Exercise session 10

Today: 13.1, 13.2, 13.3

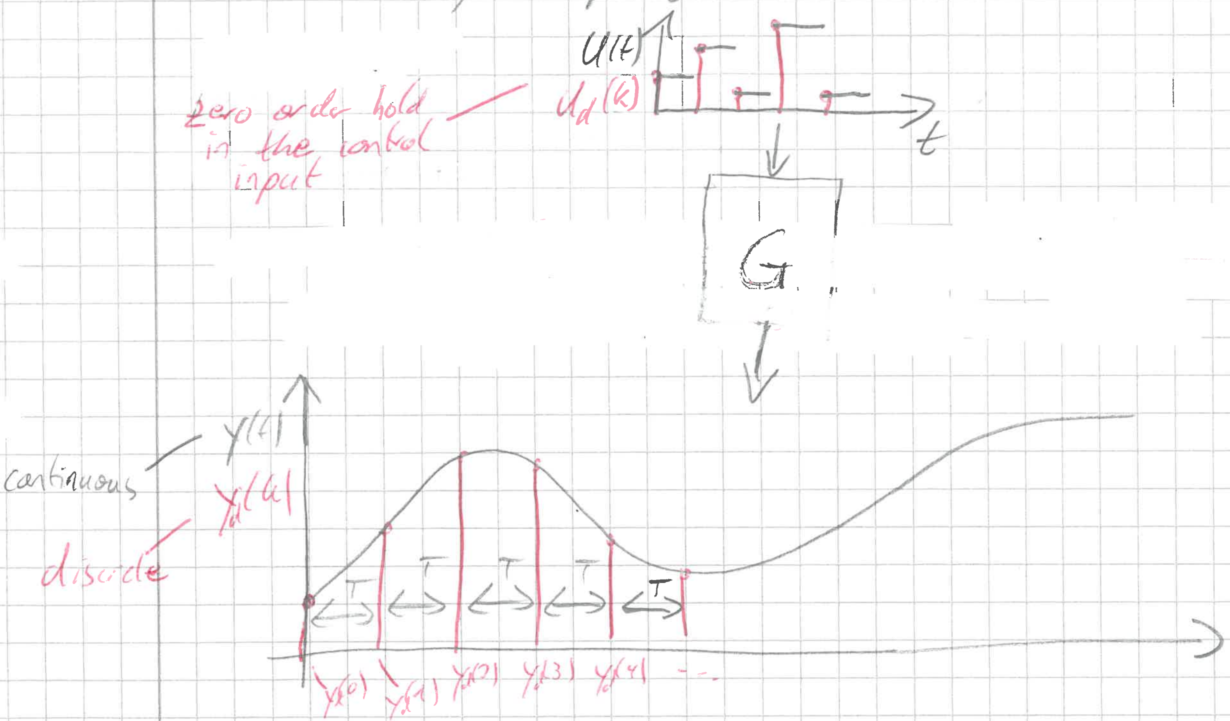
— Discrete time systems

— Model Predictive Control (including hard constraints)

advantage

Discrete time systems, are for instance obtained by sampling a continuous time system.

as in Matlab simulations



Periodic sampling means $y_d(k) = y(k \cdot T)$, where T is the sampling period.

Now given

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

and assuming that $u(t) = \text{constant}$ for all $t \in [kT, (k+1)T)$

we get:

$$x_d(k+1) = F\, x_d(k) + G\, u_d(k)$$
$$y_d(k) = C\, x_d(k)$$

with $F = e^{AT}$

$$G = \int_0^T e^{A\tau} B \, d\tau$$

// can be derived by looking at the solution $x(t) = e^{At} x(0) + \int_0^t e^{A(t-s)} Bu(s)\, ds$ at times $t = kT$

Note e^{AT} is the matrix exponential

$\Rightarrow$ expm(AT) in Matlab

with $e^{AT} = I + \sum_{i=1}^{\infty} \frac{A^i T^i}{i!}$

or $e^{AT} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}\big|_{t=T}$

Controllability: Controllable iff

$R = [G \;\; FG \; \dots \; F^{n-1}G]$ has full rank

Observability: Observable iff

$$O = \begin{bmatrix} C \\ CF \\ \vdots \\ CF^{n-1} \end{bmatrix}$$ has full rank

stability: stable iff

$|\lambda_i| < 1 \quad \forall i$

13.1)

Consider

rotational matrix

$$\dot{x}(t) = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) = Ax + Bu$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) = Cx$$

The sampled system is

$$x_d(k+1) = F x_d(k) + G u_d(k)$$
$$y_d(k) = C x_d(k)$$

with

$$F = e^{AT} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}\Big|_{t=T}$$
$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} s & -\omega \\ \omega & s \end{bmatrix}^{-1}\right\}\Big|_{t=T}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\begin{bmatrix} s & \omega \\ -\omega & s \end{bmatrix}\right\}\Big|_{t=T}$$
$$= \begin{bmatrix} \cos(\omega T) & \sin(\omega T) \\ -\sin(\omega T) & \cos(\omega T) \end{bmatrix}$$

$$G = \int_0^T \begin{bmatrix} \cos(\omega\tau) & \sin(\omega\tau) \\ -\sin(\omega\tau) & \cos(\omega\tau) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\tau$$
$$= \int_0^T \begin{bmatrix} \cos(\omega\tau) \\ -\sin(\omega\tau) \end{bmatrix} d\tau = \frac{1}{\omega}\begin{bmatrix} \sin(\omega\tau) \\ \cos(\omega\tau) \end{bmatrix}_0^T$$
$$= \frac{1}{\omega}\begin{bmatrix} \sin(\omega T) \\ \cos(\omega T) - 1 \end{bmatrix}$$

What if $T = \frac{n \cdot 2\pi}{\omega}$ for $n = 0, 1, \ldots$?

When is the system observable?

Form $O = \begin{bmatrix} C \\ CF \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cos(\omega T) & \sin(\omega T) \end{bmatrix}$

needs to have $\text{rank}(O) = 2$!

we need hence

$\Rightarrow \sin(\omega T) \neq 0$

Hence $\omega T \neq n \cdot \pi$

$\Rightarrow \boxed{T \neq \frac{n \cdot \pi}{\omega}}$

Model Predictive Control

Assume k to be the current time.

We use a finite prediction horizon N

to predict $x(k+1), x(k+2), \dots, x(k+N)$

by using $x(k+1) = Fx(k) + Gu(k)$ recursively

and solve

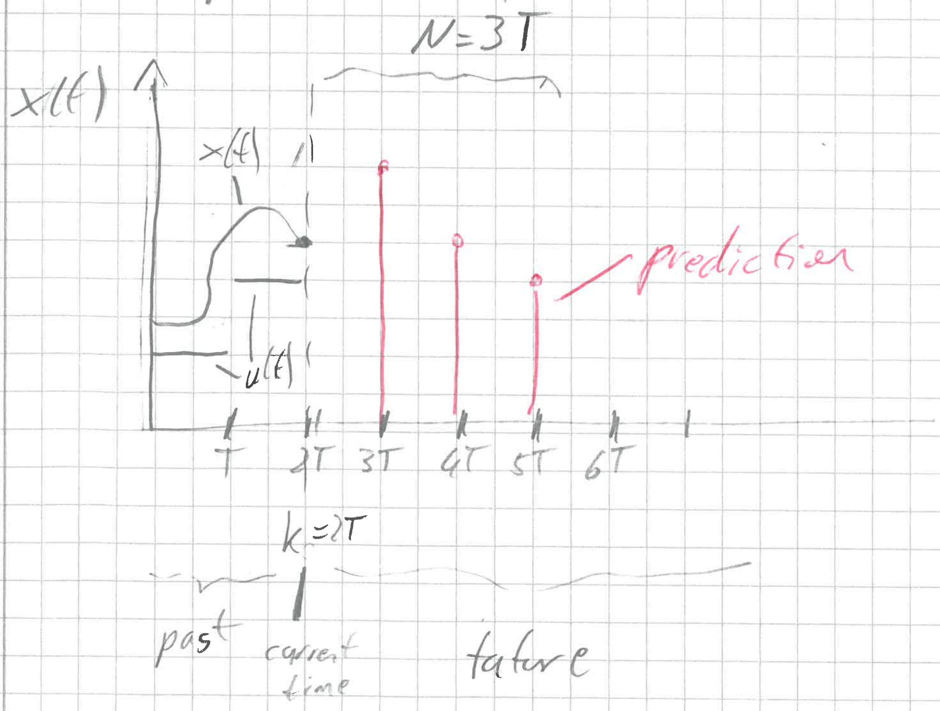
$$\min_{\bar{u}} \sum_{i=k+1}^{k+N} x^T(i)\, Q_1\, x(i) + u^T(i-1)\, Q_2\, u(i-1)$$

with $\bar{u} = \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+N-1) \end{bmatrix}$

at every sampling step. We then apply

$\bar{u}(1) = u(k)$ to the system and repeat

the procedure by setting $k = k+1$.



13.2) Consider

$$x(k+1) = \begin{bmatrix} 1.7 & 1 \\ -0.7 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0.9 \\ -0.6 \end{bmatrix} u(k) = F x(k) + G u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) = C x(k)$$

and determine

$$u(k) = -F_y\, y(k) + F_r\, r(k)$$

by minimizing

$$J = \sum_{i=1}^{3} \underbrace{\left(r(k+i) - y(k+i)\right)^2}_{\text{reference tracking}} + 0.1 \underbrace{u(k+i-1)^2}_{\text{input minimization}}$$

with respect to $\bar{u}$.

① $\bar{r} = \begin{bmatrix} r(k+1) \\ r(k+2) \\ r(k+3) \end{bmatrix}$ ← We assume to know this in advance

$\bar{y} = \begin{bmatrix} y(k+1) \\ y(k+2) \\ y(k+3) \end{bmatrix}$ ← estimated by the discrete state-space model

$\bar{u} = \begin{bmatrix} u(k) \\ u(k+1) \\ u(k+2) \end{bmatrix}$ ← to be determined

Then $J = (\bar{r} - \bar{y})^T (\bar{r} - \bar{y}) + 0.1\, \bar{u}^T \bar{u}$

② Find analytical expression for $\bar{y}$

$$x(k) = x(k)$$

$$x(k+1) = F x(k) + G u(k)$$

$$x(k+2) = F^2 x(k) + F G u(k) + G u(k+1)$$

$$x(k+3) = F^3 x(k) + F^2 G u(k) + F G u(k+1) + G u(k+2)$$

$y(k) = Cx(k)$

$y(k+1) = Cx(k+1)$

$\vdots$

Hence we get

$$\bar{Y} = \begin{bmatrix} y(k+1) \\ y(k+2) \\ y(k+3) \end{bmatrix} = \begin{bmatrix} CF \\ CF^2 \\ CF^3 \end{bmatrix} x(k) + \begin{bmatrix} CG & 0 & 0 \\ CFG & CG & 0 \\ CF^2G & CFG & CG \end{bmatrix} \bar{u}$$

$$= P x(k) + Q \bar{u} \qquad P \in \mathbb{R}^{N \times n}, \quad Q \in \mathbb{R}^{N \times N \cdot m}$$

Hence $J = (\bar{r} - Px(k) - Q\bar{u})^T(\bar{r} - Px(k) - Q\bar{u}) + 0.1\bar{u}^T\bar{u}$

$$= \bar{r}^T\bar{r} - 2\bar{r}^T P x(k) - 2\bar{r}^T Q\bar{u} + x^T(k)P^TPx(k)$$
$$+ 2x^T(k)P^TQ\bar{u} + \bar{u}^TQ^TQ\bar{u} + 0.1\bar{u}^T\bar{u}$$

(3) minimize only consider terms containing $\bar{u}$

$$\frac{\partial J}{d\bar{u}} = \frac{\partial}{d\bar{u}}\left(-2\bar{r}^TQ\bar{u} + 2x^T(k)P^TQ\bar{u} + \bar{u}^TQ^TQ\bar{u} + 0.1\bar{u}^T\bar{u}\right)$$

$$(*) \quad = -2\bar{r}^TQ + 2x^T(k)P^TQ + 2\bar{u}^T\underbrace{Q^TQ}_{\text{symmetric}} + 0.2\bar{u}^T$$

make sure that the dimensions are correct

Setting $(*) = 0$ gives

$$2\bar{r}^TQ - 2x^T(k)P^TQ = \bar{u}^T(2Q^TQ + 0.2I)$$

$$\Leftrightarrow (2\bar{r}^TQ - 2x^T(k)P^TQ)(2Q^TQ + 0.2I)^{-1} = \bar{u}^T \qquad \text{recall } (AB)^T = B^TA^T$$

$$\Leftrightarrow (2Q^TQ + 0.2I)^{-1}(2Q^T\bar{r} - 2Q^TPx(k)) = \bar{u}$$

$$\Leftrightarrow (Q^TQ + 0.1I)^{-1}Q^T(\bar{r} - Px(k)) = \bar{u}_{\lambda}$$

Inserting values, we get

$$(**) \quad u(k) = -\begin{bmatrix} 1.7 & 1.1 \end{bmatrix} x(k) + \begin{bmatrix} 0.9 & 0.1 & 0.01 \end{bmatrix} \bar{r}$$

$\Rightarrow$ Note that $\bar{u} = \begin{bmatrix} u(k) \\ u(k+1) \\ u(k+2) \end{bmatrix}$

Hence we only calculated the first elmt of $\bar{u}$ which is applied to the system until the procedure is repeated at the next sampling step.

Now: express $(**)$ as $u(k) = F_y y_{past} + F_r \bar{r}$

We know that (from the system model)

$$x_1(k) = y(k)$$

and

$$x_2(k) = -0.7\, x_1(k-1) - 0.6\, u(k-1)$$

$$= -0.7\, y(k-1) - 0.6\, u(k-1)$$

Hence $(**)$ becomes

$$u(k) = -1.7\, y(k) + 0.77\, y(k-1) + 0.66\, u(k-1)$$
$$+ 0.9\, r(k+1) + 0.1\, r(k+2) + 0.01\, r(k+3)$$

13.3)

Consider $y(k+1) = -y(k) + 2u(k)$
$= F\,y(k) + G\,u(k)$

OBS: Scalar System!

$\boxed{\begin{array}{l} F = -1 \\ G = 2 \end{array}}$

The MPC formulation is given as

$$(*) \quad \min_{u} \sum_{i=k}^{k+N_p} y(i)^2 + \sum_{i=k}^{k+N_p-1} u(i)^2$$

$$(**) \quad s.t \quad -1 \le u(k) \le 1$$

If $N_p = 1$, translate this into

$$\min_{u} u^T H u + h^T u$$
$$s.t \quad Lu \le b$$

1 Rewrite $(*)$ as:

$$\min_{u(k)} y(k)^2 + y(k+1)^2 + u(k)^2$$

$$\Leftrightarrow \min_{u(k)} y(k)^2 + (F y(k) + G u(k))^2 + u(k)^2$$

$$\Leftrightarrow \min_{u(k)} y(k)^2 + F^2 y(k)^2 + 2F y(k) G u(k) + G^2 u(k)^2 + u(k)^2$$

$$\Leftrightarrow \min_{u(k)} 2y(k)^2 - 4y(k)u(k) + 5u(k)^2$$

$$\Leftrightarrow \min_{u(k)} -4y(k)u(k) + 5u(k)^2$$

Hence we have $H = 5$ and $h^T = -4y(k)$

5 From $(**)$ we get

$$u(k) < 1$$

and $-u(k) < 1$

and hence $L = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$