EL2520
Control Theory and Practice

Lecture 3: Robustness

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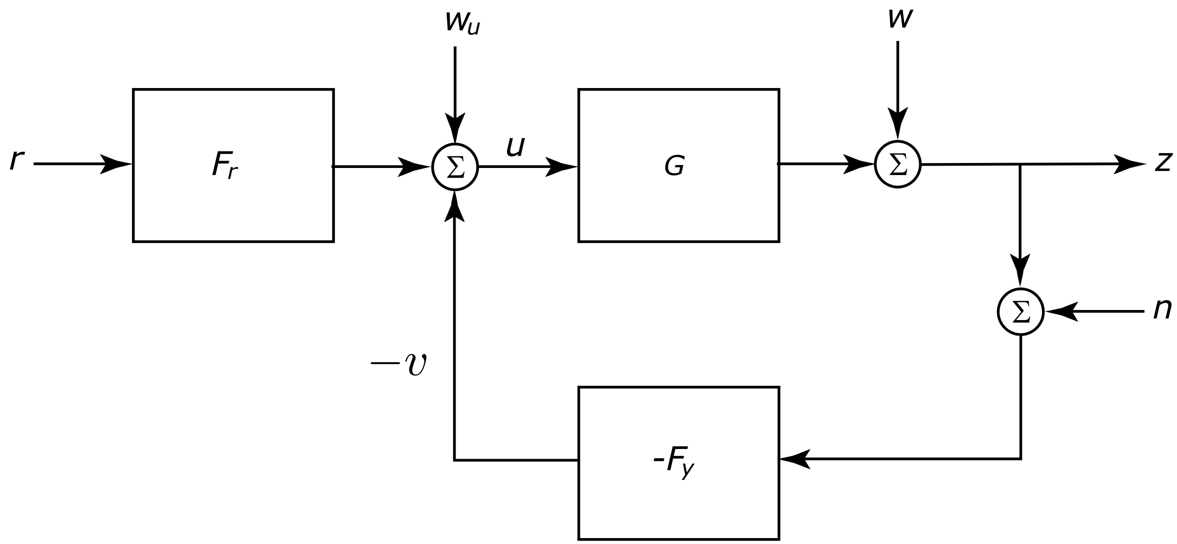
Summary Lecture 2



$$z = \underbrace{\frac{1}{1+GF_y}}_{S} w + \underbrace{\frac{G}{1+GF_y}}_{SG} w_u + \underbrace{\frac{GF_r}{1+GF_y}}_{G_c} r - \underbrace{\frac{GF_y}{1+GF_y}}_{T} n$$

$$u = SF_r r - SF_y(w+n) + Sw_u$$

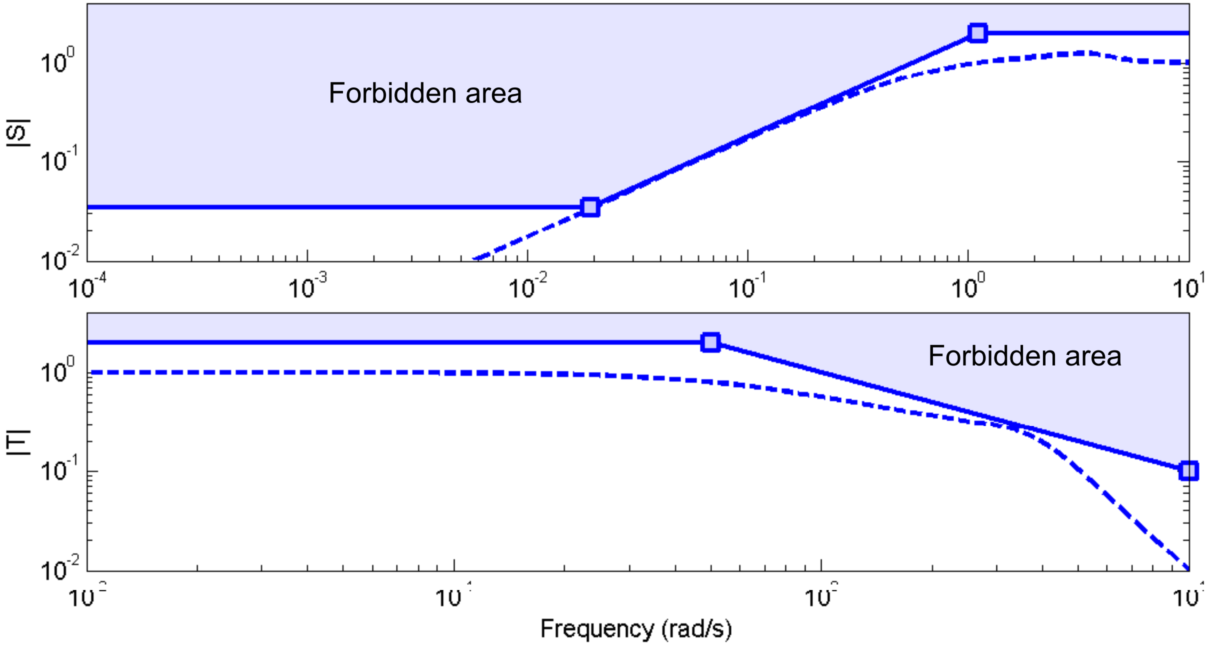
- 6 transfer-functions we want to shape using F_y and F_r
- Most important: sensitivity S, complementary sensitivity T and closed-loop transfer function G_c

Summary cont'd

- Internal stability if S, SG, SF_y, F_r all stable
  - cancelling poles and zeros in RHP (between controller and plant) always results in internal instability!
- Disturbance sensitivity $z = Sw$; feedback attenuates disturbances for frequencies where $|S(i\omega)| < 1$
- Effect of feedback on uncertainty of closed-loop: $\Delta_{G_c} = S\Delta_G$
- Noise: $z = Tn$
- Assume true system $\tilde{G} = G(1 + \Delta_G)$. Robust stability if $\|T\Delta_G\|_\infty < 1$
- Thus, make |S| small for disturbances and insensitivity to model uncertainty, |T| small for measurement noise and robust stability

Sensitivity shaping

$$\begin{aligned} |S(i\omega)| &\le |W_S^{-1}(i\omega)| \\ |T(i\omega)| &\le |W_T^{-1}(i\omega)| \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \|SW_S\|_\infty &\le 1 \\ \|TW_T\|_\infty &\le 1 \end{aligned}$$



Todays Learning Outcomes

You should

- understand what robustness implies
- be able to quantify uncertainty using model sets
- be able to analyze robust stability and robust performance, given a model set, for SISO systems

Reference

Based on Chapter 7 in "Multivariable Feedback Control",

S. Skogestad, I. Postlethwaite, Wiley (but slides + ch.6 in course book + Lecture notes 3 suffices)

Classes of uncertainty

Parametric uncertainty:

- Model structure known, but some parameters are uncertain

Dynamic uncertainty:

- Some (often high frequency) dynamics is missing, either
  by lack of knowledge/information or in order to get a simpler model

Often, we have a combination of the two.

- Convenient to represent in "lumped" form

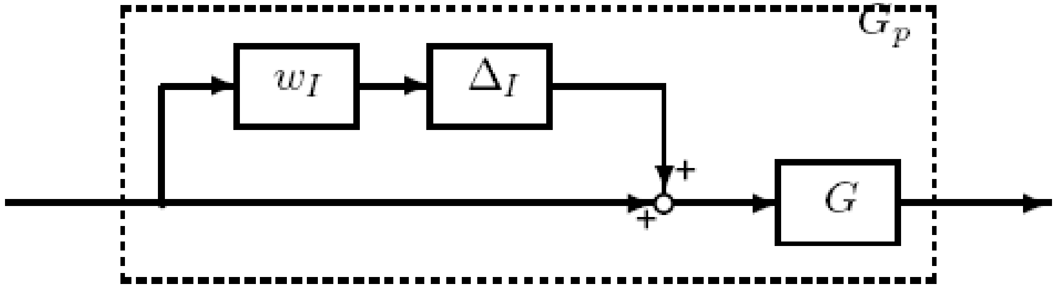
Multiplicative uncertainty

The true system is assumed to belong to the set:

$$\Pi_I = \{G_p(s) = G(s)(1 + W_I(s)\Delta_I(s)) \mid \|\Delta_I\|_\infty \le 1\}$$

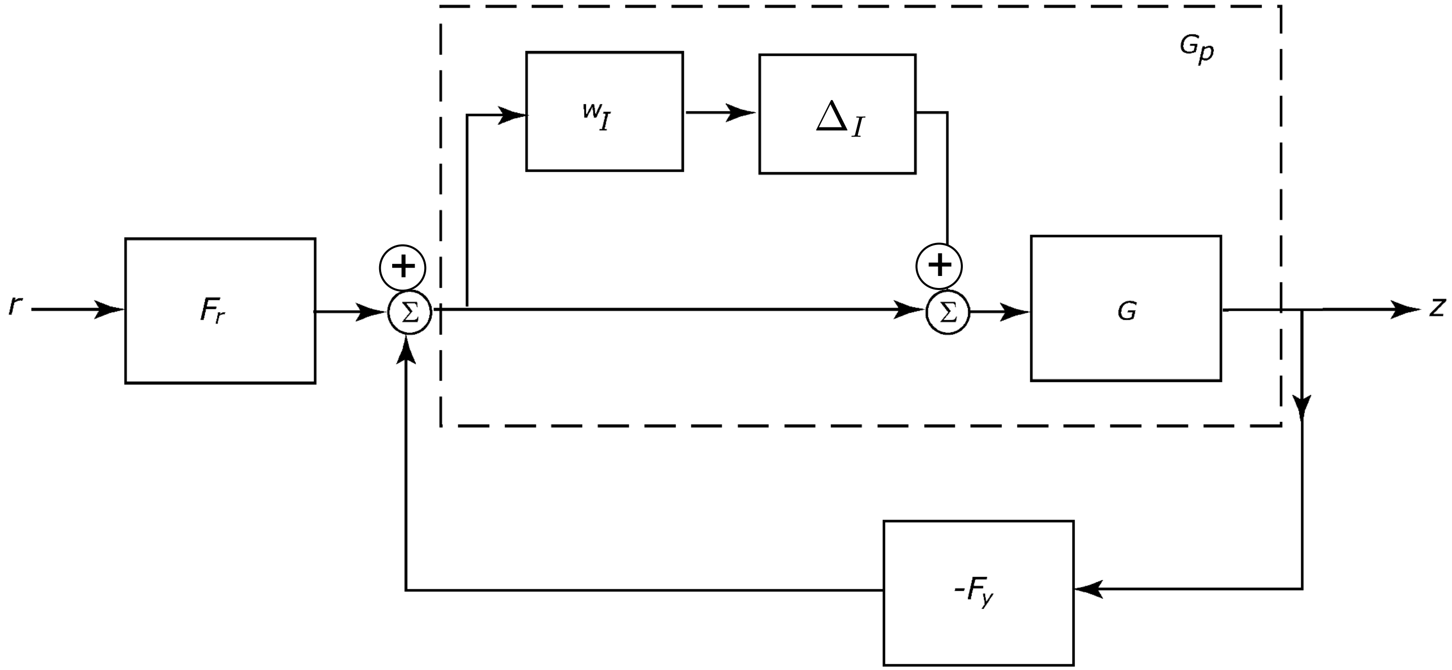
Here,

- Π_I is a *family* of possible behaviours of the physical plant
- Δ_I is *any* stable transfer function with gain less than one
- W_I is the *multiplicative uncertainty weight*



Robust stability: closed-loop stability for all $G_p \in \Pi_I$

Robust stability: multiplicative uncertainty



The global system is stable if w_I and T stable, Δ_I stable, and

$$\|W_I T\|_\infty \leq 1 \quad \Leftrightarrow \quad |T(i\omega)| \leq |W_I^{-1}(i\omega)| \qquad \forall\omega$$

- follows from Small Gain Theorem

Example: uncertain gain

Consider the set of possible plants:

$$\Pi_I = \{G_p : G_p(s) = kG(s), \quad k \in [1-\delta, 1+\delta]\}$$

Standard form representation (with real Δ_I):

$$W_I(s) = \delta$$

Example: uncertain zero location

Nominal plant: $G(s) = (s_0 + s)G_0(s)$

Set of possible plants:

$$\Pi_I = \{G_p : G(s) = (s_0' + s)G_0(s), s_0' \in [-\delta + s_0, s_0 + \delta]\}$$

Standard form representation (with real Δ_I):

$$W_I(s) = \frac{\delta}{s_0 + s}$$

Alternative approach to obtain weight

Note that the multiplicative uncertainty class

$$\Pi_I = \{G_p(s) = G(s)(1 + W_I(s)\Delta_I(s)) \mid \|\Delta_I\|_\infty \leq 1\}$$

can be re-written as

$$\Pi_I = \left\{G_p(s) \;\middle|\; |W_I^{-1}G^{-1}(G_p - G)| \leq 1\right\}$$

Thus, the uncertainty about the system captured by W_I if

$$|W_I(i\omega)| \geq \left|\frac{G_p(i\omega) - G(i\omega)}{G(i\omega)}\right| \qquad \forall G_p \in \Pi_i$$

(must first pick a nominal model G)

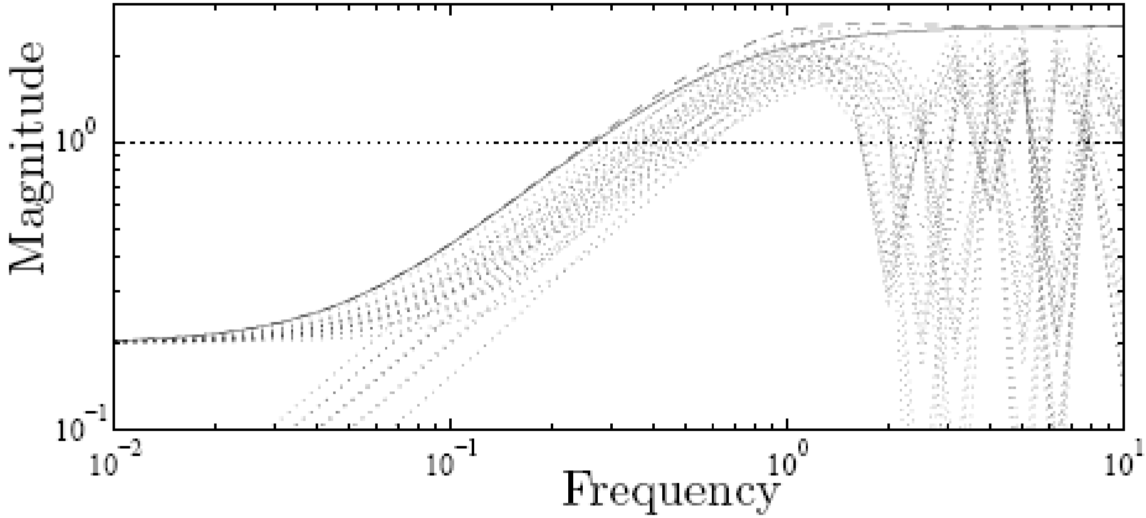
Example 1

Consider the uncertain system: $G_p(s) = \dfrac{k}{\tau s + 1} e^{-\theta s}, \quad k, \theta, \tau \in [2, 3]$

with nominal plant $G(s) = \dfrac{\bar{k}}{\bar{\tau} s + 1}$

Sample uncertainties (dotted) and corresponding W_I (dashed)

$$\left| \frac{G_p - G}{G} \right|$$



Example 2

Consider the following nominal plant and controller

$$G(s) = \frac{3(1-2s)}{(5s+1)(10s+1)}, \qquad K(s) = K_c \frac{12.7s+1}{12.7s}$$

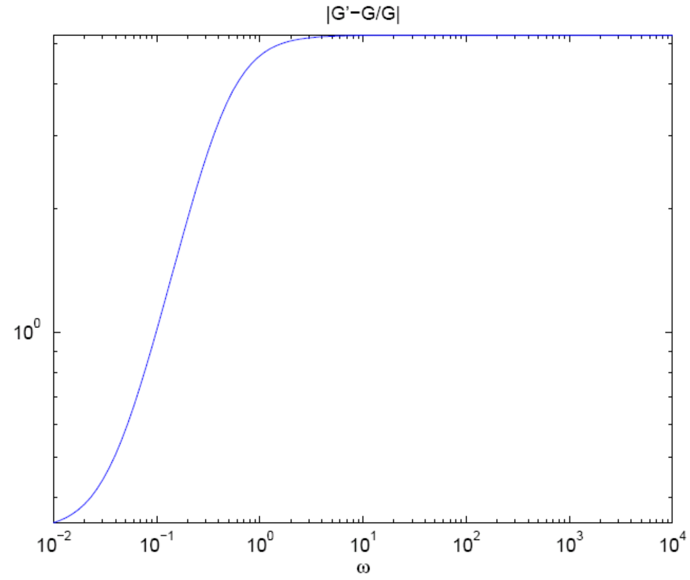
and assume that one "extreme" possible plant is

$$G'(s) = \frac{4(1-3s)}{(4s+1)^2}$$

Is the closed-loop robustly stable?

Example 2

Relative error

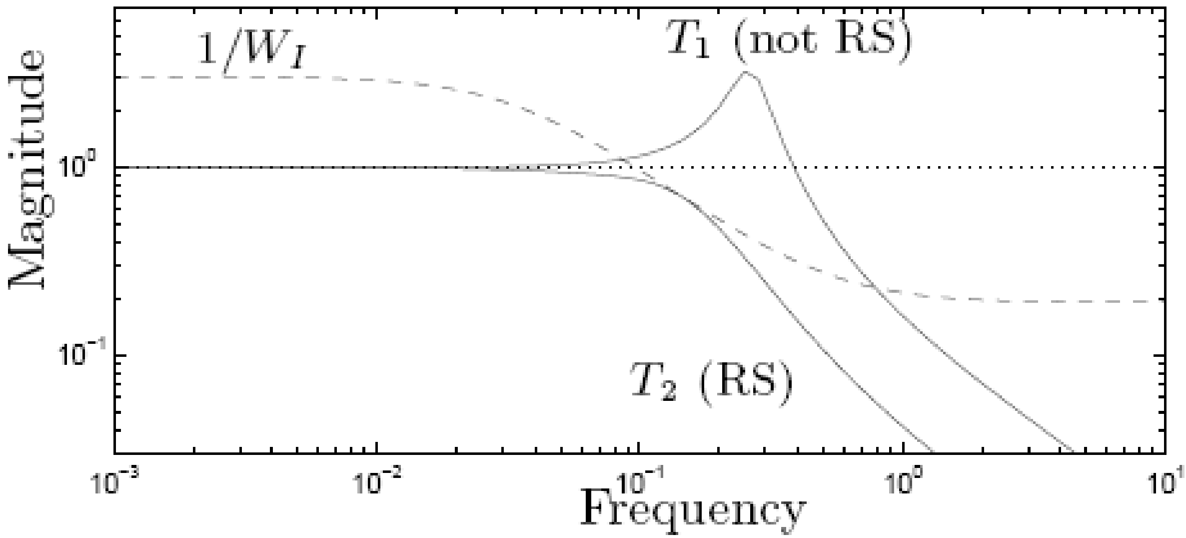


The relative error is around 0.33 for low frequencies and 5.25 at high frequencies. Fitted weight

$$W_I(s) = \frac{10s + 0.33}{(10/5.25)s + 1}$$

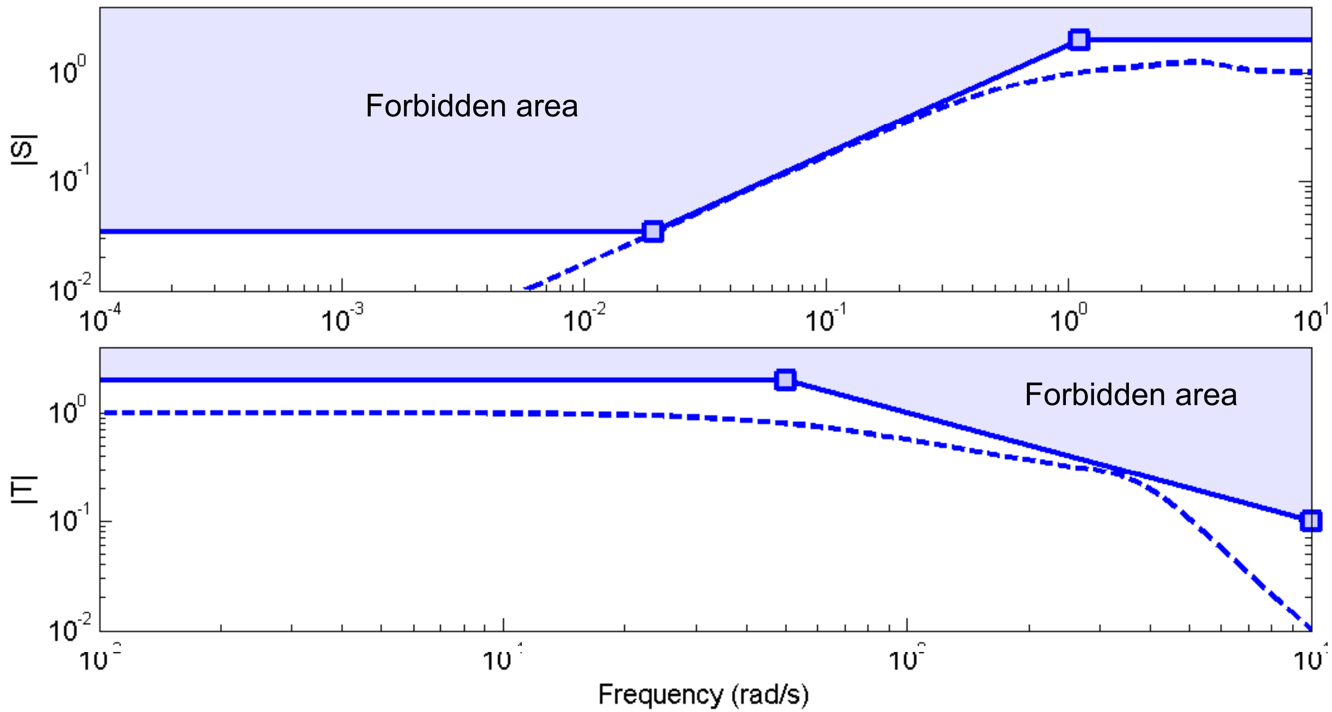
Example 2

Uncertainty weight W_I and complementary sensitivities for two sets of controller parameters



The system with K_c=1.13 (T1) is not robustly stable, whereas it is robustly stable with K_c=0.31 (T2).

Frequency domain specifications



$$|S(i\omega)| \leq |W_S^{-1}(i\omega)|$$
$$|T(i\omega)| \leq |W_T^{-1}(i\omega)|$$

Can we choose weights w_S, w_T ("forbidden areas") freely?

– No, there are many constraints and limitations!

Robust performance

Nominal performance specified in terms of sensitivity function

$$|W_P S| \leq 1 \quad \forall\omega$$

Robust performance

$$|W_P S_p| \leq 1 \qquad \text{for all } \omega \text{ and all } S_p$$

We have

$$W_P S_p = W_P \frac{1}{1 + L_p} = \frac{W_P}{1 + L + W_I \Delta L}$$

Worst-case Δ is such that 1+L and $W_I \Delta L$ point in opposite directions

$$|W_P S_p| \leq \frac{|W_P|}{|1 + L| - |W_I L|} = \frac{|W_P S|}{1 - |W_I T|} \quad \forall\omega$$

Robust performance cont’d

Robust performance

$$|W_P S_p| = \frac{|W_P S|}{1 - |W_I T|} \leq 1$$

Can be expressed as

$$|W_P S| + |W_I T| \leq 1 \quad \forall \omega$$

Sometimes approximated by the *mixed* sensitivity constraint

$$\left\| \begin{pmatrix} W_P S \\ W_I T \end{pmatrix} \right\|_\infty \leq 1$$

Robust stability and performance

In summary

$$\begin{aligned} \text{nominal performance} \quad & |W_P S| \leq 1 \quad \forall\omega \\ \text{robust stability} \quad & |W_I T| \leq 1 \quad \forall\omega \\ \text{robust performance} \quad & |W_P S| + |W_I T| \leq 1 \quad \forall\omega \end{aligned}$$

Note that nominal performance and robust stability implies

$$|W_P S| + |W_I T| \leq 2 \quad \forall\omega$$

(i.e. robust performance cannot be "too bad").

Holds only in SISO case!

Summary

Robustness

- insensitivity to model errors

Can guarantee robustness if we model (or bound) uncertainty

- sometimes need to ”pull out” uncertainty by hand
- sometimes, can fall back onto standard forms
  (e.g. multiplicative input uncertainty)

Robustness typically introduces new constraints on T

Robust performance: satisfy bounds on sensitivity function S for all plants in the model set