AUTOMATIC CONTROL

KTH

EL2450 Hybrid and Embedded Control Systems

Exam 08:00–13:00, June 7, 2017

Aid:

Lecture notes (slides) from the course and textbook from basic course ("Reglerteknik" by Glad & Ljung or similar text approved by course responsible). Mathematical handbook (e.g., "Beta Mathematics Handbook" by Råde & Westergren). Other textbooks, handbooks, exercises, solutions, calculators etc. may **not** be used unless previously approved by course responsible.

Observandum:

- Name and social security number (*personnummer*) on every page.
- Only one solution per page and write on one side per sheet.
- Each answer has to be motivated.
- Specify the total number of handed in pages on the cover.
- Each subproblem is marked with its maximum credit.

Grading:

Grade A: ≥ 43, Grade B: ≥ 38

Grade C: ≥ 33, Grade D: ≥ 28

Grade E: ≥ 23, Grade Fx: ≥ 21

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Lycka till!

1. Consider the following system:

$$\dot{x} = \begin{cases} A_1 x, & c^\top x \geq 0 \\ A_2 x, & c^\top x < 0 \end{cases} \tag{1}$$

where $x \in \mathbb{R}^2$, $c = [c_1, c_2] \in \mathbb{R}^2$, with $c_1 \neq 0$, $c_2 \neq 0$, and

$$A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix}.$$

(a) [2p] Model formally the system as a Hybrid Automaton

$$H = (Q, X, \mathsf{Init}, f, D, E, G, R).$$

(b) [4p] Prove that it is impossible to find a common quadratic Lyapunov function for the equilibrium $x = 0$ of the switching system (1).

(c) [2p] Prove that there exist Lyapunov functions V_1 and V_2 that render the individual subsystems $\dot{x} = A_1 x_1$ and $\dot{x} = A_2 x_2$, respectively, asymptotically stable.

(d) [2p] Prove that the equilibrium $x = 0$ is asymptotically stable for the switching system (1).

2. Consider the following system:

$$\dot{x}_1(t) = -x_1(t) + x_2^2(t),$$
$$\dot{x}_2(t) = -x_1(t)x_2(t) + u(t),$$

with x_1, $x_2 \in \mathbb{R}$, $u \in \mathbb{R}$ and $t \geq 0$.

(a) [1p] Suppose that the linear feedback control law $u(t) = -\gamma x_2(t)$ with $\gamma > 0$ is designed for the system. Let $x(t) = [x_1(t), x_2(t)]^\top \in \mathbb{R}^2$ and consider the Lyapunov function candidate $V(x) = \frac{1}{2}\|x\|^2$. Prove that under the aforementioned control law, the system converges to origin asymptotically.

(b) [1p] In order to implement the controller on a digital platform, the state of the system is sampled *aperiodically* at a sequence of time instants $\{t_k\}$, $k \in \mathbb{N}$, and the control signal is now given by:

$$u(t) = -\gamma x_2(t_k), \quad t \in [t_k, t_{k+1}). \tag{3}$$

Write the closed loop equations of the system in terms of the states $x_1(t)$, $x_2(t)$ and the state error $e_2(t)$, where $e_2(t) = x_2(t_k) - x_2(t)$, $t \in [t_k, t_{k+1})$.

(c) [2p] Let $e(t) = [0, e_2(t)]^\top \in \mathbb{R}^2$. By using the same Lyapunov function candidate as in (a) find a relation between the norms $\|e(t)\|$ of the error and $\|x(t)\|$ of the state such that the event-based control law (3) renders the origin asymptotically stable. Determine also the event-triggered condition under which the control updates are calculated. *Useful property:* $\alpha_1 z_1 + \alpha_2 z_2 \geq \min\{\alpha_1, \alpha_2\}(z_1 + z_2)$ *for all* α_1, α_2, z_1, $z_2 > 0$.

In the following tasks, you are going to show that the closed-loop system with the event-triggered condition that you have computed does not exhibit Zeno behavior. For the rest of this exercise, consider $t \in [t_k, t_{k+1})$.

(d) [2p] Observing that $\dot{e}_2(t) = -\dot{x}_2(t)$, write an upper bound for $\dot{e}_2(t)$ that depends only on $\|x(t)\|$ and on constant factors. Using the property that $\|x(t)\|$ is monotonically decreasing, rewrite the upper bound so that it depends only on $\|x(t_k)\|$ and on constant factors. Then, observing that $e_2(t) = \int_{t_k}^{t} \dot{e}_2(\tau)d\tau$, write an upper bound for $e_2(t)$ that depends only on $\|x(t_k)\|$, $(t - t_k)$ and on constant factors.

(e) [2p] Observing that $\frac{d\|x(t)\|}{dt} = \frac{x(t)^\top \dot{x}(t)}{\|x(t)\|}$, and using the property that $\|x(t)\|$ is monotonically decreasing, write a lower bound for $\frac{d\|x(t)\|}{dt}$ that depends only on $\|x(t_k)\|$. Then, observing that $\|x(t)\| = \|x(t_k)\| + \int_{t_k}^{t} \frac{d\|x(\tau)\|}{d\tau}d\tau$, write a lower bound for $\|x(t)\|$ that depends only on $\|x(t_k)\|$, $(t - t_k)$ and on constant factors.

(f) [2p] Using the results in (d) and (e), find a lower bound for $(t - t_k)$ to violate the condition that you have computed in (c). Your lower bound should only depend on initial conditions (i.e., $x(0)$) and constant parameters. Conclude that the closed-loop system does not exhibit Zeno behavior.

3. (a) [3p] Consider the transition systems $\mathcal{T}_1$ (left), $\mathcal{T}_2$ (right) in Figure 1. Are they bisimilar? Justify your answer.

   (b) [3p] Find the minimal quotient transition system which is bisimilar to transition system $\mathcal{T}_1$ (left) in Figure 1.

   (c) [2p] Construct a transition system which is bisimilar to $\mathcal{T}_3$ in Figure 2 (left) and has no self loops. Provide the bisimulation relation and justify your answer.

   (d) [1p] Draw a transition system which is bisimilar to $\mathcal{T}_4$ in Figure 2 (center) and has no self loops.

   (e) [1p] Does there exist a *finite* transition system which simulates $\mathcal{T}_5$ in Figure 2 (right) and has *at most one* outgoing transition from each state?

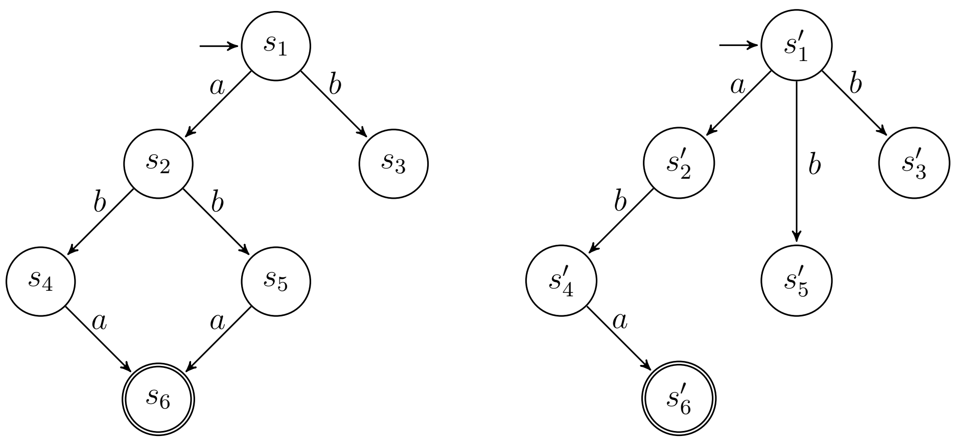

Figure 1: Transition systems $\mathcal{T}_1$ (left) and $\mathcal{T}_2$ (right).

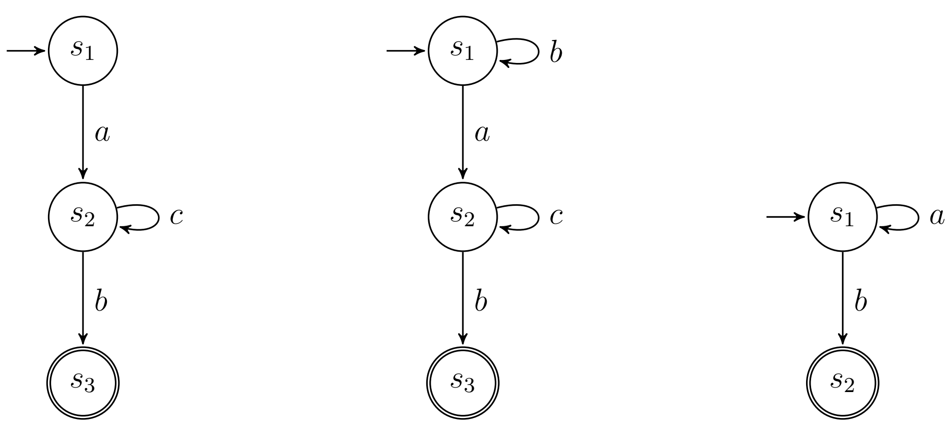

Figure 2: Transition systems $\mathcal{T}_3$ (left), $\mathcal{T}_4$ (center), and $\mathcal{T}_5$ (right).

4. Consider the Timed Automaton $\mathcal{T}$ depicted in Figure 3.

   (a) [3p] Model formally the Timed Automaton as a Hybrid Automaton:
   $$\mathcal{H} = (Q, X, \mathsf{Init}, f, \mathsf{Act}, D, E, G, R).$$

   (b) [3p] Model formally the Timed Automaton as a Transition System:
   $$\mathcal{TS} = (S, \Sigma, \rightarrow, S_0).$$

   (c) [3p] Decide whether the following states are reachable from the state $(q_1, 0, 0, 0)$:
   (i) $(q_3, 3, 1, 2)$
   (ii) $(q_2, 0, 0, 3)$
   (iii) $(q_3, 1.5, 0, 1.5)$
   (iv) $(q_2, 1, 1, 3)$.

   (d) [1p] Describe a procedure under which the reachability of a general Timed Automaton can be performed.

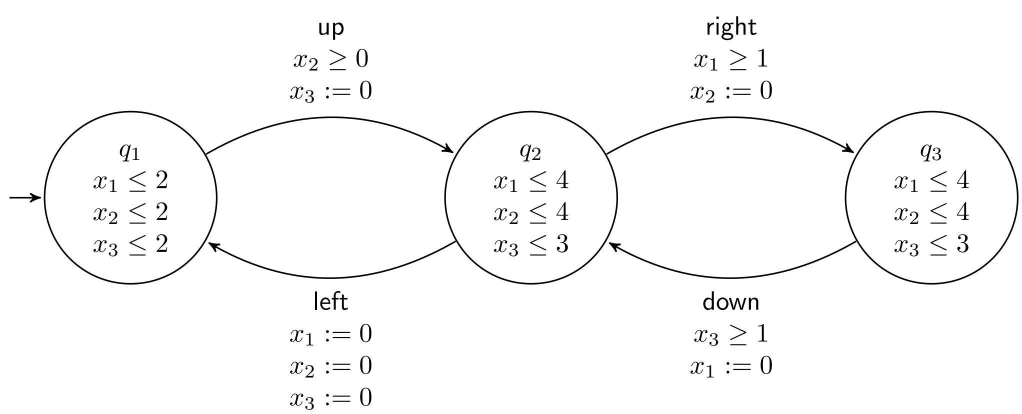

Figure 3: The timed automaton $\mathcal{T}$.

	C_i	T_i	D_i
J_1	1	3	3
J_2	2	4	4

Table 1:

5. Consider the two tasks J_1 and J_2 with computation times, periods and deadlines defined by Table 1. The tasks are executed on a preemptive CPU.

   (a) [2p] Compute the utilization factor and the scheduling length.

   (b) [3p] Are the tasks schedulable under the EDF algorithm? What is the worst-case response time for each task?

   (c) [3p] Are the tasks schedulable under the RM algorithm? What is the worst-case response time for each task?

   (d) [2p] We consider now the possibility of having aperiodic tasks. In order to handle aperiodic tasks, we run a polling server J_s with computation time $C_s = 3$ and period $T_s = 6$. Assume that an aperiodic task with computation time $C_a = 2$ asks for the CPU at time $t = 5$. Plot the time evolution when a polling server is used together with the two tasks J_1 and J_2 under the RM algorithm.