EL2520 - Control Theory and Practice - Advanced Course
Solution/Answers – 2018-05-29

1. (a) (i) The determinant is $\det G = \frac{2(s-1)}{(s+2)^2}$ and the denominator is the LCD for all minors. Hence there are two poles in $s = -2$ and one zero at $s = 1$. The zero direction is defined as $y_z^H G(z) = 1$, such that $|y_z| = 1$ and we get $y_z^H = \frac{1}{\sqrt{5}}[-2\ 1]$. (ii) With two poles we need 2 states in a minimal state space realization.

   (b) The singular values are given by $\sigma_i(i\omega) = \sqrt{\lambda(G(i\omega)^H G(i\omega))} = \frac{1}{\sqrt{4\omega^2+1}}\sqrt{\lambda(G(0)^H G)}$ (here one can also use the fact that the singular values of a symmetric matrix equals the magnitude of the eigenvalues). This gives $\bar{\sigma} = \frac{3}{\sqrt{4\omega^2+1}}$ and $\underline{\sigma} = \frac{1}{\sqrt{4\omega^2+1}}$. We have $\|G\|_\infty = \sup_\omega \bar{\sigma}(G) = 3$.

   (c) The closed-loop eigenvalue is $2 - K$ and the system is stable for $-1 < 2 - K < 1$ which gives $1 < K < 3$.

2. (a) Since $\bar{\sigma}(S) > |1 - \bar{\sigma}(T)|$ we can not have $\bar{\sigma}(S) < 0.5$ when $\bar{\sigma}(T) < 0.5$ (and vice versa) at any frequency. Hence we can not have the magnitude of both weights > 2 at the same frequency and get $\|W_S S\|_\infty < 1$ and $\|W_T T\|_\infty < 1$. Here both weights are larger than 2 around frequency 0.3 rad/s and hence the objective is infeasible.

   (b) For internal stability we require that the sensitivity function S has a RHP zero at the same position as $G(s)$ has a RHP pole. In this case we have a pole at $s = 1$ in $G(s)$, but the only zero of $S(s)$ is at $s = 0$. Hence the proposed controller does not give internal stability since there is a pole-zero cancellation in the RHP between the plant and the controller.

   (c) Here $G(s)$ has a RHP zero at $s = 1$. For any RHP zero $s = z > 0$ we require $|y_z^H G_d(z)| < 1$ where y_z is the corresponding zero direction. We get $y_z^H = \frac{1}{\sqrt{13}}[3\ \ -2]$ and $G_d(z) = \frac{1}{11}[2\ 3]^T$ which gives $|y_z^H G_d(z)| = 0 < 1$. Since there are no other limitations, the objective is feasible.

3. (a) (i) We get $T = K/(s - 1 + K)$ and $W_T T = \frac{2K}{3}\frac{s+1}{s-1+K}$. Closed-loop stability for $K > 1$. The peak-value of $|W_T T(i\omega)|$, corresponding to $\|W_T T\|_\infty$, occurs at $\omega = 0$ if $1 < K < 2$ and at $\omega = \infty$ if $K > 2$. The peak values are $2K/(3(-1 + K))$ and $2K/3$, respectively. Hence, the minimum is obtained for $K = 2$ corresponding to $\|W_T T\|_\infty = 4/3$. (ii) Since there is a RHP pole at $s = p = 1$ and $|W_T(p)| = \frac{4}{3} > 1$ it is not possible to achieve $\|W_T T\|_\infty < 1$.

   (b) (i) We have $y = GS_I w_u = SGw_u$ and hence we require $\|W_1 SG\|_\infty < 1$ with $W_1 = \frac{1}{2}\frac{s+1}{s}$ which enforces $\bar{\sigma}(SG(i0.5)) < 1$ and $\bar{\sigma}(SG(0)) =$

0. Robust stability with relative uncertainty at the input is ensured if $\|W_2T_I\|_\infty < 1$ where $W_2 = 0.2$, corredsponding to the relative uncertainty. To get a scalar objective function value we stack the two objectives into one matrix to obtain

$$\min_{F_y} \left\| \begin{matrix} W_1SG \\ W_2T_I \end{matrix} \right\|_\infty$$

(ii) For SG we choose $z_{e1} = W_1y = W_1SGw_u$. For T_I we have that $u = u_c + w_u$ where u_c is the output of the controller F_y, and hence $u_c = F_yG(I + F_yG)^{-1}w_u = T_Iw_u$. Hence we choose $z_{e2} = W_2u_c = W_2T_Iw_u$. Hence, with $w_e - w_u$ we get in closed=loop

$$z_e = \begin{pmatrix} W_1SG \\ W_2T_I \end{pmatrix} w_e$$

Now the signal minimization problem given corresponds to minimizing the $\mathcal{H}_\infty$-norm of the transfer-function above.

4. (a) The system is on state space form

$$\dot{y} = 2u(t)$$

(i) This is an LQ problem with $A = 0, B = 2, M = 1, Q_1 = 1, Q_2 = 0.1$. The Riccati equation is then

$$1 - 40P^2 = 0$$

and the solution $P > 0$ is $P = 1/(2\sqrt{10})$. The optimal feedback gain is then $L = 20P = \sqrt{10}$. Thus, optimal $K_P = \sqrt{10}$. (ii) We have $A = 0$, $R_1 = 0$, $R_2 = 0.1$, $C = 1$ for the Riccati equation for the Kalman filter, yielding

$$10P^2 = 0$$

and hence $P = 0$. Thus, the optimal estimate is

$$\dot{\hat{y}} = 2u(t)$$

(Note that there is no feedback from the measurement. This will by some not be considered a Kalman filter, as it often is a requirement that a process disturbance, i.e., $R_1 \neq 0$, is included in the model in which case we would get $P \neq 0$.) The estimator does not influence the optimal gain in (i) due to the separation principle.

(b) (i) The state-space model is $\dot{y} = 2u(t)$ and hence, in discrete time, $A = e^{0T} = 1$ and $B = \int_0^T 2e^{0T}dt = 2T = 1$. Thus, the discrete time state space model is

$$y_{k+1} = y_k + u_k$$

(ii) With $N = 1$ we get the objective function to be minimized

$$V = Q_y y_k^2 + Q_y(y_k + u_k)^2 + u_k^2 + u_{k+1}^2 = 2Q_y y_k^2 + 2Q_y y_k u_k + (Q_y + 1)u_k^2 + u_{k+1}^2$$

Since we can not influence y_k (the current value of y), we can remove the terms with only y_k. Also, u_{k+1}^2 can be removed since obviously the optimum value will be $u_{k+1} = 0$. Thus,

$$V = (Q_y + 1)u_k^2 + 4Q_y y_k u_k$$

and hence $H = Q_y + 1$ and $h = 4Q_y y_k$. The constraint $|u_k| < 1$ can be written as $u_k < 1$ and $-u_k < 1$, i.e., $L = [1\ \ -1]^T$ and $b = [1\ 1]^T$.

5. (a) Because we employ the Small Gain Theorem for analyzing stability of the loop with Δ included, and then require all blocks to be stable.

(b) Writing the closed-loop on M-Δ-form we get $M = -W_{iI}S_I$ and hence from the SGT we get the sufficient robust stability condition

$$\|W_{iI}S_I\|_\infty < 1$$

and nominal stability.

(c) We get

$$T_I = \frac{K_p}{K_p + s + 1}\ ; \quad S_I = \frac{s+1}{K_p + s + 1}$$

and nominal stability for $K_p > -1$. For the uncertainty weights we get

$$k = (1 + W_I\Delta_I) \ \Rightarrow\ |W_I\Delta_I| < |1 - k| < 0.75$$

and

$$k = (1 + W_{iI}\Delta_{iI})^{-1} \ \Rightarrow\ |W_{iI}\Delta_{iI}| < |\frac{1}{k} - 1| < 3$$

With these weights we get

$$\|W_I\Delta_I T_I\|_\infty = 0.75\left|\frac{K_p}{K_p + 1}\right|$$

(peaks at $\omega = 0$ since only one pole and no zeros) and hence RS for all $K_p > -4/7$. For $-1 < K_p < 0$ we get

$$\|W_{iI}\Delta_{iI}S_I\|_\infty = 3\frac{1}{K_p + 1} > 3$$

(peaks at $\omega = 0$ since pole smaller magnitude than zero) while with $K_p > 0$ we get

$$\|W_{iI}\Delta_{iI}S_I\|_\infty = 3$$

(peaks at $\omega = \infty$ since zero smaller magnitude than pole) and hence we can not guarantee RS with any value of K_p with the inverse multiplicative uncertainty.

(d) For the relative input uncertainty we get

$$W_I \Delta_I = \frac{1-\alpha}{s+\alpha}$$

and the robustness condition becomes

$$\|W_I \Delta_I T_I\|_\infty = \left|\frac{K_p}{K_p+1}\right| \left|\frac{1-\alpha}{\alpha}\right| \leq \left|\frac{3K_p}{1+K_p}\right| < 1$$

which then gives $-0.25 < K_p < 0.5$ for robust stability.
For the inverse uncertainty we get

$$W_{iI} \Delta_{iI} = \frac{\alpha-1}{s+1}$$

and the robustness condition

$$\|W_{iI} \Delta_{iI} S_I\|_\infty = \frac{|\alpha-1|}{|1+K_p|} \leq \frac{1}{1+K_p} < 1$$

which gives that we have robust stability for all $K_p > 0$.

Thus, we see that it is important to choose the "correct" uncertainty description for robustness analysis, as we otherwise may get overly conservative results.