HOMEWORK-5

NUMERICAL ALGORITHMS

1

We know that $f(t)-p(t) = \frac{1}{n!} f^{n}(\theta).(t-t_1)(t-t_2)\cdots(t-t_n)$

Now consider $w(t) = (t-t_1)(t-t_2)\cdots(t-t_n)$

at $t: t_i < t < t_{i+1}$ (for any i)

AND:

$\Rightarrow (t-t_1) = (t-t_i)+(t_i-t_{i-1})+(t_{i-1}-t_{i-2})+\cdots+(t_2-t_1)$

$\left[\begin{array}{l} (\because \text{negative \& positive terms cancel out each other}) \\ \text{and } (t_i - t_{i-1}) = h \text{ (Interval size)} \end{array} \right]$

$\Rightarrow (t-t_1) = (t-t_i) + (i-1)h$

$\Rightarrow (t-t_1) \leq (t_{i+1}-t_i) + (i-1)h$

$\left[\begin{array}{l} \because (t \cancel{-t_i}) \leq t_{i+1} \\ \Rightarrow (t-t_i) \leq (t_{i+1}-t_i) \end{array} \right]$

$\Rightarrow (t-t_1) \leq ih$

Similarly $(t-t_2) \leq (i-1)h$

$\vdots$

$(t-t_{i-1}) \leq 2h$

- All these terms are positive and $\because t > t_i$
  $\Rightarrow t > t_j \ (\forall j < i)$

$\Rightarrow$ Similarly, $|(t-t_n)| = (t_n - t) \quad (\because t_n > t)$

$|(t-t_n)| = (t_n - t_{n-1}) + (t_{n-1} - t_{n-2}) + \cdots + (t_{i+2} - t_{i+1}) + (t_{i+1} - t)$

$= (n-i-1)h + (t_{i+1} - t)$

$$(t_n - t) = (n-i-1)h + (t_{i+1} - t) \leq (n-i-1)h + (t_{i+1} - t_i)$$
$$\leq (n-i-1)h + h$$
$$\leq (n-i)h$$

$$\Rightarrow (t_{n-1} - t) \leq (n-i-1)h$$
$$\vdots$$
$$(t_{i+2} - t) \leq 2h$$

$$\Rightarrow w(t) = (t-t_1)(t-t_2) \cdots (t-t_i)(t-t_{i+1}) \cdots (t-t_n)$$
$$|w(t)| = (t-t_1)(t-t_2) \cdots (t-t_i)(t_{i+1}-t) \cdots (t_n - t)$$
$$\leq ih \cdot (i-1)h \cdots 2h\,(t-t_i)(t_{i+1}-t) \cdot 2h \cdots (n-i)h$$
$$\leq i!\, h^{i-1} (t-t_i)(t_{i+1}-t) \cdot (n-i)!\, h^{n-i-1}$$
$$|w(t)| \leq (t-t_i)(t_{i+1}-t) \cdot i!(n-i)!\, h^{n-2}$$

Now max value of $(t-t_i)(t_{i+1}-t)$ would be at middle point, $t = \frac{(t_i + t_{i+1})}{2}$

$$\Rightarrow (t-t_i)(t_{i+1}-t) = \left(\frac{t_i + t_{i+1}}{2} - t_i\right)\left(t_{i+1} - \frac{t_i + t_{i+1}}{2}\right)$$
$$= \left(\frac{t_{i+1} - t_i}{2}\right)\left(\frac{t_{i+1} - t_i}{2}\right)$$
$$= \frac{(t_{i+1} - t_i)^2}{4} = \frac{h^2}{4}$$

$$\Rightarrow |w(t)| \leq \frac{h^2}{4} \cdot i!(n-i)!\, h^{n-2}$$

$$|w(t)| \leq \frac{i!(n-i)!\, h^n}{4}$$

$$\Rightarrow \; |f(t)-p(t)| = \left|\frac{1}{n!} f^{n}(\theta)\; w(t)\right|$$

$$= \frac{1}{n!}\, |f^{n}(\theta)|\; |w(t)|$$

and $\quad |f^{n}(\theta)| \leq M$

$$\Rightarrow \; |f(t)-p(t)| \leq \frac{M}{n!} \cdot \frac{i!\,(n-i)!\, h^{n}}{4} \leq \frac{M h^{n}}{4\left(\frac{n!}{i!\,(n-i)!}\right)}$$

$$\boxed{\Rightarrow \; |f(t)-p(t)| \leq \frac{M h^{n}}{4\binom{n}{i}}}$$

②

Using method of undetermined coefficients, we have (For three nodes x_1, x_2, x_3 and equal node weights)

$$\Rightarrow Q_3(x^0) = w\sum_{i=1}^{n} f(x) = w\sum_{i=1}^{n} 1 = b-a = \int_a^b 1\,dx$$

$$= 3w = 1-(-1)$$

$$\boxed{\Rightarrow w = 2/3}$$

$$\Rightarrow Q_3(x^1) = w\sum_{i=1}^{3} f(x_i)$$

$$\Rightarrow wx_1 + wx_2 + wx_3 = \frac{(b^2-a^2)}{2} = \int_a^b x\,dx$$

$$\Rightarrow (x_1+x_2+x_3)w = 0$$

$$\Rightarrow x_1+x_2+x_3 = 0 \quad \text{——} ①$$

$$\Rightarrow Q_3(x^2) = wx_1^2 + wx_2^2 + wx_3^2 = \frac{b^3-a^3}{3} = \frac{1+1}{3} = \frac{2}{3}$$

$$\Rightarrow \frac{2}{3}(x_1^2+x_2^2+x_3^2) = \frac{2}{3}$$

$$\Rightarrow x_1^2+x_2^2+x_3^2 = 1 \quad \text{——} ②$$

$$\Rightarrow Q_3(x^3) = w(x_1^3+x_2^3+x_3^3) = \frac{b^4-a^4}{4} = \frac{1-1}{4} = 0$$

$$\Rightarrow x_1^3+x_2^3+x_3^3 = 0 \quad \text{——} ③$$

from ① $\Rightarrow$ $x_1 = -(x_2+x_3)$ —— ④

from ② $\Rightarrow$ $[-(x_2+x_3)]^2 + x_2^2 + x_3^2 = 1$

$$\Rightarrow 2x_2^2 + 2x_3^2 + 2x_3x_2 = 1 \quad \text{——} ⑤$$

$\Rightarrow \quad x_3^2 + x_2^2 + x_2 x_3 = 1/2 \quad — (5)$

from (3): $[(-(x_2+x_3))]^3 + x_2^3 + x_3^3 = 0$

$\Rightarrow -\cancel{x_2^3} - \cancel{x_3^3} - 3x_3 x_2 (x_3 + x_2) + \cancel{x_2^3} + \cancel{x_3^3} = 0$

$\Rightarrow \quad 3x_2 x_3 (x_2 + x_3) = 0 \quad — (6)$

$\Rightarrow$ Here, we can have 3 cases, $x_2 = 0$ or $x_3 = 0$ or $x_2 = -x_3$

Considering all cases and (Putting in (5))

CASE - I

$\rightarrow \boxed{x_2 = 0}$

$\Rightarrow x_3^2 = 1/2$ (from (5))

$\boxed{\Rightarrow x_3 = \pm 1/\sqrt{2}}$

and $x_1 = -(x_2 + x_3)$

$= -x_3$

$\boxed{x_1 = \mp 1/\sqrt{2}}$

CASE - II

$\boxed{\Rightarrow x_3 = 0}$

$\Rightarrow x_2^2 = 1/2$ (from (5))

$\Rightarrow \boxed{x_2 = \pm 1/\sqrt{2}}$

$x_1 = -(x_2 + x_3)$

$\boxed{x_1 = \mp 1/\sqrt{2}}$

$\rightarrow$ CASE - III $[x_2 = -x_3]$

$\Rightarrow x_3^2 + x_3^2 - x_3^2 = 1/2$ (from (5))

$\Rightarrow x_3^2 = 1/2 \Rightarrow \boxed{x_3 = \pm 1/\sqrt{2}}$

$\Rightarrow \boxed{x_2 = \mp 1/\sqrt{2}}$

$\boxed{x_1 = -(x_2 + x_3) = 0}$

⇒ In all the cases, we are getting the values of nodes

as $+\frac{1}{\sqrt{2}}, 0, -1/\sqrt{2}$.

And, the degree of this rule is ≥ 3.

because we are ~~fit~~ exactly integrating any polynomial of degree ≤ 3.

⇒ Thus, in general also, any n point rule where we determi-ned nodes as in this way would have a degree of $\geq (n)$.

where n = number of nodes.

③

(a)

→ Using monomial basis:

$\phi_1(t) = 1$
$\phi_2(t) = t$
$\phi_3(t) = t^2$
$\phi_4(t) = t^3$

Let the polynomial (cubic) on interval $[t_i, t_{i+1}]$ be:

$$p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \text{ (cubic polynomial)}$$

Then,

$$p(t_i) = f(t_i) = a_0 + a_1 t_i + a_2 t_i^2 + a_3 t_i^3 = y_i \quad \text{— ①}$$

$$p(t_{i+1}) = f(t_{i+1}) = a_0 + a_1 t_{i+1} + a_2 t_{i+1}^2 + a_3 t_{i+1}^3 = y_{i+1} \quad \text{— ②}$$

And,

$$p'(t) = a_1 + a_2 2t + a_3 \cdot 3t^2$$

$$\Rightarrow p'(t_i) = f'(t_i) = m_i = a_1 + a_2 \cdot 2t_i + a_3 \cdot 3t_i^2 \quad \text{— ③}$$

$$p'(t_{i+1}) = f'(t_{i+1}) = m_{i+1} = a_1 + a_2 \cdot 2t_{i+1} + a_3 \cdot 3t_{i+1}^2 \quad \text{— ④}$$

Thus $p(t)$ fits $f(t_i)$ and $f'(t_i)$ exactly at t_i, t_{i+1}

And, we can find the coefficients a_0, a_1, a_2, a_3 by solving the above system of equations, which can be given as:

$$\begin{bmatrix} 1 & t_i & t_i^2 & t_i^3 \\ 1 & t_{i+1} & t_{i+1}^2 & t_{i+1}^3 \\ 0 & 1 & 2t_i & 3t_i^2 \\ 0 & 1 & 2t_{i+1} & 3t_{i+1}^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} f(t_i) = y_i \\ f(t_{i+1}) = y_{i+1} \\ f'(t_i) = m_i \\ f'(t_{i+1}) = m_{i+1} \end{bmatrix}$$

(3)

(a)

→ Using newton's basis :

Let the polynomial be: $p(t) = a_1\phi_1(t) + a_2\phi_2(t) + a_3\phi_3(t) + a_4\phi_4(t)$

$\Rightarrow p'(t) = a_1\phi_1'(t) + a_2\phi_2'(t) + a_3\phi_3'(t) + a_4\phi_4'(t)$.

⇒ we need $\phi_j(t_i) = 0$ for $i < j$ and for $j = 1, 2$

and $\phi_j(t_i) = 0$ for $j = 3, 4$ for any i

and $\phi_j'(t_i) = 0$ for $j > i$ for $j = 1, 2$.

⇒ Thus choose ϕ as the newton's basis for first two functions :

$\phi_1(t) = 1$ $\qquad \phi_2(t) = (t - t_{i-1})$ $\qquad$ [On interval (t_{i-1}, t_i)]

and choose

$\phi_3(t) = (t - t_{i-1})(t - t_i)$ → since we want it to be zero at both t_i, t_{i-1}

$\phi_4(t) = (t - t_{i-1})^2 (t - t_i)$

$$\Rightarrow \phi_3'(t) = (2t - t_{i-1} - t_i) = \begin{cases} t_i - t_{i-1}, & \text{at } t = t_i \\ t_{i-1} - t_i, & \text{at } t = t_{i-1} \end{cases}$$

$$\phi_4'(t) = 2(t - t_{i-1})(t - t_i) + (t - t_{i-1})^2$$

$$= \begin{cases} 0 & \text{at } t = t_{i-1}. \\ (t_i - t_{i-1})^2 & \text{at } t = t_i \end{cases}$$

⇒ we want $p(t_i) = y_i$, $p(t_{i-1}) = y_{i-1}$

$p'(t_i) = m_i$, $p'(t_{i-1}) = m_{i-1}$

⇒ The coefficients then can be found using the lower triangular system as:

$$\Rightarrow \begin{bmatrix} \phi_1(t_{i-1}) & \phi_2(t_{i-1}) & \phi_3(t_{i-1}) & \phi_4(t_{i-1}) \\ \phi_1(t_i) & \phi_2(t_i) & \phi_3(t_i) & \phi_4(t_i) \\ \phi_1'(t_{i-1}) & \phi_2'(t_{i-1}) & \phi_3'(t_{i-1}) & \phi_4'(t_{i-1}) \\ \phi_1'(t_i) & \phi_2'(t_i) & \phi_3'(t_i) & \phi_4'(t_i) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} y_{i-1} \\ y_i \\ m_{i-1} \\ m_i \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & t_i - t_{i-1} & 0 & 0 \\ 0 & 1 & t_{i-1} - t_i & 0 \\ 0 & 1 & t_i - t_{i-1} & (t_i - t_{i-1})^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} y_{i-1} \\ y_i \\ m_{i-1} \\ m_i \end{bmatrix}$$

⇒ We can find the coefficients using the above system and then evaluate using the coefficients.

⇒ For evaluation, we first find the interval where tnew belongs (using linear search/ binary search) and then use the coefficients found for that interval to calculate:

$$\boxed{p(t_{new}) = a_1 \phi_1(t_{new}) + a_2 \phi_2(t_{new}) + a_3 \phi_3(t_{new}) + a_4 \phi_4(t_{new})}$$

Using Newton's Basis for coding, since it is conditioned better than monomial Basis

③

(b) Given three points (t_{i-1}, y_{i-1}), (t_i, y_i), (t_{i+1}, y_{i+1})

⇒ fitting curve between them using Lagrange's interpolation:

$$p(t) = \frac{y_{i-1}(t-t_i)(t-t_{i+1})}{(t_{i-1}-t_i)(t_{i-1}-t_{i+1})} + \frac{y_i(t-t_{i-1})(t-t_{i+1})}{(t_i-t_{i-1})(t_i-t_{i+1})} + \frac{y_{i+1}(t-t_{i-1})(t-t_i)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)}$$

⇒ Differentiating it:

$$\boxed{p'(t) = \frac{y_{i-1}(2t-t_i-t_{i+1})}{(t_{i-1}-t_i)(t_{i-1}-t_{i+1})} + \frac{y_i(2t-t_{i-1}-t_{i+1})}{(t_i-t_{i-1})(t_i-t_{i+1})} + \frac{y_{i+1}(2t-t_{i-1}-t_i)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)}}$$

⇒ And this would work even when the intervals are unequally spaced.

$$p'(t_i) = \frac{y_{i-1}(t_i-t_{i+1})}{(t_{i-1}-t_i)(t_{i-1}-t_{i+1})} + \frac{y_i(2t_i-t_{i-1}-t_{i+1})}{(t_i-t_{i-1})(t_i-t_{i+1})} + \frac{y_{i+1}(t_i-t_{i-1})}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)}$$

$$\boxed{\Rightarrow p'(t_i) = \frac{y_{i-1}(t_i-t_{i+1})}{(t_{i-1}-t_i)(t_{i-1}-t_{i+1})} + \frac{y_i}{(t_i-t_{i+1})} + \frac{y_i}{(t_i-t_{i-1})} + \frac{y_{i+1}(t_i-t_{i-1})}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)}} \quad ①$$

→ Eqⁿ (1) would work for all (t_i), ~~ex~~

$$\left[\because \frac{y_i(2t_i - t_{i-1} - t_{i+1})}{(t_i - t_{i-1})(t_i - t_{i+1})} = \frac{y_i(t_i - t_{i-1})}{(t_i - t_{i-1})(t_i - t_{i+1})} + \frac{y_i(t_i - t_{i+1})}{(t_i - t_{i-1})(t_i - t_{i+1})}\right.$$

$$\left. = \frac{y_i}{(t_i - t_{i+1})} + \frac{y_i}{(t_i - t_{i-1})}\right]$$

But, for ~~to~~, first (t_0) and last (t_n) points, we need to put

~~t_0~~ t_0 as t_{i-1}, t_1 as t_i and t_2 as t_{i+1}.

→ for first and last points we need to find polynomial with ~~as~~ points (t_0, t_1, t_2) and (t_{n-2}, t_{n-1}, t_n)

$$p(t) = \frac{y_0(t - t_1)(t - t_2)}{(t_0 - t_1)(t_0 - t_2)} + \frac{y_1(t - t_0)(t - t_2)}{(t_1 - t_0)(t_1 - t_2)} + \frac{y_2(t - t_0)(t - t_1)}{(t_2 - t_0)(t_2 - t_1)}$$

$$\Rightarrow p'(t_0) = \frac{y_0(2t_0 - t_1 - t_2)}{(t_0 - t_1)(t_0 - t_2)} + \frac{y_1(2t_0 - t_0 - t_2)}{(t_1 - t_0)(t_1 - t_2)} + \frac{y_2(2t_0 - t_0 - t_1)}{(t_2 - t_0)(t_2 - t_1)}$$

$$\boxed{p'(t_0) = \frac{y_0}{(t_0 - t_1)} + \frac{y_0}{(t_0 - t_2)} + \frac{y_1(t_0 - t_2)}{(t_1 - t_0)(t_1 - t_2)} + \frac{y_2(t_0 - t_1)}{(t_2 - t_0)(t_2 - t_1)}} \quad (2)$$

This can be used for ~~dt~~ finding differential at first point.

→ for last point, t_n:

$$p(t) = \frac{y_{n-2}(t - t_{n-1})(t - t_n)}{(t_{n-2} - t_{n-1})(t_{n-2} - t_n)} + \frac{y_{n-1}(t - t_{n-2})(t - t_n)}{(t_{n-1} - t_{n-2})(t_{n-1} - t_n)} + \frac{y_n(t - t_{n-2})(t - t_{n-1})}{(t_n - t_{n-2})(t_n - t_{n-1})}$$

Then:

$$p'(t_n) = \frac{y_{n-2}(2t_n - t_{n-1} - t_n)}{(t_{n-2}-t_{n-1})(t_{n-2}-t_n)} + \frac{y_{n-1}(2t_n - t_{n-2} - t_n)}{(t_{n-1}-t_{n-2})(t_{n-1}-t_n)}$$

$$+ \frac{y_n(2t_n - t_{n-2} - t_{n-1})}{(t_n - t_{n-2})(t_n - t_{n-1})}$$

$$\boxed{p'(t_n) = \frac{y_{n-2}(t_n - t_{n-1})}{(t_{n-2}-t_{n-1})(t_{n-2}-t_n)} + \frac{y_{n-1}(t_n - t_{n-2})}{(t_{n-1}-t_{n-2})(t_{n-1}-t_n)} + \frac{y_n}{(t_n-t_{n-2})} + \frac{y_n}{(t_n-t_{n-1})}} \quad (3)$$

$\Rightarrow$ The above can be used for finding differential at t_n.

\# The above differentials are also accurate on $O(h^2)$.

$\because$ for 3 pts, $p(t) = f(t) + \frac{f'''(\theta)}{n!}(t-t_{i-1})(t-t_i)(t-t_{i+1})$ $\hookrightarrow O(H^3)$

$\Rightarrow p'(t) = f'(t) + \frac{f'''(\theta)}{n!} \cdot O(t-t_i)^2$ $\hookrightarrow O(H^2)$

$\Rightarrow$ $O(H^2)$ accurate, where H = max spacing between pts