Exercise Session 5

Topics:
- RGA (relative gain array)
- → Decentralized and Decoupled Control
- Robustness in MIMO systems

→ 10.1(iv), 10.3, 10.4, 10.6

10.3) Consider a system with two in- and outputs

$$Y = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} u$$

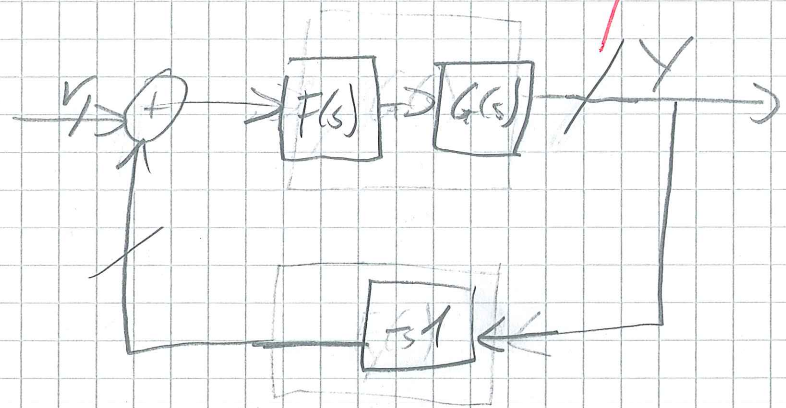
which is controlled by a decentralized controller

$$F_Y = \begin{bmatrix} F_1(s) & 0 \\ 0 & F_2(s) \end{bmatrix}$$

a) Draw a block diagram:

1. version: MIMO block diagram

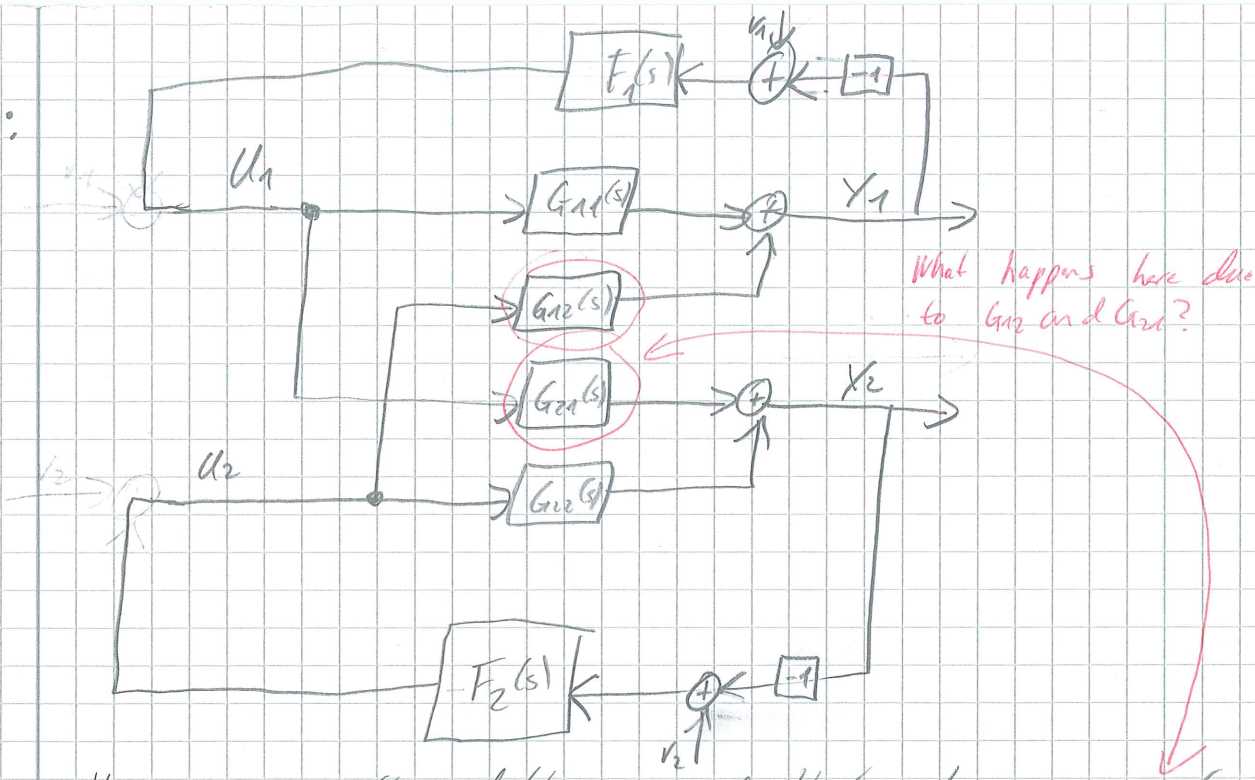
indicates a vector



with $Y = G(s)F(s)(r-y)$

→ Looks similar to SISO control, but the internal behavior is hidden in G and F. ⇒ cross couplings

2nd version:



Here we see the difference and effects of cross couplings

b) If $G_{12}(s) = G_{21}(s) = 0$, then the cross couplings disappear and we can design two single controller for:

1. $u_1 \rightarrow Y_1$
2. $u_2 \rightarrow Y_2$

according to some given specifications (e.g. rise time, overshoot, steady-state error, ...)

$\rightarrow$ We get a <u>decoupled</u> system.
(two separate SISO systems!)

The next natural question is: How to deal with these cross couplings? For example:

Relative Gain Array (RGA):

$$\boxed{\text{A measure of cross couplings}}$$

→ cross couplings naturally arise in MIMO systems
(without them MIMO control similar to SISO control)

→ For square systems, we define the RGA as:
↳ number of inputs = number of outputs

Assume $y = G u$

For non-square systems use the pseudo inverse (see book)

then $RGA(G) = G .* (G^{-1})^T$
element-wise multiplication

- sum of rows/colums is 1
- If G diagonal, then $RGA(G) = I$
  (meaning: perfect decoupling)

We use the RGA for a decentralized control approach (one controller for one input-output pair)

Hence $F_y = \begin{bmatrix} F_{y1} & & 0 \\ & F_{y2} & \\ 0 & & \ddots \end{bmatrix}$ is diagonal

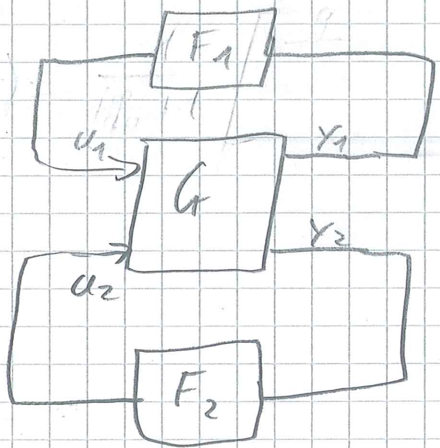
after renumbering inputs and outputs.

To find this input-output pair, we use the RGA and two pairing rules:

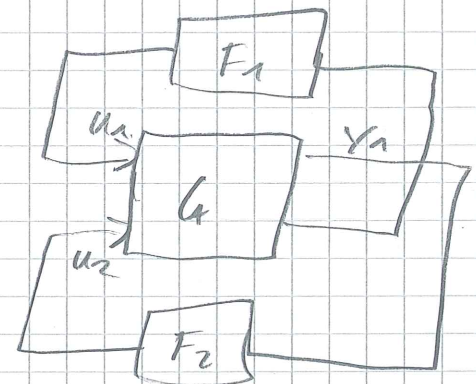
- Pair u_j and y_i if $RGA(G(i\omega_c))|_{ij} \approx 1$
- Avoid to pair u_j and y_i if $RGA(G(0))|_{ij} < 0$

If not stability issues may arise

To illustrate the situation for a 2x2 system:
We need to decide if



or



10.1 ii) Given $G(s) = \frac{1}{\frac{s}{20}+1}\begin{bmatrix} \frac{8}{s+1} & 2 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$

$$G^{-1}(s) = \frac{1}{\det G(s)}\begin{bmatrix} G_{22}(s) & -G_{12}(s) \\ -G_{21}(s) & G_{11}(s) \end{bmatrix}$$

with

$$\det G(s) = \frac{1}{\left(\frac{s}{20}+1\right)^2}\left(\frac{36}{s+1} - 12\right)$$

$$= \frac{1}{\left(\frac{s}{20}+1\right)^2(s+1)}\left(36 - 12s - 12\right)$$

$$= \frac{1}{\left(\frac{s}{20}+1\right)^2(s+1)}\left(24 - 12s\right) = \frac{12(2-s)}{\left(\frac{s}{20}+1\right)^2(s+1)}$$

Hence

$$G^{-1}(s) = \frac{\left(\frac{s}{20}+1\right)^{\cancel{2}}(s+1)}{12(2-s)} \frac{1}{\cancel{\left(\frac{s}{20}+1\right)}} \begin{bmatrix} 4 & -2 \\ -6 & \frac{9}{s+1} \end{bmatrix}$$

$$G^{-1}(s)^T = \frac{\left(\frac{s}{20}+1\right)(s+1)}{12(2-s)} \begin{bmatrix} 4 & -6 \\ -2 & \frac{9}{s+1} \end{bmatrix}$$

Now calculate the RGA element-wise:

$$RGA(G(s))_{11} = \frac{1}{\cancel{\frac{s}{20}+1}} \frac{9}{\cancel{s+1}} \frac{\cancel{\left(\frac{s}{20}+1\right)}\cancel{(s+1)}}{12(2-s)} \cdot 4$$

$$= \frac{3}{2-s}$$

Furthermore

$$RGA(G(s))_{12} = \frac{1}{\cancel{\frac{s}{20}+1}} \cdot 2 \cdot \frac{\cancel{\left(\frac{s}{20}+1\right)}(s+1)}{12(2-s)} \cdot (-6) = \frac{-(s+1)}{2-s}$$

$$RGA(G(s))_{21} = \frac{1}{\cancel{\frac{s}{20}+1}} \cdot 6 \cdot \frac{\cancel{\left(\frac{s}{20}+1\right)}(s+1)}{12(2-s)} \cdot (-2) = \frac{-(s+1)}{2-s}$$

$$RGA(G(s))_{22} = \frac{1}{\cancel{\frac{s}{20}+1}} \cdot 4 \cdot \frac{\cancel{\left(\frac{s}{20}+1\right)}\cancel{(s+1)}}{12(2-s)} \cdot \frac{9}{\cancel{s+1}} \quad \frac{3}{2-s}$$

Now evaluate at $s = i \cdot 0$ and $s = i \cdot \omega_c$:

$s = 0 \cdot i$

$$RGA(G(0)) = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \Rightarrow \text{avoid pairing } u_2 \to y_1 \text{ and } u_1 \to y_2$$

$s = i\omega_c$

Now we chose $\omega_c = 20 \frac{rad}{s}$

$$RGA(G(i\omega_c))_{11} = \frac{3}{2-20i} \cdot \frac{(2+20i)}{(2+20i)} = \frac{6+60i}{404}$$

contradictory

$$RGA(G(i\omega_c))_{12} = \frac{-(20i+1)}{2-20i} \frac{(2+20i)}{(2+20i)} = \frac{-60i+398}{404}$$

$$\Rightarrow RGA(G(i\omega_c)) = \begin{bmatrix} \frac{6+60i}{404} & \frac{398-60i}{404} \\ \frac{398-60i}{404} & \frac{6+60i}{404} \end{bmatrix}$$

suggest to pair $u_1 \to y_2$ and $u_2 \to y_1$

since the off-diagonal elements are closest to 1. (draw these points in the complex plane!)

This means, that decentralized control might work poorly for the chosen crossover frequency.

10.6) Decentralized vs. Decoupled control

Consider $y = G(s)\,u$

$$= \begin{bmatrix} \frac{1}{s+1} & \frac{0.5}{(s+1)^2} \\ 0 & \frac{0.5}{s+1} \end{bmatrix} u$$

$\Rightarrow$ Note that there is no coupling between $u_1 \rightarrow y_2$ (element $G_{21}(s) = 0$)

a) Determine poles and zeros

Poles:
- $\det(G(s)) = \frac{0.5}{(s+1)^2}$ (order 2 minor)
- $\frac{1}{s+1}$, $\frac{0.5}{(s+1)^2}$, $\frac{0.5}{s+1}$ (order 1 minors)

$\Rightarrow$ The least common denominator is $(s+1)^2$ and hence there is a double pole at $s=-1$

Zeros:

The maximal minor is $\frac{0.5}{(s+1)^2}$ and hence there are no zeros.

Note: it is already "normed"

b) Determine the RGA at $\omega = 0$ and suggest a pairing rule!

$\Rightarrow$ Due to $G_{21}(s) = 0$ we can already assume what will happen...

At $\omega = 0$ we get

$$G(i\cdot 0) = \begin{bmatrix} 1 & 0.5 \\ 0 & 0.5 \end{bmatrix}$$

and hence

$$G^{-1}(i\cdot 0) = 2\cdot\begin{bmatrix} 0.5 & -0.5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

$$\left(G^{-1}(i\cdot 0)\right)^T = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

$$RGA(G(i0)) = \underset{\text{pointwise multiplication}}{G \mathbin{.*} (G^{-1})^T} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

→ no negative pairings, hence no restriction how to couple inputs and outputs

For $\omega_c = 1\ \mathrm{rad/s}$, we also get

$$RGA(G(i\omega_c)) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which tells us to couple $u_1 \rightarrow y_1$
and $u_2 \rightarrow y_2$

c) Compute two PI controller:

$$K_i\left(1+\frac{1}{sT_i}\right) \qquad i=1,2$$

with $\boxed{\omega_c = 1\,\frac{rad}{s}}$ and $\boxed{\phi_m = 90°}$ for each pairing.

Approach: 1) adjust $\arg(L(i\omega_c)) = -90°$ by T_i
2) adjust $|L(i\omega_c)| = 1$ by K_i

$u_1 \to y_1$: $$L_1(s) \stackrel{!}{=} G_{11}(s) \cdot F_1(s) = \frac{1}{s+1} \cdot K_1\left(1+\frac{1}{sT_1}\right)$$
$$= \frac{K_1}{s+1}\left(\frac{sT_1+1}{sT_1}\right)$$

1) $$\arg(L_1(i\omega_c)) = -\arctan\left(\frac{\omega_c}{1}\right) + \arctan\left(\frac{\omega_c T_1}{1}\right) - \frac{\pi}{2}$$
$$= -\arctan(1) + \arctan(T_i) - 90$$
$$= -45 + \arctan(T_i) - 90$$

$$\phi_m = 180 + \arg(L_1(i\omega_c)) = 90$$ requirement

$$\Leftrightarrow \quad 45 + \arctan(T_i) = 90$$

$$\Leftrightarrow \quad T_i = 1$$

2) $$|L(i\omega_c)| = \frac{K_1}{\sqrt{\omega_c^2+1^2}} \frac{\sqrt{(\omega_c T_1)^2+1^2}}{\sqrt{(\omega_c T_1)^2}} \stackrel{\omega_c=1,\ T_1=1}{=} K_1 = 1$$ requirement

(easy choice)

$u_2 \to y_2$

$$L_2(s) = \frac{0.5}{s+1} K_2 \left(\frac{sT_2 + 1}{sT_2} \right)$$

$$\arg(L_2(i\omega_c)) = \arg(L_1(i\omega_c))$$

and hence $T_2 = T_1$

to achive the right gain

$$|L_2(i\omega_c)| = \frac{0.5 K_2}{\sqrt{\omega_c^2 + 1^2}} \frac{\sqrt{(\omega_c T_2)^2 + 1^2}}{\sqrt{(\omega_c T_2)^2}} = 0.5 K_2 \overset{!}{=} 1$$

$$K_2 = 2$$

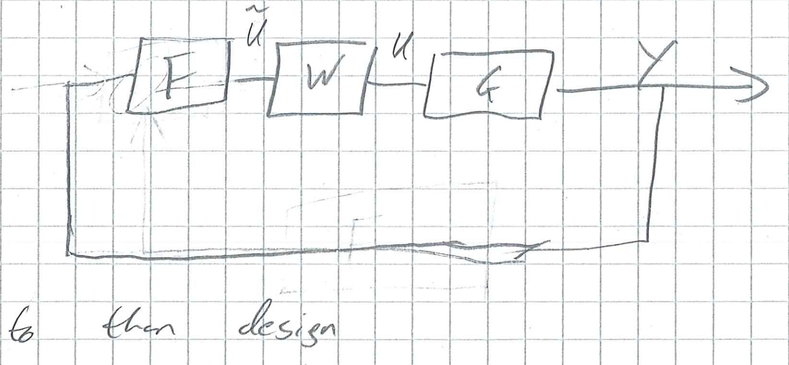
d) Determine a decoupling

$$U(s) = \begin{bmatrix} 1 & W_{12}(s) \\ W_{21}(s) & 1 \end{bmatrix} \tilde{U}(s) = W(s)\tilde{U}(s)$$

The idea is to get

$$Y(s) = G(s) W(s) \tilde{U}(s)$$

s.t. $\boxed{G(s)W(s) \text{ is diagonal s.t.}}$



to then design

$$\tilde{U} = F y$$

The full controller is then given by,

$$U = W F y$$

Now the calculations:

$$G(s)W(s) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} 1 & W_{12} \\ W_{21} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} G_{11} + G_{12}W_{21} & G_{11}W_{12} + G_{12} \\ G_{21} + G_{22}W_{21} & G_{21}W_{12} + G_{22} \end{bmatrix}$$

We need:

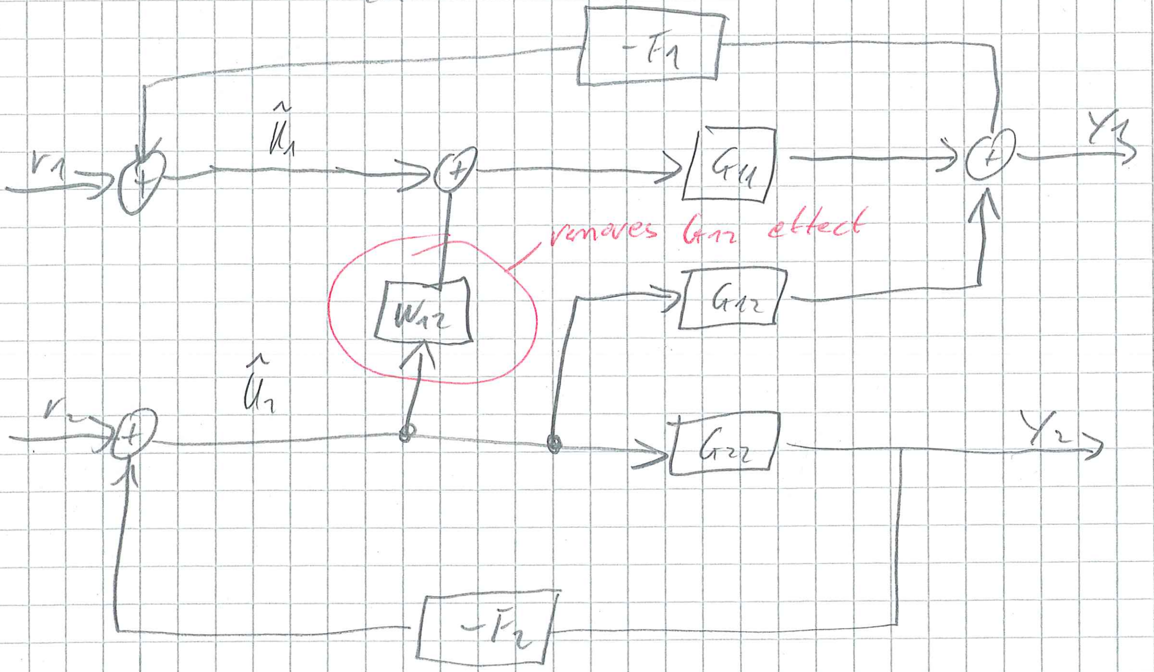
$$\text{I} \quad G_{11}W_{12} + G_{12} = 0 \qquad \Rightarrow \qquad W_{12} = -G_{11}^{-1}G_{12}$$

$$\text{II} \quad G_{21} + G_{22}W_{21} = 0 \qquad\qquad W_{21} = -G_{22}^{-1}G_{21}$$

might give problems with ~~zer~~ RHP poles/zeros

We get $W_{12} = -\frac{0.5}{s+1}$

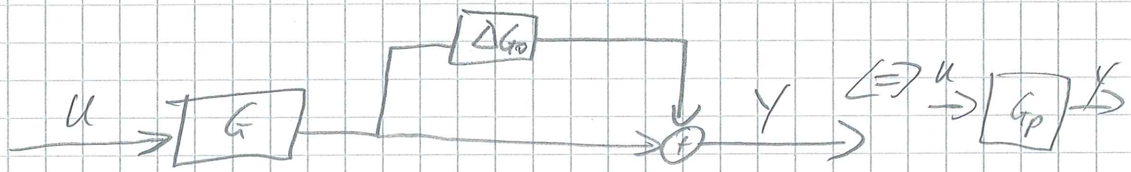
$W_{21} = 0$



Excursion: Uncertainties (Robustness)

For the next exercise we consider

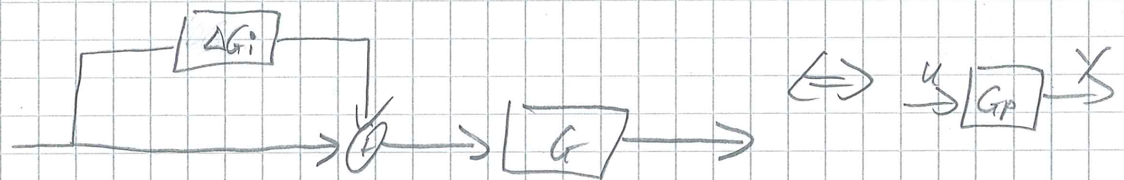
multiplicative $\boxed{\text{output}}$ uncertainty



with $G_p = (I + \Delta G_o) G$ and we need $\| \Delta G_o T \|_\infty < 1$

Recall $T = (I + GF_y)^{-1} GF_y$

and multiplicative $\boxed{\text{input}}$ uncertainty



with $G_p = G(I + \Delta G_i)$

and we need $\| \Delta G_i T_I \|_\infty < 1$

with $T_I = (I + F_y G)^{-1} F_y G$

These results stem from the application of the small-gain theorem! (see lectures)

10.4) Consider the plant model

$$(*)\quad G(s) = \frac{1}{s+1}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

while the true process is given by

$$(**)\quad G_p(s) = \frac{1}{s+1}\begin{bmatrix} 2 & \alpha \\ 0 & 2\alpha \end{bmatrix}$$

uncertainties with respect to the second input

a) Design an inverse based controller

$$F(s) = \frac{k}{s} G^{-1}(s)$$

with k such that closed loop with $(*)$ has the same poles as the plant $(*)$

The closed-loop is

$$G_{cl}(s) = (I + GF)^{-1} GF \checkmark$$

with $GF = G \cdot G^{-1} \frac{k}{s} = \frac{k}{s} I$

such that $G_{cl}(s) = \frac{k}{s(1+\frac{k}{s})} I = \frac{k}{s+k} I$

Note that $(*)$ has two poles at -1, hence choosing $k=1$ gives the same speed of G and closed-loop.

Note: $G_{cl}(s)$ has also two poles at -1.

b) Remember that we model uncertainties as

$$G_p = (I + \Delta G_0) G$$

multiplicative output uncertainty

with stability if $\boxed{\|\Delta G_0 T\|_\infty = \sup_\omega \bar{\sigma}(\Delta G_0 T) < 1}$

① Calc. ΔG_0
② Calc. T
③ Calc. $\bar{\sigma}(\Delta G_0 T)$
④ Form supremum
⑤ Find suitable α!

1 We have to calculate ΔG_0 by considering

$$G_p = (I + \Delta G_0) G$$

$$\Leftrightarrow \quad G_p G^{-1} - I = \Delta G_0$$

We have $G^{-1} = \begin{bmatrix} 0.5s + 0.5 & -0.25s - 0.25 \\ 0 & 0.5s + 0.5 \end{bmatrix}$

$$\Leftrightarrow \quad \Delta G_0 = \frac{1}{s+1} \begin{bmatrix} 2 & \alpha \\ 0 & 2\alpha \end{bmatrix} \begin{bmatrix} 0.5s + 0.5 & -0.25s - 0.25 \\ 0 & 0.5s + 0.5 \end{bmatrix} - I$$

$$\Leftrightarrow \quad \Delta G_0 = \frac{1}{s+1} \begin{bmatrix} s+1 & -0.5s - 0.5 + 0.5\alpha s + 0.5\alpha \\ 0 & \alpha s + \alpha \end{bmatrix} - I$$

$$= \frac{1}{s+1} \begin{bmatrix} 0 & (s+1)(\frac{1}{2}\alpha - \frac{1}{2}) \\ 0 & (s+1)(\alpha - 1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{2}\alpha - \frac{1}{2} \\ 0 & \alpha - 1 \end{bmatrix}$$

② Next, calculate $T = G_{cc} = (I+GF)^{-1}GF$

only if $F_y = F_r$

$$= \frac{1}{s+1} I$$

remembe $k=1$

We now have $\Delta G_0 T = \frac{1}{s+1}\begin{bmatrix} 0 & \frac{1}{2}\alpha - \frac{1}{2} \\ 0 & \alpha - 1 \end{bmatrix}$

Recall;
we need: $\|\Delta G_0 T\|_\infty = \sup_\omega \bar{\sigma}(\Delta G_0 T) < 1$

③ Calculate $\bar{\sigma}(\Delta G_0(i\omega)T(i\omega))$:

$$(\Delta G_0 T)^* \Delta G_0 T = \frac{1}{|1+i\omega|^2}\begin{bmatrix} 0 & 0 \\ \frac{1}{2}\alpha - \frac{1}{2} & \alpha - 1 \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{2}\alpha - \frac{1}{2} \\ 0 & \alpha - 1 \end{bmatrix}$$

$$= \frac{1}{|1+i\omega|^2}\begin{bmatrix} 0 & 0 \\ 0 & \left(\frac{1}{2}\alpha - \frac{1}{2}\right)^2 + (\alpha - 1)^2 \end{bmatrix}$$

$$= \frac{(\alpha - 1)^2}{|1+i\omega|^2}\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1.25 \end{bmatrix}}$$

the Maximum eigenvalue of this matrix is 1.25 and the minimal eigenvalue is 0

Hence $\bar{\sigma}(\Delta G_0 T) = \sqrt{\frac{1.25(\alpha - 1)^2}{|1+i\omega|^2}}$

$$= \frac{\sqrt{1.25}\,|\alpha - 1|}{|1+i\omega|}$$

$|\alpha - 1|$?

4 $\|\Delta G_0 T\|_\infty = \sup_\omega \left| \frac{\sqrt{1.25}\,|\alpha - 1|}{|1 + i\omega|} \right| = \sqrt{1.25}\,|\alpha - 1|$

abs. value gives 2 cases

5 $\sqrt{1.25}\,|\alpha - 1| < 1$

i) $\sqrt{1.25}(\alpha - 1) < 1$

$\alpha < \frac{1}{\sqrt{1.25}} + 1$

ii) $\sqrt{1.25}(1 - \alpha) < 1$

$1 - \frac{1}{\sqrt{1.25}} < \alpha$

$$\Rightarrow \boxed{0.1056 < \alpha < 1.8944}$$

c) We now consider multiplicative input uncertainty

Repeat the steps from b)

$G_p = G(I + \Delta G_i)$ and we again

have $\|\Delta G_i T_I\|_\infty < 1$

comes from the derivation of the stability criterion

with $T_I = (I + FG)^{-1} FG = T = \frac{1}{s+1} I$

since F is a diagonal controller no difference

Now calculate ΔG_i

$$\Rightarrow \Delta G_i = G^{-1} G_p - I$$

$$= \begin{bmatrix} 0.5s + 0.5 & -0.25s - 0.25 \\ 0 & 0.5s + 0.5 \end{bmatrix} \begin{bmatrix} 2 & \alpha \\ 0 & 2\alpha \end{bmatrix} \frac{1}{s+1} - I$$

$$= \begin{bmatrix} s+1 & 0.5s\alpha + 0.5\alpha - 0.5s\alpha - 0.5\alpha \\ 0 & \alpha s + \alpha \end{bmatrix} \frac{1}{s+1} - I$$

$$= \frac{1}{s+1}\begin{bmatrix} 0 & 0 \\ 0 & (\alpha-1)(s+1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & \alpha-1 \end{bmatrix}$$

$$\Delta G_i T_I = \frac{\alpha-1}{s+1}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow (\Delta G_i T_I)^* \Delta G_i T_I = \frac{(\alpha-1)^2}{|s+1|^2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

maximum eigenvalue is 1

$$\bar{\sigma}(\Delta G_i T_I) = \frac{|\alpha-1|}{|s+1|}$$

$$\sup \bar{\sigma}(\Delta G_i T_I) = |\alpha-1|$$

2 cases again

i) $\alpha - 1 < 1$

$\alpha < 2$

ii) $-\alpha+1 < 1$

$0 < \alpha$

and hence $0 < \alpha < 2$

$\Rightarrow$ Modeling makes a difference

d) Calculate the closed-loop poles

We have

$$G_{cl} = \underbrace{(I + G_p F)^{-1}}_{1} \underbrace{G_p F}_{2} \quad \text{with} \quad F = \frac{1}{s} G^{-1}(s)$$

1 $$I + G_p F = I + \frac{1}{s+1}\begin{bmatrix} 2 & \alpha \\ 0 & 2\alpha \end{bmatrix}\begin{bmatrix} 0.5s+0.5 & -0.25s-0.25 \\ 0 & 0.5s+0.5 \end{bmatrix}\frac{1}{s}$$

$$= \frac{1}{s(s+1)}\left(\begin{bmatrix} s(s+1) & 0 \\ 0 & s(s+1) \end{bmatrix} + \begin{bmatrix} s+1 & (s+1)(\frac{1}{2}\alpha - \frac{1}{2}) \\ 0 & \alpha(s+1) \end{bmatrix}\right)$$

$$= \frac{1}{s}\begin{bmatrix} s+1 & \frac{1}{2}\alpha - \frac{1}{2} \\ 0 & \alpha + s \end{bmatrix}$$

$$\det(I + G_p F) = \left(\frac{s+1}{s}\right)\left(\frac{\alpha+s}{s}\right) = \frac{(s+1)(\alpha+s)}{s^2}$$

$$(I + G_p F)^{-1} = \frac{\cancel{s^2}}{(s+1)(\alpha+s)}\begin{bmatrix} \alpha+s & \frac{1}{2} - \frac{1}{2}\alpha \\ 0 & s+1 \end{bmatrix} \cdot \frac{1}{\cancel{s}}$$

$$(I + G_p F)^{-1} G_p F = \frac{\cancel{s}}{(s+1)(\alpha+s)}\begin{bmatrix} \alpha+s & \frac{1}{2} - \frac{1}{2}\alpha \\ 0 & s+1 \end{bmatrix} \cdot \frac{1}{(s+1)\cancel{s}}\begin{bmatrix} s+1 & (s+1)(\frac{1}{2}\alpha - \frac{1}{2}) \\ 0 & \alpha(s+1) \end{bmatrix}$$

$$= \frac{1}{(s+1)^2(\alpha+s)}\begin{bmatrix} (\alpha+s)(s+1) & (\alpha+s)(s+1)(\frac{1}{2}\alpha - \frac{1}{2}) - (\frac{1}{2}\alpha - \frac{1}{2})(s+1)\alpha \\ 0 & \alpha(s+1)^2 \end{bmatrix}$$

$$= \frac{1}{(s+1)(\alpha+s)}\begin{bmatrix} \alpha+s & (\frac{1}{2}\alpha - \frac{1}{2})(\alpha + s - \cancel{\alpha}) \\ 0 & \alpha(s+1) \end{bmatrix}$$

$$= \frac{1}{(s+1)(\alpha+s)}\begin{bmatrix} \alpha+s & \left(\frac{\alpha-1}{2}\right)s \\ 0 & \alpha(s+1) \end{bmatrix}$$

Calculate the least common denominator of the minors:

$\rightarrow \quad \frac{1}{s+1}, \quad \frac{\left(\frac{\alpha-1}{2}\right)s}{(s+1)(\alpha+s)}, \quad 0, \quad \frac{\alpha}{\alpha+s}$ (1st order)

$\rightarrow \quad \frac{1}{(s+1)^{\cancel{2}}(\alpha+s)^{\cancel{2}}}\left((\cancel{\alpha+s})\,\alpha\,(\cancel{s+1})\right)$ (2nd order)

LCD is $(s+1)(\alpha+s)$ and hence we have the poles $s=-1$ and $s=-\alpha$

The closed-loop is stable for $\alpha>0$, which is not as conservative as the previous results.