EL2520 - Control Theory and Practice - Advanced Course
Solution/Answers – 2018-08-20

1a. (i) The minors of the system matrix G are

$$\frac{-s+1}{(s+1)^2}\ \frac{1}{s+1}\ \frac{3(-s+1)}{(s+1)^2}\ \frac{-s+1}{(s+1)^2}\ \frac{(-s+1)^2(3s+2)}{(s+1)^4}$$

The pole polynomial $(s+1)^4 = 0$ implies four poles in -1. The zero polynomial $(-s+1)^2(3s+2) = 0$ implies one zero in $s = -2/3$ and two zeros in $s = 1$. With 4 poles, at least four states are needed in a state-space realization

(ii) The steady-state RGA reveals negative relative gains on the diagonal, and hence we should avoid pairing on the diagonal. At $\omega = 1$ the absolute values of the relative gains for the off-diagonals are 1.18 which implies weak interactions around the bandwidth. The off-diagonal pairing is hence the preferred choice.

1b. We have

$$\begin{aligned}\dot{y} &= 5y(t) + u(t)u \\ y &= x\end{aligned}$$

We can determine a LQ controller by solving the Riccati equation

$$A^T S + SA + M^T Q_1 M - SBQ_2^{-1}B^T S = 0\,,$$

where $A = 5$, $B = 1$, $M = 1$, $Q_1 = 1$, $Q_2 = K$. We get

$$10S + 1 - \frac{S^2}{K} = 0 \Rightarrow S = 5K + \sqrt{25K^2 + K}$$

The controller is a state feedback $u = -Ly$ given by

$$L = Q_2^{-1}B^T S = 5 + sqrt25 + (1/K)$$

The closed-loop pole is $5 - L$, and as $K \to 0$ the pole approaches $-\infty$ while for $K \to \infty$ we the pole approaches -5.

2a. For any control system, we have that $\bar{\sigma}(S) + \bar{\sigma}(T) \geq 1\ \forall\omega$. In the figure, this is not satisfied for some frequencies, e.g., at $\omega = 0.3$ we have both singular values below 0.3 and hence the sum os less than one which is not possible.

2b. Since $G(s)$ has a transmission zero at $s = 3$ in the RHP, the closed-loop transfer-function must have the same zero to ensure internal stability.

Since the given closed-loop is minimum-phase with no transmission zeros, the controller must cancel the RHP pole in G and hence the system will become internally unstable. Thus, this is not an appropriate choice.

We can modify the desired G_c, such that the RHP zero of G is not canceled, i.e.,

$$G_c = \frac{(-s+3)}{(s+3)(\tau s+1)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

This system has the same singular values as the given closed-loop transfer-function since $(-s+3)/(s+3)$ is all pass, i.e., has magnitude 1 at all frequencies.

We derive the corresponding controller F

$$F = -G^{-1}(G_c - I)^{-1} G_c = -\frac{10s+1}{\tau s^2 + (3\tau+2)s} \begin{pmatrix} 2 & -1 \\ -(s+1) & 2 \end{pmatrix}$$

The system is internally stable for all $\tau > 0$.

3a. We have that $|G_d| > 1$ for $\omega < 0.39$, and hence we need disturbance attenuation up to this frequency. Since $|G| > |G_d|$ for all frequencies, we can keep $|y| < 1$ without violating the input constraint $|u| < 1$ for all frequencies. The system has a RHP zero at $s = 0.25$ which limits the bandwidth of the sensitivity to 0.25, or 0.125 if we only allow a maximum peak in $|S|$ of 2. Thus, it is not possible to achieve the required bandwidth of 0.39. To determine the maximum disturbance we can attenuate, we determine $|G_d(i0.25)| = 1.49$ and since $|S(i0.25)| \geq 1$ we require $|d| < 1/1.49 = 0.67$ to enable acceptable attenutation. A more conservative estimate is obtained if we consider the maximum bandwidth 0.125 for which we get $|d| < 1/|G_d(i0.125)| = 0.4$.

3b. *i)* The transfer function between disturbance d and output y is the sensitivity $S(i\omega)$. The transfer function between measurement noise n and output y is the complementary sensitivity $T(i\omega)$. The requirements can be formulated as

$$\begin{aligned} |S(i\omega)| &< 0.1 \qquad \omega < 0.5\,\text{rad/s} \\ |S(0)| &< 0.01 \\ |T(i\omega)| &< 0.2 \qquad \omega > 2\,\text{rad/s} \end{aligned}$$

ii) In terms of loop gain $L = GF_y$, the sensitivity and complementary sensitivity can be approximated as

$$\begin{aligned} |S(i\omega)| \approx |L^{-1}(i\omega)| \qquad &\Rightarrow |L(i\omega)| > 10 \qquad \omega < 0.5\,\text{rad/s} \\ |L(0) > 1| > 100 & \\ |T(i\omega)| \approx |L(i\omega)| \qquad &\Rightarrow |L(i\omega)| < 0.2 \qquad \omega > 2\,\text{rad/s} \end{aligned}$$

iii) A sketch of the loop gain L is not included in this solution. However L has a lower bound of 10 for $\omega < 0.5$ and upper bound of 0.2 for $\omega > 2$, therefore L is required to have a negative slope of around -2 for frequencies $0.5 < \omega < 2$ in which the crossover will be. This limits the achievable phase margin of the system and may even not be feasible due to the stability requirement.

iv) The requirement can be translated in $|S(i\omega)| < |W_S^{-1}(i\omega)|\,\forall\omega$. We consider a weight

$$W_S(s) = \frac{s + M\omega_B}{M(s + A)}$$

and we require $|W_S(0) > 100|$, $\omega_B > 0.5$ and choose $M = 2$. This gives $A < 0.005$.

4a. With the prediction horizon $N_p = 1$, the objective function becomes

$$V = Q_y y_{k+1}^2 + Q_y y_k^2 + u_k^2$$

With the state-space model inserted

$$\begin{aligned} V &= Q_y(-y_k + 4u_k)^2 + Q_y y_k^2 + u_k^2 = \\ &= Q_y y_k^2 + Q_y y_k^2 + (4Q_y + 1)u_k^2 - 8Q_y y_k u_k \end{aligned}$$

As the value of y_k is assumed given, the first two terms of the objective function will not affect the optimal input sequence and can be removed. Thus, $H = (4Q_y + 1)$, and $h = -8Q_y y_k$.
The constraint gives

$$Lu_k \leq b$$

where $L = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

4b. The gain of the saturation is given by

$$||\text{sat}(\cdot)||_\infty = \sup_{\text{u}\neq 0} \frac{||\text{sat(u)}||_2}{||\text{u}||_2} = 1\,.$$

A sufficient condition for stability of the closed loop can be derived with the small gain theorem $||\text{sat}(\cdot)||_\infty ||\text{M}||_\infty \leq 1$, where M is the transfer function from u_p to u

$$\begin{aligned} u &= W_s(u_p - u) - F_y y = W_s(u_p - u) - F_y G u_p \\ u &= \frac{W_s - F_y G}{1 + W_s} u_p = M u_p \end{aligned}$$

After substitutions, we get

$$M(s) = \frac{(1 - 10K)s + 1}{(s+1)(10s+1)}$$

We need then to guarantee $||M||_\infty = \sup_\omega |M(i\omega)| \leq 1$. Since $|M(0)| = 1$, the condition is verified if the frequency breakpoint of the zero is higher than the frequency breakpoint of the first pole, i.e.,

$$\frac{1}{|1 - 10K|} \geq \frac{1}{10}$$

which gives $-0.9 \leq K \leq 1.1$.

5a. By writing the system on $M-\Delta$ form we determine

$$M = F_y(I + GF_y)^{-1}$$

Thus, from the SGT the criteria for RS becomes $F_y(I + GF_y)^{-1}$ stable and

$$\|F_y(I + GF_y)^{-1}\Delta_a\|_\infty < 1$$

The latter is satisfied if

$$\|Fy(1 + GF_y)^{-1}\|_\infty < 1/\delta$$

(b) The SGT is in general conservative since it does utilize phase information. However, the Δ block can have arbitrary phase and therefore the SGT is not conservative if Δ describes the uncertainty precisely.

(c) By writing the system on $M-\Delta$ form we find

$$M = (I + F_yG)^{-1} = S_I$$

which is the sensitivity function on the input side. The robustness criterion is thus internal stability and

$$\|S_I\Delta_I\|_\infty < 1$$

(d) With $p = 0$ we get $G(s) = 1/s$ and

$$G_0(s) = G(s)(1 + \Delta_I(s))^{-1} \Rightarrow \frac{1}{s-p} = \frac{1}{s(1+\Delta_I)} \Rightarrow \Delta_I = -\frac{p}{s}$$

The sensitivity function for the the case with $p = 0$ becomes

$$S_I = S = \frac{s}{s+K}$$

The criterion for stability is then

$$\left\| \frac{s}{s+K}\frac{p}{s} \right\|_\infty < 1 \quad \forall p$$

which gives

$$K > p \; \forall p \Rightarrow K > 1$$

The closed loop system is

$$G_c = \frac{K}{s-p+K}$$

and we see that we must require $K > 1$ for stability for all $p \in [-1, 1]$. The robustness criterion thus give the exact bounds in this case, that is, not conservativeness despite the use of the SGT.