1 Problem

Consider the following system:

$$\dot{x}_1(t) = -x_1(t) + x_2^2(t), \tag{1a}$$

$$\dot{x}_2(t) = -x_1(t)x_2(t) + u(t), \tag{1b}$$

with x_1, $x_2 \in \mathbb{R}$, $u \in \mathbb{R}$ and $t \geq 0$.

(a) [1p] Suppose that the linear feedback control law $u(t) = -\gamma x_2(t)$ with $\gamma > 0$ is designed for the system. Let $x(t) = [x_1(t), x_2(t)]^\top \in \mathbb{R}^2$ and consider the Lyapunov function candidate $V(x) = \frac{1}{2}\|x\|^2$. Prove that under the aforementioned control law, the system converges to origin asymptotically.

(b) [1p] In order to implement the controller on a digital platform, the state of the system is sampled *aperiodically* at a sequence of time instants $\{t_k\}$, $k \in \mathbb{N}$, and the control signal is now given by:

$$u(t) = -\gamma x_2(t_k), \quad t \in [t_k, t_{k+1}). \tag{2}$$

Write the closed loop equations of the system in terms of the states $x_1(t)$, $x_2(t)$ and the state error $e_2(t)$, where $e_2(t) = x_2(t_k) - x_2(t)$, $t \in [t_k, t_{k+1})$.

(c) [2p] Let $e(t) = [0, e_2(t)]^\top \in \mathbb{R}^2$. By using the same Lyapunov function candidate as in (a) find a relation between the norms $\|e(t)\|$ of the error and $\|x(t)\|$ of the state such that the event-based control law (2) renders the origin asymptotically stable. Determine also the event-triggered condition under which the control updates are calculated. *Useful property:* $\alpha_1 z_1 + \alpha_2 z_2 \geq \min\{\alpha_1, \alpha_2\}(z_1 + z_2)$ *for all* α_1, α_2, z_1, $z_2 > 0$.

In the following tasks, you are going to show that the closed-loop system with the event-triggered condition that you have computed does not exhibit Zeno behavior. For the rest of this exercise, consider $t \in [t_k, t_{k+1})$.

(d) [2p] Observing that $\dot{e}_2(t) = -\dot{x}_2(t)$, write an upper bound for $\dot{e}_2(t)$ that depends only on $\|x(t)\|$ and on constant factors. Using the property that $\|x(t)\|$ is monotonically decreasing, rewrite the upper bound so that it depends only on $\|x(t_k)\|$ and on constant factors. Then, observing that $e_2(t) = \int_{t_k}^{t} \dot{e}_2(\tau)\,\mathrm{d}\tau$, write an upper bound for $e_2(t)$ that depends only on $\|x(t_k)\|$, $(t - t_k)$ and on constant factors.

(e) [2p] Observing that $\frac{\mathrm{d}\|x(t)\|}{\mathrm{d}t} = \frac{x(t)^\top \dot{x}(t)}{\|x(t)\|}$, and using the property that $\|x(t)\|$ is monotonically decreasing, write a lower bound for $\frac{\mathrm{d}\|x(t)\|}{\mathrm{d}t}$ that depends only on $\|x(t_k)\|$. Then, observing that $\|x(t)\| = \|x(t_k)\| + \int_{t_k}^{t} \frac{\mathrm{d}\|x(\tau)\|}{\mathrm{d}\tau}\,\mathrm{d}\tau$, write a lower bound for $\|x(t)\|$ that depends only on $\|x(t_k)\|$, $(t - t_k)$ and on constant factors.

(f) [2p] Using the results in (d) and (e), find a lower bound for $(t - t_k)$ to violate the condition that you have computed in (d). Your lower bound should only depend on initial conditions (i.e., $x(0)$) and constant parameters. Conclude that the closed-loop system does not exhibit Zeno behavior.

(a) Under the given control law, the closed loop system is written as:

$$\dot{x}_1(t) = -x_1(t) + x_2^2(t), \tag{3a}$$
$$\dot{x}_2(t) = -x_1(t)x_2(t) - \gamma\, x_2(t). \tag{3b}$$

By computing the time derivative of V along the trajectories of the system (3) we get:

$$\begin{aligned} \dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1(-x_1 + x_2^2) + x_2(-x_1x_2 - \gamma\, x_2) \\ &= -x_1^2 + x_1^2x_2 - x_1^2x_2 - \gamma\, x_2^2 \\ &= -x_1^2 - \gamma\, x_2^2. \end{aligned} \tag{4}$$

Thus, we have that $\dot{V} < 0$ which implies that the system is asymptotically stable.

(b) By substituting $u(t) = -\gamma x_2(t_k) = -\gamma\,[e_2(t) + x_2(t)]$ in (1) we closed loop system:

$$\dot{x}_1(t) = -x_1(t) + x_2^2(t), \tag{5a}$$
$$\dot{x}_2(t) = -x_1(t)x_2(t) - \gamma e_2(t) - \gamma x_2(t). \tag{5b}$$

(c) By computing the time derivative of V along the trajectories of the system (5) is written as:

$$\begin{aligned} \dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1(-x_1 + x_2^2) + x_2(-x_1x_2 - \gamma e_2 - \gamma x_2) \\ &= -x_1^2 + x_1^2x_2 - x_1^2x_2 - x_2\gamma e_2 - \gamma x_2^2 \\ &= -x_1^2 - \gamma x_2^2 - \gamma x_2 e_2 \\ &\le -\min\{1,\gamma\}\|x\|^2 + \gamma|x_2||e_2| \\ &\le -\min\{1,\gamma\}\|x\|^2 + \gamma\|x\|\|e\| \\ &= \gamma\|x\|\left[-\tfrac{\min\{1,\gamma\}}{\gamma}\|x\| + \|e\|\right]. \end{aligned} \tag{6}$$

Thus, the required condition that renders $\dot{V} < 0$ is:

$$\|e\| \le \sigma\|x\|, \sigma \in \left(0, \tfrac{\min\{1,\gamma\}}{\gamma}\right).$$

Therefore, under the aperiodic control law (2) the closed loop system is asymptotically stable. The event-triggered condition is given by:

$$t_{k+1} = \inf_t \{t > t_k : \|e(t)\| = \sigma\|x(t)\|\}.$$

(d) We can compute

$$\begin{aligned} \dot{e}_2(t) = -\dot{x}_2(t) &= -x_1(t)x_2(t) + u(t) \\ &= -x_1(t)x_2(t) - \gamma x_2(t_k) \\ &\le \|x(t)\|\|x(t)\| + \gamma\|x(t_k)\| \\ &\le \|x(t_k)\|^2 + \gamma\|x(t_k)\|. \end{aligned} \tag{7}$$

Simply integrate (7) to have

$$|e_2(t)| \le \|x(t_k)\|\,[\|x(t_k)\| + \gamma]\,(t - t_k).$$

(e) We can compute

$$\begin{aligned}
\frac{\mathrm{d}\|x(t)\|}{\mathrm{d}t} &= \frac{1}{\|x(t)\|}\left[-x_1^2(t) + x_2(t)u(t)\right] \\
&= \frac{1}{\|x(t)\|}\left[-x_1^2(t) - \gamma x_2(t)x_2(t_k)\right] \\
&\geq \frac{1}{\|x(t)\|}\left[-\|x(t)\|^2 - \gamma\|x(t)\|\|x(t_k)\|\right] \\
&\geq -\|x(t)\| - \gamma\|x(t_k)\| \\
&\geq -(1+\gamma)\|x(t_k)\|.
\end{aligned} \tag{8}$$

Simply integrate (8) to have

$$\begin{aligned}
\|x(t)\| &\geq \|x(t_k)\| - (1+\gamma)\|x(t_k)\|(t-t_k) \\
&= \|x(t_k)\|\left[1-(1+\gamma)(t-t_k)\right].
\end{aligned} \tag{9}$$

(f) From (d) and (e), we can see that to violate the condition $\|e(t)\| < \sigma\|x(t)\|$ it is necessary that

$$\|x(t_k)\|\left[\|x(t_k)\| + \gamma\right](t-t_k) \geq \sigma\|x(t_k)\|\left[1-(1+\gamma)(t-t_k)\right].$$

Solving for $(t - t_k)$, we have

$$(t-t_k) \geq \frac{\sigma}{\|x(t_k)\| + \gamma + \gamma\sigma + \sigma}\ .$$

Using the monotonicity of $\|x(t)\|$, we have

$$(t-t_k) \geq \frac{1}{\|x(t_0)\| + \gamma + \gamma\sigma + \sigma} > 0.$$

Since this lower bound is positive, there cannot be accumulation points of the update times t_k. Hence, the closed-loop system does not exhibit Zeno behavior.