6 EL2520 Lecture notes 6: Performance Limitations in Multivariable Control

In this lecture we will extend the results on performance specifications and performance limitations in SISO systems, covered in Lecture 4, to the more general case of MIMO systems. We start by considering internal stability for MIMO systems since stability is a pre-requisite for any consideration of performance.

6.1 Internal Stability

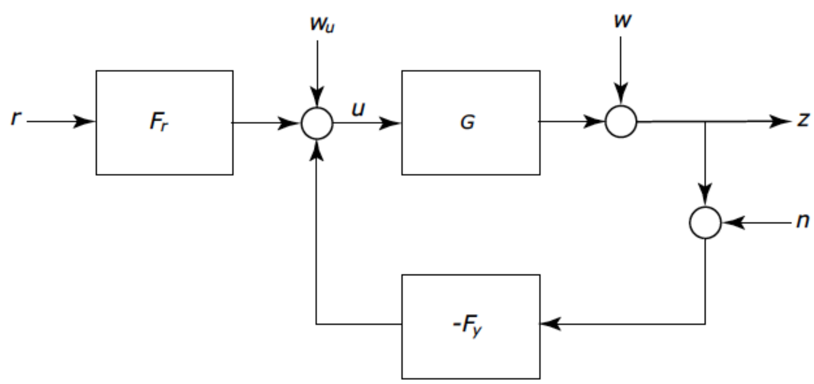

Figure 6.1: Two-degree of freedom control system.

Consider the closed-loop system in Figure 6.1. Recall that a system is internally stable if it is input-output stable from any input to any output. It suffices to consider one input and one output on either side of the two blocks in the loop in Figure 6.1. Let us for instance consider the inputs w_u and w and the outputs u and z, for which we derive the transfer-functions

$$z = \underbrace{(I + GF_y)^{-1}}_{S} w + \underbrace{(I + GF_y)^{-1}G}_{SG} w_u \quad (1)$$

$$u = \underbrace{(I + F_yG)^{-1}F_y}_{S_uF_y} w + \underbrace{(I + F_yG)^{-1}}_{S_u} w_u \quad (2)$$

Thus, we have internal stability if S, SG, S_u, S_uF_y all stable. Note that we must also require the pre-filter $F_r(s)$ to be stable since it is outside the feedback loop. Note that cancellations of poles and zeros in the RHP between $G(s)$ and $F_y(s)$ always will cause instability of at least one of these transfer-functions and hence internal instability. On the other hand, if there are no cancellations in the RHP, then it suffices to check stability of one the four transfer-functions (in addition to $F_r(s)$).

We finally remark that the fact that we are not allowed to cancel zeros or poles in the RHP implies that these always will appear as corresponding RHP zeros of $T(s)$ and $S(s)$, respectively, just like in the SISO case.

6.2 Performance Specifications

Consider again the closed-loop system in Figure 6.1. Let us for now limit our discussion to the problem of attenuating disturbances w and measurement noise n in the output z. From the block diagram we derive

$$z = (I + GF_y)^{-1}w - (I + GF_y)^{-1}GF_y n = Sw - Tn$$

Thus, we should make the sensitivity function S "small" for disturbance attenuation and the complementary sensitivity T "small" to avoid noise being amplified in the output. At each frequency ω we have have (see also Lec 5)

$$\underline{\sigma}(S(i\omega)) \leq \frac{|z|}{|w|} \leq \bar{\sigma}(S(i\omega)) \ ; \quad \underline{\sigma}(T(i\omega)) \leq \frac{|z|}{|n|} \leq \bar{\sigma}(T(i\omega))$$

depending on the direction of w and n. Thus, to bound the output $|z|$ in the presence of disturbances w and noise n in any direction we must bound $\bar{\sigma}(S)$ and $\bar{\sigma}(T)$. Similar to what we did for SISO systems, it is then natural to specify a frequency dependent bound on the sensitivity and complementary sensitivity, respectively

$$\bar{\sigma}(S(i\omega)) \leq |W_S^{-1}(i\omega)| \ \forall\omega \quad \Leftrightarrow \quad \|W_S S\|_\infty \leq 1$$

$$\bar{\sigma}(T(i\omega)) \leq |W_T^{-1}(i\omega)| \ \forall\omega \quad \Leftrightarrow \quad \|W_T T\|_\infty \leq 1$$

Thus, we have performance specifications in terms of the $\mathcal{H}_\infty$ norm of the weighted sensitivity functions, just like we derived for SISO systems earlier. The bounds are illustrated in Figure 6.2.

The control design problem then essentially consists of defining the weights W_S and W_T and then solving the resulting optimization problem[1]. Since the objective of the optimization is to make $\|W_S S\|_\infty \leq 1$, $\|W_T T\|_\infty \leq 1$, it is important to choose the weights W_S and W_T so that they not only reflect our control objectives, but also reflect any fundamental limitations and conflicts that must be satisfied. We therefore next discuss limitations and conflicts that must be respected when choosing the weights W_S amd W_T.

6.3 S+T=I

We have

$$S + T = (I + GF_y)^{-1} + (I + GF_y)^{-1}GF_y = I$$

[1]In many case, we may want to consider also other transfer-functions than S and T, but more about that in Lecture 7. For now we focus on S and T.

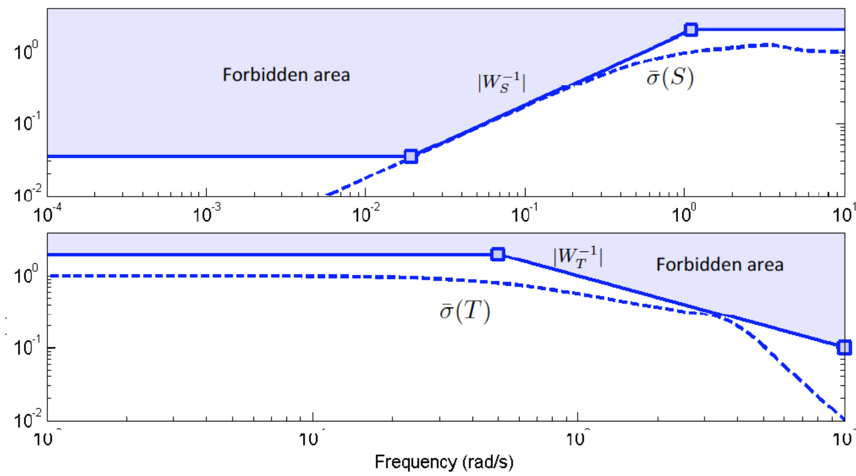

Figure 6.2: Performance bounds for maximum singular values of S and T, respectively. The bounds corresponds to the performance objective $\|W_S S\|_\infty <$ and $\|W_T T\|_\infty < 1$.

To understand what limitations this impose on $\bar{\sigma}(S)$ and $\bar{\sigma}(T)$, and thereby our choice of W_S and W_T, we employ Fan's Theorem:

$$\sigma_i(A + B) \geq \sigma_i(A) - \bar{\sigma}(B)$$

Consider $\sigma_i = \bar{\sigma}$, i.e., the maximum singular value, and let $A = S, B = T$. Fan's Thm then yields

$$\bar{\sigma}(S) \leq 1 + \bar{\sigma}(T)$$

Similarly, let $A = I, B = -T$ or $A = -I, B = T$ which combined yields

$$\bar{\sigma}(S) \geq |1 - \bar{\sigma}(T)|$$

and thus

$$|1 - \bar{\sigma}(T)| \leq \bar{\sigma}(S) \leq 1 + \bar{\sigma}(T)$$

From this we conclude that we can not make both $\bar{\sigma}(S)$ and $\bar{\sigma}(T)$ small at the same frequency. Hence, we can not choose both $|W_S|$ and $|W_T|$ large at the same frequency. Also, if one is much larger than one at some frequency, then so is the other

$$\bar{\sigma}(S) >> 1 \quad \Leftrightarrow \quad \bar{\sigma}(T) >> 1$$

6.4 Extension of Bode Sensitivity Integral

Recall the Bode Sensitivity integral presented in Lecture 4. It can be shown that the determinant of a MIMO sensitivity function $\det S$ has the properties of a SISO sensitivity function. Application of Bode Sensitivity integral to $\det S$ then yields

$$\int_0^\infty \ln|\det S(i\omega)|d\omega = \sum_j \int_0^\infty \ln \sigma_j(S(i\omega))d\omega = \pi \sum_k \Re(p_i)$$

where we have used the fact that $|\det(S)| = \prod_j \sigma_j(S)$. The interpretation of the sensitivity integral is that we must trade-off sensitivity reduction at one frequency by a similar sensitivity increase at another frequency, but a similar trade-off can be made between different directions of the sensitivity function. That is, reducing sensitivity in one direction can be compensated for by an increased sensitivity in another direction.

The main conclusion to draw in terms of choice of the weight W_S is that it can not be chosen large at all frequencies and in all directions. Note that a scalar weight W_S often is preferred, which implies that all directions are given the same weight and the trade-off must then be made over frequencies in a similar way as for SISO systems.

6.5 RHP Zeros and Poles

We here show that the limitations from RHP zeros and poles, as derived in Lecture 4, carries over to the MIMO case more or less directly when considering the maximum singular values of S and T.

Theorem: Assume $G(s)$ has a RHP zero at $s = z > 0$. Then

$$\|W_S S\|_\infty \geq |W_S(z)|$$

Proof: By definition $G(z)$ is rank deficient, i.e., there exist a vector y_z such that

$$y_z^H G(z) = 0 \;\Rightarrow\; y_z^H T(z) = 0$$

Here y_z is the output zero direction as defined above. Now, $S + T = I$ and hence

$$y_z^H S(z) = y_z^H \;\Rightarrow\; S^H(z) y_z = y_z$$

which implies that

$$\bar{\sigma}(S(z)) \geq 1$$

where we have used the fact that $\bar{\sigma}(S^H) = \bar{\sigma}(S)$. Applying the Maximum Modulus Thm (see Lec 4) to the weighted sensitivity yields

$$\|W_S S\|_\infty \geq \bar{\sigma}(W_S(z)S(z)) \geq |W_S(z)|$$

where have assumed a scalar weight $W_S(s)$. Thus, we have the same restriction on W_S as derived for SISO systems in Lecture 4, i.e., the weight must satisfy the constraint

$$|W_S(z)| < 1$$

Consider next limitations imposed by RHP poles

Theorem: Assume $G(s)$ as a RHP pole at $s = p$. Then

$$\|W_T T\|_\infty \geq |W_T(p)|$$

Proof: As above, but with

$$S(p) y_p = 0 \;\Rightarrow\; T(p) y_p = y_p$$

Again, this is the same restriction as derived for SISO systems in Lecture 4 and we require that the weight on the complementary sensitivity is chosen so that

$$|W_T(p)| < 1$$

6.6 Requirements for Disturbance Attenuation

Consider a scalar disturbance d such that the disturbance on the output is

$$w = g_d(s)d \;\Rightarrow\; z = S(s)g_d(s)d \;, \quad |d| < 1 \;\forall w$$

Note that we assume the problem has been scaled such that the expected magnitude of the disturbance is less than 1. Assume we also have scaled the output z such that the performance requirement is $|z| < 1 \;\forall\omega$. This then gives the requirement

$$\bar{\sigma}(Sg_d) < 1 \;\forall\omega \;\Rightarrow\; \|Sg_d\|_\infty < 1$$

Define the disturbance direction as

$$y_d(i\omega) = \frac{g_d(i\omega)}{|g_d(i\omega)|}$$

The requirement then becomes

$$\bar{\sigma}(Sy_d) < \frac{1}{|g_d|} \quad \forall\omega$$

Thus, the requirement on the sensitivity S is only in the direction y_d. Perform a singular value decomposition (SVD) of S at a given frequency and consider in particular the high-gain and low-gain directions of S

$$S\bar{v} = \bar{\sigma}(S)\bar{u} \;, \quad S\underline{v} = \underline{\sigma}(S)\underline{u}$$

Now, if the disturbance direction y_d is completely aligned with $\bar{v}$ we get the requirement

$$\bar{\sigma}(S) < \frac{1}{|g_d|}$$

However, if y_d is aligned with the weak direction $\underline{v}$ we get the requirement

$$\underline{\sigma}(S) < \frac{1}{|g_d|}$$

Thus, the requirement imposed by a disturbance on the sensitivity function is highly dependent on which direction the disturbance acts in.

6.7 Disturbances and RHP Zeros

From the above we have seen that RHP zeros in the open-loop plant $G(s)$ impose limitations on the achievable maximum singular value of the sensitivity function $\bar{\sigma}(S)$. However, we also noted that a given disturbance may not be aligned with the worst direction of the sensitivity function and that we therefore in principle may achieve acceptable disturbance attenuation even if $\bar{\sigma}(S) > 1$ at a given frequency. The directions of the sensitivity S depends on the specific controller used. We here show that it is possible to determine if a RHP zero makes it fundamentally infeasible to get acceptable attenuation of a given disturbance with any controller.

Assume $G(s)$ has a RHP zero at $s = z$, then as discussed above

$$y_z^H S(z) = y_z^H$$

From the Maximum Modulus Thm we then get

$$\|Sg_d\|_\infty \geq \|y_z^H S g_d\|_\infty \geq |y_z^H g_d(z)|$$

Thus, for acceptable disturbance attenuation to be possible we must require

$$|y_z^H g_d(z)| < 1$$

Otherwise, no controller exists that will provide $\|Sg_d\|_\infty < 1$, i.e., acceptable disturbance attenuation.

Note that the size of the inner product $y_z^H g_d(z)$ depends on how the two vectors y_z and $g_d(z)$ are aligned. Two extreme cases are

- $y_z \perp g_d(z) \;\Rightarrow\; y_z^H g_d(z) = 0$, i.e., no apparant limitation from the RHP zero.
- $y_z \parallel g_d(z) \;\Rightarrow\; |y_z^H g_d(z)| = |g_d(z)|$, i.e., can in principle not atteunate disturbances for frequencies above $\omega = z$.

Example: Consider the 2×2 system

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{1}{s+2} \\ \frac{s+3}{s+1} & \frac{2}{s+2} \end{pmatrix}$$

which has a zero at $s = 1$. To determine y_z, consider

$$G(1) = \begin{pmatrix} 1 & 1/3 \\ 2 & 2/3 \end{pmatrix} \quad \Rightarrow \quad y_z^H = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 & 1 \end{pmatrix}$$

There are two disturbances affecting the output z. The first disturbance d_1 has the transfer-function

$$z = g_{d1}(s) d_1 \;; \quad g_{d1} = \frac{2}{s+1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

From this we compute $|y_z^H g_{d1}(1)| = 1/\sqrt{5} < 1$, and hence the RHP zero does not prevent acceptable disturbance attenuation. For the second disturbance d_2 we have

$$z = g_{d2}(s) d_2 \;; \quad g_{d2} = \frac{2}{s+1} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

and we get $|y_z^H g_{d2}(1)| = 3/\sqrt{5} > 1$. Thus, it is not possible to achieve acceptable disturbance attenuation of this disturbance with any controller due to the existence of the RHP zero at $s = 1$.