EL2450 Hybrid and Embedded Control

Lecture 3: Analysis of sampled control

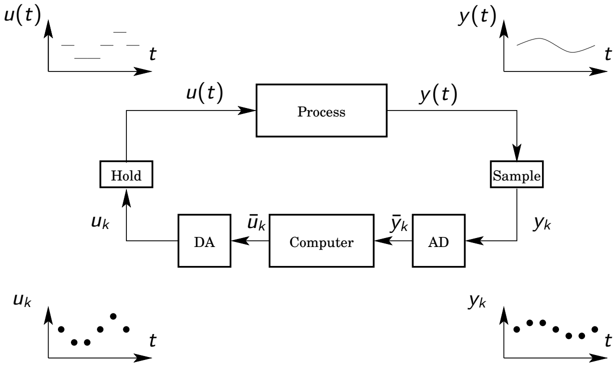
- Stability
- Reachability and observability
- Observers
- State and output feedback

Today's Goal

You should be able to

- Determine stability, reachability and observability for discrete-time systems
- Design state observers
- Design sampled-data control based on state and output feedback
- Design deadbeat controllers

Sampled Control System



To understand closed-loop behavior, we need tools to study properties of discrete-time control systems

Stability

The solution $x^*(k)$ of $x(k+1) = f(x(k))$ is *stable* if for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that for all solutions $x(k)$

$$\|x(0) - x^*(0)\| < \delta \quad \Rightarrow \quad \|x(k) - x^*(k)\| < \epsilon, \quad k = 0, 1, \ldots$$

The solution is *asymptotically stable* if it is stable and

$$\|x(0) - x^*(0)\| < \delta \quad \Rightarrow \quad \|x(k) - x^*(k)\| \to 0, \text{ as } k \to \infty$$

Linear Systems A linear system $x(k+1) = \Phi x(k)$ is asymptotically stable if and only if $|\lambda_i(\Phi)| < 1$, $i = 1, \ldots, n$.

Stability-LTI Systems

For $x(k+1) = \Phi x(k)$, $x^*(k+1) = \Phi x^*(k)$ and
$\tilde{x}(k) = x(k) - x^*(k)$, then $\tilde{x}(k+1) = \Phi \tilde{x}(k)$, $\tilde{x}(0) = x(0) - x^*(0)$.

If $x^*(k)$ is stable, all other solutions of $x(k+1) = \Phi x(k)$ stable.
For LTI systems, stability is a property of the system and not of a particular solution.

Solution $(x(k) = \Phi^k x(0))$ can be transformed to linear combination of terms $p_i(k)\lambda_i^k(\Phi)$, where $p_i(k)$ polynomials in k of order one less than the multiplicity of $\lambda_i(\Phi)$. Thus asymptotic stability equivalent to $|\lambda_i(\Phi)| < 1$, $i = 1, \ldots, n$.

Lyapunov Function

A continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is a *Lyapunov function* for $x(k+1) = f(x(k)), f(0) = 0$ if

1. $V(0) = 0$ and $V(x) > 0$, $\forall x \neq 0$
2. $\Delta V(x) := V(f(x)) - V(x) \leq 0$, $\forall x \neq 0$

Linear Systems $V(x) = x^T P x$, P positive definite, is a Lyapunov function for $x(k+1) = \Phi x(k)$, if (and only if)

$$\Phi^T P \Phi - P = -Q, \qquad Q \text{ positive semidefinite}$$

because
$\Delta V(x) = V(\Phi x) - V(x) = x^T(\Phi^T P \Phi - P)x = -x^T Q x \leq 0.$

Stability Test

The solution $x^*(k) = 0$ for $x(k+1) = f(x(k))$ is stable if there exists a Lyapunov function. It is asymptotically stable if, moreover, $\Delta V(x)$ is negative definite.

Linear Systems A linear system $x(k+1) = \Phi x(k)$ is asymptotically stable if (and only if) for any positive definite Q, there exists positive definite P such that

$$\Phi^T P \Phi - P = -Q$$

Reachability

$$x(k+1) = f(x(k), u(k))$$

is *reachable* if for any x_0, x_1 there exists a finite integer $N > 0$ and a control sequence $u(k)$, $k = 0, 1, \ldots, N-1$, such that $x(0) = x_0$ and $x(N) = x_1$.

Linear Systems

For $x(k+1) = \Phi x(k) + \Gamma u(k)$, we have

$$
\begin{aligned}
x(n) &= \Phi^n x(0) + \Phi^{n-1}\Gamma u(0) + \cdots + \Gamma u(n-1) \\
&= \Phi^n x(0) + \underbrace{\begin{bmatrix} \Gamma & \Phi\Gamma & \dots & \Phi^{n-1}\Gamma \end{bmatrix}}_{W_c} \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix}
\end{aligned}
$$

Hence, the system is reachable if and only if W_c is invertible.

Example

$$x(k+1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \quad \Rightarrow \quad W_c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Thus the system is not reachable.
Note that still, for any $x(0)$ there exist $u(0), u(1)$ such that $x(2) = 0$.

Observability

$$x(k+1) = f(x(k))$$
$$y(k) = h(x(k))$$

is *observable* if there exists $N < \infty$ such that $x(0) = x_0$ can be determined from $y(0), \ldots, y(N)$.

Linear Systems

$$\begin{aligned} x(k+1) &= \Phi x(k) \\ y(k) &= Cx(k) \end{aligned}$$

gives $y(0) = Cx(0)$, $y(1) = C\Phi x(0)$, $\ldots$:

$$W_o x(0) := \begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{bmatrix} x(0) = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix}$$

Hence, the system is observable if and only if W_o is invertible.

Example

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x(k)$$
$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(k)$$

is not observable since

$$\operatorname{rank} W_o = \operatorname{rank} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = 1 < 2$$

Note that both $x(0) = (0, 1)$ and $(1, 0)$ give $y(0) = 1$, $y(1) = 2$.

Observers

How obtain estimates of $x(k)$ from current and past y and u?
Direct calculations of $x(k)$ based on

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= C x(k) \end{aligned}$$

give

$$\underbrace{\begin{bmatrix} y(k-n+1) \\ y(k-n+2) \\ \vdots \\ y(k) \end{bmatrix}}_{Y_k} = W_o x(k-n+1) + \begin{bmatrix} 0 & 0 & \dots & 0 \\ C\Gamma & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ C\Phi^{n-2}\Gamma & C\Phi^{n-3}\Gamma & \dots & C\Gamma \end{bmatrix} \underbrace{\begin{bmatrix} u(k-n+1) \\ u(k-n+2) \\ \vdots \\ u(k-1) \end{bmatrix}}_{U_{k-1}}$$

Static Observer

This leads to the formula (multiply by W_o^{-1} and express $x(k)$ as function of $x(k-n+1)$)

$$x(k) = QY_k + RU_{k-1}$$

where the matrices Q, R depend only on Φ, Γ, C.

Note

- $x(k)$ is simply a linear combination of old y and u
- Method can be sensitive to noise and disturbances

Example

For ZOH-sampled $G(s) = 1/s^2$ with $h = 1$, we have

$$\Phi = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\begin{aligned} y(k) = x_1(k) &= x_1(k-1) + x_2(k-1) + u(k-1)/2 \\ &= y(k-1) + [x_2(k) - u(k-1)] + u(k-1)/2 \end{aligned}$$

$$\begin{aligned} x_1(k) &= y(k) \\ x_2(k) &= y(k) - y(k-1) + u(k-1)/2 \end{aligned}$$

- Disturbance in y hits x directly

Dynamic Observer

Denote by $\hat{x}(k+1|k)$ an estimate of $x(k+1)$ based on measurements $\ldots, y(k-1), y(k)$ up to time k.
A dynamic observer is given by

$$\hat{x}(k+1|k) = \Phi\hat{x}(k|k-1) + \Gamma u(k) + K[y(k) - C\hat{x}(k|k-1)]$$

The reconstruction error $\tilde{x}(k+1|k) = x(k+1) - \hat{x}(k+1|k) \to 0$ as $k \to \infty$ if K is chosen such that $|\lambda_i(\Phi - KC)| < 1$, because

$$\tilde{x}(k+1|k) = (\Phi - KC)\tilde{x}(k|k-1)$$

Example

Consider $G(s) = 1/s^2, h = 1$ again. Then, $\Phi - KC = \begin{bmatrix} 1 - k_1 & 1 \\ -k_2 & 1 \end{bmatrix}$
with characteristic equation $z^2 + (k_1 - 2)z + 1 - k_1 + k_2 = 0$.
$k_1 = 1$, $k_2 = 1/4$ give zeros in $1/2$. Then,

$$\begin{bmatrix} \hat{x}_1(k+1|k) \\ \hat{x}_2(k+1|k) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1(k|k-1) \\ \hat{x}_2(k|k-1) \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 1 \\ 1/4 \end{bmatrix} [y(k) - \hat{x}_1(k|k-1)]$$

Reduced-Order Observer

The dynamic observer has a unit delay from y to $\hat{x}$ (Check!)
This can be avoided by considering

$$\begin{aligned}\hat{x}(k+1|k+1) = \Phi\hat{x}(k|k) + \Gamma u(k)\\ + K[y(k+1) - C(\Phi\hat{x}(k|k) + \Gamma u(k))]\end{aligned}$$

where $y(k+1)$ is the current measurement.
The reconstruction error $\tilde{x} = x - \hat{x}$ fulfills

$$\tilde{x}(k+1|k+1) = (I - KC)\Phi\tilde{x}(k|k)$$

so K should be chosen such that $|\lambda_i[(I - KC)\Phi]| < 1$.

$$
\begin{aligned}
y(k+1) - C\hat{x}(k+1|k+1) &= C\tilde{x}(k+1|k+1) \\
&= (I - CK)C\Phi\tilde{x}(k|k)
\end{aligned}
$$

If we have p (independent) outputs (rank $C = p$), then K ($n \times p$ matrix) can be chosen such that $I - CK = 0$, where CK is a $p \times p$ matrix. Then,

$$
C\hat{x}(k+1|k+1) = y(k+1)
$$

hence, y is estimated with no error.
The exact knowledge of y can be used to reduce the order of the observer by p (see example). The resulting (reduced-order) filter is called a *Luenberger observer*.

Example

For the double integrator we get

$$\begin{bmatrix} \hat{x}_1(k+1|k+1) \\ \hat{x}_2(k+1|k+1) \end{bmatrix} = \begin{bmatrix} 1-k_1 & 1-k_1 \\ -k_2 & 1-k_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1(k|k) \\ \hat{x}_2(k|k) \end{bmatrix} + \begin{bmatrix} (1-k_1)/2 \\ 1-k_2/2 \end{bmatrix} u(k) + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} y(k+1)$$

$CK = I$ implies $k_1 = 1$. Then, the reduced-order observer becomes ($\hat{x}_1(k+1|k+1) = y(k+1)$ and)

$$\hat{x}_2(k+1|k+1) = (1-k_2)\hat{x}_2(k|k) + (1-k_2/2)u(k) + k_2(y(k+1) - y(k))$$

$k_2 = 1/2$ gives eigenvalue in $1/2$.

Feedback Control

Recall the main reasons for using feedback control:

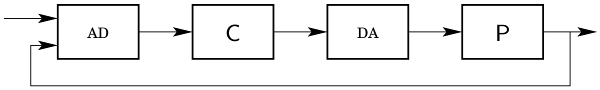
- Counteract process uncertainties and variations
- Track reference trajectories
- Attenuate disturbances
- Reduce influence of measurement noise

Disadvantages with feedback control include higher system design complexity (sensor, actuator, computer)

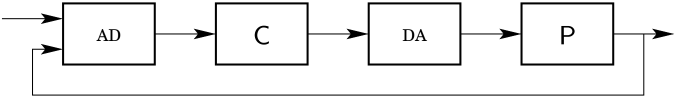
State Feedback and Output Feedback

$$x(k+1) = f(x(k), u(k))$$

Output feedback: $u(k)$ is based on the output $y(k) = h(x(k))$
State feedback: $u(k)$ is based on the state $x(k)$
Control objectives: Stabilization to equilibrium or tracking of certain $r(k)$



Example: Control of Double Integrator



Sampling

$$G(s) = \frac{1}{s^2}$$

with $h = 1$ gives

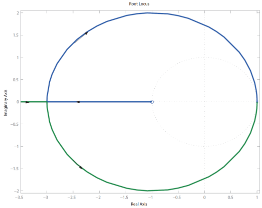
$$H(q) = \frac{q+1}{2(q-1)^2}$$

P Control

Consider proportional control based on output y:
$u(k) = K(r(k) - y(k))$, $K > 0$. Gives characteristic equation

$$z^2 + (K/2 - 2)z + 1 + K/2 = 0$$

which has zeros outside unit circle for all $K > 0$:



Hence, system not stabilizable with P control

PD Control

Consider proportional-derivative control:

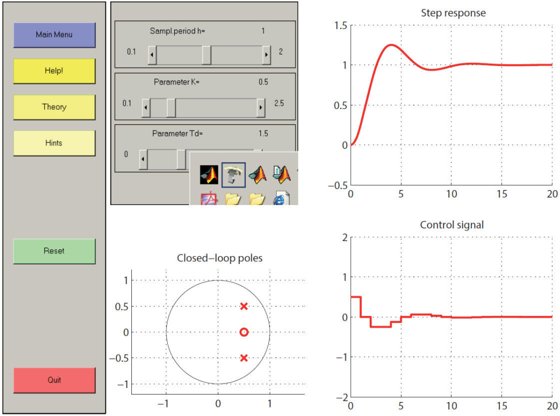
$$u(k) = K[r(k) - y(k) + 2T_d \frac{q-1}{q+1}(r(k) - y(k))]$$

Then,

$$y(k) = \frac{K/2[(1+2T_d)q + 1 - 2T_d]}{q^2 + (T_d K + K/2 - 2)q + 1 - T_d K + K/2} r(k)$$

where K, T_d can be chosen to give a stable closed-loop poles

$K = 0.5$, $T_d = 1.5$ give poles $0.5 \pm 0.5i$ and zero 0.5



From CCSDEMO

State Feedback

Consider the process $x(k+1) = \Phi x(k) + \Gamma u(k)$ under state feedback $u(k) = -Lx(k)$. Closed-loop dynamics is given by

$$x(k+1) = (\Phi - \Gamma L)x(k)$$

so L should be chosen such that $\lambda_i(\Phi - \Gamma L)$ are suitably placed inside unit circle.

Example For the double integrator, closed-loop characteristic polynomial equals $p(z) = z^2 + p_1 z + p_2$ if

$$L = \begin{bmatrix} \dfrac{1 + p_1 + p_2}{h^2} & \dfrac{3 + p_1 - p_2}{2h} \end{bmatrix}$$

Note that u becomes large if h is small.

Pole placement

Previous example indicates a design procedure for pole placement. Can be extended to higher order but calculations become tedius.

- More systematic methods exist.
- Ackermann's formula:

$$L = (0 \dots 0 \quad 1) W_c^{-1} p(\Phi)$$

  where $p(z)$ is the desired closed-loop characteristic polynomial.
- Use the formula to calculate gains in the previous example.

Deadbeat Control

If the desired closed-loop poles are put at the origin, then $p(z) = z^n$.
From Cayley-Hamilton it follows that $\Phi_c := \Phi - \Gamma L$ fulfills $\Phi_c^n = 0$.
Hence, for any $x(0)$

$$x(n) = \Phi_c^n x(0) = 0$$

This control strategy is called *deadbeat control*.
It drives the state into the origin in at most n steps.

Note There is no correspondence to deadbeat control in continuous time.
What happens in continuous time if we try to drive the state to the origin in zero time? Can we drive the state in time nh?

Example

For the double integrator, closed-loop characteristic polynomial equals $p(z) = z^2$ if

$$L = \begin{bmatrix} \frac{1}{h^2} & \frac{3}{2h} \end{bmatrix}$$

Then,

$$\begin{aligned} u(0) &= -\frac{1}{h^2}x_1(0) - \frac{3}{2h}x_2(0) \\ u(h) &= -\frac{1}{h^2}x_1(h) - \frac{3}{2h}x_2(h) \end{aligned}$$

This control strategy drives the state to the origin in two steps.

Output Feedback

If the state x is not measured, a dynamic observer

$$\hat{x}(k+1|k) = \Phi\hat{x}(k|k-1) + \Gamma u(k) + K[y(k) - C\hat{x}(k|k-1)]$$

can be used together with the control law

$$u(k) = -L\hat{x}(k|k-1)$$

With $\tilde{x} = x - \hat{x}$, the closed-loop system is

$$\begin{bmatrix} x(k+1) \\ \tilde{x}(k+1|k) \end{bmatrix} = \begin{bmatrix} \Phi - \Gamma L & \Gamma L \\ 0 & \Phi - KC \end{bmatrix} \begin{bmatrix} x(k) \\ \tilde{x}(k|k-1) \end{bmatrix}$$

Dynamics of order $2n$ determined by $\Phi - \Gamma L$ and $\Phi - KC$.

Duality

$$x(k+1) = \Phi x(k) + \Gamma u(k),\ \ y(k) = Cx(k)$$

with W_c, W_o. Then $\exists L$ s.t. $\Phi - \Gamma L$ has prescribed eigenvalues if (and only if) W_c is invertible.

Existence of K s.t. $\Phi - KC$ has prescribed eigenvalues equivalent to existence of K^T s.t. $\Phi^T - C^T K^T$ has prescribed eigenvalues. This happens if (and only if)

$$\underbrace{\begin{bmatrix} C^T & \Phi^T C^T & \dots & (\Phi^{n-1})^T C^T \end{bmatrix}}_{W_o^T}$$

is invertible.

Pulse-Transfer Function of Controller

The nth-order controller is given by

$$\begin{aligned}\hat{x}(k+1|k) &= (\Phi - KC - \Gamma L)\hat{x}(k|k-1) + Ky(k) \\ u(k) &= -L\hat{x}(k|k-1)\end{aligned}$$

which gives the input–output relation

$$u(k) = -L(qI - \Phi + KC + \Gamma L)^{-1}Ky(k)$$

Internal model principle: note that the controller contains a model of the process (Φ, Γ, C) together with $2n$ tuning parameters in K and L.

Servo Problem

We can add a reference command r to the previous formulation:

$$u(k) = -L\hat{x}(k) + \ell_r r(k)$$

Then, the closed-loop system is

$$\begin{bmatrix} x(k+1) \\ \tilde{x}(k+1) \end{bmatrix} = \begin{bmatrix} \Phi - \Gamma L & \Gamma L \\ 0 & \Phi - KC \end{bmatrix} \begin{bmatrix} x(k) \\ \tilde{x}(k) \end{bmatrix} + \begin{bmatrix} \Gamma \ell_r \\ 0 \end{bmatrix} r(k)$$

$$y(k) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \tilde{x}(k) \end{bmatrix}$$

On input–output form:

$$
\begin{aligned}
y(k) &= \begin{bmatrix} C & 0 \end{bmatrix} \left(qI - \begin{bmatrix} \Phi - \Gamma L & \Gamma L \\ 0 & \Phi - KC \end{bmatrix} \right)^{-1} \begin{bmatrix} \Gamma \ell_r \\ 0 \end{bmatrix} r(k) \\
&= C(qI - \Phi + \Gamma L)^{-1} \Gamma \ell_r r(k)
\end{aligned}
$$

Note

- Observer does not influence the input–output relation

Polynomial Representation of Servo Problem

We can do the previous calculations in polynomial form:

$$y(k) = \frac{B(q)}{A(q)} u(k), \qquad u(k) = \frac{S(q)}{R(q)} (\ell_r r(k) - y(k))$$

gives

$$(A(q)R(q) + B(q)S(q))y(k) = B(q)S(q)\ell_r r(k)$$

If the desired closed-loop system with respect to $r(t)$ is $y(k) = [B_m(q)/A_m(q)]r(k)$:

$$\frac{B(q)S(q)\ell_r}{A(q)R(q) + B(q)S(q)} = \frac{B_m(q)}{A_m(q)}$$

which gives the control parameters R, S, ℓ_r

Next Lecture

Computer realization of controllers

- Approximation of continuous-time designs
- Digital PID structures
- Choice of sampling time