EL2520
Control Theory and Practice

Lecture 2: The closed-loop system

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Goals

After this lecture, you should:

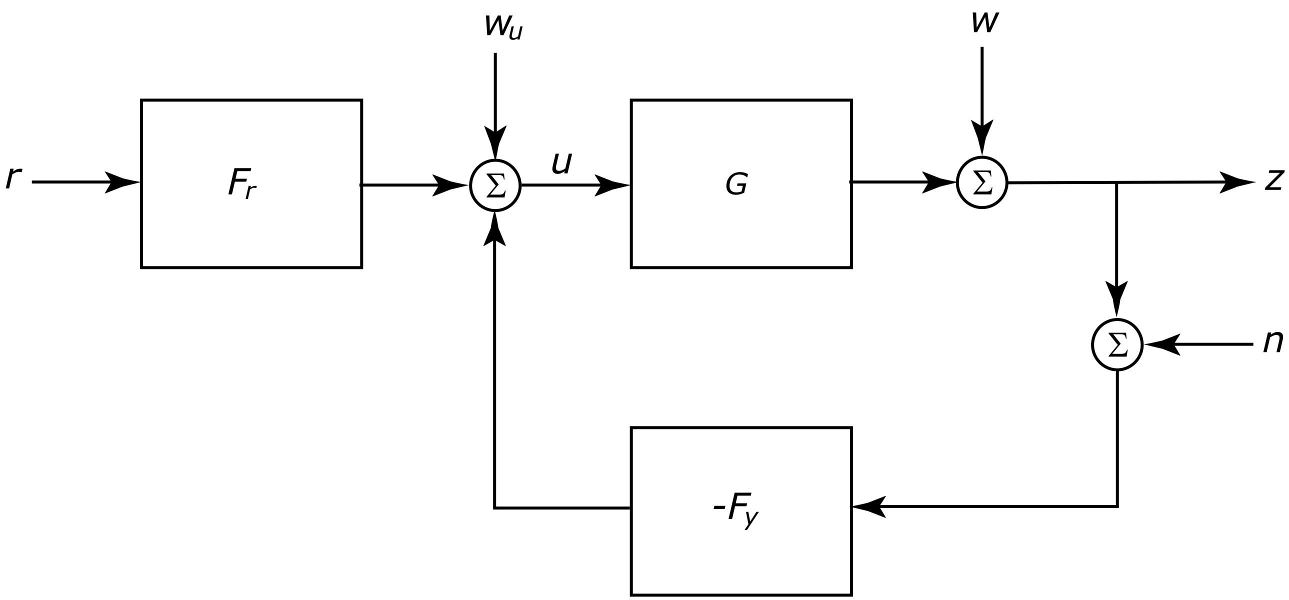
- Know that the closed-loop is characterized by 7 transfer functions
  - several objectives, but trade-offs exist, e.g. $S+T=1$.
  - *internal stability* requires all 7 to be stable

- Be able to determine, analyze and design desired sensitivity functions
  - sensitivity function for disturbance rejection
  - complementary sensitivity function for robust stability and noise
  - control problem formulated as objective of making norm of key transfer-functions small.

Material: course book Chapter 6, lecture notes 2.

Contents

The closed-loop system



Controller:	feedback F_y, feedforward F_r
Disturbances:	w, w_u drives system from desired state
Measurement noise:	corrupts information about z

Aim: find controller such that z follows r, with limited use of u

The design problem

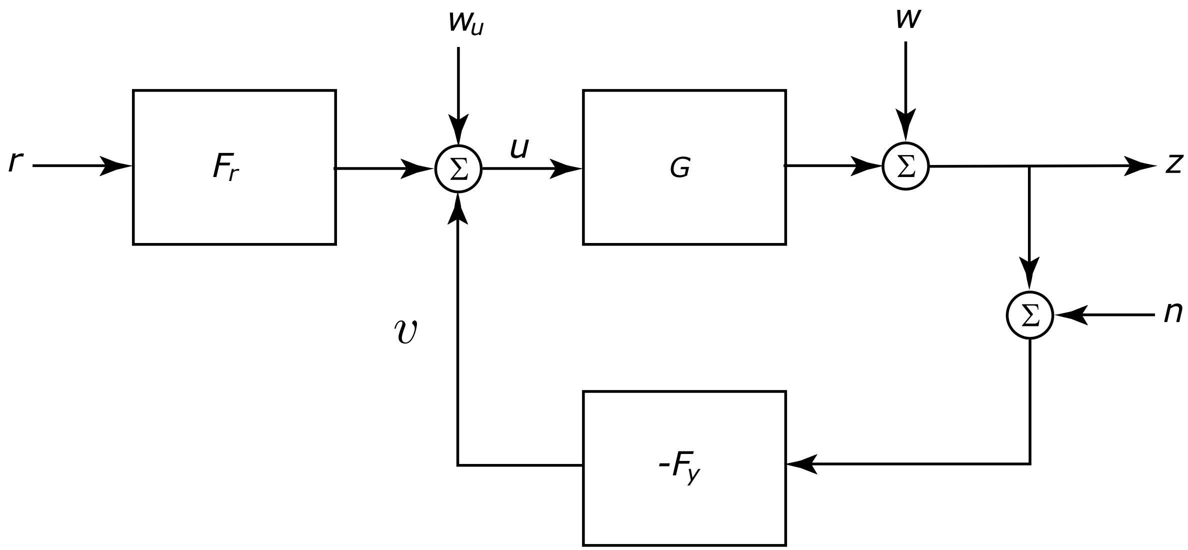
Design problem: find a controller that

a) Attenuates the effect of disturbances
b) Does not inject too much measurement noise into the system
c) Makes the closed loop insensitive to process variations
d) Makes the output follow command signal

Often convenient with two-degree of freedom controller (can design setpoint tracking independent of disturbance attenuation).

Use feedback to deal with a,b,c; use feedforward to deal with d.

Closed Loop Transfer Functions



$$z = w + G(\,w_u + F_r r - F_y(z+n)\,) \Rightarrow$$

$$z = \underbrace{\frac{1}{1+GF_y}}_{S} w + \underbrace{\frac{G}{1+GF_y}}_{SG} w_u + \underbrace{\frac{GF_r}{1+GF_y}}_{G_c} r - \underbrace{\frac{GF_y}{1+GF_y}}_{T} n$$

Similarly, we find

$$u = SF_r r - SF_y(w+n) + Sw_u$$

SISO: Closed-loop characterized by *seven* transfer functions

Transfer functions and observations

$$S = \frac{1}{1+GF_y} \qquad (w \to z,\ w_u \to u) \quad \text{sensitivity function}$$

$$T = \frac{GF_y}{1+GF_y} \qquad (n \to z) \quad \text{complementary sensitivity}$$

$$G_c = \frac{GF_r}{1+GF_y} \qquad (r \to z) \quad \text{closed loop system}$$

$$SG = \frac{G}{1+GF_y} \qquad (w_u \to z)$$

$$SF_y = \frac{F_y}{1+GF_y} \qquad (n \to u)$$

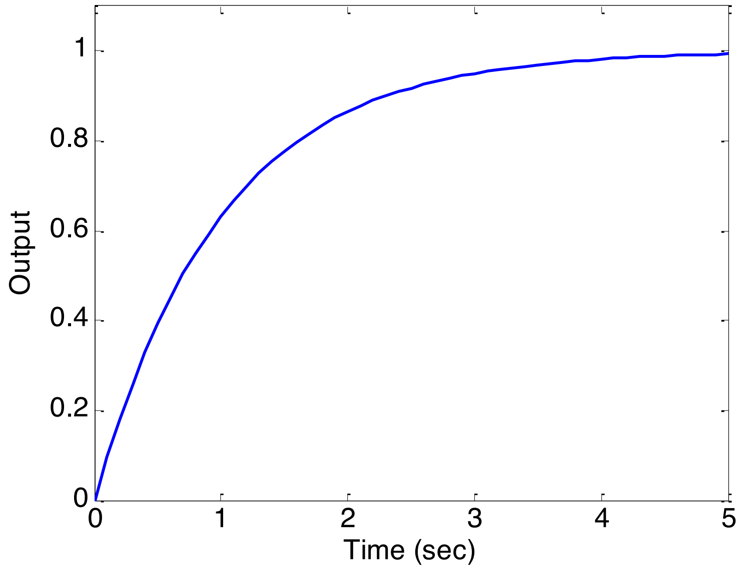
$$SF_r = \frac{F_r}{1+GF_y} \qquad (r \to u) \qquad TF_r = \frac{F_r G F_y}{1+GF_y} \qquad (r \to v)$$

Observation: need to look at all! Many tradeoffs (e.g. S+T=1)

A warning!

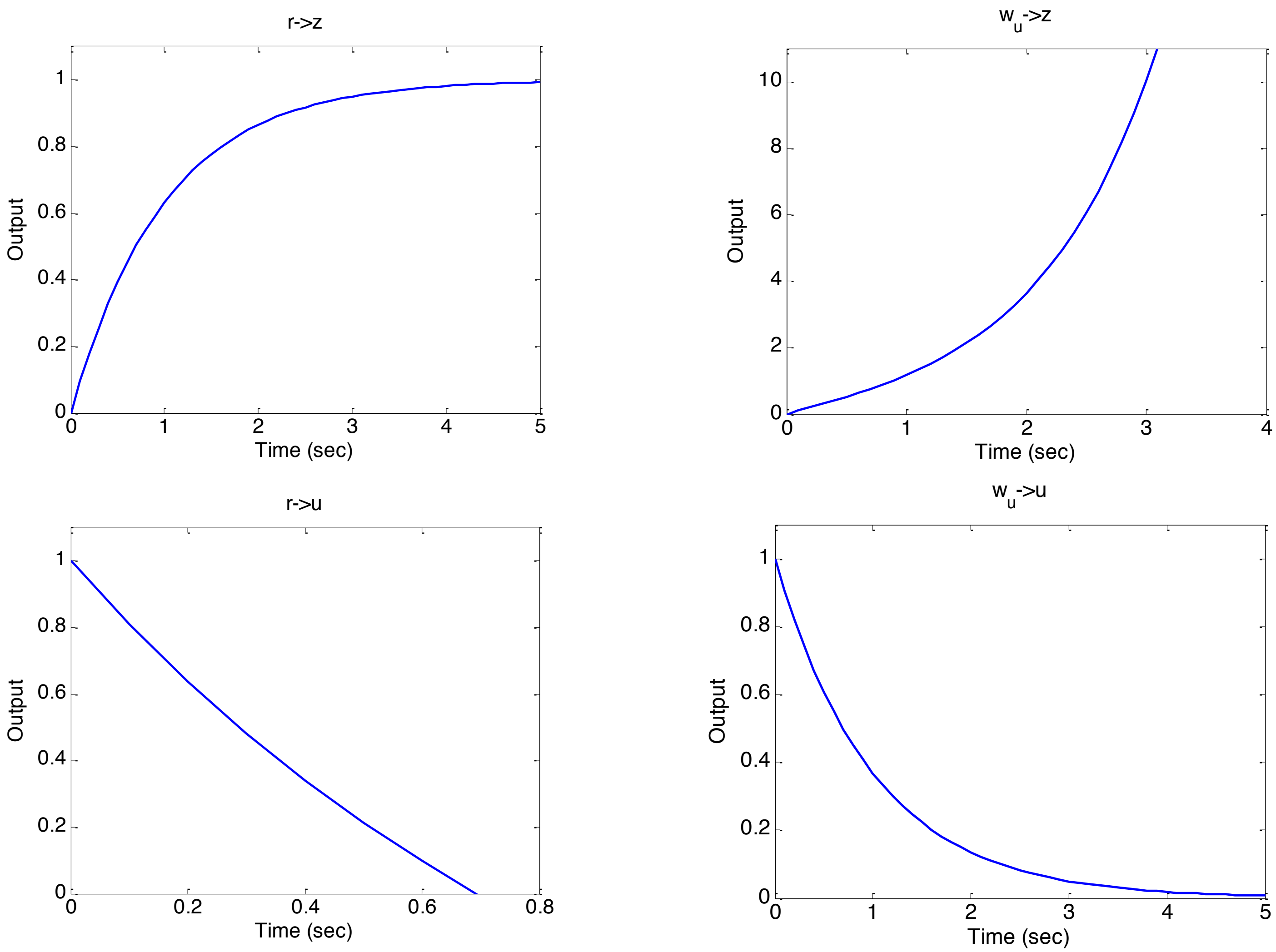
Individual time responses may look good

$$r \rightarrow z$$



but you need to verify that *all* transfer functions are as desired!

Four responses, same controller



What is going on?

Process: $$G = \frac{1}{s-1}$$

Controller: $$F_y = F_r = \frac{s-1}{s}$$

Transfer functions:

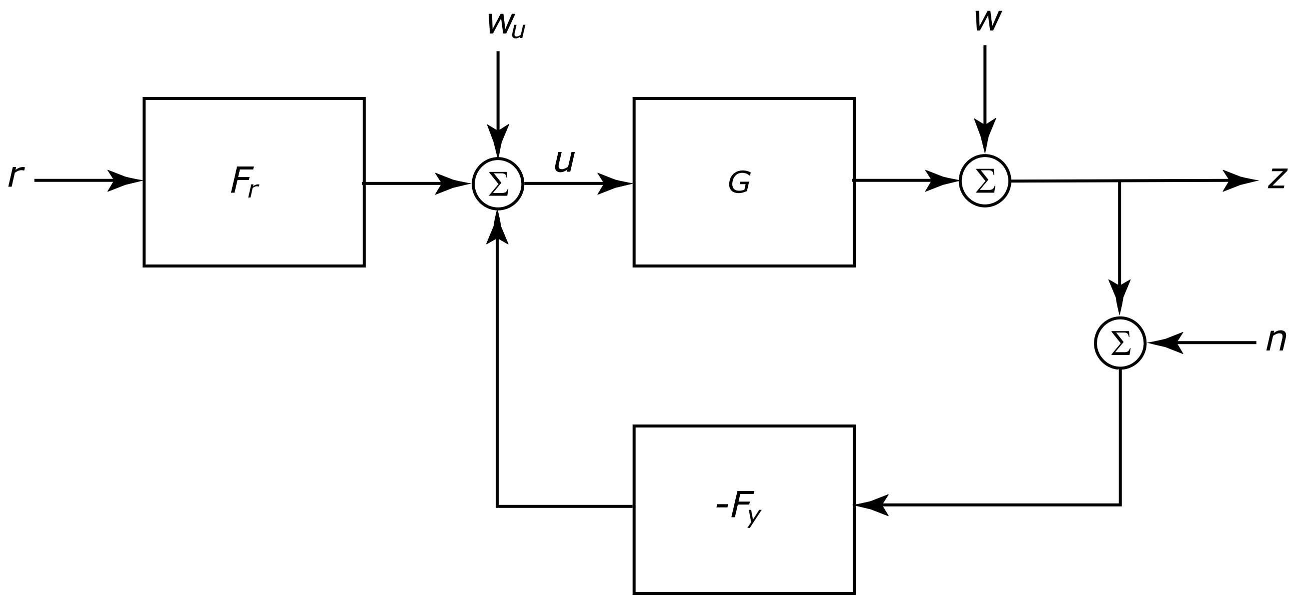
$$T = \frac{1/s}{1+1/s} = \frac{1}{s+1} \qquad \text{(stable)}$$

$$SG = \frac{s}{(s+1)(s-1)} \qquad \text{(unstable!)}$$

$$SF_y = \frac{(s-1)}{(s+1)} \qquad \text{(stable)}$$

$$S = \frac{s}{(s+1)} \qquad \text{(stable)}$$

Internal stability



Definition. The closed loop system above is *internally stable* if it is input-output stable.

Here inputs are all external signals; outputs are all internal signals

Theorem. If G is SISO, the closed-loop system is internally stable if and only if $S,\ SG,\ SF_y,\ F_r$ are all stable

Sensitivity functions

Sensitivity and complementary sensitivity are particularly important:

- S determines attenuation of disturbances and sensitivity to model uncertainty
- T determines sensitivity to noise and robustness to unmodelled dynamics

Both connected to classical stability margins (gain, phase margin) in SISO case

First trade-off: **S+T=1** - cannot make both zero at the same time.

Disturbance rejection

The transfer function from w to z in open loop is

$$G^{\mathsf{ol}}_{w \to z} = 1$$

while the closed-loop counter-part is

$$G^{\mathsf{cl}}_{w \to z} = \frac{1}{1 + GF_y}$$

Thus

$$\frac{G^{CL}_{w \to z}}{G^{OL}_{w \to z}} = \frac{1}{1 + GF_y} = S$$

S quantifies disturbance attenuation from feedback

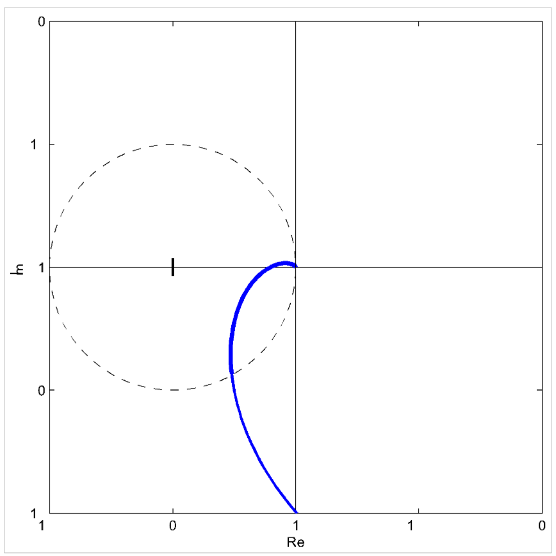
Disturbance at frequencies with

- $|S(i\omega)| < 1$ attenuated by feedback
- $|S(i\omega)| > 1$ amplified by feedback

Nyquist curve interpretation

$|S(i\omega)| = |L(i\omega) + 1|^{-1}$, where $L = GF_y$, is inverse distance from Nyquist curve to the point -1 on the negative real axis

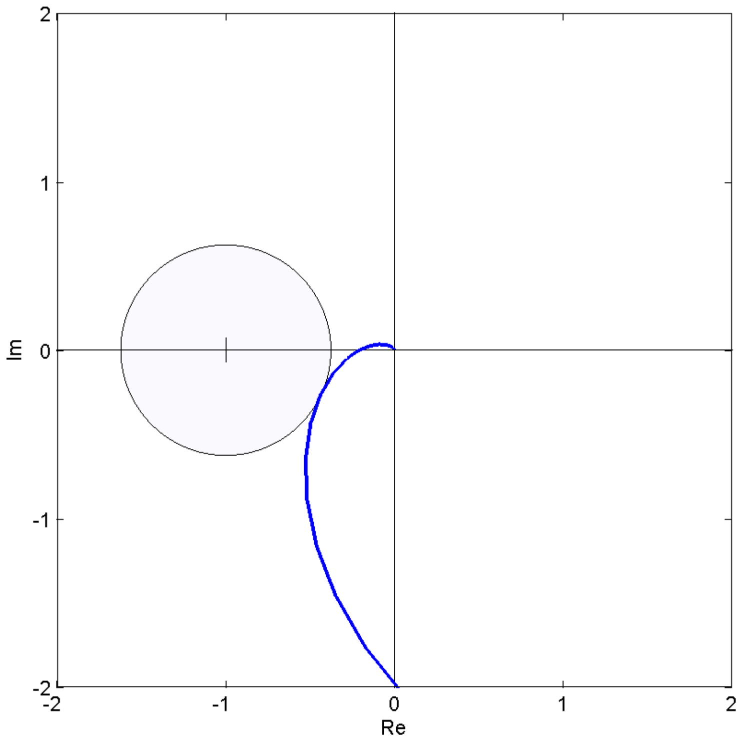
Disturbance amplified at frequencies where Nyquist curve is inside unit circle centered at the -1 point.



Observation: cannot avoid circle with radius one if pole excess is 2 or more, i.e., must amplify disturbances at some frequencies.

Maximum sensitivity and M_s-circles

Specification $|S(i\omega)| \leq M_s$: loop gain must stay outside circle with radius M_s^{-1}



Reasonable values: $1.2 < M_s < 2$

(picture shows M_s=2)

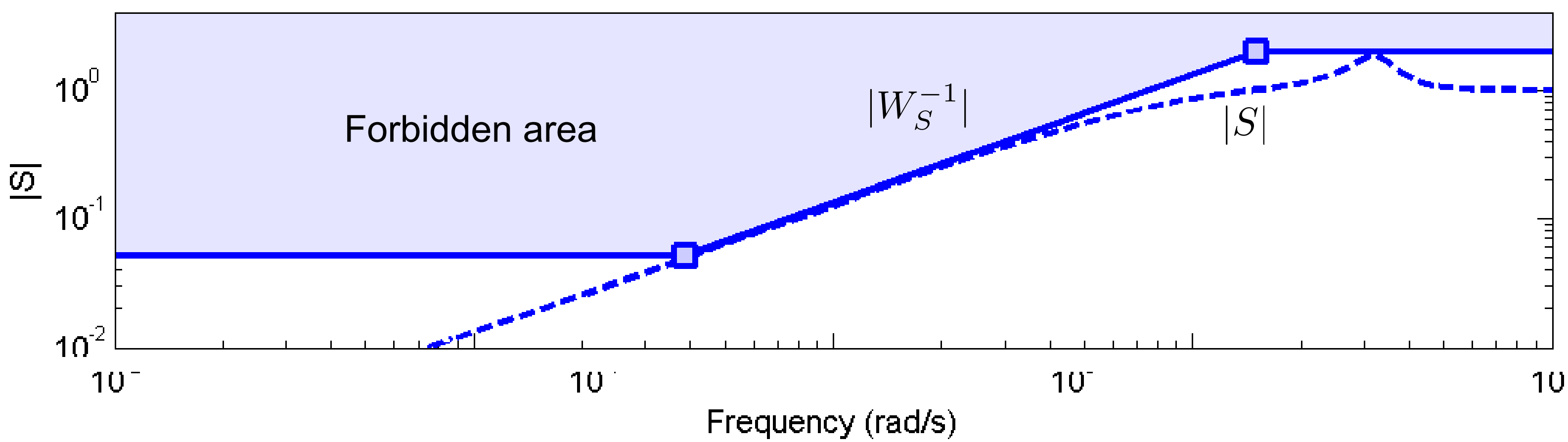
Sensitivity shaping

Observations:

- Can’t attenuate disturbances at all frequencies (if pole excess is 2)
- Need to limit $|S(i\omega)|$ at frequencies with significant disturbances

Reasonable design specification

$$|S(i\omega)| \leq |W_S^{-1}(i\omega)| \ \forall\omega \quad \Leftrightarrow \quad \|W_S S\|_\infty \leq 1$$



Sensitivity to uncertainties

The response of z to r is nominally (without uncertainty)

$$z = G_{\mathrm{cl}} r = \frac{G F_r}{1 + G F_y} r$$

If there is a model error, so that the true open-loop system is

$$\tilde{G} = (1 + \Delta_G) G$$

then the true response to r is

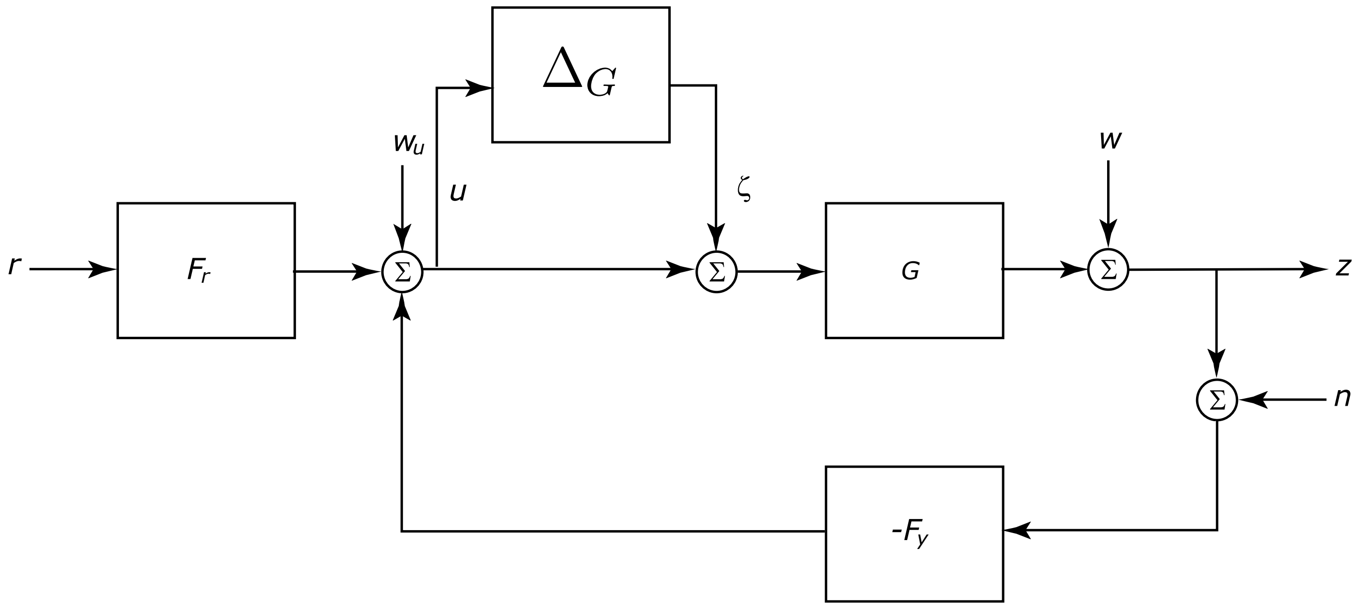
$$\tilde{z} = (1 + \tilde{S} \Delta_G) z$$

Thus, sensitivity function also quantifies “attenuation” of model uncertainty

Robust stability

Uncertainty also affect stability of closed-loop system.

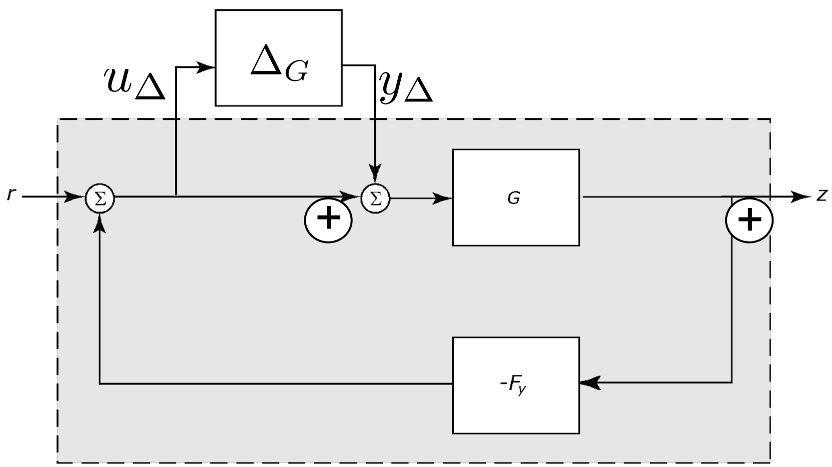
Assume true system is given by $\tilde{G} = (1 + \Delta_G)G$



What linear Δ_G can be tolerated without risking stability?

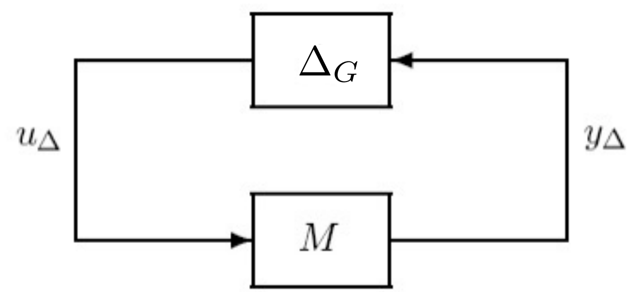
Robust stability

Assume all exogenous inputs (r, w, w_u, n) zero, and re-write



$$u_\Delta = T y_\Delta = M y_\Delta$$

$\Rightarrow$



Robust stability

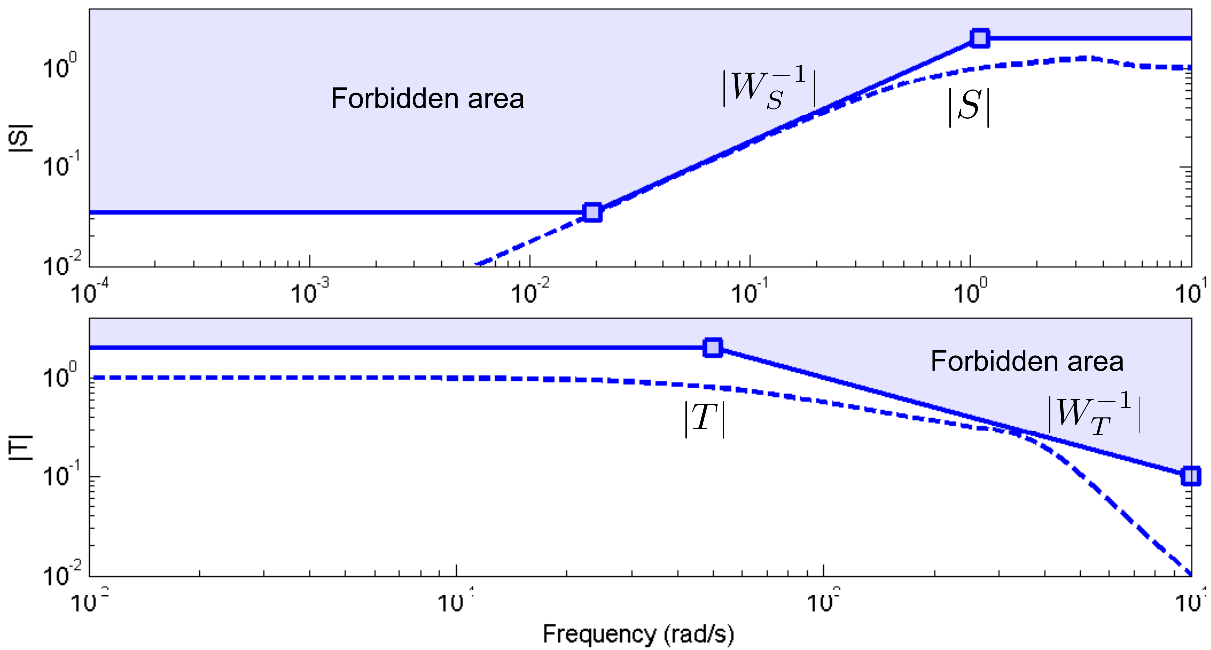
Assume Δ_G stable and nominal-system T internally stable.
If $\|T\Delta_G\|_\infty < 1$, then the system is input-output stable.

$$|T(i\omega)| < \frac{1}{|\Delta_G(i\omega)|}\ \forall\omega \quad \Leftrightarrow \quad \|T\Delta_G\|_\infty < 1$$

Proof: Small Gain Theorem.

Sensitivity shaping

Reasonable design criterion: make sure that both the sensitivity S and the complementary sensitivity T avoid "forbidden areas"



$$|S(i\omega)| \leq |W_S^{-1}(i\omega)| \qquad \Leftrightarrow \qquad \|SW_S\|_\infty \leq 1$$

$$|T(i\omega)| \leq |W_T^{-1}(i\omega)| \qquad \qquad \|TW_T\|_\infty \leq 1$$

Summary

– Closed-loop system characterized by 7 transfer functions
  - Need to consider all!
  - All must be stable for internal stability.
– Sensitivity and complementary especially important
  - S: disturbance attenuation, "performance sensitivity"
  - T: noise attenuation, robust stability
– Control system design via "sensitivity shaping"
– Conflicts and limitations
  - S+T=1
  - $|S(i\omega)| \geq 1$ for some ω (disturbance amplification!)
  - More on this in the next two lectures!