EL2520 - Control Theory and Practice - Advanced Course
Solution/Answers – 2017-06-05

1. (a) (i) The determinant is $\det G = \frac{s}{(5s+1)^2}$ and the denominator is the LCD for all minors. Hence there are two poles in $s = -0.2$ and one zero at $s = 0$. (ii) The zero at $s = 0$ implies that the system is singular at steady-state and hence we can not control the two outputs independently at steady-state. (The correlation between the two outputs is given by direction orthogonal to the zero output direction $y_z^H = [-1\ 1]$, i.e., $y = [1\ 1]$).

   (b) (i) We get $S = \frac{s}{(1+K)s-K}$ and $w_P S = 0.5\frac{s+1}{(1+K)s-K}$. The closed loop is stable for $K \in [-1, 0]$. As noted in the hint, the peak value of $|w_P S(i\omega)|$ will occur at $\omega = 0$ or $\omega = \infty$ (depending on the size of the pole relative to the zero). At $\omega = 0$ we get $|w_P S| = -0.5/K$ and at $\omega = \infty$ we get $|w_P S| = 0.5/(1 + K)$. Thus, we get the peak at $\omega = 0$ for $K \in [-0.5, 0]$ and at $\omega = \infty$ for $K \in [-1, -0.5]$. For both intervals the minimum is obtained for $K = -0.5$, which hence is the optimal gain. The corresponding $\|w_P S\|_\infty = 1$ and hence we just satisfy the performance requirement. (ii) We know that $S(z) = 1$ for RHP zero in G at $s = z$. Then, from Maximum Modulus Thm we get that $\|w_P S\|_\infty > |w_P(z)|$. Here $z = 1$ and $|w_P(1)| = 4/3 > 1$ and hence it is not possible to achieve $\|w_P S\|_\infty < 1$ with any controller.

2. (a) The RGA is $G \times (G^{-1})^T$. For 2×2 systems $\lambda_{11} = \lambda_{22} = \frac{1}{1-\frac{G_{12}G_{21}}{G_{11}G_{22}}}$ and $\lambda_{12} = \lambda_{21} = 1 - \lambda_{11}$. At $\omega = 0$ we get $\lambda_{11} = -10$ and $\lambda_{12} = 11$. Since we should not pair on negative steady-state RGA elements, this suggests the off-diagonal pairing as the only viable option. At the expected bandwidth $\omega = 1$ we get $\lambda_{11}(i1) = 1/(1 - 1.1(i + 1))$ and $\lambda_{12} = 1 - \lambda_{11}$ then gives $|\lambda_{12}(i1)| = 1.41$ which is close to 1 and hence implies that there are relatively weak interactions around the bandwidth and we should expect decentralized control to work well with the off-diagonal pairing.

   (b) The transfer matrix $G(s)$ has a RHP zero at $s = 1$. For internal stability, this zero needs to be retained in the complementary sensitivity $T(s)$. We have

   $$T = I - S = \frac{1}{10s+1}\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}$$

   which has no zeros whatsoever. Thus, the controller has cancelled the RHP zero by a corresponding RHP pole and the closed-loop is not internally stable.

   (c) All RHP poles in $G(s)$ must be retained as RHP zeros in $S(s)$ to satisfy internal stability. Here G is unstable, i.e., has RHP poles, while S has no RHP zeros, and hence a RHP pole zero cancellation has occured between the controller and plant and the closed loop is not internally stable.

3. (a) We have $y = 2\hat{y}$, $u = \hat{u}/3$, $d = 10\hat{d}$. This gives

$$y = 2 \cdot 3\hat{G}u + 2 \cdot 0.1\hat{G}_d d = \frac{6}{2s+1}e^{-4s}u + \frac{1.6}{(2s+1)(5s+1)}e^{-2s}d$$

(b) The control has a delay of 4 time units, which gives an approximate upper bound $\omega_B < 1/4$ for the sensitivity function S. To determine the frequency range where we need sensitivity reduction, i.e., $|S| < 1$, we consider frequencies for which $|G_d| > 1$. Since G_d has low pass characteristics (no zeros,only real poles), it suffices to check $|G_d(i\omega)| = 1.6/(\sqrt{4\omega^2+1}\sqrt{25\omega^2+1})$ for $\omega = 0.25$ to get $|G_d(i0.25)| = 0.89$ which is smaller than 1 and it suffices to have $|S| < 1$ for $\omega < 0.25$ and the time delay should hence not represent a problem. We also need to check whether we have sufficient input for disturbance attenuation. We consider first the input required for perfect disturbance attenuation, which is $u = |G^{-1}G_d| = |(4/15)/(\sqrt{25\omega^2+1})|$ which is less than 1 at all frequences and hence we sufficient input for disturbance attenuation. In summary, we should be able to obtain acceptable disturbance attenuation using feedback only.

(c) The requirements are $\|SG_d\|_\infty < 1$, $\|F_y SG_d\|_\infty < 1, \|0.3T\|_\infty < 1$ and we stack all of these into one matrix that we minimize the norm of

$$\min_u \left\| \begin{vmatrix} SG_d \\ F_y SG_d \\ 0.3T \end{vmatrix} \right\|_\infty$$

Wee have $y = SGdd$ and $u = F_y SG_d$ and hence we should have d as an input and $z = [y\ u]^T$ as an output of the extended system for the first two criteria. Unfortunately, it is difficult to find an output which has the transfer function T from the input d. Thus, we have to add another input, e.g., measurement noise n with gain 0.3 since then $y = 0.3Tn$. The disadvantage of adding the second input n is that we then also include the transfer function from n to u in the objective.

4. (a) Here $G(s)$ has a RHP zero at $s = 1$ and the corresponding output direction is $y_z = \frac{1}{\sqrt{2}}[-1\ 1]^T$. Then, based on the fact that $S(z) = 1$ and the Maximum Mod Thm, we get $|y_z^H G_d(z)| < 1$ as a requirement for $\|SG_d\|_\infty < 1$, i.e., acceptable disturbance attenuation. For d_1 we get $|y_z^H G_{d1}(1)| = 0$ and hence no problems with attenuation due to the RHP zero. For d_2 we get $|y_z^H G_{d2}(1)| = 6/\sqrt{2} > 1$ and hence it is not possible to get acceptable attenuation of disturbance d_2 due to

the existence of the RHP zero. (The first disturbance is othogonal to the zero direction, while the second one is alligned with the zero direction).

(b) (i) We get $F = e^{AT} = 1$ and $G = \int_0^T e^{A\tau} B d\tau = T$. The eigenvalue is 1 which corresponds to a pure integrator, and this corresponds well with the eigenvalue at 0 in the continuous time system.
(ii) With $N = 1$, $J = k_1(x(k+1) - r(k+1))^2 + q_2 u^2(k)$. We have $x(k+1) = x(k) + Tu(k)$ and inserting this in J gives

$$J = q_1(x(k) + Tu(k) - r(k+1))^2 + q_2 u^2(k)$$

Minimizing this with respect to $u(k)$s corresponds to a quadratic program and is hence convex. Taking the derivative $dJ/du(k) = 2q_1(x(k) + Tu(k) - r(k+1))T + 2q_2 u(k)$ and setting it to zero yields the control law

$$u(k) = \frac{Tq_1(r(k+1) - x(k))}{T^2 q_1 + q_2}$$

The closed-loop is then

$$x(k+1) = x(k) + T\frac{Tq_1(r(k+1) - x(k))}{T^2 q_1 + q_2}$$

and the eigenvalue is then $1 - T^2 q_1/(T^2 q_1 + q_2)$ which is always in the range $[0, 1]$, and hence the closed loop stable, for all positive q_1, q_2. As $q_1 \to \infty$ we get that the eigenvalue goes to zero which corresponds to bringing the system to steady-state in one sample (known as dead beat control).

5. (a) Since the uncertainty implies that a pole may cross the imaginary axis we would need an unstable perturbation in the multiplicative set and it is then not possible to employ the small gain theorem to derive a robust stability criterion (the standard robust stability criterion assumes both nominal stability and a stable perturbation).

(b) Employ the inverse multiplicative uncertainty $G_p = G(1+W_{iI}\Delta_{iI})^{-1}$. With this description is it possible to represent poles crossing the imaginary axis even with a stable perturbation $\Delta_{iI}(s)$. By drawing the description in a block-diagram, then rewriting the diagram on $M - \Delta$ form we find that $M = W_{iI}S$ and according to the SGT we then have robust stability if $W_{iI}(s)S(s)$ stable, $\Delta_{iI}(s)$ and $\|W_{iI}S\Delta_{iI}\|_\infty < 1$. With $\|\Delta_{iI}\|_\infty < 1$ we get the RS condition $\|W_{iI}S\|_\infty < 1$.

(c) We can write

$$2\frac{(s+1)(s+z)}{(5s+1)^2(s+p)} = 2\frac{(s+1)}{(5s+1)^2}(1 + \Delta_G(s))^{-1}$$

which gives

$$1 + \Delta_G = \frac{s+p}{s+z} \quad \Rightarrow \quad \Delta_G = \frac{p-z}{s+z}$$

The maximum amplitude of Δ_G is obtained for $p - z = 4$ and $z = 1$ (somewhat conservative choice). Thus the uncertainty can be represented by an inverse multiplicative uncertainty with $\|\Delta_{iI}\|_\infty < 1$ and

$$W_{iI}(s) = \frac{4}{s+1}$$

We then get that the RS condition $\|SW_{iI}(s)\|_\infty < 1$ implies that

$$|S| \leq \frac{|i\omega + 1|}{4} \quad \forall \omega$$

which implies $|S| < 1$ for $\omega < \sqrt{15}$, hence we need a bandwidth of at least $\sqrt{15} = 3.873\ rad/s$ to guarantee robust stability.
(Remark: it would obviously have been better to use $z = 2$ for the nominal model, as this is the center value).