EE – Automatic Control
March 7, 2019

EL2520

CONTROL THEORY AND PRACTICE ADVANCED COURSE

Year 2019

EXERCISES

Exercises

March 7, 2019

Introduction

This compendium contains the exercises which will be solved during the exercise sessions, as well as a number of recommended extra exercises. The exercises are partially taken from the exercise compendium that was used in the course until 2013. That compendium collected a number of exercises from KTH, Uppsala University, Linköpings Tekniska Högskola and Chalmers, was mostly written in Swedish, and had a substantially broader scope than this document. We hope and believe that this updated and more focused exercise compendium is better aligned with the learning aims of the current course offering at KTH and that it will improve the overall learning experience. Nevertheless, the joint intellectual contributions of professors and students from KTH, Uppsala, Linköping and Chalmers to this exercise material is greatly acknowledged.

1 Repetition: Loop Shaping

1.1 Open-loop shaping from closed-loop specifications

Consider the system

$$G(s) = \frac{100}{(s+1)(s+2)(s+10)}$$

and assume that we want to achieve certain properties of the closed-loop system with respect to disturbance attenuation. Therefore, we want S to have a maximal peak of $M_S = 2$ and a closed-loop bandwidth of $\omega_{BS} = 5$. Furthermore, we want to have a static error $e_\infty = 0$.

(a) Translate the requirements given in terms of the closed-loop system into requirements in terms of the open-loop system.

(b) Design a control law that achieves the above objectives. Verify that the closed-loop system satisfies the requirements.

2 Signal norms

2.1 Signal Norms

Determine the signal norms $\|y\|_\infty$ and $\|y\|_2$ of the following continuous-time signals:

(a)

$$y(t) = \begin{cases} a\sin(t), & t > 0 \\ 0, & t \leq 0 \end{cases}$$

(b)

$$y(t) = \begin{cases} \frac{1}{t}, & t > 1 \\ 0, & t \leq 1 \end{cases}$$

(c)

$$y(t) = \begin{cases} e^{-t}(1-e^{-t}), & t > 0 \\ 0, & t \leq 0 \end{cases}$$

(d)

$$y(t) = \begin{cases} \begin{bmatrix} e^{-t} \\ ae^{-t} \end{bmatrix} & t > 0 \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & t \leq 0 \end{cases}$$

2.2 Signal Norm Comparison

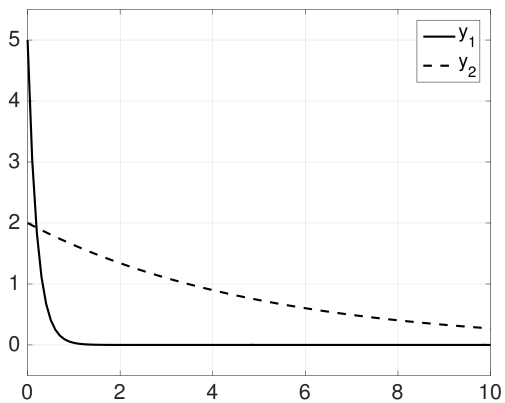
Consider the two signals

$$y_1(t) = \begin{cases} 5e^{-5t}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

$$y_2(t) = \begin{cases} 2e^{-0.2t}, & t > 0 \\ 0, & t \leq 0 \end{cases},$$

which are shown in the figure below.

Signals y_1 and y_2



(a) Which of these two signals has the largest L_∞-norm and which one has the largest L_2-norm? Answer without calculations.

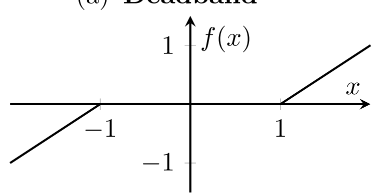
(b) Calculate the norms $\|y_1\|_\infty$, $\|y_2\|_\infty$, $\|y_1\|_2$, and $\|y_2\|_2$ to verify the discussion in (a).

3 Gains of static nonlinearities

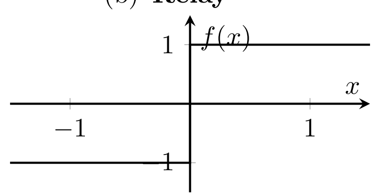
3.1 Static gain of nonlinearities

Determine the gain of the following static nonlinearities.

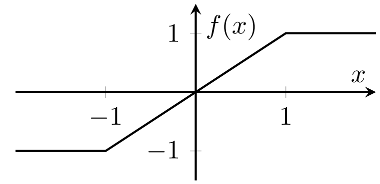
(a) **Deadband**



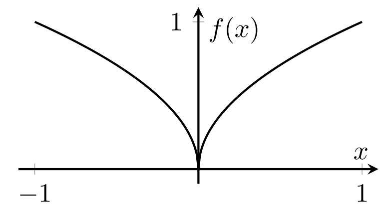
(b) **Relay**



(c) **Saturation**



(d) **Cusp** $\sqrt{|x|}$



4 Gains of scalar linear systems

4.1 Gain of first-order linear systems

Compute the gain $\|G\|$ of the first-order linear system

$$G(s) = \frac{K}{1+sT}$$

for all parameter values K and T.

4.2 Gain of second-order systems

Consider the linear system

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}.$$

Compute the gain $\|G\|$ of the system for all parameter values $\omega_0 > 0$ and $\zeta > 0$.

4.3 Closed-loop disturbance gain

The system

$$y = \frac{1}{2s+1}(u + d_u)$$

where u is the control input and d_u is a disturbance on the input, is controlled by the PI-controller

$$u = -10\frac{2s+1}{2s}y$$

(a) Draw a block-diagram for the closed-loop system and derive the closed-loop transfer-functions from the disturbance d_u to the control u and output y, respectively.

(b) What is the maximum amplification of a disturbance d_u in the output y? What is the corresponding worst-case disturbance $d_u(t)$?

5 Gains of multivariable linear systems

5.1 Gain of static multivariable systems

Consider a static linear system on the form

$$y = Mu$$

For the particular instances of M listed below, compute the gain of the system and determine the input directions that yield the largest and smallest amplification of the input, respectively.

a)

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

b)

$$M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

c)

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

5.2 Gain of a multivariable system

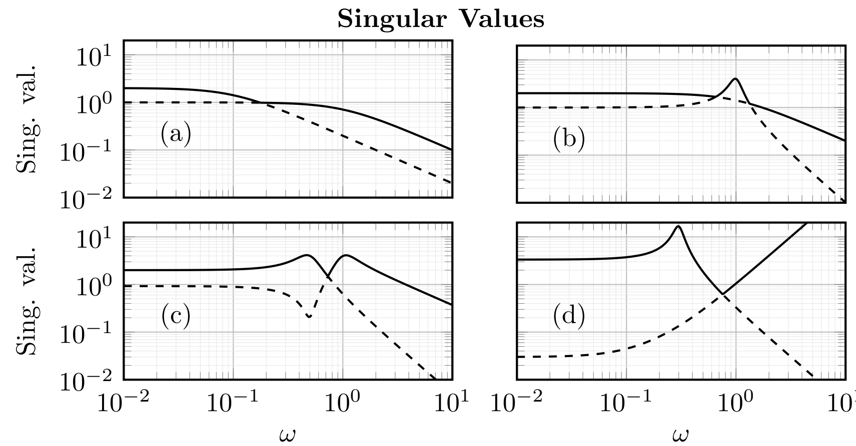
Consider the multivariable system with transfer matrix

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{-1}{s+1} \\ \frac{-1}{s+1} & \frac{1}{s+1} \end{bmatrix}.$$

Determine the gain of the system, and find the worst-case input direction.

5.3 Gain of linear systems from singular value plots

Determine the gain and the critical frequency of an input that is maximally amplified for four linear systems whose singular value plots are given below.



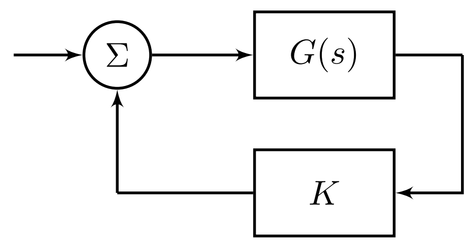
6 The small-gain theorem

6.1 Conservatism of the small gain theorem

Analyze the stability of the linear system depicted in the block diagram below where

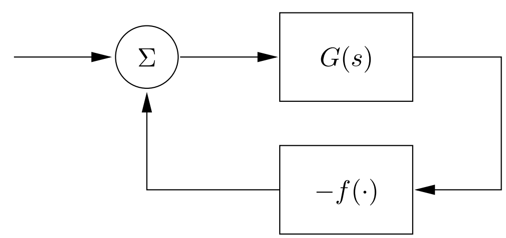
$$G(s) = \frac{a}{s+a}.$$

First use the small gain theorem and then compute the poles of the closed-loop system. Comment on potential limitations and conservatism of the small gain theorem!

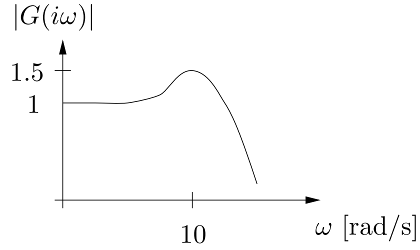


6.2 Small gain analysis of nonlinear control loops

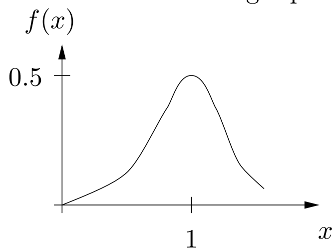
Consider the following feedback interconnection of two systems



where $G(s)$ is a stable linear system whose frequency response magnitude is



and $f(\cdot)$ is an static amplifier with the following input-output characteristic



Is the closed-loop system stable?

6.3 Small-gain analysis of saturated feedback loops

Consider the linear system

$$G(s) = \frac{2}{s^2+2s+2}$$

. The control input passes through a saturated valve:

$$\tilde{u} = \begin{cases} 1 & \text{if } u > 2 \\ \frac{1}{2}u & \text{if } |u| < 2 \\ -1 & \text{if } u < -2 \end{cases}$$

The output is thus given by

$$y(t) = G(p)\tilde{u}(t).$$

The control input is determined by a proportional controller $u(t) = -Ky(t)$. For which values of K does the small gain theorem guarantee stability of the closed-loop system?

7 Internal stability

7.1 Canceling a RHP-zero and internal stability

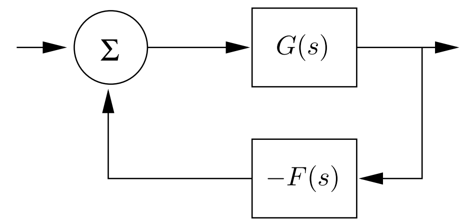
A system

$$G(s) = \frac{s-1}{s+1}$$

is controlled by the feedback controller

$$F(s) = \frac{s+2}{s-1}$$

as depicted in the block diagram below.



Compute the closed loop transfer function G_c, the sensitivity function S and the complementary sensitivity function T. Are they stable? Is the system internally stable?

8 Robustness and limitations

8.1 Robustness and the Nyquist criterion

Assume that we have designed the controller F for the scalar system G such that the loop gain $L = GF$ and the corresponding closed loop system are stable. Due to a modeling error, the real loop gain turns out to be L_p. Define the relative model error as

$$L_p = (1 + \Delta_L)L$$

and assume that Δ_L is stable. Use the Nyquist criterion to derive a condition on Δ_L guaranteeing stability for the true system.

8.2 Limitations due to RHP-zeros

We are given the system

$$G(s) = \frac{s-3}{s+1}$$

and we want the complementary sensitivity function to be

$$T(s) = \frac{5}{s+5}.$$

a) Design a feedback controller $F_r = F_y = F$ which yield T as above. Will the controller work?

b) Suggest an alternative T, which has the same bandwidth 5 rad/s, but yield a stable closed loop with $F_r = F_y = F$.

c) What is the corresponding sensitivity function?

d) Could a two-degree of freedom controller with different F_r and F_y be a good alternative? More specifically, could we choose F_r and F_y such that none of them have a pole in s=3, and such that the closed loop transfer function become

$$G_c = \frac{GF_r}{1 + GF_y} = \frac{5}{s+5}?$$

8.3 Limitations due to RHP-zeros and time delays

A time-continuous system has a zero at $s = 3$ and a time-delay of 1.0 seconds. What is the highest realistic crossover frequency if the open loop systems amplitude curve is monotonically decreasing.

8.4 Time-domain limitations

The engineer Civerth was given the task to design a SISO control system with the specifications:

a) The stationary control error, when the reference signal is a ramp, should be zero (that is, the error cofficient $e_1 = 0$).

b) The step response for the closed loop system should not have an overshoot.

Did he succeed?
Investigate by treating the subproblems a) - c) below.

a) Rewrite the requirement $e_1 = 0$ as a condition on the sensitivity function $S(s)$.

b) Assume that the condition in part a) holds. Show that, for a step in the reference signal

$$\int_0^\infty e(t)dt = \lim_{s\to 0} E(s) = 0$$

c) What does the relation in b) tell about the possible overshoot in the step response?

8.5 RHP zeros and Disturbance Attenuation

Consider the system

$$y = \frac{s-1}{(s+1)^2}u + 0.5\frac{s-10}{(s+1)^2}d$$

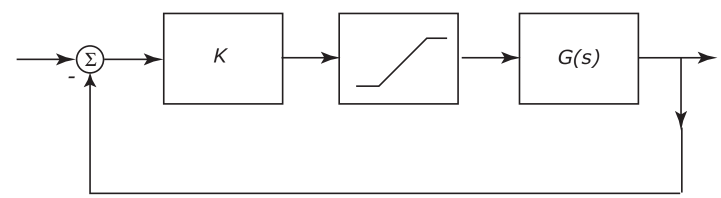
where y is the output, u the control input and d a disturbance. Assume we want to use feedback to keep $|y| < 0.5$ for all disturbances of magnitude $|d| < 1$. Is this feasible? You can consider sinusoidal disturbances frequency by frequency, recalling from the H_∞ norm of linear stable systems that the worst case amplification is obtained for a sinusoidal input.

8.6 Modeling for small-gain analysis

The small-gain theorem is an important tool for robustness analysis, but it is also useful for nonlinear systems analysis. In order to avoid unnecessarily conservative results it is important how we go from an exact description of the nonlinearity to the gain description. As an example of this, consider the system

$$G(s) = \frac{1}{s+\varepsilon}$$

where $\varepsilon > 0$ is a small number. We wish to speed up the system using proportional control with gain K. The control signal is limited to $[-1, 1]$, and the closed-loop system is described by the system below.

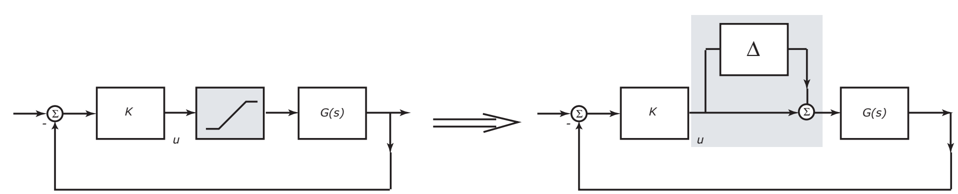


a) First, we consider the saturation as an uncertainty Δ. Show that the saturation

$$\mathrm{sat}(x) = \begin{cases} 1 & \text{if } x \geq 1 \\ x & \text{if } -1 \leq x \leq 1 \\ -1 & \text{if } x \leq -1 \end{cases}$$

has gain 1, and use the small gain theorem to derive a condition on K that guarantees stability of the closed-loop system. What is the largest value of K for which your criterion guarantees stability?

b) An alternative approach is to represent the saturation as an additive uncertainty, see the figure below. Determine an expression for how Δ depends on u and show that its gain is equal to one. Use the small gain theorem to derive a stability criterion on the closed-loop system. What values of the control gain K guarantee stability using this approach?



8.7 Small-gain analysis of time-delay systems

There is a strong current interest in wireless automation, where wireless sensors replace traditional solutions to reduce cabling cost and increase flexibility. However, due to harsh radio conditions sensor messages sent over the wireless interface may be corrupted or lost. One way to deal with this is to let the system resend the data until it succeeds. This introduces an uncertain (communication) delay in the system between the plant and controller. In this problem, we will investigate the effects of such delays using the tools from the course.

a) Consider a linear system $G_0(s)$ where the output is sensed using a wireless sensor discussed above. We model the sensor as an uncertain delay, include this delay in the system model, and consider

$$G_p(s) = G_0(s)e^{-Ls}$$

where $L \in [0, L_{\max}]$. Show that the closed-loop control system under output feedback $u = -F_y y$ can be represented as in Figure 1 (left), and verify that the associated transfer function Δ satisfies $\|\Delta\|_\infty \leq 2$

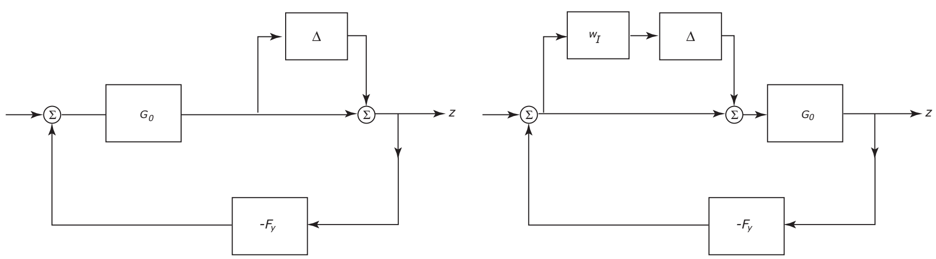


Figure 1: Uncertainty model for problem 8.7a.(left) and 8.7c.(right)

b) Use the small-gain theorem to derive a criterion for robust stability of the closed-loop system for the uncertainty model in a, exploiting only the norm bound on the Δ block.

c) An alternative approach that has been discussed in class, is to consider the multiplicative input uncertainty description shown in Figure 1(right). Robust stability is then guaranteed if

$$|T(i\omega)| \cdot |w_I(i\omega)| \leq 1 \qquad \forall \omega$$

where w_I bounds the relative error between the nominal model and all possible perturbed plants.

$$|w_I(i\omega)| \geq \left| \frac{G_p - G_0}{G_0} \right| \qquad \forall G_p \tag{1}$$

Derive an expression for how the relative error between G_p and G_0 depends on frequency, and show that this condition is less conservative than the condition derived in b) for all frequencies $\omega \leq L_{\max}^{-1}$. (4p)

d) When $L_{\max} = 1$, the weight

$$w_I(s) = 2\frac{s}{s+1}$$

satisfies the bound (1). Figure 2 shows the complementary sensitivity for two different controllers. Which of these guarantee robust stability of the closed-loop system?

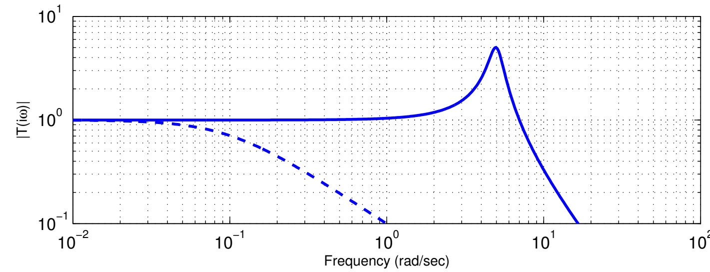


Figure 2: Complementary sensitivities for the two control designs in 5d).

8.8 SISO Controllability Analysis

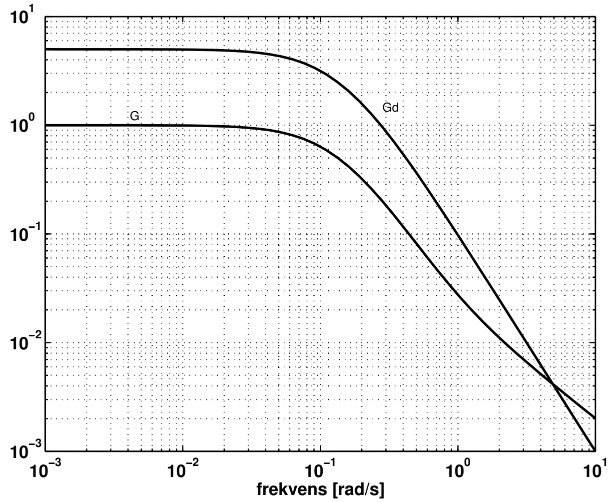
Given the system

$$\hat{y} = \hat{G}\hat{u} + \hat{G}_d\hat{d}$$

with

$$\hat{G}(s) = \frac{-s+1}{(5s+1)(10s+1)}; \quad \hat{G}_d(s) = \frac{5e^{-s}}{(5s+1)(10s+1)}$$

and $\hat{u} \in [-10, 10]$, $\hat{d} \in [-3, 3]$. Assume that the normal equilibrium point is $\hat{u} = 0$, $\hat{d} = 0$, $\hat{y} = 0$ and that the goal of the controller is to achieve $\hat{y} \in [-0.5, 0.5]$. The amplitude curves $\hat{G}$ and $\hat{G}_d$ are shown below.



(a) Determine the scaled system

$$y = Gu + G_d d$$

such that u and d are both $\in [-1, 1]$ and so that an acceptable control error lies between $y \in [-1, 1]$.

(b) Investigate if an acceptable control law can in theory be derived through using feedback control. If this is not the case, suggest an appropriate way to make the system controllable.

9 MIMO basics

9.1 Deriving a transfer matrix from a physical model

A simplified description of a alternating current generator is as follows: The rotor winding is fed with the magnetizing current I_m and the rotor is driven by the mechanical momentum M. In the stator winding an alternating voltage with peak value e is then generated. The generator load is purely resistive with resistance R. The angular velocity of the generator is ω. The system can then be described by the simplified equations:

$$\begin{aligned} e &= R \cdot I_f \ (I_f = \text{peak value of current}) \\ J\dot{\omega} &= M - M_e \ (\text{forcing moment - electromotive force}) \\ M_e &= K_e \cdot \omega \cdot I_f \\ e &= C_e \cdot I_m \cdot \omega \end{aligned}$$

where J, K_e and C_e are constants. We consider e and ω as outputs, M and I_m as inputs and R as a disturbance. Find a state space description of the generator and linearize it about the working point

$$\omega_0 = R_0 = I_{m0} = M = 1$$

and let

$$K_e = C_e = 1, \ J = 1.$$

Finally derive the transfer matrix from

$$u = \begin{bmatrix} \Delta M \\ \Delta I_m \\ \Delta R \end{bmatrix} \text{ to } y = \begin{bmatrix} \Delta\omega \\ \Delta e \end{bmatrix}.$$

Where $\Delta\cdot$ are deviation variables from the working point.

9.2 Going from transfer matrix to state space

Find a state space description of the system

$$G(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s^2+s+1)} \end{bmatrix}.$$

9.3 Stability and stabilizability

Consider the discrete-time system

$$x(t+1) = \begin{bmatrix} -1 & 1.5 \\ -3 & 3.5 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0.5 \\ -2 & 0.5 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

a) Is the system stable?

b) Can the system be stabilized using state feedback?

c) Can the system be stabilized using only the input u_1?

9.4 Poles and order of minimal realization

Determine the poles (with multiplicity) of the system

$$G(s) = \frac{1}{s+1} \begin{bmatrix} 1-s & \frac{1}{3} - s \\ 2-s & 1-s \end{bmatrix}$$

What is the order of a minimal state space realization?

9.5 Poles and order of minimal realization

Determine the poles and zeros (with multiplicity) of the system

$$G(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{1}{s+2} \\ \frac{-s+3}{s+1} & \frac{2}{s+2} \end{bmatrix}.$$

Explain why we can have poles and zeros at the same location without canceling.

9.6 Limitations in MIMO systems

Consider the system

$$y = \frac{10}{s+1}\begin{pmatrix} 2 & 2s \\ 1 & 1 \end{pmatrix} u + \frac{3}{s+1}\begin{pmatrix} 1 \\ 2 \end{pmatrix} d_1 + \frac{2}{s+1}\begin{pmatrix} 4 \\ 2 \end{pmatrix} d_2$$

Which of the two disturbances d_1 and d_2 do you expect to be most difficult to attenuate? Motivate!

9.7 Internal Stability in MIMO systems

An open-loop unstable system is controlled by a feedback controller that yields the closed-loop sensitivity function

$$S(s) = \frac{1}{s+1}\begin{pmatrix} s+0.1 & -0.1 \\ -0.1 & s+0.1 \end{pmatrix}$$

Is the closed-loop system stable? Motivate!

10 Decentralized and decoupled control

10.1 Computing the RGA

Consider the systems

(i) $G(s) = \begin{bmatrix} \frac{1}{10s+1} & \frac{2}{2s+1} \\ \frac{1}{10s+1} & \frac{s-1}{2s+1} \end{bmatrix}$

(ii) $G(s) = \begin{bmatrix} \frac{1}{10s+1} & \frac{-2}{2s+1} \\ \frac{1}{10s+1} & \frac{s-1}{2s+1} \end{bmatrix}$

(iii) $G(s) = \frac{1}{0.1s+1}\begin{bmatrix} \frac{0.6}{s+1} & -0.4 \\ 0.3 & 0.6 \end{bmatrix}$

(iv) $G(s) = \frac{1}{s/20+1}\begin{bmatrix} \frac{9}{s+1} & 2 \\ 6 & 4 \end{bmatrix}$

(v) $G(s) = \begin{bmatrix} \frac{1}{s+2} & \frac{10}{s+2} \\ \frac{1}{s+5} & \frac{5}{s+3} \end{bmatrix}$

(vi) $G(s) = \begin{bmatrix} \frac{4}{s+4} & \frac{16}{s+8} \\ \frac{10}{s+8} & \frac{4}{s+4} \end{bmatrix}$

(a) Evaluate $\mathrm{RGA}(G(0))$ for the systems above.

(b) Evaluate $\mathrm{RGA}(G(i\omega_c))$ for the systems above with ω_c given for the respective system as

(i) $\omega_c = 0.1$

(ii) $\omega_c = 0.1$

(iii) $\omega_c = 10$

(iv) $\omega_c = 20$

(v)

(vi) $\omega_c = 3$

10.2 RGA properties

In this exercise we will prove some useful properties of the RGA matrix.

(a) Prove that the sum of any row or column of an RGA matrix equals 1 for all invertible 2×2 transfer matrices.

(b) Prove that for any 2×2 invertible transfer matrix

$$G(s) = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix}$$

the RGA matrix is given by

$$\mathrm{RGA}(G(s)) = \begin{bmatrix} c(s) & 1-c(s) \\ 1-c(s) & c(s) \end{bmatrix}, \quad c(s) = \frac{g_{11}(s)g_{22}(s)}{g_{11}(s)g_{22}(s) - g_{12}(s)g_{21}(s)}$$

(c) Prove that the sum of any row or column of an RGA matrix equals 1 for all invertible $n \times n$ transfer matrices.

Note that it is usually easy to check wheter or not the rows & columns sum to 1. This is an excellent way to check your $\mathrm{RGA}(G(s))$ calculations for errors.

10.3 Decentralized Control of Diagonal Plant

Consider the control system where the plant

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$$

is controlled by a decentralized diagonal controller

$$F = \begin{bmatrix} F_1(s) & 0 \\ 0 & F_2(s) \end{bmatrix}.$$

(a) Draw a block diagram explicitly illustrating the internal connections in the system. That is, the block diagram should contain a block for each nonzero element in $G(s)$ and $F(s)$.

(b) Assume $G_{12}(s) = G_{21}(s) = 0$ so that the system becomes diagonal. How is such a MIMO system related to SISO systems?

10.4 Decoupled robustness - modeling issues

Consider a process with the nominal model

$$G(s) = \frac{1}{s+1}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Assume that the gain of the second input is uncertain such that the true process is given by

$$G_p(s) = \frac{1}{s+1}\begin{bmatrix} 2 & \alpha \\ 0 & 2\alpha \end{bmatrix}$$

where α represent the uncertain gain.

a) Use the nominal model and design a decoupling controller

$$F(s) = \frac{k}{s}G^{-1}(s).$$

Chose k such that the closed loop system is as fast as the unregulated process.

b) Model the uncertainty as a relative error on the output. Use a suitable robustness criterion to determine for which α stability is guaranteed.

c) Model the uncertainty as a relative error on the input. Use a suitable robustness criterion to determine for which α stability is guaranteed. Is there a difference between the modeling approaches?

d) Compute the poles of the true closed loop system and determine for which values of α the system stable.

10.5 Decentralized & Decoupled Control

A multivariable system consists of two coupled parallel tanks, and has the dynamics

$$\dot{x} = \begin{bmatrix} -1-\alpha & \alpha \\ \alpha & -1-\alpha \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$$

The coefficient α depends on the size of an opening between the two tanks. Set the parameter $\alpha = 0.5$. Assume the aim is to design a control system.

a) Assume that decentralized control is to be used. It may be possible to let u_1 be connected to y_1 and u_2 connected to y_2. A second alternative is to instead make the connections $u_1 - y_2$ and $u_2 - y_1$. Use RGA for the static situation to determine which of the two alternatives that is preferable.

b) Determine a decoupling matrix $W_1(s)$ such that the system $\tilde{G}(s) = G(s)W_1(s)$ is decoupled for $\omega = 0$.

c) Assume that the system is to be controlled by a diagonal proportional regulator

$$U(s) = W_1(s)\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}(R(s) - Y(s))$$

where the regulator gains are set to $K_1 = K_2 = 10$. Determine the poles of the closed loop system, for two cases:

(i) The regulator is applied without decoupling

(ii) The regulator is applied with the decoupling of part b).

10.6 Decentralized and Decoupled Control of a Three Tank Process

A process consisting of three tanks as in the figure is described by the system

$$Y(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{0.5}{(s+1)^2} \\ 0 & \frac{0.5}{s+1} \end{bmatrix} U(s)$$

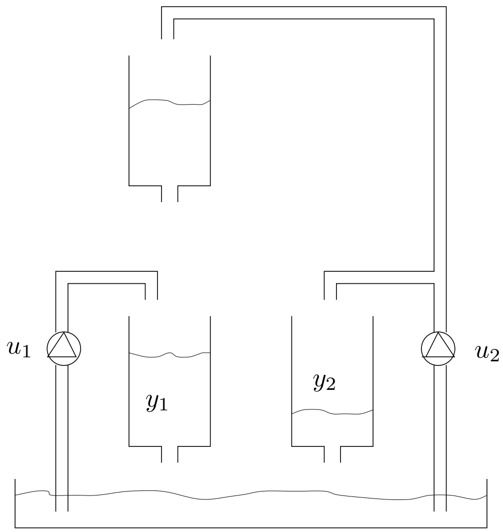
a) Determine the systems poles and zeros with multiplicity.

b) Determine the RGA for the frequency $\omega = 0$ and $\omega_c = 1$rad/s. Then suggest a suitable pairing for decentralized control.

c) Assume PI-control is to be used. Compute the controller parameters for the two regulators

$$K_i\left(1 + \frac{1}{sT_i}\right), \quad i = 1, 2$$

such that the crossover frequency becomes $\omega_c = 1$ rad/s and the phase margin is 90° for the SISO loops corresponding to the pairing determined in b).



d) Determine a dynamic decoupling

$$U(s) = \begin{bmatrix} 1 & W_{12}(s) \\ W_{21}(s) & 1 \end{bmatrix} \tilde{U}(s)$$

and draw a block diagram only containing single variable transfer functions to illustrate the structure of this dynamic decoupling.

11 $\mathcal{H}_\infty$-control and $\mathcal{H}_2$-control

11.1 Transforming problem to H_∞ form

Consider the system

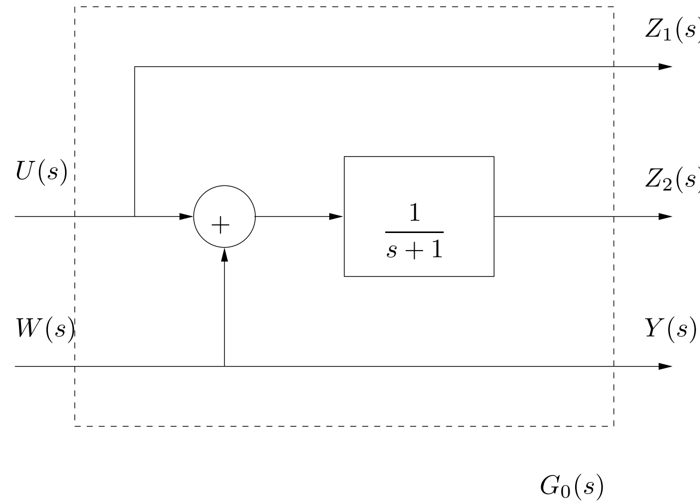
$$Y(s) = \frac{1}{s+1} U(s)$$

We want to determine a controller such that the closed loop fulfills the following conditions on S, T and G_{wu}:

$$|S(i\omega)| < \gamma\omega$$
$$|T(i\omega)| < 2\gamma$$
$$|G_{wu}(i\omega)| < 0.2\gamma$$

Find the equations and conditions that determine such a controller.

11.2 $\mathcal{H}_\infty$-optimal feed forward control



Consider the extended system $G_0(s)$ given in the block diagram.

a) Show that a state space description of G_0 is given by

$$\dot{x}(t) = -x(t) + u(t) + w(t)$$
$$z(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = w(t)$$

b) An observer-based controller is given by

$$\dot{\hat{x}}(t) = -\hat{x}(t) + u(t) + y(t)$$
$$u(t) = -L\hat{x}(t)$$

Determine L such that $\|G_{ec}\|_\infty \leq 1$ and such that the closed loop is internally stable where $G_{ec}(s)$ is the closed loop transfer function from $W(s)$ to $Z(s)$.

c) Consider the same observer-based controller as above. Determine the L which minimizes $\|G_{ec}\|_\infty$ and yields an internally stable system. Derive the transfer function of the controller from $Y(s)$ to $U(s)$ for the optimal L.

d) Let $U(s) = -KY(s)$, that is, consider a proportional controller. Determine the K that minimizes $\|G_{ec}\|_\infty$, where $G_{ec}(s)$ is the closed-loop transfer function from $W(s)$ to $Z(s)$ under the proportional control. What is the minimum of $\|G_{ec}\|_\infty$?

11.3 Effect of weights on $\mathcal{H}_\infty$-control

Assume that the system is given by

$$Z(s) = G(s)U(s) + W(s)$$
$$Y(s) = Z(s) + N(s)$$

We would like to control z using an $\mathcal{H}_\infty$-optimal controller, i.e., a feedback law $U(s) = -F_y(s)Y(s)$ minimizing $\|G_{ec}\|_\infty$, where

$$G_{ec}(s) = \begin{bmatrix} W_U(s)G_{wu}(s) \\ -W_T(s)T(s) \\ W_S(s)S(s) \end{bmatrix}$$

Here, $G_{wu}(s)$ is the transfer function from w to u, T is the complementary sensitivity function and S is the sensitivity function. The following weights have been proposed:

$$W_S(s) = \frac{s+\omega_S}{s}, \quad W_T(s) = \frac{s+\omega_T}{\omega_T}\frac{1}{1+\epsilon s} \quad \text{and} \quad W_U(s) = 1,$$

where $\epsilon \ll 1/\omega_T$.

a) If we increase ω_S, should we expect a better or worse attenuation of low frequent process disturbances?

b) If we increase ω_T, should we expect an increase or a decrease of the closed loop bandwidth?

11.4 Robustness and $\mathcal{H}_\infty$-control

Assume that we have the system

$$Z(s) = G(s)U(s) + W(s)$$
$$Y(s) = Z(s) + N(s)$$

We will control z using an $\mathcal{H}_\infty$-optimal controller, i.e., a feedback law $u(t) = -F_y(p)y(t)$ minimizing $\|G_{ec}\|_\infty$,where

$$G_{ec}(s) = \begin{bmatrix} W_U(s)G_{wu}(s) \\ -W_T(s)T(s) \\ W_S(s)S(s) \end{bmatrix}$$

Here, $G_{wu}(s)$ is the transfer function from w to u, T is the complementary sensitivity and S is the sensitivity.

a) The following weight for T is proposed

$$W_T(s) = \frac{s+\omega_T}{\omega_T}\frac{1}{1+\epsilon s},$$

where $\epsilon \ll 1/\omega_T$. How should we expect the robustness of the closed loop to be affected if we increase ω_T?

b) Let the weight on S be W_S. One of the weight functions below results in integral action in the controller. Which one?

$$W_S(s) = 1, \quad W_S(s) = \frac{s+\omega_S}{s}, \qquad W_S(s) = \frac{s}{s+\omega_S}$$

11.5 $\mathcal{H}_\infty$ and robustness

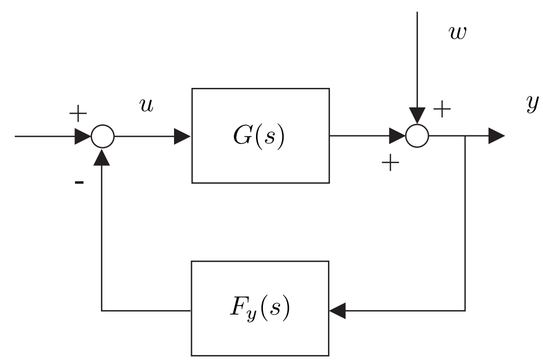
An $\mathcal{H}_\infty$-controller F_y (see figure) has been determined for the system

$$G(s) = \frac{1+s}{1+0.3s+2s^2}$$

such that $\|G_{ec}\|_\infty < \gamma$, where $\gamma = 2.5$ and the weights are

$$G_{ec} = \begin{bmatrix} W_uG_{wu} & -W_TT & W_SS \end{bmatrix}^T$$
$$W_u = \text{constant}$$
$$W_T = \frac{s+3}{1+0.1s}$$
$$W_S = \frac{s+3}{s}$$

S is the sensitivity of the closed loop, T is the complementary sensitivity and G_{wu} is the transfer function from the disturbance w to the control signal u.



a) What is the order of F_y?

b) One of the specifications of the system was that disturbances w with frequency lower than 0.1 rad/s should be damped by at least a factor 10. Is this fulfilled for the nominal system (i.e., with G from above being a perfect model)?

c) To keep the order of the controller low, G was simplified during the modeling and some fast dynamics were neglected. A better model is

$$G(s) = \frac{1+s}{1+0.3s+2s^2}\frac{1}{1+0.1s}$$

Assuming this is the true system, will the closed loop be stable with the controller F_y determined for the simpler model?

11.6 Specifications and $\mathcal{H}_\infty$ weight

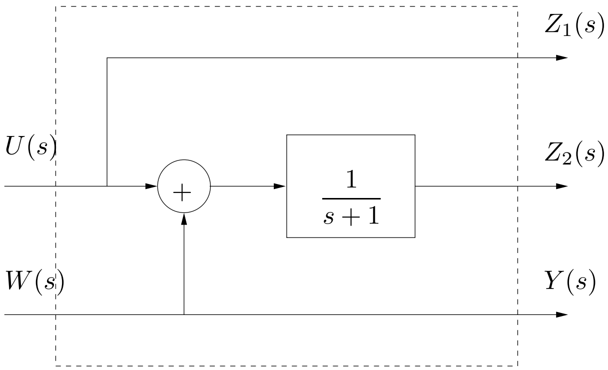
A multivariable system should attenuate all system disturbances (w) with at least a factor 10 for frequencies below 0.1 rad/s. Measurement nose (n) should be damped by at least a factor 10 for frequencies above 2 rad/s. Constant disturbances should be damped with at least a factor 100 at stationarity.

a) Formulate conditions on the singular values of S and T which guarantee the above requirements.

b) Translate the conditions to conditions on the loop gain GF.

c) Formulate the conditions using $\|\cdot\|_\infty$ and the weight-functions W_S and W_T.

11.7 $\mathcal{H}_2$ optimal feed forward



$G_0(s)$

Consider the extended system $G_0(s)$ given in the block diagram above.

a) Show that

$$\begin{aligned}\frac{dx(t)}{dt} &= -x(t) + u(t) + w(t)\\ z(t) &= \begin{bmatrix}0\\1\end{bmatrix} x(t) + \begin{bmatrix}1\\0\end{bmatrix} u(t)\\ y(t) &= w(t)\end{aligned}$$

is a statespace description of $G_0(s)$.

b) An observer based controller is given by

$$\begin{aligned}\frac{d\hat{x}(t)}{dt} &= -\hat{x}(t) + u(t) + y(t)\\ u(t) &= -L\hat{x}(t)\end{aligned}$$

Determine L such that $\|G_{ec}\|_2$ is minimized, where $G_{ec}(s)$ is the closed loop transfer function from $W(s)$ to $Z(s)$. Also determine the controllers transfer function from $Y(s)$ to $U(s)$ for this optimal L.

c) What is the value of $\|G_{ec}\|_2$ for the obtimal controller determined above? *Hint:* $\|G_{ec}\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{tr}\left(G_{ec}(i\omega)G_{ec}^T(-i\omega)\right) d\omega$

d) Let $U(s) = -KY(s)$, i.e., use proportional control. Determine the value of K that minimize $\|G_{ec}\|_2$, where $G_{ec}(s)$ is the closed loop transfer function from $W(s)$ to $Z(s)$ for proportional control. What is the smallest value of $\|G_{ec}\|_2$?

11.8 Translating time-domain specifications into an $\mathcal{H}_\infty$ formulation

Given the scaled linear system

$$y = G_1(s)u_1 + G_2(s)u_2 + G_d(s)d$$

where

$$G_1(s) = \frac{s-4}{10s+1}\ ; \quad G_2(s) = \frac{e^{-2s}}{10s+1}\ ; \quad G_d(s) = 3\frac{e^{-s}}{10s+1}$$

and the inputs u_1, u_2 and the disturbance d are all $\in [-1,1]$. The aim is to keep the output $y \in [-1,1]$ in the presence of disturbance d.

(a) Determine if it, in theory, is possible to achieve acceptable control performance according to the above specifications, and if this can be done using only one of the two available inputs. Which input should be used?

(b) Determine a feedback controller, based on measuring y, that satisifies the performance requirements with the chosen input from (a).

(c) Formulate an H_∞-optimal control problem that reflects the performance requirement for disturbance attenuation, the limited control input u and a requirement of robust stability in the presence of 20% uncertainty at the output. You should also determine the inputs and outputs of an extended system that reflects the objective.

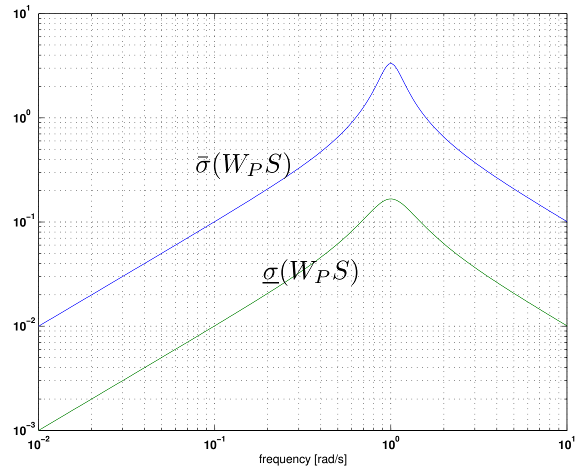
11.9 $\mathcal{H}_\infty$ vs. $\mathcal{H}_2$

Two alternative controllers have been designed for a linear system based on minimizing the norm of the weighted sensitivity

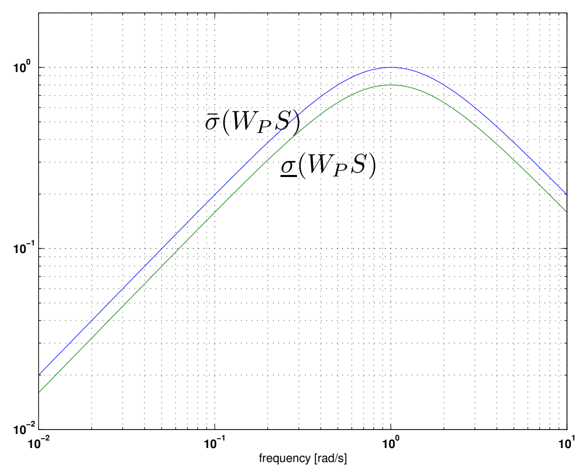
$$\|W_P S\|_m$$

using H_2-optimal control ($m = 2$) and H_∞-optimal control ($m = \infty$), respectively. The resulting singular values of the weighted sensitivity functions are shown below.

A



B

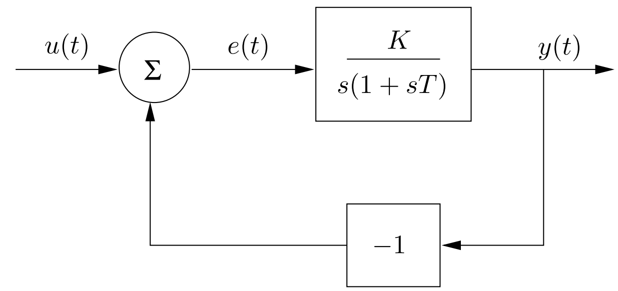


Which of the two controllers, A and B, correspond to H_2- and H_∞-optimization respectively? Motivate!

12 Kalman filters and Linear Quadratic control

12.1 On Kalman Filtering

The figure below shows a block diagram of a DC-servo



The input signal $u(t)$ is a stationary stochastic process with spectrum

$$\Phi_u(\omega) = \frac{\omega_0^2}{\omega_0^2 + \omega^2}$$

a) Calculate the spectrum of $e(t)$ expressed in terms of ω_0, K and T.

b) Which choice of K makes $Ee^2(t)$ as small as possible?

12.2 LQ control

Consider the double integrator

$$\ddot{y}(t) = u(t).$$

We would like to design a controller which minimizes the criterion

$$\int_0^\infty (y^2(t) + \eta \cdot u^2(t))dt \quad ; \quad \eta > 0$$

Assume that y and $\dot{y}$ can both be measured. Where are the closed-loop poles located under the optimal control. What happens with the control signal as η is decreased?

12.3 LQG control

Consider the system

$$Z(s) = \frac{1}{s+1}U(s) + \frac{1}{s+1}V(s)$$
$$Y(s) = Z(s) + E(s)$$

where v and e are unit disturbances with spectra

$$\Phi_v(\omega) \equiv r_1 \quad \text{and} \quad \Phi_e(\omega) \equiv 1$$

We would like to minimize the criterion

$$E\{q_1 z^2(t) + u^2(t)\}$$

a) Determine the loop gain under the optimal control.

b) What is the difference between r_1's and q_1's influence on the loop gain?

c) Sketch the loop gain.What happens when $r_1 \to \infty$ and when $q_1 \to \infty$?

12.4 LQG control

Consider the system

$$\begin{aligned} Z(s) &= \frac{1}{s+1}U(s) + \frac{1}{s+1}\nu(s) \\ Y(s) &= Z(s) + E(s) \end{aligned}$$

where ν is very low-frequent noise.

$$\nu = \frac{1}{s+\varepsilon}V(s)$$

v and e are unit noise $\Phi_v(\omega) \equiv \Phi_e(\omega) \equiv 1$.

Determine the LQG controller which minimizes

$$Ez^2 + Eu^2$$

when $\varepsilon \to 0$. What is the static gain of the sensitivity function?

13 Discrete-time systems and Model Predictive Control

13.1 Sampling a continuous system

The system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{aligned}$$

is sampled with the interval T. Determine the system matrices for the sampled system on state space form. For which values of T is the sampled system observable?

13.2 Receding horizon control

A system is given as

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 1.7 & 1 \\ -0.7 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0.9 \\ -0.6 \end{bmatrix} u_k \\ y_k &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_k \end{aligned}$$

Determine a control law

$$u_k = -F_y(z)y_k + F_r(z)r_k$$

by solving the MPC problem with cost function

$$J = \sum_{i=1}^{3} (r_{k+i} - \hat{y}_{k+i})^2 + 0.1 \cdot (u_{k+i-1})^2$$

where the state space model is used for the predictions.

13.3 Transforming a MPC problem into a QP

A system with constraints on the input is to be controlled with an MPC controller. The model is

$$y_{k+1} = -y_k + 2u_k$$

The MPC optimization problem is formulated as

$$\min_u \left[\sum_{i=k}^{k+N_P} y_i^2 + \sum_{i=k}^{k+N_P-1} u_i^2 \right]$$

subject to the constraint

$$-1 < u < 1$$

For the horizon $N_P = 1$, translate the MPC problem into a Quadratic Programming (QP) problem

$$\min_u u^T H u + h^T u \; ; \quad \texttt{subject to} \quad Lu \leq b$$

to be solved at each sample. That is, determine H, h, L and b. Note that the output y_k at the current sampling time is assumed given, while the input u_k is to be computed.

Solutions

Solutions to most problems are given below.

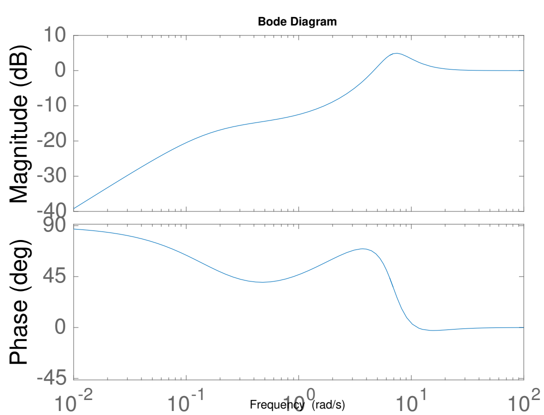
1 Repetition: Loop Shaping

1.1 Solution

(a) The open-loop system is $L(s) = G(s)K(s)$ where $K(s)$ is the controller to be designed. It approximately holds that $\omega_c = \omega_{BS} = 5$rad/s where ω_c is the crossover frequency. Furthermore, we have the approximation $\phi_m \geq 2\arcsin(\frac{1}{2M_S}) = 29$ where ϕ_m is the phase margin. Note that this is a minimum requirement, hence we choose $\phi_m = 50$ to have some margin. A steady-state error of 0 implies that $S(0) = 0$ or equivalently $L(0) = \infty$ (can be derived by the final value theorem), which implies that $L(s)$ must contain an integrator.

(b) When $K(s) = 1$, we have that $L(s) = G(s)$ with $\omega_c = 2.7$ rad/s and $\phi_m = 41.8$ (check in MATLAB with the margin command or by "hand"-calculations), which does not satisfy the specifications derived in (a). Hence, we calculate first arg($G(i5)$)=-173.45. In order to achieve $\phi_m = 50$ at $\omega_c = 5$, we need to increase the phase by 43.45 at $\omega_c = 5$. To have some margin (used later for the lag-element to account for the static error), we increase the phase by 48 (instead of 43.45). The controller that achieves this is $K_1(s) = \frac{0.7245*s+1}{s+10}$ and hence the phase margin objective has been accounted for. Next, we use the lag element $K_2(s) = \frac{s+0.2}{s}$ to account for the steady state error of 0. Finally, we adjust the crossover frequency by the element $K_3(s) = \frac{1}{G(i5)K_1(i5)K_2(i5)} = 9.1263$. The final controller is $K(s) = K_1(s)K_2(s)K_3(s)$ and the sensitivity function is shown below.

Closed loop sensitivity function S



Note that $M_S = 1.76$ and $\omega_{BS} \approx 4.8$ rad/s.

2 Signal norms

2.1 Solution

a) $\|y\|_\infty = |a|, \qquad \|y\|_2 = \infty$

b) $\|y\|_\infty = 1, \qquad \|y\|_2 = 1$

c) $\|y\|_\infty = \dfrac{1}{4}, \qquad \|y\|_2 = \sqrt{\dfrac{1}{12}}$

d) $\|y\|_\infty = \sqrt{1+a^2}, \qquad \|y\|_2 = \sqrt{\dfrac{1+a^2}{2}}$

2.2 Solution

(a) It holds that $\|y_1\|_\infty > \|y_2\|_\infty$ since y_1 has a higher peak than the signal y_2. Note that the L_∞-norm is also called peak norm. Furthermore, it holds that $\|y_1\|_2 < \|y_2\|_2$ since the area under the curve of the signal y_2 is higher than the one of y_1. The L_2-norm is also called energy-norm.

(b) Calculations show $\|y_1\|_\infty = 5$, $\|y_2\|_\infty = 2$, $\|y_1\|_2 = \sqrt{2.5}$, and $\|y_2\|_2 = \sqrt{10}$

3 Gains of static nonlinearities

3.1 Solution

a) $\|f(x)\| = 1$

b) $\|f(x)\| = \infty$

c) $\|f(x)\| = 1$

d) $\|f(x)\| = \infty$

4 Gains of scalar linear systems

4.1 Solution

For the gain to be well defined we must have a stable system. Thus, all poles must have a strictly negative real part and we hence need $T \geq 0$. For equality the system reduces and becomes purely static. Since

$$|G(i\omega)| = \frac{K}{\sqrt{\omega^2 T^2 + 1}}$$

is a strictly decreasing function for $\omega \geq 0$, we must have

$$\|G\| = \sup_{\omega} |G(i\omega)| = |G(0)| = |K|$$

and the gain is thus $|K|$ when it exist, i.e., for $T \geq 0$.

4.2 Solution

$$\|G\| = 1, \qquad \zeta > \frac{1}{\sqrt{2}}$$

$$\|G\| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}, \qquad 0 < \zeta < \frac{1}{\sqrt{2}}$$

4.3 Solution

(a) Block-diagram not shown here.

$$y = G(1+GF)^{-1}d_u = GSd_u = \frac{0.5s}{(s+5)(s+0.5)}d_u\,; \quad u = (1+GF)^{-1}d_u = Sd_u = \frac{s}{s+5}d_u$$

(b)

$$\|G(1+GF)^{-1}\|_\infty = \sup_{\omega} |0.5i\omega/(i\omega+5)(i\omega+0.5)| = 0.0909$$

and the corresponding frequency is $\omega = 1.58$. Thus, worst case amplification is 0.0909 and the corresponding disturbance is $d_u(t) = a\sin(1.58t)$ where a is any amplitude.

5 Gains of multivariable linear systems

5.1 Solution

a) In this case we can find the gain and directions of the system by direct inspection since the output is just a rescaled version of the input. That is, the first output is the first input rescaled by a factor 1 and the second output is the second input rescaled with a factor 2. Thus, we realize that $\|M\| = 2$ since we can at most magnify the input by a factor 2 which is achieved by putting all of the "energy" in the second input, i.e., the input direction $v_1 = [0\ 1]^T$. The direction associated with the smallest amplification is similarly achieved by putting all the input "energy" in the first input, i.e., $v_2 = [1\ 0]^T$.

In this case, we could find the gain and direction by inspection. However this is not always easy. A structured way to find these quantities is by computing the singular value decomposition (SVD) of M. We hence also compute the SVD of M, i.e., $M = U\Sigma V^*$. We begin to determine the singular values by computing the eigenvalues λ_i to M^*M. That is, we solve for λ_i in the characteristic polynomial given by:

$$\det(\lambda I - M^*M) = \det \begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda-4 \end{bmatrix} = (\lambda-1)(\lambda-4) = 0$$

The solutions are trivially $\lambda_1 = 4$, $\lambda_2 = 1$. The singular values are now given by the square root of λ_i. We get the largest singular value to be $\overline{\sigma} = 2$, and the smallest to be $\underline{\sigma} = 1$. Hence, we get

$$\Sigma = \operatorname{diag}\{\sigma_i\} = \begin{bmatrix} \overline{\sigma} & 0 \\ 0 & \underline{\sigma} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

The unitary matrix U is the matrix of right (or column) eigenvectors u_i to MM^*. For square matrices, the eigenvalues are the same for M^*M and MM^* (for non square systems they are still the same except for a few extra zero valued eigenvalues in the larger matrix) so we need to solve the equations

$$MM^*u_1 = \lambda_1 u_1, \quad MM^*u_2 = \lambda_2 u_2, \text{ normalized such that } |u_1| = |u_2| = 1$$

which if we insert M and λ_i becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} u_1 = 4u_1, \quad \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} u_2 = 1u_2$$

which has the (normalized) solutions $u_1 = [0\ 1]^T$ and $u_2 = [1\ 0]^T$. Hence we have

$$U = [u_1\ u_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Similarly, V is the right (or column) eigenvectors for M^*M which by analogous calculations as above yield

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Now, the gain $\|M\|$ equals the largest singular value, i.e., $\|M\| = 2$ and the corresponding input direction is the column-vector of V which correspond to this singular value (Note that the standard notation for the input in control is usually u so we might be tricked into thinking that the columns of U is the input directions while they in fact correspond to the output directions.). Normally we sort the the singular values in Σ such that the largest is the top-left one. This means that the first column in V is the input direction corresponding to the largest gain of the system and the last column correspond to the direction with least gain. Hence, the input direction with the largest gain is $v_1 = [0\ 1]^T$ and the input direction with the smallest gain is $v_2 = [1\ 0]^T$. In Matlab we could solve this exercise by writing

```
M = [1 0;0 2];
[U,Sigma,V] = svd(M);
v1 = V(:,1)
v2 = V(:,2)
gainM = S(1,1)
```

which if executed yields the following output

```
>> v1=V(:,1)

v1 =

     0
     1

>> v2=V(:,2)

v2 =

     1
     0

>> gainM=Sigma(1,1)

gainM =

     2
```

Which correspond well to the results we derived above.

b) Calculating the SVD yields

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}, \quad \Sigma = \begin{bmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}\end{bmatrix}, \quad V = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Here we see that all singular values are equal. Hence all input directions will have the same gain.

c) This is a non-square system but we can still determine its SVD which turns out to be

$$U = \begin{bmatrix}\sqrt{2} & 0 & \sqrt{2}\\ \sqrt{2} & 0 & -\sqrt{2}\\ 0 & -1 & 0\end{bmatrix}, \quad \Sigma = \begin{bmatrix}2 & 0\\ 0 & \sqrt{2}\\ 0 & 0\end{bmatrix}, \quad V = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

In this case the largest singular value is $\overline{\sigma} = 2$ with the corresponding input direction

$$v_1 = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}.$$

The input direction with the smallest gain is

$$v_2 = \begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}.$$

5.2 Solution

To find the gain and directions we compute the SVD of the plant. The transfer matrix can be written as

$$G(s) = \frac{1}{s+1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Hence, we get

$$G(s)^*G(s) = \frac{1}{|s+1|^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}^2$$

Note that $1/|s+1|^2$ is just a scaling factor

5.3 Solution

The gains correspond to the peak-values of the plots.

a) $\|G\| = 2$

b) $\|G\| = 4$

c) $\|G\| = 4$

d) In this plot we cannot read the peak value but we can conclude that $\|G\| > 20$

6 The small-gain theorem

6.1 Solution

I. $a > 0 \Rightarrow$ The linear system is stable. Small gain theorem $\Rightarrow$ Stable if $|K| < 1$. Linear theory $\Rightarrow$ Stable if $K < 1$.

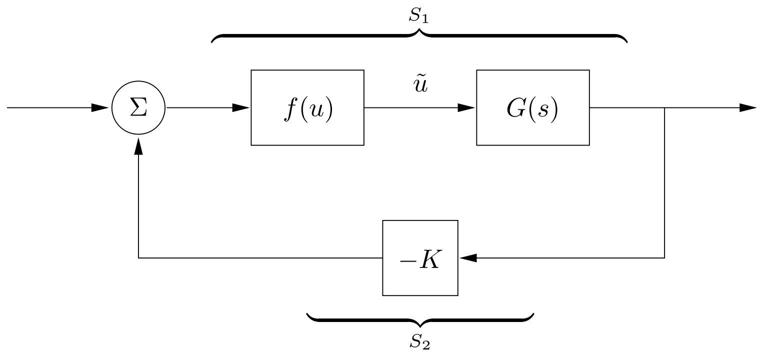
II. $a < 0 \Rightarrow$ The linear system is unstable $\Rightarrow$ We cannot use the small gain theorem. Linear theory $\Rightarrow$ Stable if:
$a(1-K) > 0$ (Pole is determined from $s + a(1-K) = 0$)

$$\begin{array}{llll}\text{If } a > 0: & 1-K > 0 & \Rightarrow & K < 1\\ \text{If } a < 0: & 1-K < 0 & \Rightarrow & K > 1\end{array}$$

6.2 Solution

The gain of f is 0.5 and the gain of G is 1.5.
Since $0.5 \cdot 1.5 < 1$, the system is stable by the small gain theorem.

6.3 Solution



According to the small-gain theorem the above system is stable if $\|S_1\| \cdot \|S_2\| < 1$.

$$\text{We} \quad \text{have:} \begin{cases} \|S_1\| \leq \|f(u)\| \cdot \|G\| \\ \|S_2\| = K \end{cases}$$

where

$$\|G\| = \sup_{\omega} |G(i\omega)| = \sup_{\omega} \frac{2}{\sqrt{(2-\omega^2)^2 + 4\omega^2}} = 1 \quad (\text{for } \omega = 0)$$

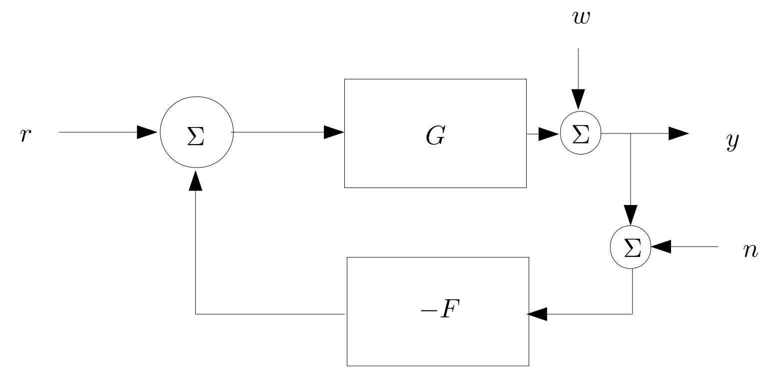
$$\begin{aligned} \|f(u)\|^2 = \frac{\|\tilde{u}\|_2^2}{\|u\|_2^2} = \frac{\int_{-\infty}^{\infty} (f(u(t))^2 dt}{\|u\|_2^2} &\leq \left[|f(u(t))| \leq \frac{1}{2}|u(t)| \right] \\ \leq \frac{\frac{1}{4}\|u\|_2^2}{\|u\|_2^2} = \frac{1}{4} \quad &\Rightarrow \quad \|f(u)\| \leq \frac{1}{2} \end{aligned}$$

$$\|S_c\| \cdot \|S_2\| \leq \frac{1}{2} \cdot K < 1$$

That is, we must choose $K < 2$ to guarantee stability (input-output)

7 Internal stability

7.1 Solution



The transfer functions

$$G = \frac{s-1}{s+2}, \quad F = \frac{s+2}{s-1}$$

yield

$$\begin{aligned} Y = G(R - F(Y+N)) + W \;\Rightarrow\; (1+GF)Y = GR - GFN + W \\ \Rightarrow\; Y = (1+GF)^{-1}GR - (1+GF)^{-1}GFN + (1+GF)^{-1}W \end{aligned}$$

Here we have

$$G_c = G_{ry} = (1+GF)^{-1}G = \frac{s-1}{2s+3}$$

$$S = G_{wy} = (1+GF)^{-1} = \frac{s+1}{2s+3}$$

$$T = 1 - S = \frac{s+2}{2s+3}$$

Which are all stable. Do we have internal stability?
We check the following four transfer functions

$$H_{11} = (1+FG)^{-1} = \frac{s+1}{2s+3}$$

$$H_{12} = (1+FG)^{-1}F = \frac{(s+2)(s+1)}{(s-1)(2s+3)}$$

$$H_{21} = (1+GF)^{-1}G = \frac{s-1}{2s+3}$$

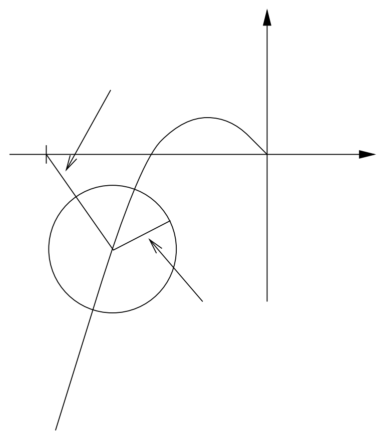
$$H_{22} = (1+GF)^{-1} = \frac{s+1}{2s+3}$$

The system is not internally stable since H_{12} is unstable.

8 Robustness and limitations

8.1 Solution

Since L and $L/(1+L)$ are stable, the simplified Nyquist criterion may be used. The true loop gain is $L_p = L + \tilde{L}$, $\tilde{L} = \Delta_L L$. A typical Nyquist plot is given in the figure below



The worst case is when L_p is the point on the circle closest to -1. We get the condition

$$
\begin{aligned}
&|1+L| - |\tilde{L}| > 0 \quad \forall \omega \\
\Leftrightarrow \; &|\Delta_L| < \frac{|1+L|}{|L|} = \frac{1}{|T|} \quad \forall \omega
\end{aligned}
$$

which is the robustness criterion in the scalar case.

8.2 Solution

a) With a one-degree of freedom controller the complementary sensitivity function becomes

$$T(s) = \frac{F(s)G(s)}{1+F(s)G(s)}$$

To determine the controller $F(s)$ which yield the desired $T(s)$ we express $F(s)$ as a function of $G(s)$ and $T(s)$ as follows. The above expression yields

$$T(s) + T(s)F(s)G(s) = F(s)G(s)$$

and

$$T(s) = G(s)(1 - T(s))F(s)$$

so

$$F(s) = G^{-1}(s)\frac{T(s)}{1-T(s)}.$$

With the given $T(s)$ and $G(s)$ we get

$$F(s) = \frac{5(s+1)}{s(s-3)}.$$

The RHP-zero is canceled when we look at the transfer function from reference to output. However, for internal stability, all transfer functions between any two signals in the loop must be stable. In this case we get the transfer funtion from the reference to the output to be

$$U(s) = \frac{5(s+1)}{(s-3)(s+5)}R(s)$$

with a pole at $s = 3$. We conclude that the closed loop is internally unstable.

b) The bandwidth 5 rad/s can be achieved if we keep the RHP-zero and place a pole at $s = -3$

$$T(s) = \frac{5(s-3)}{(s+3)(s+5)}$$

i.e., we multiply the desired T with an all-pass filter which don't affect the gain. The relation

$$F(s) = G^{-1}(s)\frac{T(s)}{1-T(s)}$$

yields

$$F(s) = \frac{5(s+1)}{(s^2+3s+30)}.$$

In this case no cancellation occurs and the closed loop system turns out to be internally stable.

c)

$$S(s) = 1 - T(s) = \frac{s^2+3s+30}{s^2+8s+15}$$

which has the character of a low-pass filter with $S(0) = 2$ which is undesirable since we the amplify low-frequency disturbances which we rather like to attenuate.

d) Since $G_c = \frac{GF_r}{1+GF_y}$ and G_c lacks a zero in $s = 3$ the factor $(s-3)$ in $G(s)$ must be canceled. This yields stability problems.

8.3 Solution

The transfer function for such a system can be expressed as

$$G(s) = e^{-s}(3-s)\bar{G}(s)$$

or

$$G(s) = e^{-s}\frac{(3-s)}{(3+s)}(3+s)\bar{G}(s)$$

The argument of the frequency response function is now

$$\arg G(i\omega) = -\omega - 2\arctan\frac{\omega}{3} + \arg((3+i\omega)\bar{G}(i\omega)).$$

According to the specifications, the amplitude is decreasing monotonically and due to Bode's relations this implies that

$$\arg((3+i\omega)\bar{G}(i\omega)) \leq 0.$$

Thus,

$$\arg G(i\omega) \leq -\omega - 2\arctan\frac{\omega}{3}$$

and that the phase margin satisfies

$$\varphi_m = \pi + \arg G(i\omega_c) \leq \pi - \omega_c - 2\arctan\frac{\omega_c}{3}$$

Let us study the limiting case when $\varphi_m = 0$ and assume equality in the limit above. Then

$$0 = \pi - \omega_c - 2\arctan\frac{\omega_c}{3}$$

which mean that

$$\omega_c \approx 2$$

The highest possible crossover frequency is hence about 2 rad/s.

8.4 Solution

a) As $e(t) = S(p)r(t) + \ldots$, we have for $r(t)$ a ramp

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sS(s)\frac{1}{s^2}$$

and it must hold

$$S(0) = 0, \quad \frac{dS(s)}{ds}_{|s=0} = 0$$

b) Use Laplace transform (the reference signal is a step function in this case)

$$\int_0^\infty e(t)\ dt = \lim_{s\to 0}\int_0^\infty e^{-st}e(t)\ dt = \lim_{s\to 0} E(s) = \lim_{s\to 0} S(s)\frac{1}{s} = \lim_{s\to 0}\frac{dS(s)}{ds} = 0$$

c) As

$$\int_0^\infty e(t)dt = 0$$

the control error $e(t)$ must take both positive and negative values. Hence there must be an overshoot in the step response. (Civerth, who is a well trained engineer, realized that there is no way to satisfy the two design objectives simultaneously.)

8.5 Solution

The zero at $s = 1$ in the transfer-function from u to y implies that the closed-loop sensitivity $|S(i1)| \geq 1$. The closed-loop transfer-function from the disturbance d to the output y is $y = SG_d d$ where G_d is the corresponding open-loop transfer-function. Thus, to keep $|y| < 0.5$ for $|d| < 1$ at frequency $\omega = 1$ we require $|G_d(i1)| < 0.5$. However, we find that $|G_d(i1)| = 2.51$ and hence the desired disturbance attenuation can not be achieved using feedback control. So what could be the solution?

8.6 Solution

a) The gain of a system M is

$$||M|| = \sup_u \frac{||M(u)||}{||u||}$$

where sup means the least upper limit (supremum). For the nonlinearity in the problem, an input signal that attains maximum gain is $u = 1$ (In fact, any signal with $|u(t)| \leq 1$ can be used here). With this choice of input signal, the gain is calculated to 1.

Next step is to use the small-gain theorem. The two linear blocks can be combined into $-GK = -\frac{K}{s+\epsilon}$. For this system, the gain is at maximum for $\omega = 0$ and is then $\frac{K}{\epsilon}$ (compare to the SISO case treated in an earlier exercise). Because all involved transfer functions are stable, the low gain theorem can be used.

$$|\frac{K}{\epsilon}||1| < 1$$

which gives $K < \epsilon$ for stability. This criterion thus limits the controller to have almost zero gain.

Answer: $K < \epsilon$ gives stability.

b) For the two block diagrams to be equal, it is required that

$$\mathrm{sat}(u) = u + \Delta(u)$$

This gives

$$\Delta(u) = \begin{cases} 1-u & \text{if } u \geq 1 \\ 0 & \text{if } -1 \leq u \leq 1 \\ -1-u & \text{if } u \leq -1 \end{cases}$$

The gain for Δ is calculated by finding an input signal that maximizes the gain. Using $u = c \gg 1$ gives

$$||\Delta|| = \sup_c \frac{|1-c|}{|c|} = 1$$

The loop gain outside the Δ block is now calculated.
Alternative 1: go from output (left side of Δ) to input (right side of Δ). The gain is $K, -G$ then breakout causing $(I+KG)^{-1}$, adding up to $-KG(I+KG)^{-1}$. Alternative 2: let the signal on the input to Δ be a and the output b. Then $a = -KG(a+b)$ and thus $a = -(I+KG)^{-1}KGb = -KG(I+KG)^{-1}$, using the push through rule in the last equality.
The loop gain seen from the Δ block is $T_i = -KG(I+KG)^{-1}$.

$$T_i = \frac{K}{s+K+\epsilon}.$$

The gain for T_i is $||T_i|| = \frac{K}{K+\epsilon}$ attained for $\omega = 0$. Applying the small-gain theorem gives

$$|\frac{K}{K+\epsilon}||1| < 1$$

which with positive K gives $K < K+\epsilon$ which is always true. Stability can then be guaranteed for all $K > 0$.
Answer: The system is stable for all $K > 0$.

8.7 Solution

a) From the block diagram, we see that we should write G_p on the form $G_p = G_0(1+\Delta)$, which means that $\Delta(s) = e^{-sL} - 1$.

To verify that $|\Delta(i\omega)| \leq 2$, use Euler's formula $e^{ix} = \cos(x) + i\sin(x)$. Then, $\Delta(i\omega) = \cos(\omega L) - i\sin(\omega L) - 1$ so $|\Delta(i\omega)|^2 = (\cos(\omega L) - 1)^2 + (\sin(\omega L))^2 = 2 - 2\cos(\omega L) \leq 4$. Thus $|\Delta(i\omega)| \leq 2$.

b) Let u_Δ be the input to the Δ-block and y_Δ be its output. To get the system on the standard $M-\Delta$-form from the lectures, we need to determine the transfer function from y_Δ to u_Δ. We find

$u_\Delta = -G_0F_y(y_\Delta + u_\Delta)$, *i.e.*

$$u_\Delta = -(1+G_0F_y)^{-1}G_0F_y y_\Delta = -Ty_\Delta$$

The small gain now requires that $|\Delta(i\omega)||T(i\omega)| \leq 1$ for all frequencies. If we only use the norm bound on Δ, we need to require that

$$|T(i\omega)| \leq \frac{1}{2} \qquad \forall\omega$$

(a very conservative requirement!)

c) Since

$$\frac{G_p - G_0}{G_0} = e^{-sL} - 1$$

we can the derivations used in 4a) to see that the relative error is given by

$$e^{-i\omega L} - 1 = 1 - \cos(\omega L) + i\sin(\omega L)$$

For the worst-case delay $L_{\max}$, the relative error is less than 2 for all frequencies $\omega \leq \pi/L_{\max}$. Moreover, in this frequency interval cosine is monotone decreasing so the relative error is smaller for delays L with $L \leq L_{\max}$. Hence, the maximum delay value $L = L_{\max}$ is the worst-case.

d) The stability criterion requires that

$$|T(i\omega)| \leq |w_I^{-1}(i\omega)|$$

The magnitude of $w_I^{-1}(i\omega)$ has slope -1 at low frequencies, a breakpoint around 1 rad/s, and a high-frequency gain of 1/2. Superimposing this bound on the Bode diagrams of the complementary sensitivity gives that the controller corresponding to the dashed line is guaranteed to be robustly stable, while the controller corresponding to the full line does not satisfy the bound. Hence, robust stability of the design corresponding to the solid line cannot be guaranteed with the suggested criterion.

8.8 Solution

(a) With $y = 2\hat{y}$, $u = 0.1\hat{u}$ and $d = \hat{d}/3$ we get the corresponding scaled transfer function

$$G(s) = 20\hat{G}(s)\ ; \quad G_d(s) = 6\hat{G}_d(s).$$

(b) The limitations that the controller has to deal with come from the disturbance and we need to have effective control for the scaled system where the transfer function is $|G_d(i\omega)| > 1$. From $G_d(s)$, given in the figure, we find that $|G_d| > 1$ for $\omega < \omega_d$ with $\omega_d = 0.75\ rad/s$. The bandwidth requirements are hence $\omega_{BS} > 0.75$.

More limitations come from zeros in the RHP, delays and input limitations as follows.

The system has a RHP zero at $s = 1$, which gives a bandwidth requirement of $\omega_{BS} < 1$ (or $\omega_{BS} < 0.5$ when we want $M_S < 2$). This means that in theory it is possible to handle disturbances and get an acceptable control error despite of the RHP zero.

The system has no time delays (here only time delays of the system G are of interest) and hence there are no limitations.

Finally, we need to investigate if we have enough input to deal with disturbance, this means that we need $|G| > |G_d| - 1$ for all frequencies. We see directly, that this is not possible for low frequencies, i.e, see $G(0) = 20$ and $G_d(0) = 30$.

An appropriate way to deal and improve the controllability of the system is to improve the equipment for the control signal, e.g., get a more powerful motor. In other words, the process G needs to be changed.

9 MIMO basics

9.1 Solution

$$\begin{pmatrix} \Delta\omega \\ \Delta e \end{pmatrix} = \frac{1}{s+2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & s+1 & 1 \end{pmatrix} \begin{pmatrix} \Delta M \\ \Delta Im \\ \Delta R \end{pmatrix}$$

9.2 Solution

First find a common denominator

$$\begin{aligned} Y(s) &= \frac{(s^2+s+1)}{(s+1)(s+2)(s^2+s+1)} U_1(s) + \frac{(s+2)(s+3)}{(s+1)(s+2)(s^2+s+1)} U_2(s) \\ &= \frac{(s^2+s+1)}{(s^4+4s^3+6s^2+5s+2)} U_1(s) + \frac{(s^2+5s+6)}{(s^4+4s^3+6s^2+5s+2)} U_2(s) \end{aligned}$$

We can now use the obsevable canonical form

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -4 & 1 & 0 & 0 \\ -6 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(t) \end{aligned}$$

9.3 Solution

a) The eigenvalues of the A-matrix are 0.5 and 2. The system is thus unstable.

b) The controllability matrix is

$$S = \begin{bmatrix} -1 & 0.5 & -2 & 0.25 \\ -2 & 0.5 & -4 & 0.25 \end{bmatrix}$$

and it has full rank since the rows are linearly independent. The system is thus controllable and the poles can be moved arbitrarily by state feedback. Hence, we can stabilize the system with state feedback.

c) This correspond to $B = \begin{bmatrix} -1 & -2 \end{bmatrix}^T$, which yields

$$S = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix}.$$

This matrix does not have full rank and we cannot move the poles arbitrary. However, transforming the state space model to diagonal form shows that the uncontrollable mode is stable and we can hence stabilize the system using only u_1.

9.4 Solution

a) The transfer matrix has the determinant (maximal minor)

$$\frac{(1-s)^2}{(s+1)^2} - \frac{(1/3-s)(2-s)}{(s+1)^2} = \frac{1/3s+1/3}{(s+1)^2} = \frac{1}{3(s+1)}$$

and the minors

$$\frac{1-s}{s+1}, \quad \frac{1/3-s}{s+1}, \quad \frac{2-s}{s+1}, \quad \frac{1-s}{s+1}$$

The system thus has the pole $\{-1\}$ and no zero. The maximal state space realization has one state.

9.5 Solution

a) The transfer matrix has the determinant (maximal minor)

$$\frac{4}{(s+1)(s+2)} - \frac{(3-s)}{(s+1)(s+2)} = \frac{s+1}{(s+1)(s+2)}$$

and the first order minors are the four elements of $G(s)$. We hence have poles at -1 and -2 and a zero at -1. Note that we have a pole and a zero at the same location, which would in SISO systems lead to a cancellation. In MIMO systems, however, poles and zeors may occur at the same location when the directions of the poles and zeros are different.

9.6 Solution

$G(s)$ has a RHP zero at $s = 1$ and $y_z^H G(z) = 0$ gives the zero direction $y_z^H = 1/\sqrt{5}\begin{bmatrix}-1 & 2\end{bmatrix}$. Then, the requirement acceptable disturbance attenuation being feasible is $|y_z^H G_d(z)| < 1$. For disturbance d_1 we get $y_z^H G_{d1}(z) = 2.01 > 1$, and for d_2 $y_z^H G_{d2}(z) = 0 < 1$. Hence, it is not possible to attenuate disturbance d_1 but the there are no given limitations that hinders acceptable attenuation of d_2.

9.7 Solution

Any RHP pole in the open-loop should appear as a RHP zero in $S(s)$ for internal stability. Here the open-loop has RHP poles, but $S(s)$ has no RHP zeros and hence the system is not internally stable.

10 Decentralized and decoupled control

10.1 Solution

(a) We want to evaluate the expression

$$G(0) \circ (G(0)^{-1})^T$$

We start by evaluating $G(s)$ at $s = 0$ to get

$$G(0)$$

We invert $G(0)$ to get

$$G(0)^{-1}$$

We take the transpose of $G(0)^{-1}$ to get

$$(G(0)^{-1})^T$$

Entrywise multiplication yields

$$\mathrm{RGA}(G(0))$$

For the given systems we get

(i)

$$G(0) = \begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix}.$$

$$G(0)^{-1} = \frac{1}{3}\begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix}$$

$$(G(0)^{-1})^T = \frac{1}{3}\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}$$

$$\mathrm{RGA}(G(0)) = \begin{bmatrix}1\cdot\frac{1}{3} & 2\cdot\frac{1}{3}\\ 1\cdot\frac{2}{3} & -1\cdot\frac{-1}{3}\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$$

(ii)

$$G(0) = \begin{bmatrix}1 & -2\\ 1 & -1\end{bmatrix}.$$

$$G(0)^{-1} = \begin{bmatrix}-1 & 2\\ -1 & 1\end{bmatrix}$$

$$(G(0)^{-1})^T = \begin{bmatrix}-1 & -1\\ 2 & 1\end{bmatrix}$$

$$\mathrm{RGA}(G(0)) = \begin{bmatrix}1\cdot -1 & -2\cdot -1\\ 1\cdot 2 & -1\cdot 1\end{bmatrix} = \begin{bmatrix}-1 & 2\\ 2 & -1\end{bmatrix}$$

(iii)

$$G(0) = \begin{bmatrix}0.6 & -0.4\\ 0.3 & 0.6\end{bmatrix}.$$

$$G(0)^{-1} = \frac{1}{0.48}\begin{bmatrix}0.6 & 0.4\\ -0.3 & 0.6\end{bmatrix}$$

$$(G(0)^{-1})^T = \frac{1}{0.48}\begin{bmatrix}0.6 & -0.3\\ 0.4 & 0.6\end{bmatrix}$$

$$\mathrm{RGA}(G(0)) = \begin{bmatrix}0.6\cdot\frac{0.6}{0.48} & -0.4\cdot\frac{-0.3}{0.48}\\ 0.3\cdot\frac{0.4}{0.48} & 0.6\cdot\frac{0.6}{0.48}\end{bmatrix} = \frac{1}{4}\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$$

(b) (i) We want to evaluate the expression

$$G(s) \circ (G(s)^{-1})^T$$

at $s = i\omega_c$. One can either evalute $G(s = i\omega_c)$ initially and work with complex matrices or one can save the evaluation to the end. Here we will do the latter. We have

$$G(s) = \begin{bmatrix}\frac{1}{10s+1} & \frac{2}{2s+1}\\ \frac{1}{10s+1} & \frac{s-1}{2s+1}\end{bmatrix}$$

Inverting $G(s)$ yields

$$\begin{aligned} G(s)^{-1} &= \frac{(10s+1)(2s+1)}{s-3}\begin{bmatrix}\frac{s-1}{2s+1} & \frac{-2}{2s+1}\\ \frac{-1}{10s+1} & \frac{1}{10s+1}\end{bmatrix}\\ &= \frac{1}{s-3}\begin{bmatrix}(s-1)(10s+1) & -2(10s+1)\\ -(2s+1) & 2s+1\end{bmatrix}\end{aligned}$$

The transpose becomes

$$(G(s)^{-1})^T = \frac{1}{s-3}\begin{bmatrix}(s-1)(10s+1) & -(2s+1)\\ -2(10s+1) & 2s+1\end{bmatrix}$$

$$\begin{aligned}\text{RGA}(G(s)) &= \frac{1}{s-3}\begin{bmatrix}\frac{1}{10s+1}\cdot(s-1)(10s+1) & \frac{2}{2s+1}\cdot -(2s+1)\\ \frac{1}{10s+1}\cdot -2(10s+1) & \frac{s-1}{2s+1}\cdot(2s+1)\end{bmatrix}\\ &= \frac{1}{s-3}\begin{bmatrix}s-1 & -2\\ -2 & s-1\end{bmatrix}\end{aligned}$$

We now evaluate this for $s = i\omega_c$ to get

$$\begin{aligned}\text{RGA}(G(i\omega_c = 0.1i)) &= \frac{-(3+0.1i)}{9+0.01}\begin{bmatrix}0.1i-1 & -2\\ -2 & 0.1i-1\end{bmatrix}\\ &= \frac{1}{9.01}\begin{bmatrix}3.01-0.2i & 6+0.2i\\ 6+0.2i & 3.01-0.2i\end{bmatrix}\end{aligned}$$

(ii) Following the solution in (i) but for the system

$$G(s) = \begin{bmatrix}\frac{1}{10s+1} & \frac{-2}{2s+1}\\ \frac{1}{10s+1} & \frac{s-1}{2s+1}\end{bmatrix}$$

we get

$$\begin{aligned}G(s)^{-1} &= \frac{(10s+1)(2s+1)}{s+1}\begin{bmatrix}\frac{s-1}{2s+1} & \frac{2}{2s+1}\\ \frac{-1}{10s+1} & \frac{1}{10s+1}\end{bmatrix}\\ &= \frac{1}{s+1}\begin{bmatrix}(s-1)(10s+1) & 2(10s+1)\\ -(2s+1) & 2s+1\end{bmatrix}\end{aligned}$$

The transpose becomes

$$(G(s)^{-1})^T = \frac{1}{s+1}\begin{bmatrix}(s-1)(10s+1) & -(2s+1)\\ 2(10s+1) & 2s+1\end{bmatrix}$$

$$\begin{aligned}\text{RGA}(G(s)) &= \frac{1}{s+1}\begin{bmatrix}\frac{1}{10s+1}\cdot(s-1)(10s+1) & \frac{-2}{2s+1}\cdot -(2s+1)\\ \frac{1}{10s+1}\cdot 2(10s+1) & \frac{s-1}{2s+1}\cdot(2s+1)\end{bmatrix}\\ &= \frac{1}{s+1}\begin{bmatrix}s-1 & 2\\ 2 & s-1\end{bmatrix}\end{aligned}$$

We now evaluate this for $s = i\omega_c$ to get

$$\begin{aligned}\text{RGA}(G(i\omega_c = 0.1i)) &= \frac{1-0.1i}{1+0.01}\begin{bmatrix}0.1i-1 & 2\\ 2 & 0.1i-1\end{bmatrix}\\ &= \frac{1}{1.01}\begin{bmatrix}-0.99+0.2i & 2-0.2i\\ 2-0.2i & -0.99+0.2i\end{bmatrix}\end{aligned}$$

(iii) For this system we instead start with evaluating $G(i\omega_c)$. We get

$$G(i\omega_c = 10i) = \frac{1}{1+i}\begin{bmatrix}\frac{0.6}{1+10i} & -0.4\\ 0.3 & 0.6\end{bmatrix} = \begin{bmatrix}-0.0267-0.0327i & -0.2+0.2i\\ 0.15-0.15i & 0.3-0.3i\end{bmatrix}$$

Inverting this yields

$$G(10i)^{-1} = \begin{bmatrix}3.1897+5.7759i & 2.1264+3.8506i\\ -1.5948-2.8879i & 0.6034-0.2586i\end{bmatrix}$$

The transpose become

$$(G(10i)^{-1})^T = \begin{bmatrix}3.1897+5.7759i & -1.5948-2.8879i\\ 2.1264+3.8506i & 0.6034-0.2586i\end{bmatrix}$$

Finally, entrywise multiplication yield

$$\text{RGA}(G(10i)) = \begin{bmatrix}0.1034-0.2586i & 0.8966+0.2586i\\ 0.8966+0.2586i & 0.1034-0.2586i\end{bmatrix}$$

10.2 Solution

(a) We write

$$G(s) = \begin{bmatrix}g_{11}(s) & g_{12}(s)\\ g_{21}(s) & g_{22}(s)\end{bmatrix}$$

and evaluate the inverse to get

$$G^{-1}(s) = \frac{1}{g_{11}(s)g_{22}(s) - g_{12}(s)g_{21}(s)}\begin{bmatrix}g_{22}(s) & -g_{12}(s)\\ -g_{21}(s) & g_{11}(s)\end{bmatrix}$$

Taking the transpose yields

$$G^{-1}(s) = \frac{1}{g_{11}(s)g_{22}(s) - g_{12}(s)g_{21}(s)}\begin{bmatrix}g_{22}(s) & -g_{21}(s)\\ -g_{12}(s) & g_{11}(s)\end{bmatrix}$$

We now evalueate the RGA$(G(s))$ matrix to get

$$\text{RGA}(G(s)) = \begin{bmatrix}\frac{g_{11}(s)g_{22}(s)}{g_{11}(s)g_{22}(s)-g_{12}(s)g_{21}(s)} & \frac{-g_{12}(s)g_{21}(s)}{g_{11}(s)g_{22}(s)-g_{12}(s)g_{21}(s)}\\ \frac{-g_{11}(s)g_{22}(s)}{g_{11}(s)g_{22}(s)-g_{12}(s)g_{21}(s)} & \frac{g_{12}(s)g_{21}(s)}{g_{11}(s)g_{22}(s)-g_{12}(s)g_{21}(s)}\end{bmatrix}$$

Finally we see that the first row sums to

$$\begin{aligned}&\frac{g_{11}(s)g_{22}(s)}{g_{11}(s)g_{22}(s) - g_{12}(s)g_{21}(s)} + \frac{-g_{12}(s)g_{21}(s)}{g_{11}(s)g_{22}(s) - g_{12}(s)g_{21}(s)}\\ &= \frac{g_{11}(s)g_{22}(s) - g_{12}(s)g_{21}(s)}{g_{11}(s)g_{22}(s) - g_{12}(s)g_{21}(s)} = 1\end{aligned}$$

Clearly, all other row and column sums are also equal to 1.
□

(b) From (a) we know that the top left element of the RGA$(G(s))$ matrix is

$$c(s) = \frac{g_{11}(s)g_{22}(s)}{g_{11}(s)g_{22}(s) - g_{12}(s)g_{21}(s)}.$$

Let $b(s)$ be the top right element of the RGA($G(s)$) matrix. Since the top row must sum to 1, we get

$$b(s) + c(s) = 1 \Rightarrow b(s) = 1 - c(s).$$

The same argument can be applied to prove that the bottom left element equals 1-c(s) using the first column sum. Finally, let $d(s)$ be the bottom right element. The second column must sum to 1, hence

$$1 - c(s) + d(s) = 1 \Rightarrow d(s) = c(s)$$

□

(c) We write

$$G(s) = \begin{bmatrix} g_{1,1}(s) & \cdots & g_{1,n}(s) \\ \vdots & \ddots & \vdots \\ g_{n,1}(s) & \cdots & g_{n,n}(s) \end{bmatrix}$$

and

$$G^{-1}(s) = \begin{bmatrix} \hat{g}_{1,1}(s) & \cdots & \hat{g}_{1,n}(s) \\ \vdots & \ddots & \vdots \\ \hat{g}_{n,1}(s) & \cdots & \hat{g}_{n,n}(s) \end{bmatrix}$$

From linear algebra we know that

$$G(s)G^{-1}(s) = I = G^{-1}(s)G(s).$$

Using the rules for matrix multiplication we get that the element $I_{i,i} = 1$ can be written as

$$\sum_{k=1}^{n} g_{i,k}(s)\hat{g}_{k,i}(s) = 1$$

or

$$\sum_{k=1}^{n} \hat{g}_{i,k}(s)g_{k,i}(s) = 1$$

depending on the order of the multiplication. Take the transpose of $G^{-1}(s)$

$$(G^{-1}(s))^T = \begin{bmatrix} \hat{g}_{1,1}(s) & \cdots & \hat{g}_{n,1}(s) \\ \vdots & \ddots & \vdots \\ \hat{g}_{1,n}(s) & \cdots & \hat{g}_{n,n}(s) \end{bmatrix}$$

By evaluating the RGA($G(s)$) matrix we get

$$\text{RGA}(G(s)) = \begin{bmatrix} g_{1,1}(s)\hat{g}_{1,1}(s) & \cdots & g_{1,n}(s)\hat{g}_{n,1}(s) \\ \vdots & \ddots & \vdots \\ g_{n,1}(s)\hat{g}_{1,n}(s) & \cdots & g_{n,n}(s)\hat{g}_{n,n}(s) \end{bmatrix}$$

Clearly, the i:th row-sum can be written as

$$\sum_{k=1}^{n} g_{i,k}(s)\hat{g}_{k,i}(s)$$

which equal the sum of the i:th diagonal element of $G(s)G^{-1}(s)$ and hence must be equal to 1. Likewise, the j:th column sum can be written as

$$\sum_{k=1}^{n} \hat{g}_{i,k}(s)g_{k,i}(s)$$

which equal the j:th diagonal element of $G^{-1}(s)G(s)$ and hence must be equal to 1. Since i and j were chosen arbitrarily it must be true for all $i, j \in 1, \ldots, n$.
□

10.3 Solution

10.4 Solution

a) The controller becomes

$$F(s) = \frac{k(s+1)}{s} \begin{bmatrix} 1/2 & -1/4 \\ 0 & 1/2 \end{bmatrix}$$

This yield the loop gain $GF = \frac{k}{s}I$ and the closed loop system

$$G_c = (I + GF)^{-1}GF = \frac{k}{s+k}I.$$

The open loop system has two poles in -1 and G_c two poles in $-k$. Choose $k = 1$ to make the systems equally fast.

b) In this case the true systems is given by

$$G_p = (I + \Delta_o)G$$

where Δ_o is the relative output uncertainty. The robustness criterion is $\|\Delta_o T\|_\infty < 1$ where

$$T = (I + GF)^{-1}GF = \frac{1}{s+1}I.$$

Here we have

$$\Delta_o = (G_p - G)G^{-1} = (\alpha - 1)\begin{bmatrix} 0 & 1/2 \\ 0 & 1 \end{bmatrix}.$$

Let $H = \Delta_o T$. We seek

$$\|H\|_\infty = \max_\omega \bar{\sigma}\left(H(i\omega)\right) = \max_\omega \sqrt{\lambda_{max}\left(H^*(i\omega)H(i\omega)\right)}$$

where $\lambda_{max}(H^*(i\omega)H(i\omega))$ is the larges eigenvalue of $H^*(i\omega)H(i\omega) = H^T(-i\omega)H(i\omega)$. This comes from

$$\lambda_{max}\left(H^T(-i\omega)H(i\omega)\right) = \frac{(\alpha-1)^2}{1+\omega^2}\lambda_{max}\left(\begin{bmatrix}0 & 0\\ 1/2 & 1\end{bmatrix}\begin{bmatrix}0 & 1/2\\ 0 & 1\end{bmatrix}\right)$$
$$= \frac{(\alpha-1)^2}{1+\omega^2}\frac{5}{4}.$$

Hence we get $\|H\|_\infty = |\alpha - 1\frac{\sqrt{5}}{2}|$ and the robustness criterion guarantee stability for all α fulfilling

$$1 - \frac{2}{\sqrt{5}} < \alpha < 1 + \frac{2}{\sqrt{5}}$$
$$0.11 < \alpha < 1.89$$

c) We now describe the true system as

$$G_p = G(I + \Delta_I)$$

where Δ_I is the relative input uncertainty. The robustness criterion in this case is $\|\Delta_I T_I\|_\infty < 1$ where $T_I = (I + FG)^{-1}FG$. Note the differences between the definition of T and T_I! However, in this case we have $T_I = T$ since $FG = GF = \frac{k}{s}I$. The input uncertainty is

$$\Delta_I = G^{-1}(G_p - G) = (\alpha - 1)\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

From this it follows that

$$\Delta_I T_I = \frac{\alpha-1}{s+1}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

which yield $\|\Delta_I T_I\|_\infty = |\alpha - 1|$. The robustness criterion now guarantee closed loop stability for all α fulfilling

$$0 < \alpha < 2.$$

We note a difference compared to the previous exercise. We conclude that the modelling makes a difference for multivariable systems!

d) The closed loop system is

$$(I + G_pF)^{-1}G_pF = \frac{1}{(s+1)(s+\alpha)}\begin{bmatrix}s+\alpha & \frac{\alpha-1}{2}s\\ 0 & \alpha(s+1)\end{bmatrix}.$$

This system has poles at -1 and $-\alpha$ so the system is stable for all $\alpha > 0$! Note that the robustness criterion is a "worst case" condition and do not take the structure of the uncertainty into account.

10.5 Solution

a) For $\alpha = 0.5$ one gets

$$G(0) = \begin{bmatrix}0.75 & 0.25\\ 0.25 & 0.75\end{bmatrix}$$

leading to

$$RGA(G(0)) = \begin{bmatrix}1.125 & -0.125\\ -0.125 & 1.125\end{bmatrix}$$

The second alternative (to make the pairing $u_1 - y_2$ and $u_2 - y_1$) will hence give nonnegative elements in RGA, and should therefore be avoided.

b) The requirement on static decoupling gives

$$W_1 = G^{-1}(0) = \begin{bmatrix}1.5 & -0.5\\ -0.5 & 1.5\end{bmatrix}$$

c) The open loop system is

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

and the regulator is

$$u = W_1K(r - y)$$

with $W_1 = I$ when there is no decoupling, and $K = 10I$. The closed loop system is

$$\dot{x} = Ax + BW_1K(r - Cx)$$
$$= (A - BW_1KC)x + BW_1Kr$$
$$= (A - W_1K)x + W_1Kr$$

where we have used that $B = I$, $C = I$. The closed loop poles are the eigenvalues of $A - W_1K$.

(i)

$$A - W_1K = \begin{bmatrix}-1.5 & 0.5\\ 0.5 & -1.5\end{bmatrix} - \begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix} = \begin{bmatrix}-11.5 & 0.5\\ 0.5 & -11.5\end{bmatrix}$$

which has eigenvalues in -11.5 ± 0.5, thas is in $s = -11$, $s = -12$.

(ii)

$$A - W_1K = \begin{bmatrix}-1.5 & 0.5\\ 0.5 & -1.5\end{bmatrix} - \begin{bmatrix}1.5 & -0.5\\ -0.5 & 1.5\end{bmatrix} = \begin{bmatrix}-16.5 & 5.5\\ 5.5 & -16.5\end{bmatrix}$$

which has eigenvalues in -16.5 ± 5.5, thas is in $s = -11$, $s = -22$.

10.6 Solution

a) The minors of the transfer matrix is given by

$$\frac{1}{s+1}, \quad \frac{0.5}{s+1}, \quad \frac{0.5}{(s+1)^2}$$

The smallest common denominator is $(s+1)^2$.The system thus have a double pole in -1 The maximum minor to the transfer matrix is $0.5/(s+1)^2$ which is already normed to have the pole polynomial $(s+1)^2$ as denominator. Hence, the system lacks zeros.

b) We get

$$\text{RGA} = G(0) \circ (G^{-1}(0))^T = \begin{bmatrix} 1 & 0.5 \\ 0 & 0.5 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We conclude that we should pair u_1 with y_1 and u_2 with y_2.

c) The SISO loop gain where y_1 is controlled by u_1 is

$$L_1(s) = K_1 \left(1 + \frac{1}{sT_1}\right) \frac{1}{s+1}$$

which for $s = i\omega$ has the argument

$$\arg(L_1(i\omega)) = -\frac{\pi}{2} + \arctan(\omega T_1) - \arctan \omega$$

The desired phase margin 90° at $\omega = 1$ yield the condition

$$\arg(L_1(i1)) = -\frac{\pi}{2}$$

which is fulfilled for $T_1 = 1$. This choice of T_1 yield

$$L(s) = K_1 \left(1 + \frac{1}{s}\right) \frac{1}{s+1} = \frac{K_1}{s}$$

To get the crossover frequency $\omega_c = 1$ we must have $|L(i1)| = 1$ which is fulfilled for $K_1 = 1$. A similar derivation yield $T_2 = 1$ and $K_2 = 2$.

d) We have

$$\begin{bmatrix} \frac{1}{s+1} & \frac{0.5}{(s+1)^2} \\ 0 & \frac{0.5}{s+1} \end{bmatrix} \begin{bmatrix} 1 & W_{12}(s) \\ W_{21}(s) & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} + \frac{0.5}{(s+1)^2} W_{21}(s) & \frac{1}{s+1} W_{12}(s) + \frac{0.5}{(s+1)^2} \\ \frac{0.5}{s+1} W_{21}(s) & \frac{0.5}{s+1} \end{bmatrix}$$

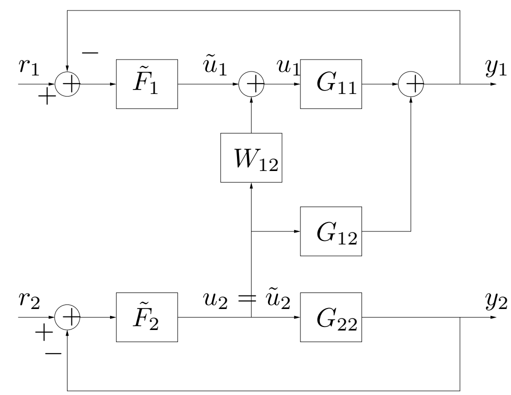
which need to be diagonal in order to decouple the system. This implies the following condition on W_{12} and W_{21}:

$$\frac{1}{s+1} W_{12}(s) + \frac{0.5}{(s+1)^2} = 0, \qquad \frac{0.5}{s+1} W_{21}(s) = 0$$

with the solution $W_{21}(s) = 0$ and $W_{12}(s) = -0.5/(s+1)$. To draw the block diagram we note

$$\begin{aligned} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} &= \begin{bmatrix} 1 & -\frac{0.5}{s+1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{U}_1(s) \\ \tilde{U}_2(s) \end{bmatrix} \\ &= \begin{bmatrix} \tilde{U}_1(s) - \frac{0.5}{s+1} U_2(s) \\ \tilde{U}_2(s) \end{bmatrix} \end{aligned}$$

The block diagram is given below



11 $\mathcal{H}_\infty$-control and $\mathcal{H}_2$-control

11.1 Solution

11.2 Solution

a) Taking the Laplace transform of the state space equations yields

$$\begin{aligned} sX(s) &= -X(s) + U(s) + W(s) \\ Z(s) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} X(s) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U(s) \\ Y(s) &= W(s) \end{aligned}$$

from which it follows that $X(s) = \frac{1}{s+1}(U(s) + W(s))$ and that $Z_1(s) = U(s)$, $Z_2(s) = X(s) = \frac{1}{s+1}(U(s) + W(s))$, and $Y(s) = W(s)$.

b) We first check that the state space description is on the correct form ((10.7) in the Swedish course book) and that the condition $D^T \begin{bmatrix} M & D \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$ is fulfilled. Then note that the problem is an $\mathcal{H}_\infty$-optimal control problem. Solve the Riccati equation

$$-S - S + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} + S(\gamma^{-2} - 1)S = 0$$

for $\gamma = 1$. The solution is $S = 1/2$ which is positive and therefore positive semidefinite.This yields $L = S = 1/2$. Here $-1 - 1 \times L = -3/2$ is stable and $L = 1/2$ is the desired solution.

c) Study the same problem as above but with 1 changed to γ. That is, determine L such that $\|G_{ec}\|_\infty \leq \gamma$ with an internally stable closed loop. Then find the smallest γ for which a solution exist. Solve the Riccati equation

$$-S - S + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} + S(\gamma^{-2} - 1)S = 0$$

For arbitrary γ. If $\gamma \neq 1$ the solutions is

$$S = \frac{1}{\gamma^{-2} - 1} \pm \frac{\sqrt{2 - \gamma^{-2}}}{|\gamma^{-2} - 1|}$$

There is a real solution iff $2 - \gamma^{-2} \geq 0$ which is equivalent to $\gamma \geq 1/\sqrt{2}$. For $\gamma = 1/\sqrt{2}$ we get $S = 1$ which is positive semidefinite. This yilds $L = S = 1$. For this L we get $A - BL = -1 - 1 \times L = -2$, which is stable. Hence $L = 1$ is the solution that minimizes $\|G_{ec}\|_\infty$ and yields an internally stable closed loop. The controller transfer function is derived by taking the Laplace transform of the controller state-space equations

$$s\hat{X}(s) = -\hat{X}(s) + U(s) + Y(s); \qquad U(s) = -\hat{X}(s)$$

Simplifications give that $U(s) = -\frac{1}{s+2}Y(s)$.

d) The closed-loop transfer function can be calculated from

$$\begin{aligned} Z(s) &= \begin{bmatrix} 1 & 0 \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} U(s) \\ W(s) \end{bmatrix} \\ Y(s) &= W(s) \\ U(s) &= -KY(s) = -KW(s) \end{aligned}$$

This implies that

$$Z(s) = \underbrace{\begin{bmatrix} -K \\ \frac{1-K}{s+1} \end{bmatrix}}_{G_{ec}(s)} W(s)$$

Furthermore we have

$$\bar{\sigma}^2\left(G_{ec}(i\omega)\right) = G_{ce}^T(-i\omega)G_{ec}(i\omega) = K^2 + \frac{(1-K)^2}{\omega^2 + 1}$$

It follows that

$$\|G_{ec}\|_\infty^2 = \sup_\omega \bar{\sigma}^2\left(G_{ec}(i\omega)\right) = K^2 + (1-K)^2 =: f(K)$$

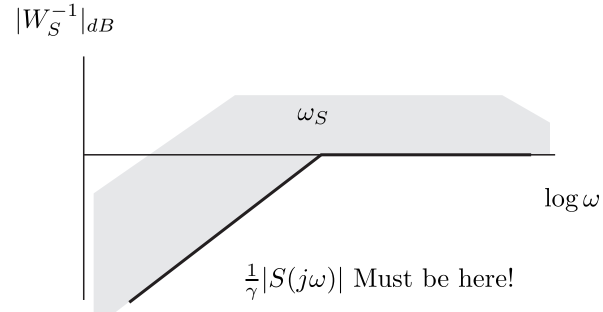
Furthermore $f'(K) = 4K - 2 = 0$ for $K = 1/2$ and $f''(K) = 4 > 0$, so $f(K)$ is minimized by $K = 1/2$. Since $f(1/2) = 1/2$ we find that

$$\min_K \|G_{ec}\|_\infty = \frac{1}{\sqrt{2}}$$

11.3 Solution

$$\|G_{ec}\|_\infty < \gamma \Rightarrow \begin{cases} |W_S(j\omega)S(j\omega)| & < & \gamma & \forall\omega \\ |W_T(j\omega)T(j\omega)| & < & \gamma & \forall\omega \\ |W_U(j\omega)G_{wu}(j\omega)| & < & \gamma & \forall\omega \end{cases}$$

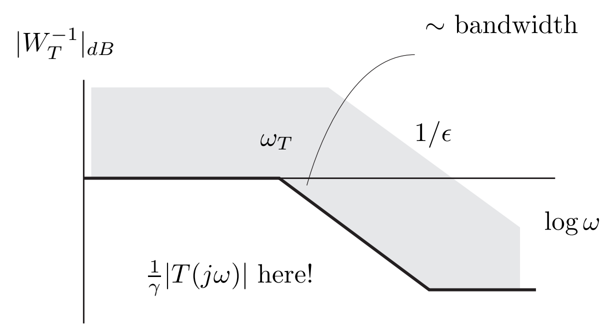
Hence we have $|S(j\omega)| < \gamma|W_S^{-1}(j\omega)|$ and $|T(j\omega)| < \gamma|W_T^{-1}(j\omega)|$.



a)

The smaller $\frac{1}{\gamma}|S(j\omega)|$ is for low frequencies, the better the attenuation of low frequency disturbances. From the figure we see that ω_S increases $\Rightarrow$ better attenuation.

b) We have $W_T^{-1} = \dfrac{\omega_T(1 + \epsilon s)}{s + \omega_T}$



If ω_T increases the bandwidth may increase.

11.4 Solution

If $\|G_{ec}\| < \gamma$ we have

$$\begin{gathered} |W_S(i\omega)S(i\omega)| < \gamma \ \ \forall\, \omega \\ |W_T(i\omega)T(i\omega)| < \gamma \ \ \forall\, \omega \\ \Longleftrightarrow \\ |S(i\omega)| < \gamma|W_S^{-1}(i\omega)| \\ |T(i\omega)| < \gamma|W_T^{-1}(i\omega)| \end{gathered}$$

a)

$$W_T^{-1} = \frac{1+\epsilon s}{1+\frac{s}{\omega_T}}$$

In the Bode plot $\frac{1}{\gamma}|T(i\omega)|$ will be in between ω_T and $1/\epsilon$.
If we increase ω_T, $|T|$ may increase betwen ω_T and $1/\epsilon \Longrightarrow$ worse robustness.
We have $|T| < \frac{1}{|\Delta G|}$ (Equation (6.33) in the Swedish course book) $\Longrightarrow$ stable.

b) Only

$$W_S(s) = \frac{s+\omega_S}{s}$$

yields $|W_S^{-1}(i\omega)| \to 0$ when $\omega \to 0 \Longrightarrow |S(i\omega)| \to 0 \Longrightarrow$ low-frequent process noise is eliminated.

11.5 Solution

11.6 Solution

11.7 Solution

a) Laplace transformation of the state space equations yield

$$\begin{aligned}
sX(s) &= -X(s) + U(s) + W(s) \\
Z(s) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} X(s) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U(s) \\
Y(s) &= W(s)
\end{aligned}$$

from which it follow that $X(s) = \frac{1}{s+1}(U(s)+W(s))$ and that $Z_1(s) = U(s)$, $Z_1(s) = X(s) = \frac{1}{s+1}(U(s)+W(s))$, and $Y(s) = W(s)$.

b) First check that the state space description is on the correct form ((10.7) in the Swedish course book) and check that the condition $D^T[M \ D] = [0 \ I]$ is fulfilled. Then note that the problem is an $\mathcal{H}_2$-problem for which we need to solve the Ricatti equation

$$-S - S + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} - S^2 = 0$$

to find the optimal state feedback. The solution is $S = \sqrt{2}-1$. Hence $L = S = \sqrt{2}-1$. The transfer function is derived by Laplace transforming the controller equations

$$s\hat{X}(s) = -\hat{X}(s) + U(s) + Y(s); \qquad U(s) = -(\sqrt{2}-1)\hat{X}(s)$$

which yield $U(s) = -\frac{\sqrt{2}-1}{s+\sqrt{2}}Y(s)$.

c)

$$\begin{aligned}
Z(s) &= \begin{bmatrix} 1 & 0 \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} U(s) \\ W(s) \end{bmatrix} \\
Y(s) &= W(s) \\
U(s) &= -\frac{\sqrt{2}-1}{s+\sqrt{2}}Y(s) = -\frac{\sqrt{2}-1}{s+\sqrt{2}}W(s)
\end{aligned}$$

This implies that

$$Z(s) = \underbrace{\frac{1}{s+\sqrt{2}} \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}}_{G_{ec}(s)} W(s)$$

Furthermore it holds that

$$\begin{aligned}
\|G_{ec}\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}\left(G_{ec}(i\omega)G_{ec}^T(-i\omega)\right) d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_{ec}^T(-i\omega)G_{ec}(i\omega) d\omega \\
&= \frac{2-\sqrt{2}}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\left(\frac{\omega}{\sqrt{2}}\right)^2+1} d\omega \\
&= \frac{2-\sqrt{2}}{2\pi}\left[\sqrt{2}\arctan\frac{\omega}{\sqrt{2}}\right]_{-\infty}^{\infty} = \sqrt{2}-1
\end{aligned}$$

Hence $\|G_{ec}\|_2 = \sqrt{\sqrt{2}-1}$.

d) The closed loop transfer function is calculated from

$$\begin{aligned}
Z(s) &= \begin{bmatrix} 1 & 0 \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} U(s) \\ W(s) \end{bmatrix} \\
Y(s) &= W(s) \\
U(s) &= -KY(s) = -KW(s)
\end{aligned}$$

Hence

$$Z(s) = \underbrace{\begin{bmatrix} -K \\ \frac{1-K}{s+1} \end{bmatrix}}_{G_{ec}(s)} W(s)$$

Furthermore we have

$$\begin{aligned}
\|G_{ec}\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}\left(G_{ec}(i\omega)G_{ec}^T(-i\omega)\right) d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_{ec}^T(-i\omega)G_{ec}(i\omega) d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[K^2 + \frac{(1-K)^2}{\omega^2+1}\right] d\omega
\end{aligned}$$

The integral is finite only if $K = 0$. Hence $K = 0$ minimize $\|G_{ec}\|_2^2$. The value of the integral for $K = 0$ is given by

$$\frac{1}{2\pi}\left[\arctan\omega\right]_{-\infty}^{\infty} = \frac{1}{2}$$

So for proportional control the minimum value of the norm is $\|G_{ec}\|_2 = \frac{1}{\sqrt{2}}$.

11.8 Solution

(a) Disturbance attenuation is needed for frequencies where $|G_d(i\omega)| > 1$, and $|G_d| = 3/\sqrt{100\omega^2+1}$ which equals one for $\omega_d = \sqrt{0.08} = 0.28$. The limitations are partly due to the RHP zero in G_1 which gives that disturbance attenuation can only be achieved up to $\omega = z = 4$ (or $\omega = z/2 = 2$ if we want to limit the peak of S to 2), and the delay in G_2 which gives that attenuation can only be achieved approximately up to $\omega = 2/\theta = 1$. Neither of these are thus a problem since they exceed the required ω_d. However, we also need sufficient input to counteract the disturbance, i.e., $|G| > |G_d| \forall\omega < \omega_d$ for perfect disturbance attenuation (for acceptable attenuation it suffices that $|G| > |G_d| - 1$). We see that $|G_1| > |G_d| \forall\omega$ which $|G_2| < |G_d| - 1 \forall\omega$ and hence we can use u_1 only, but not u_2 only.

(b) Requirement is $|SG_d| < 1 \forall\omega$. We try with a simple P-controller $F_y = K_c$, which gives $S = 1/(1 + G_1 K_c) = (10s+1)/(10s+1+K_c(s-4))$ and $SG_d = 3e^{-s}/((10+K_c)s + (1-4K_c))$ which is stable and with magnitude $< 1\forall\omega$ if $-10 < K_c < -0.5$.

(c) The requirement for $|y| < 1$ with $|d| < 1$ translates to $\|SG_d\|_\infty < 1$, the requirement for $|u| < 1$ with $|d| < 1$ translates into $\|F_y SG_d\|_\infty < 1$ and the robust stability requirement $\|0.2T\|_\infty > 1$. Stacked, this gives the optimization problem

$$\min_u \left\| \begin{array}{c} SG_d \\ F_y SG_d \\ 0.2T \end{array} \right\|_\infty$$

We have $y = SGdd$ and $u = F_y SG_d$ and hence we should have d as an input and $z = [y\ u]^T$ as an output of the extended system for the first two criteria. Unfortunately, it is difficult to find an output which has the transfer function T from the input d. Thus, we have to add another input, e.g., measurement noise n with gain 0.2 since then $y = 0.2Tn$. The disadvantage of adding the second input n is that we then also include the transfer function from n to u in the objective.

11.9 Solution

The H_∞-optimal controller is the one that minimizes $||W_p S||_\infty = \sup_\omega \bar{\sigma}(W_p S)$. Since for controller A we have $\sup_\omega \bar{\sigma}(W_p S) \simeq 3$, while for controller B $\sup_\omega \bar{\sigma}(W_p S) \simeq 1$, we can conclude that B is the H_∞-optimal controller.

12 Kalman filters and Linear Quadratic control

12.1 Solution

1. The block diagram yields

$$E(s) = G(s)U(s)$$

where

$$G(s) = \frac{s(1+sT)}{s(1+sT)+K}$$

which is stable for all $K > 0$. The spectrum of e is given by (5.15b) in the book

$$\Phi_e(\omega) = |G(i\omega)|^2\Phi_u(\omega)$$

$$|G(i\omega)|^2 = \left|\frac{i\omega - \omega^2 T}{K + i\omega - \omega^2 T}\right|^2 = \frac{\omega^2 + \omega^4 T^2}{(K-\omega^2 T)^2 + \omega^2}$$

yields

$$\Phi_e(\omega) = \frac{(\omega^2 + \omega^4 T^2)}{((K-\omega^2 T)^2 + \omega^2)} \frac{\omega_0^2}{(\omega^2 + \omega_0^2)}$$

2.

12.2 Solution

The double integrator

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

$$V = \int_0^\infty (y^2 + \eta \cdot u^2)\, dt \qquad \Rightarrow \qquad Q_1 = 1,\ Q_2 = \eta$$

Controlled quantity: $\quad z = y = Cx \quad \Rightarrow \quad M = C.$

$$A^T\bar{S} + \bar{S}A + M^T Q_1 M - \bar{S}BQ_2^{-1}B^T\bar{S} = 0$$

Let

$$\bar{S} = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} \Rightarrow$$

The above can be stated as

$$\begin{pmatrix} 0 & 0 \\ s_1 & s_2 \end{pmatrix} + \begin{pmatrix} 0 & s_1 \\ 0 & s_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{\eta} \cdot \begin{pmatrix} s_2^2 & s_2 s_3 \\ s_2 s_3 & s_3^2 \end{pmatrix} = 0$$

$$\begin{cases} 1 - \frac{s_2^2}{\eta} = 0 \\ s_1 - \frac{s_2 s_3}{\eta} = 0 \\ 2s_2 - \frac{s_3^2}{\eta} = 0 \end{cases}, \quad \bar{S} \text{ positive definite} \quad \Rightarrow \quad \begin{cases} s_1 = \sqrt{2} \cdot \eta^{1/4} \\ s_2 = \eta^{1/2} \\ s_3 = \sqrt{2} \cdot \eta^{3/4} \end{cases}$$

$$L = Q_2^{-1} B^T \bar{S} = \frac{1}{\eta} \cdot (0 \quad 1) \begin{pmatrix} \sqrt{2}\eta^{1/4} & \eta^{1/2} \\ \eta^{1/2} & \sqrt{2} \cdot \eta^{3/4} \end{pmatrix} =$$

$$= \frac{1}{\eta} \cdot (\eta^{1/2} \quad \sqrt{2}\eta^{3/4}) = (\eta^{-1/2} \quad \sqrt{2} \cdot \eta^{-1/4})$$

Now let $\mu = \eta^{-1/4}$. Then $L = (\mu^2 \quad \sqrt{2} \cdot \mu)$

$$u = -Lx \Rightarrow \dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot (\mu^2 \quad \sqrt{2} \cdot \mu) \cdot x =$$

$$= \begin{pmatrix} 0 & 1 \\ -\mu^2 & -\sqrt{2}\mu \end{pmatrix} x$$

$$0 = \begin{vmatrix} s & -1 \\ \mu^2 & s + \sqrt{2} \cdot \mu \end{vmatrix} = s^2 + \sqrt{2}\mu s + \mu^2$$

$$s = -\frac{\mu}{\sqrt{2}} \pm \sqrt{\frac{\mu^2}{2} - \mu^2} = -\frac{\mu}{\sqrt{2}} \pm i \cdot \frac{\mu}{\sqrt{2}} =$$

$$= -\frac{\mu}{\sqrt{2}} \cdot (1 \pm i) = -\frac{1}{\sqrt{2} \cdot \eta^{1/4}} \cdot (1 \pm i)$$

If η is decreased the poles distance to the origin will increase. This implies that $|L|$ increase. A greater $|L|$ implies that $u(t)$ will increase. Compare with the criterion.

12.3 Solution

$$\begin{aligned} z &= \frac{1}{p+1}u + \frac{1}{p+1}v & \Rightarrow \quad & G(s) = \frac{1}{s+1} \\ y &= z + e, & & \Phi_v(\omega) \equiv r_1, \quad \Phi_e(\omega) \equiv 1 \end{aligned}$$

$$\text{minimize } J = E\{q_1 z^2(t) + u^2(t)\}$$

a) Introduce the state $x_1 = z$ such that the system can be written as

$$\begin{cases} \dot{x}_1 &= -x_1 + u + v \\ y &= x_1 + e \\ z &= x_1 \end{cases}$$

That is $A = -1, B = 1, M = 1, Q_1 = q_1$ and $Q_2 = 1$. $Q_{12} = 0$ is assumed. We cannot measure the state but we measure y. According to the separation principle we minimize J if we

1. Estimate $\hat{x}(t)$ using a Kalmanfilter.
2. Use the feedback $u(t) = -L\hat{x}(t)$, where L is chosen using LQ-methodology.

i.e.,

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

where $K = PC^T R_2^{-1}$ and P is the positive semidefinite symmetric solution to the equation

$$AP + PA^T + NR_1N^T - PC^T R_2^{-1} CP = 0$$

which in our case becomes the scalar equation

$$P^2 + 2P - r_1 = 0$$

with solution

$$P = -1 + \sqrt{1 + r_1}$$

. In other words

$$K = -1 + \sqrt{1 + r_1}.$$

Use the feedback $u = -L\hat{x}$ where $L = Q_2^{-1} B^T S$, S is the solution to

$$A^T S + SA + M^T Q_1 M - SBQ_2^{-1} B^T S = 0$$

which with $M = 1, Q_1 = q_1$ and $Q_2 = 1$ yield

$$L = S = -1 + \sqrt{1 + q_1}.$$

The loop gain is then

$$\begin{aligned} G(s)F_y(s) &= \frac{1}{s+1} L \frac{1}{1 + s + L + K} K = \\ &= \frac{(-1 + \sqrt{1+r_1})(-1 + \sqrt{1+q_1})}{(s+1)(s - 1 + \sqrt{1+r_1} + \sqrt{1+q_1})} \end{aligned}$$

b) r_1 and q_1 has the same effect on the loop gain due to symmetry.

c)

$$G(s)F_y(s) = \frac{(-1 + \sqrt{1+r_1})(-1 + \sqrt{1+q_1})}{(s+1)(s - 1 + \sqrt{1+r_1} + \sqrt{1+q_1})}$$

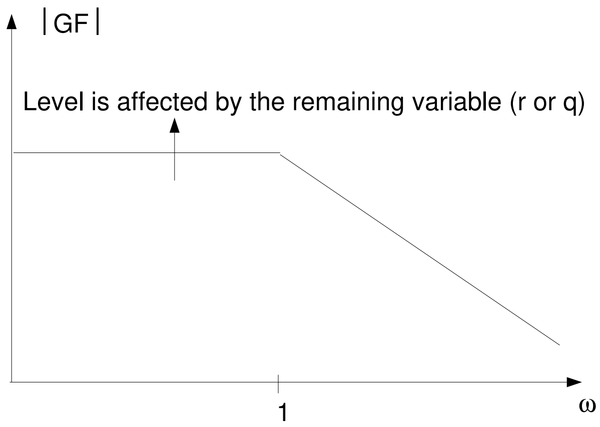
Let r_1 or $q_1 \to \infty$. What happens?

$r_1 \to \infty \Rightarrow G(s)F_y(s) = \frac{(-1+\sqrt{1+q_1})}{(s+1)\frac{(s-1+\sqrt{1+r_1}+\sqrt{1+q_1})}{-1+\sqrt{1+r_1}}} \to \frac{-1+\sqrt{1+q_1}}{s+1}$

Similarily

$$\lim_{q_1 \to \infty} G(s)F_y(s) = \frac{-1 + \sqrt{1+r_1}}{s+1}$$

By varying q_1 and/or r_1 the loop gain may be shaped according to the sketch



12.4 Solution

i) Determine a Kalman filter: $\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$

ii) Use state feedback: $u = -L\hat{x}$, with L according to LQ-theory

i) + ii) yield the regulator $F_y = L(sI - A + BL + KC)^{-1}K$.

i) Statespace description:

Let $x_1 = z,\ x_2 = \nu,\ v_1 = v,\ v_2 = e$ and $x = (x_1, x_2)^T$. We then get

$$\begin{cases} \dot{x} = \underbrace{\begin{pmatrix} -1 & 1 \\ 0 & -\varepsilon \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{B} u + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{N} v_1 \\ y = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{C} x + v_2 \end{cases}$$

Furthermore we have

$$\begin{aligned} R_1 &= \Phi_{v_1}(\omega) = \Phi_v(\omega) = 1 \\ R_2 &= \Phi_{v_2}(\omega) = \Phi_e(\omega) = 1 \\ R_{12} &= \Phi_{v_1 v_2} = 0 \end{aligned}$$

The Kalman filter is given by: $K = PC^T R_2^{-1}$ with P from

$$AP + PA^T + NR_1N^T - PC^TR_2^{-1}CP = 0$$

this equation is solved by letting $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$. In solving the Riccati equation (for P) we can disregard ε and let $\varepsilon = 0$.

This yields $P = \begin{pmatrix} \sqrt{3}-1 & 1 \\ 1 & \sqrt{3} \end{pmatrix}$ and

$K = PC^TR_2^{-1} = \begin{pmatrix} \sqrt{3}-1 \\ 1 \end{pmatrix}$

ii) Determine L such that

$$\min_L \int_0^\infty [x_1^2(t) + u^2(t)]dt = \min_L \int_0^\infty [y^TQ_1y + u^TQ_2u]dt$$

where $Q_1 = Q_2 = 1$.

The optimal L is given by: $L = Q_2^{-1}B^TS$ where S fulfill

$$A^TS + SA + C^TQ_1C - SBQ_2^{-1}B^TS = 0.$$

This equation is solved by letting $S = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}$. Here we cannot let $\varepsilon = 0$ initially. The Riccati equation for S becomes

$$\begin{aligned} 0 &= \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -\varepsilon \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & -\varepsilon \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} \\ &\quad + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} \end{bmatrix} \\ L &= \begin{bmatrix} s_{11} & s_{12} \end{bmatrix} \end{aligned}$$

Evaluate the elements. This yields the following equations

$$\begin{aligned} &\text{11 elements :} && 0 = -2s_{11} + 1 - s_{11}^2 \Rightarrow s_{11} = \sqrt{2} - 1 \\ &\text{12 elements :} && 0 = s_{11} - \varepsilon s_{12} - s_{12} - s_{11}s_{12} \Longrightarrow \\ & && s_{12} = \frac{s_{11}}{1 + s_{11} + \varepsilon} = \frac{\sqrt{2}-1}{\sqrt{2}+\varepsilon} \longrightarrow 1 - \frac{1}{\sqrt{2}},\ \varepsilon \longrightarrow 0 \\ &\text{22 elements :} && 0 = 2s_{12} - 2\varepsilon s_{22} - s_{12}^2 \Longrightarrow s_{22} = \frac{1}{2\varepsilon}(2s_{12} - s_{12}^2) \approx \frac{1}{4\varepsilon}, \quad \text{små } \varepsilon \end{aligned}$$

This yield $L = Q_2^{-1}B^TS = \begin{pmatrix} \sqrt{2}-1 & 1 - \frac{1}{\sqrt{2}} \end{pmatrix}$

The LQG-controller becomes

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}) \\ u = -L\hat{x} \end{cases}$$

with K and L according to the above.

The static gain of the sensitivity:

i) and ii) yields

$$\begin{aligned} F_y &= L(sI - A + BL + KC)^{-1}K \underset{s=0}{=} \\ &= \begin{pmatrix} l_1 & l_2 \end{pmatrix} \begin{pmatrix} 1 + l_1 + k_1 & -1 + l_2 \\ k_2 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \ldots = \\ &= \frac{l_1 + l_2}{1 - l_2} = 1 \quad \Rightarrow \\ &\Rightarrow S(0) = \frac{1}{1 + F_y(0)G(0)} = \frac{1}{1+1} = \frac{1}{2} \end{aligned}$$

13 Discrete-time systems and Model Predictive Control

13.1 Solution

The sampled system is on the form

$$\begin{aligned} x_{k+1} &= Fx_k + Gu_k \\ y_k &= Hx_k \end{aligned}$$

with

$$F = e^{AT}, \quad G = \int_0^T e^{At} B dt, \quad H = C$$

We determine e^{At} using the inverse Laplace transform

$$\begin{aligned} e^{At} &= \mathcal{L}^{-1}\left\{ \begin{bmatrix} s & -\omega \\ \omega & s \end{bmatrix}^{-1} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s^2+\omega^2} \begin{bmatrix} s & \omega \\ -\omega & s \end{bmatrix} \right\} \\ &= \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} \end{aligned}$$

So we get

$$F = \begin{bmatrix} \cos\omega T & \sin\omega T \\ -\sin\omega T & \cos\omega T \end{bmatrix}$$

and

$$G = \int_0^T \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt = \int_0^T \begin{bmatrix} \cos\omega t \\ -\sin\omega t \end{bmatrix} dt = \frac{1}{\omega} \begin{bmatrix} \sin\omega T \\ \cos\omega T - 1 \end{bmatrix}.$$

The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ \cos\omega T & \sin\omega T \end{bmatrix} \Rightarrow \det\mathcal{O} = \sin\omega T = 0 \Leftrightarrow T = \frac{k\pi}{\omega}, \ k \in \mathbf{Z}$$

Hence, the sampled system is observable if $T \neq |\frac{k\pi}{\omega}|$ for $k \in \mathbf{Z}$

13.2 Solution

13.3 Solution

With the prediction horizon $Np = 1$, the objective function becomes

$$V = y_k^2 + y_{k+1}^2 + u_k^2$$

With the state-space model inserted

$$V = 2y_k^2 - 4y_k u_k + 5u_k^2$$

As the value of y_k is assumed given, the first term of the objective function will not affect the optimal input sequence and can be removed. Thus, $H = 5$ and $h = -4y_k$.

The constraints give

$$\underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{L} u_k \leq \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{b}$$