EL2520 - Control Theory and Practice - Advanced Course
Solution/Answers – 2016-06-02

1. (a)
$$\|z\|_2^2 = \int_0^\infty e^{-t} + e^{-2t}dt = [-e^{-t} - 0.5e^{-2t}]_0^\infty = 1.5$$
and hence $\|z\|_2 = \sqrt{1.5}$.

   (b) The worst-case amplification is given by the maximum singular value. The singular values $\sigma_i = \sqrt{\lambda_i(G^TG))}$ and since all elements have the same dynamics in this case we get $\bar{\sigma} = \frac{1}{\sqrt{w^2+1}}3$ which has a maximum $\sup_\omega \bar{\sigma} = 3$ at frequency $\omega = 0$. The corresponding input direction is given by the corresponding eigenvector of G^TG which in this case is
$$\bar{v} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
and by applying this input to G we get the output direction
$$\bar{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

   (c) The determinant $\det G(s) = \frac{-1}{s(2s+1)}$ and the least common denominator of all elements and the determinant is then $s(s+1)(2s+1)$, thus poles at $s = 0, s = -1, s = -0.5$. Zeros from $\det G(s) = \frac{-(s+1)}{s(s+1)(2s+1)}$ which gives a zero at $s = -1$ (which does not cancel pole due to differing directions). Three poles implies that the minimal realization contains 3 states.

2. (a) (i) The loop-gain is $GF_y = K\frac{1-2s}{s}$ The phase is -135^o at $\omega = 0.5$ and $|GF_y(i0.5)/K| = \sqrt{8}$ and hence $K = 1/\sqrt{8}$. We have
$$S = \frac{1}{1+GF_y} = \frac{s}{s+K(1-2s)}; \quad T = \frac{K(1-2s)}{s+K(1-2s)}$$
It is easily seen that both $|S|$ and $|T|$ reach a maximum at high frequencies (one zero and one pole, zero smaller than pole)
$$\|S\|_\infty = \frac{1}{-2K+1} = 3.42 \; ; \quad \|T\|_\infty = \frac{2K}{-2K+1} = 2.41$$
(ii) The sensitivity S is zero at zero frequency and reaches one at $\omega \approx 0.35$, thus it attenuates disturbances at the output up to this frequency. The complementary sensitivity T is 1 at low frequencies and increases for frequencies above $\omega = 0.1$. Thus, the feedback amplifies the noise for frequencies above 0.1.

(b) (i) The Riccati equation becomes $2S + \eta - 2S^2 = 0$, which gives $S = (1 + \sqrt{1 + 2\eta})/2$. The feedback gain is then $\ell = \sqrt{2}S$. With $\eta = 0$ this gives $\ell = \sqrt{2}$ which implies that the closed-loop pole is in -1.

(ii) The optimal is given by the feedback control $u = -\ell\hat{x}$, where ℓ is given above and $\hat{x}$ comes from a Kalman filter with gain $k = \sqrt{2}P$. P is the positive solution to the Riccati equation $2P + \rho - 2P^2 = 0$ which gives $P = (1 + \sqrt{1 + 2\rho})/2$. The controller $F_y(s)$ is then $F_y(s) = \ell k/(s - 1 + (k + \ell)\sqrt{2})$.

3. (a) (i) Scale the system so that all signals are between -1 and 1. Gives

$$\hat{Y} = \frac{4}{0.5}\frac{1}{10s + 1}\hat{D} = \frac{8}{10s + 1}\hat{D} = G_d(s)\hat{D}$$

The amplitude of $|G_d(i\omega)| = \frac{8}{\sqrt{(100\omega^2+1)}}$ is larger than 1 for $\omega < 0.8$, thus we need to attenuate disturbances up to this frequency to keep y within the given bounds (approximate since we assume sinusoidal stationary disturbances). (ii) For real RHP zeros at $s = z$, we require $|G_d(z)| < 1$ to keep the specifications. We get $|G_d(1)| = 0.8$, and thus it is feasible. However, we also know from (i) that we need $|S| < 1$ for $\omega < 0.8$, and this will be hard to achieve since $|S(1)| > 1$ and pushing the bandwidth to 0.8 will probably imply a significant peak in $|S|$ due to the zero at $s = 1$ (peak at least 2 with bandwidth at 0.5).

(b) We have $y = Sd, y = Tn$ and the RS condition is $\|T\Delta_G\|_\infty < 1$.

(i) We need $\bar{\sigma}(S)(i\omega) < 1/20, \omega < 0.1$ and $\bar{\sigma}(S)(0) < 0.01$. Similarily, $\bar{\sigma}(T)(i\omega) < 0.1, \omega > 1$. The robustness condition gives that $\bar{\sigma}(T)$ should be less than 10 for $\omega < 1$ and less than 2 at higher frequencies, and these are less strict than the requirements from disturbances and noise.

(ii) For low frequencies, approximately up to where $\bar{\sigma}(S) = 1$, we have that $\bar{\sigma}(S) \approx 1/\underline{\sigma}(L)$. Likewise, for frequencies where $\bar{\sigma}(T) < 1$ we have $\bar{\sigma}(T) \approx \bar{\sigma}(L)$. Thus, $\underline{\sigma}(L)(0) > 100, \underline{\sigma}(L) > 20, \omega < 0.1$, $\bar{\sigma}(L) < 0.1, \omega > 1$. (drawing not shown here).

(iii) We see from (ii) that the loop-gain has to drop at least from 20 to 0.1 over one decade $\omega \in [0.1, 1]$, corresponding to a slope less than -2 around the crossover frequencies. This will be hard to achieve while maintaining stability.

4. (a) (i) The 1, 1-element of the RGA for a 2×2 system is $\lambda_{11} = \lambda_{22} = \frac{1}{1 - \frac{g_{12}g_{21}}{g_{11}g_{22}}}$ and $\lambda_{12} = \lambda_{21} = 1 - \lambda_{11}$. We get $\lambda_{11}(0) = -10$ and according to the rule not to pair on negative RGA-elements at steady-state we get that we should pair $y_1 - u_2$ and $y_2 - u_1$. We also check the corresponding RGA elements at the expected bandwidth $\lambda_{12}(i1) = 2.2 + 4.4i$ and $|\lambda_{12}| = 4.9$ which implies that the interactions has a strong impact on the loop-gains (decreases the loop-gains

by a factor of 5). Thus, decentralized control will probably not give very good performance for this system.
(ii) The system G has a RHP zero at $s = 0.5$, and since this zero is not present in GW it implies that it must have been cancelled by a pole at $s = 0.5$ in $W(s)$. Cancelling poles in the RHP implies that the system is internally unstable.
(ii) To avoid cancellation of the RHP zero we must keep it in $G(s)W(s)$. Thus, we specify

$$G(s)W(s) = \frac{-2s+1}{(s+1)^2} I$$

which gives

$$W(s) = \frac{1}{s+1} \begin{pmatrix} -(2s+10) & 11 \\ -(2s+10) & 2s+10 \end{pmatrix}$$

(b) The closed-loop A-matrix become

$$A_{cl} = A + BL = \begin{pmatrix} -3 & 0 \\ -1 & -0.5 \end{pmatrix}$$

from which it is seen that the eigenvalues are -0.5 and -3 and hence the closed-loop is unstable (eigenvalue outside unit circle in complex plane).

5. (a) To include G_2 in G_p we set them equal and get

$$\Delta_{Go} = G_2 G_1^{-1} - I = \begin{pmatrix} 0.25 & 1.125 \\ 0 & -0.2 \end{pmatrix}$$

which has ∞-norm $\|\Delta_{Go}\|_\infty = \bar{\sigma}(\Delta_{Go}) = 1.17$. In the uncertainty set, considered when employing the small gain theorem, we then allow any model G_p with stable Δ_{Go} and $\|\Delta_{Go}\|_\infty < 1.17$.

(b) We get $T = \frac{1}{s+1} I$ and $\bar{\sigma}(T) = \frac{1}{\sqrt{\omega^2+1}}$. From the SGT we then get that $\|T\Delta_G\|_\infty = 1.17 > 1$ and we do not have robust stability.

(c) Write the closed-loop with uncertainty on the $P - \Delta$-loop form and then identify $P = F_y G(I + F_y G)^{-1} = T_I$. The robust stability condition then becomes that T_I stable (nominal stability), Δ_{Gi} stable and

$$\|T_I \Delta_{Gi}\|_\infty < 1$$

(d) The uncertainty now becomes

$$\Delta_{Gi} = G_1^{-1} G_2 - I = \begin{pmatrix} 0.25 & 0 \\ 0 & -0.25 \end{pmatrix}$$

which has $\|\Delta_{Gi}\|_\infty = 0.25$. With the controller from (b) we get $T_I = \frac{1}{s+1}$ and hence

$$\|T_I \Delta_{Gi}\|_\infty = 0.25$$

and we can guarantee robust stability (and with good margin). The reason for the difference in the conclusion in (b) and (d) is that we by modeling the uncertainty at the output in this case include a much larger set of models as compared to when we model the uncertainty at the input.