EL2520 - Control Theory and Practice - Advanced Project Lab : The Four Tank Process

Kartik Seshadri Chari
kartikc@kth.se
960807-0174

Devendra Sharma
devendra@kth.se
950815-5497

Merav Modi
merav@kth.se
940120-

22 May,2019

Abstract — **This document is a report for the Project Lab conducted under the EL2520 course. This project was divided into 2 distinct sessions. The first one involved deriving a physical model of a four-tank process for minimum phase and non-minimum phase configurations via experimentation and investigating the coupling between the tanks. Emphasis was also given to manually controlling the process to understand the performance limitations due to the non-minimum dynamics. The second instance was dedicated to testing the model-based decentralized PI controller and the robust Glover-McFarlane method.**

I. Modelling

Here, the nonlinear differential equations describing the plant will be derived. Thus, the rate of change of volume in each tank is given as:

$$A\frac{dh}{dt} = q_{\text{in}} - q_{\text{out}}$$

From Bernoulli's law, we have

$$q_{\text{out}} = a\sqrt{2gh} \quad \forall g = 981cm/s^2$$

Now, as there is a pump used, its flowrate is goverened as follows:

$$q_L = \gamma ku,\ q_U = (1-\gamma)ku \quad \forall \gamma \in [0,1]$$

$\forall$ q_L denotes the lower tank and q_U denotes the upper tank From the above equations, we can derive the nonlinear system as follows:

$$\begin{aligned}
A_1\frac{dh_1}{dt} &= -q_{out,1} + q_{out,3} + q_{L,1}\\
A_2\frac{dh_2}{dt} &= -q_{out,2} + q_{out,4} + q_{L,2}\\
A_3\frac{dh_3}{dt} &= -q_{out,3} + q_{U,2}\\
A_4\frac{dh_4}{dt} &= -q_{out,4} + q_{U,1}
\end{aligned}$$

Thus, the final system of equations turns out to be:

$$\begin{aligned}
\frac{dh_1}{dt} &= \frac{-a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_1 k_1}{A_1}u_1\\
\frac{dh_2}{dt} &= \frac{-a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_2 k_2}{A_2}u_2\\
\frac{dh_3}{dt} &= \frac{-a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2)k_2}{A_3}u_2\\
\frac{dh_4}{dt} &= \frac{-a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_1)k_1}{A_4}u_1
\end{aligned} \tag{1}$$

A. *Equilibrium Equations*

For the equilibrium condition, we take the rate of change of height as 0 to get the following set of equations:

$$\begin{aligned}
\frac{-a_1}{A_1}\sqrt{2gh_1^0} + \frac{a_3}{A_1}\sqrt{2gh_3^0} + \frac{\gamma_1 k_1}{A_1}u_1^0 &= 0\\
\frac{-a_2}{A_2}\sqrt{2gh_2^0} + \frac{a_4}{A_2}\sqrt{2gh_4^0} + \frac{\gamma_2 k_2}{A_2}u_2^0 &= 0\\
\frac{-a_3}{A_3}\sqrt{2gh_3^0} + \frac{(1-\gamma_2)k_2}{A_3}u_2^0 &= 0\\
\frac{-a_4}{A_4}\sqrt{2gh_4^0} + \frac{(1-\gamma_1)k_1}{A_4}u_1^0 &= 0\\
y_i^0 = k_c h_i^0 \quad \forall i &= \{1,2,3,4\}
\end{aligned}$$

where $h_i^0,\ u_i^0,\ y_i^0$ denote the steady state values.

B. *Linearization*

Let $\Delta u_i = u_i - u_i^0$, $\Delta h_i = h_i - h_i^0$ and $\Delta y_i = y_i - y_i^0$ denote the deviations from the equilibrium and also let

$$u = \begin{bmatrix}\Delta u_1\\ \Delta u_2\end{bmatrix}, x = \begin{bmatrix}\Delta h_1\\ \Delta h_2\\ \Delta h_3\\ \Delta h_4\end{bmatrix}, y = \begin{bmatrix}\Delta y_1\\ \Delta y_2\end{bmatrix}$$

For Linearizing the above-derived system of non-linear equation 1 around its equilibrium points, we use the Taylor Series of Expansion (neglecting HOTs) to obtain the following matrices:

$$A = \begin{bmatrix} \frac{\partial \Delta h_1}{\partial h_1} & \frac{\partial \Delta h_1}{\partial h_2} & \frac{\partial \Delta h_1}{\partial h_3} & \frac{\partial \Delta h_1}{\partial h_4} \\ \frac{\partial \Delta h_2}{\partial h_1} & \frac{\partial \Delta h_2}{\partial h_2} & \frac{\partial \Delta h_2}{\partial h_3} & \frac{\partial \Delta h_2}{\partial h_4} \\ \frac{\partial \Delta h_3}{\partial h_1} & \frac{\partial \Delta h_3}{\partial h_2} & \frac{\partial \Delta h_3}{\partial h_3} & \frac{\partial \Delta h_3}{\partial h_4} \\ \frac{\partial \Delta h_4}{\partial h_1} & \frac{\partial \Delta h_4}{\partial h_2} & \frac{\partial \Delta h_4}{\partial h_3} & \frac{\partial \Delta h_4}{\partial h_4} \end{bmatrix} \Bigg|_{h_i^0, u_i^0}$$

$$B = \begin{bmatrix} \frac{\partial \Delta h_1}{\partial u_1} & \frac{\partial \Delta h_1}{\partial u_2} \\ \frac{\partial \Delta h_2}{\partial u_1} & \frac{\partial \Delta h_2}{\partial u_2} \\ \frac{\partial \Delta h_3}{\partial u_1} & \frac{\partial \Delta h_3}{\partial u_2} \\ \frac{\partial \Delta h_4}{\partial u_1} & \frac{\partial \Delta h_4}{\partial u_2} \end{bmatrix} \Bigg|_{h_i^0, u_i^0}$$

$$C = \begin{bmatrix} \frac{\partial \Delta y_1}{\partial h_1} & \frac{\partial \Delta y_1}{\partial h_2} & \frac{\partial \Delta y_1}{\partial h_3} & \frac{\partial \Delta y_1}{\partial h_4} \\ \frac{\partial \Delta y_2}{\partial h_1} & \frac{\partial \Delta y_2}{\partial h_2} & \frac{\partial \Delta y_2}{\partial h_3} & \frac{\partial \Delta y_2}{\partial h_4} \end{bmatrix} \Bigg|_{h_i^0, u_i^0}$$

$$D = \begin{bmatrix} \frac{\partial \Delta y_1}{\partial u_1} & \frac{\partial \Delta y_1}{\partial u_2} \\ \frac{\partial \Delta y_2}{\partial u_1} & \frac{\partial \Delta y_2}{\partial u_2} \end{bmatrix} \Bigg|_{h_i^0, u_i^0}$$

After solving the above matrices, we get

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

where,

$$A = \begin{bmatrix} -\frac{1}{T_1} & 0 & \frac{A_3}{A_1 T_3} & 0 \\ 0 & -\frac{1}{T_2} & 0 & \frac{A_4}{A_2 T_4} \\ 0 & 0 & -\frac{1}{T_3} & 0 \\ 0 & 0 & 0 & -\frac{1}{T_4} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \end{bmatrix}$$

and D = 0, with $T_i = \frac{A_i}{a_i}\sqrt{\frac{2h_i^0}{g}}$.

C. Transfer Matrix

In order to obtain the Transfer Matrix, we use the following formula -

$$G(s) = C(sI - A)^{-1}B$$

i.e. G(s) =

$$\begin{bmatrix} k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{T_1}{1+sT_1} & 0 & \frac{\frac{A_3 T_1}{A_1}}{(1+sT_1)(1+sT_3)} & 0 \\ 0 & \frac{T_2}{1+sT_2} & 0 & \frac{\frac{A_4 T_2}{A_2}}{(1+sT_2)(1+sT_4)} \\ 0 & 0 & \frac{T_3}{1+sT_3} & 0 \\ 0 & 0 & 0 & \frac{T_4}{1+sT_4} \end{bmatrix}$$

$$* \begin{bmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{\gamma_1 k_1 c_1}{1+sT_1} & \frac{(1-\gamma_2)k_2 c_1}{(1+sT_3)(1+sT_1)} \\ \frac{(1-\gamma_1)k_1 c_2}{(1+sT_4)(1+sT_2)} & \frac{\gamma_2 k_2 c_2}{1+sT_2} \end{bmatrix}$$

D. Zeros of Transfer Matrix

Zeros of the Transfer matrix G(s) are given by the numerator of the det(G(s)). In other words, zeros of the Transfer matrix can be obtained by solving the equation :

$$\gamma_1\gamma_2 T_3 T_4 s^2 + \gamma_1\gamma_2(T_3+T_4)s + (\gamma_1+\gamma_2-1) = 0$$
$$\implies T_3T_4 s^2 + (T_3+T_4)s + \frac{\gamma_1+\gamma_2-1}{\gamma_1\gamma_2} = 0$$

The solution of the above quadratic equation is:

$$s_1 = -\frac{T_3+T_4}{2T_3T_4} + \frac{1}{2T_3T_4}\sqrt{(T_3+T_4)^2 - 4\frac{\gamma_1+\gamma_2-1}{\gamma_1\gamma_2}}$$
$$s_2 = -\frac{T_3+T_4}{2T_3T_4} - \frac{1}{2T_3T_4}\sqrt{(T_3+T_4)^2 - 4\frac{\gamma_1+\gamma_2-1}{\gamma_1\gamma_2}}$$

Here, as s_2 has both terms negative, the zero is on the Left Half of the s-plane. But when it comes to s_1, if the term $\frac{\gamma_1+\gamma_2-1}{\gamma_1\gamma_2}$ is positive, value of the square root term will be less than $T_3 + T_4$, which makes the zero negative. Thus, for the **Minimum Phase case**, $(\gamma_1 + \gamma_2 - 1) > 0$ i.e $\gamma_1 + \gamma_2 > 1$ Now as $\gamma_1, \gamma_2 \in [0, 1]$, $\gamma_1 + \gamma_2 \leq 2$

$$\therefore\ 1 < \gamma_1 + \gamma_2 \leq 2$$

For the **Non-minimum phase case**, one of the zeros is positive which means $(\gamma_1 + \gamma_2 - 1) < 0$. Also, as $\gamma_1, \gamma_2 \in [0, 1]$, $\gamma_1 + \gamma_2 > 0$.

$$\therefore\ 0 < \gamma_1 + \gamma_2 \leq 1$$

E. RGA Analysis

RGA(G(0)) = $G(0). * G(0)^{-1^T}$ Thus,

$$G(0) = \begin{bmatrix} \gamma_1 k_1 c_1 & (1-\gamma_2)k_2 c_1 \\ (1-\gamma_1)k_1 c_2 & \gamma_2 k_2 c_2 \end{bmatrix}$$

$$G(0)^{-1^T} = (\frac{1}{\gamma_1\gamma_2 k_1 k_2 c_1 c_2 - (1-\gamma_1)(1-\gamma_2)k_1 k_2 c_1 c_2})$$

$$\begin{bmatrix} \gamma_2 k_2 c_2 & -(1-\gamma_1)k_1 c_2 \\ -(1-\gamma_2)k_2 c_1 & \gamma_1 k_1 c_1 \end{bmatrix}$$

Hence, the RGA of $G(0)$ is given by

$$RGA(G(0)) = G(0). * G(0)^{-T}$$
$$= \begin{bmatrix} \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2-1} & \frac{\gamma_1+\gamma_2-1-\gamma_1\gamma_2}{\gamma_1+\gamma_2-1} \\ \frac{\gamma_1+\gamma_2-1-\gamma_1\gamma_2}{\gamma_1+\gamma_2-1} & \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2-1} \end{bmatrix}$$

Now, let $\lambda = \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2-1}$. Thus, we get

$$RGA(G(0)) = \begin{bmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{bmatrix}$$

For **Minimum Phase case** :
$\lambda_1 = 0.625 = \lambda_2$. Thus,

$$RGA(G_{mp}(0) = \begin{bmatrix} 1.5625 & -0.5625 \\ -0.5625 & 1.5625 \end{bmatrix}$$

For **Non-Minimum Phase case** :
$\lambda_1 = 0.375 = \lambda_2$. Thus,

$$RGA(G_{nmp}(0) = \begin{bmatrix} -0.5625 & 1.5625 \\ 1.5625 & -0.5625 \end{bmatrix}$$

F. Determination of k_1 and k_2

For determining the values of k_1 and k_2, we need to measure how fast the pump fills up a tank. Thus, for k_1, tank 1 was observed and the following experiment was conducted. We stopped the outflow from tank 1 and ensured that only the pump input was allowed into the tank. This helped us obtain the value of k_1 using the following equation:

$$\frac{dh_1}{dt} = -\cancelto{0}{\frac{a_1}{A_1}\sqrt{2gh_1}} + \cancelto{0}{\frac{a_3}{A_1}\sqrt{2gh_3}} + \frac{\gamma_1 k_1}{A_1}u_1$$

We repeated the same process with the $tank_2$ for obtaining the value of k_2.
After multiple readings using different voltages, we obtained

$$k_1 = 4.4247 cm^3/sV$$
$$k_2 = 4.1092 cm^3/sV$$

G. Determination of areas of holes

Before proceeding with the procedure, some assumptions used are as follows:

- The cross-sectional areas of the tanks are the same i.e A = 15.52 cm^2
- The areas a_1 and a_2 remain the same for both Minimum and non-minimum phase cases
- The areas a_3 and a_3 will vary for Minimum and non-minimum phase cases

So, for calculating the areas, we first considered tanks 3 and 4 as they just had a single input and single output making the calculation for a_3 and a_4 simple. We let the upper 2 tanks settle at their equilibrium points h_3^0 and h_4^0 by keeping $u_1^0 = 7.5V$ and $u_2^0 = 7.5V$.

$$\therefore\ a_3 = \frac{(1-\gamma_2)k_2}{\sqrt{2gh_3^0}}u_2^0$$
$$and\ \ a_4 = \frac{(1-\gamma_1)k_1}{\sqrt{2gh_4^0}}u_1^0$$

Now, the entire system was driven to equilibrium and all the steady-state heights(h_i^0) were measured. As we had 2 equations(one each for tank 1 and tank 2) and 2 unknown, the areas of holes 1 and 2 were calculated by solving the 2*2 linear system as follows:

$$a_1 = \frac{a_3}{\sqrt{2gh_1^0}}\sqrt{2gh_3^0} + \frac{\gamma_1 k_1}{\sqrt{2gh_1^0}}u_1^0$$
$$a_2 = \frac{a_4}{\sqrt{2gh_2^0}}\sqrt{2gh_4^0} + \frac{\gamma_2 k_2}{\sqrt{2gh_2^0}}u_2^0$$

For **Minimum Phase Case**

$$a_1 = 0.2250\ cm^2$$
$$a_2 = 0.2389\ cm^2$$
$$a_3 = 0.0633\ cm^2$$
$$a_4 = 0.0888\ cm^2$$

For **Non-Minimum Phase Case**

$$a_1 = 0.2250\ cm^2$$
$$a_2 = 0.2389\ cm^2$$
$$a_3 = 0.115\ cm^2$$
$$a_4 = 0.213\ cm^2$$