COL 726 Homework 1

10–24 January, 2020

Note: In some questions, earlier parts ask you to prove results that may be useful in later parts. If you are unable to prove the earlier results, you are still allowed to use them to complete the derivation in the later parts.

1. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions on arbitrary vector spaces, and let $h = g \circ f$. Suppose $y = f(\mathbf{x})$ and $z = g(\mathbf{y}) = h(\mathbf{x})$. Prove that $\kappa_h(\mathbf{x}) \leq \kappa_g(\mathbf{y})\kappa_f(\mathbf{x})$, where $\kappa_f, \kappa_g, \kappa_h$ are the condition numbers of f, g, h respectively.

2. Suppose we have an algorithm $\widetilde{\sin} : \mathbf{F} \to \mathbf{F}$ that computes the sine of a floating-point number with relative error $\leq \epsilon_{\text{machine}}$. Now we wish to use it to compute $(\sin x)/x$ for a real number x using floating-point arithmetic.

   (a) Find an upper bound on the absolute error of the result. State your result to first order in $\epsilon_{\text{machine}}$, i.e. you may ignore terms on the order of $O(f(x)\epsilon^2_{\text{machine}})$.

   (b) For small nonzero x, is this algorithm accurate? Is it backward stable?

3. Using only basic linear algebra properties (linear combinations, spans, linear independence), show that a square upper triangular matrix

$$\mathbf{U} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ & u_{22} & \cdots & u_{2m} \\ & & \ddots & \vdots \\ & & & u_{mm} \end{bmatrix}$$

   is invertible if and only if every diagonal entry u_{ii} is nonzero. (In other words, use only parts (a)–(d) of Trefethen and Bau's Theorem 1.3.)

4. (a) Prove that for any vector $\mathbf{x} \in \mathbb{C}^m$, we have $\|\mathbf{x}\|_1 \geq \|\mathbf{x}\|_2 \geq \|\mathbf{x}\|_\infty$ and $\frac{1}{\sqrt{m}}\|\mathbf{x}\|_1 \leq \|\mathbf{x}\|_2 \leq \sqrt{m}\|\mathbf{x}\|_\infty$.

   (b) Use the above results to derive analogous bounds on induced norms $\|\mathbf{A}\|_1, \|\mathbf{A}\|_2, \|\mathbf{A}\|_\infty$ of a matrix $\mathbf{A} \in \mathbb{C}^{m\times n}$.

5. Suppose a square, invertible matrix $\mathbf{A} \in \mathbb{C}^{m \times m}$ has singular value decomposition $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^*$, with singular values $\sigma_1 \geq \cdots \geq \sigma_m > 0$. Consider the problem of solving $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{x}$ given a vector $\mathbf{b}$. Find the worst possible condition number for this problem,

$$\sup_{\mathbf{b}, \delta\mathbf{b}} \frac{\|\delta\mathbf{x}\|/\|\mathbf{x}\|}{\|\delta\mathbf{b}\|/\|\mathbf{b}\|},$$

and give explicitly a input vector $\mathbf{b}$ and small perturbation $\delta\mathbf{b}$ that attain this bound.

6. Recall that the two solutions of a quadratic equation $ax^2 + bx + c = 0$ are given by

$$x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

   (a) Ignoring underflow and overflow, discuss two different reasons why computing this formula in floating-point arithmetic can lead to loss of accuracy. Which of these, if any, is an unavoidable result of ill-conditioning of the problem?

   (b) Design a numerical experiment illustrating this loss of accuracy: Find a parametrized family of polynomials $a(h)x^2 + b(h)x + c(h)$ such that as $h \to 0$, one of the roots tends to a known limit x^*, but the corresponding root computed in floating-point arithmetic, $\tilde{x}(h)$, fails to converge to this limit. Implement this in code and create a log-log plot of $|\tilde{x}(h) - x^*|$ vs. h in which the instability is clearly visible.

   (c) Refer to Heath Example 1.15 for an alternative quadratic formula. Using this knowledge, implement a program `quadroots` which, given the coefficients `a`, `b`, `c` of arbitrary signs and sizes, computes the pair of roots $(\mathtt{x}_-, \mathtt{x}_+)$ in as numerically stable a way as possible. Run this on your experiment from (b) and add its results onto the same plot.

Submission: Submit a zip file containing (i) a PDF of your answers for all questions and your plot(s) for Question 6, and (ii) the code for Question 6(c) in a file `quadroots.py` or `quadroots.m`.