EL2520
Control Theory and Practice

Lecture 9:
LQG (cont'd), H2-optimal control

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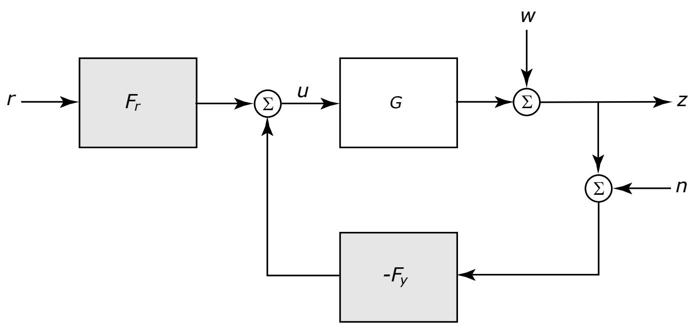
Controller Design – Signals vs Systems



$$z = Sw + Tn$$

- Design controller so that output z ''small'' in the presence of disturbance w and noise n
- Corresponds to making transfer-functions S and T ''small''
- Thus, we can either solve signal minimization problem, i.e., minimize some norm of z given inputs w and n, or solve corresponding transfer-function minimization problem.

H-infinity, LQG and H2

- $\mathcal{H}_\infty$: $$\min_{F_y} \sup_{w_e \neq 0} \frac{\|z_e\|_2}{\|w_e\|_2} = \min_{F_y} \|G_{ec}\|_\infty \; ; \quad z_e = G_{ec} w_e$$
  - minimize worst case amplification in terms of 2-norm of signals is equivalent to minimizing infinity-norm of corresponding transfer-function

- LQG: $$J = \mathbf{E}\left\{ \lim_{T \to \infty} \int_0^T [z^T Q_1 z + u^T Q_2 u] \, dt \right\}$$
  - minimize expected 2-norm of weighted output z and weighted input u when w and n white noise
  - *will show today:* special case of LQG corresponds to minimizing 2-norm of corresponding transfer-function

Today's lecture

- LQG recap and additional remarks
- A design example: radial control of DVD servo
- H2-optimal control

- Lec 10: Robust Loopshaping
- Lec 11: Case study and comparison of methods

Linear Quadratic Gaussian control

Model: linear system

$$\begin{aligned}\frac{d}{dt}x(t) &= Ax(t) + Bu(t) + Nv_1(t)\\ y(t) &= Cx(t) + v_2(t)\\ z(t) &= Mx(t)\end{aligned}$$

where v_1, v_2 are white noise with

$$\mathrm{cov}([v_1,\ v_2]) = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

Objective: minimize effect of noise on output, while punishing control cost

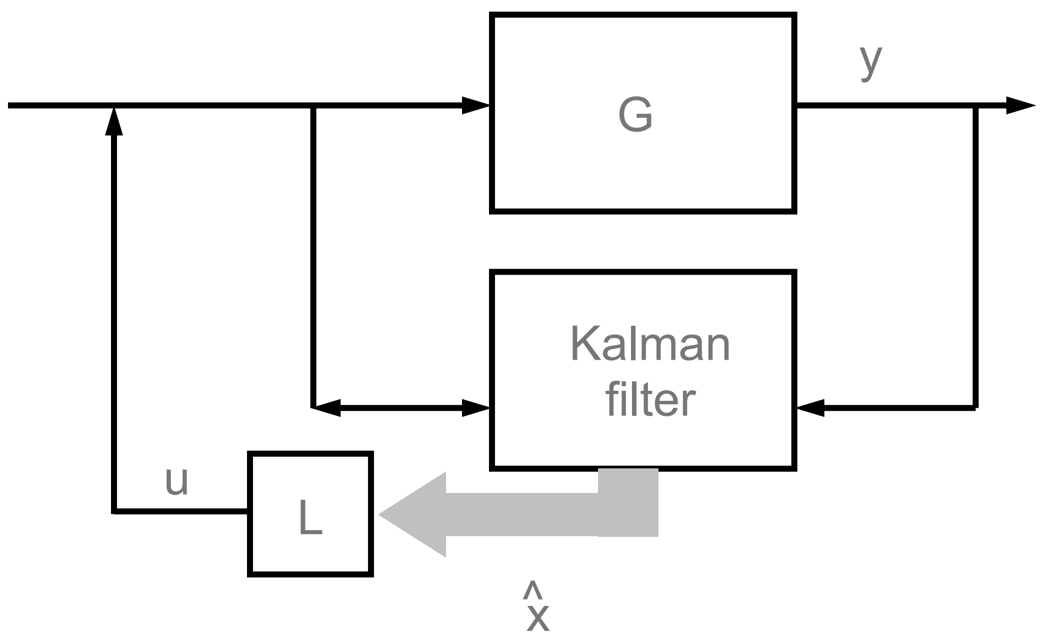
$$J = \mathbf{E}\left\{\lim_{T\to\infty}\int_0^T \left[z^T Q_1 z + u^T Q_2 u\right]\,dt\right\}$$

LQG: Linear system, Quadratic cost, Gaussian noise

Solution structure

Optimal solution satisfies *separation principle*, composed of

- Optimal linear state feedback (Linear Quadratic regulator)
- Optimal observer (Kalman filter)



The Optimal solution

State feedback

$$u(t) = -L\widehat{x}(t) = -Q_2^{-1}B^T S\widehat{x}(t)$$

where S is the solution to the algebraic Riccati equation

$$A^T S + SA + M^T Q_1 M - SBQ_2^{-1}B^T S = 0$$

Observer (Kalman filter)

$$\dot{\hat{x}} = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

where $K=(PC^T+NR_{12})R_2^{-1}$ and $P>0$ is the solution to Riccati equation

$$AP + PA^T + NR_1N^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T = 0$$

White Noise

- Inputs v_1 and v_2 are white noise signals with covariance matrices

$$E\{v_1 v_1^T\} = R_1 \ ; \quad E\{v_2 v_2^T\} = R_2$$

- The corresponding frequency spectra are

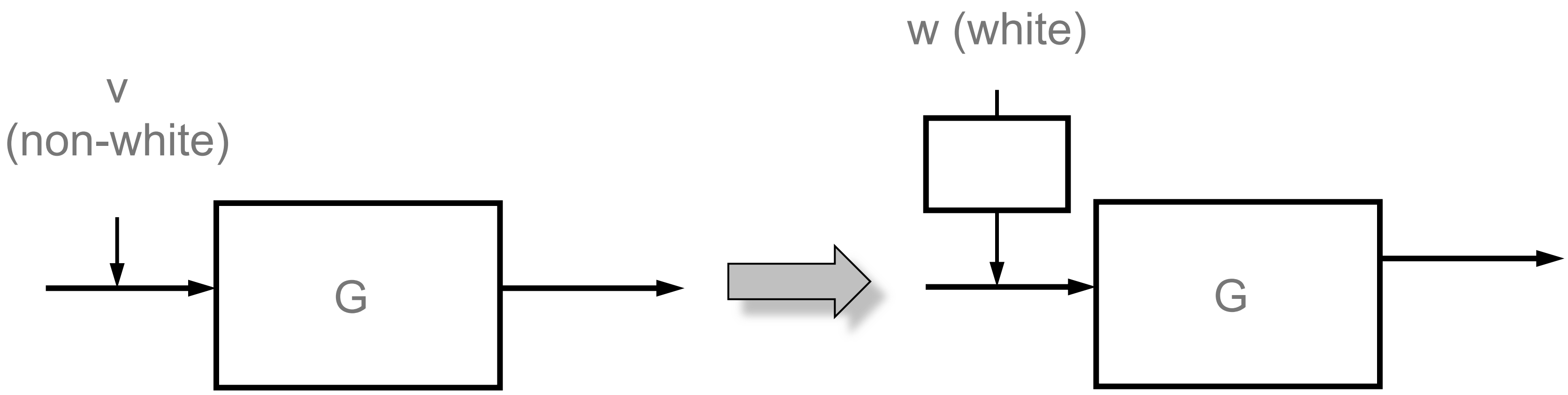
$$\Phi_{v_1}(\omega) = R_1 \ ; \quad \Phi_{v_2}(\omega) = R_2$$

  – white noise = constant spectra, i.e., same energy at all frequencies

Filtered White noise

Assumption of white noise no serious restriction:

– practically all disturbance spectra can be realized as filtered white noise
– need to augment system model with disturbance model



The servo problem

- The output z should follow a reference signal r
- Let r be the output of a linear system with white noise as input

➔ v_1 also includes the driving source of r

Extended system state: $$\begin{cases} z = M_1 x \\ r = M_2 x \end{cases}$$

Error: $e = r - z = [-M_1 \quad M_2]x = Mx$

Note: r is known, and can be included in the measurement $\overline{y} = \begin{bmatrix} y \\ r \end{bmatrix}$

Controller: $u = -F_{\overline{y}}\overline{y} = F_r r - F_y y$

Objective: minimize

$$J = \mathbf{E}\left\{ \lim_{T\to\infty} \int_0^T [e^T Q_1 e + u^T Q_2 u]\, dt \right\}$$

LQG and loop shaping

- LQG: simple to trade-off response-time vs. control effort
  - but what about sensitivity and robustness?
- These aspects can indirectly be accounted for using the noise models
  - Sensitivity function: transfer matrix w→z
  - Complementary sensitivity: transfer matrix n→z

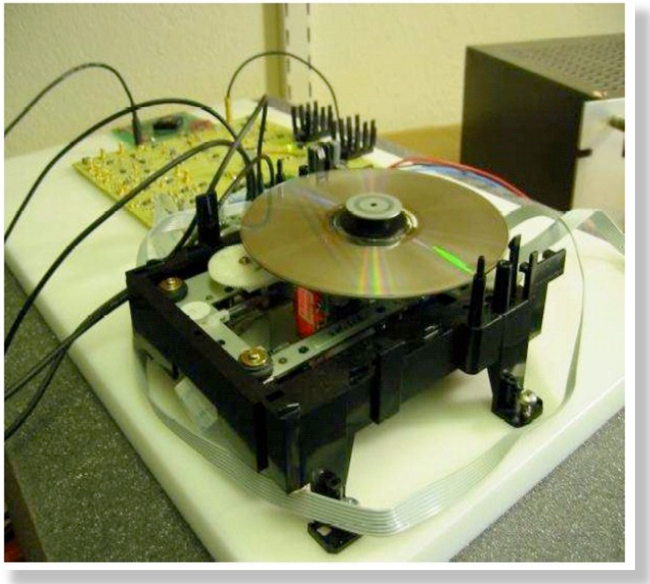
Example: S forced to be small at low frequencies by letting (some component of) w affect the output, and let w have large energy at low frequencies,

$$W(s) = \frac{1}{s+\delta} V(s)$$

(delta small, strictly positive, to ensure stabilizability)

Design example

Track following (radial control) in DVD player



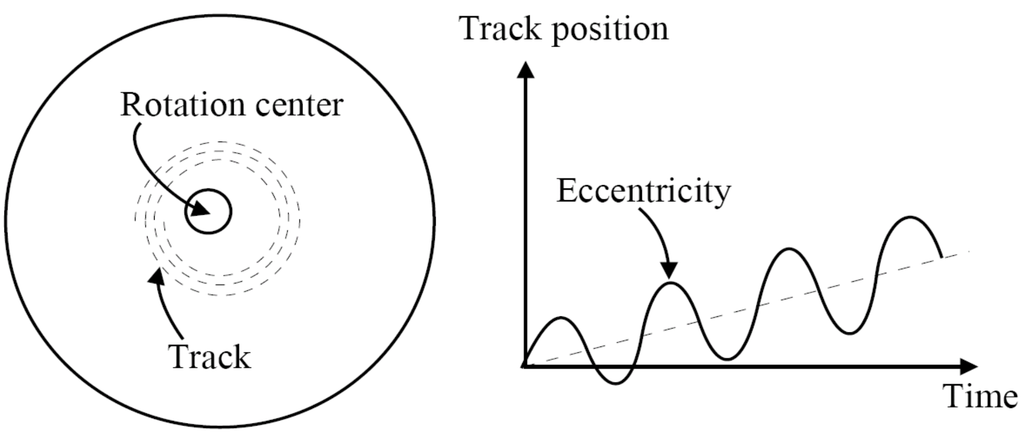
Pick up head moves 3.5 m/s and should only deviate 0.02μm from track and track is oscillating with amplitude up to 100μm per rotation (23 Hz) due to asymmetric disc

(based on laboratory exercise at LTH)

Design example

Control lens position to follow track

- high bandwidth to allow fast read/write
- key challenge: eccentricity of tracks on disk
- sinusoidal disturbance of order 100 track widths!



Model

Model identified from real system

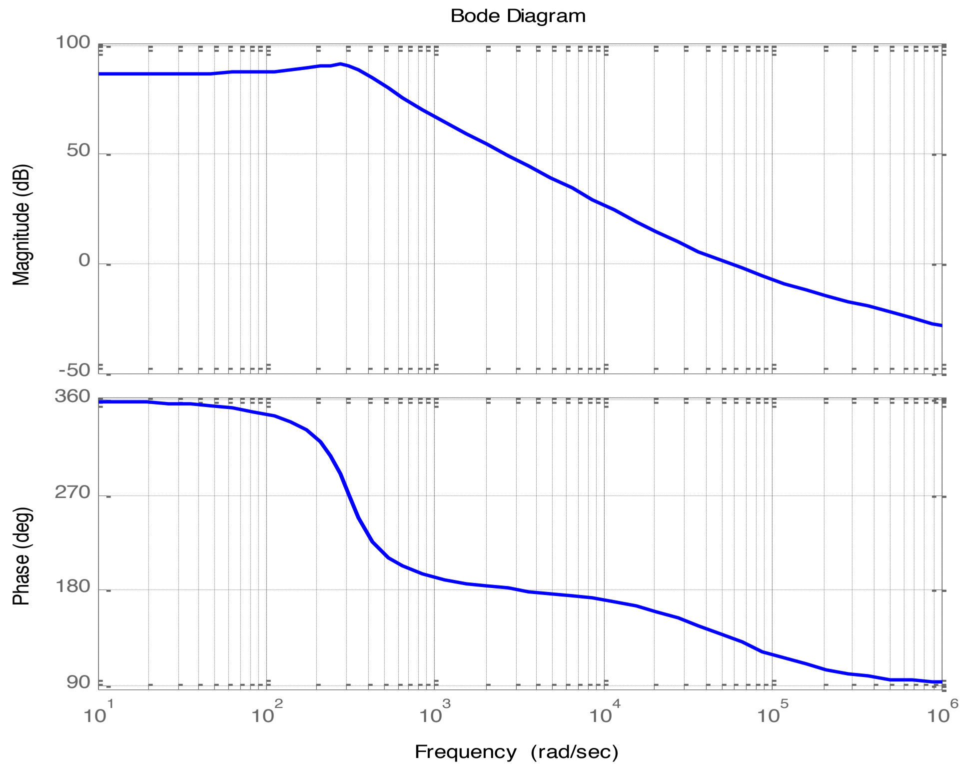
$$\dot{x} = \begin{pmatrix} 13.48 & -613.3 \\ 160.4 & -221.7 \end{pmatrix} x + \begin{pmatrix} -9.57 \\ -1046 \end{pmatrix} u$$

$$y = \begin{pmatrix} 3354 & 5.40 \end{pmatrix} x$$

with noises:

$$\dot{x} = Ax + Bu + v_1$$

$$y = Cx + v_2$$

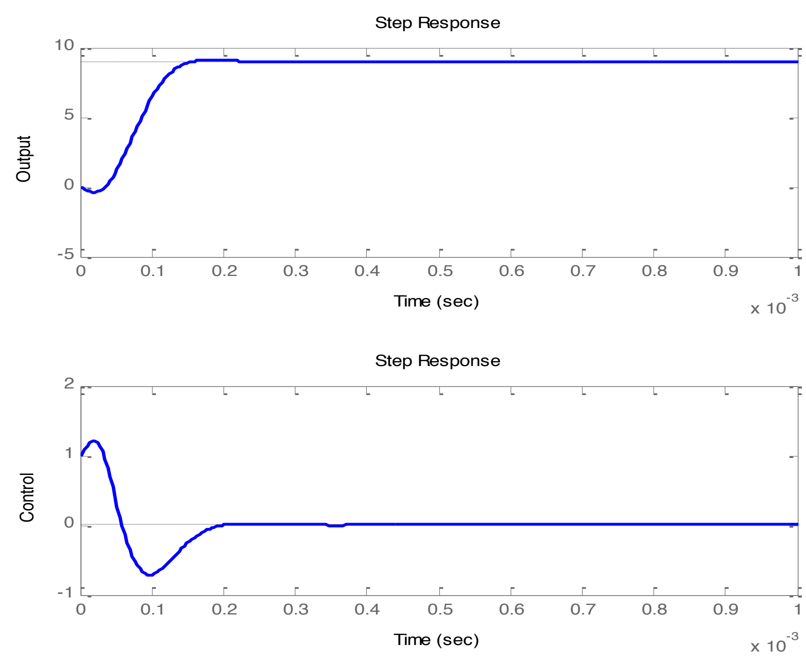


An initial design

Use

$$z = y, \quad Q_1 = 1, \quad Q_2 = 1, \quad R_1 = I, \quad R_2 = 1$$

consider response to unit step input in reference:

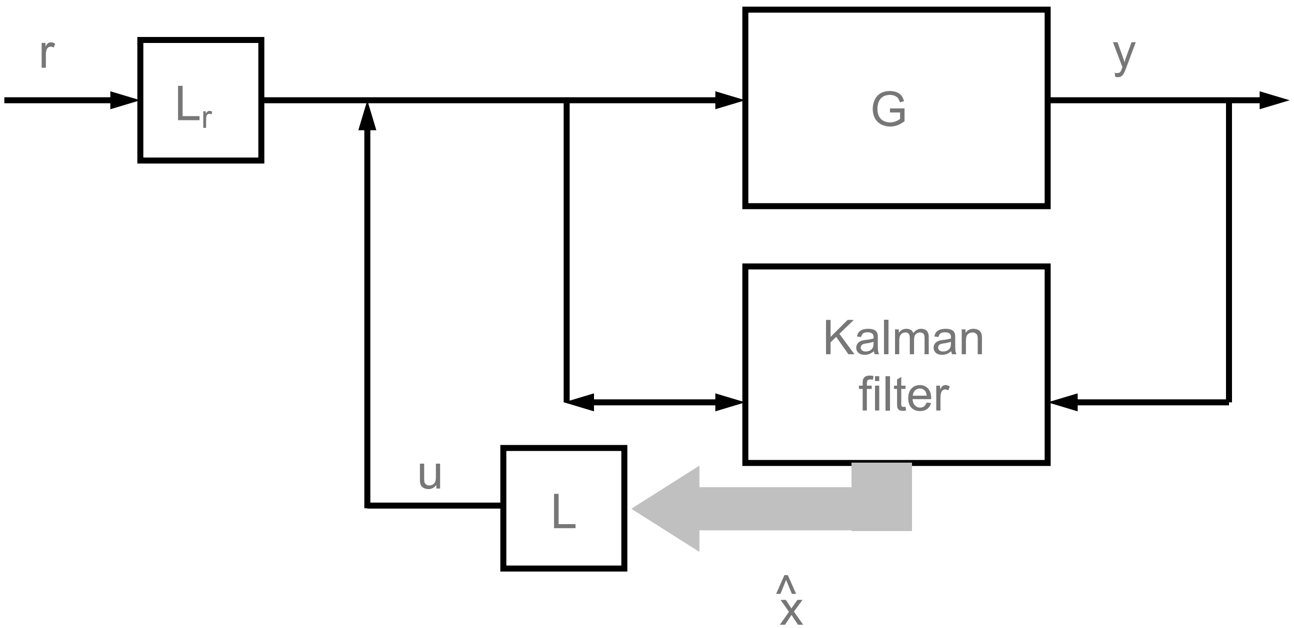


What is wrong?

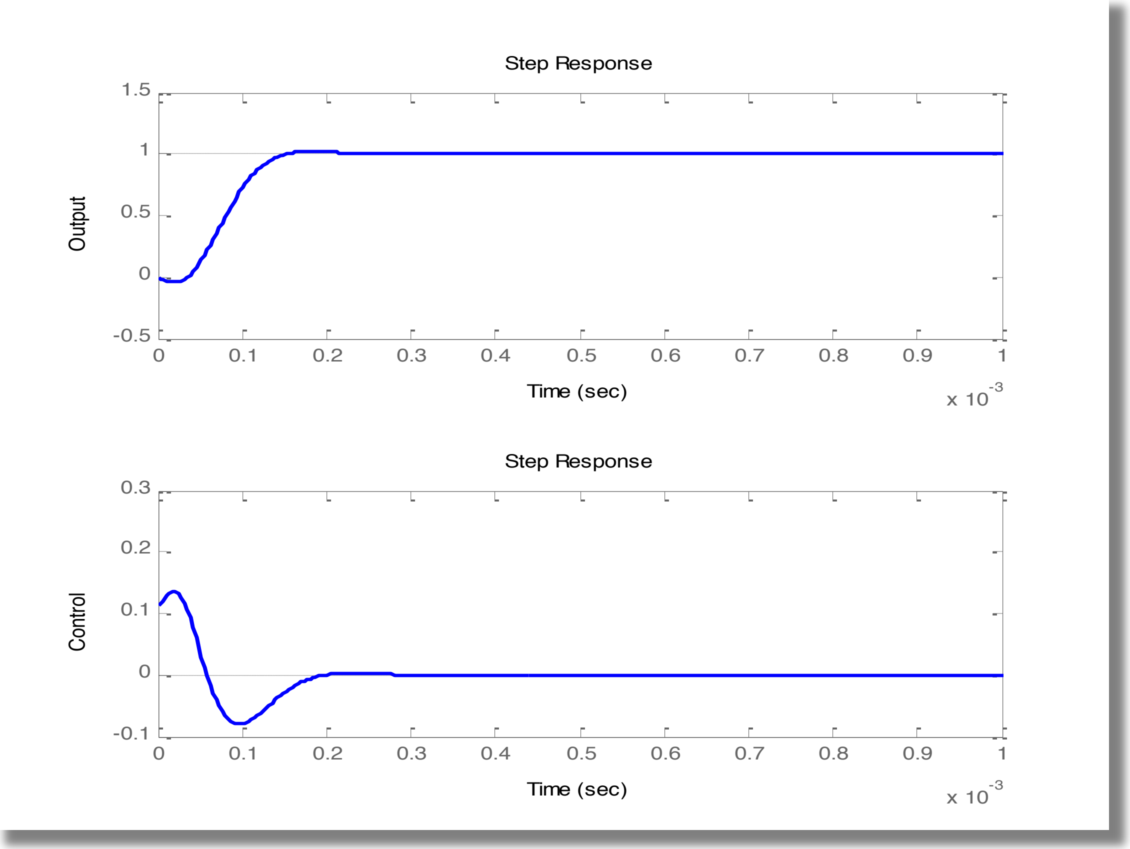
Adjusting feedforward gain

Simple solution: static adjustment of feedforward gain

$$Y(0) = G_c(0) L_r R(0) \Rightarrow L_r = 1/G_c(0)$$

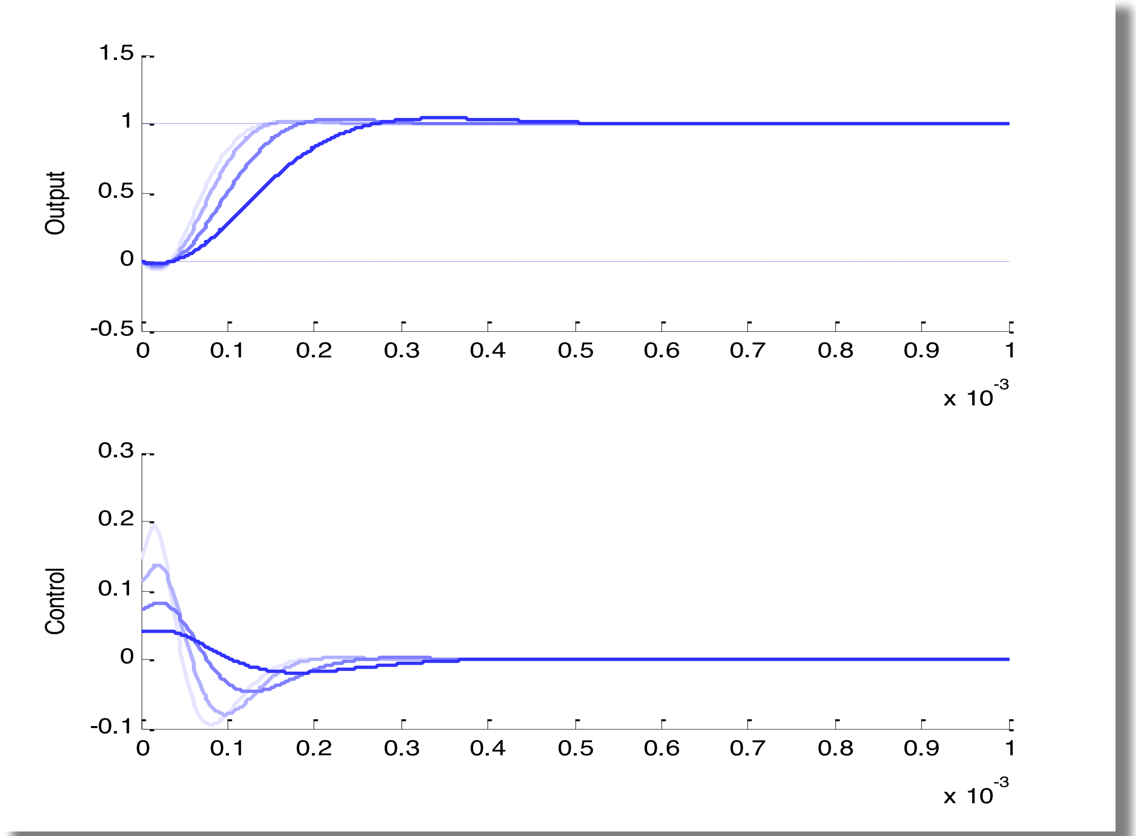


Adjusting feedforward gain



Shaping the time response

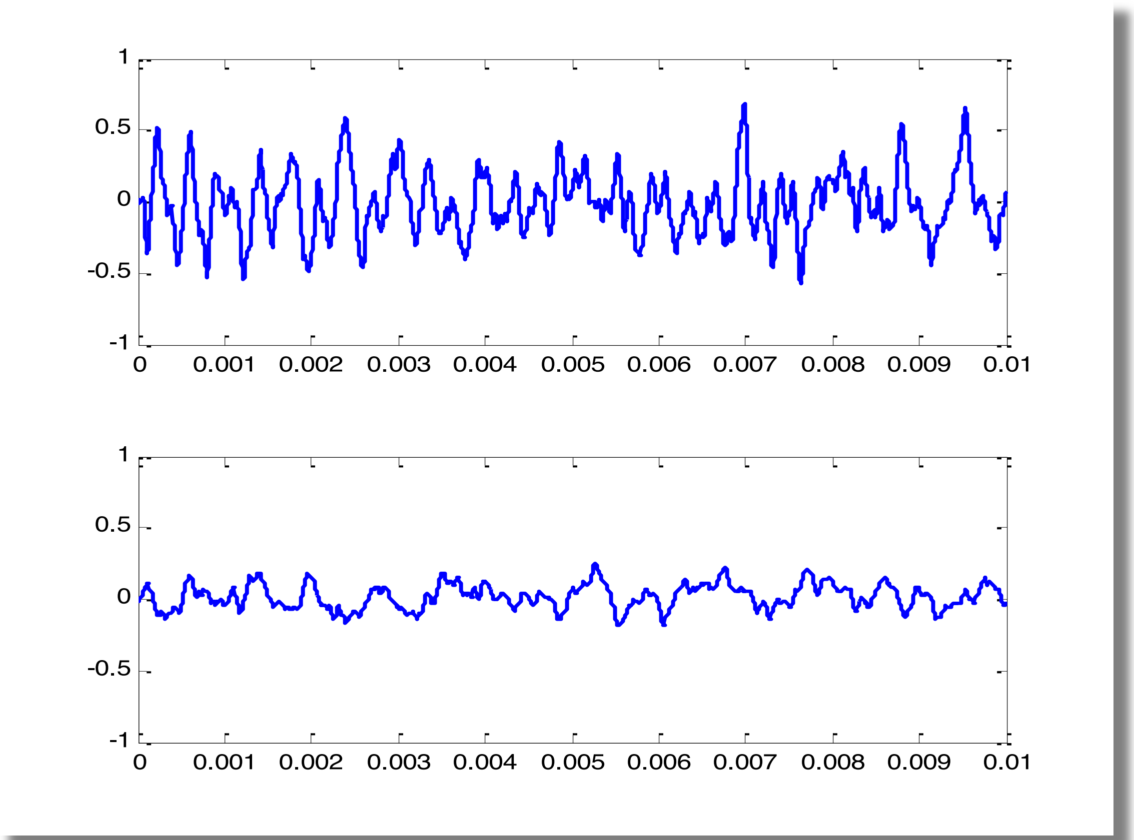
Using $Q_1=1$, $Q_2=\rho$; responses for varying ρ (which is which?)



Suppression of measurement noise

Let R_1=I, R_2=r. Time responses of z to unit variance measurement noise

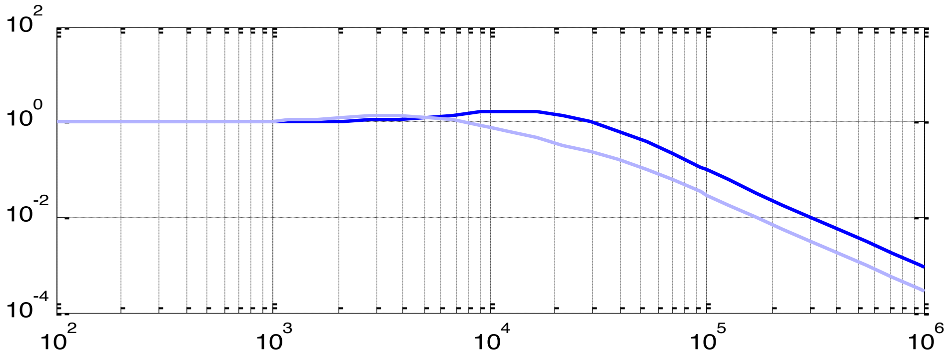
- which design corresponds to the larger value of r?



A loop shaping perspective

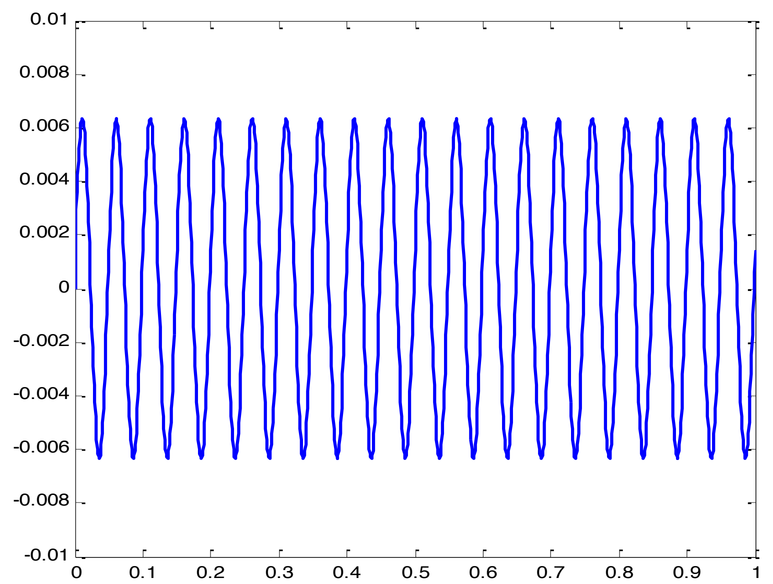
Corresponding Bode diagrams of complementary sensitivity (n to z)

- which one corresponds to the larger value of r?



Dealing with output disturbance

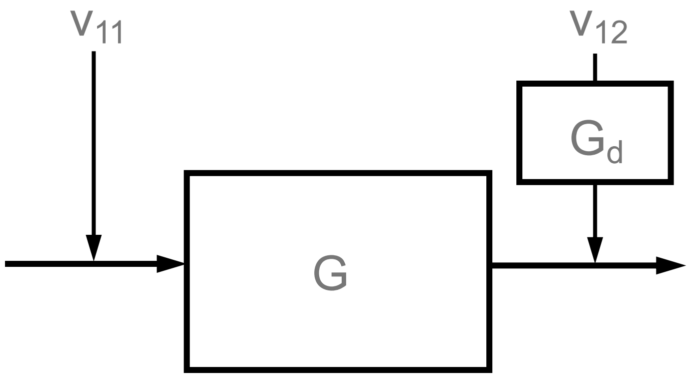
Response to sinusoidal output disturbance at 20 Hz



We would like the amplitude to be less than 1E-3.

– How can we achieve this?

Introducing disturbance model



We will use

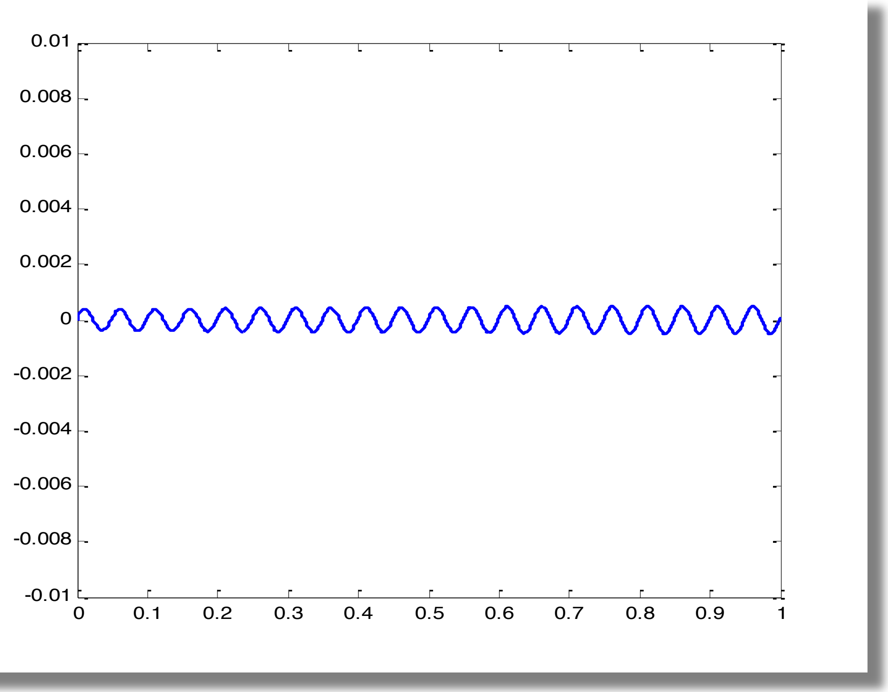
$$G_d = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

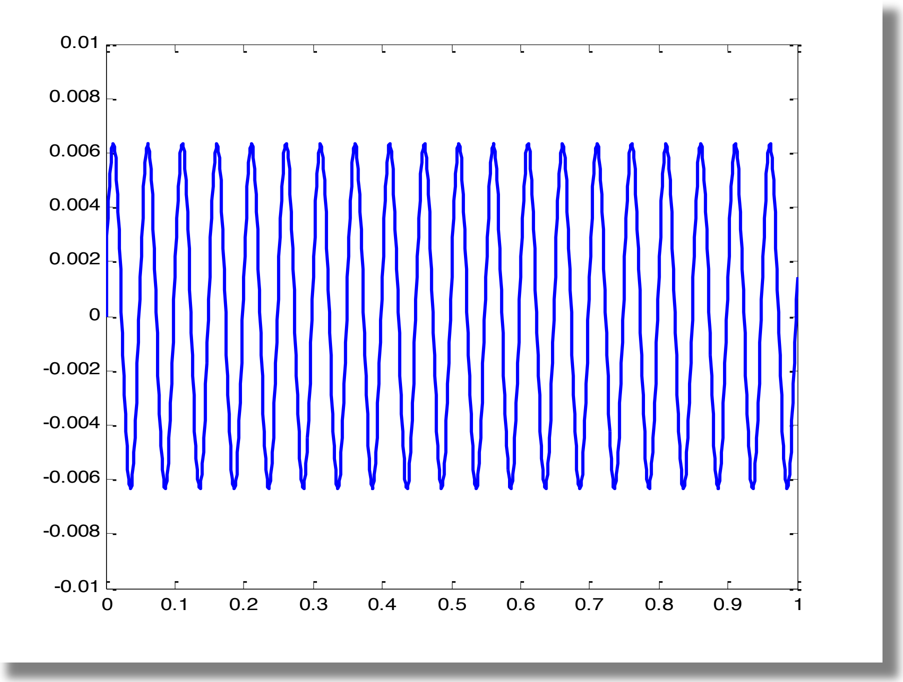
What are the appropriate values for ζ and ω_0 ?

How should we choose R_1?

Improved disturbance suppression

Disturbance response with (left) and without (right) disturbance model





Summary LQG

- Linear quadratic control review
  - separation principle, algebraic Riccati equations and noise filters
- Design example: radial control of DVD player
  - reference following
  - trading off state vs control energy
  - influencing sensitivity to noise
  - shaping the response to non-white disturbances
- Optimization in time domain, but can translate into shaping of sensitivity, complementary sensitivity etc. Systematic procedure exists: LQG-LTR (Loop Transfer Recovery). But, iterative indirect method and therefore not treated here.

Here and now: **H2-optimal control** which offers a direct method to shape closed-loop transfer-functions using the LQG machinery (white board).

H_2 and H_∞ optimal control

H_2-optimal control

$$\min_{F_y} \|G_{ec}\|_2^2 = \min_{F_y} \int_{-\infty}^{\infty} \sum_i \sigma_i^2(G_{ec}(i\omega))\, d\omega$$

(reduce all singular values at all frequencies)

H_∞-optimal control

$$\min_{F_y} \|G_{ec}\|_\infty = \min_{F_y} \sup_{\omega} \overline{\sigma}(G_{ec}(i\omega))$$

(reduce maximum singular value at worst frequency)

Design example

DC servo:

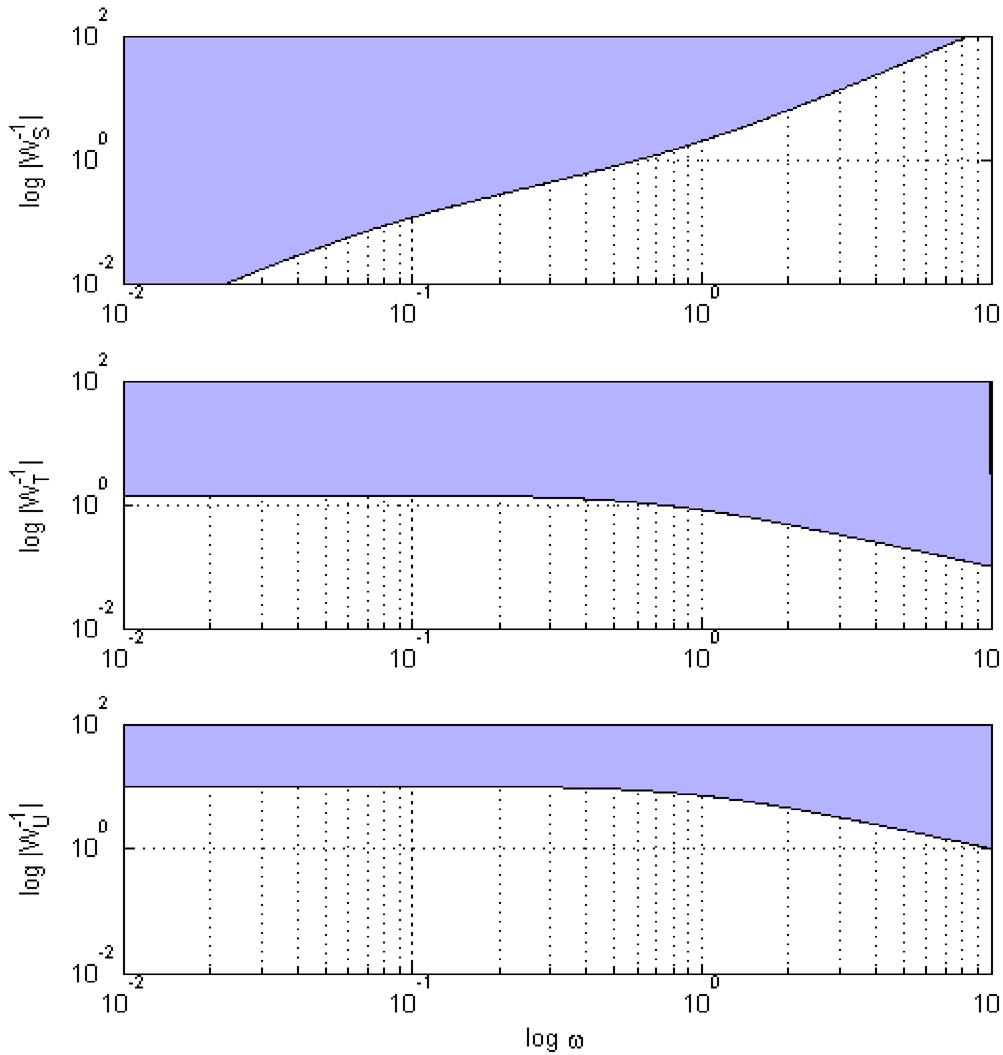
$$G(s) = \frac{1}{s(s+1)}$$

Same performance requirements as previously

Two key points:

- H_∞ optimal design allows to work directly with constraints
- The relation between H_2 and H_∞ optimal controllers

Weights

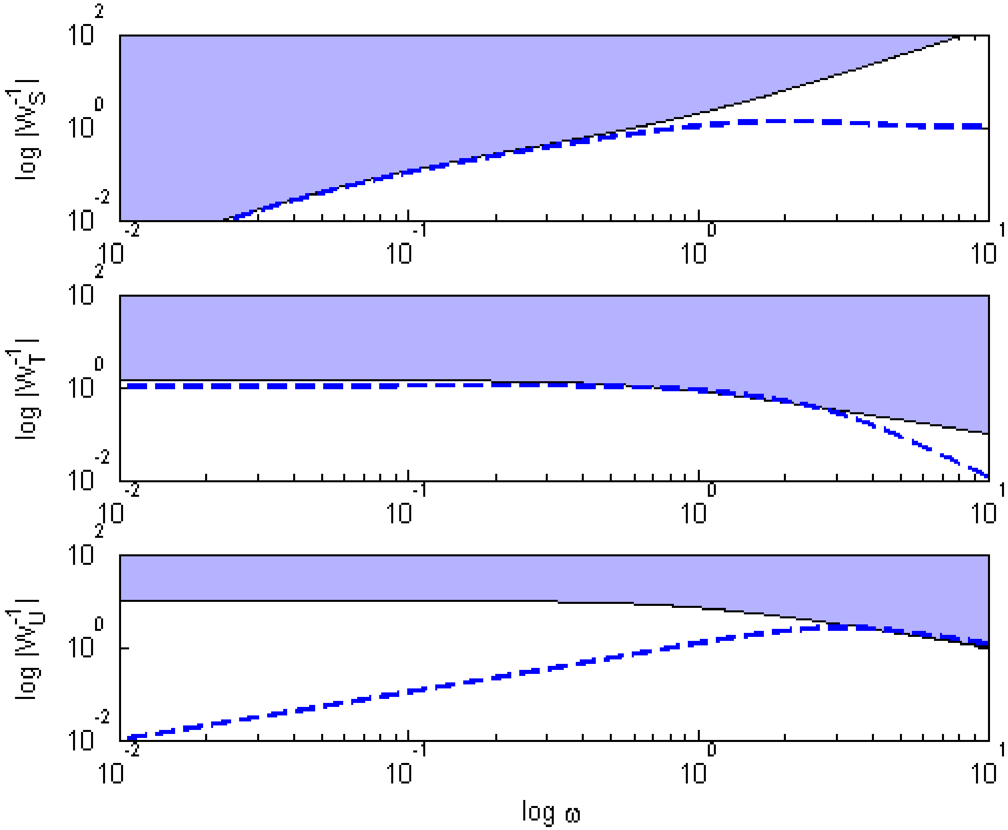


$$W_S(s) = \frac{0.71s + 0.05}{s^2(s+1)}$$

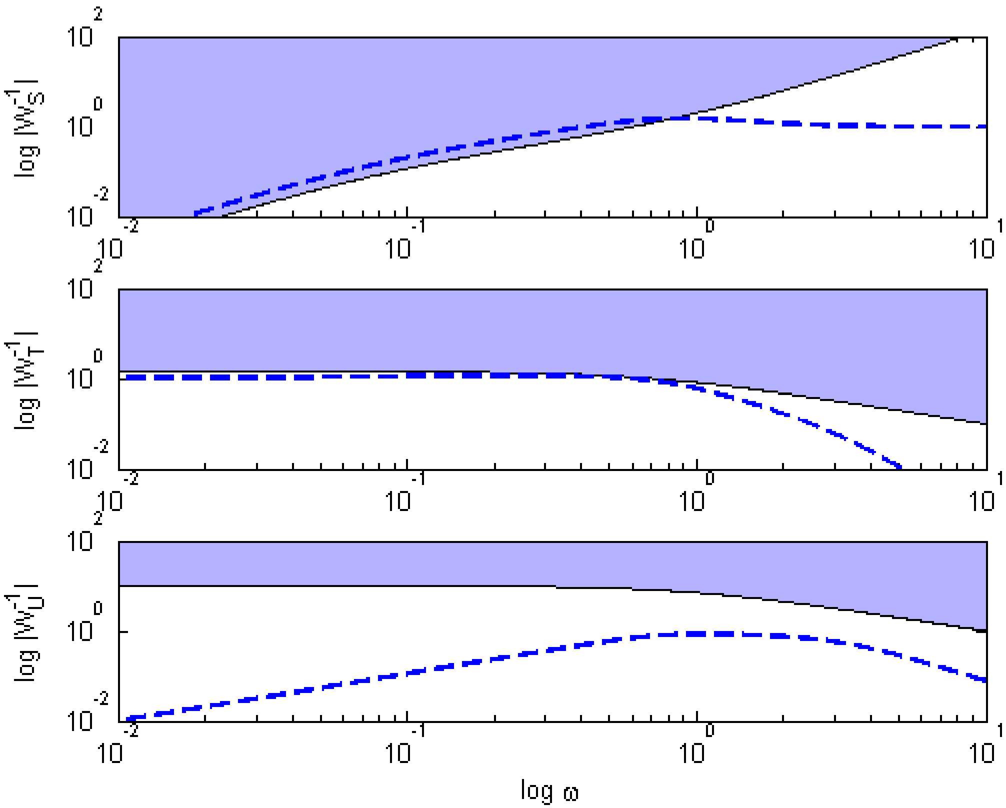
$$W_T(s) = s + 0.71$$

$$W_U(s) = \frac{10s + 10}{s + 100}$$

H_∞ optimal control



H_2-optimal controller



Quiz: why doesn't the H_2-optimal controller "meet the specs"?

Comparing the controllers