# EL2520 - Control Theory and Practice - Advanced Course

## Solution/Answers (not complete solutions) – 2016-08-19

1. (a) Poles in $s = -1, s = -1$, zeros in $s = \pm i$.

   (b) From $T = GF_y/(1 + GF_y)$ we get $F_y = (1/G)(T/(1-T)) = (2s+1)/(s(-s+1))$, and from $S = 1 - T$ we get $S = s/(s+1)$. Thus, for instance the transfer function $SF_y$ from setpoint to input is unstable with a pole in $s = 1$. Hence, the system is not internally stable (as is always the case when we cancel zeros in the RHP).

   (c) We have, for a one-degree of freedom controller, $Y = TR$ with $T = GF_y(I + GF_y)^{-1}$ which gives $F_y = G^{-1}(I - T)^{-1}T = \frac{1}{\lambda s}I$.

2. (a) (i) The system has no delays. The poles are in $s = -1$ and hence no unstable poles. The system has a zero at $s = 0.5$ which represents a bandwidth limitation. (ii) The RGA at $\omega = 0$ is

   $$\Lambda(0) = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

   Since we should never pair on negative steady-state RGA elements, this suggests pairing $u_1 - y_2$ and $u_2 - y_1$. The abolute values of the corresponding RGA elements at $\omega = 0.1$ is about 2 which is relatively close to 1 and hence this pairing should provide reasonable performance with decentralized control.

   (b) (i) The requirements are

   $$\bar{\sigma}(S) < 0.01 \text{ for } w = 0 \tag{1}$$
   $$\bar{\sigma}(S) < 0.1 \text{ for } w < 0.1 \tag{2}$$
   $$\bar{\sigma}(T) < 0.1 \text{ for } w > 10 \tag{3}$$

   (ii) which give the following requirements on the loop gains

   $$\underline{\sigma}(L) > 100 \text{ for } w = 0 \tag{4}$$
   $$\underline{\sigma}(L) > 10 \text{ for } w < 0.1 \tag{5}$$
   $$\bar{\sigma}(L) < 0.1 \text{ for } w > 10 \tag{6}$$

   The loop gain must have a slope at least $-1$ between $\omega = 0.1$ and $\omega = 10$ but this should normally not pose any problem.

3. (a) We have $|G_d(i \cdot 0.98)| = 1$ and hence we need sensitivity reduction up to the frequency $\omega_d = 0.98$. For input $u_1$ there is a delay of $T = 2$ which means that the sensitivity function $|S| \geq 1$ for $\omega > 0.5$, i.e., we can not get acceptable control with only this input. For input $u_2$ there exist no fundamental limitation, remains to check if it has

sufficient power to reject disturbances up to $\omega_d$. For perfect control we get $u_2 = -G_2^{-1}G_d d$ where $G_2^{-1}G_d = (1/4)(10s+1)/(5s+1)$ which has magnitude less than one for all frequencies, thus input $u_2$ can in theory be used to reject the disturbance $d$ completely. Answer: acceptable control can be achieved by using only input $u_2$.

(b) The sensitivity function $S = s/(s+1)$ provides the required disturbance attenuation up to frequency $\omega_d$, with some margin. The corresponding controller is given by $1 + G_2(s)F(s) = (s+1)/s$ which yields

$$F(s) = \frac{10s+1}{20s}$$

For the input we get $u_2 = FSG_d d$ with $FSG_d = (1/4)(10s+1)/((5s+1)(s+1))$ which is easily shown to have $|FSG_d| < 1 \forall\omega$

(c) The requirements are $|y| < 1$ and $|u| < 1$ for $|d| < 1$, for all frequencies. This corresponds to $|SG_d| < 1\forall\omega$ and $|FSG_d| < 1\forall\omega$, or $\|SG_d\|_\infty < 1$ and $\|FSG_d\|_\infty < 1$, and a stacked objective function reflecting these requirements is

$$J = \left\| \begin{matrix} SG_d \\ FSG_d \end{matrix} \right\|_\infty$$

4. (a) $G(s) = C(sI - A)^{-1}B$ which yields

$$G(s) = \begin{pmatrix} \frac{1}{s-3} & 0 \\ \frac{2}{s-3} & \frac{2}{s+2} \end{pmatrix}$$

(b) All matrices diagonal in the Riccati equation, hence we can solve for the two states separately. For state $x_1$, $6S_1 + 1 - S_1^2 = 0$ for which the positive solution is $S_1 = 3 + \sqrt{10}$ and $L_1 = S1$. For state $x_2$, $-4S_2 + 2 - 4S_2^2 = 0$ with positive solution $S_2 = (-1+\sqrt{3})/2$ and $L_2 = 2S_2 = -1 + \sqrt{3}$. Thus,

$$L = \begin{pmatrix} 3+\sqrt{10} & 0 \\ 0 & -1+\sqrt{3} \end{pmatrix}$$

The closed-poles are eigenvalues of $A - BL$, $s = -3.46$ and $s = -3.16$.

(c) The loop-gain, as seen from the input $u$, is

$$G_k(s) = L(sI - A)^{-1}B = \begin{pmatrix} \frac{6.16}{s-3} & 0 \\ 0 & \frac{1.46}{s+2} \end{pmatrix}$$

Hence, $T_I = G_k(I + G_k)^{-1}$ and

$$T_I = \begin{pmatrix} \frac{6.16}{s+3.16} & 0 \\ 0 & \frac{1.46}{s+3.46} \end{pmatrix}$$

(d) The closed-loop system is diagonal, corresponding to two scalar loops, and the robustness criterion for each loop is $|T\Delta_G| < 1\ \forall\omega$. Thus, the loop with the smallest $|T| = |T_I|$ can tolerate the largest uncertainty $\Delta_G$. Since $T_2$ has smaller magnitude than $T_1$ for all frequencies, we can tolerate the largest uncertainty in input 2.

(a) The complementary sensitivity is

$$T = I - S = \frac{1}{s+1}\begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{pmatrix}$$

The system is robustly stable for relative output uncertainty $\Delta_G$ if $\|T\|_\infty < 1/\|\Delta_G\|_\infty$, or $\bar{\sigma}(T) < 1/\bar{\sigma}(\Delta_G)\forall\omega$. The singular values of $T$ are given by $\sigma(T) = \sqrt{\lambda(T^H T)}$. We have

$$T^H T = \frac{1}{(s+1)^2}\begin{pmatrix} 0.82 & 0.18 \\ 0.18 & 0.82 \end{pmatrix}$$

which has eigenvalues $\lambda_i = \frac{1}{(s+1)^2}(0.64, 1)$. Hence

$$\|T\|_\infty = \sup_\omega \left| \frac{1}{i\omega + 1} \right| = 1$$

Conclusion: we can at least tolerate uncertainty on the output which is stable and such that $\|\Delta_G\|_\infty < 1$

(b) Controller with one degree of freedom, that is, the relation between setpoints and control errors is given by the sensitivity function $S$. At steady-state, for steps $R$ in the setpoint, the control errors are given by $S(0)R$. The largest and smallest control error, in terms of the 2-norm of the error $e$, is given by the eigenvalues to $S^T(0)S(0)$. The corresponding directions are given by the corresponding eigenvectors. The largest error, in a 2-norm sense, is given by $\bar{\sigma}(S(0)) = 0.2$ with direction

$$\bar{r} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The smallest error $\underline{\sigma}(S(0)) = 0$ is obtained with

$$\underline{r} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$