EL2520 - Control Theory and Practice - Advanced Course
Solution/Answers – 2017-08-18

1. (a) The determinant of $G(s)$ is -3 and the LCD of this and the elements are $(s+1)(s-1)$, hence poles at $s=-1$ and $s=1$. The zeros are given by the determinant normalized to have the pole polynomial as denominator, i.e., $\det G = -3(s+1)(s-1)/((s+1)(s-1))$ and hence we have zeros at $s=-1$ and $s=1$. With two poles we need two states in a minimal state space realization.

   (b) $\|G\|_\infty = \sup_w \bar{\sigma}(G(i\omega))$. The singular values are given by $\sqrt{\lambda(G^T G)}$ which gives $\bar{\sigma}(G) = \frac{1}{\omega^2+1}\sqrt{2}$. The supremum is at $\omega = 0$ and hence $\|G\|_\infty = \sqrt{2}$.

   (c) Any RHP poles in $G(s)$ must appear as RHP zeros in $S(s)$ for internal stability (follows from interpolation constraints). Since $S(s)$ does not have a zero at $s=1$, the system is not internally stable (a RHP pole has been cancelled).

2. (ai) For a 2×2 system, the $1,1$-element of the RGA is $\lambda_{11} = \lambda_{22} = \frac{1}{1-\frac{g_{12}g_{21}}{g_{11}g_{22}}}$ and $\lambda_{12} = \lambda_{21}$. This gives $\lambda_{11}(i0) = \frac{1}{1-2} = -1$ and $\lambda_{12} = 2$. The rule is to never pair on steady-state RGA elements, and hence $y_1 - u_2$, $y_2 - u_1$ is the only viable pairing. At bandwidth $\omega = 0.5$ we get $\lambda_{12}(i0.5) = 1 - (0.5i+1)/(0.5i-1)$ and $|\lambda_{12}(i0.5) = 1.79$ which indicates relatively weak interactions and hence decentralized control should be expected to work quite well.

   (aii) $G(s)$ has a RHP zero at $s=1$ and this can not be cancelled by $W(s)$ due to requirement of internal stability. Hence the proposed decoupler does not provide internal stability.

   (aiii) Must retain zero at $s=1$ and with decoupling this means every diagonal element must have this zero. Thus we propose

$$G(s)W(s) = D(s) = \frac{1}{(s+1)^2}\begin{pmatrix} (-s+1) & 0 \\ 0 & (-s+1) \end{pmatrix}$$

   which gives

$$W(s) = G^{-1}(s)D(s) = \begin{pmatrix} -1 & \frac{1}{s+1} \\ 2 & -1 \end{pmatrix}$$

   (b) The closed-loop Jacobian is $A = \begin{bmatrix} 1 & 1 \\ -1 & -1.1 \end{bmatrix}$ and the eigenvalues are $\lambda = -0.37, 0.27$ which both are inside the unit circle in the complex plane and hence the closed-loop is stable.

3. (a) Disturbance attenuation is needed for frequencies where $|G_d(i\omega)| > 1$, and $|G_d| = 3/\sqrt{100\omega^2+1}$ which equals one for $\omega_d = \sqrt{0.08} = 0.28$.

The limitations are partly due to the RHP zero in G_1 which gives that disturbance attenuation can only be achieved up to $\omega = z = 4$ (or $\omega = z/2 = 2$ if we want to limit the peak of S to 2), and the delay in G_2 which gives that attenuation can only be achieved approximately up to $\omega = 2/\theta = 1$. Neither of these are thus a problem since they exceed the required ω_d. However, we also need sufficient input to counteract the disturbance, i.e., $|G| > |G_d| \forall \omega < \omega_d$ for perfect disturbance attenuation (for acceptable attenuation it suffices that $|G| > |G_d| - 1$). We see that $|G_1| > |G_d| \forall \omega$ which $|G_2| < |G_d| - 1 \forall \omega$ and hence we can use u_1 only, but not u_2 only.

(b) Requirement is $|SG_d| < 1 \forall \omega$. We try with a simple P-controller $F_y = K_c$, which gives $S = 1/(1+G_1 K_c) = (10s+1)/(10s+1+K_c(s-4))$ and $SG_d = 3e^{-s}/((10+K_c)s + (1-4K_c))$ which is stable and with magnitude $< 1 \forall \omega$ if $-10 < K_c < -0.5$.

(c) The requirement for $|y| < 1$ with $|d| < 1$ translates to $\|SG_d\|_\infty < 1$, the requirement for $|u| < 1$ with $|d| < 1$ translates into $\|F_y SG_d\|_\infty < 1$ and the robust stability requirement $\|0.2T\|_\infty > 1$. Stacked, this gives the optimization problem

$$\min_u \left\| \begin{matrix} SG_d \\ F_y SG_d \\ 0.2T \end{matrix} \right\|_\infty$$

We have $y = SGdd$ and $u = F_y SG_d$ and hence we should have d as an input and $z = [y \ u]^T$ as an output of the extended system for the first two criteria. Unfortunately, it is difficult to find an output which has the transfer function T from the input d. Thus, we have to add another input, e.g., measurement noise n with gain 0.2 since then $y = 0.2Tn$. The disadvantage of adding the second input n is that we then also include the transfer function from n to u in the objective.

4. (a) $G(s)$ has a RHP zero at $s = 1$ and $y_z^H G(z) = 0$ gives the zero direction $y_z^H = 1/\sqrt{5}\begin{bmatrix}-1 & 2\end{bmatrix}$. Then, the requirement acceptable disturbance attenuation being feasible is $|y_z^H G_d(z)| < 1$. For disturbance d_1 we get $y_z^H G_{d1}(z) = 2.01 > 1$, and for d_2 $y_z^H G_{d2}(z) = 0 < 1$. Hence, it is not possible to attenuate disturbance d_1 but the there are no given limitations that hinders acceptable attenuation of d_2.

(b) Any RHP pole in the open-loop should appear as a RHP zero in $S(s)$ for internal stability. Here the open-loop has RHP poles, but $S(s)$ has no RHP zeros and hence the system is not internally stable.

(c) This is a standard LQ-problem with $M = I, Q_1 = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}, Q_2 = 2$. The optimal state feedback is given by $L = Q_2^{-1} B^T S$ where $S \geq 0$ is the solution to

$$A^T S + SA + M^T Q_1 M - SBQ_2^{-1} B^T S = 0$$

which gives $S = \sqrt{2}\begin{pmatrix} \sqrt{\sqrt{2}} & \frac{1}{\sqrt{\sqrt{2}}} \\ 1 & \end{pmatrix}$ and $L = \begin{bmatrix} 1/\sqrt{2}\sqrt{\sqrt{2}} \end{bmatrix}$.

5. (a) $G = G_0(I + \Delta_G)$ gives

$$\Delta_G = GG_0^{-1} - I = \begin{pmatrix} 0 & \frac{2}{3}\delta_1 \\ \frac{1}{2}\delta_2 & 0 \end{pmatrix}$$

(b) Two poles in $s = -2$ implies on pole in $s = -2$ for each loop, i.e., $T_{ii} = 2/(s+2)$. Since $GF_y = T/(1-T) = 2/s$ we get $K_1 = 23, \tau_{I1} = 23$ and $K_2 = 92/3, \tau_{I2} = 23$.

(c) Derivation not shown here gives the robustness criterion $\|T_I\Delta_G\|_\infty < 1$ and $T_I = F_yG_0(I + F_yG_0)^{-1}$ stable. The criterion gives that $|\delta_1| < \frac{3}{2}$ and $|\delta_2| < 2$.

(d) We get

$$FG = \begin{pmatrix} \frac{2}{s} & \frac{\delta_1}{s} \\ \frac{4\delta_2}{3s} & \frac{2}{s} \end{pmatrix}$$

and

$$S = (I + FG)^{-1} = \frac{s}{(s+2)^2 - \frac{4}{3}\delta_1\delta_2}\begin{pmatrix} s+2 & -\delta_1 \\ -\frac{4}{3}\delta_2 & s+2 \end{pmatrix}$$

for which we get stability for $\delta_1\delta_2 < 3$. It suffices to check the sensitivity for stability since there are no cancellations between F and G in the RHP. This is a less conservative criterion than obtained from the robustness criterion in (c) since we get a bound on the product of δ_1 and δ_2 rather than individual bounds. For instance, with $\delta_1 \leq 0$ we allow $\delta_2 = \infty$, but the result in (c) says $|\delta_2| < 2$ for stability. The conservativeness in (c) comes from the use of the small gain theorem which is sufficient but not necessary for stability.