EL2450 Hybrid and Embedded Control

Lecture 11: Control of hybrid systems

- Nonholonomic control
- Supervisory control
- Multi-agent control

Today's Goal

You should be able to

- identify necessary conditions for continuous feedback stabilization
- design a hybrid controller for a class of nonholonomic systems
- design supervisory controllers for linear systems
- design controllers for multi-agent systems under limited sensing and communication

Motivations for hybrid Control

Some motivations for hybrid control

- Systems that are not stabilizable by continuous feedback (e.g., nonholonomic systems)
- Systems with large modelling uncertainty (e.g., linear systems with parametric uncertainty)
- Systems with sensing and communication constraints (e.g., quantized control, large scale systems with sensing limitations)

Brockett's necessary condition for continuous stabilization

Let $\dot{x} = f(x, u)$, $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ and suppose $u = k(x), k(0) = 0$ is continuous and makes $x = 0$ (locally) asymptotically stable for $\dot{x} = f(x, k(x))$. Then the image of every neighborhood of $(0, 0)$ in $\mathbb{R}^n \times \mathbb{R}^m$ under $(x, u) \mapsto f(x, u)$ contains some neighborhood of 0 in $\mathbb{R}^n$.

- Intuition: starting near zero and applying small controls, we must be able to move in all directions
- Systems with certain constraints in motion do not satisfy this condition

Nonholonomic systems

Systems of the form

$$\dot{x} = f(x, u) = \sum_{i=1}^{m} g_i(x) u_i = G(x) u$$

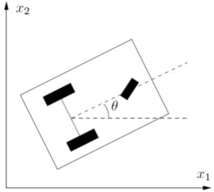
where $G \in \mathbb{R}^{n \times m}$ and $m < n$ are called nonholonomic: constraints involving both position and velocity.

Corollary of Brockett's condition: nonholonomic systems with $\operatorname{rank} G(0) = m < n$ cannot be asymptotically stabilized by continuous feedback.

Unicycle model

Unicycles are a popular model for wheeled mobile robots

$$\dot{x}_1 = u_1 \cos\theta$$
$$\dot{x}_2 = u_1 \sin\theta$$
$$\dot{\theta} = u_2$$



- u_1, u_2 linear and angular velocities, θ orientation wrt x_1 axis
- The robot cannot move sideways (nonholonomic constraint)
- Asymptotic stabilization equivalent to parking at the origin and align with x_1 axis

Nonholonomic integrator

Using the transformation (that preserves the origin)

$$
\begin{aligned}
x &= x_1 \cos\theta + x_2 \sin\theta \\
y &= \theta \\
z &= 2(x_1 \sin\theta - x_2 \cos\theta) - \theta(x_1 \cos\theta + x_2 \sin\theta) \\
u &= u_1 - u_2(x_1 \sin\theta - x_2 \cos\theta) \\
v &= u_2
\end{aligned}
$$

we get the equivalent form

$$
\begin{aligned}
\dot{x} &= u \\
\dot{y} &= v \\
\dot{z} &= xv - yu
\end{aligned}
$$

known as Brockett's *nonholonomic integrator*

A further transformation

With a further transformation

$$\begin{aligned} x &= r\cos\psi \\ y &= r\sin\psi \\ u &= \bar{u}\cos\psi - \bar{v}\sin\psi \\ v &= \bar{u}\sin\psi + \bar{v}\cos\psi \end{aligned}$$

we get (in cylindrical coordinates)

$$\begin{aligned} \dot{r} &= \bar{u} \\ \dot{\psi} &= \frac{\bar{v}}{r} \\ \dot{z} &= r\bar{v} \end{aligned}$$

which holds in this case only for $x^2 + y^2 = r^2 \neq 0$

A hybrid controller for the nonholonomic integrator

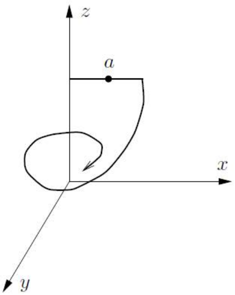
For $r(0) \neq 0$ it can be shown that if $\bar{u} = -r^2$, $\bar{v} = -z$, then $r, z \to 0$ and that $r(t) \neq 0$ for all t. Thus $x, y, z \to 0$ and thus $x_1, x_2, \theta \to 0$.

What happens when $r(0) = 0$? Apply switching logic as follows:

- Apply control that takes state away from z-axis (e.g., $u = v = 1$ or $u = v = z(0)$)
- At certain time T, switch to $\bar{u} = -r^2$, $\bar{v} = -z$
- Two discrete states and the switch between them happens at T
- Then based on the previous result the states go to the origin

Typical trajectory

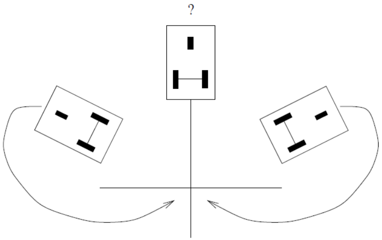
- A typical trajectory of the switching logic is on the right
- At point *a* either discrete node can be active



Intuition with respect to parking example

Unicycle stabilization related to parking at the origin.

- Ambiguity at $r(0) = 0$ leads to discontinuous controllers
- Related to need for logical choice at certain initial conditions
- How to rotate when starting at z-axis?

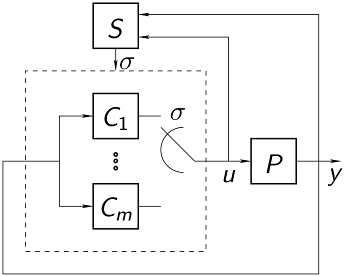


A hybrid controller for the nonholonomic integrator

- Truly hybrid controller since it depends both on state value and discrete event
- Different switching controllers for nonholonomic systems exist
- Alternative to discontinuous feedback: time-varying feedback
- Previous result can be robustified by replacing $r(0) = 0$ condition with $|r(0)| \leq \epsilon$ for some $\epsilon > 0$

Supervisory Control

- How choose switching $\sigma = \sigma(t)$ such that P has desired property?
- Let a **supervisor** decide on which controller should be active through the switching signal $\sigma : [0, \infty) \to \{1, \ldots, m\}$



Supervisory Control

- Motivated by applications where parametric uncertainty of plant/controller belongs to a compact or finite set.
- In contrast to adaptive control where parametric uncertainty evolves in a continuum.
- Example: supervisory control for linear systems with parametric uncertainty. Consider the plant

$$\begin{aligned}\dot{x} &= A_{p^*}x + B_{p^*}u \\ y &= C_{p^*}x\end{aligned}$$

where $A_p, B_p, C_p, p \in \mathcal{P}$ a given finite family of matrices, and the real $p^* \in \mathcal{P}$ unknown.

Linear supervisory Control

Define for each p the family of observers

$$\dot{x}_p = (A_p + K_p C_p)x_p + B_p u - K_p y$$
$$y_p = C_p x_p$$

and corresponding controllers $u_p = F_p x_p$, $p \in \mathcal{P}$. Key assumption: assume that K_p, F_p can be chosen such that $A_p + K_p C_p$ and $A_p + B_p F_p$ are stable.

Switching logic can be related to estimation errors of each observer, $e_p = y_p - y$, $p \in \mathcal{P}$. Define $\mu_p(t) = \int_0^t |e_p(t)|^2 dt$.

Switching logic

Design $\sigma : [0, \infty) \to \mathcal{P}$ for the control $u(t) = F_{\sigma(t)} x_{\sigma(t)}$.

Hysteresis switching logic:

- Set $\sigma(0) = \arg\min_{p \in \mathcal{P}} \mu_p(0)$
- For current value of σ equal to q, keep σ constant until

$$\min_{p \in \mathcal{P}} \mu_p(t) + h \leq \mu_q(t)$$

- Then set $\sigma(t) = \arg\min_{p \in \mathcal{P}} \mu_p(t)$

Switching logic

Results

- Result: there exists a time T where $\sigma(t) = q^*$, for all $t \geq T$ and both x and x_{q^*} go to zero.
- Proof relies on that μ_p nondecreasing and that e_{p^*} goes to zero irrespective of u, and thus μ_{p^*} remains bounded.

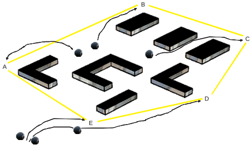
Notes on supervisory control

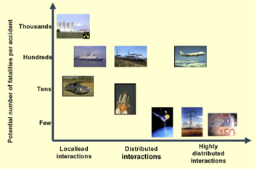
- Previous example rather special
- More advanced methods exist
- Hysteresis constant h reduces switching frequency and excludes Zeno
- Known limitations of standard adaptive control can be overcome by supervisory control

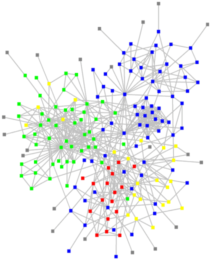
Multi-agent systems: motivation

- Many applications involve distributed control over different entities ("agents")
- Multi-robot/vehicle coordination
- Sensor networks
- Social networks
- Bio-inspired coordination
- Local control specs arise due to limitations in sensing and/or communication

Some nice figures





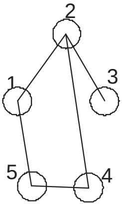


Limited Sensing and Communication aspects

- Limited Sensing: Vision based sensors, range sensors (sonars, laser scanners,...)
- Limited Communication: communication channel, bandwith, coding,...
- Limitations in communication/sensing do now allow each agent to communicate with everyone else

Graph theoretic approach

- Modelling of limitations through graphs



$G = (V, E)$

- Agents are the vertices $V = \{1, \dots, N\}$
- Edges $E \subset V \times V$ are pairs of agents that can communicate
- Neighboring set for agent i: $N_i = \{j \in V | (i, j) \in E\}$
- Connected graph: has a path between each pair of its vertices

Switching graphs

- What happens when the communication graph loses/adds edges over time?
- Time-varying sets: $N_i(t) = \{j \in V | (i, j) \in E(t)\}$
- Different $N_i(t)$ correspond to different graphs
- Different graphs correspond to different discrete nodes of a hybrid system!

The adjacency matrix and the degree matrix

- We want to associate matrices with graphs.
- Neighboring set: $N_i = \{j \in V | (i, j) \in E\}$
- Adjacency matrix

$$A = A(G) = [a_{ij}], a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{if otherwise.} \end{cases}$$

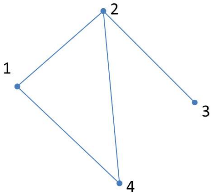
- Degree matrix

$$\Delta = \Delta(G) = \mathrm{diag}(d_1, \ldots, d_N), d_i = \sum_j a_{ij} = |N_i|$$

The Laplacian matrix and its eigenvalues

- $L = L(G) = \Delta(G) - A(G)$
- Symmetric and positive semi-definite matrix
- Eigenvalues $0 = \lambda_1(G) \leq \lambda_2(G) \leq \ldots \leq \lambda_N(G)$
- For a connected G, $L(G)$ has a simple zero eigenvalue with the corresponding eigenvector $\mathbf{1} = [1, \ldots, 1]^T$.
- Thus $\lambda_2(G) > 0$ for a connected graph.

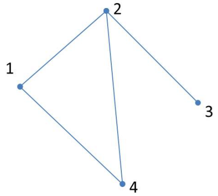
Example



$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$$\Delta = \mathrm{diag}(2,3,1,2) \; L = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$$

Example



Laplacian eigenvalues: $0, 1, 3, 4$. If the edge between $2, 3$ is deleted, then the graph becomes disconnected, with new Laplacian $L^* = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$

The eigenvalues of L^* are $0, 0, 3, 3$.

Introducing interaction: the agreement paradigm

- Consider a multiple sensor network measuring temperature
- Goal: reach the best estimate of temperature given the available limited information
- It seems that the average of all measurements is the optimal choice
- Problem: design an algorithm that converges to the average for all sensors given *relative* information exchange

Agreement algorithms for static graphs

- N agents with $\dot{x}_i = u_i, i \in V = \{1, \ldots, N\}$
- Neighboring set: $N_i = \{j \in V | (i, j) \in E\}$
- Available information for i: $(x_i - x_j), j \in N_i$
- Agreement algorithm: $u_i = -\sum_{j \in N_i} (x_i - x_j)$
- $\dot{x} = -Lx$, where L is the Laplacian matrix of the graph $G = (V, E)$
- Stack vector notation $x = [x_1, \ldots, x_N]^T$

Convergence and Performance

Theorem: If G is connected then

$$x \to \mathcal{A} \triangleq \{x \in \mathbb{R}^N | x_i = x_j, \forall i, j\},$$

and the agreement point is equal to the initial average of the agents. Moreover, the worst-case convergence rate is equal to $\lambda_2(G)$.

Proof sketch

- Proof: Performance analysis using the disagreement vector
- Let $a(t) = \frac{1}{N}\sum_i x_i(t)$ denote the average of the agents' states. Then:

$$\dot{a} = \frac{1}{N}\sum_i \dot{x}_i = -\frac{1}{N}\sum_i \sum_{j\in N_i} (x_i(t) - x_j(t)) = 0$$

- Then $a(t) = a(0) = \dfrac{1}{N}\sum_i x_i(0) \equiv a$

Disagreement dynamics

- Define $\delta_i = x_i - a$ for each $i \in V$. δ_i represents the *disagreement* of i with respect to the desired average value.
- Decomposition of x: $x(t) = a\mathbf{1} + \delta(t)$
- Disagreement dynamics: $\dot{\delta} = -L\delta$
- Using $V = \frac{1}{2}\delta^T\delta$ as a Lyapunov function candidate it can be shown that

$$||\delta(t)|| \leq ||\delta(0)||e^{-\lambda_2 t}$$

Proof sketch (cont.)

For any connected graph G:

$$\min_{z \neq 0 : \mathbf{1}^T z = 0} \frac{z^T L z}{||z||^2} = \lambda_2(G)$$

- From $\mathbf{1}^T \delta = 0$, we have $\delta^T L \delta \geq \lambda_2(G)||\delta||^2$, for all $\delta \neq 0$
- Then $\dot{V} \leq -2\lambda_2(G)V$
- Applying Comparison lemma we get $||\delta(t)|| \leq ||\delta(0)||e^{-\lambda_2 t}$

Switching graphs

- Switching control law $u_i = -\sum_{j \in N_i(t)} (x_i - x_j)$
- $G(t) \in G_c^N, \forall t \geq 0$: active graph at time t.
- G_c^N: all connected graphs with N vertices
- The system switches between possible connected graphs with N vertices. $G(t)$ remains constant between consecutive topology changes due to edge addition or loss.

Switching graphs

- We have $\dot{x}(t) = -L(G(t))x(t)$ and $a(t) = a(0) = \dfrac{1}{N}\sum_i x_i(0) = a$
- Then $\dot{\delta} = -L(G(t))\delta$ and $V = \frac{1}{2}\delta^T\delta$ can be used as a common Lyapunov function for the switched system (why?)
- It can be shown that

$$||\delta(t)|| \leq ||\delta(0)||e^{-\min\limits_{G \in G_c^N}\{\lambda_2(G)\}t}$$

Relation to supervisory control

- Assume that there exists a supervisor that can activate/deactivate certain edges
- When connectivity is lost, the supervisor tries to restore it by activating appropriate new edges

Next Lecture

Verification of hybrid systems

- Reachability of hybrid systems
- Bisimulatations of hybrid systems