EL2450 Hybrid and Embedded Control

Lecture 10: Stability of hybrid systems

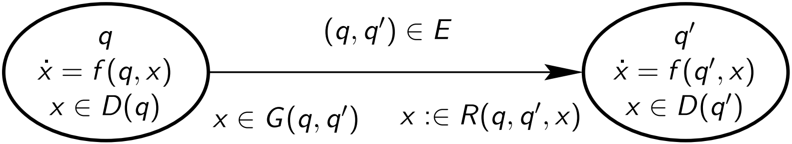
- Stability of hybrid systems
- Stability criteria for hybrid systems

Today's Goal

You should be able to

- define switched systems as a subclass of hybrid systems
- analyze stability of switched systems
- distinguish between arbitrary and constrained switching

Hybrid Automaton $H = (Q, X, \mathrm{Init}, f, D, E, G, R)$



- Q discrete state space and X continuous state space
- $\mathrm{Init} \subseteq Q \times X$ initial states
- $f : Q \times X \to X$ vector fields
- $D : Q \to 2^X$ domains
- $E \subset Q \times Q$ edges
- $G : E \to 2^X$ guards
- $R : E \times X \to 2^X$ resets

Switched Systems

Abstracted model of hybrid systems where continuous part is highlighted more

- $f_q, q \in Q$ family of functions
- $\dot{x} = f_q(x), q \in Q$ family of systems
- Example: $f_q(x) = A_q x$ linear switched systems

Switching signals

- Switched system $\dot{x}(t) = f_{\sigma(t)}(x(t))$ defined based on *switching signal* $\sigma : [0, \infty) \to Q$
- Discontinuities of $\sigma(t)$: switching times
- $\sigma(t) = \lim_{\tau \to t^+}$ for all $t \geq 0$.

Switching can be state- or time-dependent

Switched Systems

Let Ω_q, $q = 1, \ldots, m$ denote a partition of the continuous state space $\mathbb{R}^n$. A state-dependent switched system is given by

$$\dot{x} = f_q(x), \qquad x \in \Omega_q$$

Example

$x \in \mathbb{R}^2$, Ω_q quadrant q, $q = 1, \ldots, 4$, and

$$\begin{aligned} \dot{x} &= A_q x \\ x &\in \Omega_q \end{aligned}$$

Switched System as Hybrid Automaton

$$\dot{x} = f_q(x), \qquad x \in \Omega_q$$

corresponds to the hybrid automaton

- $Q = \{1, \ldots, m\}$, $X = \mathbb{R}^n$, Init $\subset \{q\} \times \Omega_q$
- $f(q, x) = f_q(x)$
- $D(q) = \Omega_q$
- $(q, q') \in E$ if $D(q)$ to $D(q')$ are "neighbors" (i.e., $\overline{D(q)} \cap \overline{D(q')} \neq \emptyset$) and there are solutions that go from $D(q)$ to $D(q')$
- $G(q, q') = \overline{D(q)} \cap \overline{D(q')}$
- $R(q, q', x) = \{x\}$

Stability

A solution x^* of a **switched system** is stable if for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that for all solutions x

$$\|x(0) - x^*(0)\| < \delta \quad \Rightarrow \quad \|x(t) - x^*(t)\| < \epsilon, \quad \forall t > 0$$

The solution is *asymptotically stable* if it is stable and

$$\|x(0) - x^*(0)\| < \delta \quad \Rightarrow \quad \|x(t) - x^*(t)\| \to 0, \text{ as } t \to \infty$$

- Can be generalized to hybrid automata
- Assumes existence of solutions that expand to infinite time (Zeno is excluded)

Lyapunov's Second Method

Let $x^* = 0$ be an equilibrium point of $\dot{x} = f(x)$. If there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$
\begin{aligned}
&V(0) = 0 \\
&V(x) > 0, \;\; \forall x \in \mathbb{R}^n \setminus \{0\} \\
&\dot{V}(x) \leq (<) 0, \;\; \forall x \in \mathbb{R}^n,
\end{aligned}
$$

then x^* is (asymptotically) stable

Lyapunov Function for Linear System

Real $\lambda_i(A) < 0$ for all i if and only if for every positive definite $Q = Q^T$ there exists a positive definite $P = P^T$ such that

$$PA + A^T P = -Q$$

A Lyapunov function for a linear system

$$\dot{x} = Ax$$

is given by

$$V(x) = x^T P x$$

In particular,

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x = -x^T Q x < 0$$

Example

$$\dot{x} = A_1 x = \begin{pmatrix} -1 & 10 \\ -100 & -1 \end{pmatrix} x$$

Then,

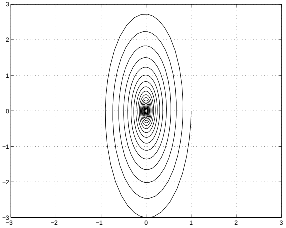
$$P = [\texttt{lyap} \text{ in Matlab}] = \begin{pmatrix} 0.2752 & -0.0225 \\ -0.0225 & 2.7478 \end{pmatrix}$$

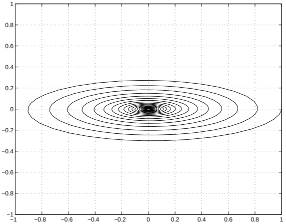
solves the Lyapunov equation $A_1 P + P A_1^T = -I$. Then, $V = x^T P x$ fulfills the three conditions in the Lyapunov theorem (check!). Hence, $x^* = 0$ is stable.

Note that $\lambda(A_1) = -1 \pm i10\sqrt{10}$

$$\dot{x} = A_1 x = \begin{pmatrix} -1 & 10 \\ -100 & -1 \end{pmatrix} x, \qquad \dot{x} = A_2 x = \begin{pmatrix} -1 & 100 \\ -10 & -1 \end{pmatrix} x$$

have the following phase portraits:



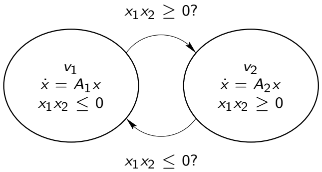


Note

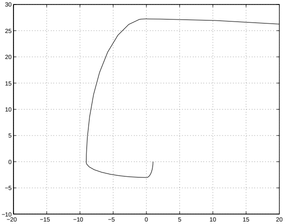
- Both systems are stable with $\lambda(A_j) = -1 \pm i10\sqrt{10}$, $j = 1, 2$
- $\exists$ Lyapunov function for each system
- What can we say about stability of linear switched system based on stability of individual subsystems?
- Answer: not much...

Example: Stable+Stable=Unstable

Consider switched system corresponding to hybrid automaton:

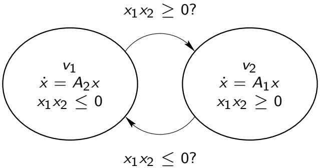


Even if A_1 and A_2 are stable, the switched system is unstable:

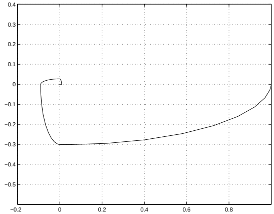


Example: Stable+Stable=Stable

Let A_1 and A_2 change place:



Then, also the switched system is stable:



Some Conclusions

- Arbitrary switching ($\sigma(t)$ can be chosen arbitrary) may destabilize a switched system even if all subsystems are stable
- Even if all subsystems are unstable, it may be still possible to stabilize the switched system by constraining the switching sequence appropriately

- Stability under arbitrary switching
- Stability under constrained switching. How to define switching sequence appropriately?

Arbitrary Switching

- In some cases σ can be chosen arbitrary and still stabilize the system, ie, the switched system is stable for any σ
- Assumption: All subsystems (asymptotically) stable. Necessary condition, why?
- Can be analyzed through Lyapunov method
- Idea: consider a single energy function for the whole system and for all possible switching sequences

Common Lyapunov Function

Consider the system

$$\dot{x} = A_\sigma x$$

where $\sigma : [0, \infty) \to \{1, \ldots, m\}$ is an arbitrary switching sequence. If there exists $P, Q > 0$ (positive definite), such that

$$PA_q + A_q^T P \leq -Q, \qquad q = 1, \ldots, m$$

then the origin is asymptotically stable

- $V(x) = x^T P x$ is a common quadratic Lyapunov function for all systems $\dot{x} = A_q x$
- Equivalent condition: there exists $P, Q_q > 0$ such that $PA_q + A_q^T P = -Q_q, \qquad q = 1, \ldots, m$

Commuting System Matrices

Consider the system $\dot{x} = A_\sigma x$, where $\sigma : [0, \infty) \to \{1, \ldots, m\}$ is an arbitrary switching sequence. If all A_q are stable and

$$A_k A_\ell = A_\ell A_k, \qquad k, \ell \in \{1, \ldots, m\}$$

then the origin is stable.

Proof for $m = 2$: If $A_1 A_2 = A_2 A_1$ then $\exp A_1 \exp A_2 = \exp A_2 \exp A_1$ (why?). Then, for time trajectory τ and $t \in [\tau_i, \tau_i']$,

$$\begin{aligned} x(t) &= \exp[A_1(t - \tau_i)] \exp[A_2(\tau_{i-1}' - \tau_{i-1})] \cdots \exp[A_1(\tau_0' - \tau_0)] x_0 \\ &= \exp\left[A_1[(t - \tau_i) + \cdots + (\tau_0' - \tau_0)]\right] \\ &\qquad \times \exp\left[A_2[(\tau_{i-1}' - \tau_{i-1}) + \cdots + (\tau_1' - \tau_1)]\right] x_0 \end{aligned}$$

Stability follows from that A_1 and A_2 are stable.

Switched Nonlinear Systems

Consider the system

$$\dot{x} = f_\sigma(x)$$

where $\sigma : [0, \infty) \to \{1, \ldots, m\}$ is an arbitrary switching sequence and $f_q(0) = 0, q \in \{1, \ldots, m\}$. If there exists a positive definite $V : \mathbb{R}^n \to \mathbb{R}$, such that

$$\frac{\partial V}{\partial x} f_q(x) < 0, \forall x \neq 0, \forall q \in \{1, \ldots, m\}$$

then the origin is asymptotically stable

Commutation Relations for Switched Nonlinear Systems

- The Lie bracket of two vector fields f_1, f_2 is defined as

$$[f_1, f_2](x) := \frac{\partial f_2(x)}{\partial x} f_1(x) - \frac{\partial f_1(x)}{\partial x} f_2(x)$$

  We say that the vector fields commute if their Lie bracket is identically zero.

- Assume that $f_q, q \in \{1, \ldots, m\}$ is a finite set of commuting vector fields and all subsystems have an asymptotically stable equilibium at the origin. Then the switched system $\dot{x} = f_\sigma(x)$ is asymptotically stable for any switching sequence.

Constrained Switching

- Stability under arbitrary switching does not hold in general (see earlier example).
- Choice of σ is constrained due to state and/or time dependencies. How to examine stability?
- How to derive appropriate σ that guarantees stability?

Multiple Lyapunov Functions

Suppose $x^* = 0$ is an equilibrium of each mode $q = 1, \ldots, m$ of the switched system

$$\dot{x} = f_\sigma(x), \quad \sigma : [0, \infty) \to \{1, \ldots, m\}$$

If there exist functions $V_1, \ldots, V_m$ such that

$$\begin{aligned} V_q(0) = 0, &\quad V_q(x) > 0, \ \forall x \in \mathbb{R}^n \setminus \{0\} \\ \dot{V}_q(x(t)) \le 0, &\quad \text{whenever } \sigma(t) = q \end{aligned}$$

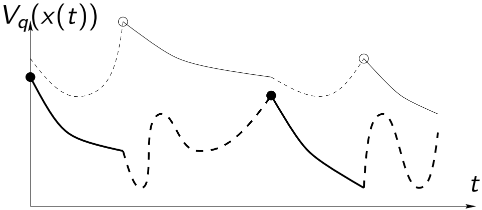
and the sequences $\{V_q(x(\tau_{i_q}))\}$, $q = 1, \ldots, m$ are non-increasing, where τ_{i_q} are the time instances when mode q becomes active, then x^* is stable.

Example

Let the origin be a stable equilibrium point for

$$\dot{x} = f_q(x), \qquad x \in \Omega_q, \qquad q = 1, 2$$

Below, $V_1(x(t))$ and $V_2(x(t))$ are shown. The active parts are solid. The sequences $\{V_q(x(\tau_{i_q}))\}$, $q = 1, 2$, are indicated

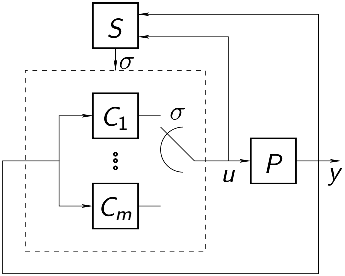


Notes on Multiple Lyapunov Functions

- Strict inequalities imply asymptotic stability for switched linear systems.
- State-dependent switching: Multiple Lyapunov functions that have the same value on the switching surface can be used.
- Alternatively, one can search for one Lyapunov function that behaves appropriately in regions of interest (ie, decreases at regions where corresponding subsystem is active and remains continuous at switching surfaces). OBS!! NOT a common Lyapunov function.
- Computational methods for both exist.

Supervisory Control

- How choose switching $\sigma = \sigma(t)$ such that $\dot{x} = f_\sigma(x)$ has desired property?
- Let a **supervisor** decide on which controller should be active through the switching signal $\sigma : [0, \infty) \to \{1, \ldots, m\}$



A Stabilizing Switching Sequence

Suppose there exist $\mu_q \geq 0$, $q \in Q$ and $\sum_{q=1}^{m} \mu_q = 1$, such that $A = \sum_{q=1}^{m} \mu_q A_q$ is stable. Then, a stabilizing switching sequence $\sigma : [0, \infty) \to Q := \{1, \ldots, m\}$ for

$$\dot{x} = A_\sigma x,$$

is given by

$$\sigma(x(t)) = \arg\min_{q \in Q} \{x^T(t)(A_q^T P + P A_q)x(t)\}$$

where $P > 0$ is the solution to $A^T P + PA = -I$.

Proof: Follows from that $\sum_{q=1}^{m} \mu_q x^T (A_q^T P + P A_q)x < 0$ and $\mu_q \geq 0$, which gives $x^T (A_{\sigma(x)}^T P + P A_{\sigma(x)})x < 0$ for any $x \neq 0$.

Notes

- Intuition: if $\exists$ stable convex combination of A_q then we can always find a mode for which the energy is decreasing at the current state.
- $V = x^T P x$ not a Common Lyapunov Function for the previous result. Why?
- Example application: Apply the previous result to a (networked) control system where either no control or a state-feedback control is applied, so that $\dot{x} = Ax$ corresponds to the open-loop system and $\dot{x} = (A - BL)x$ corresponds to the closed-loop system.

Next Lecture

Control of hybrid systems

- Supervisory control
- Stabilizing switching control