EL2520
Control Theory and Practice

Lecture 6: Fundamental Limitations in MIMO Control

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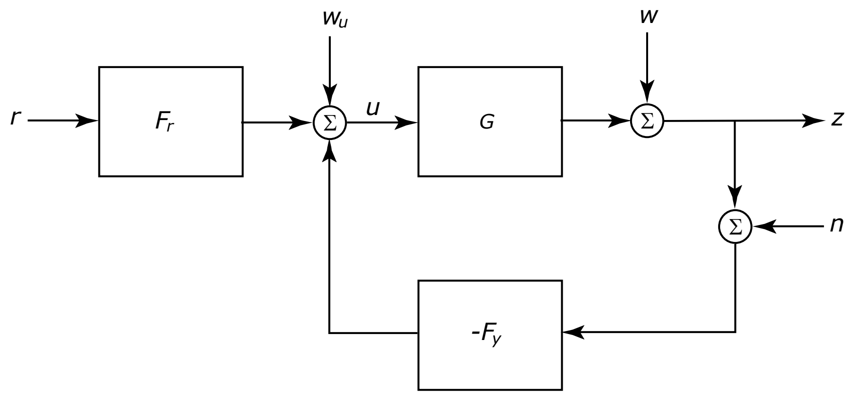
Today's Lecture

- Internal stability for MIMO systems
- Performance specifications for MIMO systems
- Performance limitations in MIMO systems (on white board)
  - S+T=I
  - Generalized Bode sensitivity integral
  - RHP zeros and poles
  - Disturbances and RHP zeros

- Next time: robust stability in MIMO systems, design of controllers satisfying performance and robustness objectives using $\mathcal{H}_\infty$-optimal control

Internal Stability



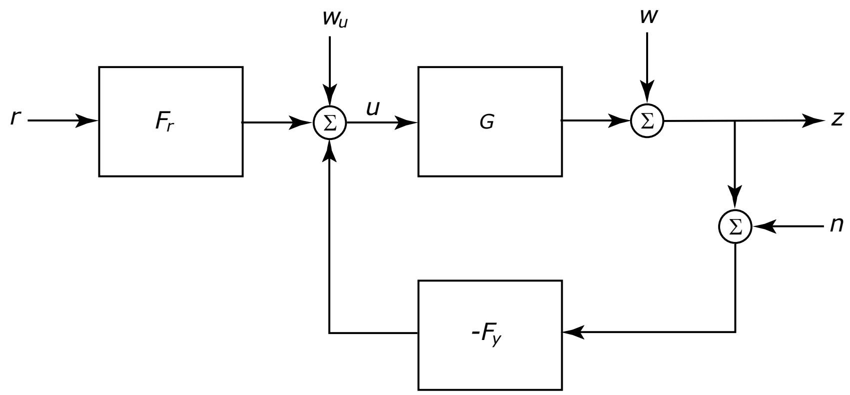
- Internally stable if input-output stable from all inputs to all outputs
- Consider one input and one output on either side of the two blocks in the loop, e.g., inputs w_u, w and outputs z, u

$$z = \underbrace{(I + GF_y)^{-1}}_{S} w + \underbrace{(I + GF_y)^{-1}}_{SG} w_u$$

$$u = \underbrace{-(I + F_yG)^{-1}F_y}_{S_uF_y} w + \underbrace{(I + F_yG)^{-1}}_{S_u} w_u$$

- Hence, internally stable if S, SG, S_u, S_uF_y (and F_r) all stable

Performance Objectives



- We have

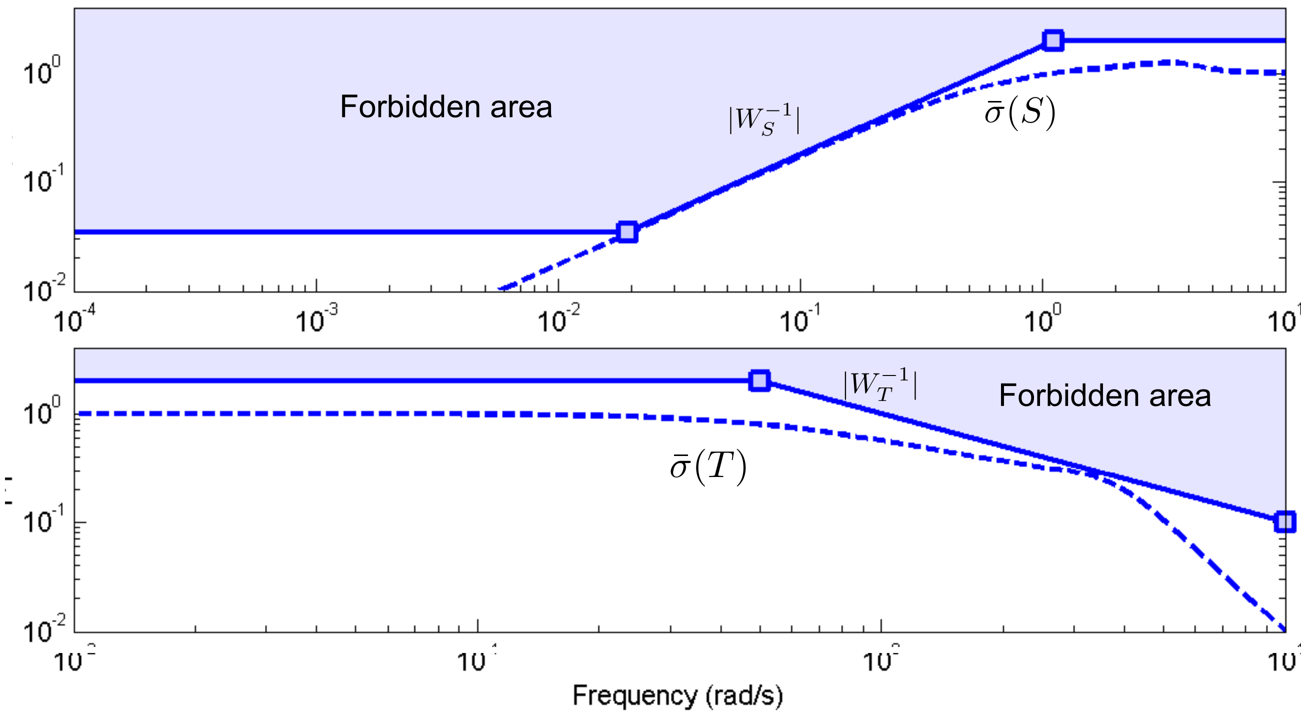
$$z = (I + GF_y)^{-1} w - (I + GF_y)^{-1} GF_y n = Sw - Tn$$

- Thus, make S "small" for disturbance attenuation and T "small" for noise damping
- At each frequency

$$\underline{\sigma}(S(i\omega)) \leq \frac{|z|}{|w|} \leq \bar{\sigma}(S(i\omega)) \; ; \quad \underline{\sigma}(T(i\omega)) \leq \frac{|z|}{|n|} \leq \bar{\sigma}(T(i\omega))$$

  – thus bound $\bar{\sigma}(S)$ and $\bar{\sigma}(T)$

Frequency Domain Specifications



$$\bar{\sigma}(S) \leq |W_S^{-1}| \quad \forall\omega \quad \Leftrightarrow \quad \|W_S S\|_\infty \leq 1 \quad ; \quad \bar{\sigma}(T) \leq |W_T^{-1}| \quad \forall\omega \quad \Leftrightarrow \quad \|W_T T\|_\infty \leq 1$$

- What fundamental limitations exist for the choices of the weights W_S and W_T ?

A Note on Scaling

- Before considering performance specifications and limitations it is useful to scale the problem so that the expected or allowed size of any signal is 1
- See Lecture notes 6 on how to scale.

S+T=I

$$\underbrace{(I+GF_y)^{-1}}_{S}+\underbrace{(I+GF_y)^{-1}GF_y}_{T}=I$$

- Fan's Thm:

$$\sigma_i(A+B) \geq \sigma_i(A) - \bar{\sigma}(B) \quad \forall i$$

- Thus,

$$|1-\bar{\sigma}(T)| \leq \bar{\sigma}(S) \leq 1+\bar{\sigma}(T)$$

- Hence, at any frequency
  - can not make both $\bar{\sigma}(S)$ and $\bar{\sigma}(T)$ small
  - peak in S implies peak in T:

$$\bar{\sigma}(S) >> 1 \ \Leftrightarrow \ \bar{\sigma}(T) >> 1$$

- Hence we can not choose $|W_S|$ and $|W_T|$ large at the same frequency

(Bode) Sensitivity Integral

- Extension of the Bode Sensitivity Integral (see lec 4) to MIMO systems yields

$$\int_0^\infty \ln|\det(S(i\omega))|d\omega = \Sigma_j \int_0^\infty \ln \sigma_j(S(i\omega))d\omega = \pi\Sigma_i \operatorname{Re}(p_i)$$

  - proof based on Cauchy integral formula
  - trade-off between frequencies as well as between directions
  - cannot make $|W_S|$ large at all frequencies and in all directions (note that a scalar weight usually is preferred and then all directions are weighted equal)

RHP Zeros

Thm: Assume G(s) has a RHP zero at s=z>0. Then

$$\|W_S S\|_\infty \geq |W_S(z)|$$

- generalization of result for SISO case (see lec 4)

Proof: By definition G(z) is rank deficient, i.e.,

$$y_z^H G(z) = 0 \;\Rightarrow\; y_z^H T(z) = 0$$

Since S=I-T we get

$$y_z^H S(z) = y_z^H \;\Rightarrow\; S^H(z) y_z = y_z$$

and since $\bar{\sigma}(S) = \bar{\sigma}(S^H)$

$$\bar{\sigma}(S(z)) \geq 1$$

RHP Zeros cont'd

Then, by Maximum Modulus Thm

$$\|W_S S\|_\infty \geq \bar{\sigma}(W_S(z)S(z)) \geq |W_S(z)|$$

where we have assumed the weight W_S is scalar

- same restriction on $\bar{\sigma}(S)$ as on $|S|$ in SISO case (see lec 4)

RHP Poles

Thm: Assume G(s) has a RHP pole at s=p>0, then

$$\|W_T T\|_\infty \geq |W_T(p)|$$

Proof: as above for RHP zeros, but with $S(p)y_p = 0 \;\Rightarrow\; T(p)y_p = y_p$

– same restriction on $\bar{\sigma}(T)$ as on $|T|$ in SISO case (see lec 4)

Requirements for Disturbance Attenuation

- Consider a scalar disturbance d such that

$$w = g_d(s)d \;\Rightarrow\; z = S(s)g_d(s)d, \quad |d| < 1\ \forall\omega$$

- Requirement $|z| < 1\ \forall\omega$ implies

$$\bar{\sigma}(Sg_d) < 1\ \forall\omega \;\Rightarrow\; \|Sg_d\|_\infty < 1$$

- Define the *disturbance direction*

$$y_d = \frac{g_d}{\|g_d\|_2}$$

- Then, requirement is

$$\bar{\sigma}(Sy_d) < \frac{1}{\|g_d\|_2}\ \forall\omega$$

  – Note: requirement on S is only in direction y_d

Requirements cont'd

- Consider high-gain and low-gain directions of sensitivity S (from SVD of S)

$$S\bar{v} = \bar{\sigma}(S)\bar{u} \; ; \quad S\underline{v} = \underline{\sigma}(S)\underline{u}$$

  - If $y_d = \bar{u} \;\Rightarrow\; \bar{\sigma}(S) < \dfrac{1}{\|g_d\|_2}$ ('worst' direction)
  - if $y_d = \underline{u} \;\Rightarrow\; \underline{\sigma}(S) < \dfrac{1}{\|g_d\|_2}$ ('best' direction)

Disturbances and RHP Zeros

- Assume G(s) has a RHP zero at s=z, then

$$y_z^H S(z) = y_z^H \;\; \Rightarrow \;\; y_z^H S(z) g_d(z) = y_z^H g_d(z)$$

- From Maximum Modulus Thm we get

$$\|S g_d\|_\infty \geq |y_z^H g_d(z)|$$

- Thus, we get requirement

$$|y_z^H g_d(z)| < 1$$

  - otherwise no controller exists that will provide acceptable performance corresponding to keeping $|z| < 1$ in the presence of disturbances $|d| < 1$

Disturbances and RHP Zeros

- "Extreme" cases
  - If $y_z \perp y_d \;\Rightarrow\; y_z^H g_d(z) = 0$ (RHP zero has no impact on disturbance attenuation)
  - if $y_z \parallel y_d \;\Rightarrow\; y_z^H g_d(z) = |g_d(z)|$ ("worst-case" alignment)

- Example:

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{1}{s+2} \\ \frac{s+3}{s+1} & \frac{2}{s+2} \end{pmatrix} \quad ; \quad \det G(s) = \frac{1-s}{(s+1)(s+2)} \;\Rightarrow\; z = 1$$

$$\Downarrow$$

$$G(1) = \begin{pmatrix} 1 & 1/3 \\ 2 & 2/3 \end{pmatrix} \;\Rightarrow\; y_z^H = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 & 1 \end{pmatrix}$$

Example cont'd

1. Disturbance d_1

$$g_{d1}(s) = \frac{2}{s+1}\begin{pmatrix}1\\1\end{pmatrix} \;\Rightarrow\; |y_z^H g_{d1}(1)| = \frac{1}{\sqrt{5}} < 1$$

2. Disturbance d_2

$$g_{d2}(s) = \frac{2}{s+1}\begin{pmatrix}-1\\1\end{pmatrix} \;\Rightarrow\; |y_z^H g_{d2}(1)| = \frac{3}{\sqrt{5}} > 1$$

– Thus, can attenuate disturbance d_1 but not d_2 such that $|z| < 1$ when $|d| < 1$ (with any controller!)

Summary

- Results on performance requirements and limitations carry more or less directly over from SISO to MIMO by considering the maximum singular values of the transfer-matrices we want to make small, e.g., S and T
- By using scalar weights W_S, W_T we impose same bound on sensitivity in all directions
- Disturbances have specific directions and therefore impose requirements on the sensitivity only in these directions
- To what extent a RHP zero imposes a limitation for disturbance attenuation depends on how the disturbance direction is aligned with the zero output direction