The Graph Data Structure

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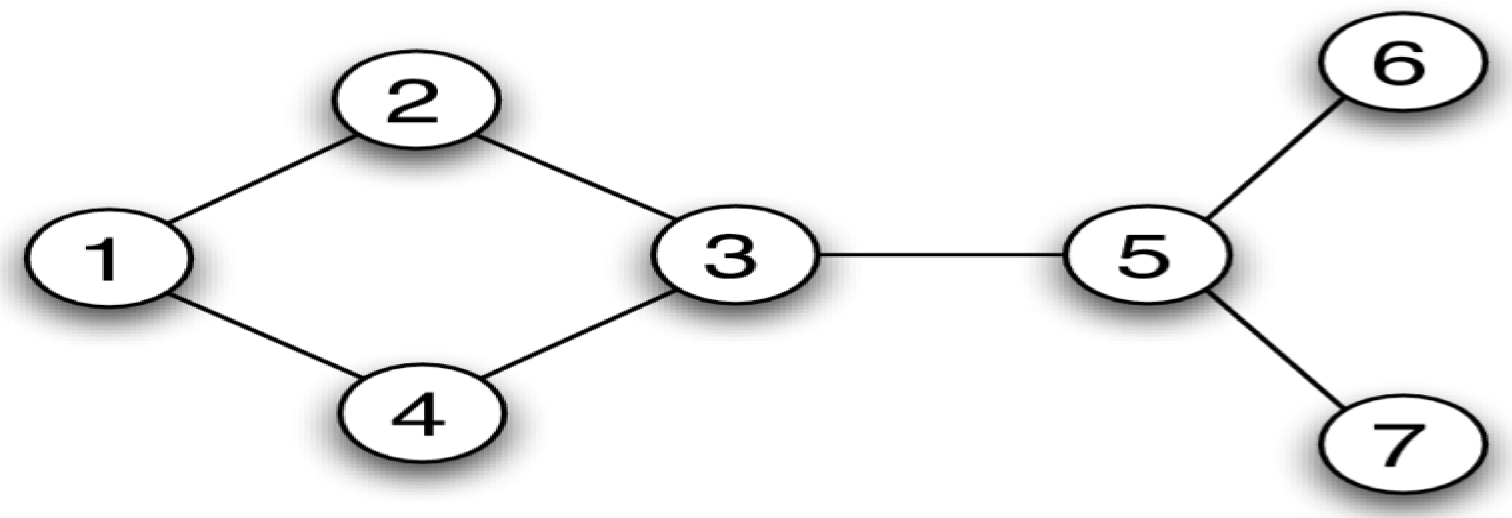
Graphs

- A graph is a data structure that is commonly represented as $G = (V, E)$, where V is a set of vertices and E is a set of edges
- Graphs are commonly used to represent a large number of real world problems
  - Railways, roadways, airline routes, transmission towers etc.
  - Routing traffic over the Internet
  - Representing game outcomes
  - Representing a problem search space

Graph Terminology

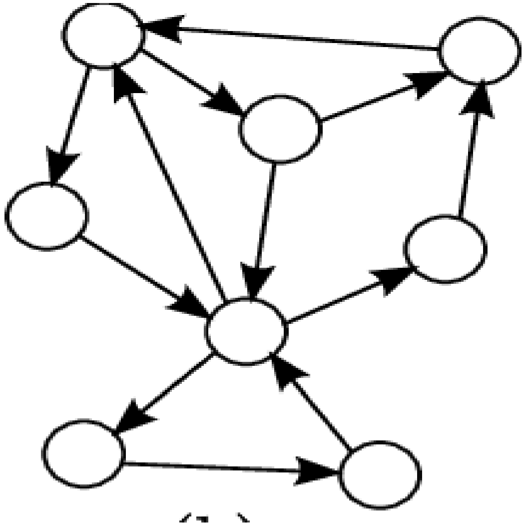
- A graph is a collection of vertices, V and a collection of edges, E
- Every edge $e \in E$ can be represented as $\{u, v\}$ where $u, v \in V$



Graph Terminology

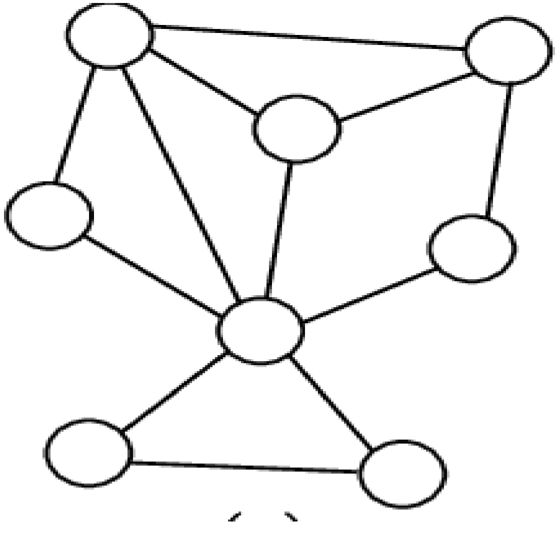
Directed Graphs

- A graph in which the edges are represented as **ordered pairs (u, v)** are called directed graphs
- Edges have direction



Undirected Graphs

- A graph in which the edges are represented as **unordered pairs (u, v)** are called undirected graphs
- Edges do not have direction



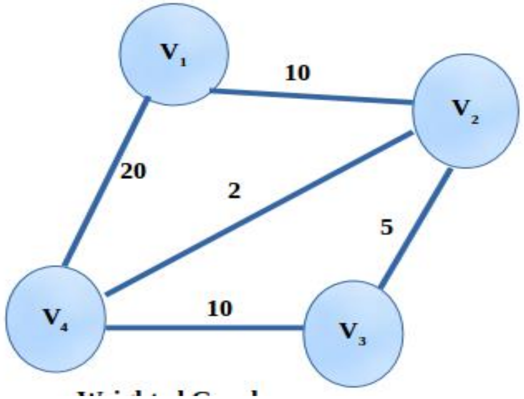
Graph Terminology

- Graphs without loops and parallel edges are often called simple graphs; non-simple graphs are sometimes called multigraphs
- For any edge *u-v* in an undirected graph, we call u a neighbor of v and vice versa, and we say that u and v are adjacent.
- The degree of a node is its number of neighbors.
- For any directed edge *u-v*, we call u a predecessor of v, and we call v a successor of u. The in-degree of a vertex is its number of predecessors; the out-degree is its number of successors

Graph Terminology

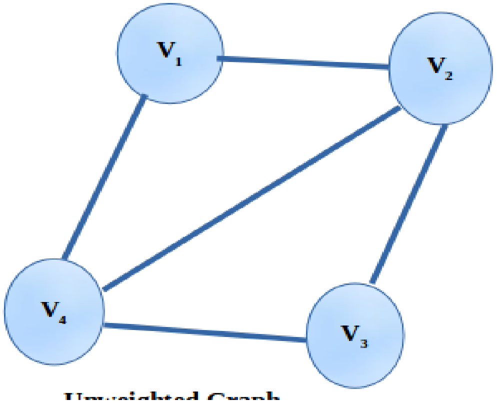
Weighted Graphs

- A graph in which the edges is assigned a numeric value (called weight) are called weighted graphs
- Can represent path length, delay etc.



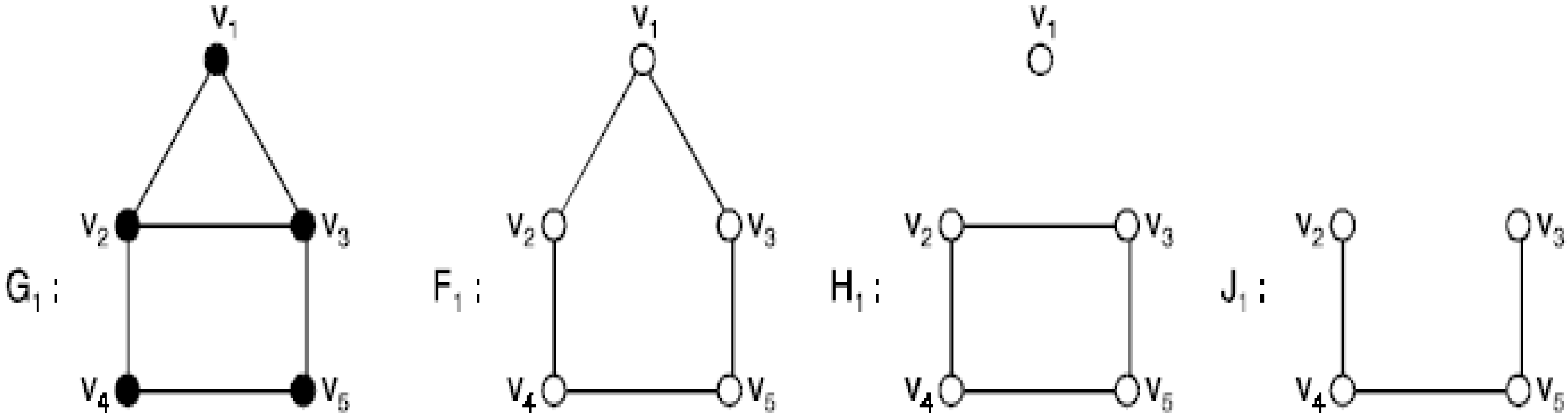
Undirected Graphs

- A graph in which the edges do not have any numeric value associated with them are called unweighted graphs



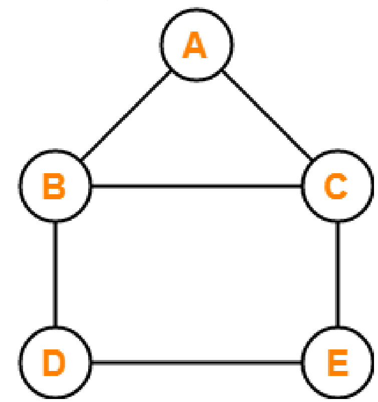
Graph Terminology

- A graph $G' = (V', E')$ is a subgraph of G = (V, E) if $V' \subseteq V$ and $\mathrm{E}' \subseteq E$
  - By definition, G is a subgraph of itself
- A proper subgraph of G is any subgraph other than G itself.



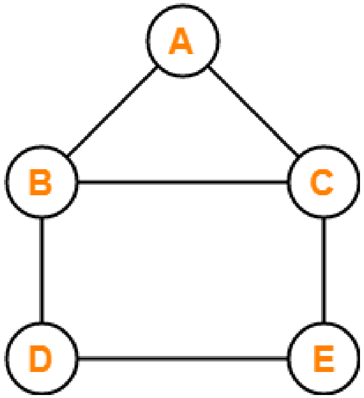
Walks, Trails, Paths

- A walk is a sequence of vertices, where each adjacent pair of vertices are adjacent in G
  - A vertex can be traversed more than once
  - An edge can be used more than once
- Eg: d b a c e d e c is a walk of length 7
- If a walk starts and ends at the same vertex it is called a closed walk
  - Otherwise it is called an open walk



Walks, Trails, Paths

- A path is a walk in which each vertex is visited at most once
  - A vertex cannot be traversed twice
  - An edge can be used more than once
- Eg: a b c e d is a path of length 4



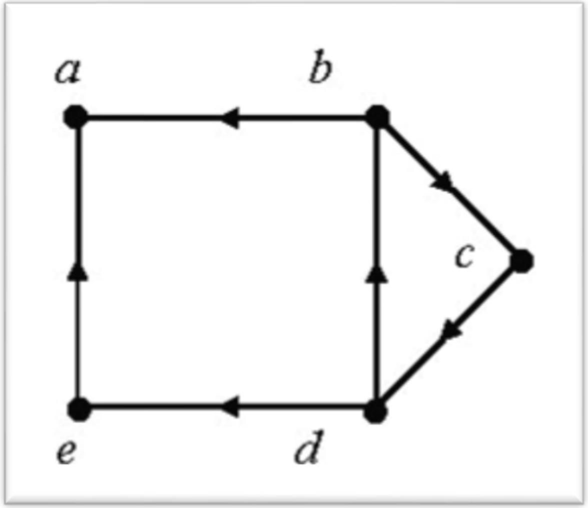
- A trail is a walk in which an edge can be traversed at most once
  - A vertex can be visited more than once
  - An edge can only be visited once

Connected Graphs

- Two vertices u and v in a graph G, v is said to be reachable from u if there exists a path between u and v.
- An undirected graph is connected if every vertex is reachable from every other vertex.
- Every undirected graph consists of one or more components, which are its maximal connected subgraphs; two vertices are in the same component if and only if there is a path between them

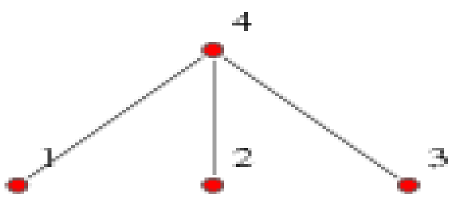
Connected Graphs

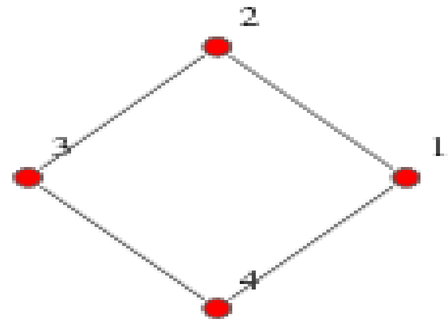
- A directed graph G is said to be strongly connected if there exists a path between any two vertices u and v
  - Note that the path needs to be a directed path
- The given graph is not strongly connected (why?)
- A directed graph G is said to be weakly connected if the graph is not strongly connected, but the underlying undirected graph is connected

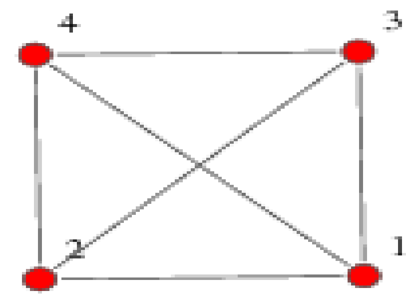


Adjacency Matrix

- An adjacency matrix is a |V| x |V| matrix
  – A[i, j] = 1 if there is an edge from vertex u to vertex v
  – Otherwise, A[i, j] = 0







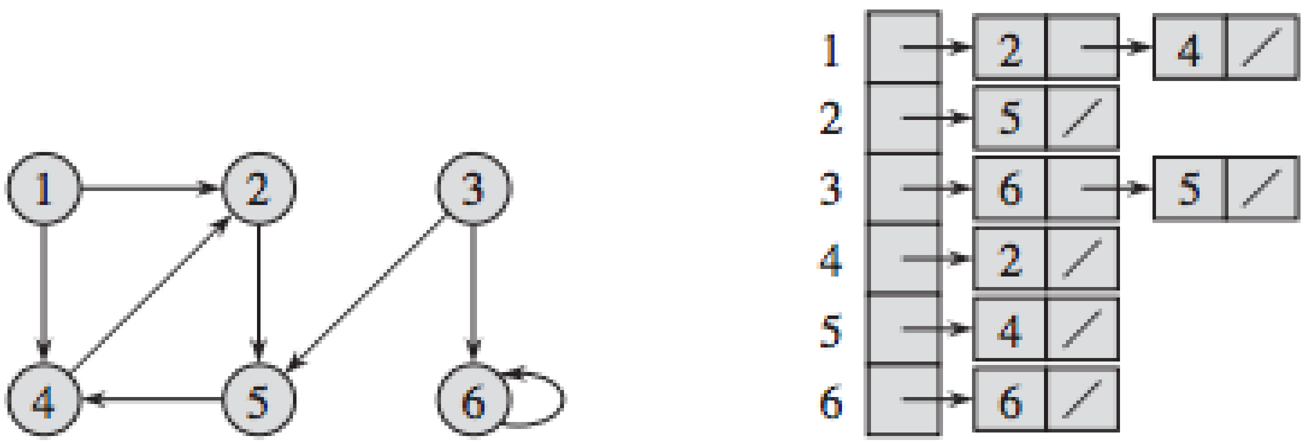
$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Adjacency Matrix

- For undirected graphs, the adjacency matrix is always symmetric, meaning A[u, v] = A[v, u] for all vertices u and v and the diagonal entries A[u, u] are all zeros.
- For directed graphs, the adjacency matrix may or may not be symmetric, and the diagonal entries may or may not be zero.
- Given an adjacency matrix
  - We can decide in $\theta(1)$ time whether two vertices are connected by an edge
  - We can also list all the neighbors of a vertex in $\theta(V)$ time.
  - Adjacency Matrices require $\theta(V^2)$ space, regardless of how many edges the graph actually has
- Wastage of space and time for sparse graphs

Adjacency List

- An adjacency list is an array of lists, each containing the neighbors of one of the vertices
  - For undirected graphs, each edge u-v is stored twice; for directed graphs, each edge u-v is stored only once



(a) (b)

Adjacency List

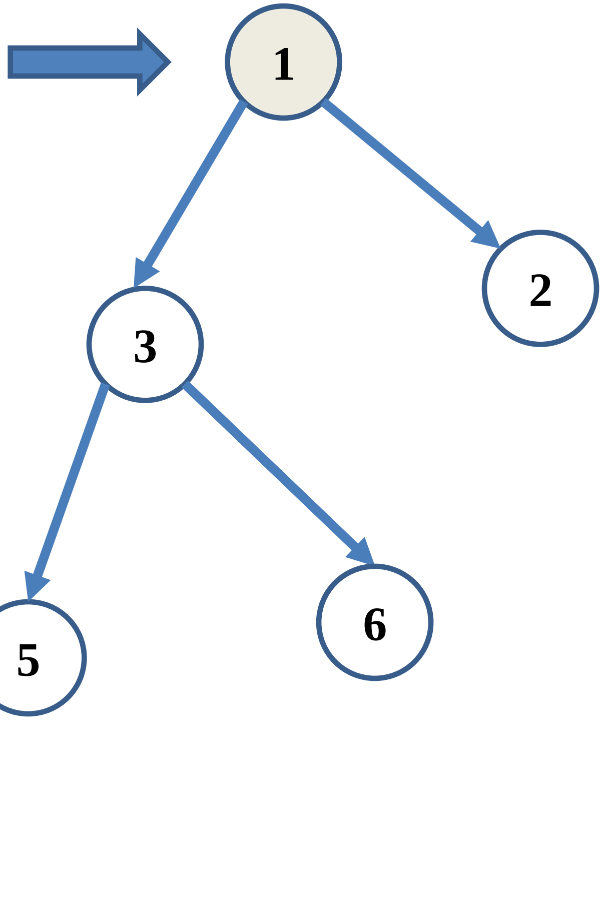
- Given an adjacency list
  - We can list the neighbors of a node v in O(1 + deg(v)) time
  - We can determine whether u-v is an edge in O(1 + deg(u)) time
  - We can traverse the graph in O(V + E) time
  - Adjacency lists require a space complexity of O(V + E)
- Adjacency lists are usually preferred for representing sparse graphs, but dense graphs can be more efficiently represented using adjacency matrices

Depth First Search

- Depth First Search (DFS) is a graph traversal algorithm
- As the name suggests, DFS explores one path in a graph completely before exploring a new one
- When DFS hits a dead end on a path, it backtracks and starts exploring a new path from the previous node
- This behaviour is suggestive of a LIFO data structure – STACK
- DFS can be implemented easily with recursion which uses an implicit stack

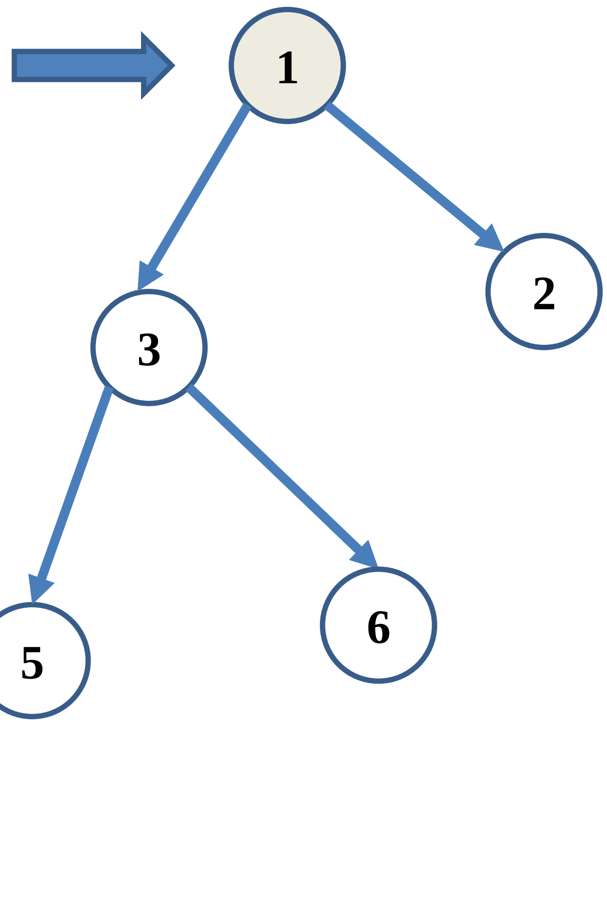
Depth First Search

DFS is a graph traversal algorithm



Depth First Search

DFS explores one path completely before moving on to another



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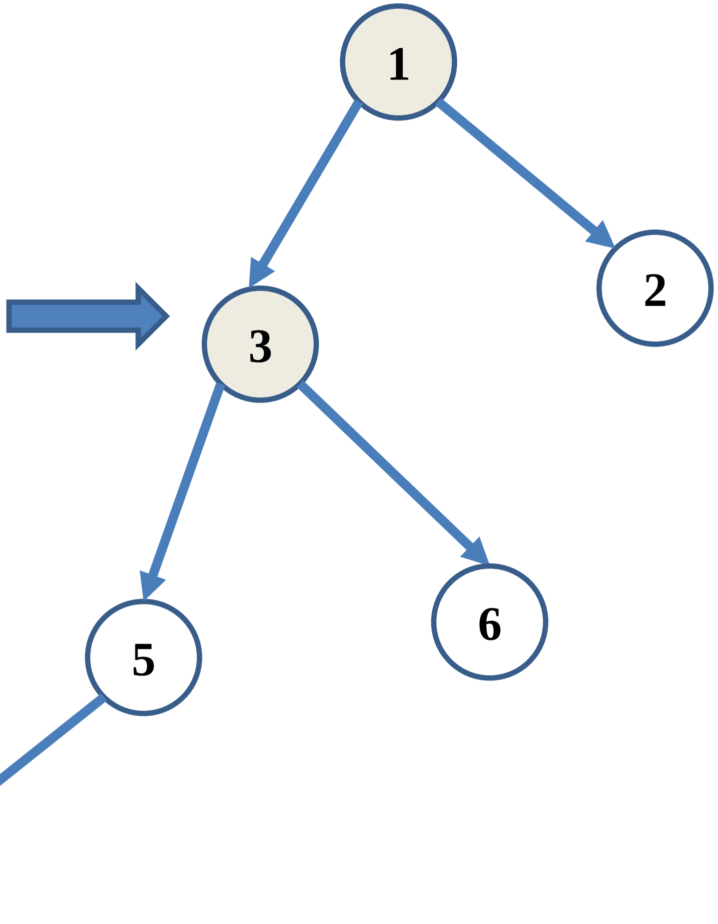

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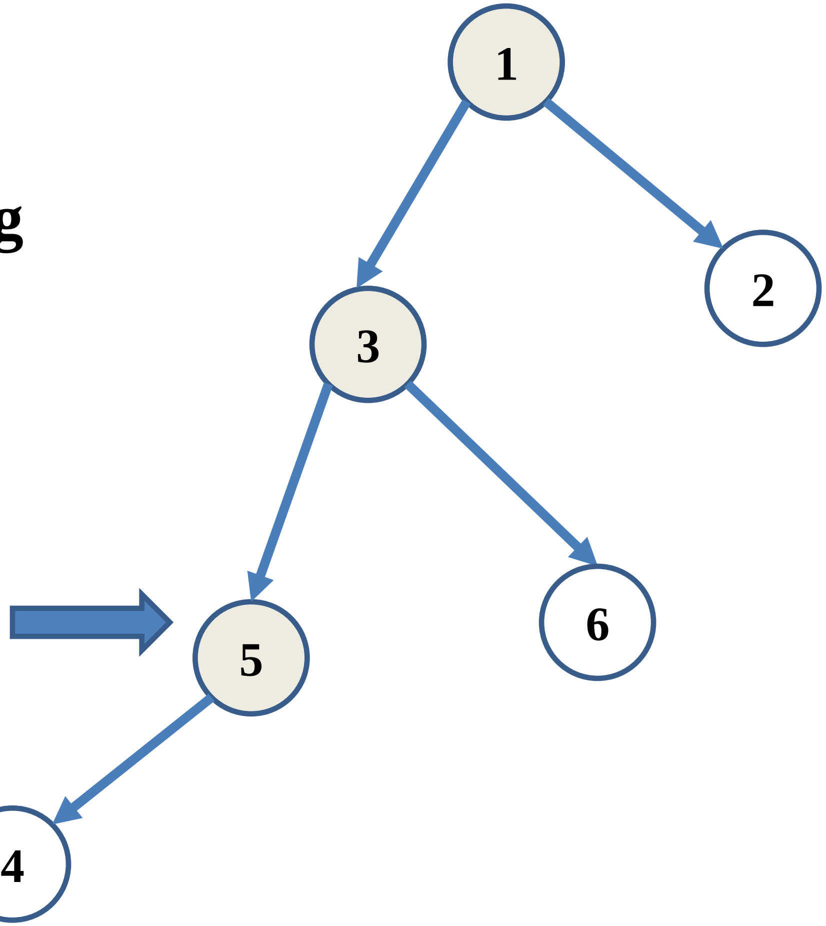

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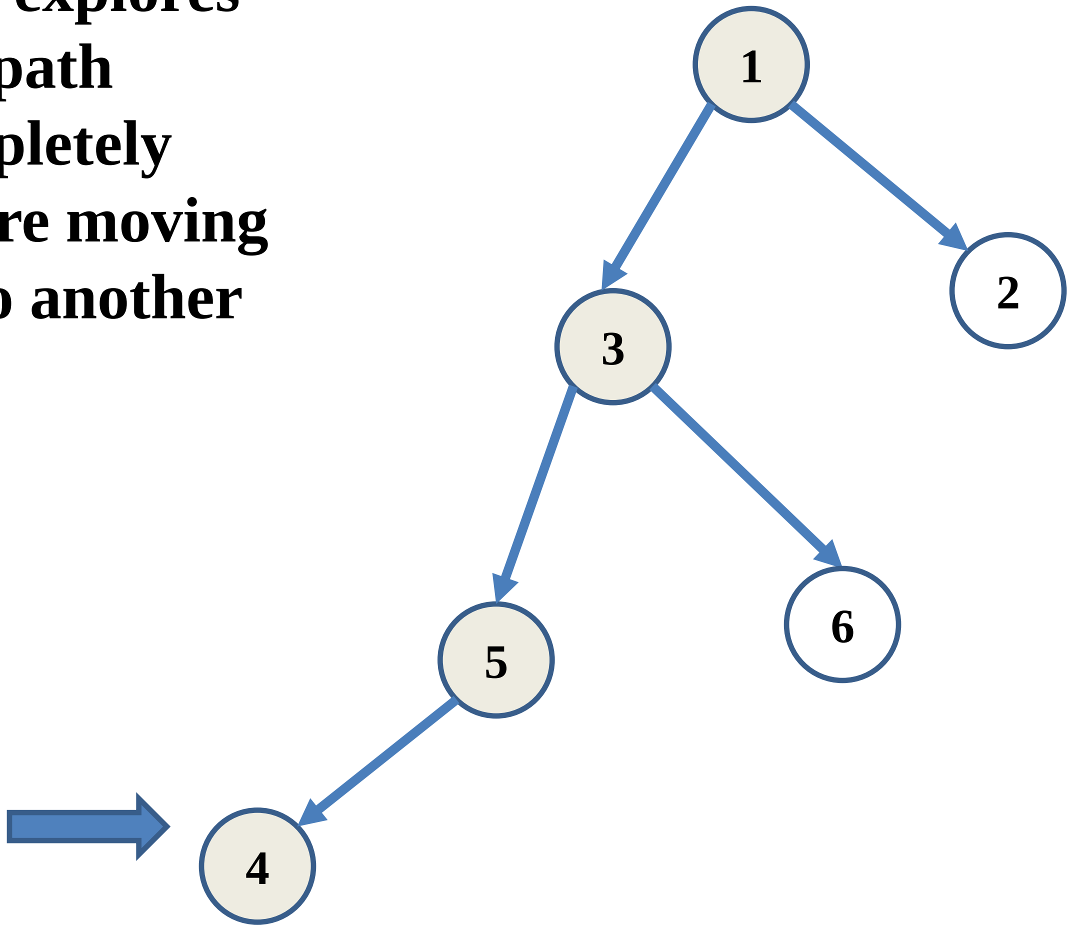

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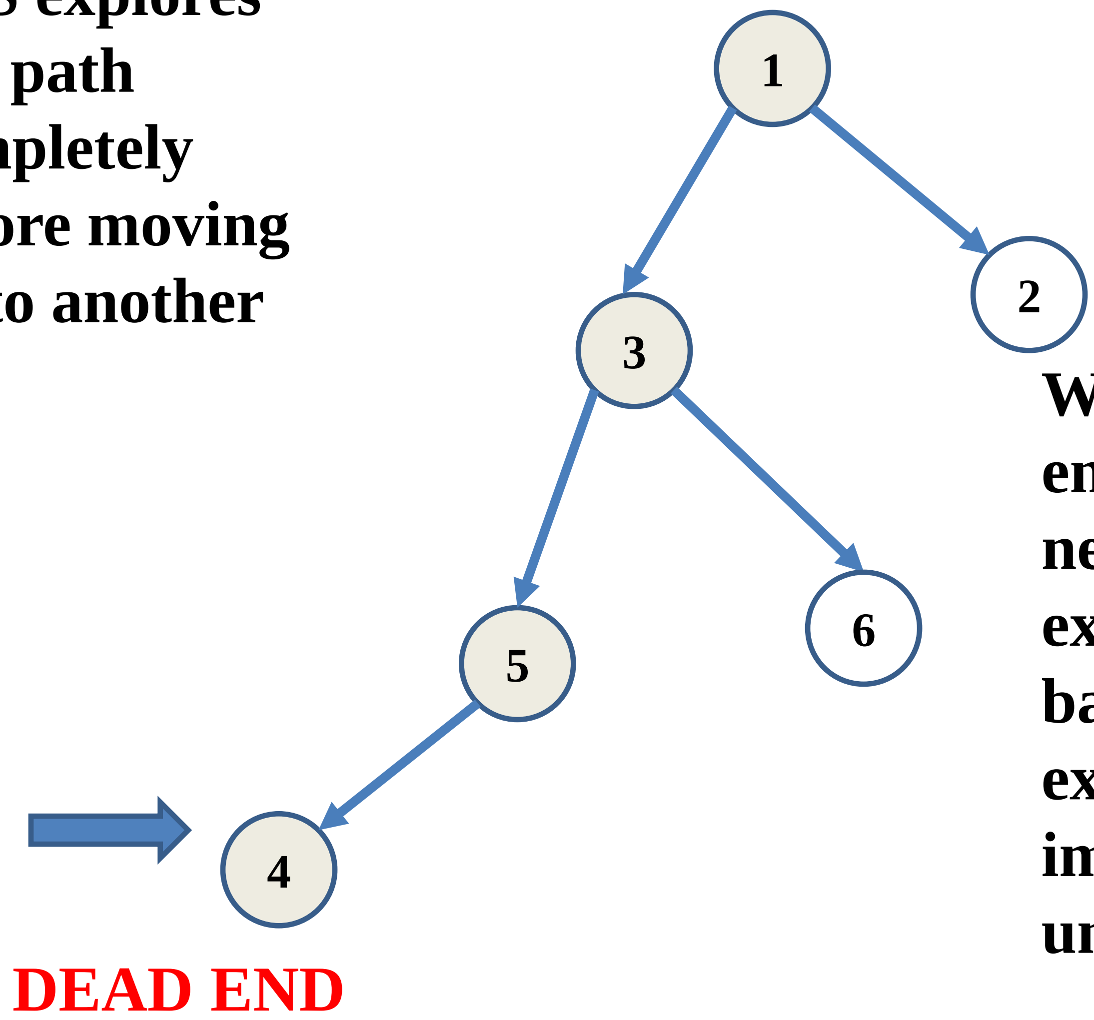

When DFS hits a dead end (no further neighbours to be explored), it backtracks and explores the immediately previous unexplored path

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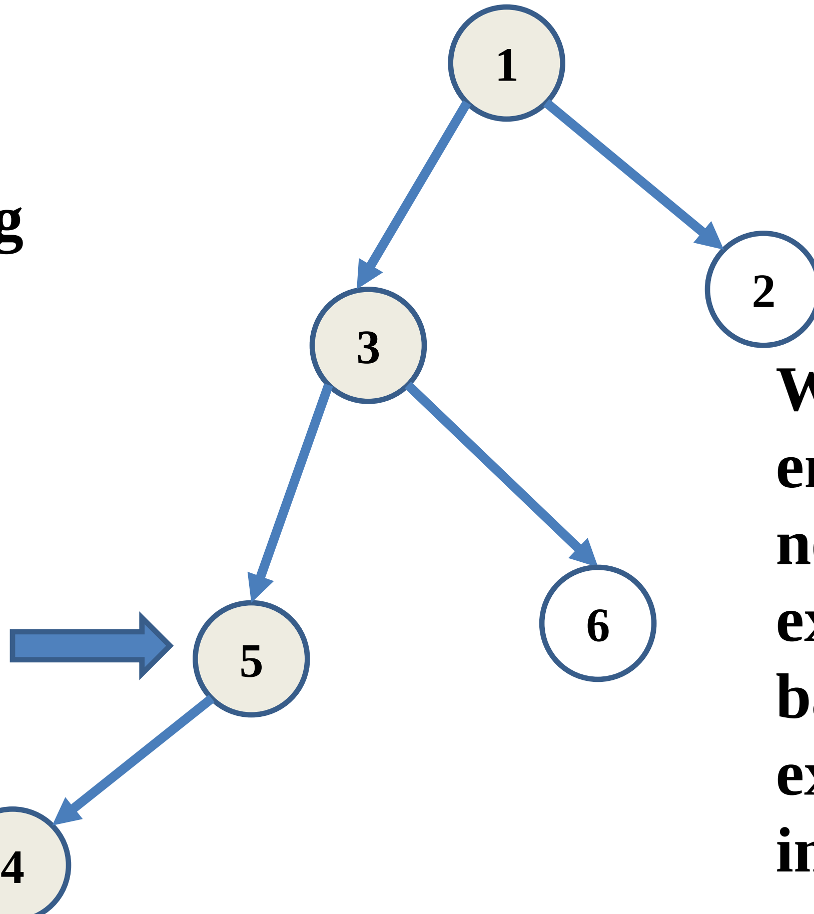

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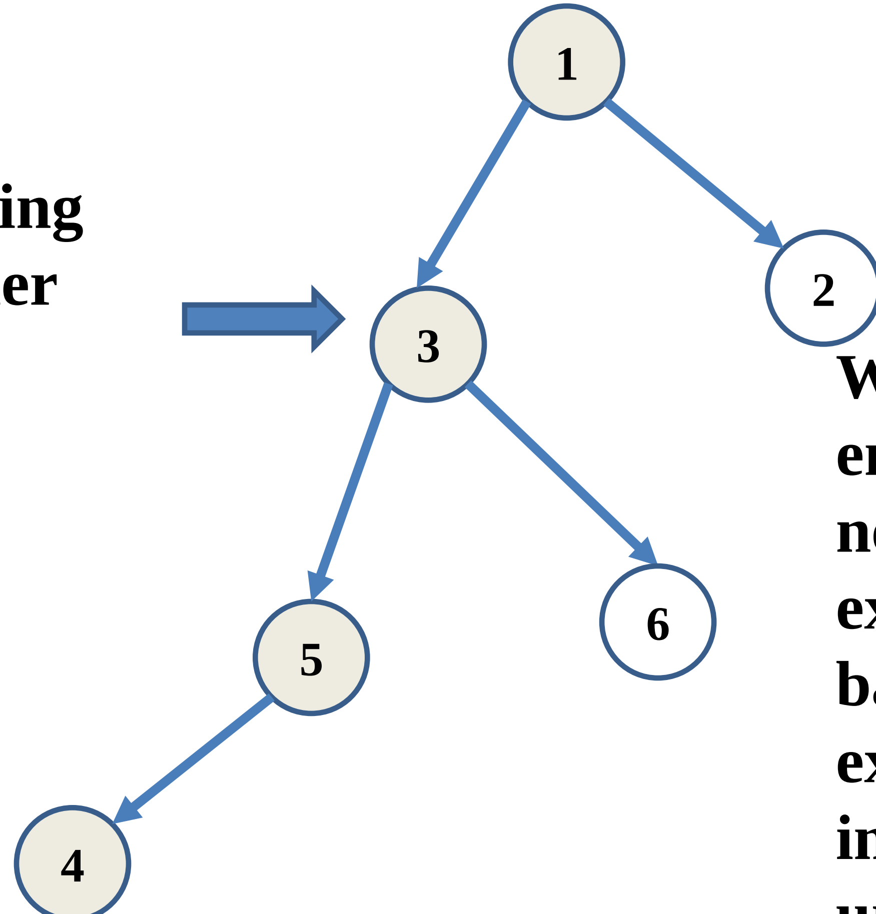

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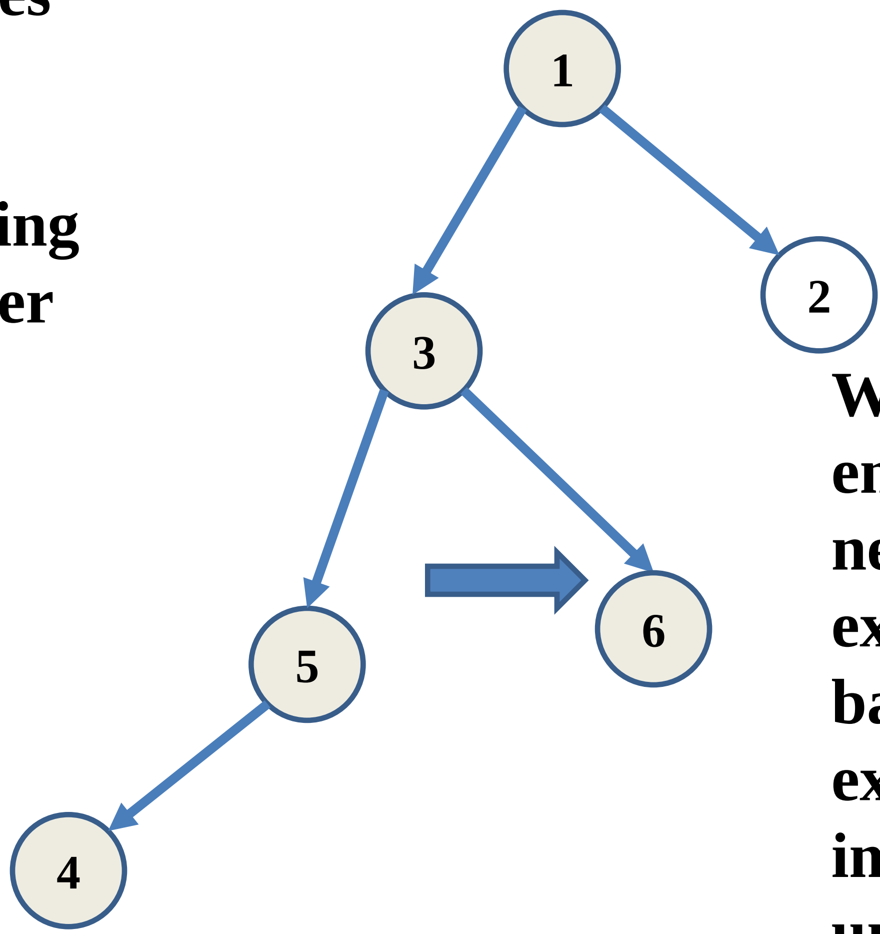

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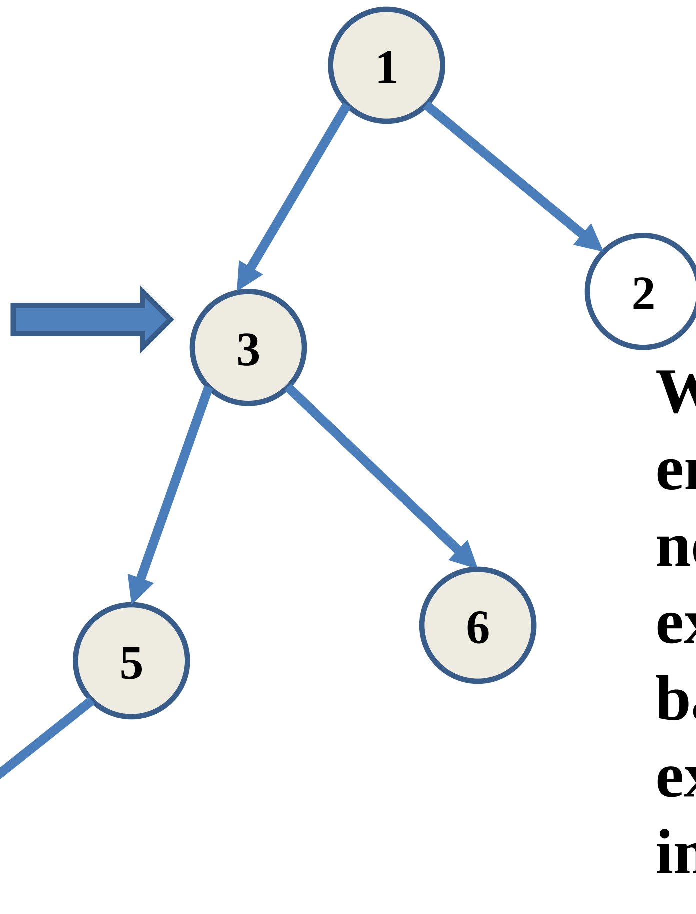

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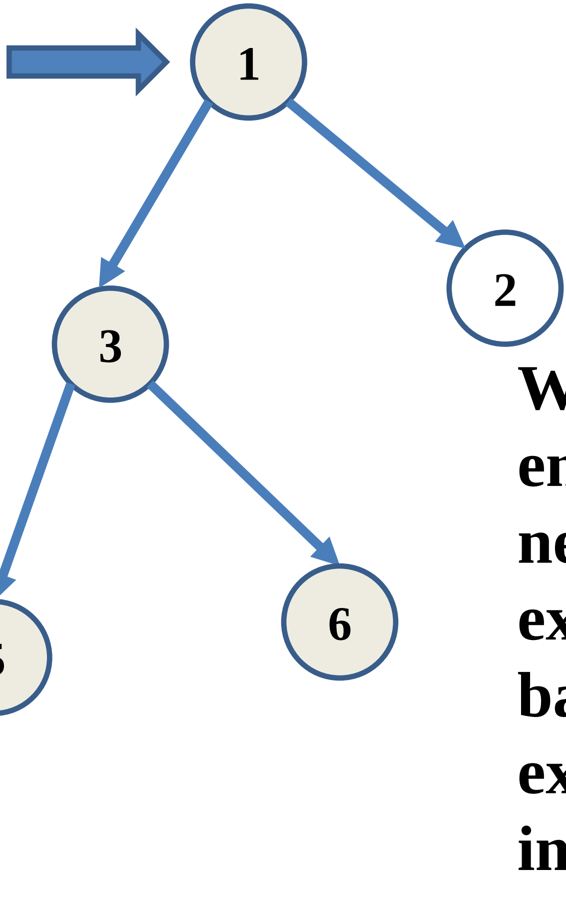

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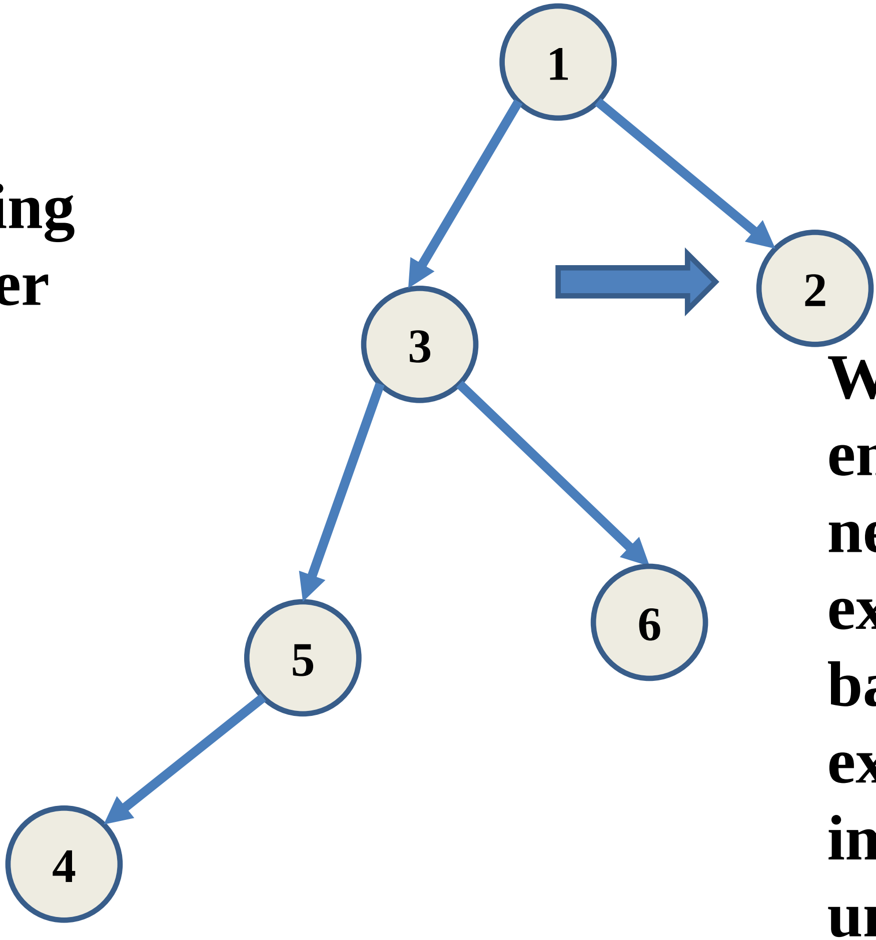

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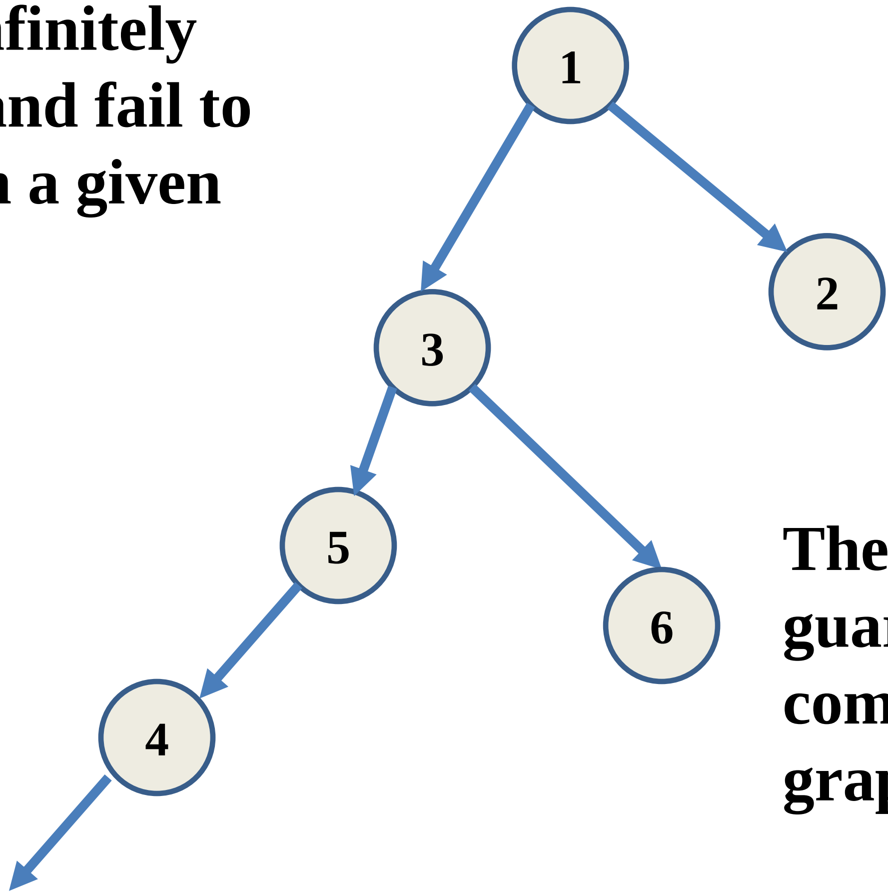
Completeness of DFS

DFS could get stuck exploring infinitely long paths and fail to terminate in a given time

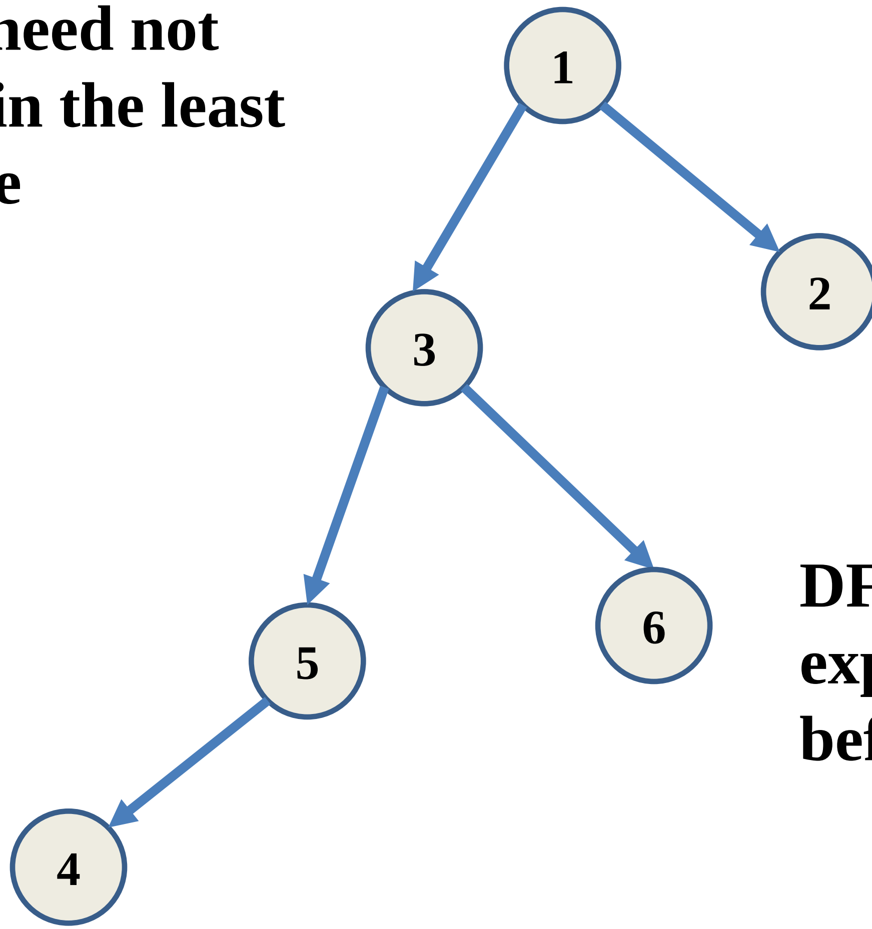


Potentially continues to ∞

The DFS algorithm is guaranteed to be complete for finite graphs

Optimality of DFS

DFS algorithm is not optimal - it need not find a node in the least possible time



Node to be found

DFS could end up exploring 1,3,5,4 and 6 before reaching 2

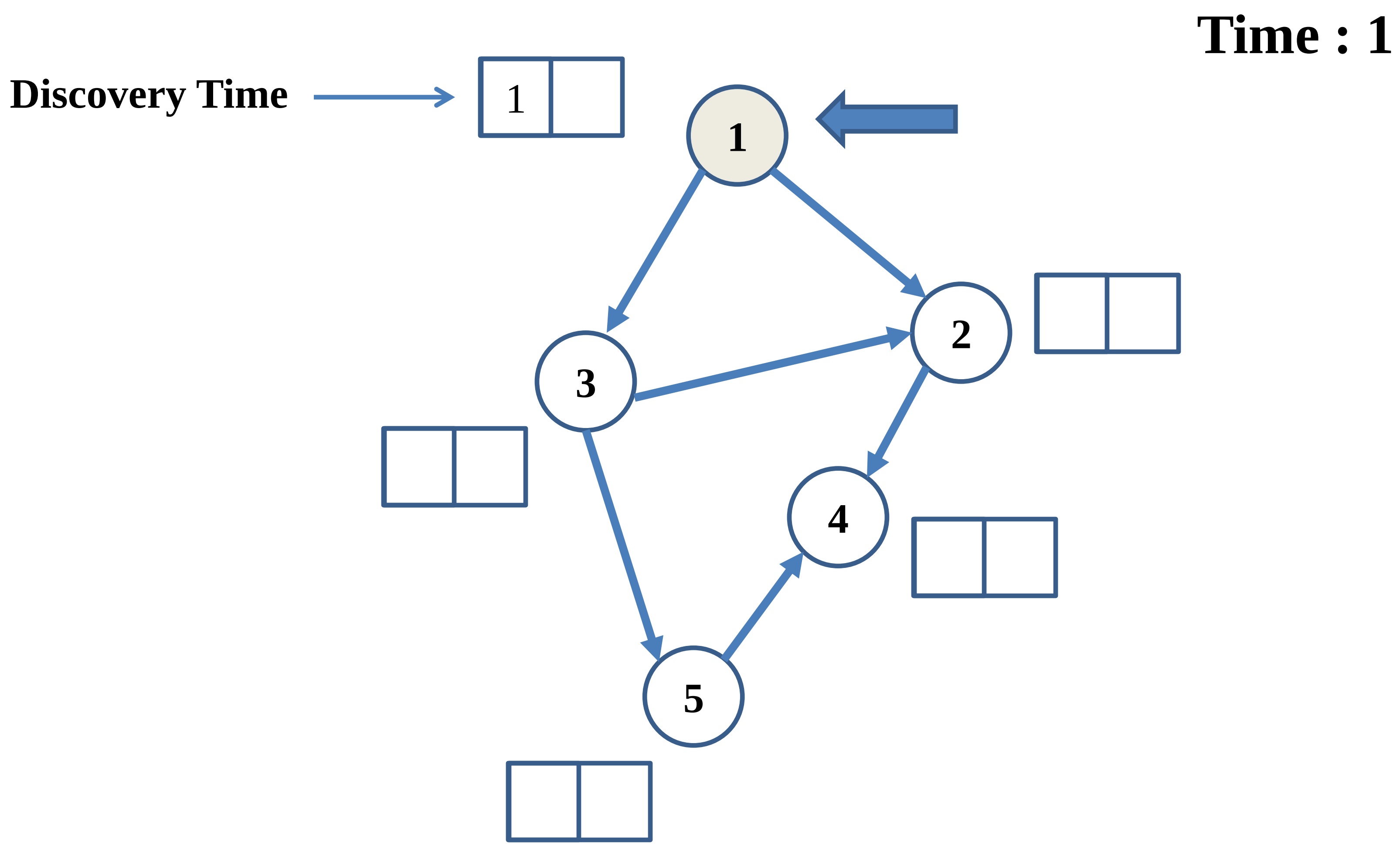
Applications of DFS

- Cycle detection
- Path Detection
- Finding strongly connected components
- Job Scheduling with dependencies (Topological Sort)

Associated Notations

- **Nodes are assigned colours**
  - WHITE : The node has not been visited
  - GRAY : The node has been visited, but all of its branches have not been visited completely
  - BLACK : A node and its branches have been explored completely
- Often, two other values are associated with a node
  - **Discovery Time (d)** : The time at which the node becomes gray
  - **Finishing Time (f)** : The time at which the node becomes black
- **Predecessor Subgraph** : Each time a node is visited, its parent is noted to construct the **DFS tree / forest**

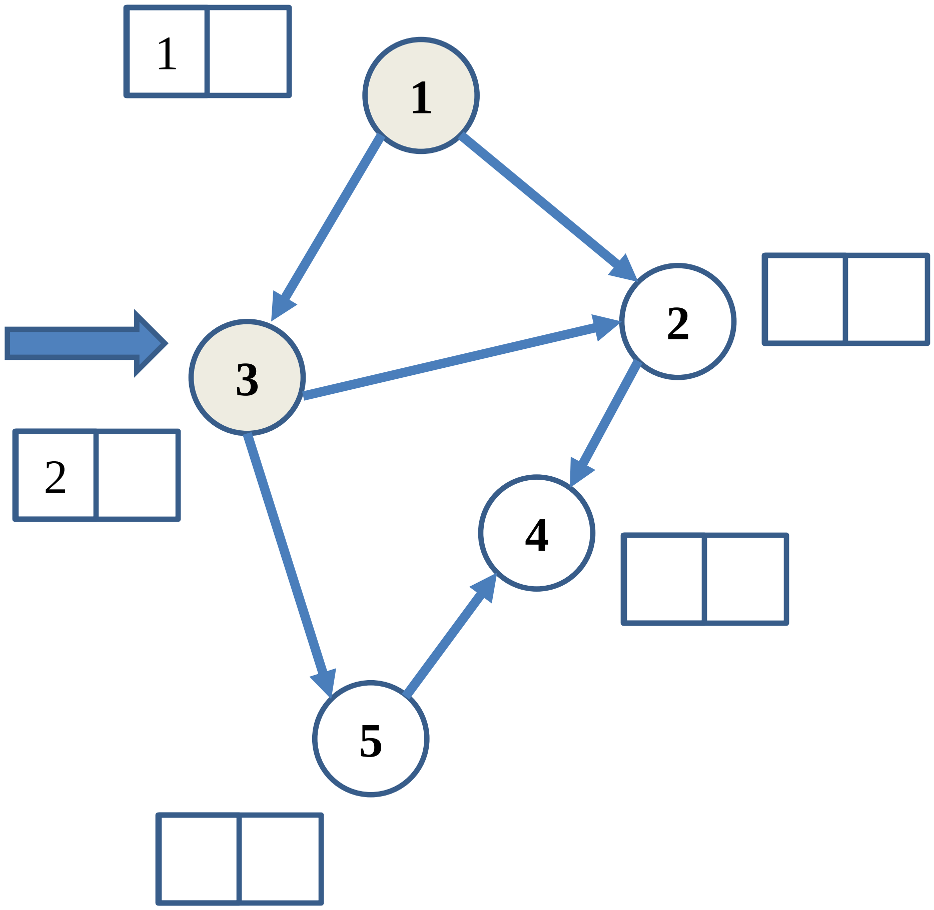
DFS in Action



DFS in Action

Time : 2

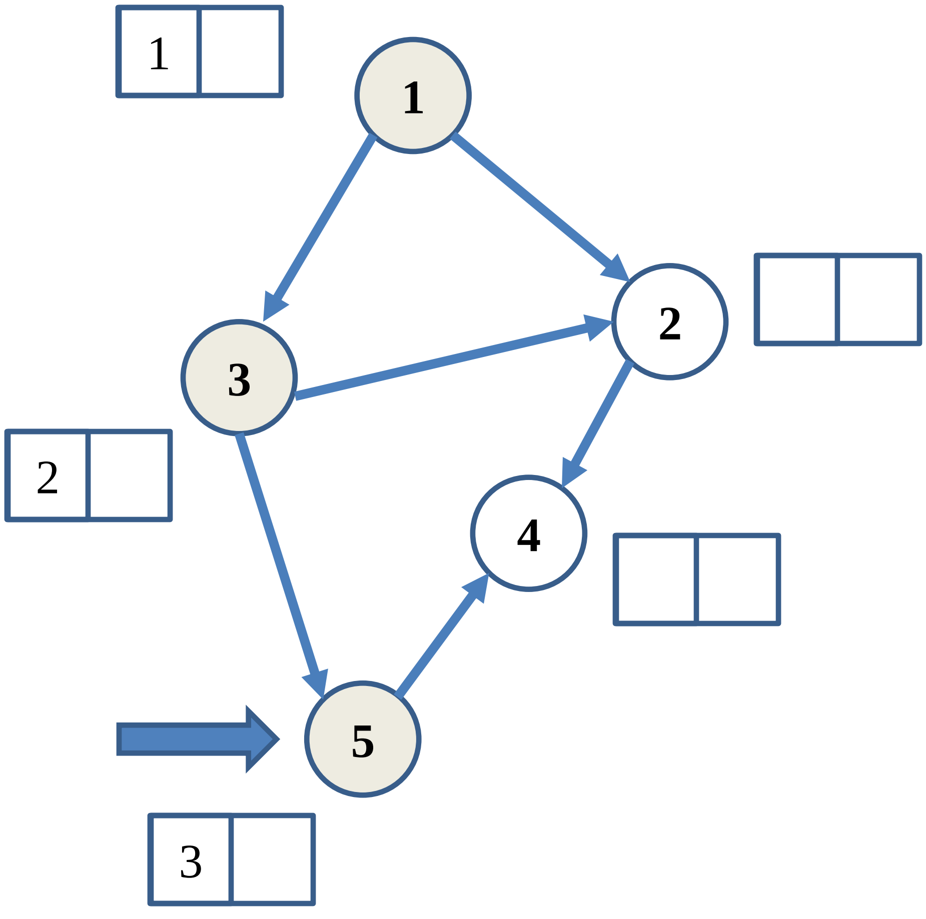
Π(3) = 1



DFS in Action

Time : 3

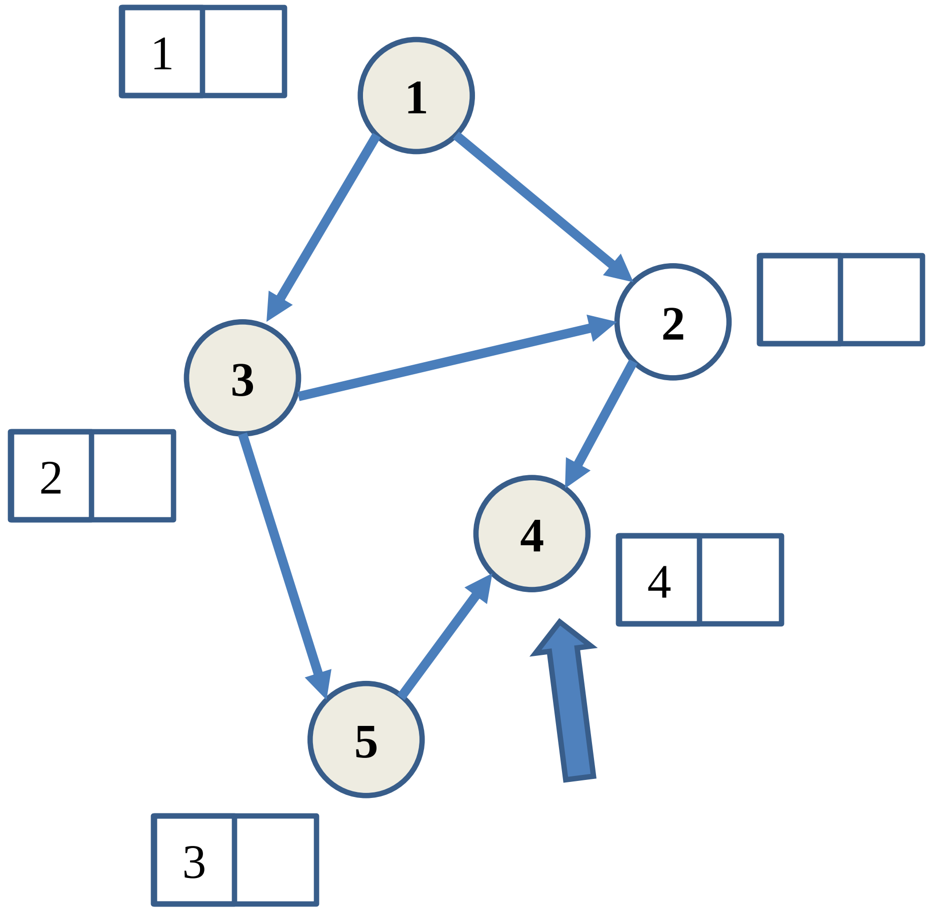
Π(3) = 1
Π(5) = 3



DFS in Action

Time : 4

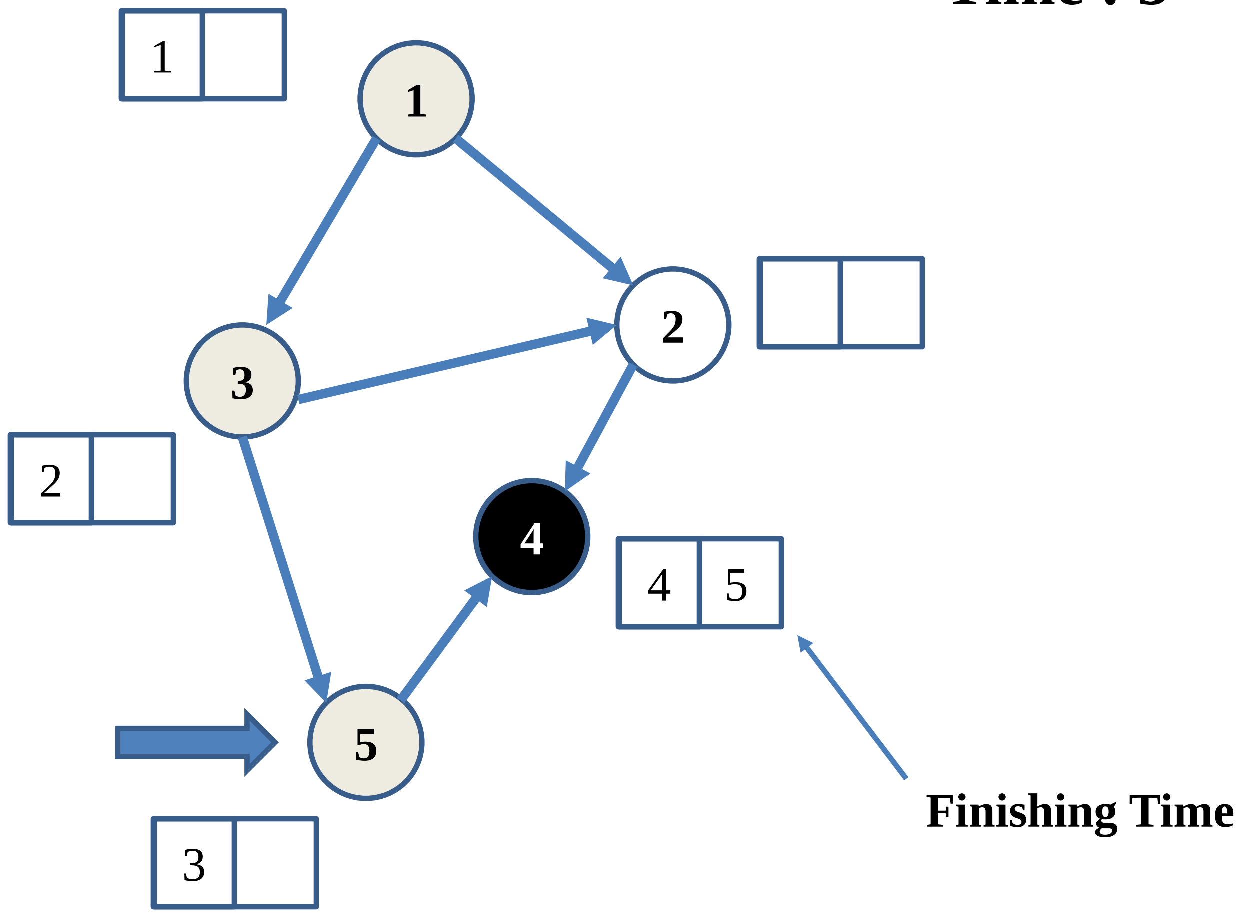
Π(3) = 1
Π(5) = 3
Π(4) = 5



DFS in Action

Time : 5

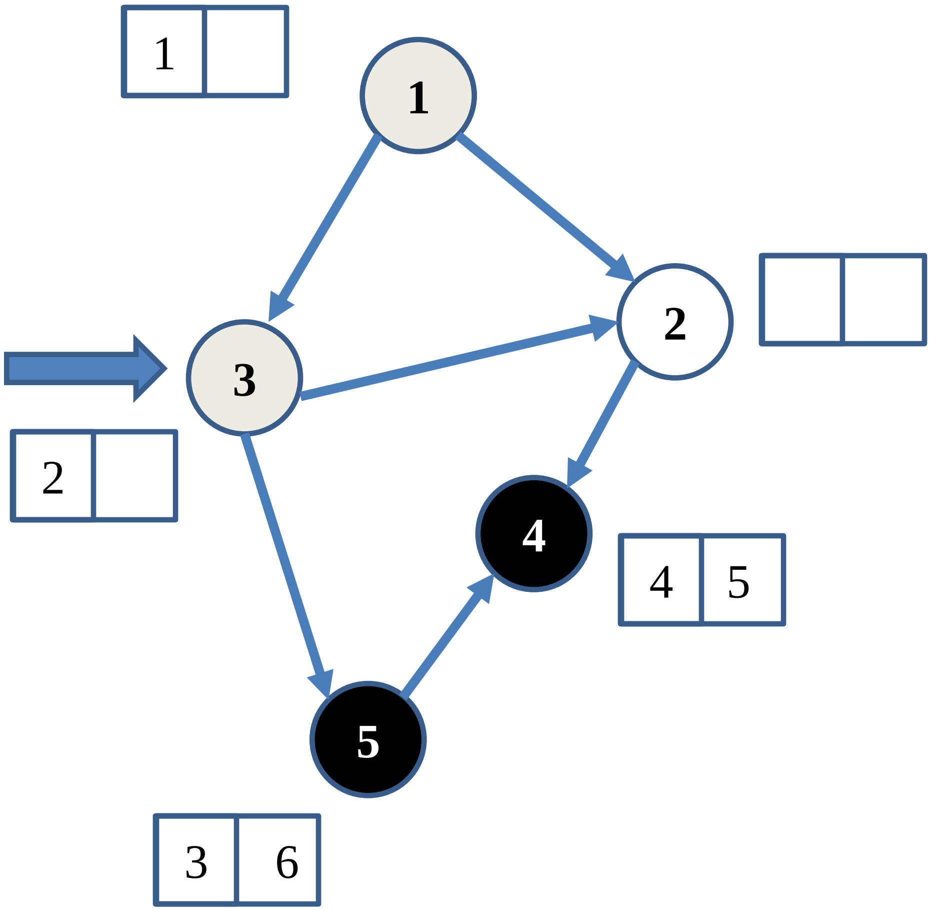
Π(3) = 1
Π(5) = 3
Π(4) = 5



DFS in Action

Time : 6

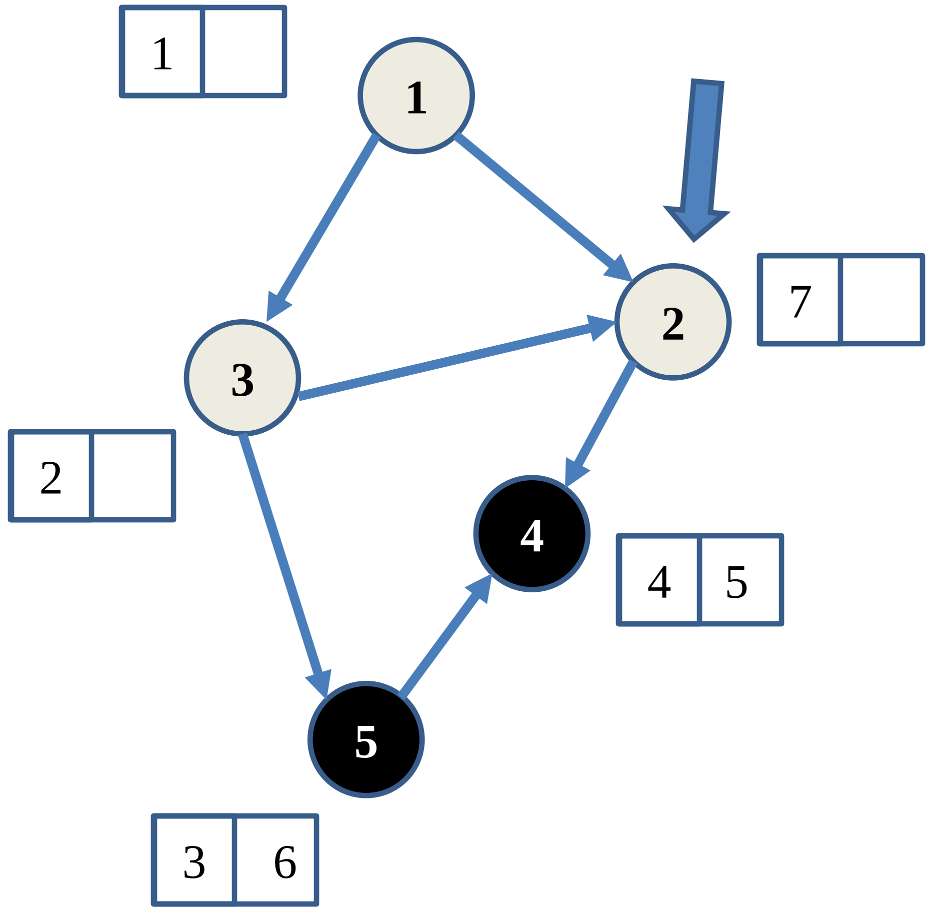
Π(3) = 1
Π(5) = 3
Π(4) = 5



DFS in Action

Time : 7

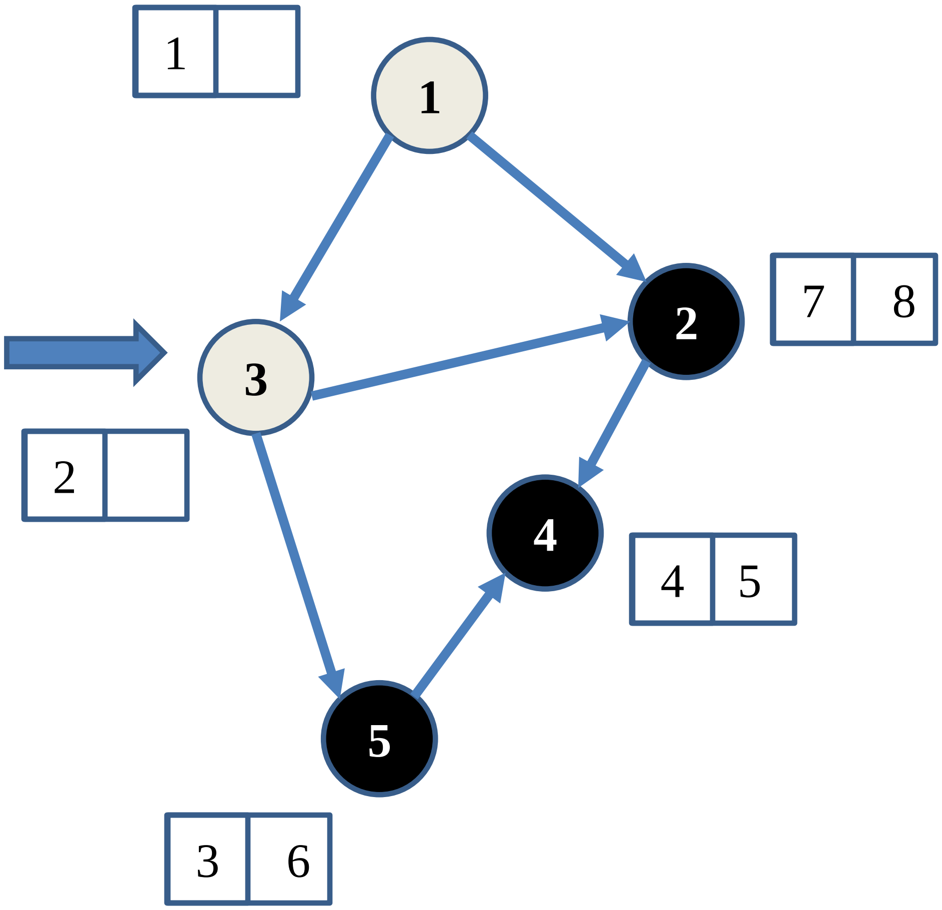
Π(3) = 1
Π(5) = 3
Π(4) = 5
Π(2) = 3



DFS in Action

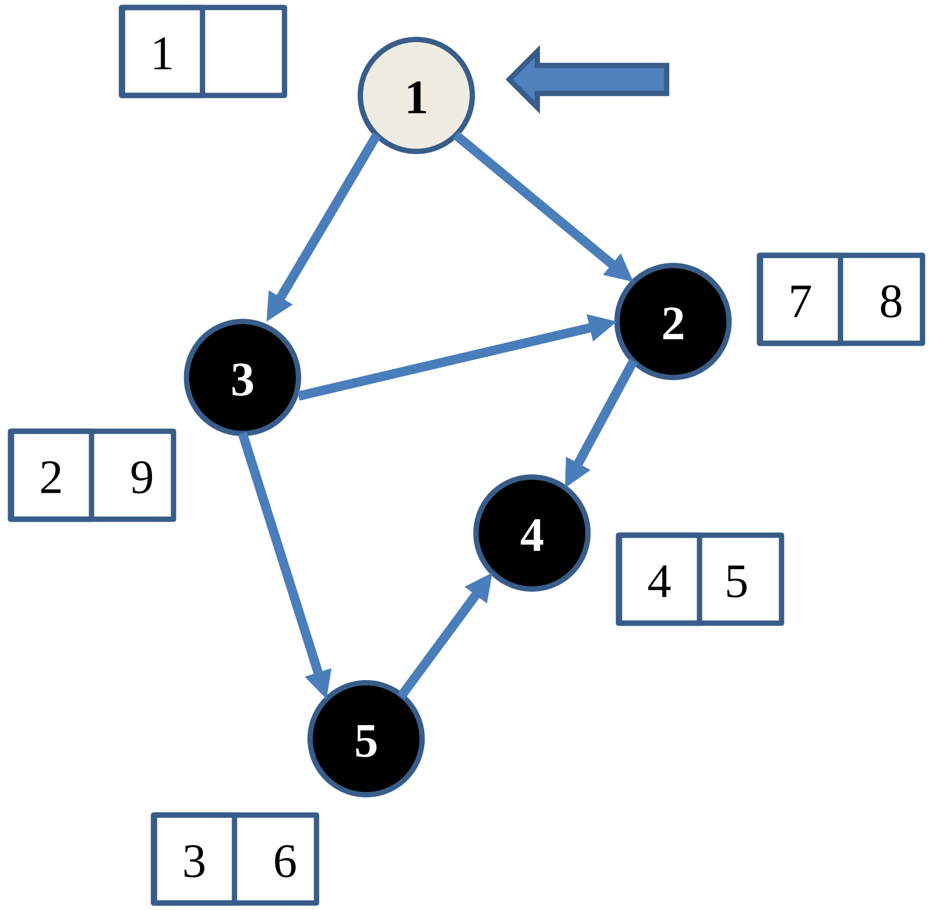
Time : 8

Π(3) = 1
Π(5) = 3
Π(4) = 5
Π(2) = 3



DFS in Action

Time : 9

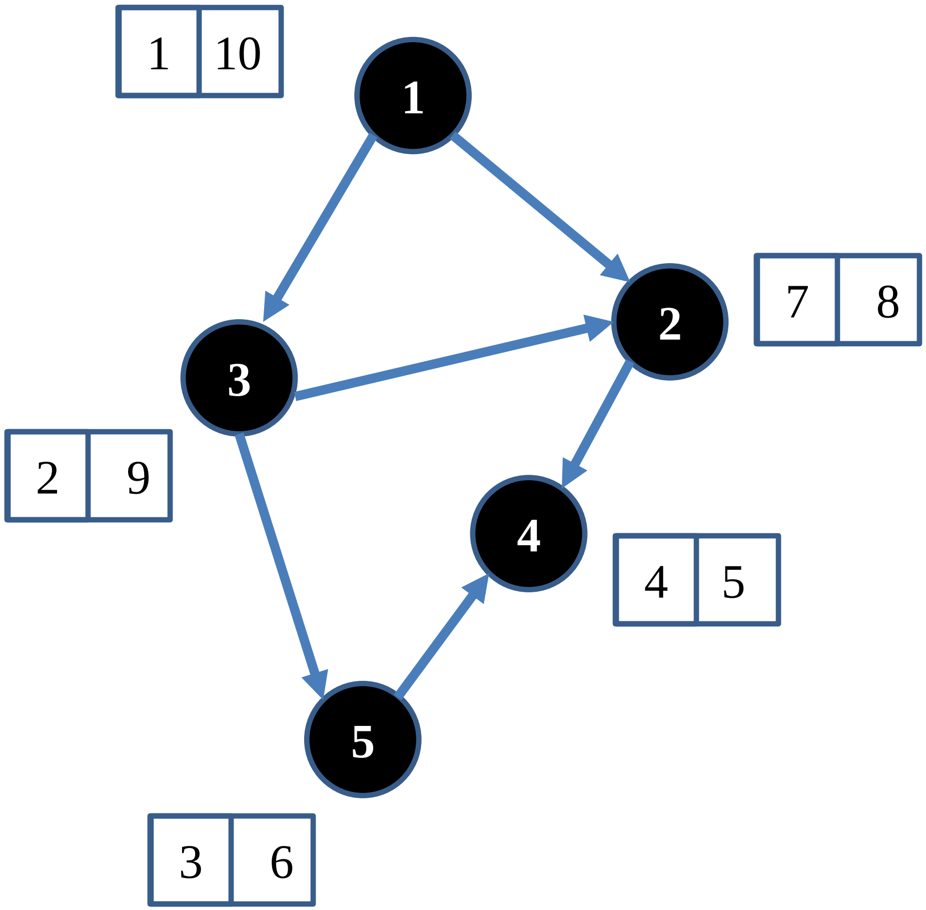


Π(3) = 1
Π(5) = 3
Π(4) = 5
Π(2) = 3

DFS in Action

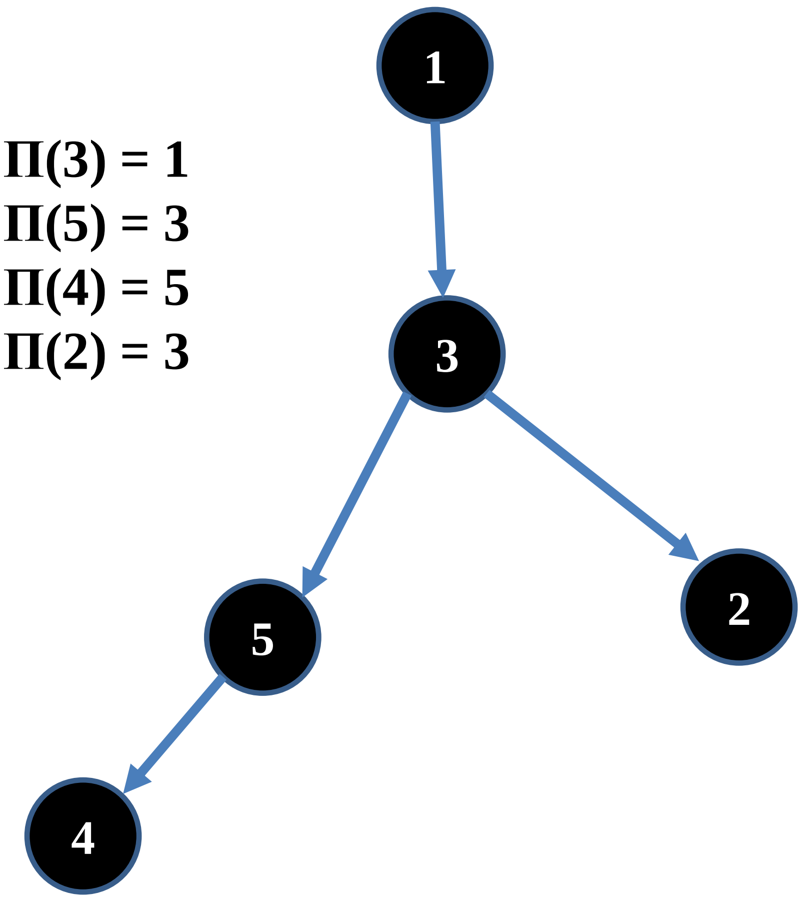
Time : 10

Π(3) = 1
Π(5) = 3
Π(4) = 5
Π(2) = 3



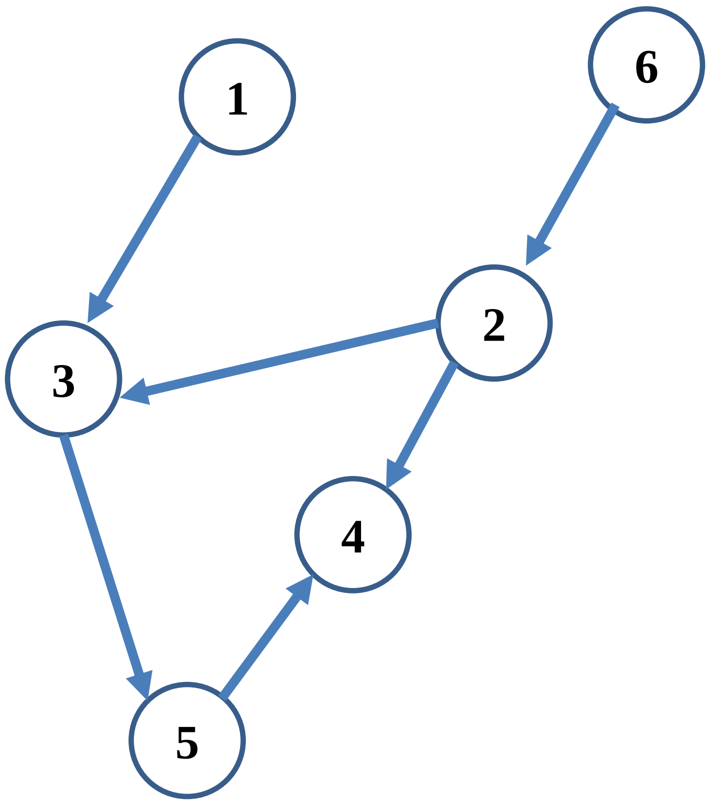
DFS Tree

Π(3) = 1
Π(5) = 3
Π(4) = 5
Π(2) = 3



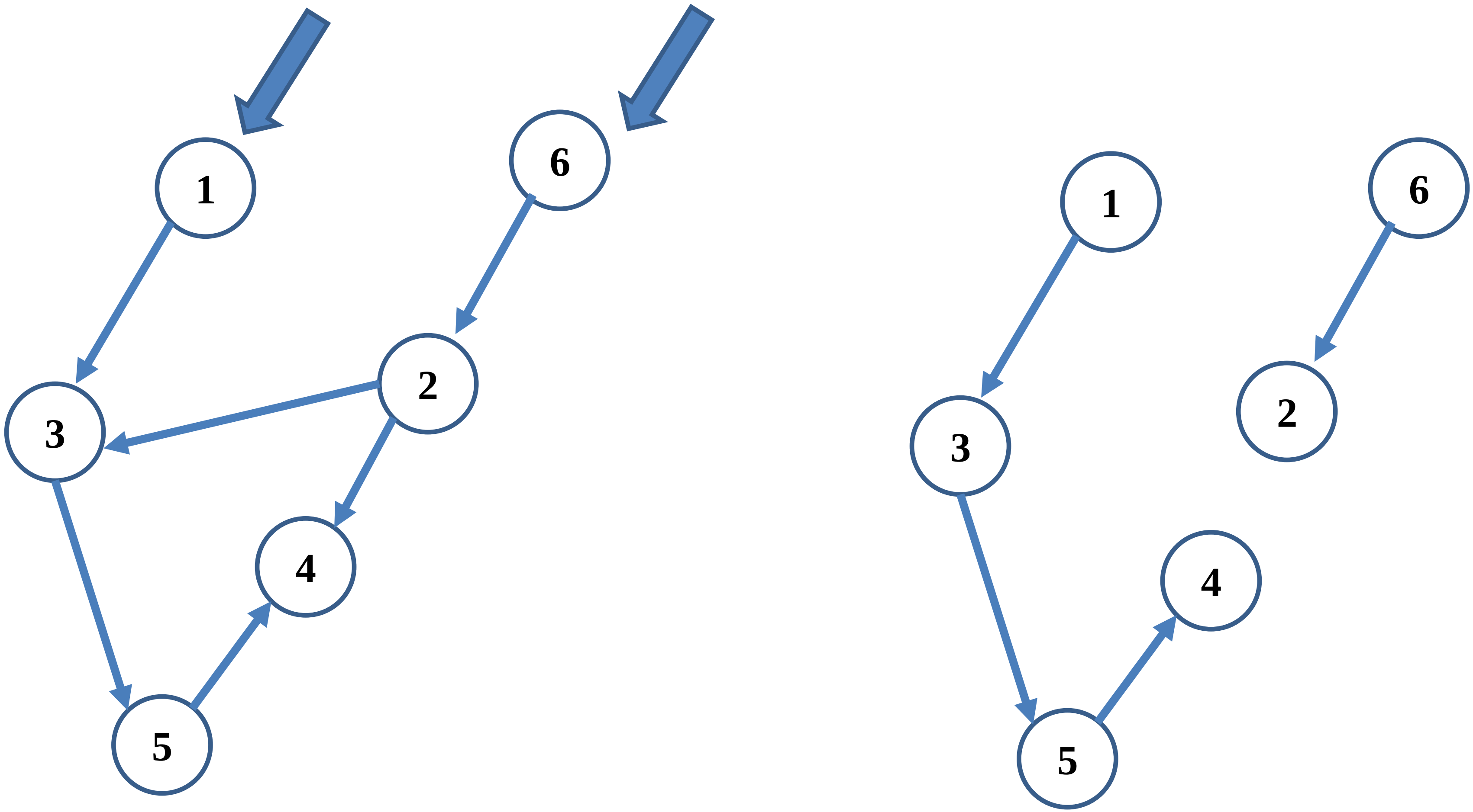
- DFS need not always form a single tree for a graph
- Sometimes, DFS may need to be performed multiple times from different starting nodes to discover all the nodes in the graph
- This gives rise to a number of DFS trees, collectively called a **DFS Forest**

Need for Multiple DFS Runs



- Nodes 2 and 6 are not reachable from 1
- Nodes 1 is not reachable from 6
- There is no starting node such that all the vertices in the graph are reachable
- We need to run DFS from at least two vertices – say 1 and 6

DFS Forest



If DFS is run from node 1 and then from node 6, we get two DFS trees (a DFS forest)

DFS Algorithm

DFS(G)

for each *vertex u* **in** *V[G]*

colour[u] = WHITE

Π[u]= NIL

time=0

for each *vertex u* **in** *V[G]*

if *colour[u]= WHITE*

DFS_VISIT(u)

DFS_VISIT Algorithm

```
DFS_VISIT(u)
    colour[u] = GRAY
    time = time+1
    d[u] = time
    for each v in Adj[u]
        if colour[v]=WHITE
            Π[v] = u
            DFS_VISIT(v)
    colour[u] = BLACK
    time = time+1
    f[u] = time
```

Time Complexity of DFS

- Every node is explored EXACTLY ONCE --- $\Theta(|V|)$
- For every node u, DFS explores all the edges in Adj[u]
- When summed over all the nodes in the graph, this amounts to the number of edges in the graph

$$\sum_{u \in V} Adj[u] = \Theta(|V|)$$

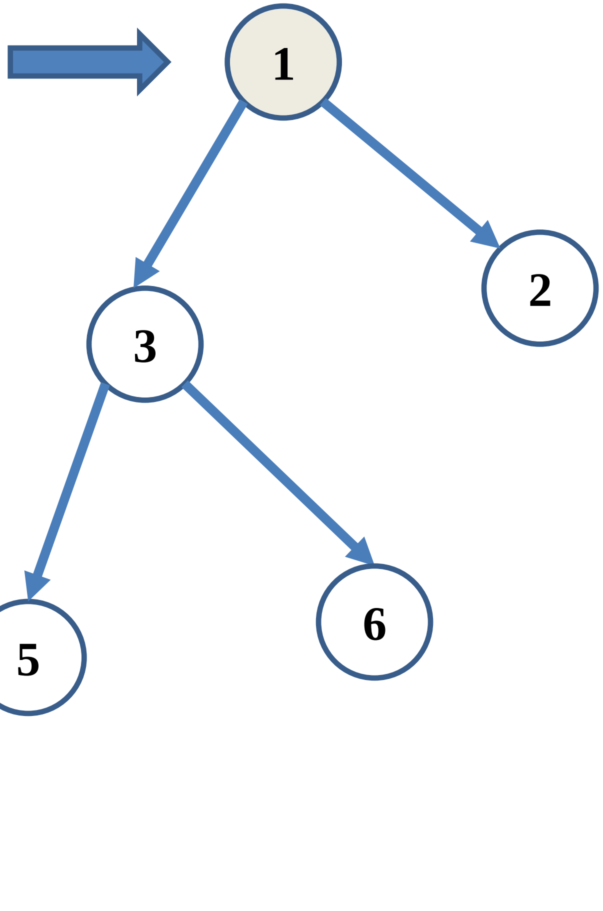
- Thus, **total complexity of DFS is $\boldsymbol{\Theta(|V| + |E|)}$**

Breadth First Search

- Breadth First Search (BFS) is another graph traversal algorithm
- As the name suggests, BFS explores all the neighbours of a node before exploring any other node
- When BFS hits a dead end on a path, it backtracks and starts exploring a new path from the previous node
- This behaviour is suggestive of a FIFO data structure – QUEUE

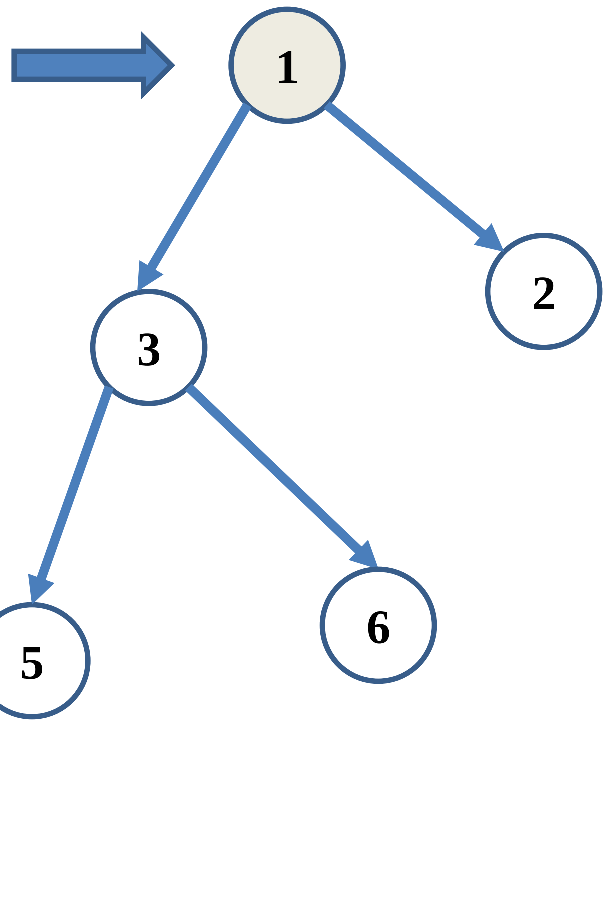
Breadth First Search

BFS is a graph traversal algorithm



Breadth First Search

BFS explores all neighbours of a given node before exploring any other node



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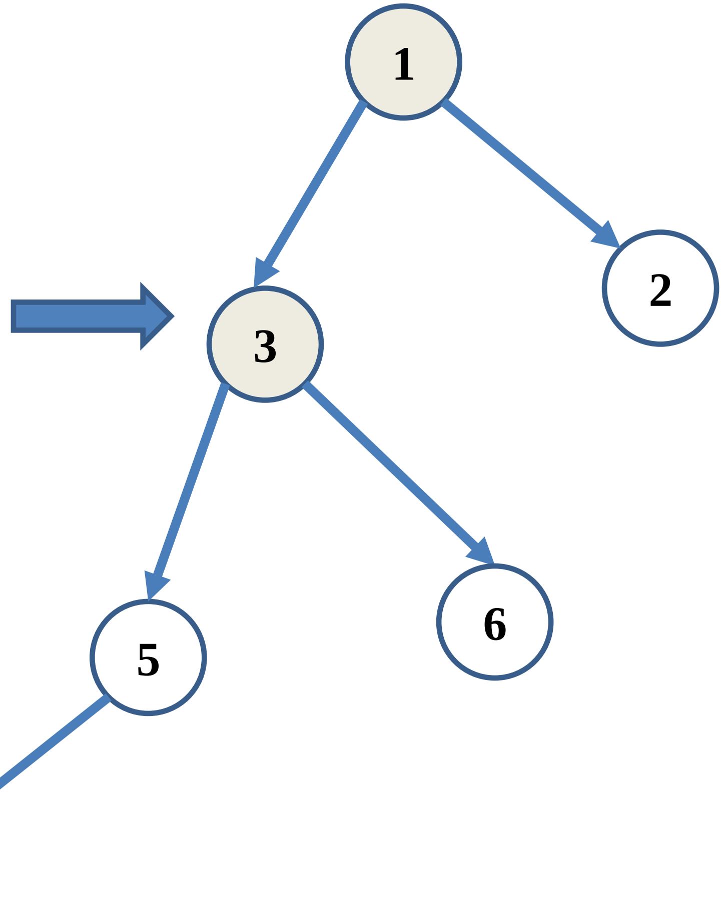

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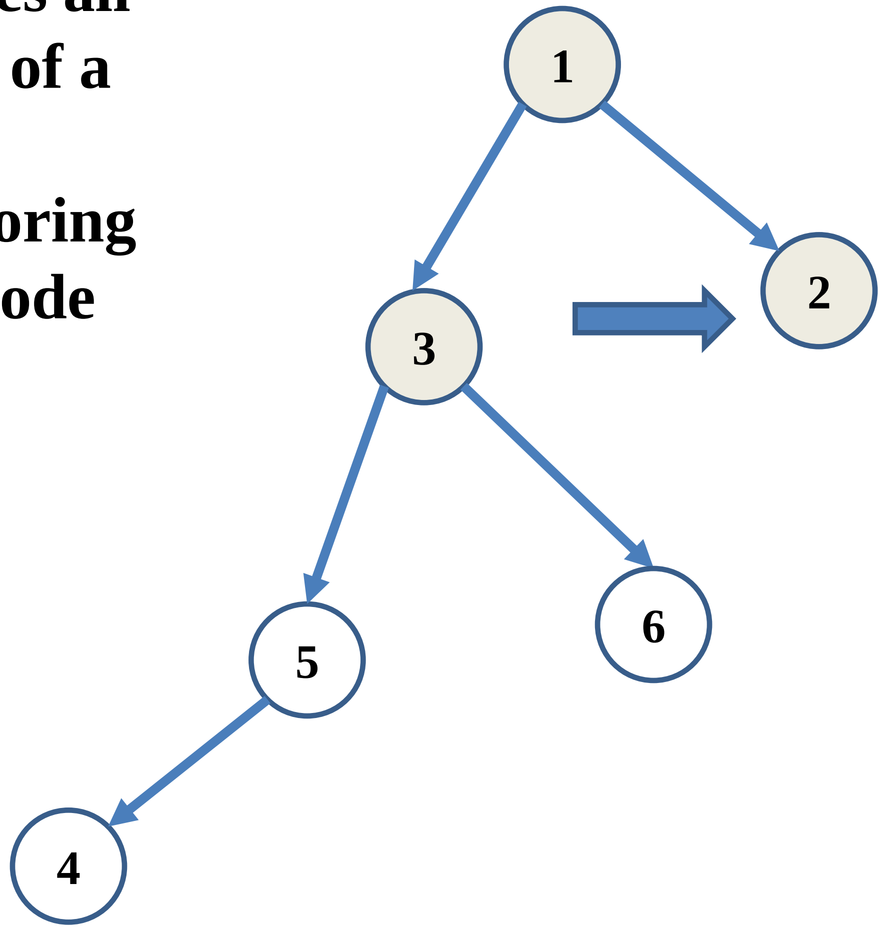

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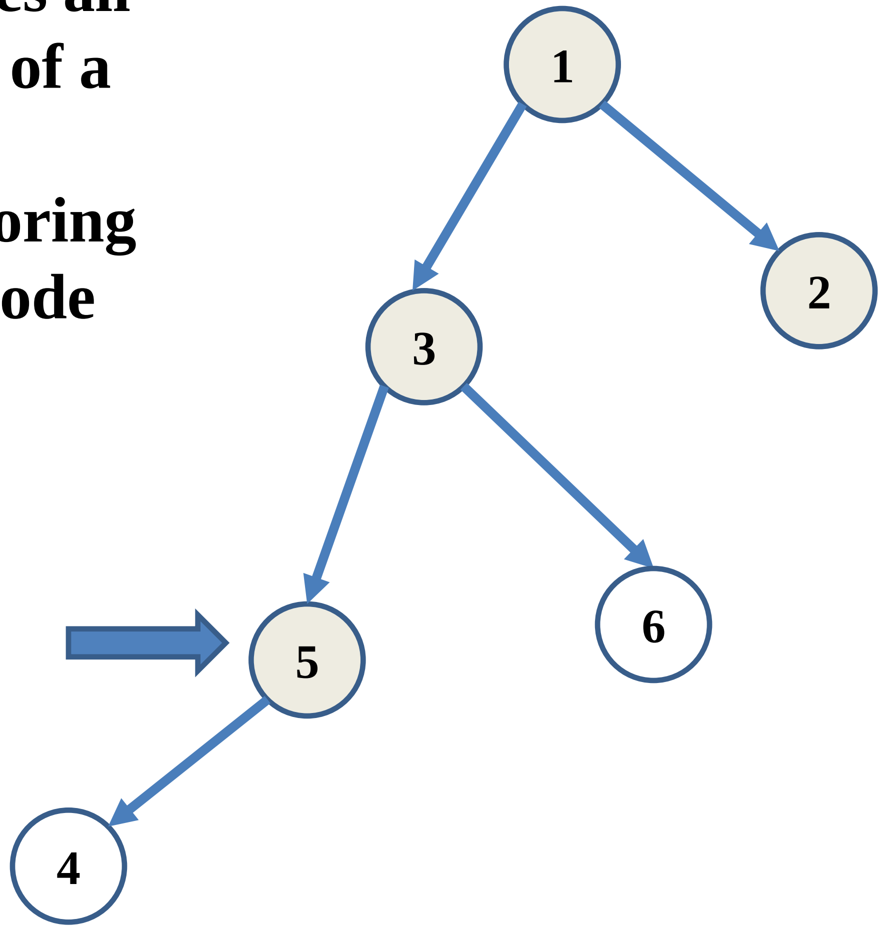
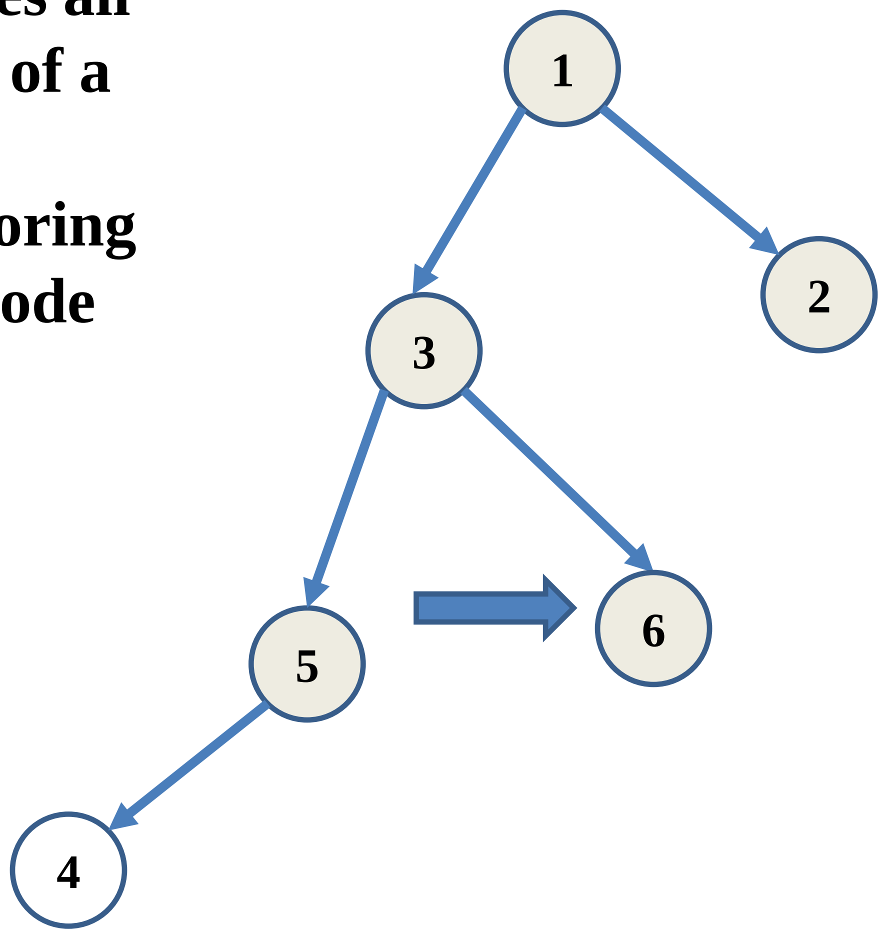


Breadth First Search

BFS explores all neighbours of a given node before exploring any other node



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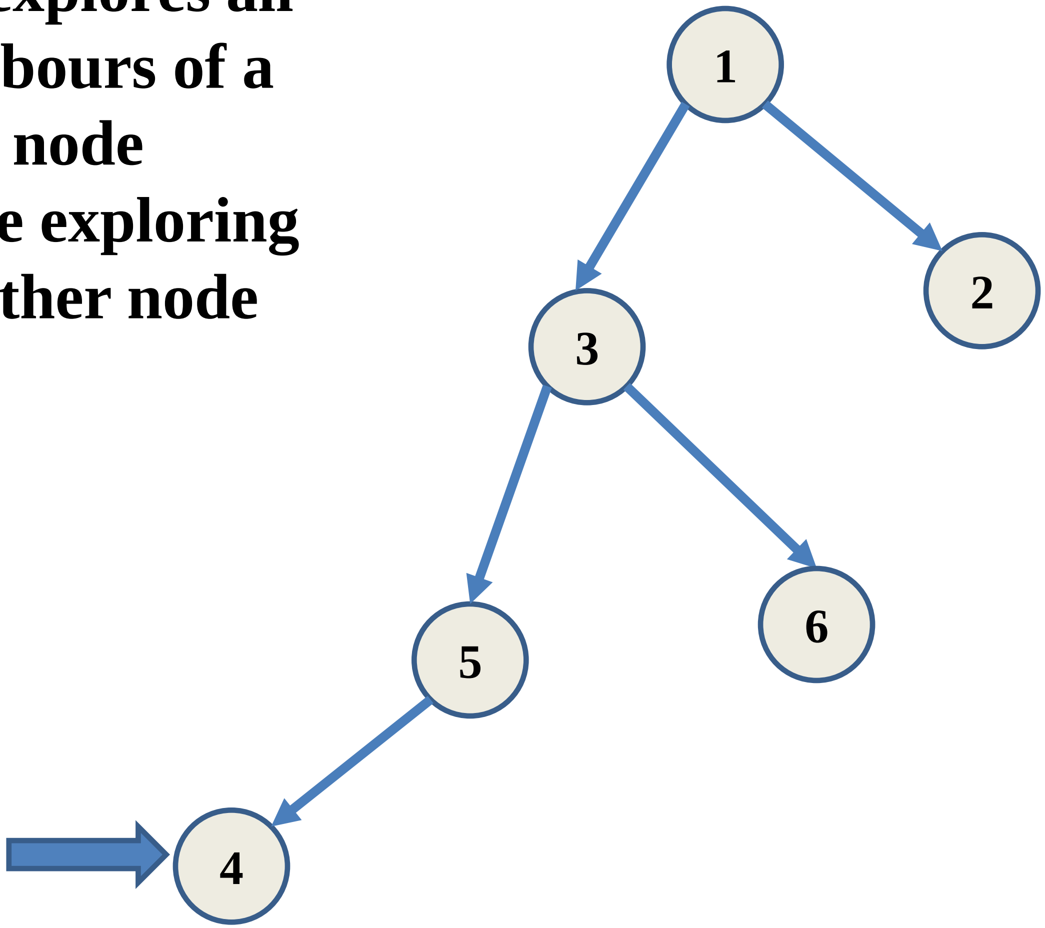
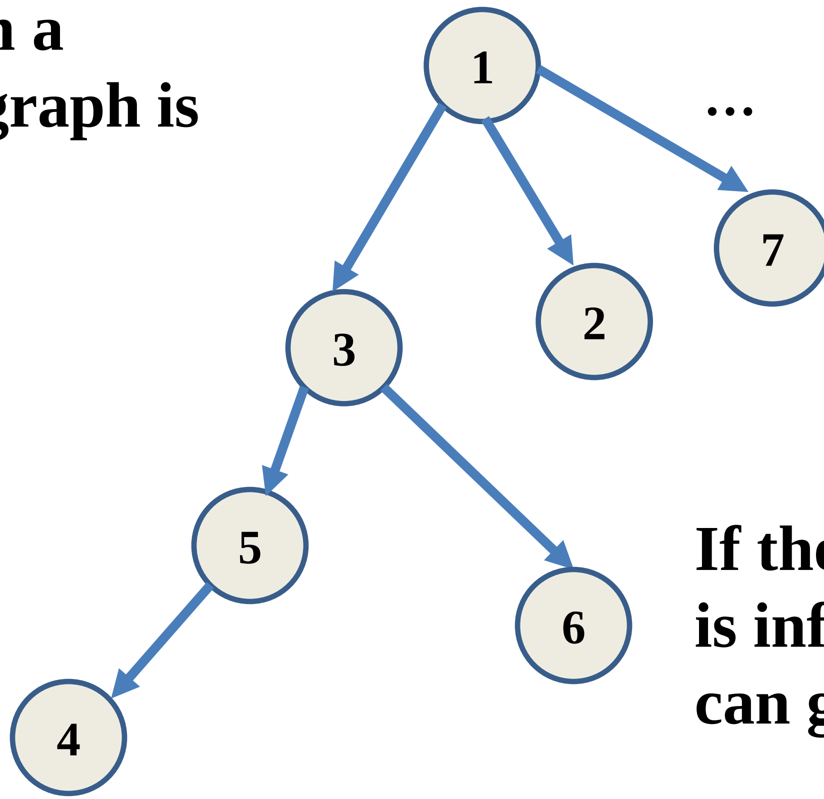


Completeness of BFS

BFS algorithm terminates with a solution if the graph is finite

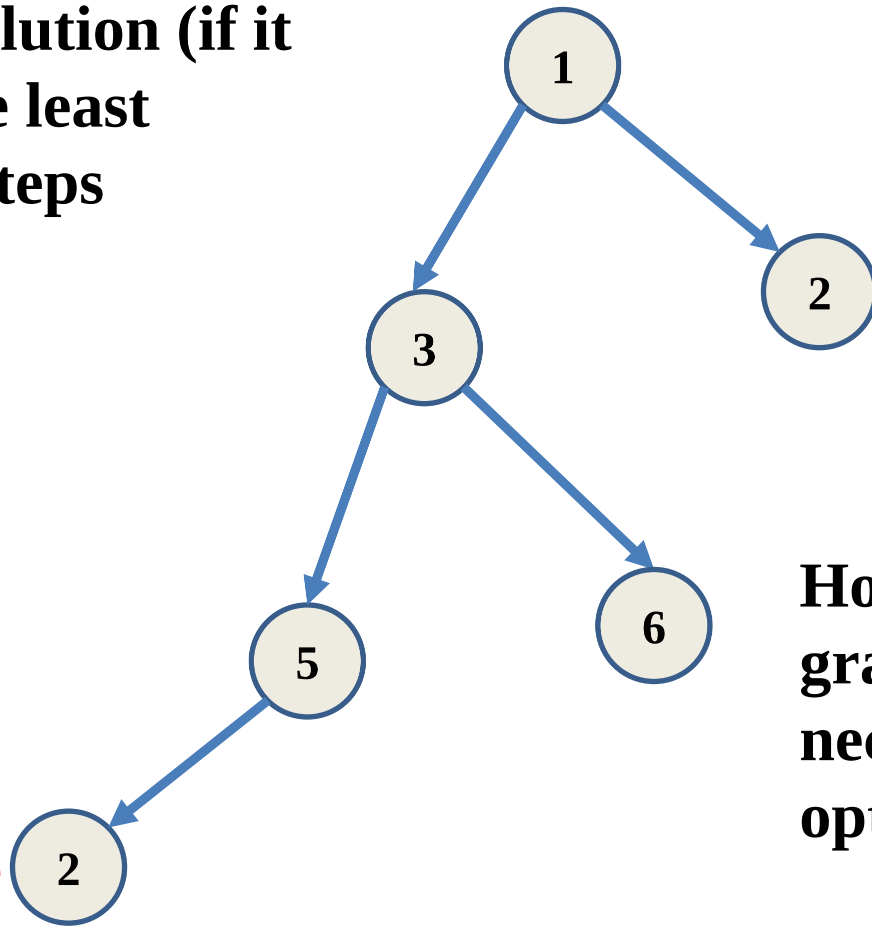
Potentially continues to ∞



If the branching factor is infinite, then BFS can get stuck in a loop

Optimality of BFS

BFS algorithm will find the a solution (if it exists) in the least number of steps



However, in a weighted graph, BFS need not necessarily give an optimal solution

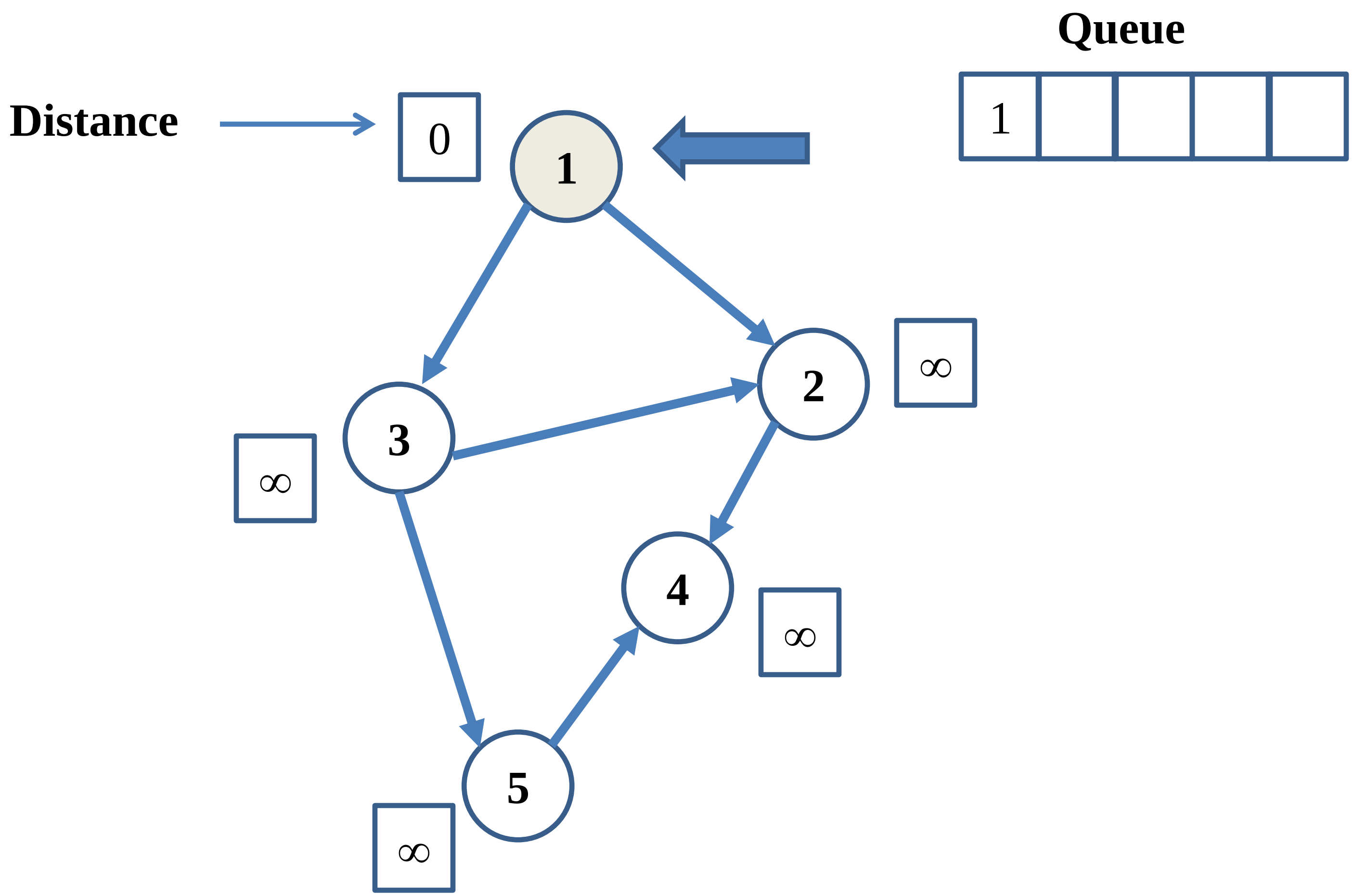
Applications of BFS

- Cycle detection
- Path Detection
- Finding strongly connected components

Associated Notations

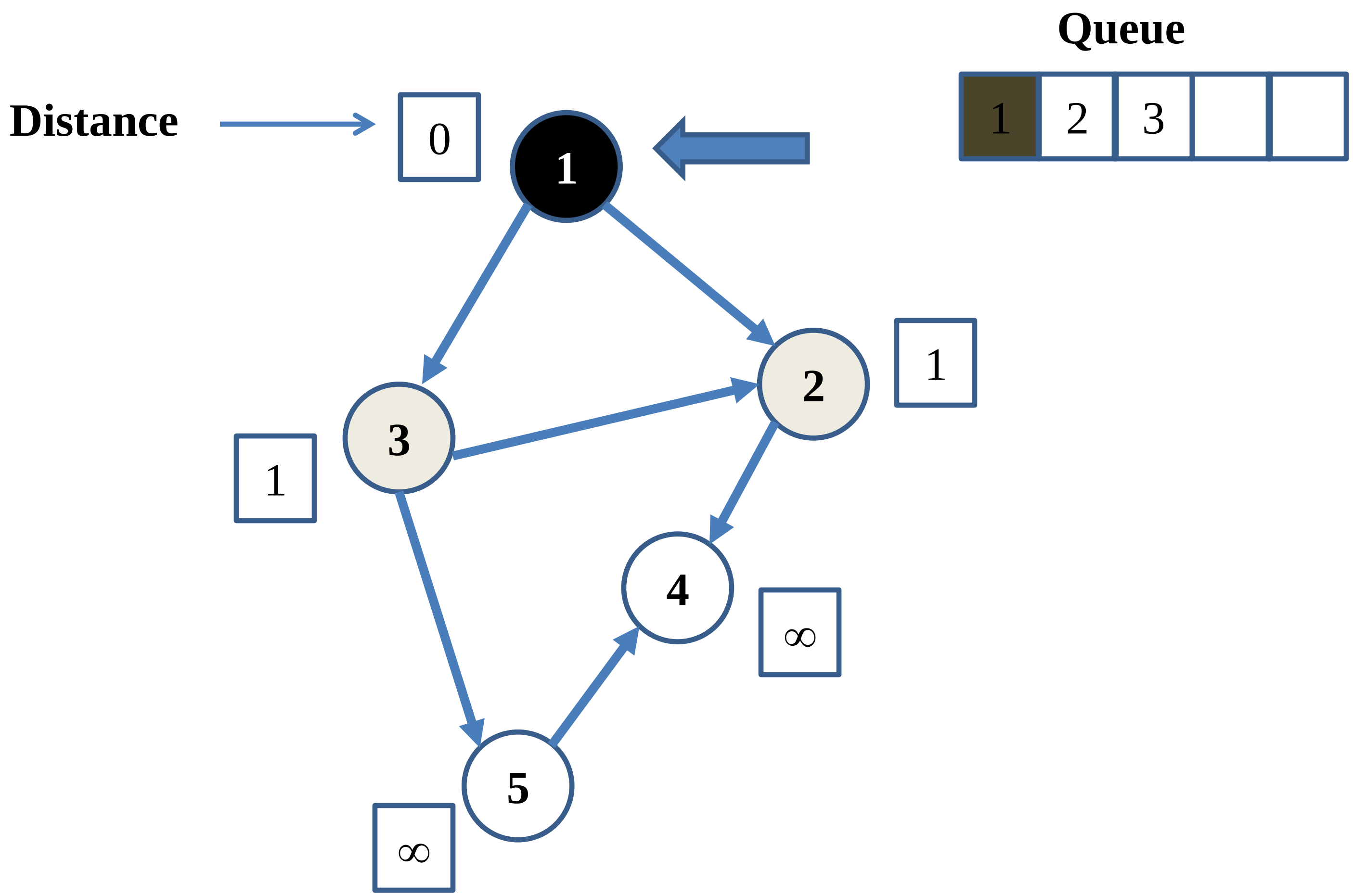
- **Nodes are assigned colours**
  - WHITE : The node has not been visited
  - GRAY : The node has been visited, but all of its branches have not been visited completely
  - BLACK : A node and its branches have been explored completely
- Every node is also assigned a distance value which is the number of steps it took to reach that node from the source vertex

BFS in Action



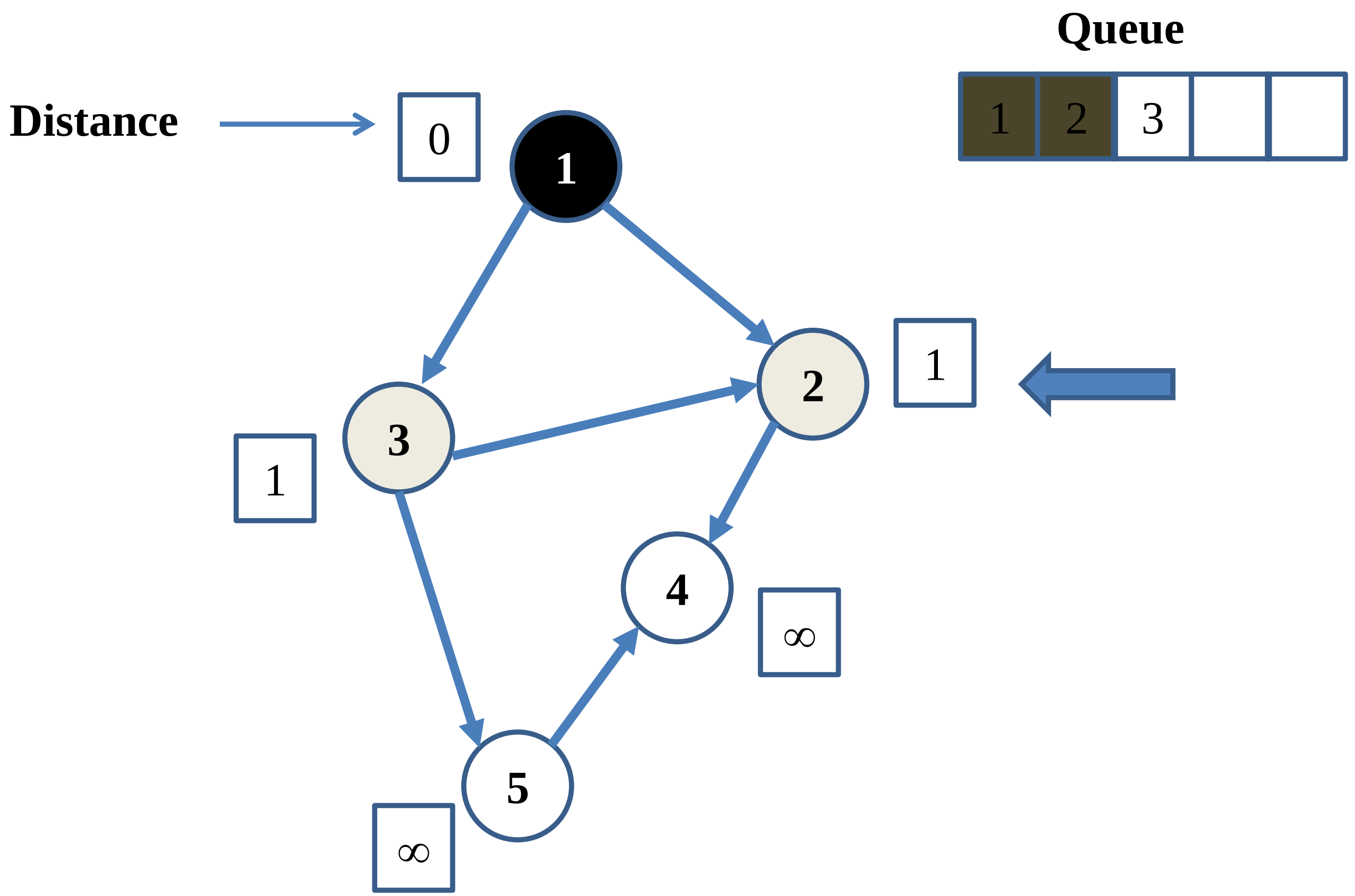
Π(1) = NIL

BFS in Action



$\Pi(1)$ = NIL
$\Pi(2) = 1$
$\Pi(3) = 1$

BFS in Action



Π(1) = NIL
Π(2) = 1
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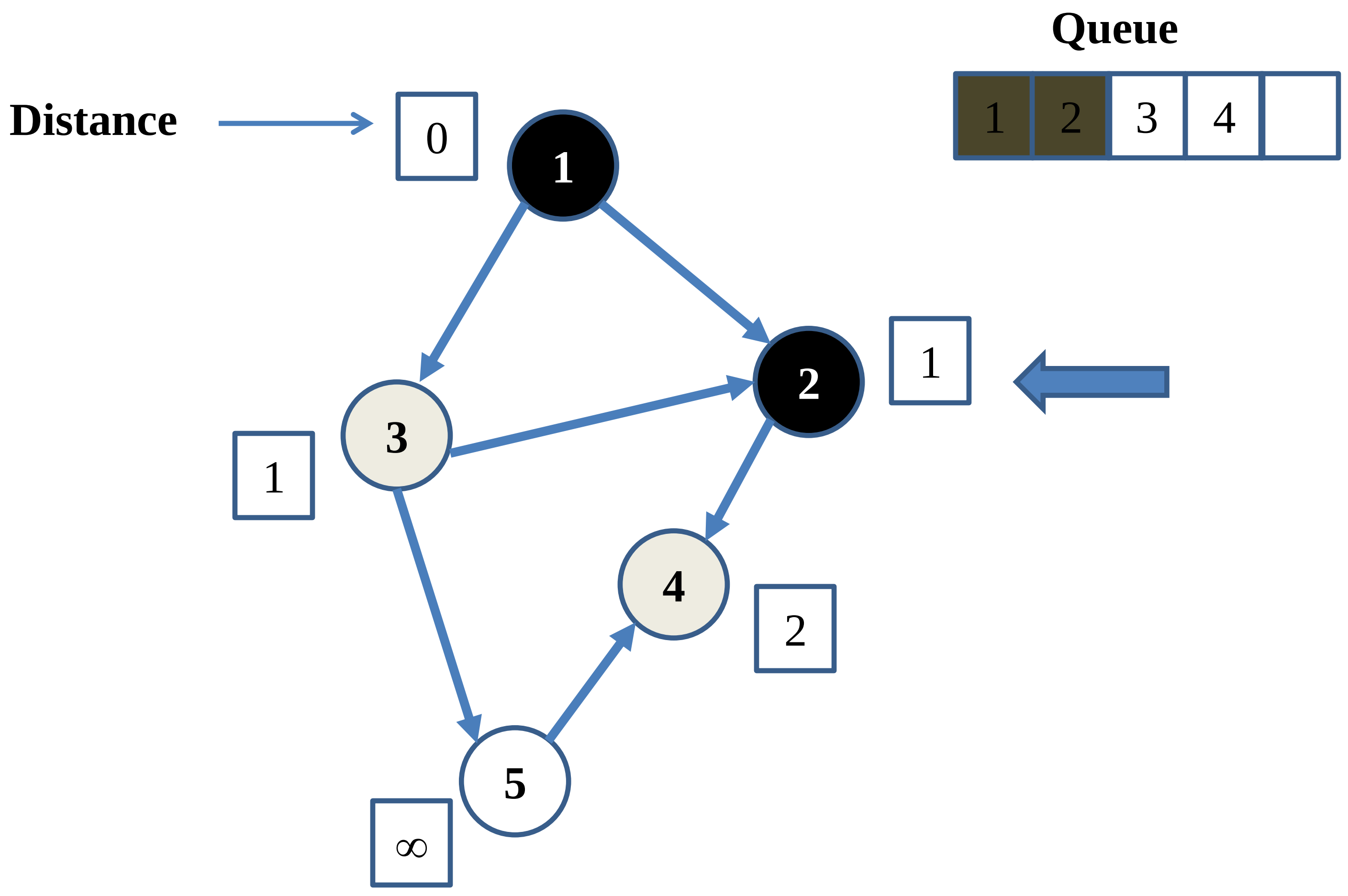
BFS in Action



Π(1) = NIL
Π(2) = 1
Π(3) = 1
Π(4) = 2

BFS in Action

Π(1) = NIL
Π(2) = 1
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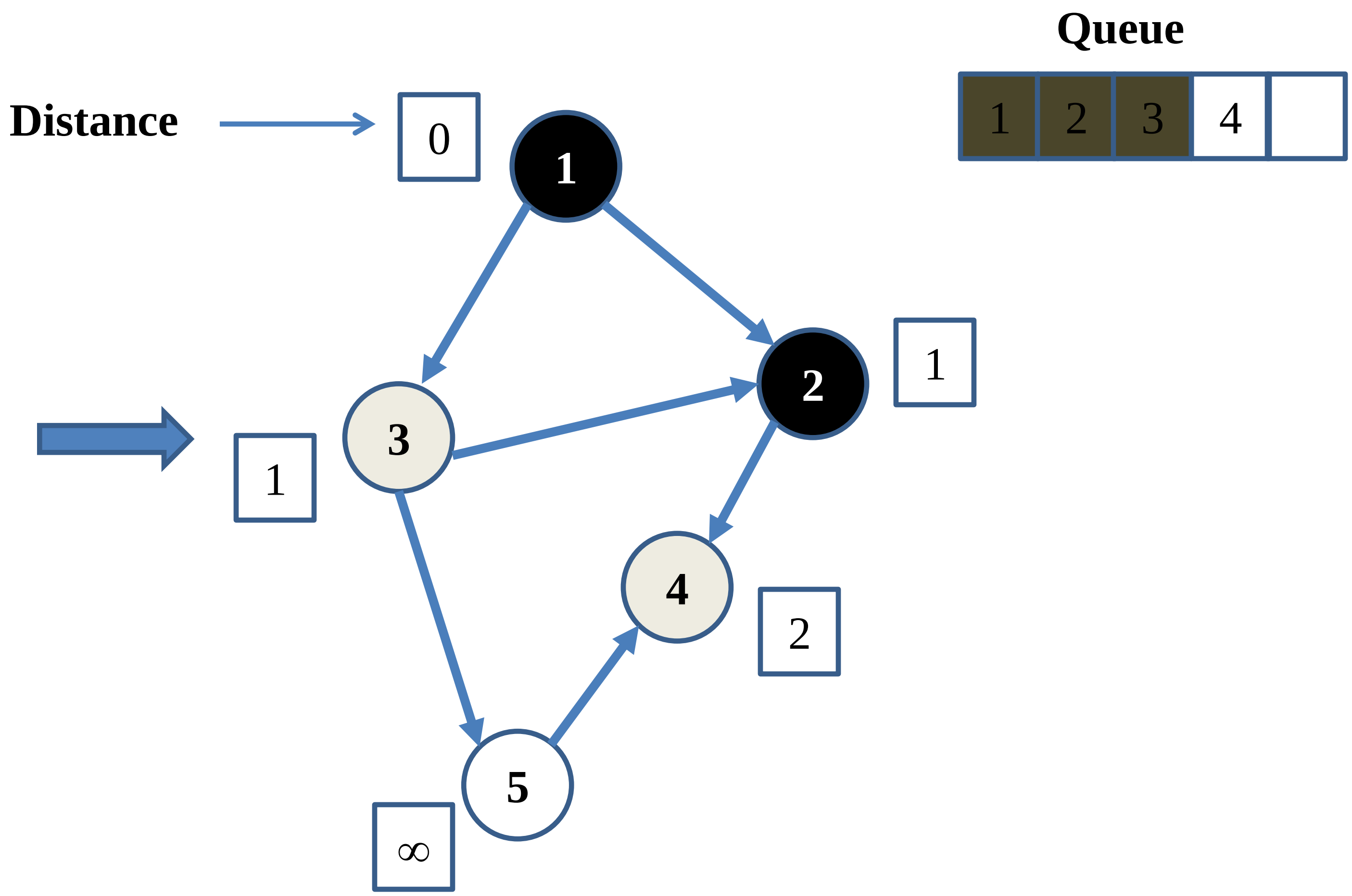
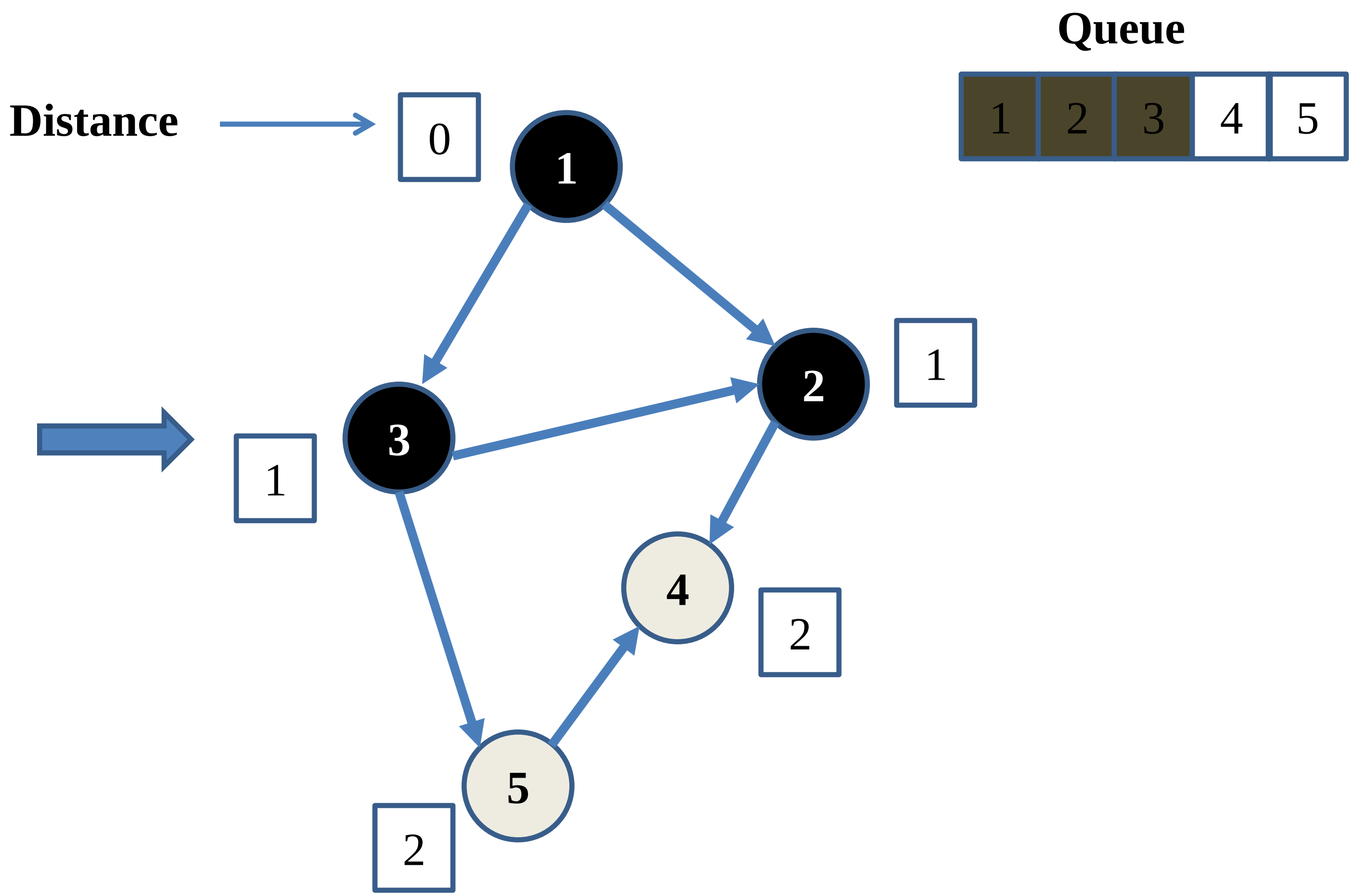
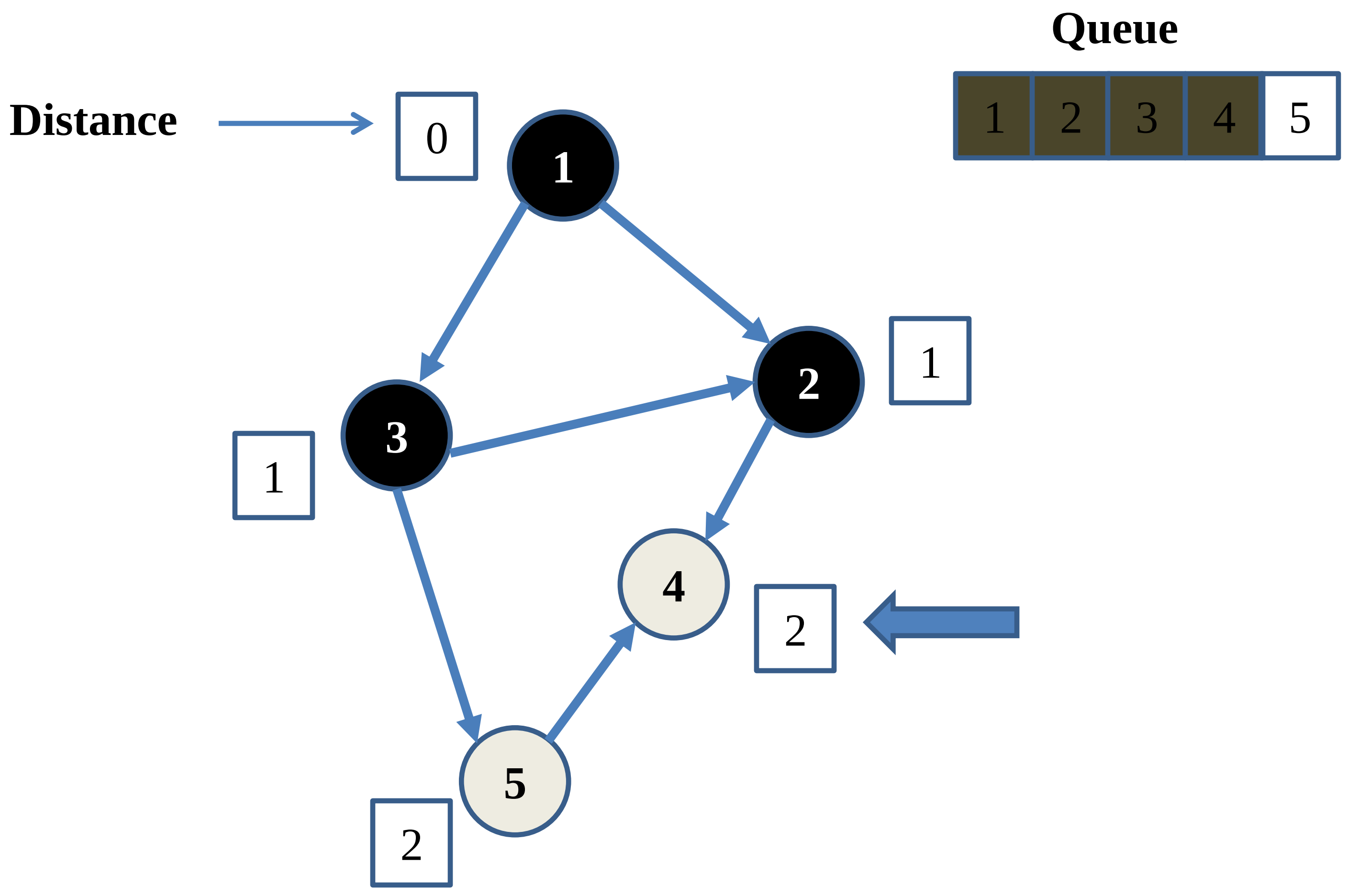


BFS in Action

Π(1) = NIL
Π(2) = 1
Π(3) = 1
Π(4) = 2
Π(5) = 3



BFS in Action



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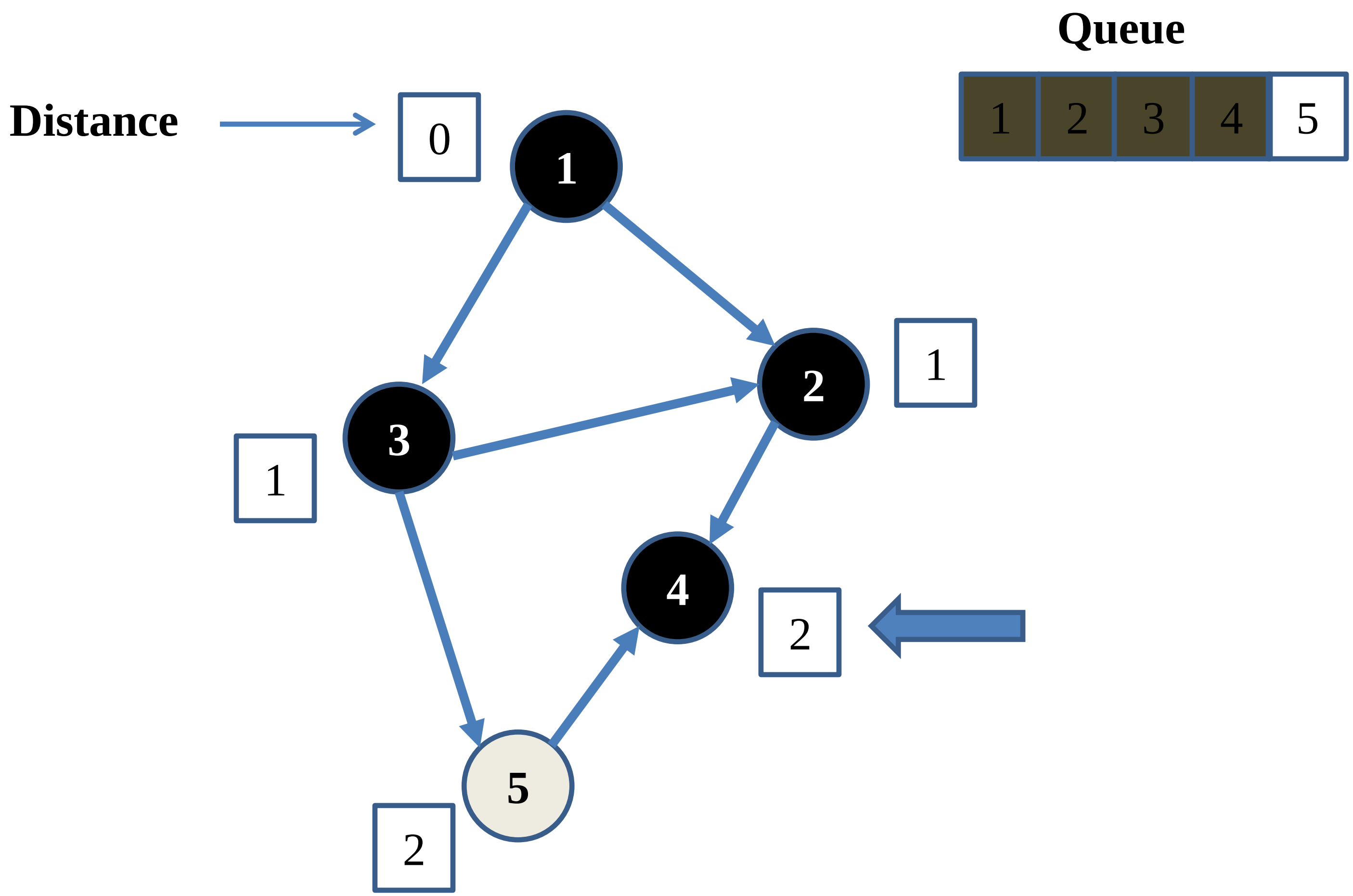
BFS in Action



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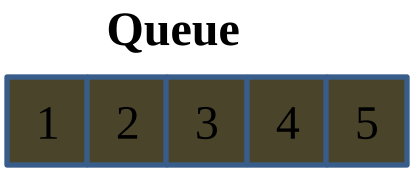
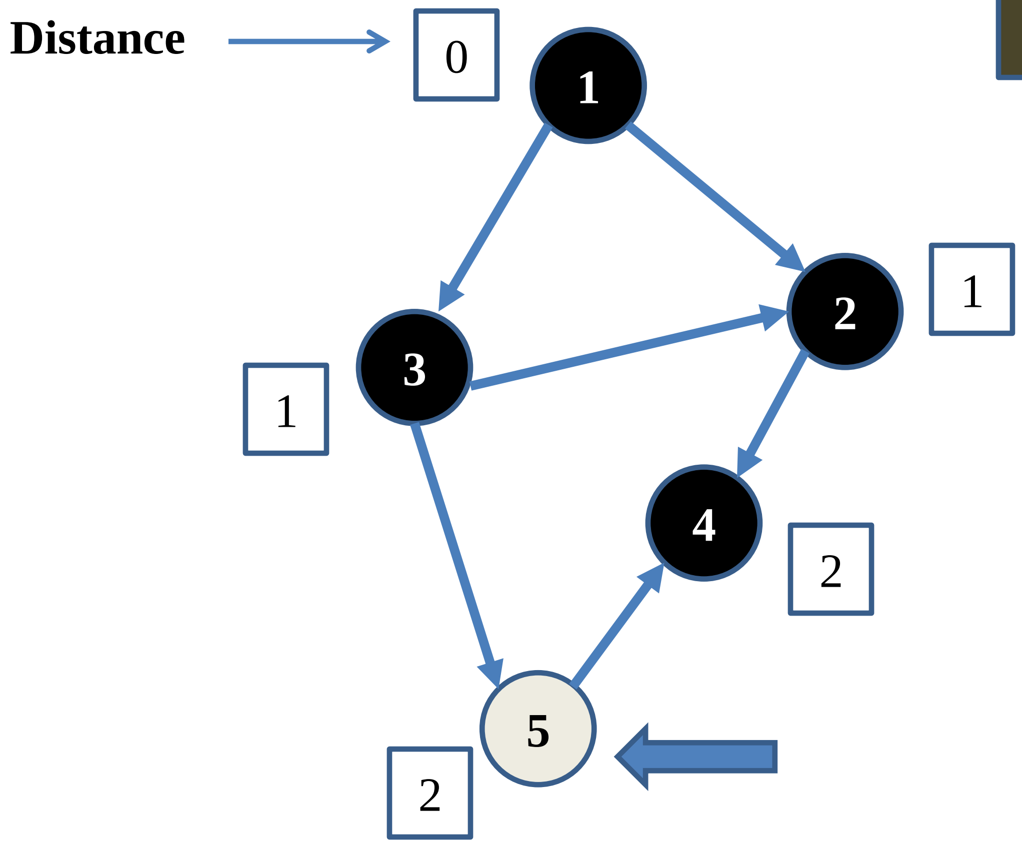
BFS in Action





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BFS in Action

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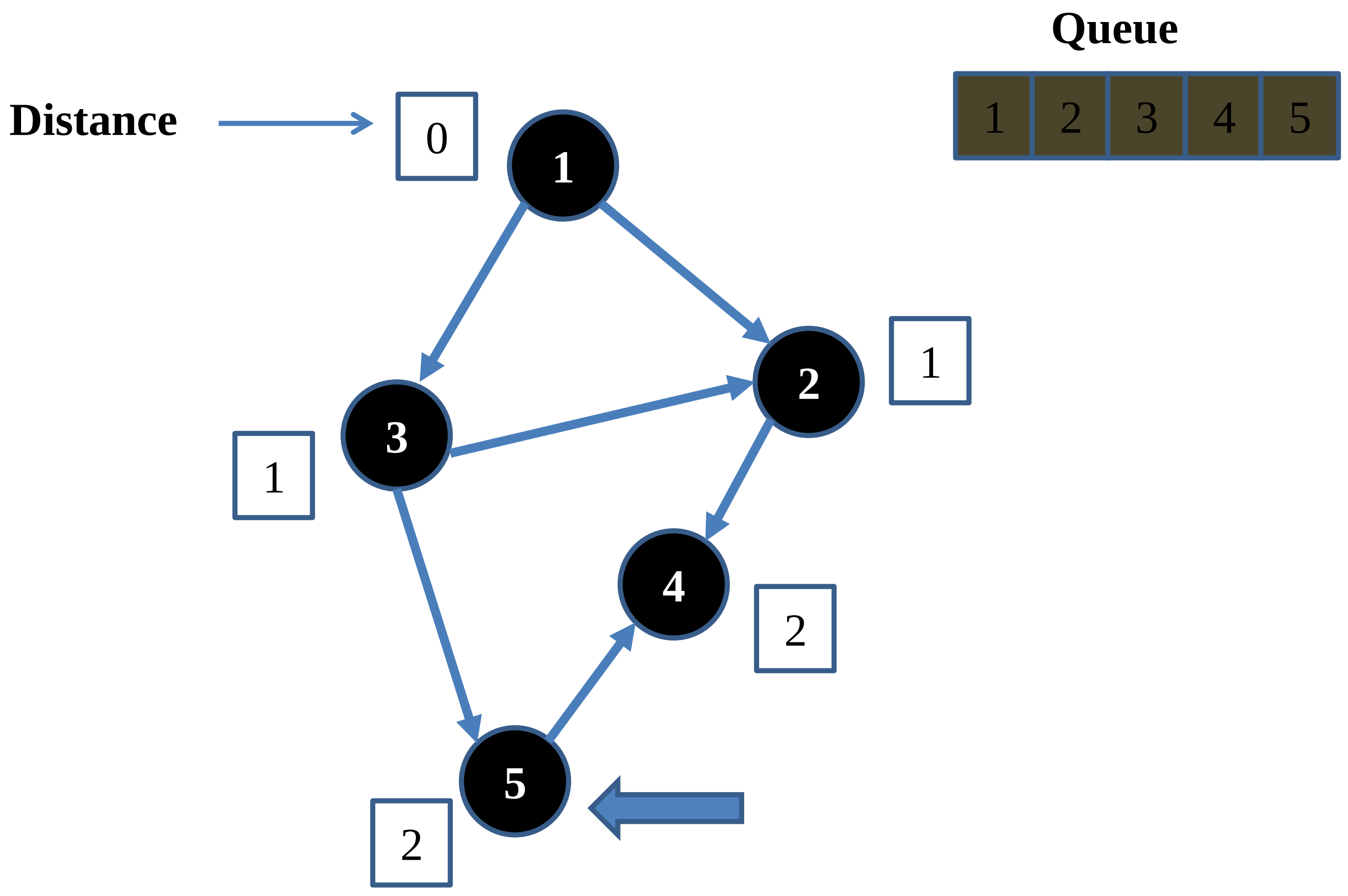
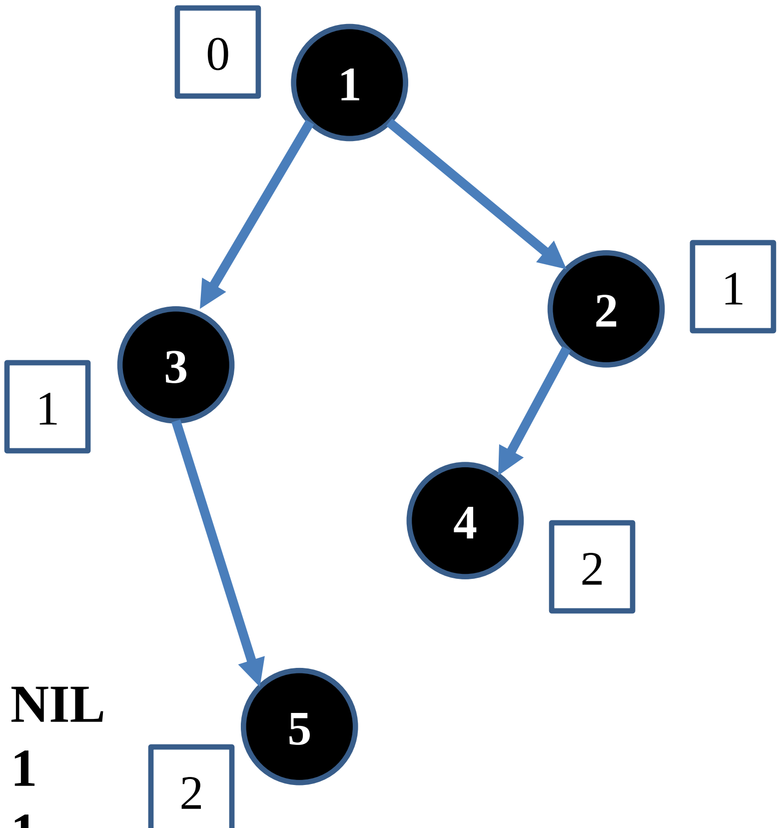


BFS Tree



- The breadth first tree gives the nodes reachable from the source node
- For an unweighted graph, it also shows the shortest path from the source vertex to every other vertex in the graph

Π(1) = NIL
Π(2) = 1
Π(3) = 1
Π(4) = 2
Π(5) = 3

BFS Algorithm

BFS(G, s)

  for each *vertex u* ***in*** *V[G] – {s}*

    colour[u] = WHITE

    Π[u]= NIL

    d[u]= ∞

  colour[s] = GRAY

  d[s] = 0

  Π[u]= NIL

  Q = φ

  ENQUEUE (Q, s)

BFS Algorithm

while $Q \neq \phi$

time = time+1

d[u] = time

for each *v* ***in*** *Adj[u]*

    if *colour[v]=WHITE*

        Π[v] = u

        DFS_VISIT(v)

colour[u] = BLACK

time = time+1

f[u] = time

Time Complexity of BFS

- Every node is explored EXACTLY ONCE --- Θ(|V|)
- For every node u, BFS explores all the edges in Adj[u]
- When summed over all the nodes in the graph, this amounts to the number of edges in the graph

$$\sum_{u \in V} Adj[u] = \Theta(|V|)$$

- Thus, **total complexity of BFS is also Θ(|V| + |E|)**

Thank You