AUTOMATIC CONTROL

KTH

EL2450 Hybrid and Embedded Control Systems

Exam –, March , 2017

Aid:

Lecture notes (slides) from the course, compendium ("reading material") and textbook from basic course ("Reglerteknik" by Glad & Ljung or similar text approved by course responsible). Mathematical handbook (e.g., "Beta Mathematics Handbook" by Råde & Westergren). Other textbooks, handbooks, exercises, solutions, calculators etc. may **not** be used unless previously approved by course responsible.

Observandum:

- Name and social security number (*personnummer*) on every page.
- Only one solution per page and write on one side per sheet.
- Each answer has to be motivated.
- Specify the total number of handed in pages on the cover.
- Each subproblem is marked with its maximum credit.

Grading:

Grade A: ≥ 43, Grade B: ≥ 38

Grade C: ≥ 33, Grade D: ≥ 28

Grade E: ≥ 23, Grade Fx: ≥ 21

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Lycka till!

1. Consider the system

$$\begin{aligned}\dot{x}_1 =& x_2,\\ \dot{x}_2 =& g\sin(x_1) - u\cos(x_1),\\ x_1(0) = x_2(0) &= u(0) = 0,\end{aligned}$$

where $x_1 \in [-\pi, \pi]$ is an angle, $x_2 \in \mathbb{R}$ is the corresponding angular velocity, u is the acceleration of the system, representing the control input, and $g > 0$ is the gravity acceleration. The goal is to drive $x = [x_1, x_2]^\top$ to the desired configuration $x_{\text{des}} = [\pi, 0]^\top$. To this end, we use a greedy controller, where we apply the maximum and minimum acceleration $u = u_{\max}$ and $u = -u_{\max}$, respectively, based on the energy of the system, which can be approximated by the function $\beta : [-\pi, \pi] \times \mathbb{R} \to \mathbb{R}$, with $\beta(x_1, x_2) = [\frac{x_2^2}{2} + g(\cos(x_1) - 1)]x_2 \cos x_1$. More specifically, we apply $u = u_{\max}$ when $\beta(x_1, x_2) \geq 0$ and $u = -u_{\max}$ when $\beta(x_1, x_2) < 0$. In order to avoid chattering, when x_1 is close to the desired configuration π (within a fixed angle θ, with $\pi > \theta > 0$), we apply a local stabilizing controller $u = \gamma_1 x_1 + \gamma_2 x_2, \gamma_1, \gamma_2 \in \mathbb{R}$, regardless of the value of β.

(a) [5p] *Using the augmented state* $z = [x^\top, u]^\top$, model the system as a hybrid automaton $H = (Q, X, \text{Init}, f, D, E, G, R)$.

(b) [5p] Consider that after appropriate state transformation and linearization, we obtain a switching system of the form

$$\dot{x} = A_q x, \tag{1}$$

with $q \in \{1, 2\}$, i.e., we only consider two of the aforementioned three states, and

$$A_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$
$$A_2 = \begin{bmatrix} a & c \\ 0 & d \end{bmatrix},$$

where $a, b, c, d \in \mathbb{R}$ are scalar constants with $a < 0, d < 0$, and c and b have the same sign. Prove that the origin is asymptotically stable, given that the equation

$$x^2 + ((b + c)^2 - 4ad)x + a^2 + 3a + 9 = 0,$$

has two real solutions, $x_1, x_2 \in \mathbb{R}$, where at least one of them is positive.

Solution

(a)

- We have three states $Q = \{q_1, q_2, q_3\}$. In q_1 we have $|x_1 - \pi| \geq \theta$, $\beta(x_1, x_2) \geq 0$ and $u = u_{\max}$. In q_2 we have $|x_1 - \pi| \geq \theta$, $\beta(x_1, x_2) < 0$ and $u = -u_{\max}$. In q_3 we have $|x_1 - \pi| < \theta$ and $u = \gamma_1 x_1 + \gamma_2 x_2$.
- The domain is $z \in X = [-\pi, \pi] \times \mathbb{R} \times [-u_{\max}, u_{\max}]$.
- The initial condition is $z(0) = [0, 0, 0]$, so $|x_1(0) - \pi| = \pi > \theta$ and $\beta(x_1(0), x_2(0)) = 0$. Hence, the initial state is q_1 and $\text{Init} = \{q_1, 0, 0, 0\}$.
- The vector field corresponding to the three states is $f = [f_1^\top, f_2^\top, f_3^\top]^\top$, with

$$
\begin{aligned}
f_1(q_1, z) &= \begin{bmatrix} x_2 \\ g\sin(x_1) - u_{\max}\cos(x_1) \\ 0 \end{bmatrix}, \\
f_2(q_2, z) &= \begin{bmatrix} x_2 \\ g\sin(x_1) + u_{\max}\cos(x_1) \\ 0 \end{bmatrix}, \\
f_3(q_3, z) &= \begin{bmatrix} x_2 \\ g\sin(x_1) - (\gamma_1 x_1 + \gamma_2 x_2)\cos(x_1) \\ \gamma_1 x_2 + \gamma_2(g\sin(x_1) - (\gamma_1 x_1 + \gamma_2 x_2)\cos(x_1)) \end{bmatrix}.
\end{aligned}
$$

- The domains are

$$
\begin{aligned}
D_1(q_1) =& \{z \in X : |x_1 - \pi| \geq \theta, \beta(x_1, x_2) \geq 0, u = u_{\max}\}, \\
D_2(q_2) =& \{z \in X : |x_1 - \pi| \geq \theta, \beta(x_1, x_2) < 0, u = -u_{\max}\}, \\
D_3(q_3) =& \{z \in X : |x_1 - \pi| < \theta, u = \gamma_1 x_1 + \gamma_2 x_2\}.
\end{aligned}
$$

- The possible edges are $E = \{(q_1, q_2), (q_1, q_3), (q_2, q_3), (q_2, q_1), (q_3, q_1), (q_3, q_2)\}$.
- The guards are

$$
\begin{aligned}
G(q_1, q_2) &= \{z \in D_1 : \beta(x_1, x_2) < 0\}, \\
G(q_2, q_1) &= \{z \in D_2 : \beta(x_1, x_2) = 0\}, \\
G(q_1, q_3) &= \{z \in D_1 : |x_1 - \pi| < \theta\}, \\
G(q_3, q_1) &= \{z \in D_3 : |x_1 - \pi| = \theta \wedge \beta(x_1, x_2) \geq 0\}, \\
G(q_2, q_3) &= \{z \in D_2 : |x_1 - \pi| < \theta\}, \\
G(q_3, q_2) &= \{z \in D_3 : |x_1 - \pi| = \theta \wedge \beta(x_1, x_2) < 0\},
\end{aligned} \tag{2}
$$

- The reset maps are

$$
\begin{aligned}
R((q_1, q_2), z) &= \{u = -u_{\max}\}, \\
R((q_2, q_1), z) &= \{u = u_{\max}\}, \\
R((q_1, q_3), z) &= \{u = \gamma_1 x_1 + \gamma_2 x_2\}, \\
R((q_2, q_3), z) &= \{u = \gamma_1 x_1 + \gamma_2 x_2\}, \\
R((q_3, q_1), z) &= \{u = u_{\max}\}, \\
R((q_3, q_2), z) &= \{u = -u_{\max}\},
\end{aligned}
$$

(b) We use the positive definite indentity matrix $P = I$ and we calculate the matrices

$$\begin{aligned}
Q_1 &= -(A_1^\top P + PA_1), \\
Q_2 &= -(A_2^\top P + PA_2), \\
Q_1 &= -\begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}, \\
Q_2 &= -\begin{bmatrix} 2a & c \\ c & 2d \end{bmatrix},
\end{aligned} \tag{3}$$

We need to prove that $\det(Q_1), \det(Q_2) > 0$, i.e.,

$$\begin{aligned}
&-b^2 - 2bc - c^2 + 4ad > 0, \Rightarrow 4ad > (b+c)^2, \\
&-c^2 + 4ad > 0 \Rightarrow 4ad > c^2
\end{aligned}$$

The given equation

$$x^2 + ((b+c)^2 - 4ad)x + a^2 + 3a + 9 = 0,$$

can be written in the form $x^2 - (x_1 + x_2)x + x_1 * x_2 = 0$, where x_1, x_2 are the solutions. The last coefficient, corresponding to the product can be shown to be positive

$$x_1 * x_2 = a^2 + 3a + 9 = \frac{1}{2} 2 * (a^2 + 3a + 9) = \frac{1}{2}((a+3)^2 + 9 + a^2) > 0. \tag{4}$$

Therefore, since one solution is positive, then the other must be positive as well, i.e., $x_1 > 0, x_2 > 0$. That means that the sum of the solutions is also positive, $x_1 + x_2 > 0$, i.e., $-(b+c)^2 + 4ad > 0$ which means that $\det(Q_1) > 0$. Note that this also implies that $\det(Q_2) > 0$, since b and c have the same sign.