Live Session, Feb. 15
Quicksort and Closest Pair of Points

CS 4102: Algorithms

Spring 2021

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Monday, Feb. 15

- Gradescope is live (link via Collab to sign up)
  - Made some cosmetic changes to programming hws
- Recommended schedule up on schedule page of website.
  - Recommended that “distancing” be done this week
- Sample HW assignment is up on webpage and GS
  - For your benefit if you want it, not required.
- Office hours going ok?
- Don’t forget Wednesday is a break day so no class!
- Today we will do quicksort and closest pair of points!
- Correction: Closest pair of points readings: CLRS 33.4
- Self-assessment “quiz”: https://forms.gle/UY83NtmQT2PBdkXT8

Quicksort and Partition

Readings: CLRS Chapter 7 (not 7.4.2)

Quicksort's Strategy

- Called on subsection of array from *first* to *last*
  - Like mergesort
- First, choose some element in the array to be the ***pivot*** element
  - Any element! Doesn't matter for correctness.
  - Often the first item. For us, the last. Or, we often move some element into the last position (to get better efficiency)
- Second, call ***partition***, which does two things:
  - Puts the pivot in its proper place, i.e. where it will be in the correctly sorted sequence
  - All elements below the pivot are less-than the pivot, and all elements above the pivot are greater-than
- Third, use quicksort recursively on both sub-lists

Quicksort is Divide and Conquer

- Divide: select pivot element p, Partition(p)
- Conquer: recursively sort left and right sublists
- Combine: Nothing!

Contrast to mergesort,
where divide is simple and combine is work

Quicksort’s Strategy (a picture)

- Use last element as pivot (or pick one and move it there)

	pivot
first	***last***

  - After call to partition…

<= pivot (unsorted)	**pivot**	**> pivot (unsorted)**
first	***split point***	***last***

  - Now sort two parts recursively and we’re done!

<= pivot (sorted)	**pivot**	**> pivot (sorted)**
first	***split point***	***last***

  - Note that splitPoint may be anywhere in *first..last*
  - Note our assumption that all keys are distinct

Quicksort Code

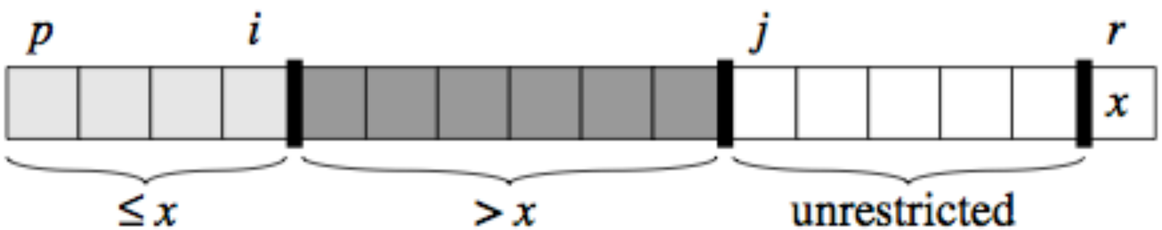
Input Parameters: *list, first, last*

Output Parameters: *list*

```
def quicksort(list, first, last):
   if first < last:
      q = partition(list, first, last)
      quicksort(list, first, q-1)
      quicksort(list, q+1, last)
   return
```

Strategy for Lomuto's Partition

- Invariant: At any point:
  - *i* indexes the right-most element <= *pivot*
  - *j-1* indexes the right-most element > *pivot*



- Strategy:
  - Look at next item *a[j]*
  - If that item > pivot, all is well!
  - If that item < pivot, increment *i* and then swap items at positions *i* and *j*
  - When done, swap pivot with item at position *i+1*
- Number of comparisons: n-1

Lomuto's Partition: Code

Input Parameters: *list, first, last*
Output Parameters: *list.*
Return value: the split point

```
def partition(list, first, last):
    pval = list[last]
    i = first-1
    for j in range(first, last): # first up to before last
        if list[j] <= pval:
            i = i+1
            (list[i], list[j]) = (list[j], list[i]) # swap!
    (list[last], list[i+1]) = (list[i+1], list[last]) # swap!
    return i+1
```

Partition this: [c t o a m b f]

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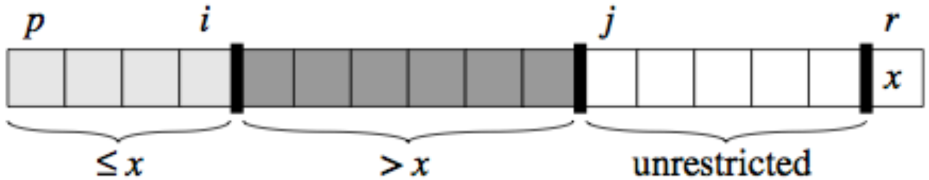

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  - Look at next item *a[j]*
  - If that item > pivot, all is well!
  - If that item < pivot, increment *i* and then swap items at positions *i* and *j*
  - When done, swap pivot with item at position *i+1*

Partition this: [c t o a m b f]

- Was that a “good” result? Correct? Desirable for some other reason?
- Let’s do a randomized partition now!

Efficiency of Quicksort

- Partition divides into two sub-lists, perhaps unequal size
  - Depends on value of pivot element
- Recurrence for Quicksort

  T(n) = partition-cost +

  T(size of 1st section) + T(size of 2nd section)
- If divides equally, $T(n) = 2\,T(n/2) + n-1$
  - Just like mergesort
  - Solve by substitution or master theorem

    $T(n) \in \Theta(n \lg n)$
- This is the best-case. But…

Worst Case of Quicksort

- What if divides in most unequal fashion possible?
  - One subsection has size 0, other has size n-1
  - T(n) = T(0) + T(n-1) + n-1
  - What if this happens every time we call partition recursively?

$$W(n) = \sum_{k=2}^{n} (k-1) \in \Theta(n^2)$$

  - Uh oh. Same as insertion sort.
    - "Sorry Prof. Hoare – we have to take back that Turing Award now!"

Quicksort's Average Case

- Good if it divides <u>equally</u>, bad if <u>most unequal</u>.
  - Remember: when subproblems size 0 and n-1
  - Can worst-case happen?
    Sure! Many cases. One is when elements already sorted. Last element is max, pivot around that. Next pivot is 2^{nd} max…
- What's the average?
  - Much closer to the best case
  - A bad-split then a good-split is closer to best-case (pp. 176-178)
  - To prove A(n), fun with recurrences!
  - The result: If all permutations are equal, then
    $A(n) \cong 1.386\ n \lg n$ (for large n)
- So very fast on average.
- And, we can take simple steps to avoid the worst case!

Avoiding Quicksort's Worst Case

- Make sure we don't pivot around max or min
  - Find a better choice and swap it with last element
  - Then partition as before
- Recall we get best case if divides equally
  - Could find median. But this costs $\Theta(n)$. Instead…
  - Choose a **random element** between first and last and swap it with the last element
  - Or, estimate the median by using the "median-of-three" method
    - Pick 3 elements (say, first, middle and last)
    - Choose median of these and swap with last. (Cost?)
    - If sorted, then this chooses real median. Best case!

Tuning Quicksort's Performance

- In practice quicksort runs fast
  - A(n) is log-linear, and the "constants" are smaller than mergesort and heapsort
  - Often used in software libraries
  - So worth tuning it to squeeze the most out of it
  - Always do something to avoid worst-case
- Sort small sub-lists with (say) insertion sort
  - For small inputs, insertion sort is fine
    - No recursion, function calls
  - Variation: don't sort small sections at all.
    After quicksort is done, sort entire array with **insertion sort**
    - It's efficient on almost-sorted arrays!

Quicksort's Space Complexity

- Looks like it's in-place, but there's a *recursion stack*
  - Depends on your definition: some people define *in-place* to **<u>not</u>** include stack space used by recursion
    - E.g. our CLRS algorithms textbook
    - Other books and people do "count" this
  - How much goes on the stack?
    - If most uneven splits, then $\Theta(n)$.
    - If splits evenly every time, then $\Theta(\lg n)$.
- Ways to reduce stack-space used due to recursion
  - Various books cover the details (not ours, though)
  - First, remove 2nd recursive call (tail-recursion)
  - Second, always do recursive call on smaller section

Summary: Quicksort

- Divide and conquer where divide does the heavy-lifting
- In worst-case, efficiency is $\Theta(n^2)$
  - But it's practical to avoid the worst-case
- On average, efficiency is $\Theta(n \lg n)$
- Better space-complexity than mergesort.
- In practice, runs fast and widely used
  - Many ways to tune its performance
- Various strategies for Partition
  - Some work better if duplicate keys
- More details? See Sedgewick's algorithms textbook
  - He's the expert! PhD on this under Donald Knuth

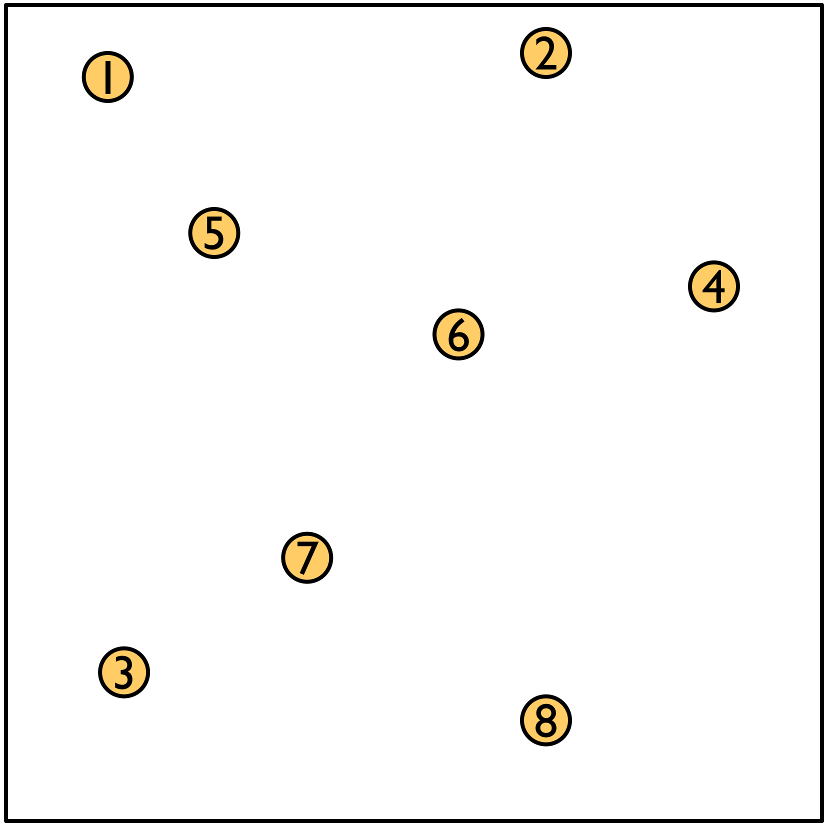
Closest Pair of Points

Readings: CLRS 33.4

Closest Pair of Points in 2D Space

Given:
A list of points

Return:
Distance of the pair of points that are closest together
(or possibly the pair too)



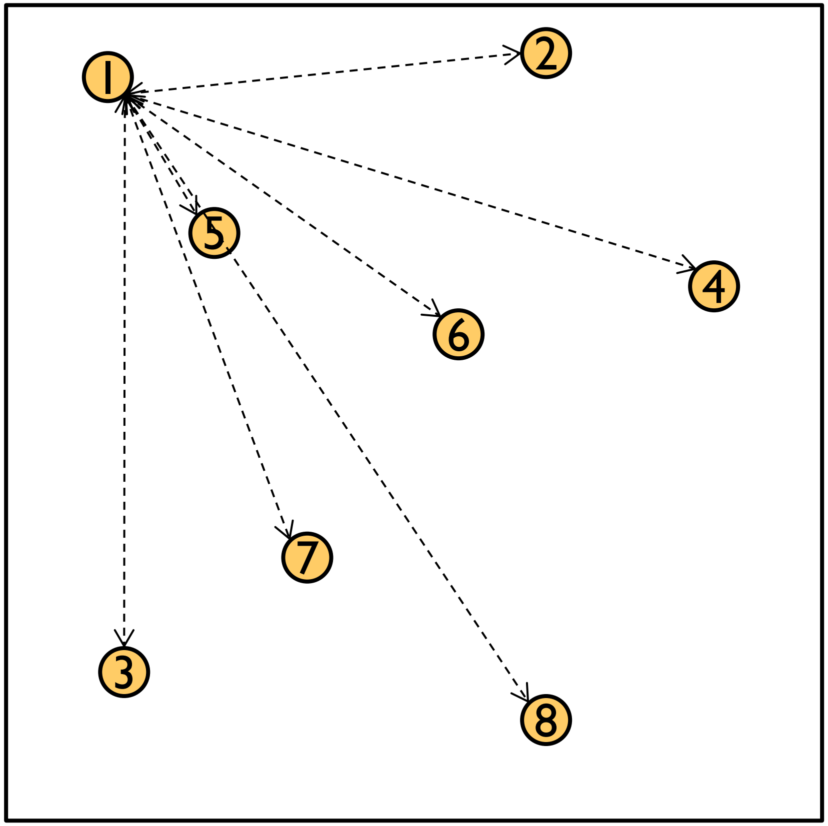
Closest Pair of Points: Naïve

Given:
A list of points

Return:
Distance of the closest pair of points

Naive Algorithm: $O(n^2)$
Test every pair of points, return the closest.

We can do better!
$\Theta(n \log n)$



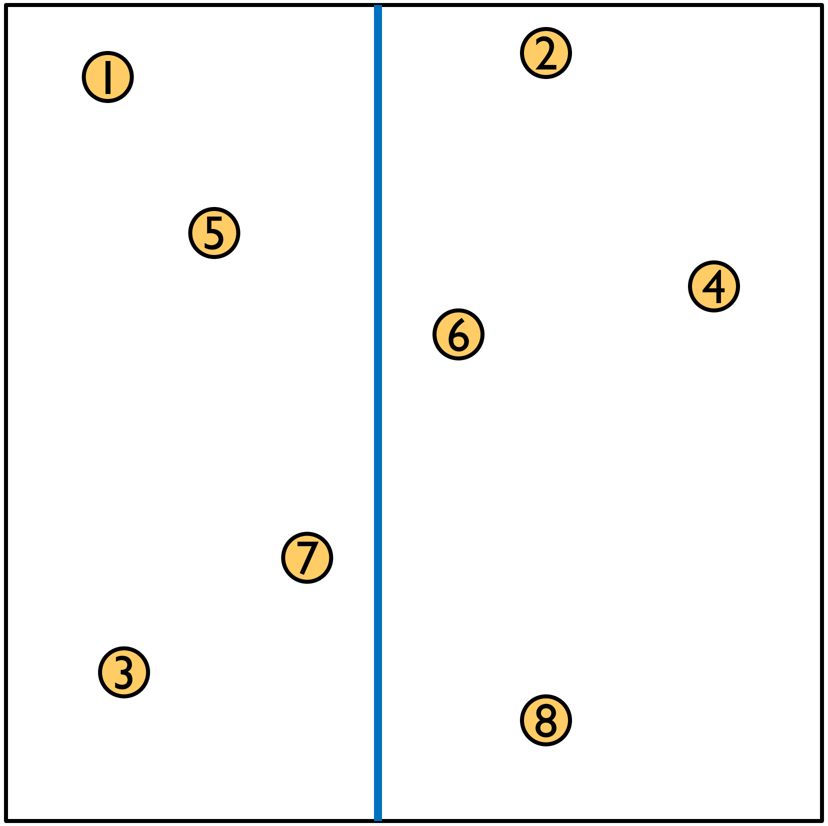
Closest Pair of Points: D&C

Divide: How?

At median x coordinate

Conquer:

Combine:

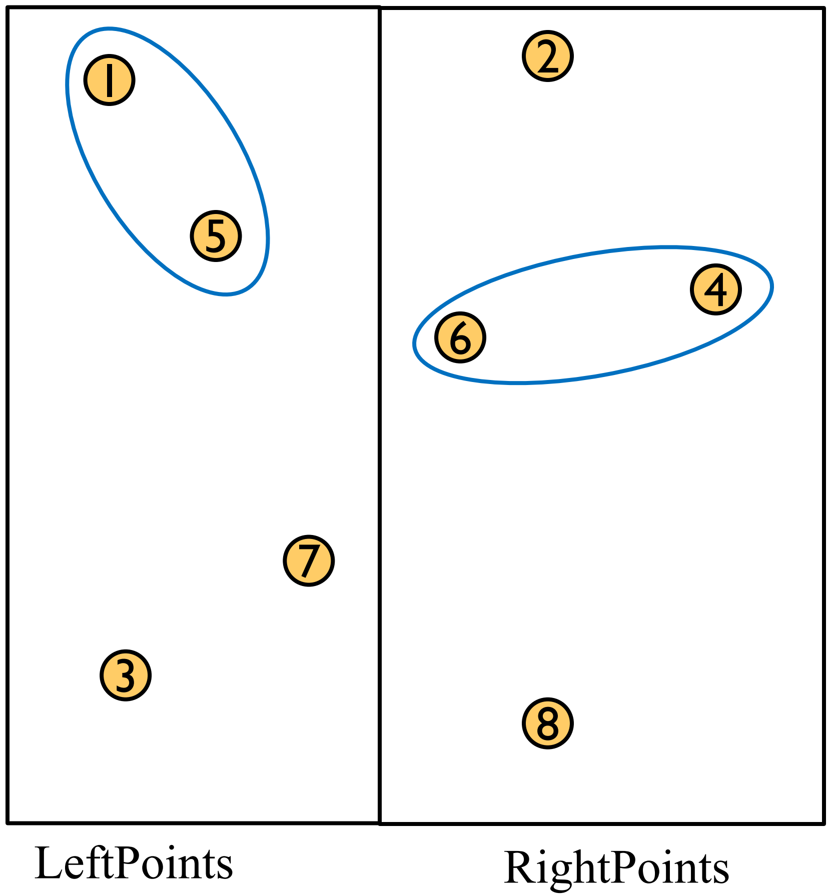


Closest Pair of Points: D&C

Divide:
At median x coordinate

Conquer:
Recursively find closest pairs from Left and Right

Combine:



Closest Pair of Points: D&C

Divide:

At median x coordinate

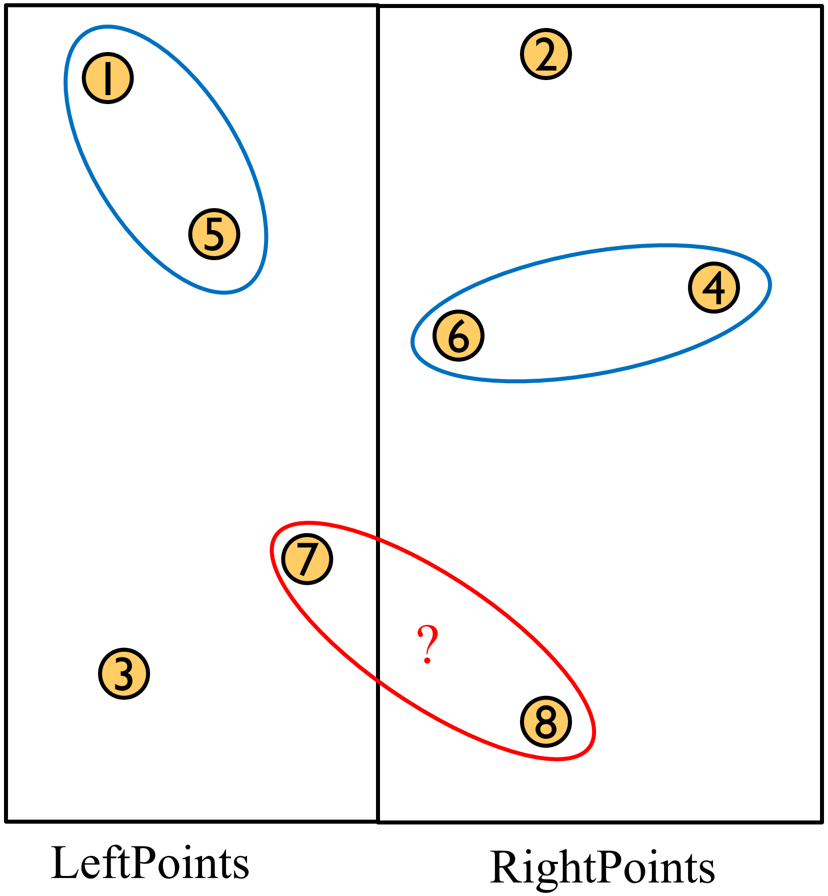
Conquer:

Recursively find closest pairs from Left and Right

Combine:

Return min of Left and Right pairs

Problem ?



Closest Pair of Points: D&C

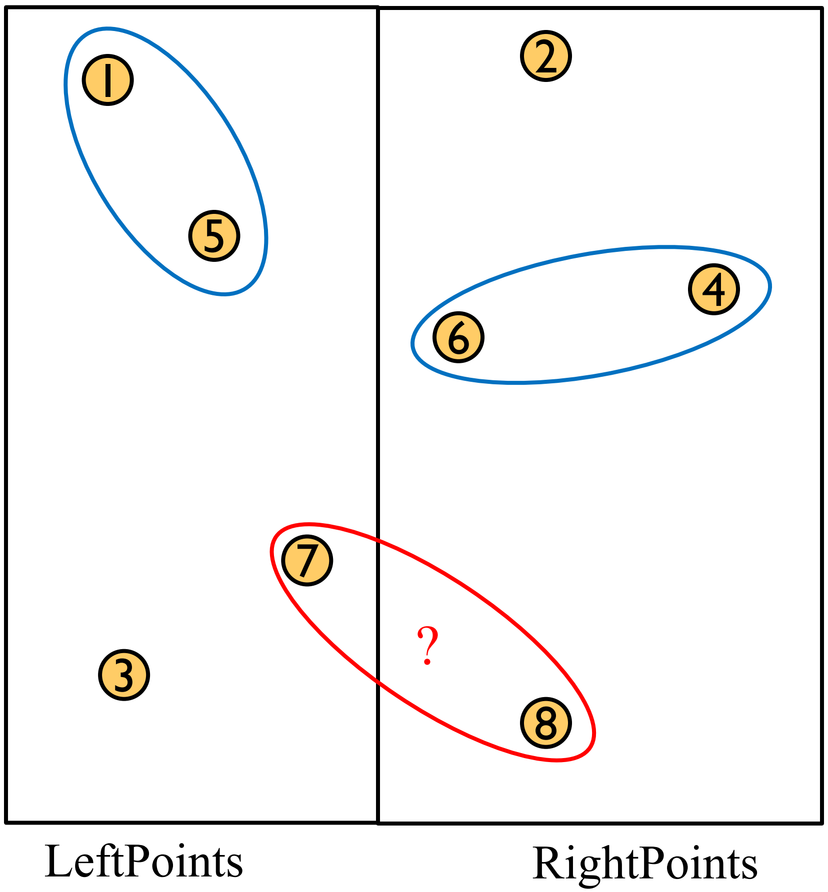
Combine:

2 Cases:

1. Closest Pair is completely in Left or Right

2. Closest Pair Spans our "Cut"

Need to test points across the cut



Spanning the Cut

Combine:

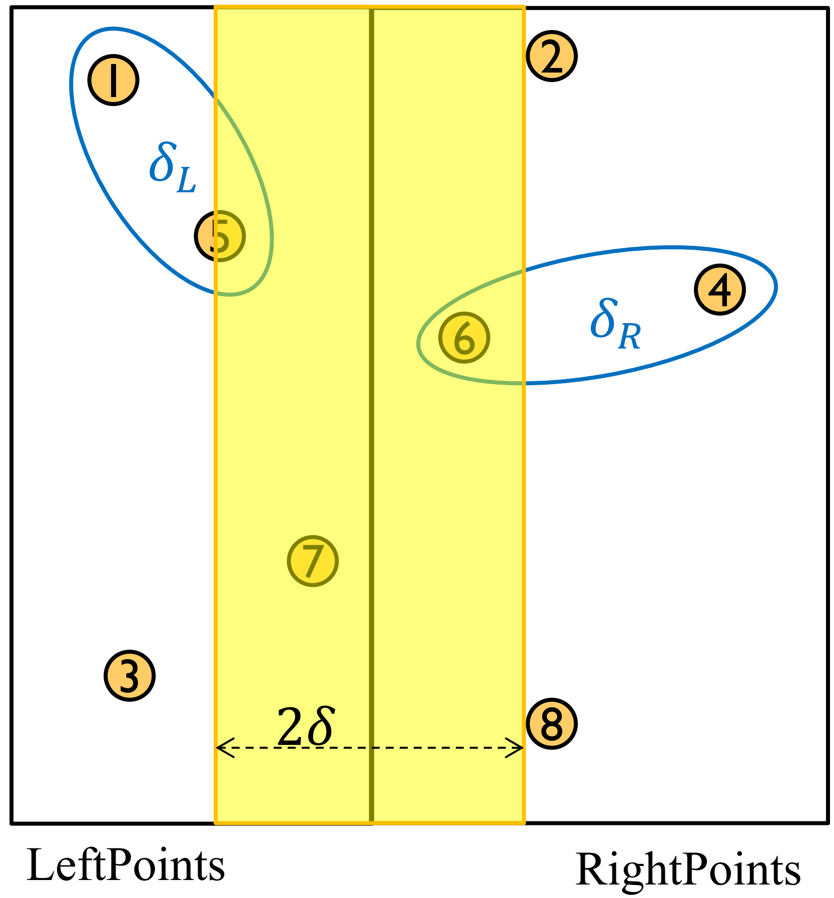
2. Closest Pair Spanned our "Cut"

Need to test points across the cut.

Bad approach: Compare all points within $\delta = \min\{\delta_L, \delta_R\}$ of the cut.

How many are there?

Define "runway" or "strip" along the cut.



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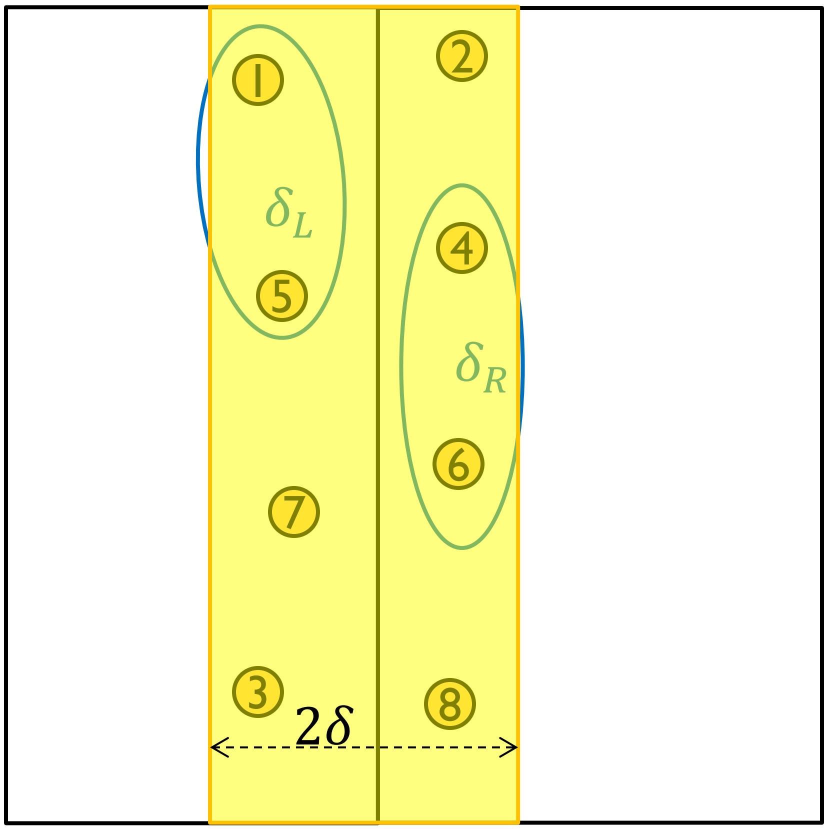
How many are there?

$$\boldsymbol{T(n) = 2T\left(\frac{n}{2}\right) + \left(\frac{n}{2}\right)^2}$$
$$\boldsymbol{= \Theta(n^2)}$$



Spanning the Cut

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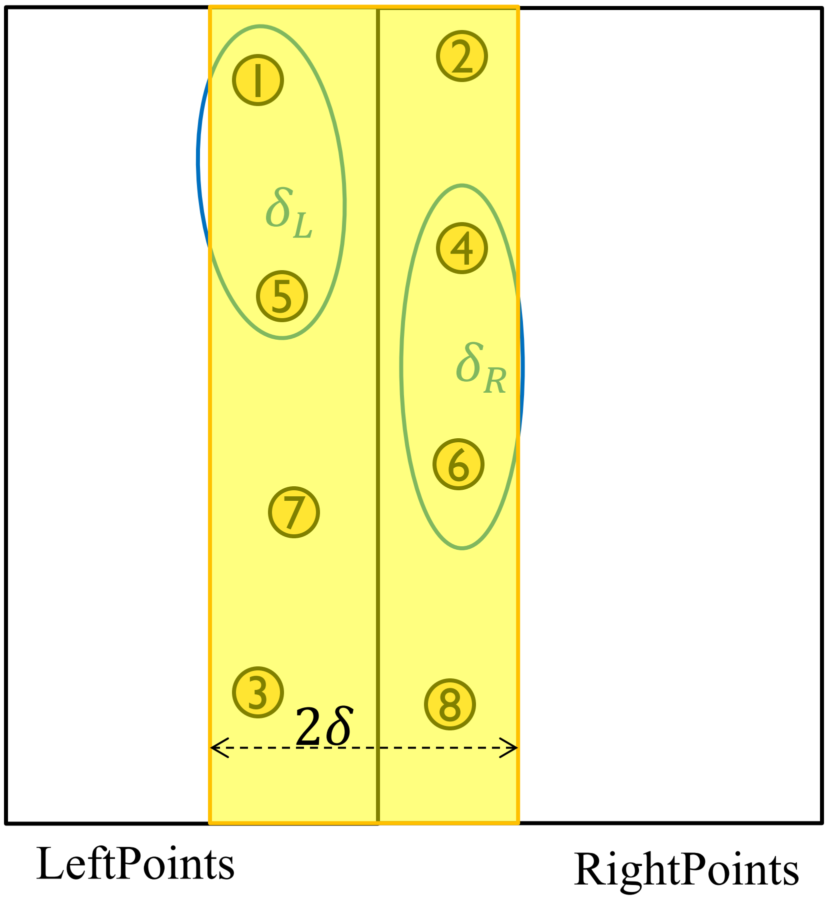
Need to test points across the cut

We don't need to test all pairs!

Don't need to test any points that are > δ from one another



Spanning the Cut

Combine:

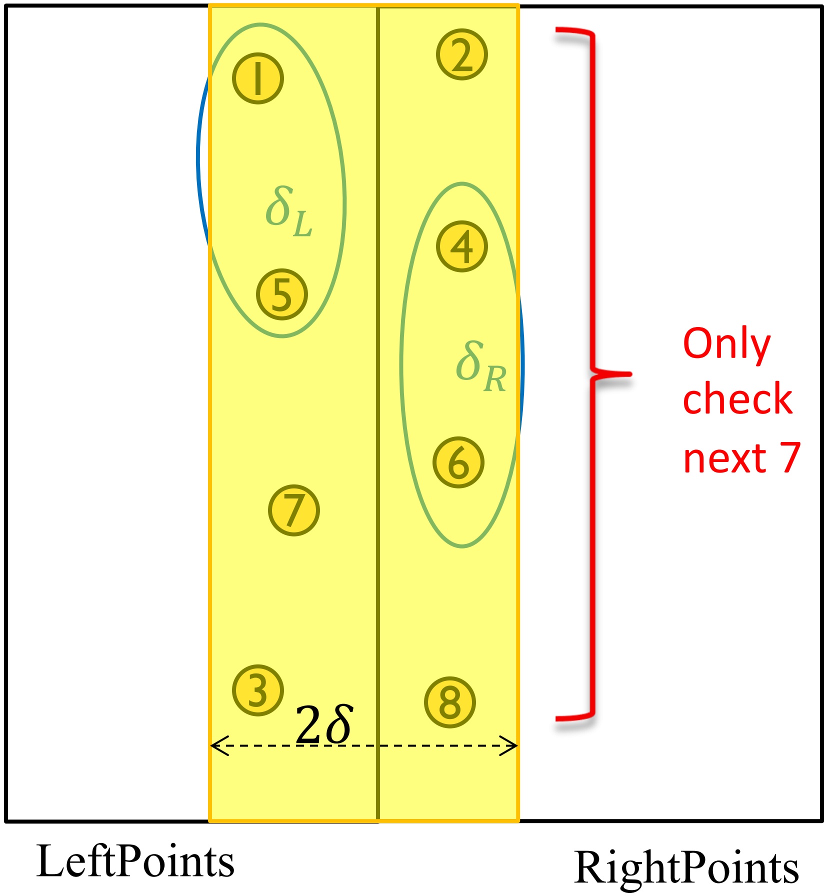
2. Closest Pair Spanned our "Cut"

Consider points in strip in increasing y-order.

For a given point p, we can *prove* the 8th point and beyond is more than δ from p.
(pp. 1041-2 in CLRS)

So for each point in strip, check next 7 points in y-order.

$\boldsymbol{\Theta(n)}$ ***Better*!**



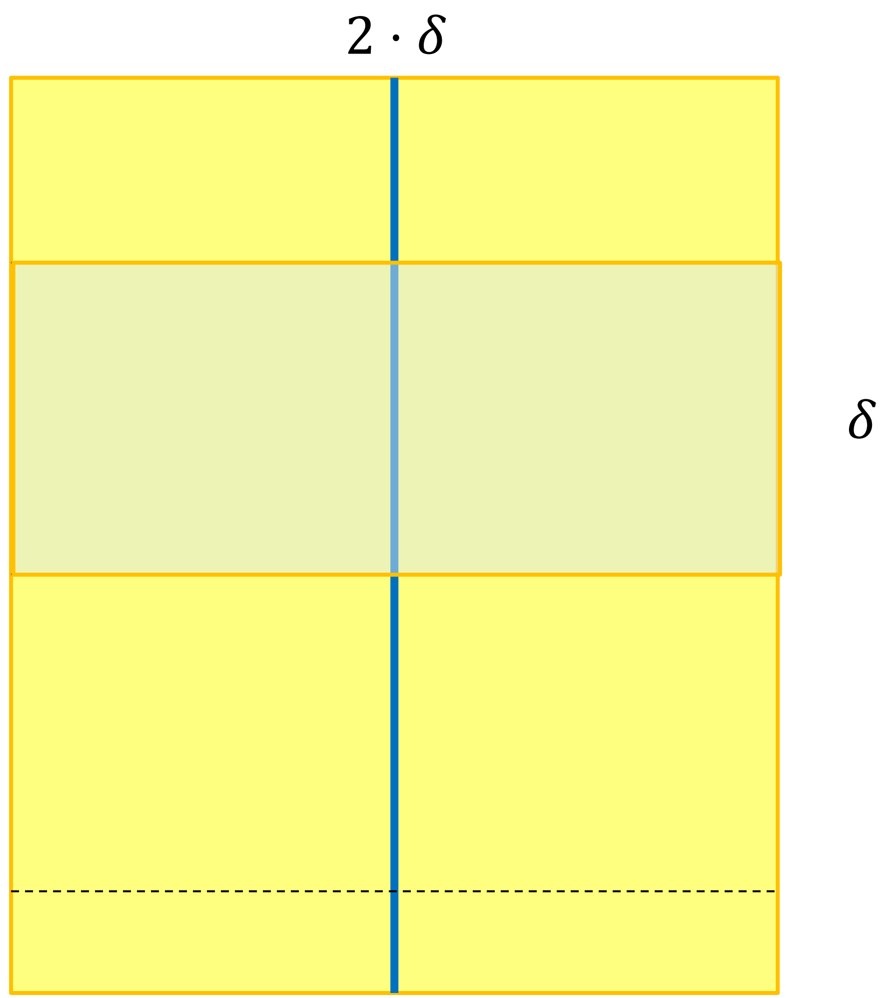
Reducing Search Space

Combine:

Need to test points across the cut

Claim #1: if two points are the closest pair that cross the cut, then you can surround them in a box that's $2 \cdot \delta$ wide by δ tall.

Let's draw some examples.

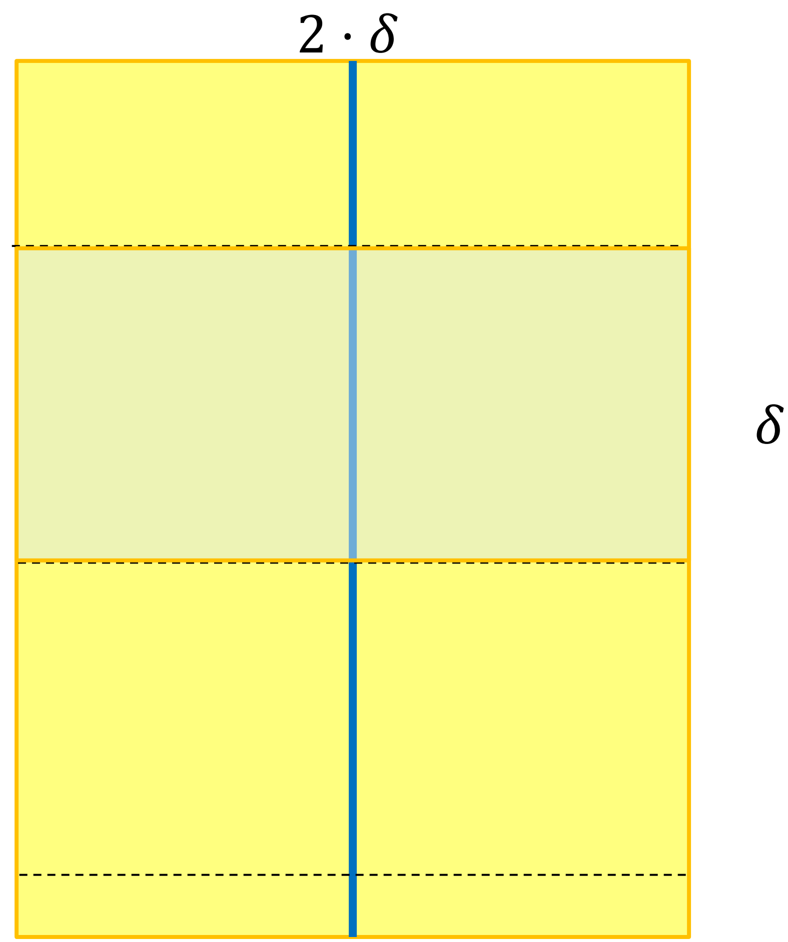


Reducing Search Space

Claim #1: if two points are the closest pair that cross the cut, then you can surround them in a box that's $2 \cdot \delta$ wide by δ tall.

Assume you're checking in increasing y-order, and you've reached the first point of the closest pair.
Do you have to look at **all points above it** to be <u>guaranteed</u> to find the other point and the minimum distance?

No!
- Imagine you drew a box with its bottom at point's y-coordinate.
- See Claim #1.
- Claim #2: only 8 points can be in the box.



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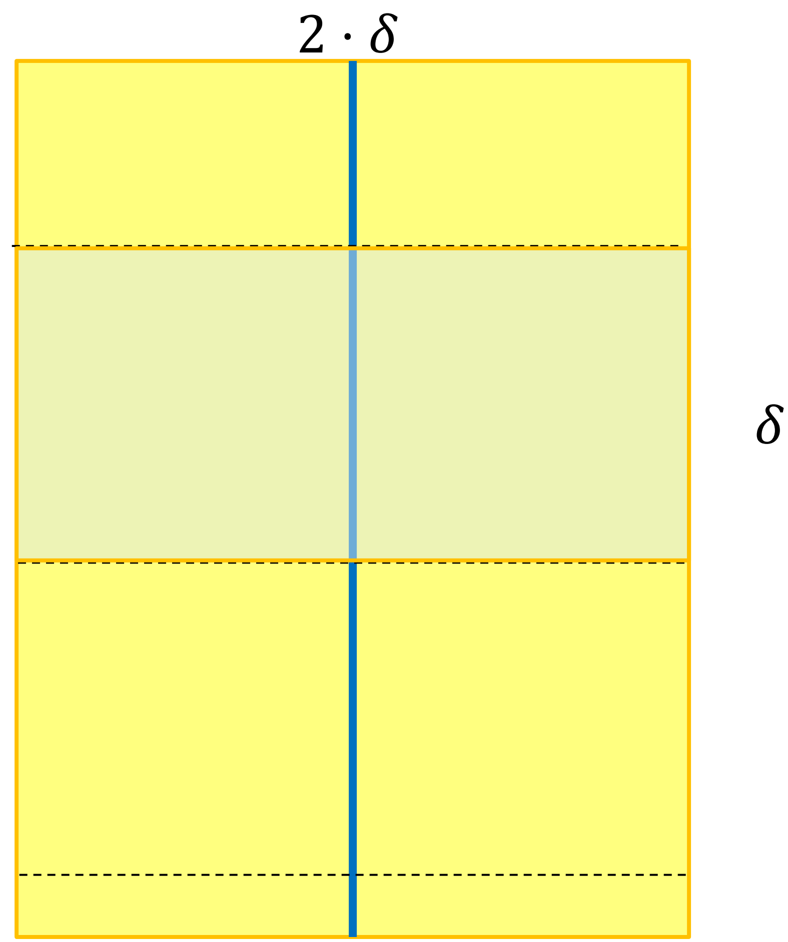

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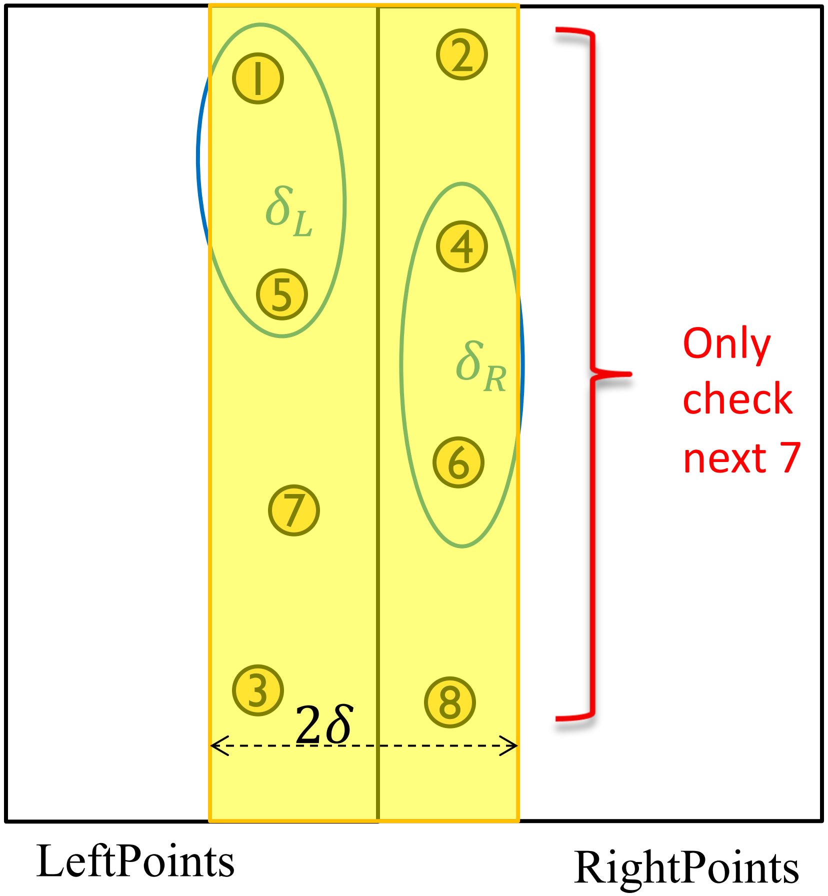


Closest Pair of Points: Divide and Conquer

Initialization: Sort points by x-coordinate (Later we'll also need to process points by y-coordinate, too.)

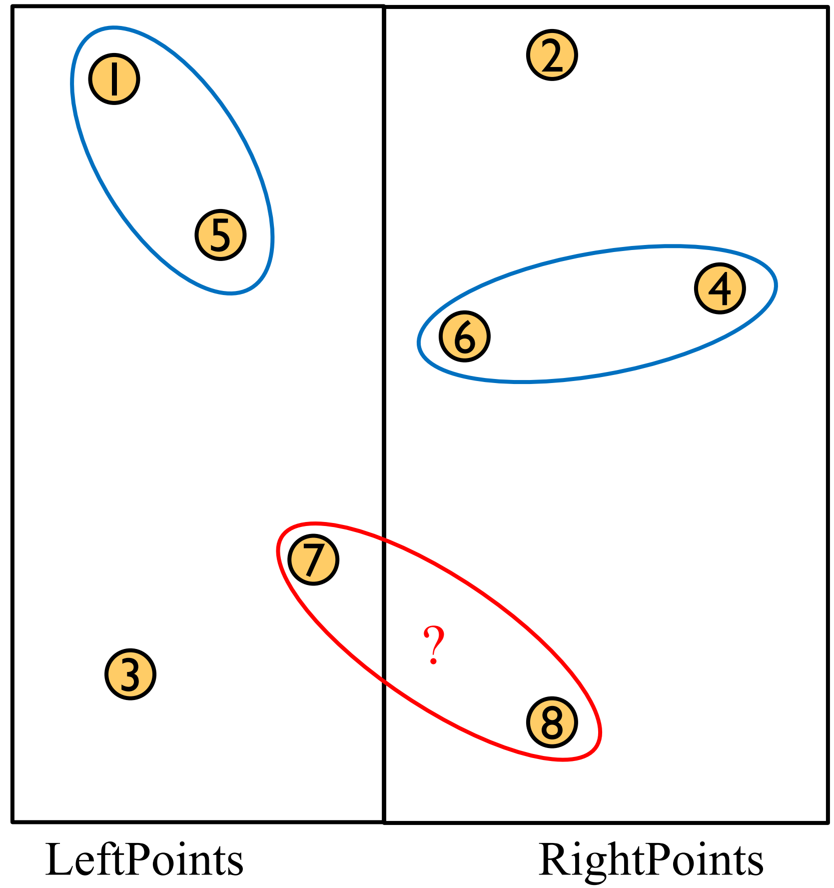
Divide: Partition points into two lists of points based on x-coordinate (split at the median x)

Conquer: Recursively compute the closest pair of points in each list

Base case?

Combine:

- Consider only points in the runway (x-coordinate within distance δ of median)
- Process runway points by y-coordinate
- Compare each point in runway to 7 points above it and save the closest pair
- Output closest pair among left, right, and runway points



Closest Pair of Points: Divide and Conquer

What is the running time?

$\Theta(n \log n)$

$T(n) = 2T(n/2) + \Theta(n)$

Case 2 of Master's Theorem

$T(n) = \Theta(n \log n)$

$\Theta(n \log n)$ **Initialization:** Sort points by x-coordinate

$$T(n) \begin{cases} \Theta(1) & \textbf{Divide: } \text{Partition points into two lists of points based on } x\text{-coordinate (split at the median } x) \\ 2T(n/2) & \textbf{Conquer: } \text{Recursively compute the closest pair of points in each list} \\ \Theta(n) & \textbf{Combine: } \text{Process runway points by } y\text{-coordinate and compare each point in runway to 7 points above it and save the closest pair} \\ \Theta(1) & \text{Output closest pair among left, right, and runway points} \end{cases}$$

Combine:

- Process runway points by y-coordinate and Compare each point in runway to 7 points above it and save the closest pair
- Output closest pair among left, right, and runway points

Summary for Closest Pair of Points

- Comparing all pairs is a brute-force fail
  - Except for small inputs
- Divide and conquer a big improvement
- Needed to find an efficient way for part of the combine step
  - Geometry came through for us here!
  - Only needed to look at constant number of points for each point in the strip
- Implementation subtleties
  - Don’t want to sort the strip by y-coordinate in each recursive call
  - In initialization, create an “index” that lets you process all points in order by y-coordinate
  - (There are other ways to address this.)