = \tilde{\mathbf{y}} - 2(\mathbf{v}^*\tilde{\mathbf{y}})\tilde{\mathbf{y}}$ for some $\tilde{\mathbf{y}}$ such that $\|\tilde{\mathbf{y}} - \mathbf{y}\|_2 = O(\|\mathbf{y}\|_2 \epsilon_{\mathrm{m}})$.

6. TBA