(a) [3p] Consider the transition systems $\mathcal{T}_1$ (left), $\mathcal{T}_2$ (right) in Figure 1. Are they bisimilar? Justify your answer.

(b) [3p] Find the minimal quotient transition system which is bisimilar to transition system $\mathcal{T}_1$ (left) in Figure 1.

(c) [2p] Construct a transition system which is bisimilar to $\mathcal{T}_3$ in Fig. 2 (left) and has no self loops. Provide the bisimulation relation and justify your answer.

(d) [1p] Draw a transition system which is bisimilar to $\mathcal{T}_4$ in Fig. 2 (center) and has no self loops.

(e) [1p] Does there exist a *finite* transition system which simulates $\mathcal{T}_5$ in Fig. 2 (right) and has *at most one* outgoing transition from each state?

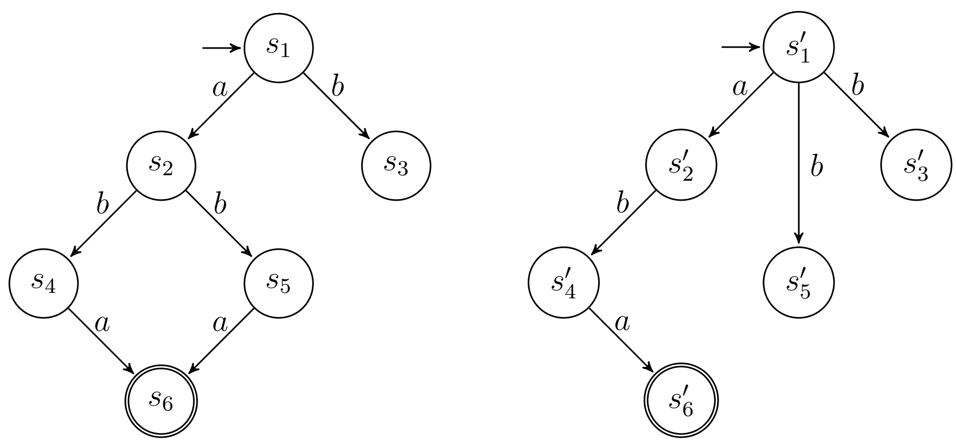


Figure 1: Transition systems $\mathcal{T}_1$ (left) and $\mathcal{T}_2$ (right).

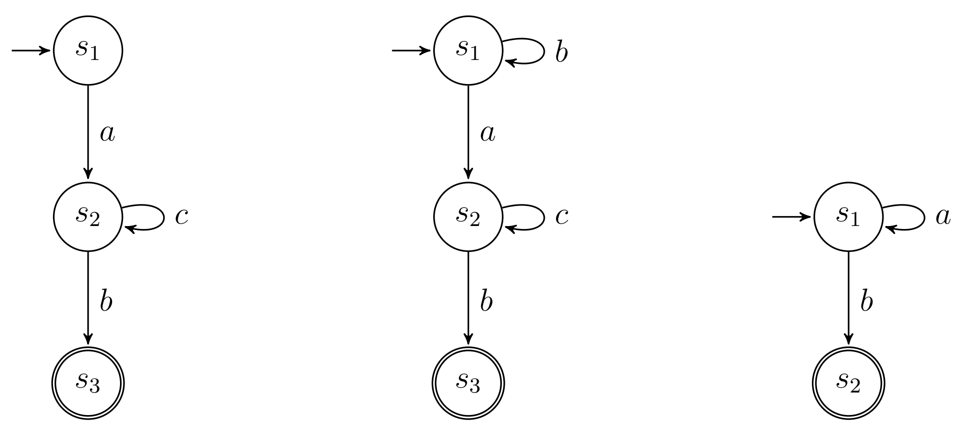


Figure 2: Transition systems $\mathcal{T}_3$ (left), $\mathcal{T}_4$ (center), and $\mathcal{T}_5$ (right).

Answers:

(a) Yes they are bisimilar with the relation $\sim \subset S_1 \times S_2$

$$\sim = \{(s_1, s_1'), (s_2, s_2'), (s_3, s_3'), (s_3, s_5'), (s_4, s_4'), (s_5, s_4'), (s_6, s_6')\},$$

which guarantees that both $\mathcal{T}_1 \sim \mathcal{T}_2$ and $\mathcal{T}_2 \sim' \mathcal{T}_1$, where

$$\sim' = \sim^{-1} = \{(s_1', s_1), (s_2', s_2), (s_3', s_3), (s_5', s_3), (s_4', s_4), (s_4', s_5), (s_6', s_6)\}.$$

Indeed we verify that $\mathcal{T}_1 \sim \mathcal{T}_2$ by checking the requirements of a simulation relation one by one:

1. *Initial states.* System $\mathcal{T}_1$ has initial state set $\{s_1\}$ and for s_1, state s_1' is an initial state of $\mathcal{T}_2$ with $s_1 \sim s_1'$.
2. *Final states.* System $\mathcal{T}_1$ has final state set $\{s_6\}$, and for the only related state s_6', i.e., with $s_6 \sim s_6'$, it holds that s_6' is a final state of $\mathcal{T}_2$.
3. *Matching transitions.* We finally check that every transition in $\mathcal{T}_1$ has a matching transition in $\mathcal{T}_2$.
   - For $s_1 \sim s_1'$ and $s_1 \xrightarrow{a} s_2$ we have that $s_1' \xrightarrow{a} s_2'$ and $s_2 \sim s_2'$.
   - For $s_1 \sim s_1'$ and $s_1 \xrightarrow{b} s_3$ we have that $s_1' \xrightarrow{b} s_3'$ and $s_3 \sim s_3'$. (or $s_1' \xrightarrow{b} s_5'$ and $s_3 \sim s_5'$).
   - For $s_2 \sim s_2'$ and $s_2 \xrightarrow{b} s_4$ we have that $s_2' \xrightarrow{b} s_4'$ and $s_4 \sim s_4'$.
   - For $s_2 \sim s_2'$ and $s_2 \xrightarrow{b} s_5$ we have that $s_2' \xrightarrow{b} s_4'$ and $s_5 \sim s_4'$.
   - For $s_4 \sim s_4'$ and $s_4 \xrightarrow{a} s_6$ we have that $s_4' \xrightarrow{a} s_6'$ and $s_6 \sim s_6'$.
   - For $s_5 \sim s_4'$ and $s_5 \xrightarrow{a} s_6$ we have that $s_4' \xrightarrow{a} s_6'$ and $s_6 \sim s_6'$.

   Analogously it is shown that $\mathcal{T}_2 \sim' \mathcal{T}_1$.

(b) We use the bisimulation quotient algorithm starting from the initial coarsest partition $\{P_1, P_2\}$ with $P_1 = \{s_1, s_2, s_3, s_4, s_5\}$ and $P_2 = \{s_6\}$. Finally, since $\text{Pre}_a(P_2) = \{s_4, s_5\}$ and $\emptyset \neq \text{Pre}_a(P_2) \neq P_1$ we split P_1 into $P_{11} = \text{Pre}_a(P_2) = \{s_4, s_5\}$ and $P_{12} = P_1 \setminus \text{Pre}_a(P_2) = \{s_1, s_2, s_3\}$. Next, since $\text{Pre}_a(P_{12}) = \{s_1\}$ and $\emptyset \neq \text{Pre}_a(P_{12}) \neq P_{12}$ we split P_{12} into $P_{121} = \text{Pre}_a(P_{12}) = \{s_1\}$ and $P_{122} = P_{12} \setminus \text{Pre}_a(P_{12}) = \{s_2, s_3\}$. Then, since $\text{Pre}_b(P_{11}) = \{s_2\}$ and $\emptyset \neq \text{Pre}_b(P_{11}) \neq P_{122}$ we split P_{122} into $P_{1221} = \text{Pre}_b(P_{11}) = \{s_2\}$ and $P_{1222} = P_{122} \setminus \text{Pre}_b(P_{11}) = \{s_3\}$, and obtain

$$S_{/\sim} = \{P_{11}, P_{121}, P_{1221}, P_{1222}, P_2\},$$

with

$$P_{11} = \{s_4, s_5\} \quad P_{121} = \{s_1\} \quad P_{1221} = \{s_2\} \quad P_{1222} = \{s_3\} \quad P_2 = \{s_6\}.$$

We next check if the algorithm is finished, i.e., that we cannot refine more. The only element which is not a singleton is P_{11}. Thus, it suffices to show that for all $P' \in S_{/\sim}$ and $\sigma \in \{a, b\}$ it holds that $P_{11} \cap \text{Pre}_\sigma(P') = \emptyset$ or P_{11}. Since only $P_{11} \cap \text{Pre}_a(P_2) \neq \emptyset$, and additionally $P_{11} \cap \text{Pre}_a(P_2) = P_{11}$ we have obtained the desired quotient system shown in Figure 3, below.

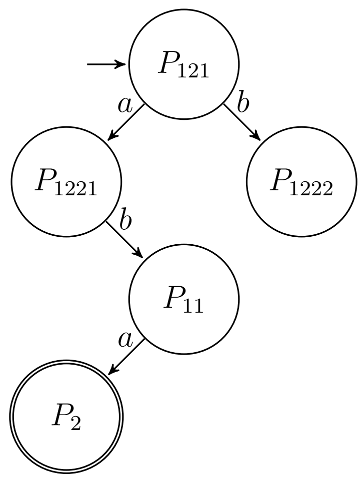


Figure 3: Bisimulation quotient of $\mathcal{T}_1$.

(c) One candidate bisimilar transition systems is $\mathcal{T}_3'$ in Figure 4 (left) (or its alternative in Figure 5). We therefore exploit the relation $\sim \subset S_3 \times S_3'$

$$\sim = \{(s_1, s_1'), (s_2, s_2'), (s_2, s_2''), (s_3, s_3')\},$$

which guarantees that both $\mathcal{T}_3 \sim \mathcal{T}_3'$ and $\mathcal{T}_3' \sim' \mathcal{T}_3$, where

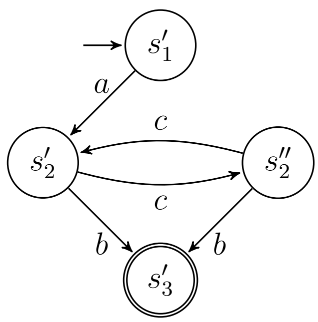
$$\sim' = \sim^{-1} = \{(s_1', s_1), (s_2', s_2), (s_2'', s_2), (s_3', s_3)\}.$$

We verify that $\mathcal{T}_3 \sim \mathcal{T}_3'$ by checking the requirements of a simulation relation one by one:

1. *Initial states.* System $\mathcal{T}_3$ has initial state set $\{s_1\}$ and for s_1, state s_1' is an initial state of $\mathcal{T}_3'$ with $s_1 \sim s_1'$.
2. *Final states.* System $\mathcal{T}_3$ has final state set $\{s_3\}$, and for the only related state s_3', i.e., with $s_3 \sim s_3'$, it holds that s_3' is a final state of $\mathcal{T}_3'$.
3. *Matching transitions.* We finally check that every transition in $\mathcal{T}_3$ has a matching transition in $\mathcal{T}_3'$.
   - For $s_1 \sim s_1'$ and $s_1 \xrightarrow{a} s_2$ we have that $s_1' \xrightarrow{a} s_2'$ and $s_2 \sim s_2'$. (or $s_1' \xrightarrow{a} s_2''$ and $s_2 \sim s_2''$ in the case of Figure 5).
   - For $s_2 \sim s_2'$ and $s_2 \xrightarrow{c} s_2$ we have that $s_2' \xrightarrow{c} s_2''$ and $s_2 \sim s_2''$.
   - For $s_2 \sim s_2''$ and $s_2 \xrightarrow{c} s_2$ we have that $s_2'' \xrightarrow{c} s_2'$ and $s_2 \sim s_2'$.
   - For $s_2 \sim s_2'$ and $s_2 \xrightarrow{b} s_3$ we have that $s_2' \xrightarrow{b} s_3'$ and $s_3 \sim s_2'$.
   - For $s_2 \sim s_2''$ and $s_2 \xrightarrow{b} s_3$ we have that $s_2'' \xrightarrow{b} s_3'$ and $s_3 \sim s_3'$.

   Analogously it is shown that $\mathcal{T}_3' \sim' \mathcal{T}_3$.

(d) Such a transition system is $\mathcal{T}_4'$ in Figure 4 (right) or its alternative with the same states in Figure 5.



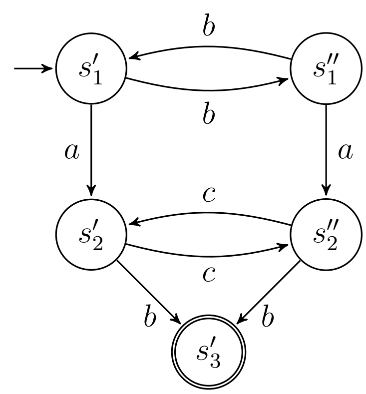
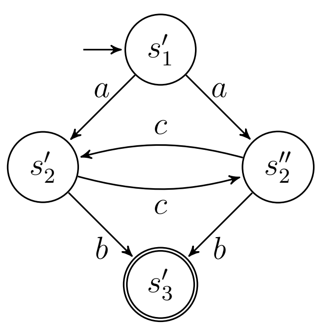


Figure 4: Transition systems $\mathcal{T}_3'$ (left) and $\mathcal{T}_4'$ (right).



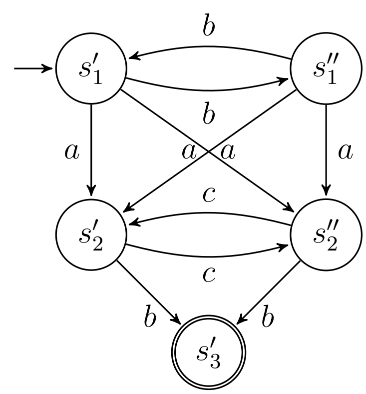


Figure 5: Alternatives for transition systems $\mathcal{T}_3'$ (one with the same states) and $\mathcal{T}_4'$ (one out of eight with the same states).

(e) No, such a system does not exist. Assuming on the contrary that such a system $\mathcal{T}_5'$ exists, it will have $k \in \mathbb{N}$ states. Consider next the sequence of transitions

$$\underbrace{s_1 \xrightarrow{a} s_1 \xrightarrow{a} \cdots \xrightarrow{a} s_1}_{k-1 \text{ transitions}} \xrightarrow{b} s_2$$

in system $\mathcal{T}_5$. Then, in order for $\mathcal{T}_5'$ to match these transitions it needs to have at least $k+1$ states, as shown in Figure 6, below. (No reason for a more rigorous explanation)

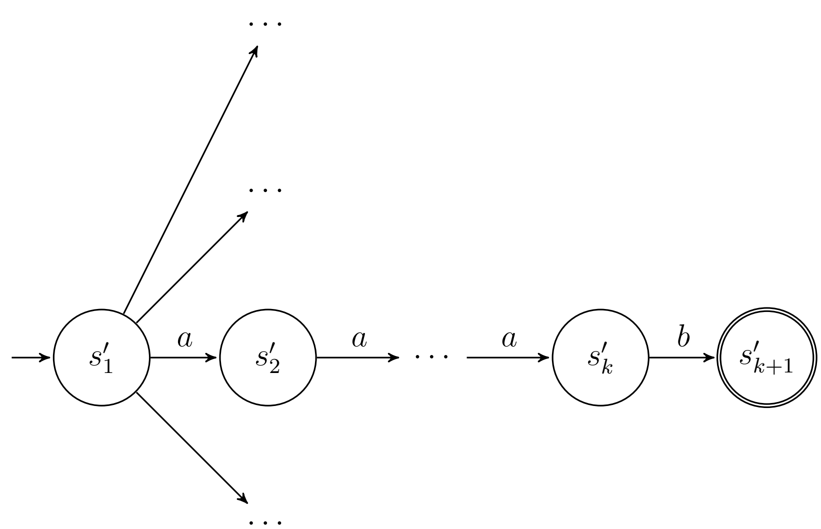


Figure 6: Transition system $\mathcal{T}_5'$.