EL2520
Control Theory and Practice

Lecture 11:
Classical and Modern Optimal Control revisited & Case Study

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Today's lecture

- Brief review of design methods covered so far, including interpretations and comparisons
- Case study: disturbance attenuation in a chemical reactor

Optimal Control

- Classical optimal control LQG
  - originally motivated by the need for methods that could deal with MIMO systems
  - formulated as optimization problem in state-space (time)
- Modern optimal control H_∞ and H_2
  - originally motivated by need to explicitly address robustness
  - formulated as optimization problem in input-output space (frequency)
  - solved in state-space
- All formulations result in controllers on the form observer + state feedback

Robust Loop shaping

- Combines classical loop shaping with H_∞-optimal control where the optimization step only addresses robust stability

Linear Quadratic Gaussian control

Model: linear system with white noise

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) + Nv_1(t)$$
$$y(t) = Cx(t) + v_2(t)$$
$$z(t) = Mx(t)$$

where v_1, v_2 are white noise with

$$\text{cov}([v_1,\ v_2]) = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

Objective: minimize effect of v on z, punish control cost

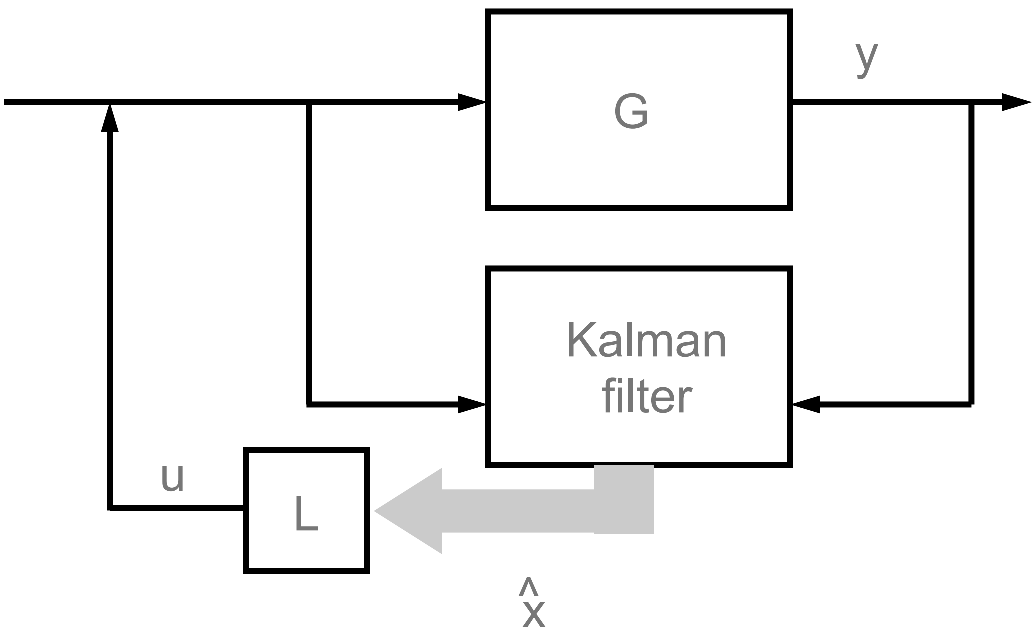
$$J = \mathbf{E}\left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T \left[z^T Q_1 z + u^T Q_2 u \right] dt \right\}$$

LQG: Linear system, Quadratic cost, Gaussian noise

Solution structure

Optimal solution satisfies *separation principle*, composed of

- Optimal linear state feedback (LQ-problem, no noise)
- Optimal observer (Kalman filter, no control)



Optimal solution

State feedback (LQ problem)

$$u(t) = -L\hat{x}(t) = -Q_2^{-1}B^T S\hat{x}(t)$$

where S>0 is the solution to the algebraic Riccati equation

$$A^T S + SA + M^T Q_1 M - SBQ_2^{-1}B^T S = 0$$

Kalman filter

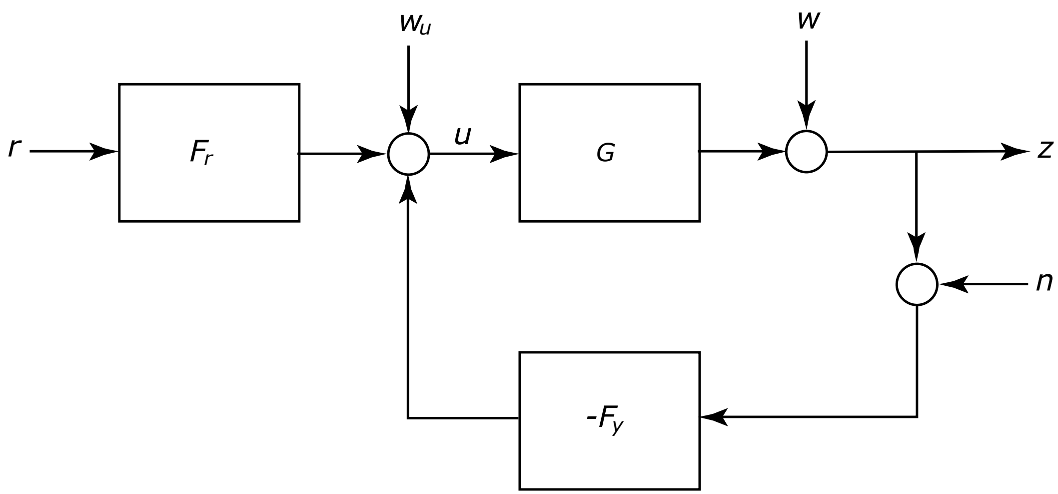
$$\dot{\hat{x}} = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t)); \quad K = (PC^T + NR_{12})R_2^{-1}$$

where P>0 is the solution to algebraic Riccati equation

$$AP + PA^T + NR_1N^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T = 0$$

- Combination of optimal state feedback, with no noise, and optimal observer, with no control, is the optimal controller to the combined problem
- Tuning parameters: Q_1, Q_2, R_1, R_2 (usually assume $R_{12} = R_{21} = 0$)

H_∞–optimal control



Aim: shape closed loop transfer-functions, e.g., S, T, G_{wu}, to achieve desired system properties

How: introduce weights W_S, W_T, W_u and determine F_y, F_r such that

$$\|W_S S\|_\infty < 1 \quad \|W_T T\|_\infty < 1 \quad \|W_u G_{wu}\|_\infty < 1$$

$$\Downarrow$$

$$\bar{\sigma}(S(i\omega)) < |w_S^{-1}(i\omega)| \quad \bar{\sigma}(T(i\omega)) < |w_T^{-1}(i\omega)| \quad \bar{\sigma}(G_{uw}(i\omega)) < |w_u^{-1}(i\omega)| \quad \forall\omega$$

where we assume $W_S = w_S I$ etc.

Selecting Weights

Weights W_S, W_T, W_u should

- reflect our requirements on performance and robustness, e.g., W_S large for frequencies where we need disturbance attenuation, W_T large where we want noise attenuation and where model uncertainty is large.
- take into account trade-offs and limitations, e.g., S+T=I, RHP poles, RHP zeros and time delays, such that $\| \cdot \|_\infty < 1$ is feasible, i.e., there exists stabilizing controller that meets specifications.

Usually a good idea to scale all signals, such that their expected / allowed magnitude is less than 1, prior to designing weights.

Controller Design – H_∞

How determine $F_y(s)$ to achieve $\|W_S S\|_\infty < 1 \quad \|W_T T\|_\infty < 1 \quad \|W_u G_{wu}\|_\infty < 1$?

- *Loop shaping*, i.e., shape loop gain $L = GF_y$

  or

- *Synthesis,* i.e., solve optimization problem

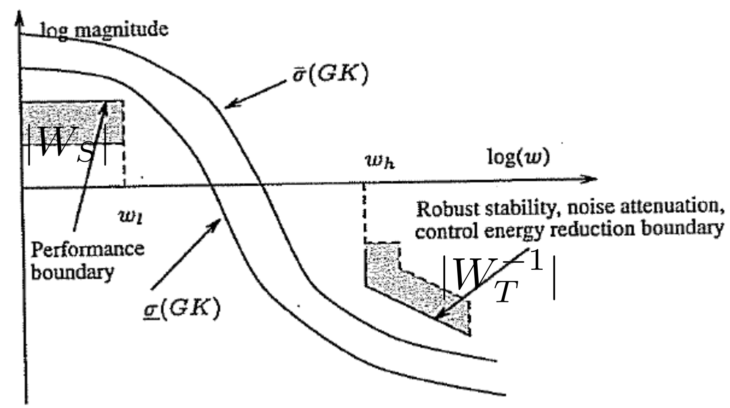
$$F_{y,opt}(s) = \arg\min_{F_y} \left\| \begin{matrix} W_S S \\ W_T T \\ W_u G_{wu} \end{matrix} \right\|_\infty \qquad (*)$$

Note that the if the ”stacked” objective above is less than 1, then we have achieved the three individual objectives

Loop Shaping

Need to translate bounds on $\bar{\sigma}(S)$ and $\bar{\sigma}(T)$ into bounds on $\sigma_i(L), \ L = GF_y$

- $\underline{\sigma}(L) >> 1 \ \Rightarrow \ \bar{\sigma}(S) \approx 1/\underline{\sigma}(L)$ and hence $\underline{\sigma}(L) > |w_S|$ where $|w_S| >> 1$
- $\bar{\sigma}(L) << 1 \ \Rightarrow \ \bar{\sigma}(T) \approx \bar{\sigma}(L)$ and hence $\bar{\sigma}(L) < |w_T^{-1}|$ where $|w_T| >> 1$



- Robust loop shaping; robustify the shaped plant by maximizing robustness margin for coprime uncertainty

H_∞ Synthesis

- Given a state space system G_0 on the form

$$\begin{aligned}\dot{x} &= Ax + Bu + Nw_e \\ z_e &= Mx + Du \qquad\qquad (**) \\ y &= Cx + w_e\end{aligned}$$

- Determine if a controller $u = -F_y(s)y$ exists such that for the resulting closed-loop system G_{ec} and given γ

$$\sup_{w_e \neq 0} \frac{\|z_e\|_2}{\|w_e\|_2} < \gamma \qquad (1)$$

- Let P>0 be a solution to the algebraic Riccati equation

$$A^T P + PA + M^T M + P(\gamma^{-2} N N^T - BB^T)P = 0$$

if $A - BB^T P$ is stable then the controller exists, otherwise not

The H$_\infty$–optimal controller

- A controller satisfying requirement (1) is then given by

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + N(y - C\hat{x}) \\ u &= -L_\infty \hat{x} \; ; \quad L_\infty = B^T P \end{aligned}$$

  i.e., state observer combined with state feedback

- The optimal controller can be found by iterating on γ untill $\gamma \approx \gamma_{min}$

- To solve the original problem $(*)$ we note that with $z_e = G_{ec} w_e$

$$\sup_{w_e \neq 0} \frac{\|z_e\|_2}{\|w_e\|_2} < \gamma \;\Leftrightarrow\; \|G_{ec}\|_\infty < \gamma$$

- Thus, select the output z_e and input w_e such that

$$z_e = G_{ec} w_e = \begin{bmatrix} W_S S \\ W_T T \\ W_u G_{wu} \end{bmatrix} w_e$$

  and determine corresponding open-loop system G_0

H_2-optimal control

- Similar to in H_∞-synthesis, we define the extended system G_0

$$\begin{aligned} \dot{x} &= Ax + Bu + Nw_e \\ z_e &= Mx + Du \\ y &= Cx + w_e \end{aligned}$$

  such that that the closed-loop transfer-matrix, with $u = -F_y y$, has transfer-function from w_e to z_e equal to the one we want to minimize norm of
- If w_e is white noise with intensity Φ_{w_e}= I, then minimizing $\|z_e\|_2$ corresponds to minimizing $\|G_{ec}\|_2$ where G_{ec} is closed-loop transfer-matrix from w_e to z_e
- Corresponds to LQG problem with $Q_1 = Q_2 = R_1 = R_2 = I$ (weights are in G_0)

H_2 and H_∞ optimal control

H_2-optimal control

$$\min_{F_y} \|G_{ec}\|_2^2 = \min_{F_y} \int_{-\infty}^{\infty} \sum_i \sigma_i^2(G_{ec}(i\omega))\, d\omega$$

(reduce all singular values at all frequencies)

H_∞-optimal control

$$\min_{F_y} \|G_{ec}\|_\infty = \min_{F_y} \sup_\omega \overline{\sigma}(G_{ec}(i\omega))$$

(reduce maximum singular value at worst frequency)

Design example

DC servo from Lecture 5:

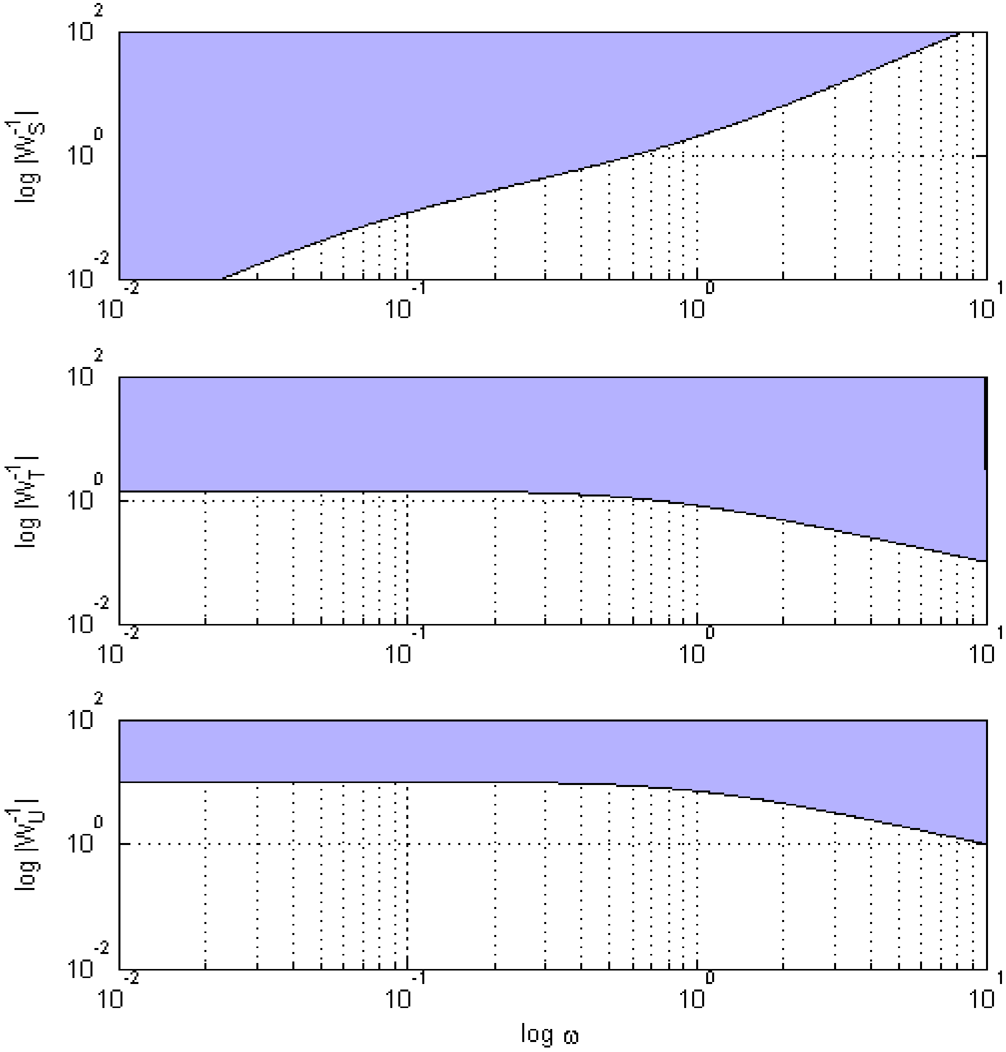
$$G(s) = \frac{1}{s(s+1)}$$

Same performance requirements as in Lec 9

Two key points:

- H_∞ optimal design allows to work directly with constraints
- The relation between H_2 and H_∞ optimal controllers

Weights

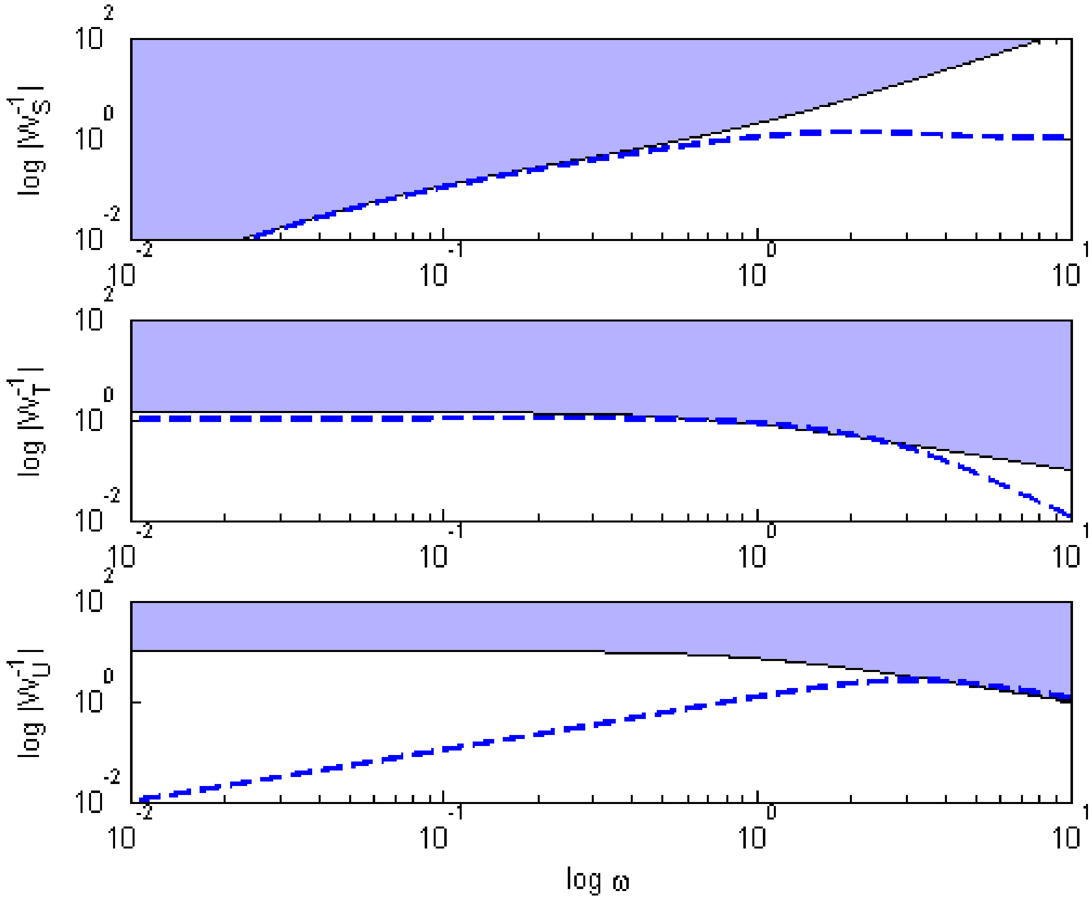


$$W_S(s) = \frac{0.71s + 0.05}{s^2(s+1)}$$

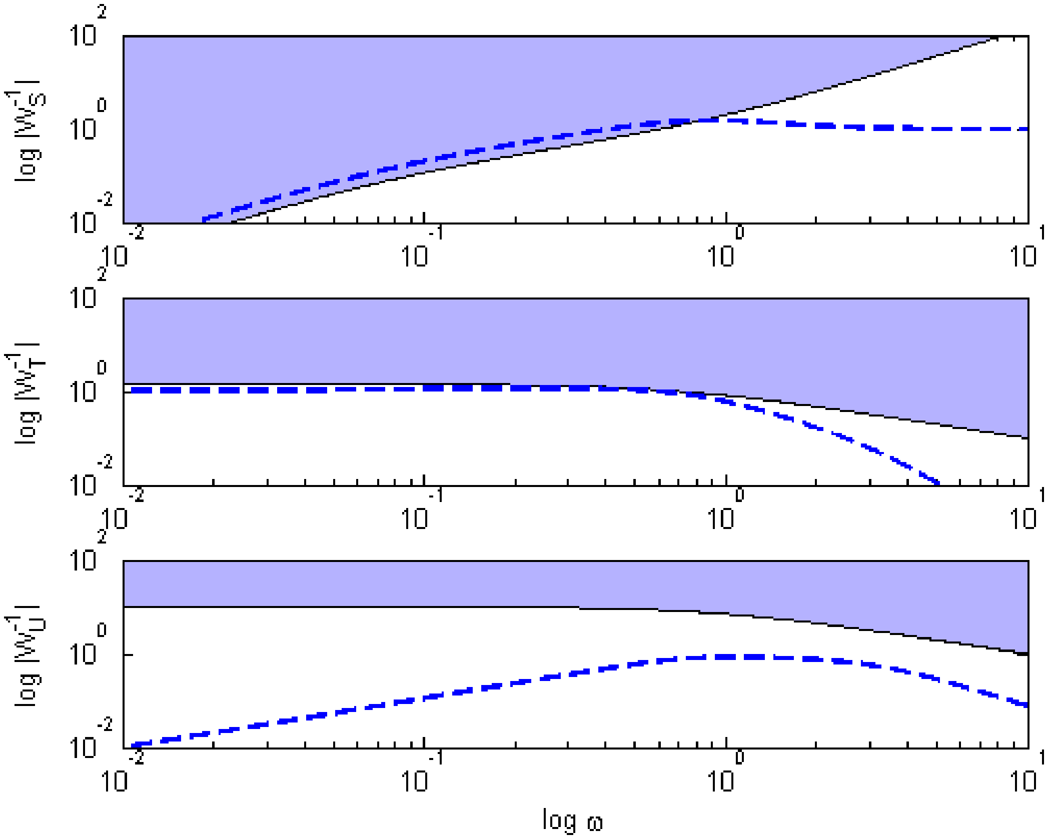
$$W_T(s) = s + 0.71$$

$$W_U(s) = \frac{10s + 10}{s + 100}$$

H_∞ optimal control

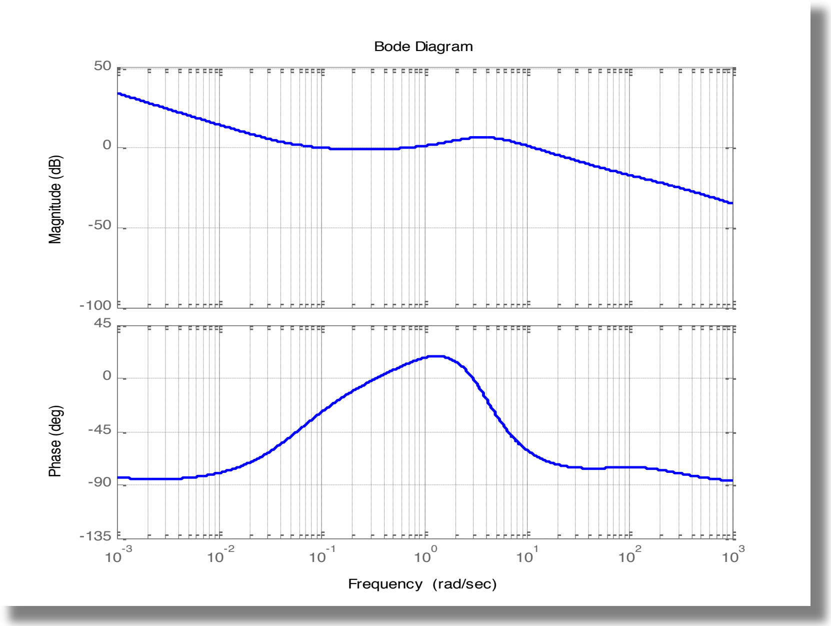


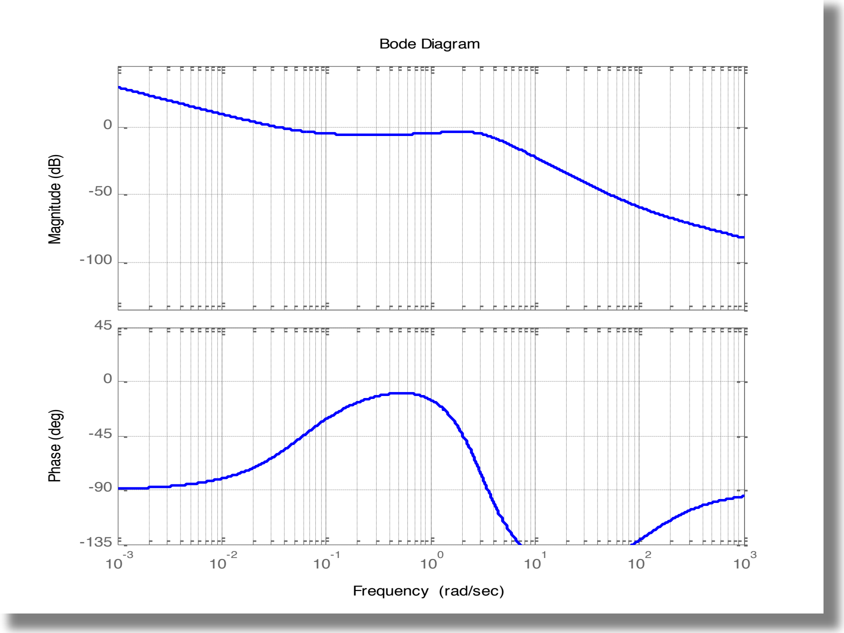
H_2-optimal controller



Quiz: why doesn't the H_2-optimal controller "meet the specs"?

Comparing the controllers





Signal vs System Optimization

- LQG is formulated as signal minimization problem, i.e., minimize weighted control error and weighted control input in the presence of (filtered) white noise disturbances
- H_2– and H_∞-optimal control usually considered as system norm minimization, e.g., minimize weighted sensitivity and weighted complementary sensitivity
- But, equivalence exists between system properties and signal properties, e.g.,

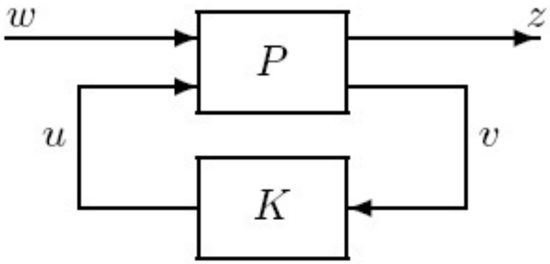
$$\sup_{w_e \neq 0} \frac{\|z_e\|_2}{\|w_e\|_2} = \|G_{ec}\|_\infty$$

- Also, equivalence exist between LQG and H_2–optimal control; minimizing $\|z_e\|_2$ with w_e white noise corresponds to LQG problem

LQG for System Optimization

- LQG may be used to obtain desired system properties, e.g., sensitivity and complementary sensitivity
- Systematic method, called Loop Transfer Recovery (LTR)
  - Main drawback: based on cancellations, and will even cancel RHP zeros and hence loss of internal stability
  - Not treated in this course
- Better to use H_2 - optimization which is a direct method, also based on using the LQG machinery

H_2/H_∞ optimal control for signal minimization



- So far: select input w and output z to reflect system transfer-function G_{ec} that we want to shape
- But: in many cases specifications may be directly on signals, e.g., keep an output less than some constraint in the presence of a bounded disturbance. Then w and z given directly by problem formulation.
- Note that weightings still will be in the frequency domain
- Case study: controller design for chemical reactor using signal minimization and sensitivity shaping, respectively

Chemical Reactor Model

- Scaled model (so that all signals should have magnitude <1 at all frequencies)

$$G(s) = \frac{1}{(15.2s+1)(3.1s+1)} \begin{pmatrix} 22.4(3.1s+1) & 59.4(8.3s+1) \\ -12.6(10.2s+1) & -60.6(12.1s+1) \end{pmatrix}$$

$$G_d(s) = \frac{1}{(15.2s+1)} \begin{pmatrix} 3.28 \\ 3.56 \end{pmatrix}$$

  - Aim: keep output deviations less than 1 in presence of disturbances with magnitude up to 1.
  - uncertainty: measurement uncertainty exceeds 100% for frequencies above 1 rad/min. Use uncertainty weight

$$W_T(s) = \frac{2s}{s+2} I$$

Controllability Analysis (see lec 6)

- Is control needed?

  Yes, since $\|G_d\|_\infty > 1$ and requirement is $\|SG_d\|_\infty < 1$

- What is required bandwidth?

  $$\|G_d(i\omega)\|_2 > 1, \ \omega < 0.35$$

  Thus, need to attenuate disturbance up to $\omega \approx 0.35$

- Any fundamental limitations?

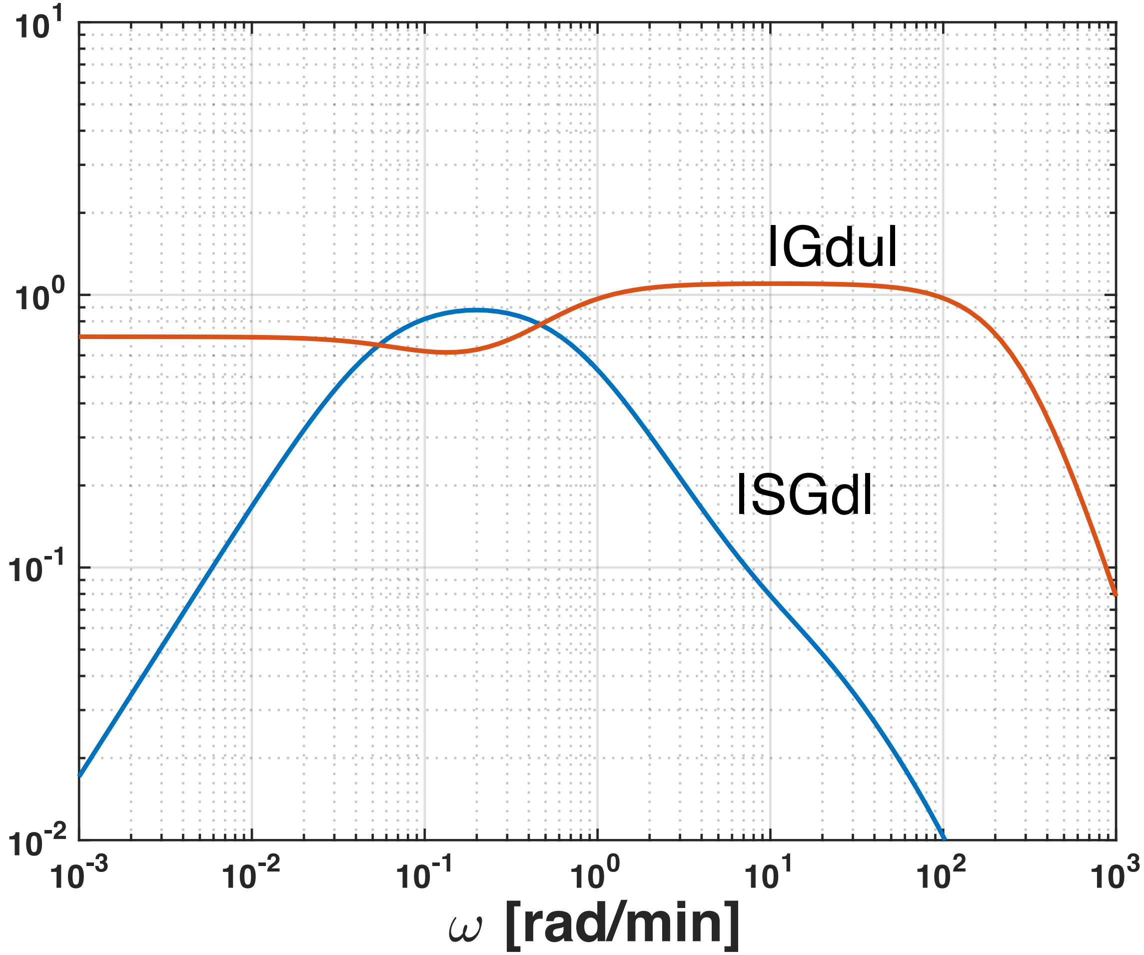
  Yes, there is a RHP zero $z = 0.62$ which gives bandwidth limitation for S $\omega_{BS} \lesssim 0.31$

- Conflict? Need to attenuate disturbances only in disturbance direction
  - zero direction $y_z = [0.816 \ 0.578]^T$
  - with $G_d(z) = [0.313 \ 0.340]^T$ we get $|y_z^H G_d(z)| = 0.45 < 1$. OK!

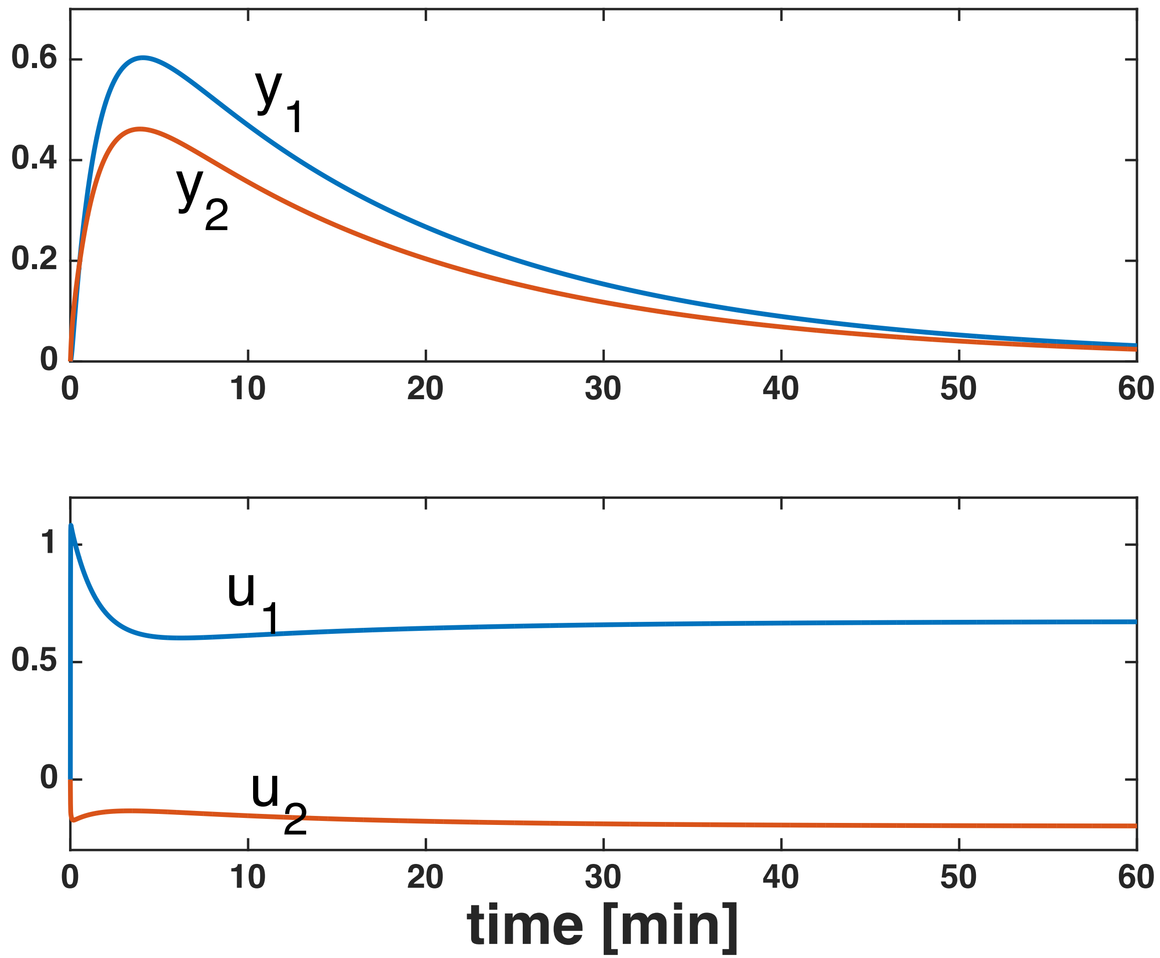
Design I: signal based, no robustness

```
WS=((s+.05)/(s+1.d-4)); % Adds integral action
WU=1;

d=icsignal(1);
u=icsignal(2);
y=icsignal(2);
Wy=icsignal(2);
Wu=icsignal(2);
P=iconnect;
P.Input=[d;u];
P.Output=[Wy;Wu;-y];
P.equation{1}=equate(y,G*u+Gd*d);
P.equation{2}=equate(Wy,WS*y);
P.equation{3}=equate(Wu,WU*u);

[C,CL,gam]=hinfsyn(P.System,2,2);

>> gam
   gam =
       1.104
```

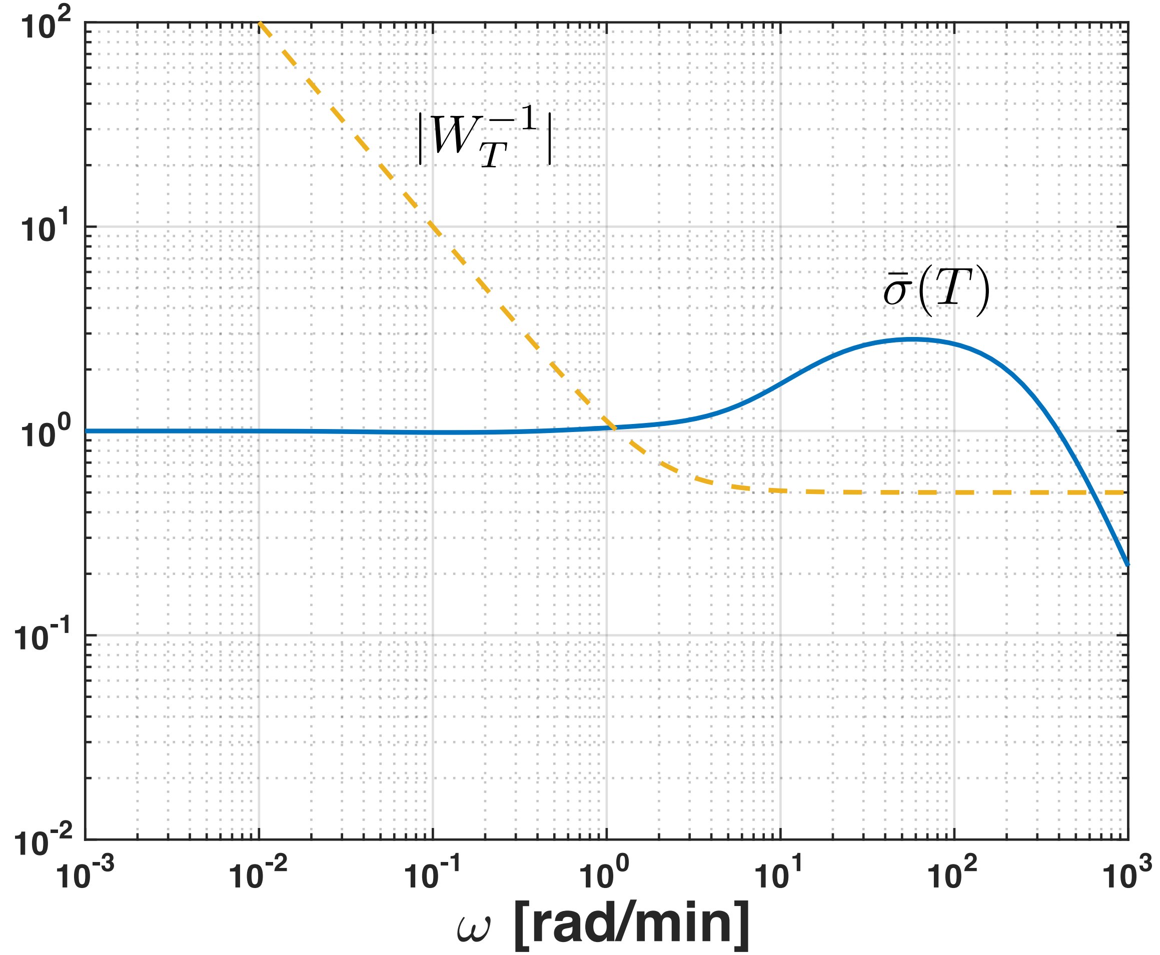
Performance



Simulation unit step disturbance



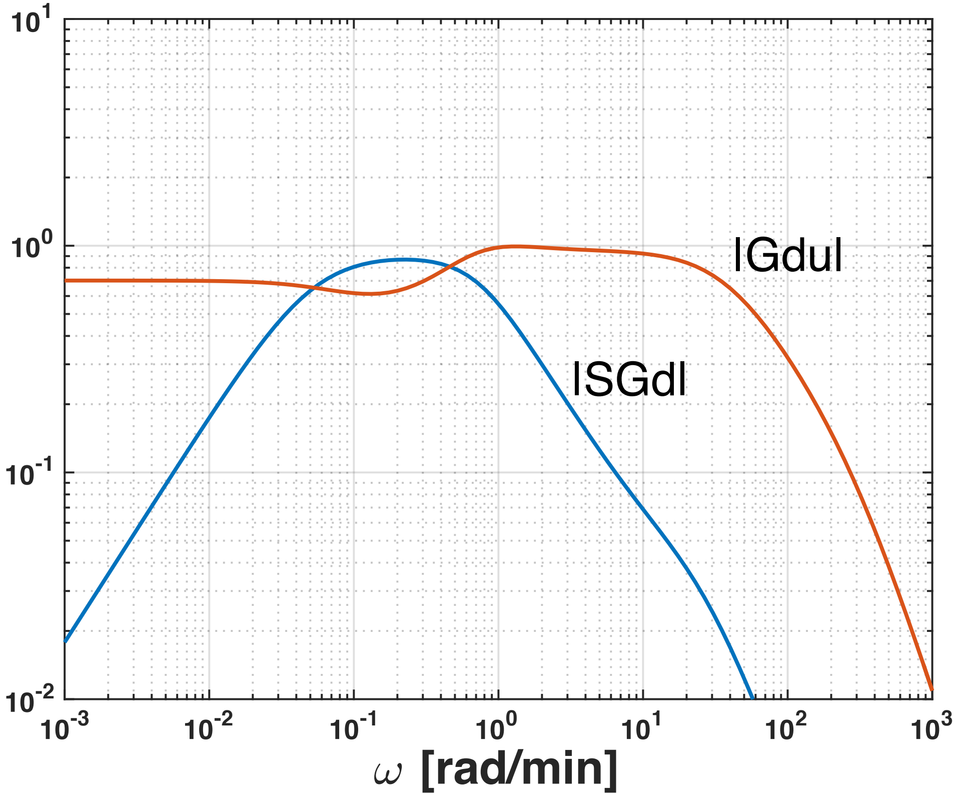
Robust Stability



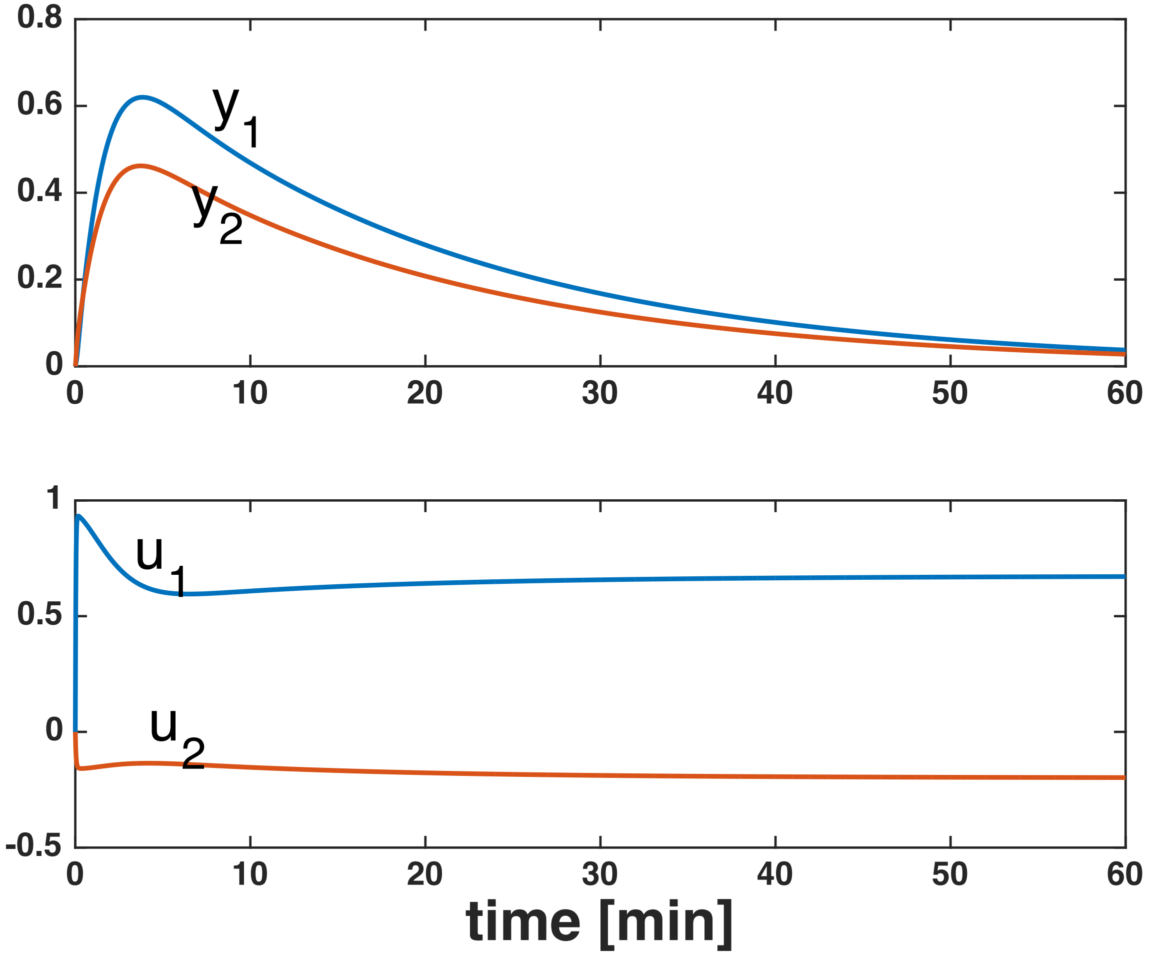
Add weight to input

- Try to push down input usage at high frequencies by introducing

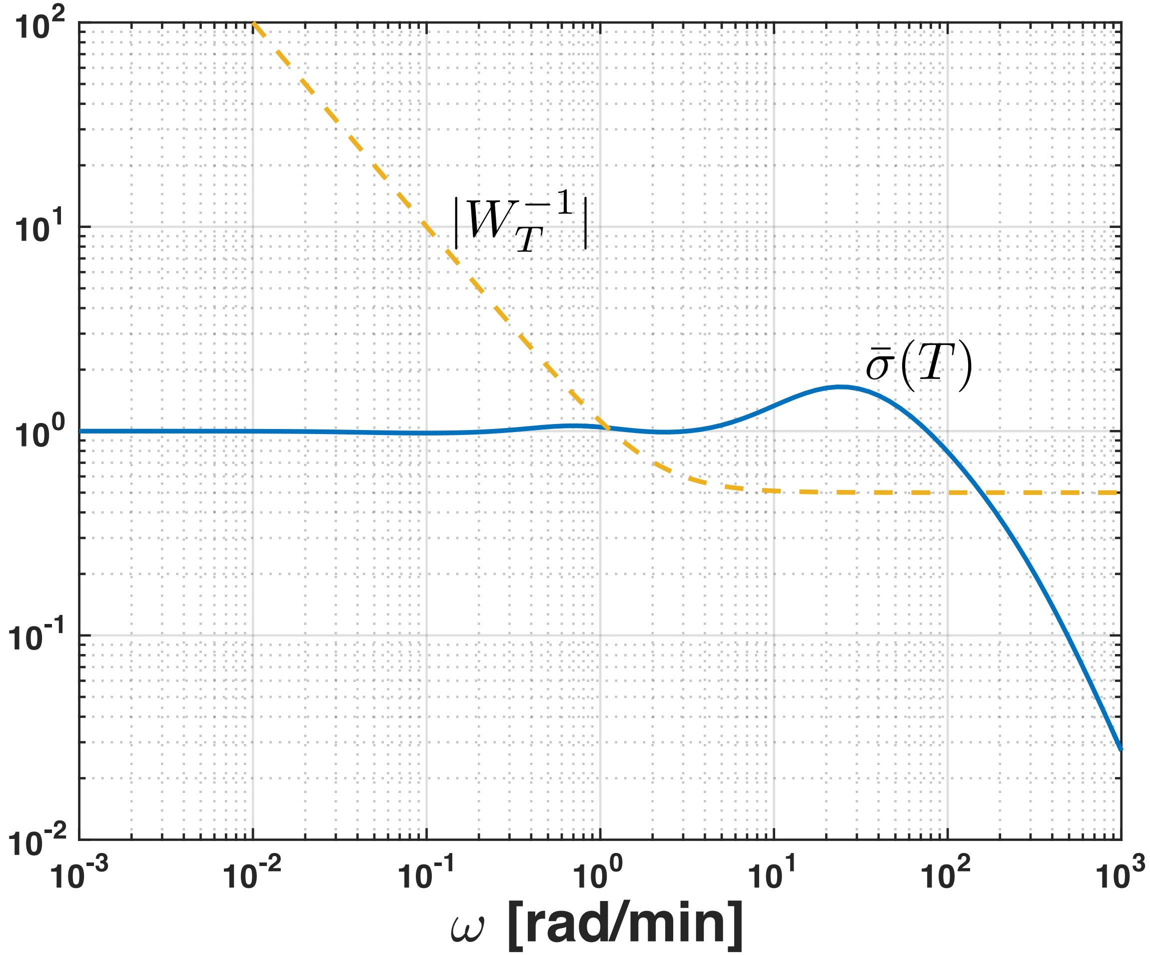
$$W_U(s) = \frac{s + 0.5}{s + 1}$$



Simulation



Robustness

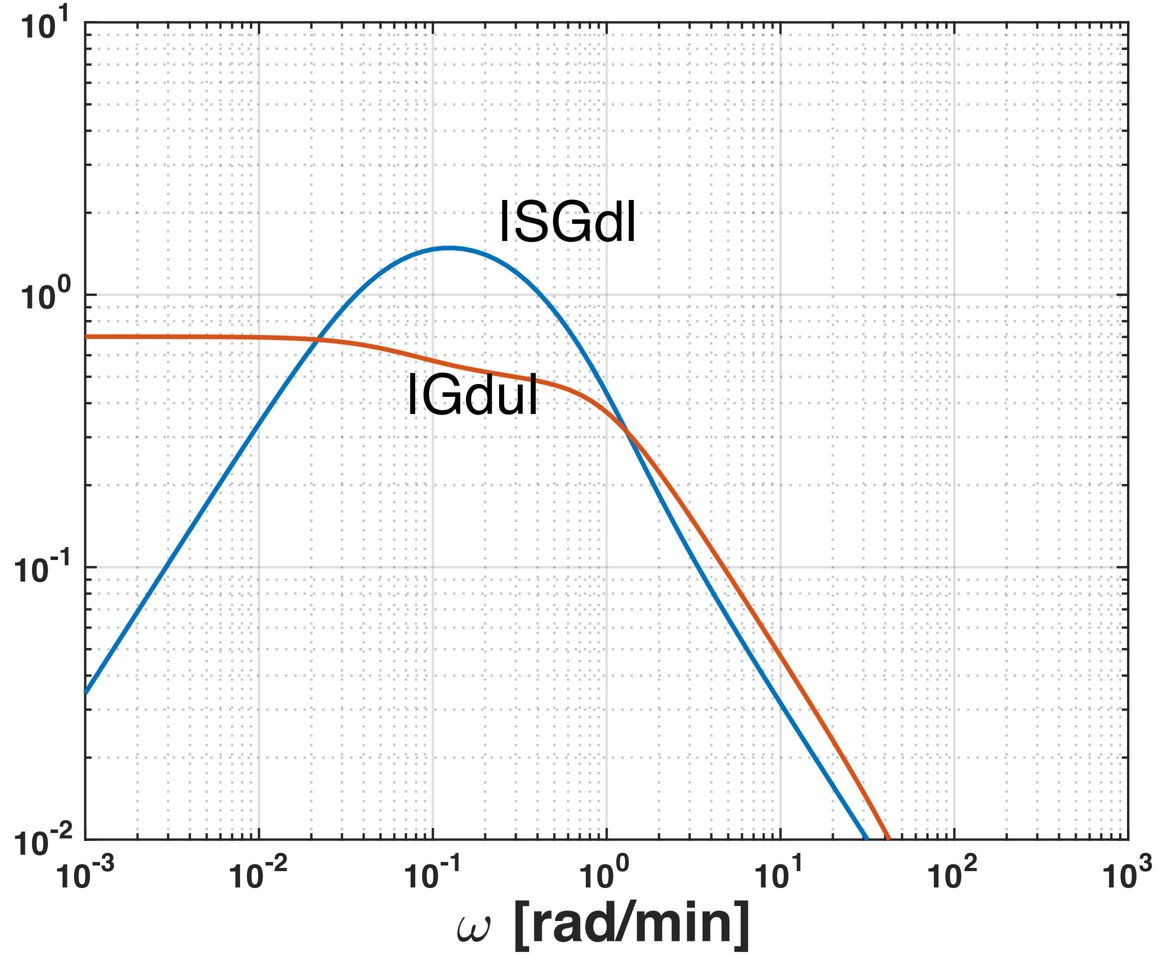


10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
$|W_T^{-1}|$
$\bar{\sigma}(T)$
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
ω [rad/min]

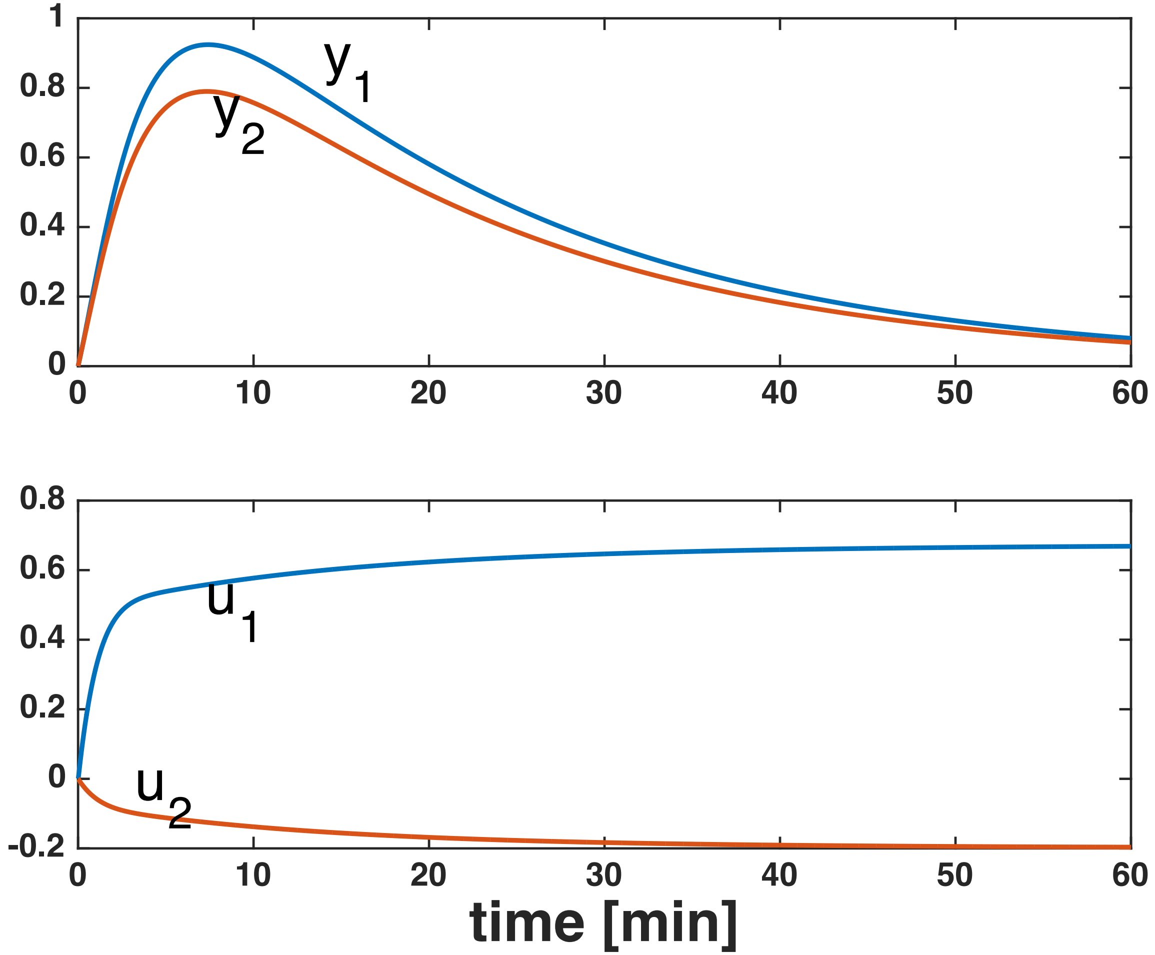
Design II: add robustness criterion

```
WS=((s+.05)/(s+1.d-4)); % Adds integral action
WU=(s+0.5)/(s+1); % to put relatively smaller weight on low frequency inputs
WT=2*s/(s+2); % For robust stability in presence of output uncertainty

d=icsignal(1);
r=icsignal(2);
u=icsignal(2);
y=icsignal(2);
Wy=icsignal(2);
Wu=icsignal(2);
Wt=icsignal(2);
P=iconnect;
P.Input=[d;r;u];
P.Output=[Wy;Wt;Wu;r-y];
P.equation{1}=equate(y,G*u+Gd*d);
P.equation{2}=equate(Wy,WS*(r-y));
P.equation{3}=equate(Wu,WU*u);
P.equation{4}=equate(Wt,WT*y);

[C,CL,gam]=hinfsyn(P.System,2,2);

>>gam
   gam = 1.796
```

Performance

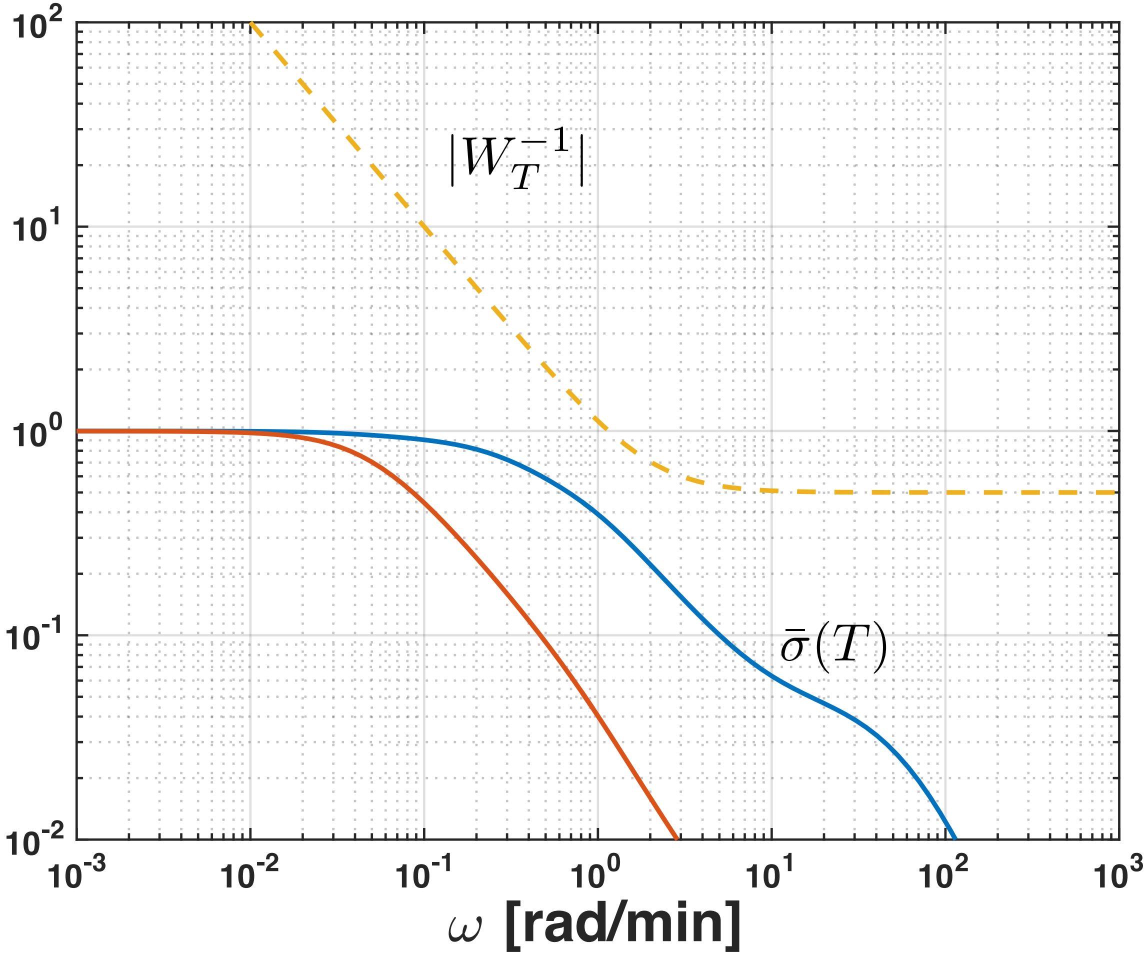


Simulation



1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
50
60
y1
y2
0.8
0.6
0.4
0.2
0
-0.2
0
10
20
30
40
50
60
u1
u2
time [min]

Robustness



$|W_T^{-1}|$
$\bar{\sigma}(T)$
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-2}
10^{-1}
10^0
10^1
10^2
10^3
ω [rad/min]

Design III: Mixed Sensitivity

```
Ws=((s+0.05)/(s+1.d-6))*eye(2);
WQ=(2*s/(s+2))*eye(2);
Wu=eye(2);

P=augw(G,Ws,Wu,WQ);

[C,CL,gam]=hinfsyn(P,2,2);
```

- Does not give satisfactory result in first try
- What is possible problem?
- Homework:
  - try to adjust weights to achieve performance and robustness requirements!
  - try to solve problem using LQG and H2, respectively