# Exercise session 1

Topics:
- Signal and System norms
- Small gain Theorem

→ 2.1, 2.2, 4.2, 6.1, 6.2

## Signal norms:

We first need a vector norm to then define a signal norm.

- Assume a vector $v \in \mathbb{R}^n$, then the Euclidean norm is:
  (vector norm)

$$\|v\| = \sqrt{\sum_{i=1}^{n} |v_i|^2} = \sqrt{v^T v}$$

where $|\cdot|$ denotes the absolute value of an element.

- Assume now a signal $y: \mathbb{R} \to \mathbb{R}^n$, then we have the following signal norms:
  - $L_\infty$ - norm (peak-norm):

$$\|y\|_\infty = \sup_{t \geq 0} \|y(t)\|$$

vector norm

  - $L_2$ - norm (energy-norm):

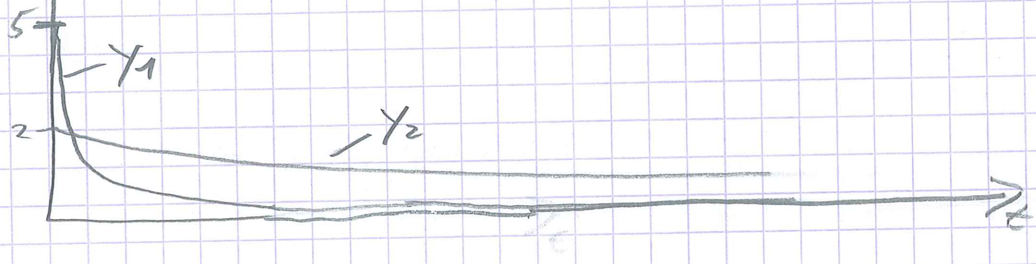
$$\|y\|_2 = \sqrt{\int_{-\infty}^{\infty} \|y(t)\|^2 \, dt}$$

→ Why do we need norms?
Why do we have different measures for norms?

2.2) Consider two signals:

$$y_1(t) = \begin{cases} 5e^{-5t} & t>0 \\ 0 & \text{otherwise} \end{cases}$$

$$y_2(t) = \begin{cases} 2e^{-0.2t} & t>0 \\ 0 & \text{otherwise} \end{cases}$$




a) $y_1$ has the largest peak value, hence $\|y_1\|_\infty > \|y_2\|_\infty$

$y_2$ has the largest area under its curve, i.e., a large energy norm so that $\|y_2\|_2 > \|y_1\|_2$

b) $\|y_1\|_\infty = \sup_{t\geq 0} |5e^{-5t}| = 5$

$\|y_2\|_\infty = \sup_{t\geq 0} |2e^{-0.2t}| = 2$

$$\|y_1\|_2 = \sqrt{\int_0^\infty \|5e^{-5t}\|^2 dt} = \sqrt{\int_0^\infty 25e^{-10t}\, dt}$$

$$= \sqrt{\left[-\frac{25}{10}e^{-10t}\right]_0^\infty} = \sqrt{2.5}$$

$$\|y_2\|_2 = \sqrt{\int_0^\infty \|2e^{-0.2t}\|^2 dt} = \sqrt{\int_0^\infty 4e^{-0.4t}\, dt}$$

$$= \sqrt{\left[-\frac{4}{0.4}e^{-0.4t}\right]_0^\infty} = \sqrt{10}$$

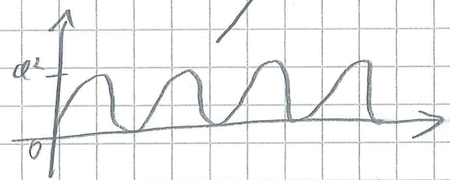
2.1) Determine the <u>system norms</u> $\|y\|_\infty$ and $\|y\|_2$ of the following signals

a) $y(t) = \begin{cases} a \sin(t) & t > 0 \\ 0 & t \leq 0 \end{cases}$

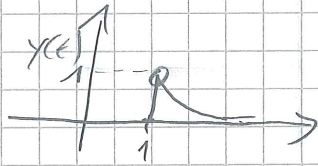

$$\|y\|_\infty = \sup_t |y(t)| = |a| \cdot \sup_{t \geq 0} |\sin(t)| = |a|$$

$$\|y\|_2 = \sqrt{\int_{-\infty}^{\infty} |y(t)|^2 \, dt} = \sqrt{\int_0^\infty a^2 \sin^2(t) \, dt} = \infty$$

→ We say that y is <u>bounded</u>, but <u>not finite-energy</u>




b) $y(t) = \begin{cases} \frac{1}{t} & t > 1 \\ 0 & t \leq 1 \end{cases}$

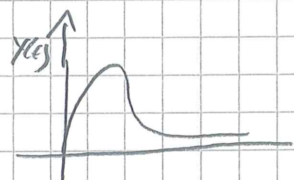



$$\|y\|_\infty = \sup_t |y(t)| = \sup_{t > 1} \left|\frac{1}{t}\right| = 1$$

$$\|y\|_2 = \sqrt{\int_{-\infty}^{\infty} |y(t)|^2 dt} = \sqrt{\int_1^\infty \frac{1}{t^2} \, dt}$$

$$= \sqrt{\left[-\frac{1}{t}\right]_1^\infty} = \sqrt{1} = 1 \quad \rightarrow \quad \text{<u>bounded and finite energy</u>}$$

c) $y(t) = \begin{cases} e^{-t}(1-e^{-t}) & t > 0 \\ 0 & t \leq 0 \end{cases}$




$$\|y\|_\infty = \sup_{t \geq 0} |y(t)| = \sup_{t \geq 0} e^{-t}(1-e^{-t}) \quad *$$

Find extremum:

$$\frac{d\, e^{-t}(1-e^{-t})}{dt} = -e^{-t}(1-e^{-t}) + e^{-t}e^{-t} \stackrel{!}{=} 0$$

$$\Leftrightarrow \quad e^{-t}(2e^{-t} - 1) \stackrel{!}{=} 0$$

$$\Rightarrow \quad t^* = -\ln\left(\frac{1}{2}\right)$$

From *: $\quad \|y\|_\infty = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}$

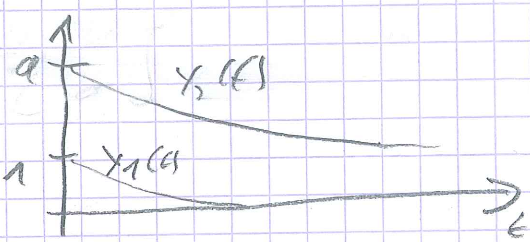
$$\|y\|_2 = \sqrt{\int_0^\infty \left[e^{-t}(1-e^{-t})\right]^2 dt} = \sqrt{\int_0^\infty e^{-2t}(1-2e^{-t}+e^{-2t})\,dt}$$

$$= \sqrt{\int_0^\infty e^{-2t}-2e^{-3t}+e^{-4t}\,dt} = \sqrt{\left[-\frac{1}{2}e^{-2t}+\frac{2}{3}e^{-3t}-\frac{1}{4}e^{-4t}\right]_0^\infty}$$

$$= \sqrt{\frac{1}{2}-\frac{2}{3}+\frac{1}{4}} = \frac{1}{\sqrt{12}}$$

d) Similar as a, b, c, but now use vector norm first and then calculate system norm.

$$y(t) = \begin{cases} \begin{bmatrix} e^{-t} \\ ae^{-t} \end{bmatrix} & t > 0 \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{otherwise} \end{cases}$$




Vector norm: $\|y(t)\| = \sqrt{y(t)^T y(t)} = \sqrt{\begin{bmatrix} e^{-t} & ae^{-t} \end{bmatrix} \begin{bmatrix} e^{-t} \\ ae^{-t} \end{bmatrix}}$

$$= \sqrt{e^{-2t}+a^2e^{-2t}}$$

Then: $\|y\|_\infty = \sup_{t>0} \|y(t)\| = \sqrt{1+a^2}$

$$\|y\|_2 = \sqrt{\int_{-\infty}^{\infty} \|y(t)\|_2^2\, dt}$$

$$= \sqrt{\int_{-\infty}^{\infty} e^{-2t}+a^2e^{-2t}\, dt}$$

$$= \sqrt{\left[-\frac{1}{2}e^{-2t}-\frac{a^2}{2}e^{-2t}\right]_0^\infty}$$

$$= \sqrt{\frac{1+a^2}{2}}$$

## System norm:

Consider a MIMO system:

$$u \longrightarrow \boxed{S(u)} \longrightarrow y$$

The **gain** is defined as $\|S\| = \sup\limits_{0 < \|u\|_2 < \infty} \dfrac{\|y\|_2}{\|u\|_2}$

$L_2$ norm

There do exist other norm definitions

(only defined for **finite-energy** signals)

$\rightarrow$ maximum energy amplification!

For scalar, linear system $Y(s) = G(s)\,U(s)$

$$\|S\| = \sup_\omega |G(i\omega)| = \|G\|_\infty$$

$\hookrightarrow$ **$H_\infty$-norm** (elementary for this course)

Input-output stable if $\|S\|_\infty < \infty$

4.2)

Short repetition:

Consider a linear, SISO system:

$$G(s) = \frac{\omega_0^2}{s^2 + 2\xi\omega_0 s + \omega_0^2}$$

From basic control course:

poles at $s_{1/2} = -\xi\omega_0 \pm \sqrt{(\xi\omega_0)^2 - \omega_0^2}$

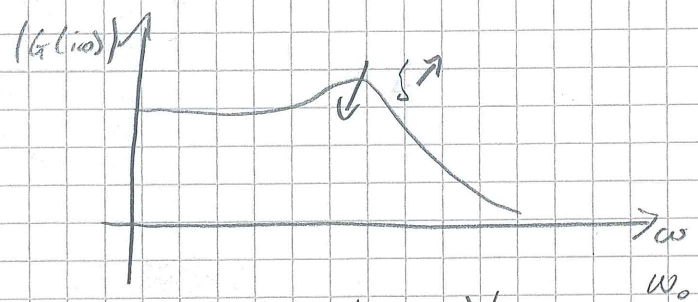
(Poles determine the step response) $= -\xi\omega_0 \pm \omega_0\sqrt{\xi^2 - 1}$

1) undamped if $\xi = 0$ $\quad s_{1/2} = \pm i\omega_0$

2) underdamped if $\xi \in (0,1)$ $\quad s_{1/2} = \omega_0\left(-\xi \pm \underbrace{\sqrt{\xi^2 - 1}}_{i\omega_d}\right)$

no imaginary parts in poles $\left\{\begin{array}{l} \text{3) critically damped if } \xi = 1 \quad s_{1/2} = -\omega_0 \\ \text{4) overdamped } \xi > 1 \end{array}\right.$

In case 2) we might get oscillations, Bode diagram can look like




Now: $\|G\|_\infty = \sup_\omega |G(i\omega)| = \sup_\omega \frac{\omega_0^2}{|-\omega^2 + 2\xi\omega_0\omega i + \omega_0^2|}$

$$= \sup_\omega \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\xi\omega_0\omega)^2}}$$

Idea: minimize the denominator

$$\frac{\partial}{\partial \omega} (\omega_0^2-\omega^2)^2 + (2\xi\omega_0\omega)^2 = 0$$

$$\Leftrightarrow -4(\omega_0^2-\omega^2)\cdot\omega + 8\xi^2\omega_0^2\omega = 0$$

$$\Leftrightarrow 4\omega\left(2\xi^2\omega_0^2 - \omega_0^2 + \omega^2\right) = 0$$

$\omega = 0$ or $\omega = \pm\sqrt{\omega_0^2(1-2\xi^2)}$

The square root has to be real (imaginary frequencies make no sense)

$\boxed{\text{Case 1: } \frac{1}{\sqrt{2}} \geq \xi}$

$$\Rightarrow 1-2\xi^2 \geq 0$$
$$\frac{1}{\sqrt{2}} \geq \xi$$

$$\|G\|_\infty = \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega_0^2(1-2\xi^2))^2 + 4\xi^2\omega_0^4(1-2\xi^2)}}$$

$$= \frac{\omega_0^2}{\sqrt{4\omega_0^4\xi^4 + 4\omega_0^4\xi^2 - 8\omega_0^4\xi^4}} = \frac{\omega_0^2}{\sqrt{4\omega_0^4\xi^2(1-\xi^2)}}$$

$$= \frac{1}{2\xi\sqrt{1-\xi^2}}$$

$$\frac{d}{d\xi}\,\xi\sqrt{1-\xi^2} = 1\cdot\sqrt{1-\xi^2} - \xi\frac{\cancel{2}\xi}{\cancel{2}\sqrt{1-\xi^2}} = 0$$

$$\Leftrightarrow 1-\xi^2-\xi^2 = 0$$

$$\Leftrightarrow \frac{1}{2} = \xi^2$$

$$\Leftrightarrow \frac{1}{\sqrt{2}} = \xi$$

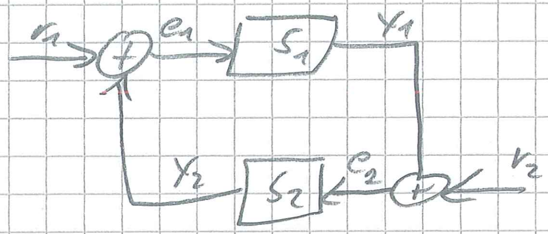
at $\xi = \frac{1}{\sqrt{2}} \Rightarrow \frac{1}{2\frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{2}}} = 1$ (Minimum at $\xi = \frac{1}{\sqrt{2}}$)

$\boxed{\text{Resonance peak!}}$

Hence $\|G\|_\infty \geq 1$

$\boxed{\text{Case } \frac{1}{\sqrt{2}} < \xi}$ evaluate at $\omega = 0 \Rightarrow \|G\|_\infty = 1$

The small gain theorem: Consider



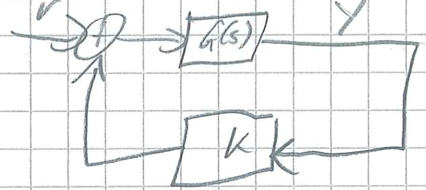

If $S_1$ and $S_2$ are input output stable ($\|S_1\| < \infty$ and $\|S_2\| < \infty$)
and $\|S_1\| \cdot \|S_2\| < 1$

$\boxed{\text{System norm}}$

Then the closed-loop with $e_1, e_2, y_1, y_2$ is input-output stable

6.1)

Assume




with

$$G(s) = \frac{a}{s+a}$$

$\boxed{\text{Case } a>0}$ $\|G\|_\infty = \sup_{\omega} |G(i\omega)| = \sup_{\omega} \frac{a}{\sqrt{\omega^2+a^2}} = 1 < \infty$ (stable)

By small gain we need $K \cdot \|G\|_\infty < 1$

$$\Rightarrow K \in [0, 1)$$

$\boxed{\text{Case } a<0}$ $G(s)$ is unstable $\Rightarrow$ no small gain since $\|G\|_\infty$ is not defined
(look at the poles of G)

Closed loop: $Y(s) = G(s)\left(R(s) + K\, Y(s)\right)$

$$\Rightarrow Y(s) = \frac{G(s)}{1 - G(s)K} R(s)$$

$$= \frac{a}{s+a-aK}$$

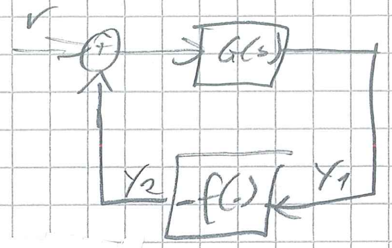
Pole at: $s = -a + aK = a(K-1)$

If $a>0$ then stable for $K-1<0$
$K<1$

$a<0$ " " " $K-1>0$
$K>1$

$\Rightarrow$ Small gain is conservative for the case $a<0$

6.2) Assume




with G and f as plotted in the compendium!

We have $\|G\|_\infty = 1.5 < \infty \Rightarrow$ stable

$S: y_2 = f(y_1)$ (Note: f is a nonlinearity and we find the "worst-case")

$$\|S\| = \sup \frac{\|f(y_1)\|_2}{\|y_1\|_2} \leq \frac{K\|y_1\|_2}{\|y_1\|_2} = K$$

What is K?

$$|y_2| = |f(y_1)| \leq \frac{1}{2}|y_1|$$

$\Rightarrow$ stable

$K = \frac{1}{2}$

Since $\frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4} < 1$, the system is stable by small gain theorem.

Homework: Do 4.1, 5.1 a, b, and 6.3