9 EL2520 Lecture notes 10: Robust Loop Shaping

In this lecture we will consider a special $\mathcal{H}_\infty$-optimal control problem, namely one in which the objective is to maximize robust stability for a certain, quite general, form of model uncertainty known as coprime uncertainty. The main reason why this problem is of interest is that it can be used to "robustify" a controller that has been designed to meet certain performance specifications but which has not addressed robust stability. For instance, in lecture 7 we considered controller design using loop shaping ideas, i.e., shaping the singular values of the loop gain $L = GF_y$, but noted that it was diffcult to address robust stability and stability margins when loopshaping for multivariable systems. In this case it is therefore relevant to first design for performance only and then use $\mathcal{H}_\infty$-optimization in a second step to make the system robustly stable. This approach is also known as Glover-MacFarlane loop shaping.

9.1 Robust Stabilization

We start by describing coprime uncertainty and then derive the corresponding robust stability condition using the Small Gain Theorem. To maximize robust stability, i.e., maximize the size of the uncertainty set that can be tolerated without loosing stability, we formulate an $\mathcal{H}_\infty$-optimal control problem which we then show can be solved explicitly. We finally illustrate the results with a simple example.

A left coprime factorization of a model $G(s)$ is

$$G(s) = M^{-1}(s)N(s) \tag{1}$$

where $M(s)$ and $N(s)$ are stable coprime transfer-functions. Two transfer-functions are coprime if they satisfy the Bezout identity

$$NU + MV = I$$

for some stable $U(s)$ and $V(s)$. Two scalar stable transfer-functions are coprime if they have no common RHP zeros. How to compute a (normalized) left coprime factorization of a transfer-matrix $G(s)$ is given in Appendix of these lecture notes.

Since $M(s)$ and $N(s)$ are stable, it implies that all RHP poles of $G(s)$ will appear as RHP zeros of $M(s)$ and all RHP zeros of $G(s)$ will appear as RHP zeros of $N(s)$. A coprime factorization is normalized if it satisfies

$$M(s)M^T(-s) + N(s)N^T(-s) = I$$

We next introduce uncertainty, in the form of stable perturbations $\Delta_M(s)$ and $\Delta_N(s)$, to $M(s)$ and $N(s)$, respectively, so that the set of plants is

$$G_p(s) = (M(s) + \Delta_M(s))^{-1}(N(s) + \Delta_N(s)) \; ; \quad \|\Delta_N \; \Delta_M\|_\infty < \epsilon \tag{2}$$

Note that this uncertainty description allows both zeros and poles to cross between the LHP and RHP and is as such quite general.

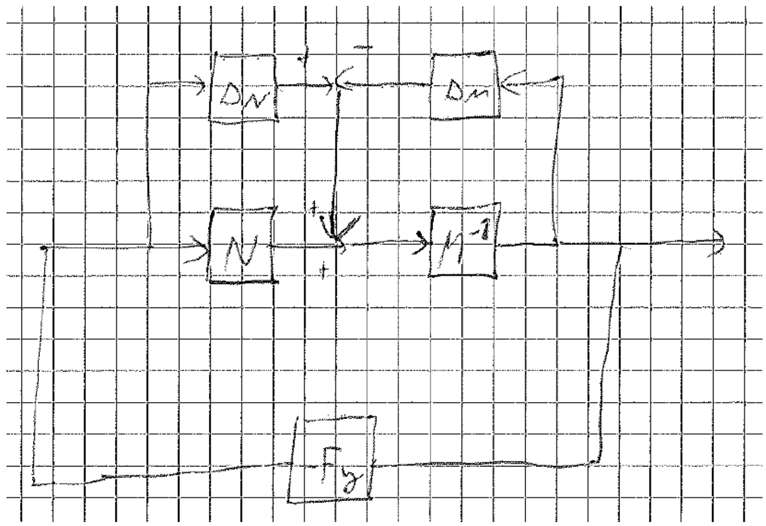

Figure 9.1: Closed-loop with left coprime factor uncertainty.

The aim of robust stabilization is to determine a controller that maximizes the robust stability region, given the uncertainty set (2), in the sense that it allows the largest possible ϵ while providing robust stability. For this purpose, we next derive a robust stability condition for a closed-loop system with the uncertain plant given by (2). The loop is illustrated in Figure 1. As usual, we rewrite the loop on the form of a P-Δ_G-loop, where P is a nominal transfer-function and Δ_G is the uncertainty, and apply the Small Gain Theorem. See also Lecture 7. Let $\Delta_G = [\Delta_N \ \Delta_M]$, then we identify P from the block diagram in Figure 1 as

$$P = -\begin{pmatrix} F_y \\ I \end{pmatrix} (I + GF_y)^{-1} M^{-1}$$

Applying the Small Gain Theorem, we can conclude that the closed-loop system is robustly stable if Δ_G is stable (Δ_N and Δ_M both stable), P is stable (nominal stability) and

$$\|\Delta_G\|_\infty \|P\|_\infty < 1$$

Since the aim of robust stabilization is to maximize the allowed $\|\Delta_G\|_\infty$, this can be formulated as minimizing $\|P\|_\infty$, i.e.,

$$\min_{F_y} \|P\|_\infty = \min_{F_y} \| \begin{pmatrix} F_y \\ I \end{pmatrix} (I + GF_y)^{-1} M^{-1} \|_\infty = \gamma_{min}$$

With the minimum achievable $\|P\|_\infty = \gamma_{min}$ we can allow $\|\Delta_G\|_\infty < 1/\gamma_{min}$.

The minimum γ_{min}, and the corresponding robustly stabilizing controller $F_y(s)$, can be computed directly from a state-space realization of $G(s)$

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \qquad (3)$$

First, solve the two algebraic Riccati equations for postitive definite matrices $X > 0, Z > 0$

$$AZ + ZA^T - ZC^TCZ + BB^T = 0$$

$$A^TX + XA - XB^TBX + C^TC = 0$$

Now, let $\lambda_m = \max_i \lambda_i(XZ)$. Then

$$\gamma = \alpha(1 + \lambda_m)^{1/2}, \ \alpha \geq 1$$

where α is a tuning parameter. Note that $\gamma = \gamma_{min}$ for the choice $\alpha = 1$. The main reason for introducing $\alpha \geq 1$ is that in many cases the maximally robustifying controller can influence too strongly on the performance, and this can be avoided to some extent by choosing α somewhat larger than 1. A typical rule of thumb is $\alpha = 1.1$. Introduce

$$R = I - \frac{1}{\gamma^2}(I + ZX)$$

Then, the optimal state feedback gain and state observer gains are, respectively,

$$L = B^TX \ ; \quad K = R^{-1}ZC^T$$

and the controller providing $\|P\|_\infty = \gamma$ is

$$\dot{\hat{x}} = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t)) \ ; \quad u(t) = -L\hat{x}(t)$$

9.2 Robust Loopshaping

Robust stabilization is usually of little interest on its own; we also need to address performance. In Glover-MacFarlane loop shaping, one first shapes the open-loop to satisfy certain performance specifications

$$\hat{G}(s) = W_2(s)G(s)W_1(s)$$

where W_1 and W_2 are post- and pre-compensators, respectively. For a discussion on loop shaping for performance, see Lecture 7. In a second step, one solves the robust stabilization problem for the shaped plant $\hat{G}(s)$. This give a robustly stabilizing controller $\hat{F}_y(s)$, and the overall controller is then

$$F_y(s) = W_1(s)\hat{F}_y(s)W_2(s)$$

In general, provided γ_{min} for the robust stabilization is small ($\gamma_{min} \approx< 4$) the controller $\hat{F}_y(s)$ will usually have a small influence on the performance. If γ_{min} is large ($\approx> 4$) then this indicates that there is a conflict between the nominal performance objectives and robust stability.

Example: Consider the system

$$z = \underbrace{\frac{200}{10s + 1}\frac{1}{(0.05s + 1)^2}}_{G(s)}u + \underbrace{\frac{100}{10s + 1}}_{G_d(s)}d$$

The model is scaled such that expected disturbances have magnitude $|d| < 1 \; \forall\omega$ and acceptable performance corresponds to keeping $|z| < 1 \; \forall\omega$. Thus, since $z = SG_d d$ acceptable performance corresponds to

$$|SG_d(i\omega)| < 1 \; \forall\omega \quad \Leftrightarrow \quad |S(i\omega)| < |G_d^{-1}(i\omega)| \; \forall\omega$$

Since $S = 1/(1 + L)$ where L is the loop-gain, we have that $|S| \approx 1/|L|$ when $|S| << 1$. Thus, we should shape the loop such that

$$|L(i\omega)| > |G_d(i\omega)| \;\; \omega < \omega_d$$

where $|G_d(i\omega)| > 1$ for $\omega < \omega_d$. A controller that satsifies this is

$$W_1 = \frac{s+2}{s}$$

The corresponding loop-gain $L = GW_1$ is shown by the dashed line in Figure 2. Also shown is a simulation of the closed-loop response to a unit step in the disturbance d. As can be seen, the controller maintains $|z| < 1$ but the response is quite oscillatory, indicating poor robust stability. We therefore next robustify the shaped loop, according to the results above, using the Matlab robust control toolbox command `ncfsyn`. The maximal robustness corresponds to $\gamma_{min} = 2.34$ which should be OK. The robustified loop gain is shown by the solid line in Figure 2, and as can be seen the robustification mainly lowers the slope of the loop gain around the crossover frequency, thereby increasing the phase margin. As can be seen from the response to the step disturbance, the performance, in terms of maximum peak and settling time, is about the same but with the oscillatory behavior removed.

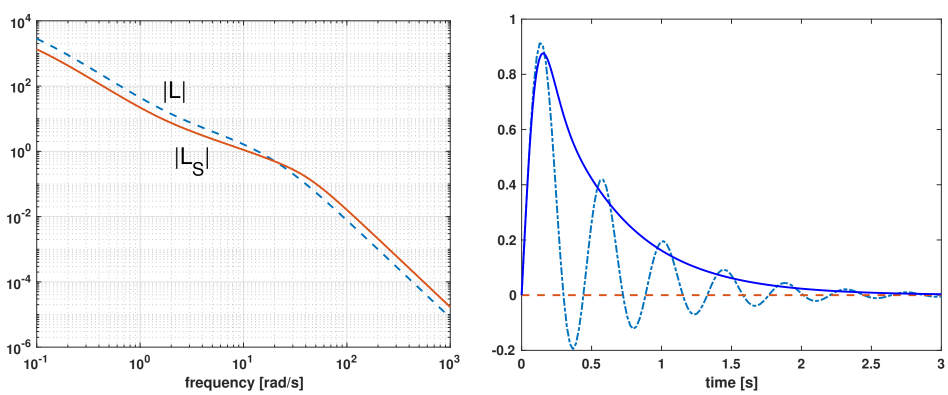

Figure 9.2: Loop gains and simulations for Example.

10 Appendix. Computing normalized left coprime factorizations

This appendix is for the interested only (not exam relevant).

Consider a state-space realization of a square transfer-matrix $G(s)$

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{4}$$

Then, the left normalized coprime factors $M(s)$ and $N(s)$, in $G(s) = M^{-1}(s)N(s)$, are given by

$$N(s) = R^{-1/2}C(sI - A - HC)^{-1}(B + HD) + R^{-1/2}D$$
$$M(s) = R^{-1/2}C(sI - A - HC)^{-1}H + R^{-1/2}$$

where $H = -(BD^T + ZC^T)R^{-1}$ and $R = I + DD^T$. Here $Z > 0$ is the positive definite solution to the algebraic Riccati equation

$$(A - BS^{-1}D^TC)Z + Z(A - BS^{-1}D^TC)^T - ZC^TR^{-1}CZ + BS^{-1}B^T = 0$$

with $S = I + D^TD$.

The result is due to Vidyasagar, *Control Systems Synthesis: A Factorization Aproach*, MIT Press (1985)