EL2450 Hybrid and Embedded Control

Lecture 2: Models of sampled systems

- Sampling of continuous-time systems
- State-space and input–output models
- Poles and zeros

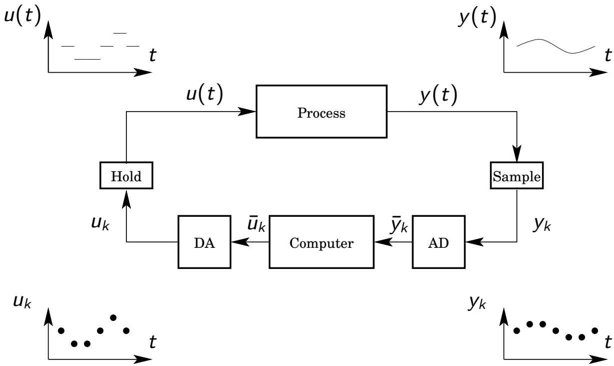
Today's Goal

You should be able to

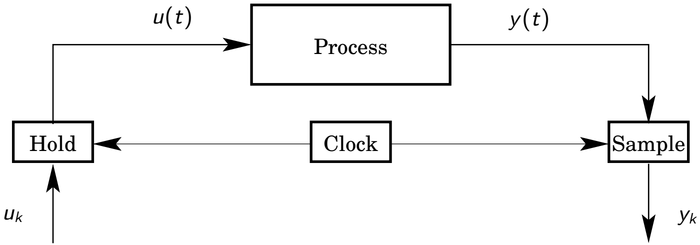
- Derive zero-order-hold sampling of systems
  (without and with delays in the loop)
- Relate SS and IO models
- Derive poles and zeros of discrete-time systems
  and relate them to their continuous-time counterparts

Computer-Controlled System

Consider the mapping from u_k to y_k:



Sampled Continuous-Time System



Process:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Sampling Continuous-Time System

For all $t \in [t_k, t_{k+1}]$

$$\begin{aligned} x(t) &= e^{A(t-t_k)}x(t_k) + \int_{t_k}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \\ &= e^{A(t-t_k)}x(t_k) + \int_{t_k}^{t} e^{A(t-\tau)}d\tau Bu(t_k) \\ &= e^{A(t-t_k)}x(t_k) + \int_{0}^{t-t_k} e^{As}dsBu(t_k), \end{aligned}$$

$$\begin{aligned} x(t_{k+1}) &= \Phi(t_{k+1}, t_k)x(t_k) + \Gamma(t_{k+1}, t_k)u(t_k) \\ y(t_k) &= Cx(t_k) + Du(t_k) \\ \Phi(t_{k+1}, t_k) &= e^{A(t_{k+1}-t_k)}, \quad \Gamma(t_{k+1}, t_k) = \int_{0}^{t_{k+1}-t_k} e^{As}dsB \end{aligned}$$

Note: Time-varying linear system

Uniform Sampling

Periodic sampling with $h > 0$ gives linear time-invariant system

$$\begin{aligned} x(kh+h) &= \Phi x(kh) + \Gamma u(kh) \\ y(kh) &= Cx(kh) + Du(kh) \end{aligned}$$

where

$$\Phi = e^{Ah}, \quad \Gamma = \int_0^h e^{As} ds B$$

Example

Sampling the system

$$\dot{x} = ax + bu, \quad a \neq 0$$

gives

$$x(kh + h) = \Phi x(kh) + \Gamma u(kh)$$

with

$$\Phi = e^{ah}, \quad \Gamma = \int_0^h e^{as} ds b = \frac{b}{a}(e^{ah} - 1)$$

Discrete-Time Systems

Suppose now on (if not otherwise stated) that $h = 1$:

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

With initial condition $x(0)$, the solution is

$$\begin{aligned} x(k) &= \Phi x(k-1) + \Gamma u(k-1) \\ &= \Phi^2 x(k-2) + \Phi\Gamma u(k-2) + \Gamma u(k-1) = \dots \\ &= \Phi^k x(0) + \sum_{j=0}^{k-1} \Phi^{k-j-1}\Gamma u(j) \end{aligned}$$

Discrete-Time Systems

With initial condition $x(0)$, the solution is

$$x(k) = \Phi^k x(0) + \sum_{j=0}^{k-1} \Phi^{k-j-1} \Gamma u(j)$$

First part depends on $x(0)$ and the second one on the input sequence. Solution properties related to eigenvalues of Φ, given by its *characteristic equation*

$$\det(\lambda I - \Phi) = 0$$

Changing Coordinates

T nonsingular matrix and $z = Tx$. Then

$$\begin{aligned} z(k+1) &= \tilde{\Phi} z(k) + \tilde{\Gamma} u(k) \\ y(k) &= \tilde{C} z(k) + \tilde{D} u(k) \end{aligned}$$

with $\tilde{\Phi} = T \Phi T^{-1}$,$\tilde{\Gamma} = T\Gamma$,$\tilde{C} = CT^{-1}$,$\tilde{D} = D$.
Key result:

$$\det(\lambda I - \Phi) = \det(\lambda I - \tilde{\Phi})$$

Coordinates can be chosen to give simple forms to system equations (revise canonical and special forms from basic control course).

Example: Diagonal Form

Φ has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Then there exists a T such that

$$T\Phi T^{-1} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Solution can be decoupled and has the form

$$z_i(k) = \lambda_i^k z_i(0) + \sum_{j=0}^{k-1} \lambda_i^{k-j-1} \beta_i u(j)$$

where $i = 1, \ldots, n$.

Example: Jordan Form

Φ has multiple eigenvalues, then generally not diagonalizable.
There exists a T such that

$$T \Phi T^{-1} = \begin{bmatrix} L_{k_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & L_{k_r}(\lambda_r) \end{bmatrix}$$

where $k_1 + \ldots + k_r = n$ and L_k is a $k \times k$ matrix with

$$L_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \ldots & 0 & \lambda & 1 \\ 0 & \ldots & 0 & 0 & \lambda \end{bmatrix}$$

Pulse Response

For linear time-invariant systems, input-output relation has the general form

$$y(k) = \sum_{m=0}^{k} h(k-m)u(m) + y_p(k)$$

where y_p relates to initial conditions. For zero initial conditions, $h(k-m)$ gives the output at k of a unit pulse injected at m. It is called *pulse-response function* of the system.

Pulse Response

We can derive the pulse response from the state-space model

$$y(k) = C\Phi^k x(0) + \sum_{j=0}^{k-1} C\Phi^{k-j-1}\Gamma u(j) + Du(k)$$

Hence, the response to a unit pulse at $k = 0$ with $x(0) = 0$ is

$$h(k) = \begin{cases} 0, & k < 0 \\ D, & k = 0 \\ C\Phi^{k-1}\Gamma, & k > 0 \end{cases}$$

Pulse-Transfer Operator

Introduce forward- and backward- shift operator q:
$qx(k) = x(k+1)$, $q^{-1}x(k) = x(k-1)$. Then,

$$qx(k) = \Phi x(k) + \Gamma u(k)$$

Hence,

$$y(k) = Cx(k) + Du(k) = \underbrace{\left[C(qI - \Phi)^{-1}\Gamma + D\right]}_{H(q)} u(k)$$

$H(q)$ is the *pulse-transfer operator*. $h(k), H(q)$ are also coordinate transformation invariant.

Pulse-Transfer Function

Similarly, with z-transform, $X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$, we obtain

$$z(X(z) - x(0)) = \Phi X(z) + \Gamma U(z)$$

so

$$\begin{aligned} Y(z) &= CX(z) + DU(z) \\ &= \underbrace{\left[C(zI - \Phi)^{-1}\Gamma + D\right]}_{H(z)} U(z) + C(zI - \Phi)^{-1} z x(0) \end{aligned}$$

$H(z) = \mathcal{Z}\{h(k)\}$ and is called the *pulse-transfer function*.

Note: $H(z)$ is equal to $H(q)$ with q replaced by z. Still they are two different mathematical objects. (Why?)

State-Space and Input–Output Models

Consider a state-space system of order n:

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

Then, $y(k) = H(q)u(k) = \dfrac{B(q)}{A(q)}u(k)$, where

$$A(q) = q^n + a_1 q^{n-1} + \cdots + a_n, \quad B(q) = b_0 q^n + b_1 q^{n-1} + \cdots + b_n$$

or equivalently

$$\begin{aligned} & y(k+n) + a_1 y(k+n-1) + \cdots + a_n y(k) \\ & \qquad = b_0 u(k+n) + b_1 u(k+n-1) + \cdots + b_n u(k) \end{aligned}$$

Sampled Double Integrator

What is $H(q)$ for a ZOH sampled $G(s) = 1/s^2$ with $h = 1$?

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$$\Phi = e^{Ah} = I + A + A^2/2 + \cdots = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

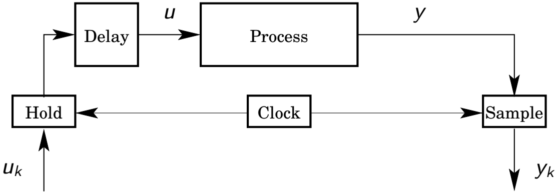
$$\Gamma = \int_0^h \begin{bmatrix} s \\ 1 \end{bmatrix} ds = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

$$\begin{aligned} H(q) &= C(qI - \Phi)^{-1}\Gamma = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q-1 & -1 \\ 0 & q-1 \end{bmatrix}^{-1} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \\ &= \frac{q+1}{2(q-1)^2} \end{aligned}$$

Note: Table 2 on page 22 in [WAA] gives $H(q) \overset{\text{ZOH samp}}{\longleftrightarrow} G(s)$

Sampling System with Time-Delay

Delayed actuation with fixed delay $\tau \in (0, h)$



Then, $\dot{x}(t) = Ax(t) + Bu(t - \tau)$. $u(t)$ changes at $kh + \tau$, so

$$
\begin{aligned}
x(kh+h) &= e^{Ah}x(kh) + \int_{kh}^{kh+h} e^{A(kh+h-s)}Bu(s-\tau)ds \\
&= e^{Ah}x(kh) + \int_{kh}^{kh+\tau} e^{A(kh+h-s)}ds\, Bu(kh-h) + \int_{kh+\tau}^{kh+h} e^{A(kh+h-s)}ds\, Bu(kh)
\end{aligned}
$$

$$x(kh+h) = \Phi x(kh) + \Gamma_0 u(kh) + \Gamma_1 u(kh-h)$$

$$\Phi = e^{Ah}, \quad \Gamma_0 = \int_0^{h-\tau} e^{As} ds B$$

$$\Gamma_1 = e^{A(h-\tau)} \int_0^{\tau} e^{As} ds B$$

Hence,

$$\begin{bmatrix} x(kh+h) \\ u(kh) \end{bmatrix} = \begin{bmatrix} \Phi & \Gamma_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(kh) \\ u(kh-h) \end{bmatrix} + \begin{bmatrix} \Gamma_0 \\ I \end{bmatrix} u(kh)$$

Note: Still LTI system, but of one order higher

Sampling System with Larger Time-Delay

What if the delay τ is larger than h?
Then, the previous approach still works, but needs a modification:
Example Suppose $\tau \in (h, 2h)$. Then,

$$\begin{bmatrix} x(kh+h) \\ u(kh-h) \\ u(kh) \end{bmatrix} = \begin{bmatrix} \Phi & \Gamma_1 & \Gamma_0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(kh) \\ u(kh-2h) \\ u(kh-h) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} u(kh)$$

for suitable Γ_1, Γ_0

Frequency Response

Recall that the frequency response of a continuous-time system $G(s)$ is given by $G(i\omega)$ for $\omega \in [0, \infty)$.
The frequency response of a sampled-data system $H(z)$ is given by $H(e^{i\omega h})$ for $\omega h \in [0, \pi]$.
Note that for $H(z)$ it suffices to study ω up to ω_N.

Poles and Zeros

- Poles are the zeros of $A(z)$
- Zeros are the zeros of $B(z)$

Equivalently

- Poles are the eigenvalues of Φ
- Zeros are $z \in \mathbf{C}$ such that $\det \begin{bmatrix} zI - \Phi & -\Gamma \\ C & D \end{bmatrix} = 0$

Physical interpretation:
Pole in p gives *mode* p^k in time response
Zero in a gives *blocking* of inputs a^k

Example: Blocking Zeros

$y(k+1) - ay(k) = u(k+1) - bu(k)$, $0 < a < 1 < b$, corresponds to

$$Y(z) = H(z)U(z) = \frac{z-b}{z-a}U(z)$$

$u(k) = b^k$ gives

$$Y(z) = \frac{z}{z-a} \overset{\mathcal{Z}^{-1}}{\rightarrow} y(k) = a^k$$

Hence, $y(k) \to 0$ and does not depend on $u(k)$.
The zero of $H(z)$ in b thus *blocks* the input $u(k) = b^k$.

Blocking Zeros

Consider

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

A zero in a means *blocking* of inputs a^k. Then for the system above, the input a^k gives zero output for $z = a$ such that $\det \begin{bmatrix} zI - \Phi & -\Gamma \\ C & D \end{bmatrix} = 0$.

Pole Locations of Sampled System

Suppose

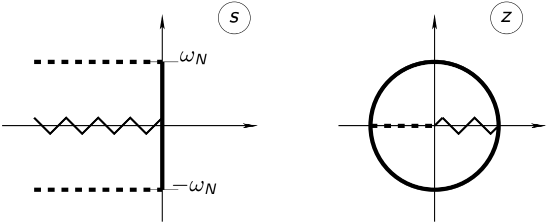
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

has poles $\lambda_i(A)$, $i = 1, \ldots, n$. Then,

$$x(kh + h) = \Phi x(kh) + \Gamma u(kh)$$
$$y(kh) = Cx(kh) + Du(kh)$$

has poles $\exp(\lambda_i(A)h)$. Follows from $\Phi = \exp(Ah)$.

Pole Mapping Through $z = \exp(sh)$



The mapping $s \mapsto \exp(sh)$ is not invertible. Several points of the s-plane are mapped into the z-plane.

The pole $z = 0$ has no counterpart in s-plane.

The left half plane of the s-domain is mapped into the unit disc of the z-plane. What might this imply?

Zero Locations of Sampled System

Zeros in s-domain is not easily mapped to z-domain.
Example:
$G(s) = \dfrac{1}{s^2}$ is sampled into $H(q) = \dfrac{q+1}{2(q-1)^2}$.
$G(s)$ has no zero, but $H(q)$ has zero in $z = -1$.
The sampling procedure gives more zeros.

Next Lecture

Analysis of sampled control

- Stability
- Reachability and observability
- Observers
- State and output feedback