Quickselect and Median of Medians: Improving Quicksort

CS 4102: Algorithms

Spring 2021

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Readings

- CLRS:
  - Chapter 9
- Wikipedia articles on Quickselect and Median of Medians

Review: Quicksort

Idea: pick a pivot element, recursively sort two sublists around that element

- Divide: select pivot element p, Partition(p)
- Conquer: recursively sort left and right sublists
- Combine: Nothing!

Partition (Divide step)

Given: a list, a pivot p

Start: unordered list

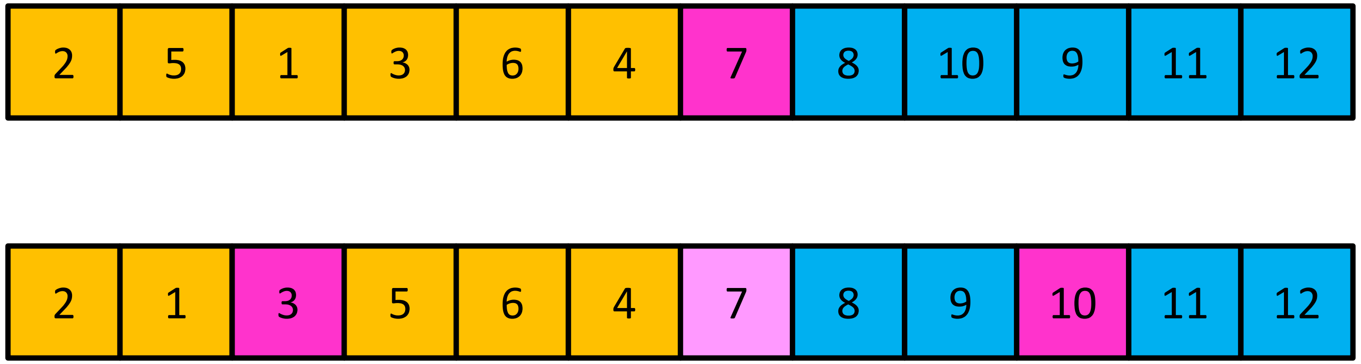
8	5	7	3	12	10	1	2	4	9	6	11

Goal: All elements $< p$ on left, all $> p$ on right

5	7	3	1	2	4	6	8	12	10	9	11

Quicksort Run Time (Best)

If the pivot is always the median:



2	5	1	3	6	4	7	8	10	9	11	12

2	1	3	5	6	4	7	8	9	10	11	12

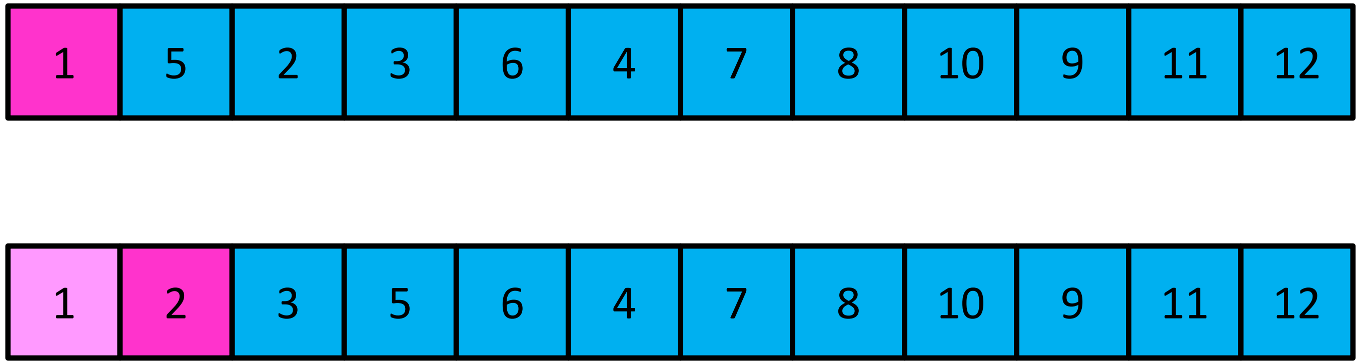
Then we divide in half each time

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$$T(n) = O(n \log n)$$

Quicksort Run Time (Worst)

If the pivot is always at the extreme:



1	5	2	3	6	4	7	8	10	9	11	12

1	2	3	5	6	4	7	8	10	9	11	12

Then we shorten by 1 each time

$$T(n) = T(n-1) + n$$

$$T(n) = O(n^2)$$

Can we Pick a Good Pivot for Quicksort?

- What makes a good Pivot for Quicksort?
  - Roughly even split between left and right
  - Ideally: the median
- Can we find a list’s median in linear time?
  - Quickselect (https://en.wikipedia.org/wiki/Quickselect)
    - Finds the median
    - Works a lot like Quicksort: needs to do a Partition
    - We need a good pivot for Quickselect for it to have good time-complexity
  - Median of Medians (https://en.wikipedia.org/wiki/Median_of_medians)
    - Can be used to find “pretty good” pivot for QS, or with Quickselect

Quickselect

- Finds i^{th} order statistic
  - i^{th} smallest element in the list
  - 1^{st} order statistic: minimum
  - n^{th} order statistic: maximum
  - $\frac{n}{2}^{\text{th}}$ order statistic: median
- CLRS, Section 9.1
  - **Selection problem**: Give list of distinct numbers and value *i*, find value *x* in list that is larger than exactly *i-1* list elements

Quickselect

Idea: pick a pivot element, partition, then recurse on the sublist containing index i

- Divide: select an element p, Partition(p)
- Conquer: if i = index of p, done!
  - if $i <$ index of p recurse left. Else recurse right
- Combine: Nothing!

(Note: just one recursive call, unlike Quicksort.)

Partition (Divide step)

Given: a list, a pivot value x

Note: now using “x” to refer to pivot value. We called it “p” in previous slides.

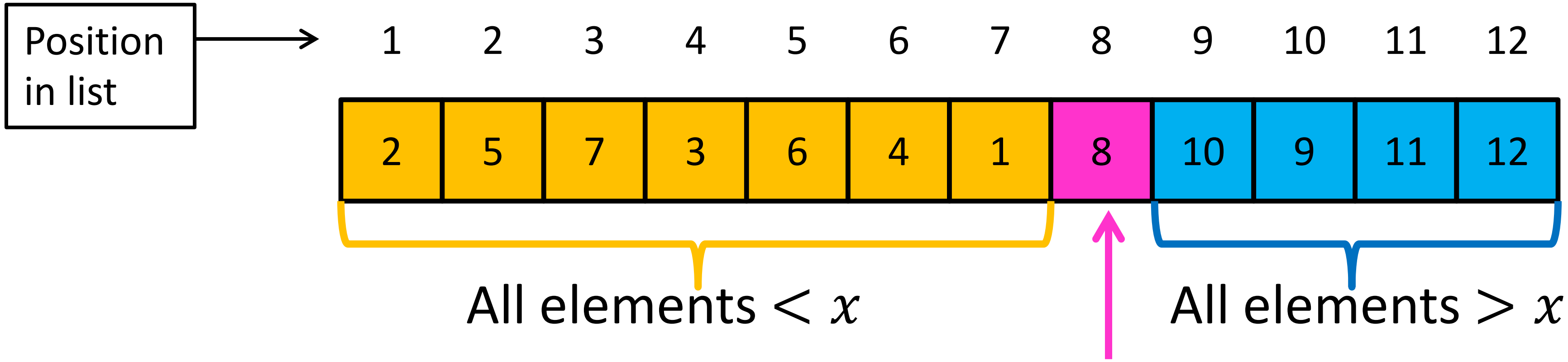
Start: unordered list

11	5	7	3	12	10	1	2	4	9	6	8

Goal: All elements $< x$ on left, all $> x$ on right

5	7	3	1	2	4	6	8	12	10	9	11

Conquer



Position in list	1	2	3	4	5	6	7	8	9	10	11	12
	2	5	7	3	6	4	1	8	10	9	11	12

All elements $< x$

All elements $> x$

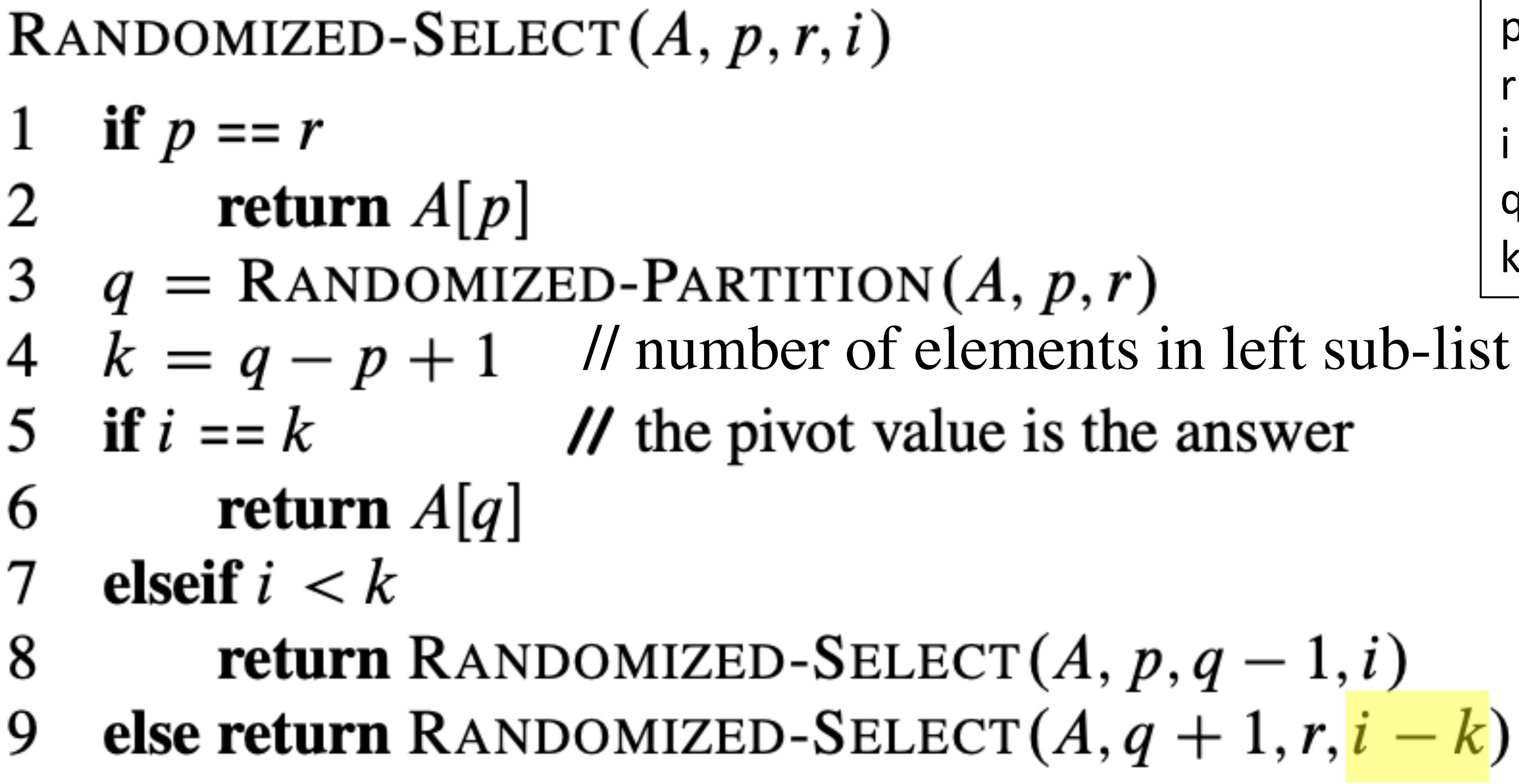
$x = 8$ put exactly where it belongs at position 8 (the split-point)

Remember: we're looking for the i^{th} order statistic

- If the split-point (8) is i we're done! The value stored at the split-point is the result.
- If $i <$ split-point, look in left sub-list (using same value i)
- If $i >$ split-point, look in right sub-list (using an adjusted value of i)
  - For example, if we wanted the 10th order statistic in the entire list, here that would be the 2nd order statistic in the right sub-list

CLRS Pseudocode for Quickselect

A – the list
p – index of first item
r – index of last item
i – find *i*th smallest item
q – pivot location
k – number on left + 1



```
RANDOMIZED-SELECT$(A, p, r, i)$
if $p == r$
    return $A[p]$
$q$ = RANDOMIZED-PARTITION$(A, p, r)$
$k = q - p + 1$      // number of elements in left sub-list + 1
if $i == k$          // the pivot value is the answer
    return $A[q]$
elseif $i < k$
    return RANDOMIZED-SELECT$(A, p, q - 1, i)$
else return RANDOMIZED-SELECT$(A, q + 1, r, i - k)$
```

// note adjustment to *i* when recursing on right side

Note: In CLRS, they're using a partition that randomly chooses the pivot element.
That's why you see "Randomized" in the names here. Ignore that for the moment.

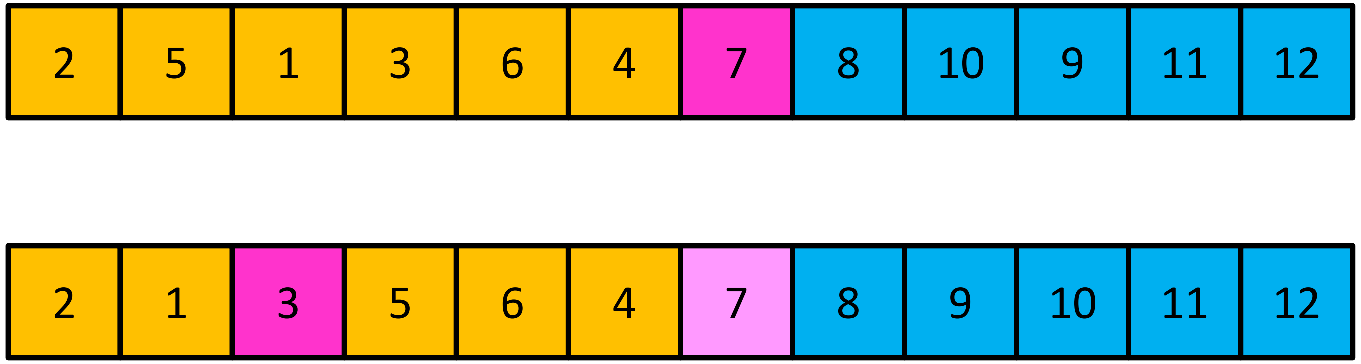
Work These Examples!

- For each of the following calls, show
  - The value of *q* after each partition,
  - Which recursive calls made

1. Select([3, 2, 9, 0, 7, 5, 6, 1], p=0, r=7, i=2)
2. Select([3, 2, 9, 0, 7, 5, 6, 1], p=0, r=7, i=5)
3. Select([3, 2, 9, 0, 7, 5, 6, 1], p=0, r=7, i=7)

Quickselect Run Time

If the pivot is always the median:



2	5	1	3	6	4	7	8	10	9	11	12

2	1	3	5	6	4	7	8	9	10	11	12

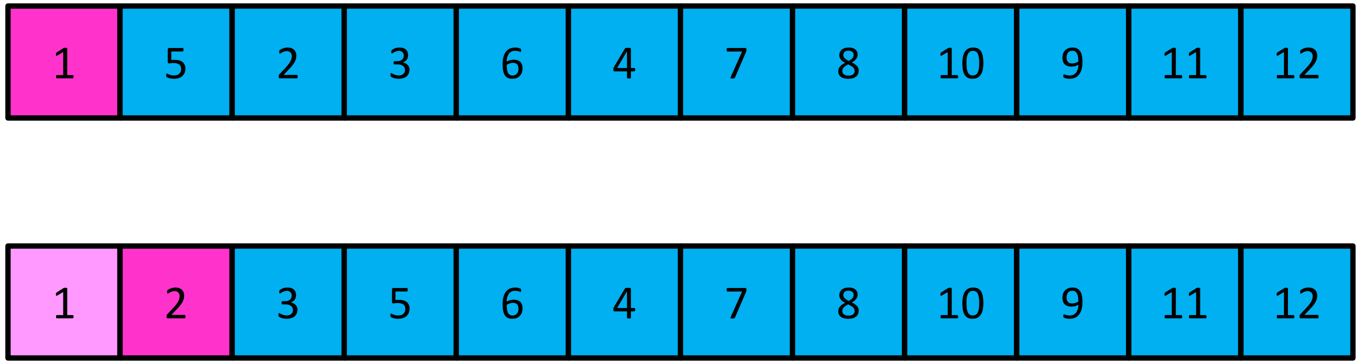
Then we divide in half each time

$$S(n) = S\left(\frac{n}{2}\right) + n$$

$$S(n) = O(n)$$

Quickselect Run Time

If the partition is always unbalanced:



1	5	2	3	6	4	7	8	10	9	11	12

1	2	3	5	6	4	7	8	10	9	11	12

Then we shorten by 1 each time

$$S(n) = S(n-1) + n$$

$$S(n) = O(n^2)$$

Good Pivot for Quickselect

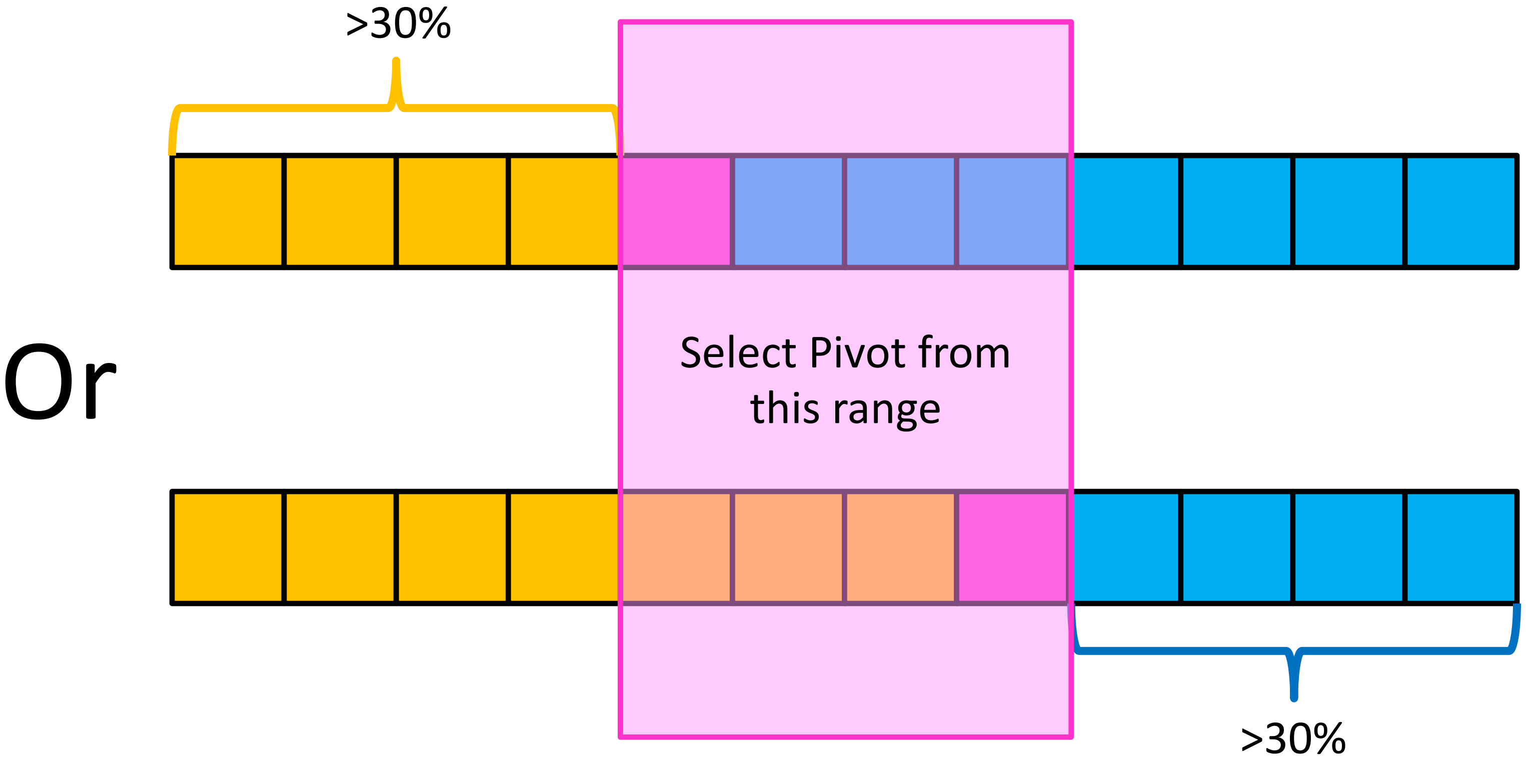
Déjà vu?

- What makes a good Pivot for Quickselect?
  - Roughly even split between left and right
  - Ideally: median

- Here’s what’s next:
  - First, **median of medians** algorithm
    - Finds something close to the median in $\Theta(n)$ time
  - Second, we can prove that when its result used with Quickselect’s partition, then Quickselect is guaranteed $\Theta(n)$
    - Because we now have a $\Theta(n)$ way to find the median, this guarantees Quicksort will be $\Theta(n \lg n)$
  - Notes:
    - We have to do all this for every call to Partition in Quicksort
    - We could just use the value returned by median of medians for Quicksort’s Partition

Pretty Good Pivot

- What makes a “pretty good” Pivot?
  - Both sides of Pivot >30%



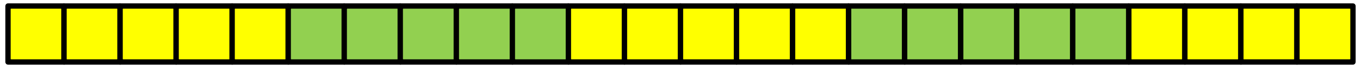
Median of Medians

- Fast way to select a “pretty good” pivot
- Guarantees pivot is greater than 30% of elements and less than 30% of the elements
  - I.e. it’s in the middle 40% (±20% of the true median)
- Idea: break list into chunks, find the median of each chunk, use the median of those medians

- CLRS, pp. 220-221
- https://en.wikipedia.org/wiki/Median_of_medians

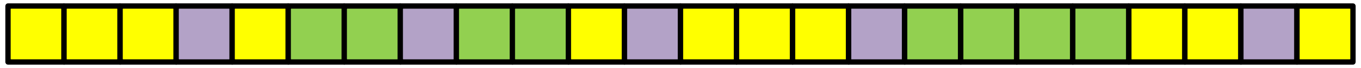
Median of Medians

1. Break list into chunks of size 5

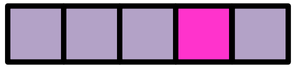


List could be long, many more than 5 chunks!

2. Find the median of each chunk
   (using insertion sort: n=5, 20 comparisons)

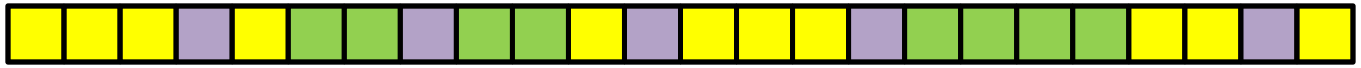


3. Return median of medians (using Quickselect, this algorithm, called recursively, on list of medians)



List could be long, many more than 5 medians!

Why is this good?

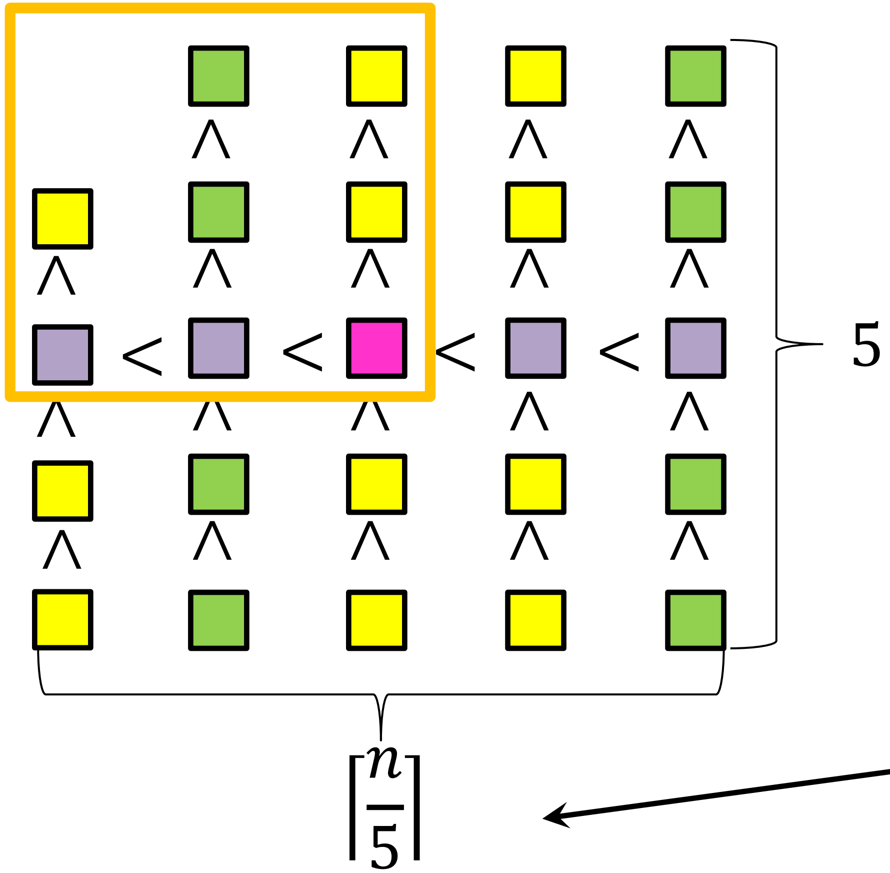


Imagine each chunk sorted, chunks ordered by their medians

MedianofMedians

is Greater than all

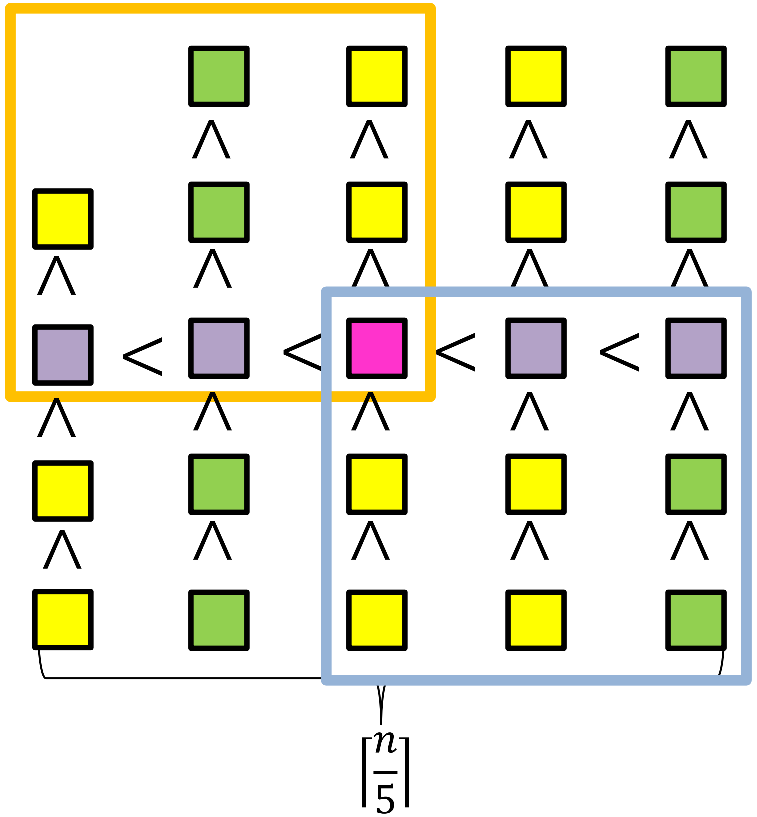
of these



List could be long, so not a small number!

Why is this good?

MedianofMedians is larger than all of these



Worried about the details of this math? See CLRS p. 221

Larger than 3 things in each (but one) list to the left

$3\left(\frac{1}{2}\cdot\left\lceil\frac{n}{5}\right\rceil-2\right)\approx\frac{3n}{10}-6$ elements $<$ ■

Similarly:

$3\left(\frac{1}{2}\cdot\left\lceil\frac{n}{5}\right\rceil-2\right)\approx\frac{3n}{10}-6$ elements $>$ ■

Run-time of Quickselect with Median of Medians

- What's the cost $S(n)$ for Quickselect with Median of Medians?
- Divide: select an element p using Median of Medians, Partition(p)

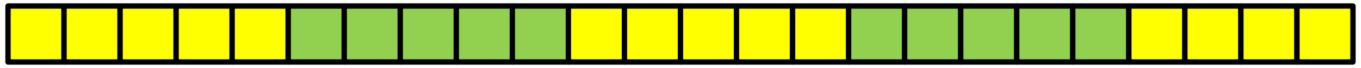
  $M(n) + \Theta(n)$
- Conquer: if $i =$ index of p, done, if $i <$ index of p recurse left. Else recurse right

  $\leq S\left(\frac{7}{10}n\right)$
- Combine: Nothing!

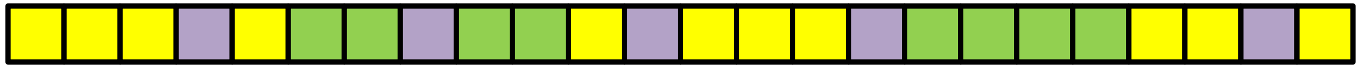
$$S(n) \leq S\left(\frac{7}{10}n\right) + M(n) + \Theta(n)$$

Run-time M(n) for Median of Medians

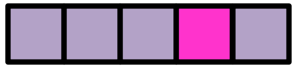
1. Break list into chunks of 5 $\Theta(n)$



2. Find the median of each chunk $\Theta(n)$



3. Return median of medians (using Quickselect)



$S\left(\frac{n}{5}\right)$

$$M(n) = S\left(\frac{n}{5}\right) + \Theta(n)$$

Quickselect

$$S(n) \le S\left(\frac{7n}{10}\right) + M(n) + \Theta(n) \qquad M(n) = S\left(\frac{n}{5}\right) + \Theta(n)$$

$$= S\left(\frac{7n}{10}\right) + S\left(\frac{n}{5}\right) + \Theta(n)$$

We can show by proof by induction that:

$S(n) = O(n)$ (next two slides)

$S(n) = \Omega(n)$

$$\therefore S(n) = \Theta(n)$$

Proof by Induction

$$T(n) = T(n/5) + T(7n/10) + c \cdot n$$

Prove $T(n) = O(n)$

Claim: $T(n) \leq 10cn$

Base Case: $T(0) = 0$

$T(1) = c \leq 10c$ which is true since $c \geq 1$

Strictly speaking, we can handle any $c > 0$, but assuming $c \geq 1$ to simplify the analysis here

Proof by Induction

$$T(n) = T(n/5) + T(7n/10) + c \cdot n$$

Inductive hypothesis: $\forall n \leq x_0 : T(n) \leq 10cn$

Inductive step:

$$T(x_0 + 1) \ = T\left(\frac{1}{5}(x_0 + 1\ )\right) + T\left(\frac{7}{10}(x_0 + 1\ )\right) + c(x_0 + 1)$$

$$\leq \left(\frac{1}{5} + \frac{7}{10}\right) 10c(x_0 + 1) + c(x_0 + 1)$$

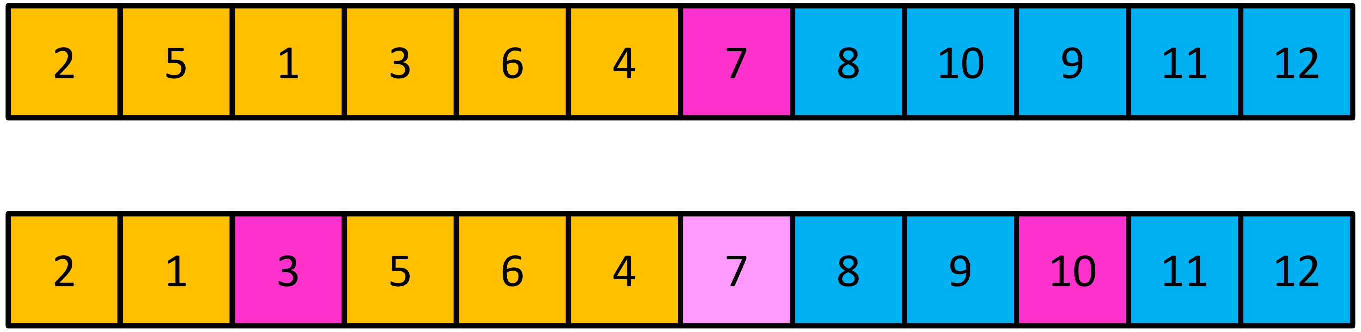
$$= 9c(x_0 + 1) + c(x_0 + 1) = 10c(x_0 + 1)$$

Compare to ‘Obvious’ Approach

- An “obvious” approach to Selection Problem:
  - Given list and value *i*: Sort list, then choose *i*-th item
  - We’ve only seen sorting algorithms that are $\Omega(n \log n)$
  - Later we’ll show this really is a lower-bound
  - So this approach is $\Theta(n \log n)$
- Therefore Quickselect is asymptotically better than this sorting-based solution for Selection Problem!

Phew! Back to Quicksort

Using Quickselect, with a median-of-medians partition, we’re guaranteed to use true median, so:



2	5	1	3	6	4	7	8	10	9	11	12

2	1	3	5	6	4	7	8	9	10	11	12

Then we divide in half each time

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$$

$$T(n) = \Theta(n \log n)$$

Is it worth it?

- Using Quickselect to pick median <u>guarantees</u> $\Theta(n \log n)$ run time
- But, this approach has very large constants
  - If you really want $\Theta(n \log n)$, better off using MergeSort
- Better approach: Choose random pivot for Quicksort
  - Very small constant (random() is a fast algorithm)
  - Can prove the *expected runtime* is $\Theta(n \log n)$
    - Why? Unbalanced partitions are very unlikely

Sorting, so far

- Sorting algorithms we have discussed:
  - Insertionsort $O(n^2)$
  - Mergesort $O(n \log n)$
  - Quicksort $O(n \log n)$
- Other sorting algorithms (next):
  - Heapsort $O(n \log n)$

Can we do better than $O(n \log n)$?

Mental Stretch

Show $\log(n!) = \Theta(n \log n)$

Hint: show $n! \leq n^n$

Hint 2: show $n! \geq \left(\frac{n}{2}\right)^{\frac{n}{2}}$

$\log n! = O(n \log n)$

$$\begin{array}{cccccccccccc} n! = & n & \cdot & (n-1) & \cdot & (n-2) & \cdot & \ldots & \cdot & 2 & \cdot & 1 \\ & \| & & \wedge & & \wedge & & & & \wedge & & \wedge \\ n^n = & n & \cdot & n & \cdot & n & \cdot & \ldots & \cdot & n & \cdot & n \end{array}$$

$$\begin{aligned} & n! \leq n^n \\ \Rightarrow\ & \log(n!) \leq \log(n^n) \\ \Rightarrow\ & \log(n!) \leq n \log n \\ \Rightarrow\ & \log(n!) = O(n \log n) \end{aligned}$$

$\log n! = \Omega(n \log n)$

$$\begin{array}{cccccccccccccc}
n! = & n & \cdot & (n-1) & \cdot & (n-2) & \cdot \ldots \cdot & \frac{n}{2} & \cdot & \left(\frac{n}{2}-1\right) & \cdot \ldots \cdot & 2 & \cdot & 1 \\
 & \vee & & \vee & & \vee & & \| & & \vee & & \vee & & \| \\
\left(\frac{n}{2}\right)^{\frac{n}{2}} = & \frac{n}{2} & \cdot & \frac{n}{2} & \cdot & \frac{n}{2} & \cdot \ldots \cdot & \frac{n}{2} & \cdot & 1 & \cdot \ldots \cdot & 1 & \cdot & 1
\end{array}$$

$$n! \geq \left(\frac{n}{2}\right)^{\frac{n}{2}}$$

$$\Rightarrow \log(n!) \geq \log\left(\left(\frac{n}{2}\right)^{\frac{n}{2}}\right)$$

$$\Rightarrow \log(n!) \geq \frac{n}{2} \log \frac{n}{2}$$

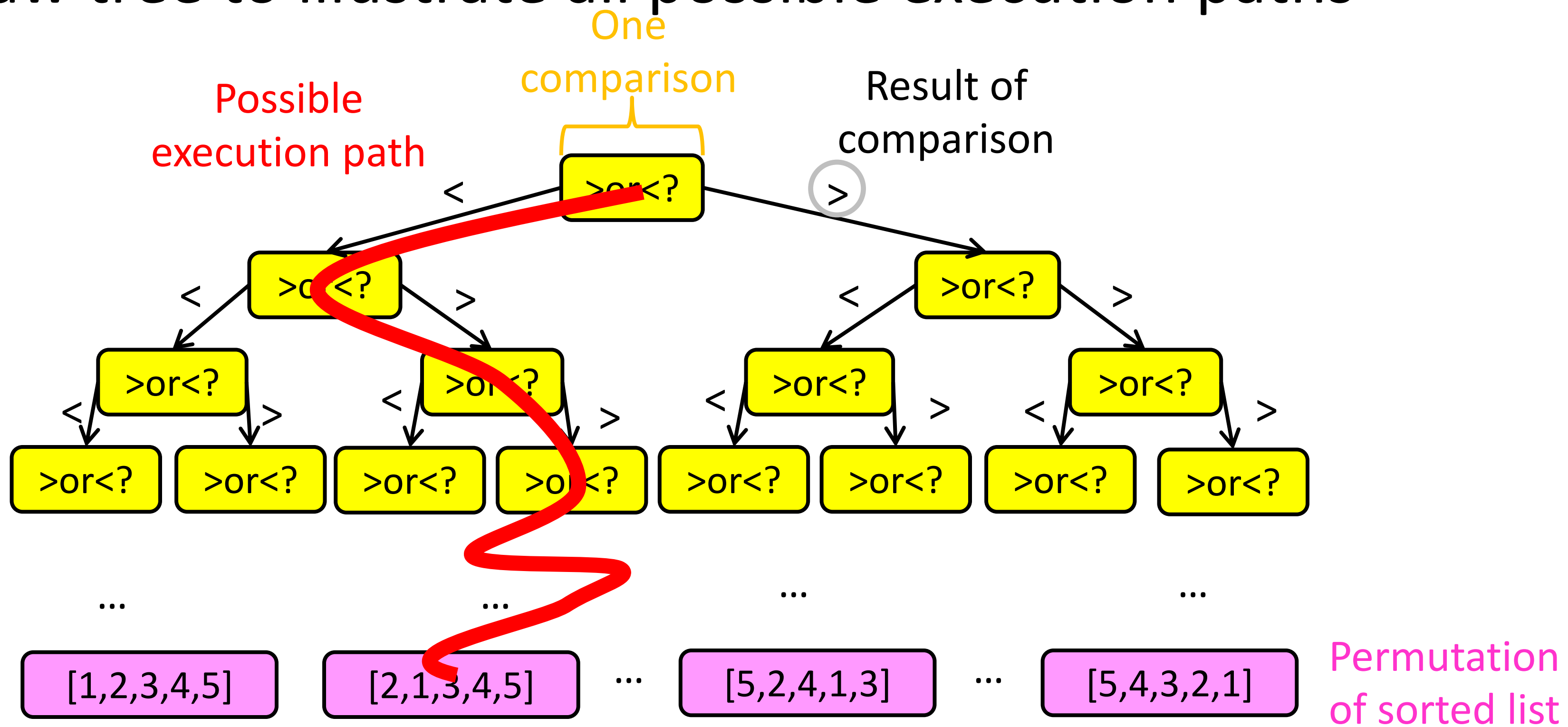
$$\Rightarrow \log(n!) = \Omega(n \log n)$$

Worst Case Lower Bounds

- Prove that there is no algorithm which can sort faster than $O(n \log n)$
- Non-existence proof!
  - Very hard to do

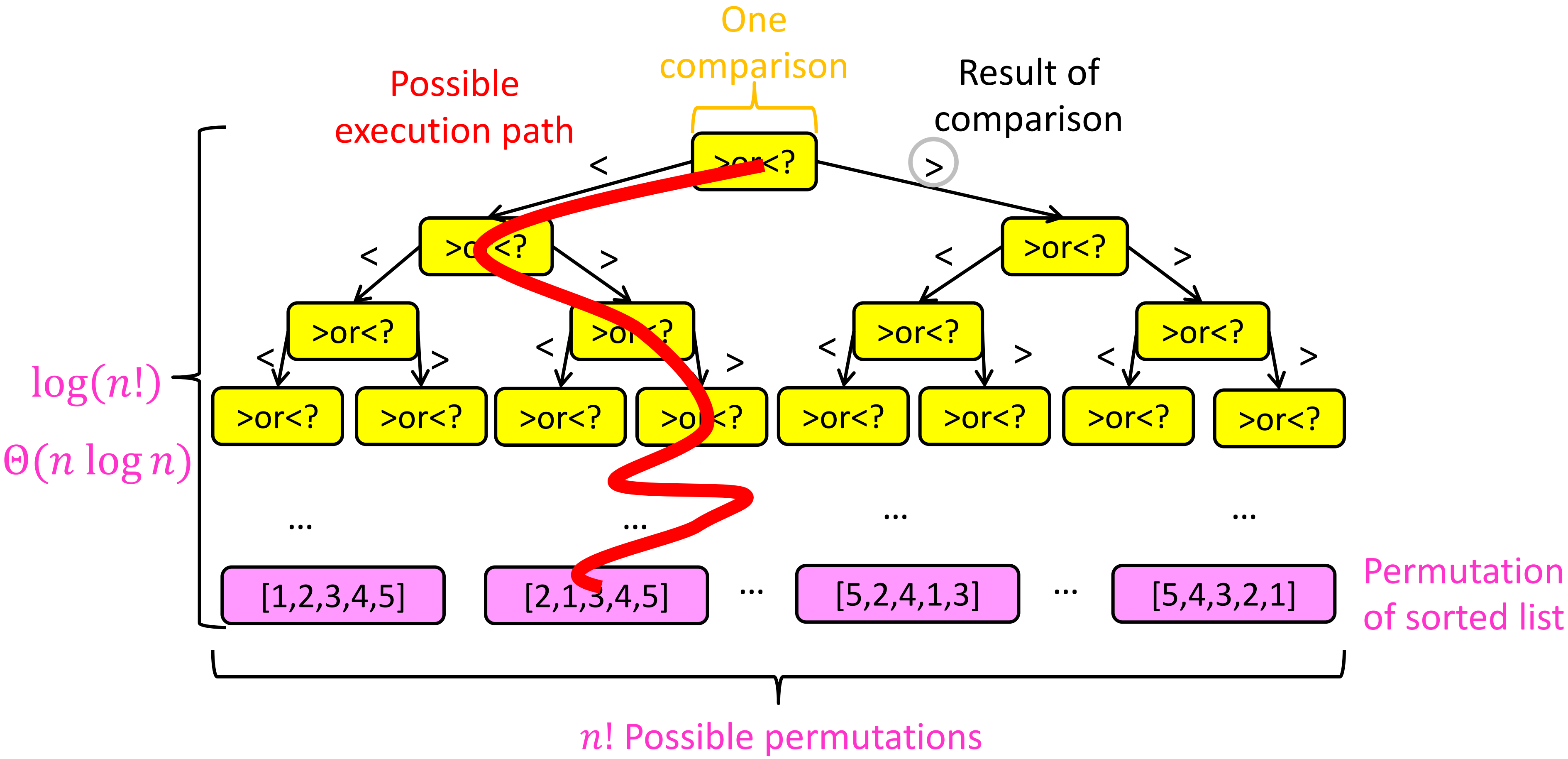
Strategy: Decision Tree

- Sorting algorithms use comparisons to figure out the order of input elements
- Draw tree to illustrate all possible execution paths



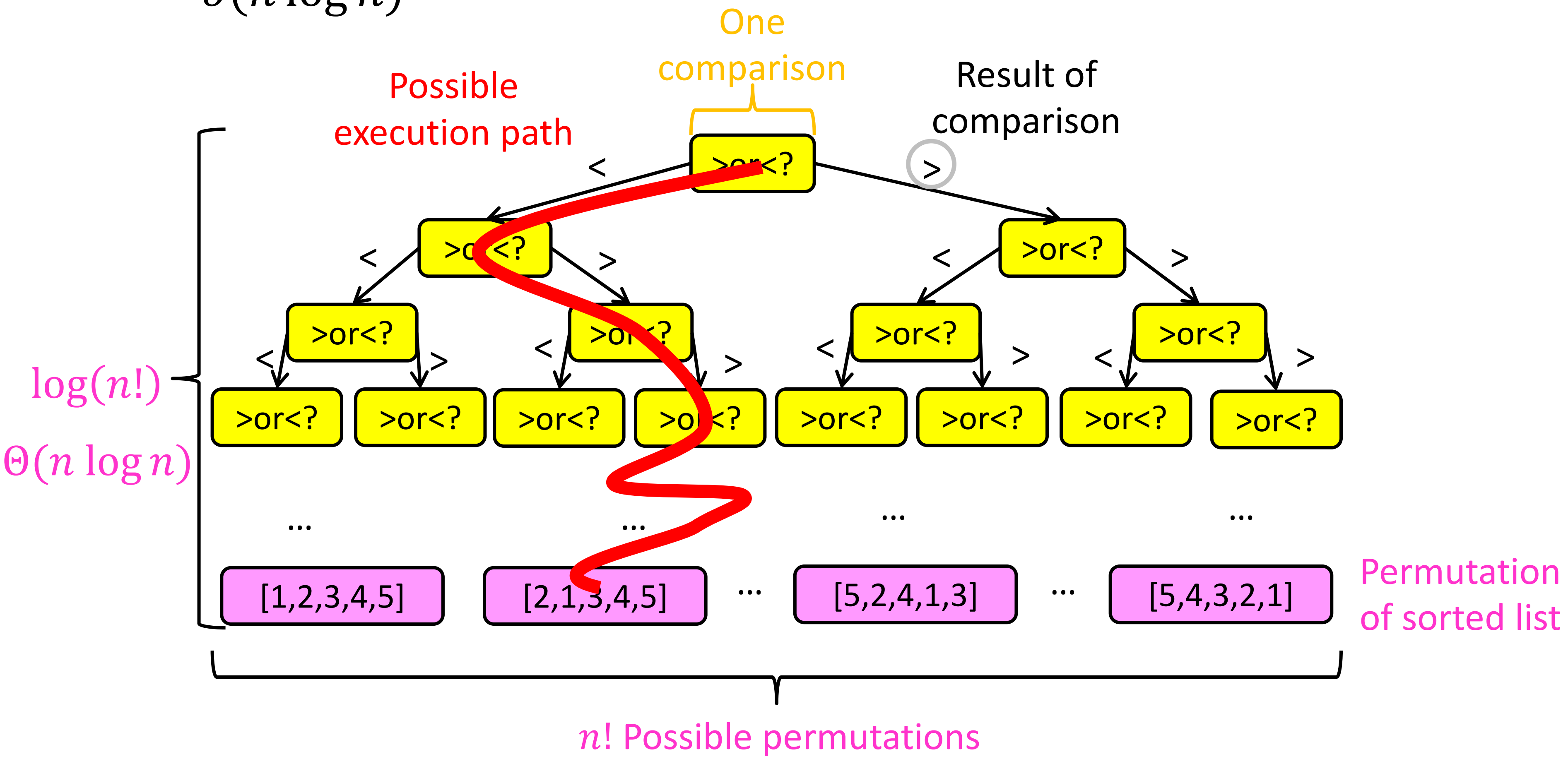
Strategy: Decision Tree

- Worst case run time is the longest execution path
- i.e., "height" of the decision tree



Strategy: Decision Tree

- Conclusion: Worst Case Optimal run time of sorting is $\Theta(n \log n)$
  - There is no (comparison-based) sorting algorithm with run time $o(n \log n)$



Sorting, so far

- Sorting algorithms we have discussed:
  - Mergesort $O(n \log n)$ Optimal!
  - Quicksort $O(n \log n)$ Optimal!
- Other sorting algorithms (will discuss):
  - Bubblesort $O(n^2)$
  - Insertionsort $O(n^2)$
  - Heapsort $O(n \log n)$ Optimal!

Speed Isn't Everything

- Important properties of sorting algorithms:
- Run Time
  - Asymptotic Complexity
  - Constants
- In Place (or In-Situ)
  - Done with only constant additional space
- Adaptive
  - Faster if list is nearly sorted
- Stable
  - Equal elements remain in original order
- Parallelizable
  - Runs faster with many computers

Mergesort

- **Divide**:
  - Break n-element list into two lists of $^n/_2$ elements
- **Conquer**:
  - If $n > 1$: Sort each sublist recursively
  - If $n = 1$: List is already sorted (base case)
- **Combine**:
  - Merge together sorted sublists into one sorted list

Run Time?

$\Theta(n \log n)$

Optimal!

In Place?	Adaptive?	Stable?
No	No	Yes! (usually)

Merge

- **Combine:** Merge sorted sublists into one sorted list
- We have:
  - 2 sorted lists (L_1, L_2)
  - 1 output list (L_{out})

```
While ($L_1$ and $L_2$ not empty):
    If $L_1[0] \leq L_2[0]$:
        $L_{out}$.append($L_1$.pop())
    Else:
        $L_{out}$.append($L_2$.pop())
$L_{out}$.append($L_1$)
$L_{out}$.append($L_2$)
```

Adaptive:
If elements are equal, leftmost comes first

Mergesort

- **Divide**:
  - Break n-element list into two lists of $^n/_2$ elements
- **Conquer**:
  - If $n > 1$: Sort each sublist recursively
  - If $n = 1$: List is already sorted (base case)
- **Combine**:
  - Merge together sorted sublists into one sorted list

Run Time?

$\Theta(n \log n)$

Optimal!

In Place?	Adaptive?	Stable?	Parallelizable?
No	No	Yes! (usually)	Yes!

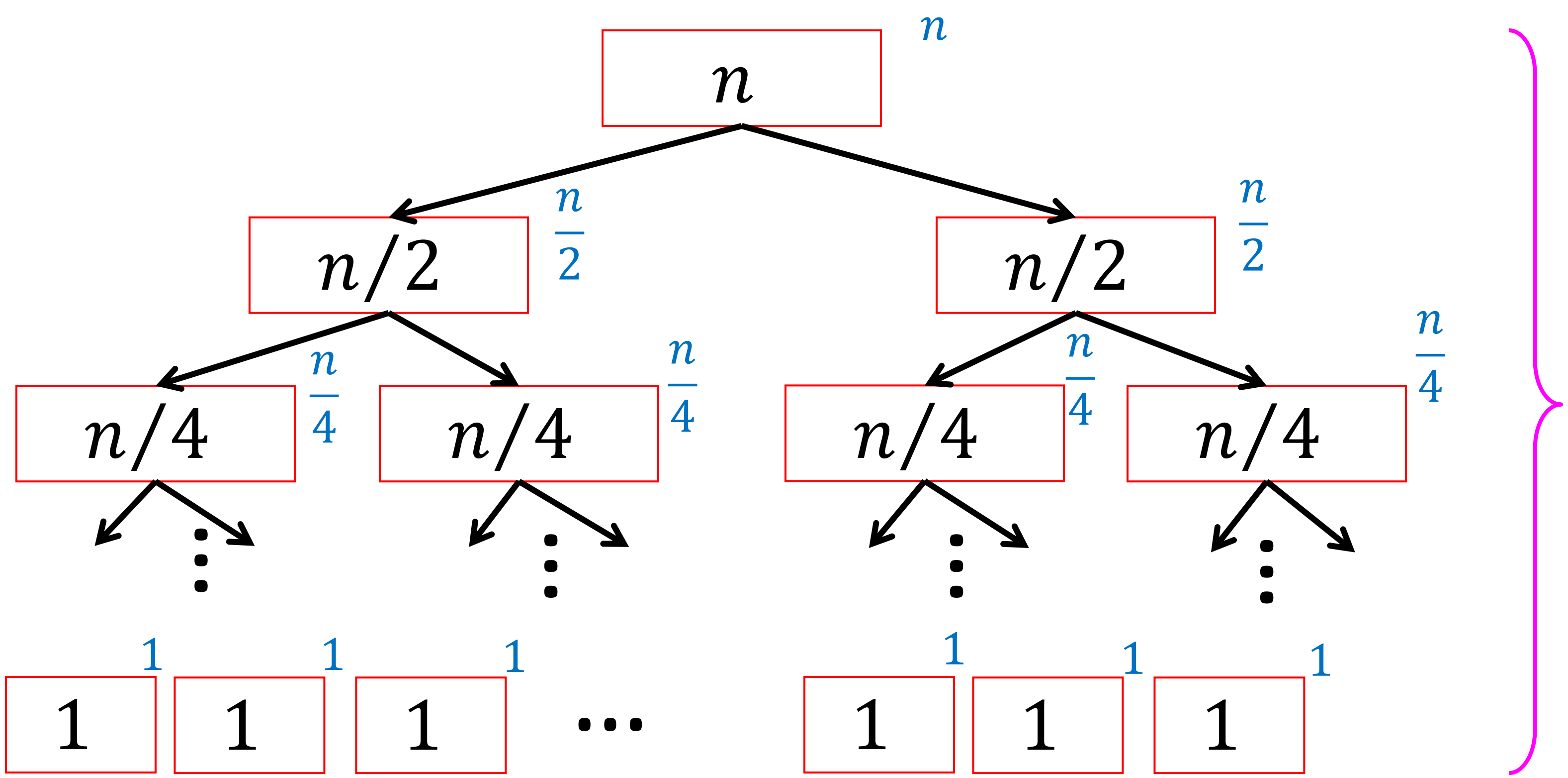
Mergesort

- **Divide:**
  - Break n-element list into two lists of ${}^{n}/_{2}$ elements
- **Conquer:**
  - If $n > 1$:
    - Sort each sublist recursively
  - If $n = 1$:
    - List is already sorted (base case)
- **Combine:**
  - Merge together sorted sublists into one sorted list

Parallelizable: Allow different machines to work on each sublist

Mergesort (Sequential)

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$



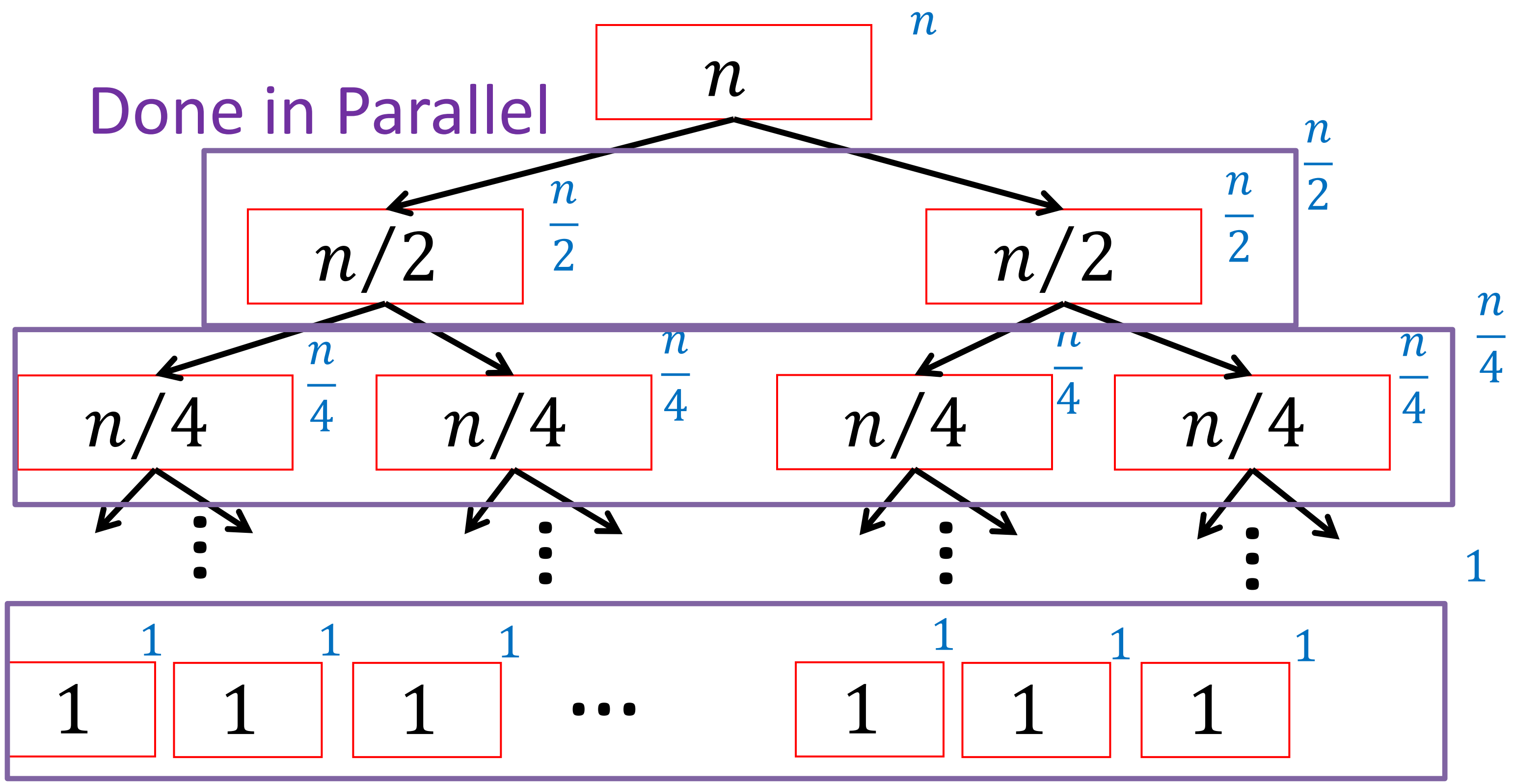
n total / level

$\log_2 n$ levels of recursion

Run Time: $\Theta(n \log n)$

Mergesort (Parallel)

$$T(n) = T\left(\frac{n}{2}\right) + n$$



Run Time: $\Theta(\log n)$

Quicksort

- Idea: pick a partition element, recursively sort two sublists around that element
- Divide: select an element p, Partition(p)
- Conquer: recursively sort left and right sublists
- Combine: Nothing!

Run Time?

$\Theta(n \log n)$

Optimal!

(almost always)

In Place?	Adaptive?	Stable?	Parallelizable?
No…	No!	No	Yes!