HOMEWORK-2

NUMERICAL ALGORITHMS

① ANSWER:

(a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

We want, $A = U \Sigma V^*$

$\Rightarrow A^*A = V\Sigma^*\Sigma V^*$

$A^*A = V\Sigma' V^*$ $\quad (\Sigma' = \Sigma^*\Sigma)$

$\Rightarrow$ This is same as the eigen decomposition,

$\rightarrow$ Then, finding eigen values of A^*A:

$(A^*A - \lambda I) = 0$

$\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$

$\begin{bmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} = 0$

$(2-\lambda)^2 - 4 = 0 \Rightarrow (2-\lambda) = \pm 2$

$\Rightarrow \lambda = 0, 4$

$\boxed{\Rightarrow \sigma = 0, 2} \Rightarrow \hat{\Sigma} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

$\rightarrow$ Eigen vectors of A^*A:

(i) $(A^*A - 4I)x = 0$

$\Rightarrow \begin{bmatrix} 2-4 & 2 \\ 2 & 2-4 \end{bmatrix} x = 0$

$\Rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} x_1 = 0 \Rightarrow \boxed{x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$\Rightarrow \left.\begin{array}{l} -2x_{11} + 2x_{12} = 0 \\ +2x_{11} + (-2)x_{12} = 0 \end{array}\right\}$ Solving these gives $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(ii) $(A^*A - 0I)x_2 = 0$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} x_2 = 0$$

$$\Rightarrow \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \left(\begin{array}{l} \text{Solving } 2x_{21} + 2x_{22} = 0 \\ \text{and } 2x_{21} + 2x_{22} = 0 \end{array} \right)$$

Let $x_{21} = 1$, $\Rightarrow x_{22} = -1$.

$$\Rightarrow \quad V = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$\because$ V needs to be orthonormal, $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

and $\quad A = U'\Sigma' V^{*'}$

$$\Rightarrow \quad A'V'\Sigma^{-1} = U\Sigma$$

$$\Rightarrow \quad U\Sigma' = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cancel{\begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix}}$$

$$\cancel{\Rightarrow \quad U'\Sigma' = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} \neq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$U' \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \quad \text{——— (1)}$$

Then we can write U as

$$U' = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\because \quad U\Sigma' = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$$

and colums of U are still orthonormal

$$\Rightarrow \quad A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \quad \left(\because \text{Taking } \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2} \text{ inside } \Sigma' \right).$$

$$\boxed{\Rightarrow \quad A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}} \quad \text{where } U = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$V^* = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

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(b) **ANSWER :** Given $A = U\Sigma V^*$

$$\begin{bmatrix} A & A \\ A & A \end{bmatrix} = \begin{bmatrix} U\Sigma V^* & U\Sigma V^* \\ U\Sigma V^* & U\Sigma V^* \end{bmatrix}$$

$$= \begin{bmatrix} U\Sigma V^* & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & U\Sigma V^* \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ U\Sigma V^* & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & U\Sigma V^* \end{bmatrix}$$

$$= \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & \Sigma \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & V^* \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \Sigma & 0 \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \Sigma \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & V^* \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix}}_{\text{This doesn't affect the result of multiplication}} + \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} 0 & \Sigma \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix}$$

$$+ \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \Sigma & 0 \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix} + \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \Sigma \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix}$$

$$= \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}\left\{ \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \Sigma \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \Sigma & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \Sigma \end{bmatrix} \right\}\begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix}$$

$$= \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} \Sigma & \Sigma \\ \Sigma & \Sigma \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix} \quad \text{——— (1)}$$

$\Rightarrow$ we can write $\begin{bmatrix} \Sigma & \Sigma \\ \Sigma & \Sigma \end{bmatrix}$ as $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} \Sigma & \Sigma \\ \Sigma & \Sigma \end{bmatrix}$$

Putting in eqn (1)

$$\Rightarrow \begin{bmatrix} A & A \\ A & A \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix}$$

$$\begin{bmatrix} A & A \\ A & A \end{bmatrix} = \begin{bmatrix} U & U \\ U & -U \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V^* & V^* \\ V^* & -V^* \end{bmatrix} \quad \text{——— (2)}$$

$\Rightarrow SVD\left(\begin{bmatrix} A & A \\ A & A \end{bmatrix}\right) \Rightarrow U'\Sigma' V'^{*}$ (Given $A = U\Sigma V^{*}$)

where $U' = \begin{bmatrix} U & U \\ U & -U \end{bmatrix}$ $V^{*} = \begin{bmatrix} V^{*} & V^{*} \\ V^{*} & -V^{*} \end{bmatrix}$

$\Sigma' = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$ $V = \begin{bmatrix} V & V \\ V & -V \end{bmatrix}$

$\Rightarrow$ In part (a):

consider $[A] = [1]$

$SVD(A) = U\Sigma V^{*}$

$\Rightarrow U = [1]$ $\Sigma = [1]$ $V^{*} = [1]$

Then $SVD\left(\begin{bmatrix} A & A \\ A & A \end{bmatrix}\right) = \begin{bmatrix} U & U \\ U & -U \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V^{*} & V^{*} \\ V^{*} & -V^{*} \end{bmatrix}$

$SVD\left(\begin{bmatrix} A & A \\ A & A \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

which is the same result as when

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

in part (a).

$\Rightarrow$ It agrees with the result in part (a).

②

(a) : ANSWER:

→ For (AB^*) to be a ~~proper~~ projector :

$$(AB^*)^2 = AB^*$$

$\Rightarrow (AB^*)(AB^*)$

$\Rightarrow A(B^*A)B^* = AB^*$

$\Rightarrow AB^*$ only if $\underline{\underline{B^*A = I}}$.

→ for (BA^*) to be a projector

$$(BA^*)^2 = BA^*$$

$\Rightarrow (BA^*)(BA^*)$

$= BA^*BA^* = \underline{\underline{BA^*}}$ only if $\underline{\underline{A^*B = I}}$

~~$\Rightarrow BA$~~

Now, consider the converse :

→ if $A^*B = I$

Then $A^*B = I$

$BA^*B = B \qquad (I \cdot B = B)$

$BA^*BA^* = BA^*$

$\boxed{\Rightarrow (BA^*)^2 = BA^*} \quad \cdot [P^2 = P]$

$\Rightarrow BA^*$ is a projector

→ if $B^*A = I$

Then $B^*A = I$

$AB^*A = A$

$AB^*AB^* = AB^*$

$\boxed{(AB^*)^2 = AB^*}$

$\Rightarrow AB^*$ is a projector $\underline{(AB^*)^2 = AB^*}$.

$\Rightarrow AB^*$ and BA^* are projectors if and only if $A^*B = B^*A = I$.

(2)

(b) ANSWER:

$\Rightarrow$ Consider an orthogonal project P, then P follows that:

(i) $P = P^*$

(ii) $P^2 = P$

$\Rightarrow (AB^*)$ must also follow these conditions,

$$\Rightarrow (AB^*) = (AB^*)^*$$

$$\Rightarrow AB^* = BA^*$$

Let SVD of $A = U_1 \Sigma_1 V_1^*$

$B = U_2 \Sigma_2 V_2^*$

$$\Rightarrow U_1 \Sigma_1 V_1^* (U_2 \Sigma_2 V_2^*)^* = U_2 \Sigma_2 V_2^* (U_1 \Sigma_1 V_1^*)^*$$

$$\Rightarrow U_1 \Sigma_1 V_1^* V_2 \Sigma_2^* U_2^* = U_2 \Sigma_2 V_2^* V_1 \Sigma_1^* U_1^*$$

($\because \Sigma$ = diagonal matrix $\Sigma^* = \Sigma$)

$$\Rightarrow U_1 \Sigma_1 V_1^* V_2 \Sigma_2 U_2^* = U_2 \Sigma_2 V_2^* \cancel{V_1} \Sigma_1 U_1^*$$

($\because \Sigma$ has non negative real entries)

Let V_1 and V_2 be ~~orthogonal~~ parallel subspaces.

Then $V_1^* V_2 = I$ (Since V_1, V_2 are orthonormal)

$\Rightarrow V_1, V_2$ differ only in sign. or:

$$\Rightarrow U_1 \Sigma_1 \Sigma_2 U_2^* = U_2 \Sigma_2 \Sigma_1 U_1^*$$

And since Σ_1, Σ_2 are both diagonal matrices then

$\Sigma_1 \Sigma_2 = \Sigma_2 \Sigma_1 = \Sigma$

$\Rightarrow U_1 \Sigma V_2^* = U_2 \Sigma U_1^*$

and let $U_1 = U_2$ (U_1 is also parallel to U_2)

Then $U_1 \Sigma U_1^* = U_1 \Sigma U_1^*$ (Thus both become equal)

$\Rightarrow$ But their singular values might be different.

$\rightarrow$ Then the assumptions included here are:

(i) V_1, V_2 are equal or differ in sign and thus are parallel subspaces.

(ii) U_1, U_2 are also equal or differ in sign.

But their singular values might differ and thus the matrices A and B are not necessarily equal.

$\Rightarrow$ Since if $A = B$, then AA^* is always an orthogonal projector.

③ QR Decomposition:

(a) $C \in \mathbb{C}^{p\times m}$, $p<m$

Since, C is full rank

$\Rightarrow$ rank(C) = rank$(CC^*) = p$

Let the Full QR decomposition of $C^* = QR$

→ And since C^* has rank (p), then it has p orthogonal vectors that can be used to generate the orthonormal basis.

$\Rightarrow$ Q has dimensions $= m\times m$

where $Q = \Big[\underbrace{Q_1}_{p}\ \ \underbrace{Q_2}_{(m-p)}\Big]\Big\}m$

and $R = \begin{bmatrix} R \\ 0 \end{bmatrix} \begin{matrix}]\,p\times p \\ ]\,m-p\times p \end{matrix} \Big\} m\times p$ (for full QR decomposition)

$\Rightarrow C^* = QR$

$C^* = [Q_1\ \ Q_2]\begin{bmatrix} R \\ 0 \end{bmatrix}$

$\begin{bmatrix} R \\ 0 \end{bmatrix} = \begin{bmatrix} Q_1^* \\ Q_2^* \end{bmatrix} C^*$

$\Rightarrow Q_2^* C^* = 0$

$\boxed{\Rightarrow\ CQ_2 = 0}$

$\Rightarrow$ null$(C) = Q_2$ having dimensions ~~$(m-p)\times$~~ $m\times(m-p)$

$\Rightarrow$ null space of $\underline{C = Q_2}$. where Q_2 is obtained by taking full QR decomposition of C^*.

③

ⓑ null(C) ∩ null(D) , $C \in \mathbb{C}^{p_1 \times m}$, $D \in \mathbb{C}^{p_2 \times m}$

$p_1, p_2 < p_1 + p_2 < m$.

Given $C \in \mathbb{C}^{p_1 \times m}$, $D \in \mathbb{C}^{p_2 \times m}$

$\Rightarrow C^* \in \mathbb{C}^{m \times p_1}$, $D^* \in \mathbb{C}^{m \times p_2}$

→ Let's consider the matrix, X obtained by augmenting matrices C^* and D^*

$$X = m\begin{array}{c} \begin{array}{cc} p_1 & p_2 \end{array} \\ \left[\begin{array}{c:c} C^* & D^* \end{array} \right] \\ \underleftrightarrow{\quad p_1+p_2 \quad} \end{array} ,$$

Now, let the full QR decomposition of X be given by:

$$X = \left[\begin{array}{c} \cancel{Q_1} \\ \cancel{Q_2} \end{array}\right] [Q_1 . Q_2] \begin{bmatrix} R \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} Q_1^* \\ Q_2^* \end{bmatrix} X = \begin{bmatrix} R \\ 0 \end{bmatrix}$$

$$\Rightarrow Q_2^* X = 0$$

$$\Rightarrow Q_2^* \left[\begin{array}{c:c} C^* & D^* \end{array} \right] = 0$$

$$\Rightarrow \begin{bmatrix} C \\ D \end{bmatrix} Q_2 = 0 \quad \text{(Taking transpose of the above equation)}$$

$$\Rightarrow \begin{bmatrix} CQ_2 \\ DQ_2 \end{bmatrix} = 0$$

$$\Rightarrow CQ_2 = 0 \quad \text{and} \quad DQ_2 = 0$$

$\Rightarrow Q_2 \in$ Null (C) and $Q_2 \in$ Null (D)

$$\boxed{\Rightarrow Q_2 \in \text{Null}(C) \cap \text{Null}(D)} , \quad Q_2 \in \mathbb{C}^{m \times m-(p_1+p_2)}$$

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③

(C) Basis for $\text{range}(A) \cap \text{null}(C)$, where $A \in \mathbb{C}^{m\times n}$
$C \in \mathbb{C}^{p\times m}$

Let the QR decomposition of $A = [\underbrace{Q_1}_{m\times n} \; \underbrace{Q_2}_{m\times(m-n)}] \begin{bmatrix} R \\ 0 \end{bmatrix}$ (Full QR decomposition)

Here, $Q_1 = \text{range}(A)$
$Q_2 = \text{null}(A)$

→ Let's consider a subspace A' orthogonal to A, such that

$Q_1^* = \text{null}(A')$ and $Q_2 = \text{range}(A')$

$\Rightarrow A' = [\underbrace{Q_2'}_{m\times(m-n)} \; \underbrace{Q_1'}_{m\times n}] \begin{bmatrix} R' \\ 0 \end{bmatrix}$ for some R'

Now consider the matrix $X = [A' \,\vdots\, C^*]$

and QR decomposition of $X = [Q_3 \; Q_4] \begin{bmatrix} R'' \\ 0 \end{bmatrix}$

$$X = [Q_3 \; Q_4] \begin{bmatrix} R'' \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} Q_3^* \\ Q_4^* \end{bmatrix} X = \begin{bmatrix} R'' \\ 0 \end{bmatrix}$$

$$\Rightarrow Q_4^* X = 0$$

$$\Rightarrow X^* Q_4 = 0$$

$$= \begin{bmatrix} (A')^* \\ C \end{bmatrix} Q_4 = 0$$

$\Rightarrow Q_4 = \text{null}(C)$ and $Q_4 = \text{null}(A')$ [From previous part]

$\Rightarrow Q_4 \in \text{null}(C) \cap \text{null}(A')$

$$\boxed{\Rightarrow Q_4 \in \text{null}(C) \cap \text{range}(A)}$$

③

ⓓ range (A) $\cap$ range(B), $A \in \mathbb{C}^{m\times n_1}$ and $B \in \mathbb{C}^{m \times n_2}$

with $n_1, n_2 < m < n_1+n_2$

Let the QR decomposition of A be $[Q_1\ Q_2]\begin{bmatrix} R_1 \\ 0 \end{bmatrix}$

and B be $[Q_3\ Q_4]\begin{bmatrix} R_2 \\ 0 \end{bmatrix}$

$\Rightarrow$ range(A) $= Q_1$ and range(B) $= Q_3$

null(A) $= Q_2$ null(B) $= Q_4$

lets assume subspaces A' and B' orthogonal to A and B

such that;

null(A') $= Q_1$ null(B') $= Q_3$

range(A') $= Q_2$ range(B') $= Q_4$

$$\Rightarrow A' = \begin{array}{c} \\ m \end{array}\begin{bmatrix} \overset{n_1}{Q_1} & Q_2 \end{bmatrix}\begin{bmatrix} 0 \\ R_1' \end{bmatrix} \text{ for some } R_1'$$

$$B' = m\begin{bmatrix} \overset{n_2}{Q_3} & Q_4 \end{bmatrix}\begin{bmatrix} 0 \\ R_2' \end{bmatrix} \text{ for some } R_2'$$

$\Rightarrow$ let $X = [A' \mid B']$ be the augmented matrix and the QR decomposition of $X = [Q_5\ Q_6]\begin{bmatrix} R' \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} Q_5^* \\ Q_6^* \end{bmatrix} X = \begin{bmatrix} R' \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} Q_5^* \\ Q_6^* \end{bmatrix} X = \begin{bmatrix} R' \\ 0 \end{bmatrix} \Rightarrow Q_6^* X = 0$$

$$\Rightarrow X^* Q_6 = 0 \Rightarrow \begin{bmatrix} A'^* \\ B'^* \end{bmatrix} Q_6 = 0$$ (from previous parts)

$$\Rightarrow (A' Q_6) = 0 \quad \text{and} \quad (B' Q_6) = 0$$

$\Rightarrow Q_6 \in$ null(A') and null(~~Q_6~~ B')

$$\boxed{\Rightarrow Q_6 \in \text{Range}(A) \cap \text{range}(B)}$$

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④ ANSWER :

ⓐ Given if x satisfies $(Ax-b,a)_W=0$, $\forall a \in range(A)$

$$\Rightarrow (Ax-b)^* W a = 0$$

$$\Rightarrow x^* A^* W a - b^* W a = 0$$

$$\Rightarrow x^* A^* W a = b^* W a$$

$$\Rightarrow a^* W^* A x = a^* W^* b$$

$$\Rightarrow a^* W A x = a^* W b \quad [\because W^* = W \text{ W is hermitian}]$$

$$\Rightarrow x = (a^* W A)^{-1} a^* W b \quad , \forall a \in range(A)$$

$$\boxed{\Rightarrow x = (A^* W A)^{-1} A^* W b}$$

which gives a unique value for x.

$\Rightarrow$ Also, since $(A^* W A)$ is invertible (because A is full rank)

$\Rightarrow$ Null $(A^* W A) = \{0\} =$ null(A)

$\Rightarrow$ This implies that x can only be unique since there is no $z \in$ Null(A) except $\{0\}$ otherwise we could have added z to x and $(x+z)$ would have been a solution here.

but since $z = \{0\}$

$\Rightarrow x + z = \hat{x}$

$\Rightarrow x$ is a unique minimizer.

Also, since it is a minimizer, then

$(Ax-b, a)_W = 0$

$\Rightarrow$ Since $a \in Range(A)$, it ~~span~~ can be written as a combination of columns of A.

$\Rightarrow (Ax-b, a)_w = 0$

$\Rightarrow (Ax-b, A)_w = 0$

$\Rightarrow (Ax-b)^*_w A = 0$

$\Rightarrow x^*A^*WA - b^*WA = 0$

$\Rightarrow x^*A^*WA = b^*WA$

$$\boxed{\Rightarrow A^*WAx = A^*Wb}$$

(4)
(b) ANSWER :

→ Using Modified Gram Scmidt.
we will follow the same procedure but instead of using 2norm and inner product, we will use weighted norm and weighted inner product.

using the same algorithm, we have :

for $j = 1 \ldots n$
  $v_j = a_j$
for $i = 1 \ldots n$
  $r_{ii} = \|\vec{v}_i\|_W$ ⟶ $\|v_i\|_W = \sqrt{v_i^* W v_i}$
  $z_{ii} = v_i / r_{ii}$
  for $j = i+1 \ldots n$
    $r_{ij} = (\vec{z}_i, v_j)_W$ ⟶ where $(z_i, v_j)_W = z_i^* W v_j$
    $v_j = v_j - r_{ij} z_i$ ⟶ subtracting component of z_i from v_j

⇒ It will generate, $A = ZR$ such that $(z_i, z_i)_W = 1$ and $(z_i, z_j)_W = 0$ for $i \neq j$.

⇒ When tested on the given example, the algorithm produced $A = ZR$, when $A = I_n$, $W = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}_n$ with errors in $\approx O(10^{-20})$ when tested on large values of n.

(4)

(c) ANSWER :

⇒ To solve this problem, we first project b along the weighted space using the projector of the weighted space. g which is ZWZ^*.

Thus $y = ZWZ^* b$

$$AX = ZWZ^* b$$

$$\Rightarrow ZRx = ZWZ^* b$$

$$\Rightarrow Rx = Z^* ZWZ^* b$$

$$Rx = Z^* (W^* Z)^* Z^* b$$

$$Rx = (Z^* WZ)^* Z^* b$$

$$Rx = I . Z^* b$$

$$\Rightarrow \boxed{Rx = Z^* b}$$

[∵ $(Z_i, Z_i)_W = 1$ and $(Z_i, Z_j)_W = 0$]

[∵ $(Z^* WZ)^*$
$= Z^* (WZ^*)^*$
$= Z^* (W^* Z^*)^*$
∵ W is hermitian]

⇒ Then z^*b can be calculated in $O(m^2)$ time since b is a vector and z is a matrix.

⇒ Also, $Rx = t$ can be solved in $O(m^2)$ time using back substitution.

⇒ Total time taken = $O(m^2)$.

⑤

(a) ANSWER :

$\rightarrow$ consider $v = \dfrac{x + \text{sign}(x_1)\|x\|_2 e_1}{\|x + \text{sign}(x_1)\|x_2\| e_1\|}$, let $x' = \text{sign}(x_1)\|x\|_2 e_1$

$$\Rightarrow v = \frac{x + x'}{\|x + x'\|}$$

There is some instability due to floating point operations,

$\Rightarrow x' = \text{sign}(x_1)\|x\|_2 \circledast e_1 = \text{sign}(x_1)\|x\|(1+\epsilon_1) \circledast e_1$

$x' = \text{sign}(x_1)\|x\|_2^{\circledast} e_1 (1+\epsilon_2)$ [Assuming that norms, vector multiplication is backward stable]

$= \text{sign}(x_1)\|x\|_2 e_1 (1+\epsilon_2)(1+\epsilon_1)$

$v = [x \oplus x'] \oslash \|x + x'\|$

$= (x + x')(1+\epsilon_3) \oslash (\|x + x'\|)(1+\epsilon_4)$

$$= \frac{(x + x')(1+\epsilon_3)(1+\epsilon_5)}{\|x + x''\|_2 (1+\epsilon_4)}$$

$$= \frac{(x + x')(1+\epsilon_3)(1+O(\epsilon_m))}{\|x + x'\|_2 (1+O(\epsilon_m))}$$

$$= \frac{[x + x'(1+\epsilon_2)(1+\epsilon_1)](1+\epsilon_3)}{\|x + x'\|_2}$$

$$= \frac{[x + x'(1+\epsilon_1+\epsilon_2+\epsilon_1\epsilon_2)](1+\epsilon)}{\|x + x'\|_2}$$

$$= \frac{[x + x'(1+O(\epsilon))][1+O(\epsilon)]}{\|x + x'\|_2}$$

$$= \frac{x + x' + x' O(\epsilon) + \cancel{x} + x' O(\epsilon) + x O(\epsilon) + x' O(\epsilon^2)}{\|x + x'\|_2}$$

$$= \frac{(x + x') + (x+x')\, O(\epsilon)}{\|x + x'\|_2}$$

$$\hat{v} = \frac{(x+x')(1+O(\epsilon))}{\|x+ x'\|_2}$$

$$\Rightarrow \|\hat{v} - v\| = \frac{(x + x')}{\|x + x'\|_2}(1+O(\epsilon)) - \frac{(x+x')}{\|x+x'\|_2}$$

$$\Rightarrow \|\hat{v} - v\| = \underline{\underline{O(\epsilon)}}$$

$$\because \quad \hat{v} - v = \frac{(x+x')}{\|x+x'\|_2} O(\epsilon)$$

$$\|\hat{v} - v\|_2 = \left\|\frac{(x+x')O(\epsilon)}{\|x+x'\|_2}\right\|_2$$

$$= \frac{\|x+x'\|_2\, O(\epsilon)}{\|x+x'\|_2}$$

$\because$ if we take the norm again, same value will be returned

$$\boxed{\Rightarrow \|\hat{v} - v\|_2 = O(\epsilon)}$$

⑤

ⓑ ANSWER :

→ From the previous part we know that $\|\check{v} - v\|_2 = O(\epsilon_m)$

→ To compute $b = y - 2(v^* y)v$ we are using $\check{v}$ and then incur floating point error, for some $\|\check{y} - y\|_2 = O(\|y\|_2\, \epsilon_m)$

let $\check{v} = v + \delta v$ (some perturbation added to v)

Then

$$\check{b} = y \ominus 2(\check{v} \circledast y)\check{v} \quad \text{with floating point arithmetic error}$$

$$= y \ominus 2(\check{v}^* y)\check{v}(1+\epsilon_1)$$

$$= [y - 2(\check{v}^* y)\check{v}(1+\epsilon_1)](1+\epsilon_2)$$

$$= y(1+\epsilon_2) - 2(\check{v}^* y)\check{v}(1+\epsilon_1)(1+\epsilon_2)$$

$$\approx y(1+\epsilon_2) - 2(\check{v}^* y)\check{v}(1+\epsilon_3) \quad \text{[ignoring higher order terms in } \epsilon\text{]}$$

$$\check{b} = [y - 2(\check{v}^* y)\check{v}](1 + O(\epsilon_m)). \quad — ①$$

consider $b' = y - 2(\check{v}^* y)\check{v}$

$$= y - 2((v+\delta v)^* y)(v + \delta v)$$

$$= y - 2(v^* y + \delta v^* y)v - 2(v^* y + \delta v^* y)\delta v$$

$$= y \underbrace{- 2(v^* y + \delta v^* y)\delta v}_{= \delta y} - 2(v^* y + \underbrace{\delta v^* y}_{= v\delta y})v \quad — ②$$

→ $\dfrac{\|2(v^* y + \delta v^* y)\delta v\|}{\|y\|} \approx O(\epsilon_m)$ [for it to be equal to δy]

$$\Rightarrow \frac{\|2(v^*y)\delta v\|}{\|y\|} + \frac{\|2(\delta v^* y)\delta v\|}{\|y\|}$$

$$\leq \frac{2\|v^*y\| \cdot \|\delta v\|}{\|y\|} + \frac{2\|\delta v^* y\| \|\delta v\|}{\|y\|}$$

$\because$ v is a unit vector $\Rightarrow \|v\|_2 = 1$

$$\leq \frac{2\cancel{\|y\|}\|\delta v\|}{\cancel{\|y\|}} + \frac{2\|\delta v\|\|y\|\|\delta v\|}{\|y\|} \quad \text{and} \quad \delta v = O(\epsilon_m)$$

$$\leq 2\epsilon_m + 2\epsilon_m$$

$$\leq O(\epsilon_m)$$

and also

$$\# \frac{\|\delta v^* y\|}{\|v^* \delta y\|} = O(\epsilon_m)$$

$$\leq \frac{\|\delta v\| \|y\|}{\|y\|} \quad (\because \text{ v is a unit vector})$$

$$\leq \|\delta v\|$$

$$\leq O(\epsilon_m) \quad \text{since } \delta v = O(\epsilon_m)$$

$$\Rightarrow b' = y - O(\epsilon_m) - 2(v^* y + O(\epsilon_m))v$$

$$b' = \hat{y} - 2(v^*\hat{y})v$$

Putting this in eq^n ①

$$\Rightarrow \hat{b} = (\hat{y} - 2(v^*\hat{y})v)(1+\epsilon_m)$$

$$\hat{b} = [\hat{y} - 2(v^*\hat{y})v] + \underbrace{\epsilon_m(\hat{y} - 2(v^*\hat{y})v)}_{\text{This is also equal to } O(\epsilon_m) \text{ with respect to } \hat{y}}$$

$\because \dfrac{\|\epsilon_m(\check{y} - 2(v^* \check{y})v)\|}{\|\check{y}\|} \leq \epsilon_m \cdot \left[\dfrac{\|\check{y}\|}{\|\check{y}\|} - \dfrac{2\|v^*\| \, \|\check{y}\| \, \|v\|}{\|\check{y}\|}\right]$

$\leq \epsilon_m [1-2]$

$\leq O(\epsilon_m)$ $\quad$ ($\because \|v\|_2 = 1$, v is a unit vector)

$\boxed{\Rightarrow \quad \check{b} = \check{y} - 2(v^* \check{y})v}$

$\because \check{b} = \check{y} - 2(v^* \check{y})v + O(\epsilon_m)$

$= \check{y} + O(\epsilon_m) - 2(v^* \check{y})v$

$\boxed{\check{b} = \check{y} - 2(v^* \check{y})v}$

⑥ ANSWER :

ⓐ given ellipse of the form, $f(x,y) = (1+a)x^2 + (1-a)y^2 + 2bxy + cx + dy + e = 0$

$\Rightarrow (1+a)x^2 + (1-a)y^2 + 2bxy + cx + dy + e = 0$

$\Rightarrow x^2 + ax^2 + y^2 - ay^2 + 2bxy + cx + dy + e = 0$

$\Rightarrow (x^2 - y^2)a + 2bxy + cx + dy + e = -(x^2+y^2)$

and $\theta = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$ & having dimensions (5×1)

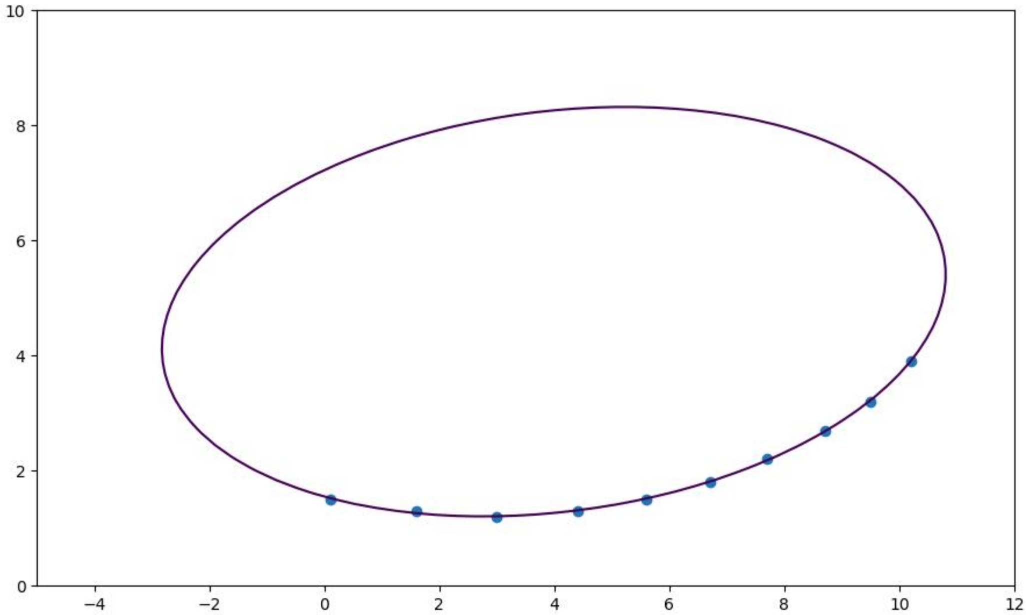
Then we have $A = \begin{bmatrix} x_1^2 - y_1^2 & 2x_1y_1 & x_1 & y_1 & 1 \\ x_2^2 - y_2^2 & 2x_2y_2 & x_2 & y_2 & 1 \\ & & \vdots & & \end{bmatrix}$ (having dimensions $n\times 5$) where n is the number of available samples.

and $b = \begin{bmatrix} -(x_1^2 + y_1^2) \\ -(x_2^2 + y_2^2) \\ \vdots \end{bmatrix}$ where b has dimensions $n\times 1$

and we need to minimize $\|A\theta - b\|_2^2$

And using the least squares minimization, we get

$\theta = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} -0.57205 \\ -0.14871 \\ -1.99915 \\ -13.78811 \\ 17.58059 \end{bmatrix}$ (using qr decomposition)



10
8
6
4
2
0
−4
−2
0
2
4
6
8
10
12

⑥

(b)

using the formulas for the above given values: (x_i, y_i)

$\rightarrow K_{b\to\theta} = \dfrac{K(A)}{\eta \cos\theta'}$ [calculating values using numpy] (for the given set of points)

and $K(A) = \|A\|_2 \|A^+\|_2 = 4823.4753$

$\eta = \dfrac{\|A\|_2 \|\theta\|_2}{\|A\theta\|_2} = 22.511325$

and $\cos\theta' = \dfrac{\|y\|}{\|b\|}$ where $y = A\theta$

$= \dfrac{\|A\theta\|_2}{\|b\|_2} = 0.9999$

$\boxed{\Rightarrow K_{b\to\theta} = 214.26977}$ (using the above formula)

$\rightarrow K_{A\to\theta} = K(A) + \dfrac{K(A)^2 \tan\theta}{\eta}$

and $\tan\theta = \dfrac{\|y-b\|_2}{\|y\|_2} = \dfrac{\|A\theta - b\|_2}{\|A\theta\|_2} = 0.0029785$

then $\boxed{K_{A\to\theta} = 7901.8472}$ (using above formula)

$\Rightarrow$ These values are calculated using linalg module in numpy.

and since these are very big values then we can say that the problem is ill-conditioned.

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(c)

→ To find condition number of θ with respect to vector v containing original data.

we have:

$$K = \frac{\|\delta\theta\| / \|\theta\|}{\|\delta v\| / \|v\|}$$ (using ∞ norm for the calculations)

$$= \frac{\|\delta\theta\| / \|\theta\|}{\|\delta b\| / \|b\|} \cdot \frac{\|\delta b\| / \|b\|}{\|\delta v\| / \|v\|}$$

$$K = K_{b\to\theta} \cdot \frac{\|\delta b\| / \|b\|}{\|\delta v\| / \|v\|} \quad \text{— (1)}$$

where v is the vector containing $[x_i, y_i]^T$

and b is a vectors that contains $[-(x_i^2+y_i^2)]^T$

⇒ Finding $\frac{\|\delta b\| / \|b\|}{\|\delta v\| / \|v\|}$ using ∞ norm

consider we changed the values x, y by δ

$$\Rightarrow \frac{\| -[(x+\delta)^2 + (y+\delta)^2] + (x^2+y^2) \|}{\|\delta v\|} \cdot \frac{\|v\|}{\|b\|}$$

$$\Rightarrow \frac{\| -(\cancel{x^2} + \delta^2 + 2x\delta + \cancel{y^2} + \delta^2 + 2y\delta - \cancel{x^2} - \cancel{y^2}) \|}{\delta} \cdot \frac{\|v\|}{\|b\|}$$ (since $\delta v = \delta$ max change in any value $= \delta$)

$$= \frac{\| -2\delta (x+y) \|}{\delta} \cdot \frac{\|v\|}{\|b\|}$$

(ignoring higher order δ)

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$$= \frac{2\delta \|x+y\|_2}{\delta} \frac{\|v\|_2}{\|b\|_2} \quad \text{(ignoring higher order } \delta\text{)}$$

$$\Rightarrow K_{v\to\theta} = K_{b\to\theta} \cdot \frac{\|\delta b\| / \|b\|}{\|\delta v\| / \|v\|}$$

$$\boxed{K_{v\to\theta} = K_{b\to\theta} \cdot \frac{2\,\|x+y\|_2 \cdot \|v\|_2}{\|b\|_2}}$$

and the value of $K_{v\to\theta} = 1324.11054$ in the 2 norm for the set of data points given.