EL2450 Hybrid and Embedded Control

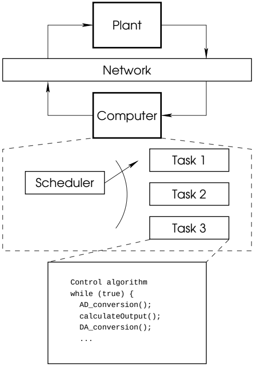
Lecture 9: Hybrid automata

- Hybrid automata as models of hybrid systems
- Dynamical properties of hybrid automata

Today's Goal

You should be able to

- specify a hybrid automaton
- define system evolution semantics
- analyze existence, uniqueness and Zenoness of solutions



$\leftarrow$ Time-triggered system

$\leftarrow$ Event-triggered system

$\leftarrow$ Event-triggered system

Hybrid Dynamics

- Dynamics are essential in reach set computations and verification process: finiteness is lost
- Need to consider all causes of system evolution!
- Mixing time- and event-triggered dynamics lead to **hybrid** dynamics
- What are the dynamics within the discrete states? What might cause discrete transitions?
- Hybrid systems are a particular class of transition systems

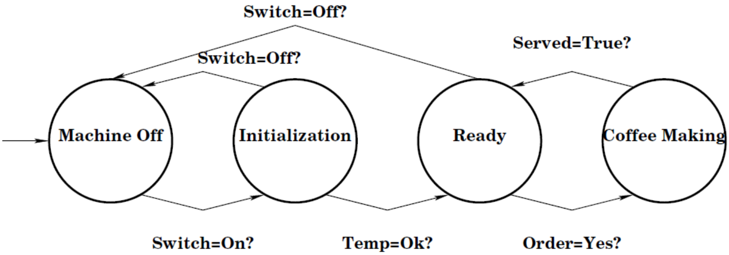
Espresso Machine Example

Task: Design the control system for an automatic espresso machine

Input and Outputs

- What are the relations between inputs and output?

Discrete Event System



- High-level abstraction
- What is in each discrete state?

Initialization

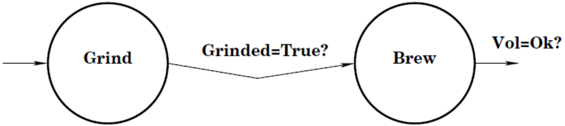
- Start initialization state when machine is switched on
- End the initialization state either
  - When machine is switched off, or
  - When the water temperature T is over a suitable temperature T^*

Ready

- Control T around T^*
- Wait for further a coffee order or that the machine is switched off

Coffee Making

Coffee making can be split into two states:



- In Grind state, coffee beans are prepared for brewing
- In Brew state, the steam is passed through the coffee
  - Pressure P and temperature T are continuously controlled
  - State terminated when coffee volume V is over a desired volume V^*

Automatic Gearbox Control Example

Task: Design the control system for an automatic gearbox

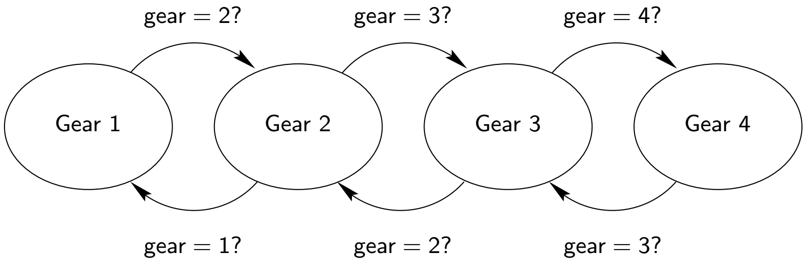
x_1 is the longitudinal position of the car and x_2 its velocity
The dynamics (for a normalized car) can be written as

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \alpha_{\text{gear}}(x_2)u$$

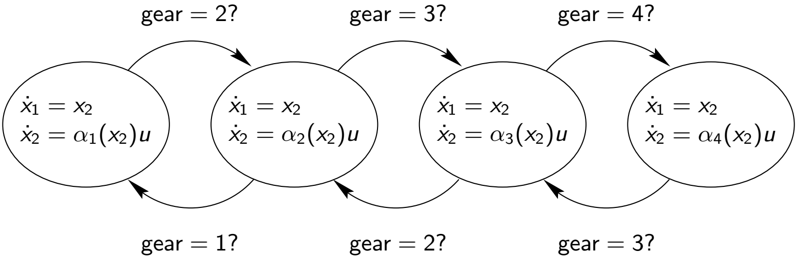
where u corresponds to the throttle position and
$\alpha_{\text{gear}}(\cdot)$ to the efficiency of a specific gear

- $u \in [0, u_{\text{max}}]$ is a real-valued control
- gear $\in \{1, 2, 3, 4\}$ is an integer-valued control

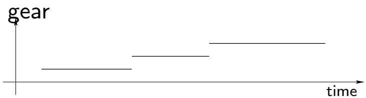
Discrete Event Model

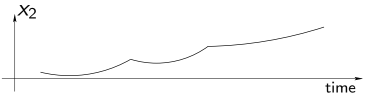


Hybrid System Model



Typical solutions:



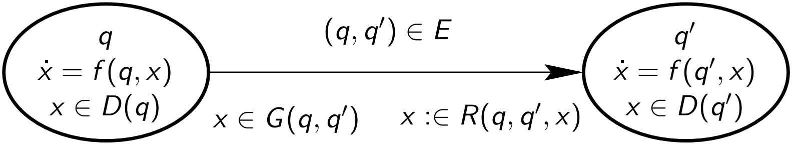


How choose controls u and gear in a suitable way?

Hybrid Automaton

- **Hybrid automaton** is a formal model of a hybrid system
- It defines the evolution of the hybrid system state

Hybrid Automaton $H = (Q, X, \mathrm{Init}, f, D, E, G, R)$

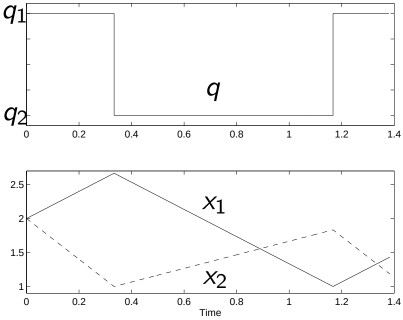


- Q discrete state space and X continuous state space
- $\mathrm{Init} \subseteq Q \times X$ initial states
- $f : Q \times X \to X$ vector fields
- $D : Q \to 2^X$ domains
- $E \subset Q \times Q$ edges
- $G : E \to 2^X$ guards
- $R : E \times X \to 2^X$ resets

Solution of Hybrid Automaton

A **solution** $\chi = (\tau, q, x)$ of H consists of

- **Time trajectory** τ: time line on which the solution is defined
- **State trajectory** (q, x): state evolution (defined on τ) of the hybrid automaton



Time Trajectory τ

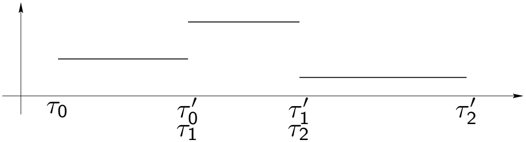
A sequence of (time) intervals

$$\tau = \{I_i\}_{i=0}^{N}$$

such that

- $I_i = [\tau_i, \tau_i']$ for all $i < N$;
- if $N < \infty$, then either $I_N = [\tau_N, \tau_N']$, or $I_N = [\tau_N, \tau_N')$; and
- $\tau_i \leq \tau_i' = \tau_{i+1}$ for all i.

Notation: $\langle \tau \rangle = \{0, 1, \ldots, N\}$

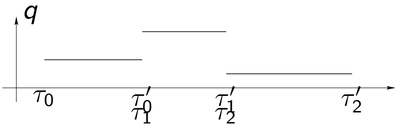


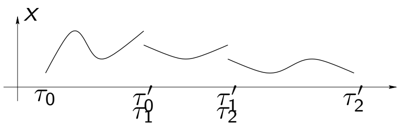
Solution $\chi = (\tau, q, x)$

Solutions accepted by H:

$\tau = \{I_i\}_{i=0}^{N}$, $q : \langle\tau\rangle \to Q$, $x = \{x^i : i \in \langle\tau\rangle\}$, $x^i : I_i \to X$ such that

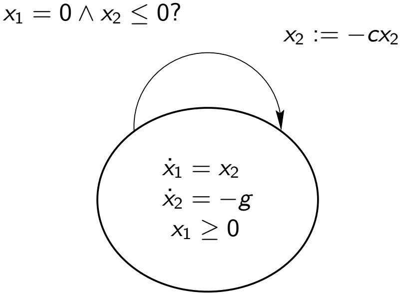
- **Initialization:** $(q(0), x^0(0)) \in \text{Init}$,
- **Time-driven:** for all $t \in [\tau_i, \tau_i')$, $\dot{x}^i(t) = f\big(q(i), x^i(t)\big)$ and $x^i(t) \in D(q(i))$
- **Event-driven:** for all $i \in \langle\tau\rangle \setminus \{N\}$, $e = (q(i), q(i+1)) \in E$, $x^i(\tau_i') \in G(e)$, and $x^{i+1}(\tau_{i+1}) \in R(e, x^i(\tau_i'))$





Example: Bouncing Ball

A ball that loses a fraction of its energy at each bounce

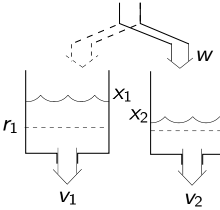


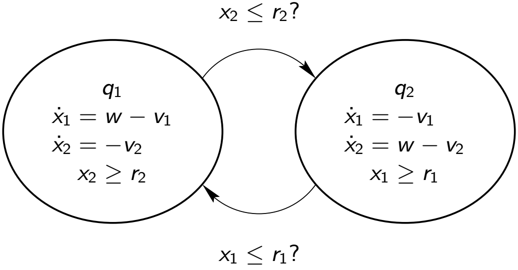
Examples: Differential Equation and Automaton

- A continuous-time system $\dot{x} = f(x)$ can be represented as a hybrid automaton with a single discrete state
- A discrete-event system can be represented as a hybrid automaton with no continuous dynamics

Example: Water Tank System

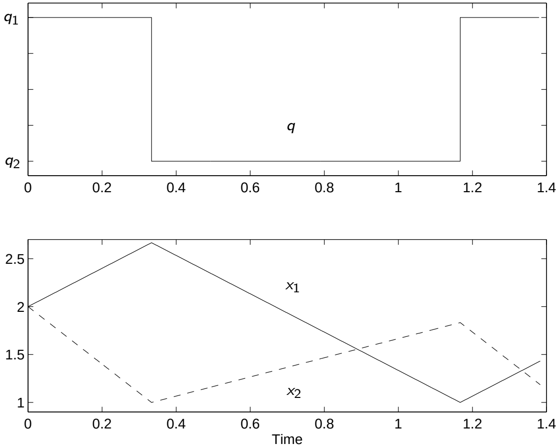
Control objective is to keep water volumes above r_1 and r_2 by switching the inflow





$$H = (Q, X, \mathrm{Init}, f, D, E, G, R)$$

- $Q = \{q_1, q_2\}$;
- $X = \mathbb{R}^2$;
- $\mathrm{Init} = Q \times \{x \in \mathbb{R}^2 : \ x_1 \geq r_1 \wedge x_2 \geq r_2\}$;
- $f(q_1, x) = (w - v_1, -v_2)$ and $f(q_2, x) = (-v_1, w - v_2)$;
- $D(q_1) = \{x \in \mathbb{R}^2 : \ x_2 \geq r_2\}$ and $D(q_2) = \{x \in \mathbb{R}^2 : \ x_1 \geq r_1\}$;
- $E = \{(q_1, q_2), (q_2, q_1)\}$;
- $G(q_1, q_2) = \{x \in \mathbb{R}^2 : x_2 \leq r_2\}$ and $G(q_2, q_1) = \{x \in \mathbb{R}^2 : x_1 \leq r_1\}$;
- $R(q_1, q_2, x) = R(q_2, q_1, x) = \{x\}$.



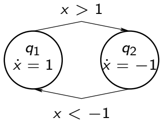
q1
q2
q
0
0.2
0.4
0.6
0.8
1
1.2
1.4
2.5
2
1.5
1
x1
x2
Time

Hybrid Automaton as a Transition System

A hybrid automaton is a transition system $T_H = (S, \Sigma, \rightarrow)$ with interacting event-driven and time-driven evolution:

- $S = Q \times X$ and $(q, x) \in S$ denotes the state
- $\Sigma = \{g\} \cup \text{Time}$ ($\text{Time} = \{t : t \geq 0\}$) where the generators $\{g\}$ cause the discrete jumps and Time the continuous evolution
- $(q, x) \rightarrow (q', x')$ defines the event-driven and time-driven transitions

Example: $S = Q \times X$, with $Q = \{q_1, q_2\}$ and $X = \mathbb{R}$
$\Sigma = \{g_1, g_2\} \cup \text{Time}$, g_1 corresponds to the event $x > 1$ and g_2 to $x < -1$



Transition Relation for Hybrid Automaton

- To each discrete state $q \in Q$, we associate a differential equation $\dot{x} = f_q(x) = f(q, x)$. Let $\phi_q(t)$ denote its solution.
- To each generator g (linked to an edge $e \in Q \times Q$), we associate a guard $G : Q \times Q \to 2^X$

The transition relation $\to$ of $T_H = (S, \Sigma, \to)$ then consists of two parts:
Time-driven: $(q, x) \xrightarrow{t} (q, y)$ provided that $x = \phi_q(0)$ and $y = \phi_q(t)$
Event-driven: $(q, x) \xrightarrow{g} (q', x')$ provided that $x \in G(q, q')$ and $x' \in R(q, q', x)$

Example: Time-driven dynamics is given by $f_{q_1} = 1$ and $f_{q_2} = -1$

Event-driven dynamics is given by $G(q_1, q_2) = \{x > 1\}$ and

$G(q_2, q_2) = \{x < -1\}$

Properties of Hybrid Automata

Liveness	For all $(q_0, x_0) \in \text{Init}$, there exists at least one (infinite) solution from (q_0, x_0)
Determinism	For all $(q_0, x_0) \in \text{Init}$, there exists at most one solution starting from (q_0, x_0)
Zenoness	τ infinite sequence and finite execution time: $\tau_\infty = \sum_{i=1}^{\infty}(\tau_i' - \tau_i) < \infty$
Stability	Stability of equilibria and other invariant sets
Reachability	Reachable states $\text{Reach} \subset Q \times X$

Liveness

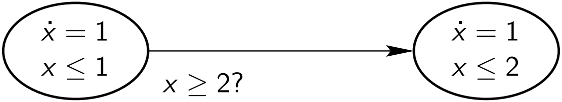
Definition
For all $(q_0, x_0) \in \mathrm{Init}$, there exists **at least one** (infinite) solution from (q_0, x_0)

Fact
A hybrid automaton is live if for all reachable states for which continuous evolution is impossible, a discrete transition is possible

- It is reasonable to expect that models for physical systems should be live
- If a hybrid automaton is not live, it can be due to over-simplifications in the model

Example

Let Init $= (q_1, 0)$. Then the following hybrid automaton is not live (blocking):



Determinism

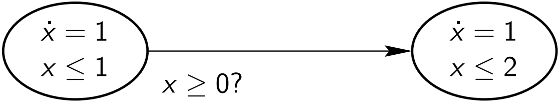
Definition
For all $(q_0, x_0) \in \text{Init}$, there exists **at most one** solution starting from (q_0, x_0)

Fact
A hybrid automaton is deterministic if there is

- no choice between continuous evolution and a discrete transition, and
- a discrete transition can never lead to multiple destinations

Example

Let Init $= (q_1, 0)$. Then the following hybrid automaton is non-deterministic:



Determinism (formally)

Let

$$\mathrm{Out}_H = \{(q,x) \in Q \times X : \forall \epsilon > 0, \exists t \in [0,\epsilon), \phi_q(0) = x, \phi_q(t) \notin D(q)\}$$

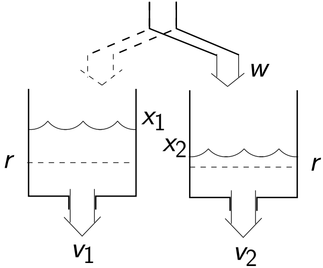
denote the set of states where continuous evolution is impossible.

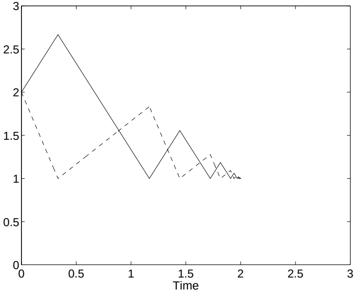
Fact H is deterministic if and only if for all reachable (q,x)

- if $x \in G(q,q')$ for some $(q,q') \in E$, then $(q,x) \in \mathrm{Out}_H$
- if $(q,q'),(q,q'') \in E$ with $q' \neq q''$, then $G(q,q') \cap G(q,q'') = \emptyset$
- if $(q,q') \in E$ and $x \in G(q,q')$, then $|R(q,q',x)| \leq 1$

Zeno Solution of Hybrid Automaton

A solution $\chi = (\tau, q, x)$ is Zeno if $\tau_\infty = \sum_{i=1}^{\infty}(\tau_i' - \tau_i) < \infty$
Example—Water tank system: If $\max\{v_1, v_2\} < w < v_1 + v_2$ then
$\tau_\infty = (x_1(0) + x_2(0) - 2r)/(v_1 + v_2 - w) < \infty$

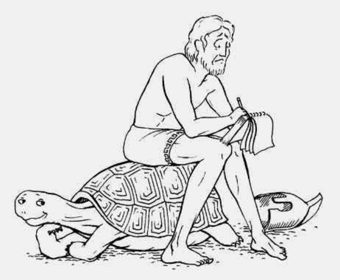




Execution is not defined for $t > \tau_\infty$

Zeno of Elea (490–430 B.C.)

- Born in southern Italy
- Met Socrates in Athens 449 B.C.
- Went back to Elea
  and into politics
- Tortured to death



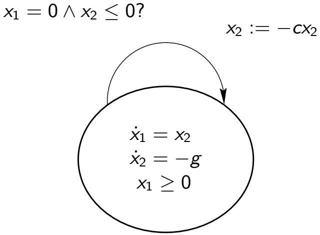
- Paradoxes "proved" that motion and time are illusions
- Led to mathematical problems not solved until 19th century

Zeno

- A solution is Zeno if it exhibits infinitely many discrete jumps in finite time
- Zeno is a truly hybrid phenomenon: it cannot be formulated for a purely discrete system without the notion of continuous time
- Zeno is due to that the model does not reflect reality with sufficient detail
- Fact: a hybrid automaton has Zeno solutions only if (Q, E) is a cyclic graph (a graph with a loop)

Example: Bouncing Ball

A ball that loses a fraction of its energy at each bounce

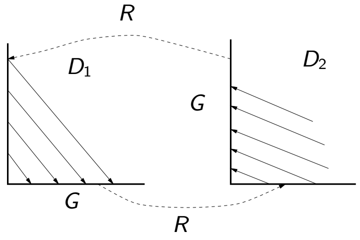


For which values of c are the solutions of the bouncing ball hybrid automaton Zeno?

Zeno State

The convergence point of a Zeno solution is denoted **Zeno state**
Zeno states lie on the intersection of guards

Example—Water tank system:



Next Lecture

Stability of hybrid systems

- Stability of hybrid systems
- Stability criteria for hybrid systems