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Phase transition in gauge theory of cuprates.

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Contents

Chapter 1

Gauge Theory for Doped Cuprates

1.1 Introduction

This project build upon the 2+1 dimensional SU(2) gauge theory of fluctuating incommensurate spin density wave (SDW) fluctuations presented previously [1]. This theory fractionalized the SDW order parameter by transforming to a rotating reference frame in spin space.

$$\boldsymbol{\sigma} \cdot \boldsymbol{S_i} = R_i \boldsymbol{\sigma} \cdot \boldsymbol{H_i} R_i^{\dagger} \tag{1.1}$$

In the above expression, the spin excitation is fractionalized into emergent fields R and H. R_i is a spacetime dependent SU(2) rotation matrix containing the orientational fluctuations of the SDW order parameter while $\boldsymbol{H_i}$ is the slowly varying spin magnetic moment in the rotated reference frame.

The fractionalization of $\boldsymbol{S}$ into $\boldsymbol{R}$ and $\boldsymbol{H}$ results in the following SU(2) gauge invariance

$$R_i \to R_i V_i^{\dagger} \quad , \quad \boldsymbol{\sigma} \cdot \boldsymbol{H_i} \to V_i \boldsymbol{\sigma} \cdot \boldsymbol{H_i} V_i^{\dagger} \tag{1.2}$$

For Hole doped cuprates, the Higgs field is parameterized as

$$\boldsymbol{H_i} = Re[\boldsymbol{\mathcal{H}}_x e^{i\boldsymbol{K}_x \cdot \boldsymbol{r}_i} + \boldsymbol{\mathcal{H}}_y e^{i\boldsymbol{K}_y \cdot \boldsymbol{r}_i}] \tag{1.3}$$

where $\boldsymbol{K}_x, \boldsymbol{K}_y$ are the incommensurate wavevectors. $\boldsymbol{K}_{x,y}$ were defined so that $\boldsymbol{K}_{x,y}$ are the wavevectors of the CDW orders. The Gauge theory to describe Phase diagram of Doped cuprates was developed for $N_h = 4$ real, adjoint Higgs scalars [1].

The allowed form of Potential for the Higgs field is severely restricted by the Lattice and Gauge Symmetry, and upto quartic order in Higgs field only 5 independent terms are allowed.

$$\begin{aligned} V(\boldsymbol{\mathcal{H}}_{x,y}) = s(\boldsymbol{\mathcal{H}}_x \cdot \boldsymbol{\mathcal{H}}_x + \boldsymbol{\mathcal{H}}_y \cdot \boldsymbol{\mathcal{H}}_y) + u_0(\boldsymbol{\mathcal{H}}_x \cdot \boldsymbol{\mathcal{H}}_x + \boldsymbol{\mathcal{H}}_y \cdot \boldsymbol{\mathcal{H}}_y)^2 + \frac{u_1}{4}(\boldsymbol{\mathcal{H}}_x \cdot \boldsymbol{\mathcal{H}}_x - \boldsymbol{\mathcal{H}}_y \cdot \boldsymbol{\mathcal{H}}_y)^2 \\ + \frac{u_2}{2}(|\boldsymbol{\mathcal{H}}_x \cdot \boldsymbol{\mathcal{H}}_x|^2 + |\boldsymbol{\mathcal{H}}_y \cdot \boldsymbol{\mathcal{H}}_y|^2) + u_3(|\boldsymbol{\mathcal{H}}_x \cdot \boldsymbol{\mathcal{H}}_y|^2 + |\boldsymbol{\mathcal{H}}_x \cdot \boldsymbol{\mathcal{H}}_y^*|^2) \end{aligned} \tag{1.4}$$

To map the phase diagram of the Higgs field with that of the hole-doped cuprate, the following order parameters were identified as gauge-invariant bilinear combinations-

$$\begin{aligned} \phi &= |\boldsymbol{\mathcal{H}}_x|^2 - |\boldsymbol{\mathcal{H}}_y|^2 \quad \text{Ising nematic order paramter} \\ \Phi_x &= \boldsymbol{\mathcal{H}}_x \cdot \boldsymbol{\mathcal{H}}_x \quad \text{CDW order paramter at wavelength } 2\boldsymbol{K}_x \\ \Phi_y &= \boldsymbol{\mathcal{H}}_y \cdot \boldsymbol{\mathcal{H}}_y \quad \text{CDW order paramter at wavelength } 2\boldsymbol{K}_y \\ \Phi_+ &= \boldsymbol{\mathcal{H}}_x \cdot \boldsymbol{\mathcal{H}}_y \quad \text{CDW order paramter at wavelength } \boldsymbol{K}_x + \boldsymbol{K}_y \\ \Phi_- &= \boldsymbol{\mathcal{H}}_x \cdot \boldsymbol{\mathcal{H}}_y^* \quad \text{CDW order paramter at wavelength } \boldsymbol{K}_x - \boldsymbol{K}_y \end{aligned} \tag{1.5}$$

We can write the components of Higgs field $\boldsymbol{\mathcal{H}}_x = (\mathcal{H}_x^1, \mathcal{H}_x^2, \mathcal{H}_x^3)$ in terms of their real and imaginary parts as

$$\mathcal{H}_x^a = H_1^a + H_2^a \quad , \quad \mathcal{H}_y^a = H_2^a + H_3^a \tag{1.6}$$

Hence the real components of the Higgs Field are H_l^a, where $a \in \{1, 2, 3\}$ is a color index and $l \in \{1, 2, 3, 4\}$ is the flavor index. In this notation, the Lagrangian can be naturally generalised to N_h flavors. The corresponding $O(N_h)$ invariant potential is given by

$$V(H) = \frac{s}{2} H_l^a H_l^a + u_0 (H_l^a H_l^a)^2 + u_1 \left(H_l^a H_m^a H_l^b H_m^b - \frac{(H_l^a H_l^a)^2}{N_h} \right) \tag{1.7}$$

In the previous work [1], the critical point was analyzed in the limit of large N_h. The aim of this work is to study the critical point for the relevant small values of N_h.

1.2 Lattice Gauge Theory

The Higgs field transforms as an adjoint of SU(2), and the Lagrangian is given by

$$\mathcal{L}_{\mathcal{H}} = \frac{1}{g^2}\boldsymbol{F_{\mu\nu}} \cdot \boldsymbol{F_{\mu\nu}} + |\partial_\mu \boldsymbol{\mathcal{H}}_m - A_\mu \times \boldsymbol{H}_m| + V(\boldsymbol{\mathcal{H}}_{x,y}) \tag{1.8}$$

The action $S = \int \mathcal{L}_{\mathcal{H}}$ upon discretization becomes

$$S = a\sum_r \left(6H_l^a(r)H_l^a(r) - \sum_\mu H_l^a(r+\mu)H_l^b(r)Tr\{\sigma^a U_\mu(r)\sigma^b U_\mu^\dagger(r)\} + a^2V(r)\right) \\ - \frac{1}{ag^2}\sum_\Box Tr\{W_c + W_{ac}\} \tag{1.9}$$

where $W_c = U_1U_2U_3^\dagger U_4^\dagger, W_{ac} = U_4U_3U_2^\dagger U_1^\dagger$ are the Wilson loop operator in the clockwise and anticlockwise direction respectively.

We want to integrate out the gauge field to obtain effective action as sum of Gauge-invariant product of Higgs fields. In the present chapter, we will work in the strong coupling regime and will only look at terms upto quartic order in Higgs Field. Denote the integral over Gauge field of the relevant part in the partition function by Z_p

$$Z_p = \int [\mathcal{D}U] \exp\left\{\sum_{r,\mu} aH_l^a(r+\mu)H_l^b(r)Tr\{\sigma^a U_\mu \sigma^b U_\mu^\dagger\} + \frac{1}{ag^2}\sum_\Box Tr\{W_c + W_{ac}\}\right\} \tag{1.10}$$

On expanding the exponential, we obtain

$$Z_p = \prod_l \int \mathrm{d}U_l \left(1 + aH_l^aH_l^bTr\{\sigma^aU_l\sigma^bU_l^\dagger\} + \frac{a^2}{2}H_l^aH_l^bH_m^cH_m^dTr\{\sigma^aU_l\sigma^bU_l^\dagger\}Tr\{\sigma^cU_l\sigma^dU_l^\dagger\}\right) \\ \times \left(1 + \frac{1}{2a^2g^4}\sum_\Box Tr^2\{W_c + W_{ac}\}\right) \tag{1.11}$$

Where the product is over all links in the lattice.
As shown in Fig 1.1, we associate the terms in the above expression with corresponding diagrams.

With this association, the only non-zero terms in the above expression are denoted by diagrams shown the Fig 1.2 as shown.

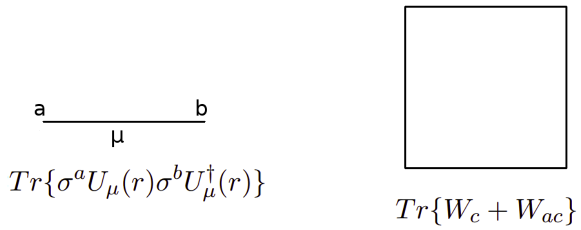
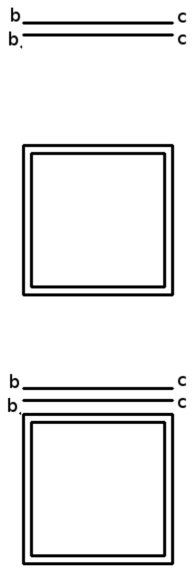


FIGURE 1.1: Diagrammatic representation of terms in expansion.



$$\int \mathrm{d}U_\mu Tr^2\{\sigma^a U_\mu \sigma^b U_\mu^\dagger\} = \frac{4}{3}$$

$$\int [\mathcal{D}U] Tr^2\{W_c + W_{ac}\} = 4$$

$$\int [\mathcal{D}U] Tr^2\{\sigma^a U_\mu \sigma^b U_\mu^\dagger\} Tr^2\{W_c + W_{ac}\} = \frac{16}{3}$$

FIGURE 1.2: Non-zero terms upto quartic order in Higgs field.

1.3 Group Inregration

To carry out the integration over the SU(2) manifold, the following parameterization was used

$$U = I_2 \cos\theta + i\sigma^1 \sin\theta \sin\psi \cos\phi + i\sigma^2 \sin\theta \cos\psi + i\sigma^3 \sin\theta \sin\psi \sin\phi \tag{1.12}$$

and the corresponding Haar measure is

$$dU = \frac{1}{2\pi^2} \int_0^\pi \mathrm{d}\theta \int_0^\pi \mathrm{d}\psi \int_0^{2\pi} \mathrm{d}\phi \sin^2\theta \sin\psi \tag{1.13}$$

1.4 Effective Action

After carrying out the integration, we obtain

$$Z_p = (1 + \frac{2}{a^2 g^4}) \prod_{l=r,\mu} \left(1 + \frac{2a^2}{3} H_l^c(r)(r+\mu) H_l^b H_m^c(r+\mu) H_m^b(r)\right) \tag{1.14}$$

Ignoring the irrelevant constant, the effective action is given by

$$S_{eff} = \sum_r \left(a^3 V(r) + 6a H_l^b(r) H_l^b(r) - \sum_\mu \frac{a^2}{3} H_l^c(r)(r+\mu) H_l^b H_m^c(r+\mu) H_m^b(r)\right) \tag{1.15}$$

It seems that while considering terms only upto quartic order in the Higgs field, the coupling strength does not play any role. This is a direct consequence of the factorization of following diagram.

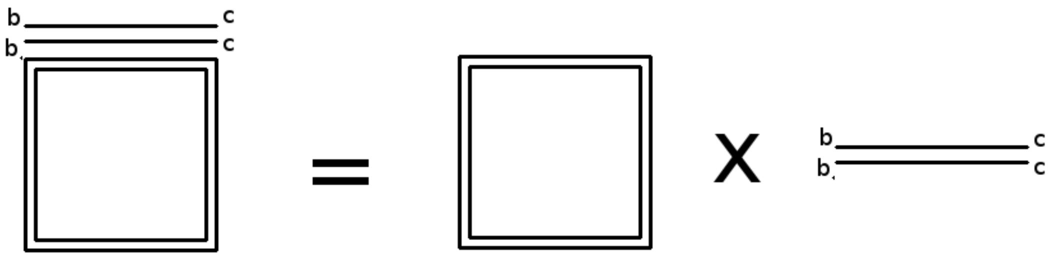


FIGURE 1.3: Factorization of quartic term.

Chapter 2

Effective Action

2.1 Quartic Order

In the last section, we say that the coefficient of the Quartic terms in the effective action had no dependence on the coupling constant g upto order $\frac{1}{g^4}$. Therefore, to understand the effect of Gauge fields on the Quartic terms, we need to evaluate higher order terms in g.

Instead, we will write down the Effective action of the Higgs Field upto Quartic order for any value of the coupling constant g. Starting from eq 1.9 , we can write Z_p as (ignoring an irrelevant proportionality constant)

$$Z_p = 1 + \frac{a^2\kappa(1/g^2)}{2} \sum_{r,\mu} H_l^a H_l^b H_m^c H_m^d \tag{2.1}$$

where the function $\kappa(1/g^2)$ is given by

$$\kappa(1/g^2) = \frac{\int [\mathcal{D}U] e^{\frac{2}{ag^2}\sum_{\Box}\mathrm{Tr}\{W\}}\, \mathrm{Tr}\Big\{\{\sigma^a U_\mu \sigma^b U_m^\dagger u\}\Big\}^2}{\int [\mathcal{D}U] e^{\frac{2}{ag^2}\sum_{\Box}\mathrm{Tr}\{W\}}} \tag{2.2}$$

thus encompassing the contribution of the Gauge Fields. In the large coupling limit, $\kappa(1/g^2) = \frac{4}{3} + \mathcal{O}(\frac{1}{g^6})$. For smaller value of g, Monte Carlo code for pure SU(2) gauge theory can be used to find the coupling constant $\kappa(1/g^2)$.

Hence, upto quartic order, the effective action is given by

$$S_{eff} = \sum_r \Big(-\frac{a^2}{2}\kappa(1/g^2)\sum_\mu H_l^a H_l^b H_m^c H_m^d + (6a + s\frac{a^3}{2})H_l^a H_l^a + a^3(u_0 - \frac{u_1}{4})(H_l^a H_l^a)^2 + a^3 u_1 H_l^a H_m^a H_l^b H_m^b \Big) \quad (2.3)$$

We will later return to this action to study its phase diagram.

2.2 Higher Order Correction

The next order non-zero terms in the effective action are of order 8. There are several terms at this order but the physically most important term is the one coupling Higgs field on a plaquette. This term correspond to the following diagram.

In the above term, the gauge fields couple to the Higgs field on the corners of the plaquette and therefore the phase of the Gauge fields determines the texture of the Higgs field. After integrating out the gauge fields, we obtain the following gauge-invariant coupling between the Higgs field.

$$\sum_{r,\mu>\nu} \frac{1}{a^2 g^4}[H_l(r)\cdot H_m(r)][H_m(r+\mu)\cdot H_n(r+\mu)][H_n(r+\mu+\nu)\cdot H_p(r+\mu+\nu)][H_p(r+\nu)\cdot H_l(r+\nu)] \quad (2.4)$$

However, after integrating out the gauge fields, we lose the information all the information about the gauge fields. To retain the necessary information about the Gauge fields, we can couple the plaquette action to a Z_2 Ising gauge theory on the lattice by modifying the Wilson action as

$$-\frac{2}{ag^2}\sum_{\square} Tr\{W_c\} \rightarrow -2\sum_{\square}\left(\frac{1}{ag^2} + \lambda\sigma_i\sigma_j\sigma_k\sigma_l\right) Tr\{W_c\} \quad (2.5)$$

where λ is large enough to ensure that there is a one-one mapping between the phase of the Gauge Field $\{U_\mu\}$ and the ising Gauge field $\{\sigma_i\}$. This will allow us to identify the topological phase transition even after integrating out the Gauge fields on the links. The eigth order term gets modified to

$$\sum_{r,\mu>\nu}\left(\frac{1}{a^2g^4} + \lambda\sigma_i\sigma_j\sigma_k\sigma_l\right)[H_q(i)\cdot H_m(i)][H_m(j)\cdot H_n(j)][H_n(k)\cdot H_p(k)][H_p(l)\cdot H_q(l)] \quad (2.6)$$

where $(i, j, k, l) = (r, r + \mu, r + \mu + \nu, r + \nu)$.

Chapter 3

Numerical Method

In this chapter I will outline some ideas for the numerical study of the effective action given by equation (2.3) using Monte Carlo Method. Our aim is to obtain the expectation value of the order parameters given by the following formula.

$$\langle f(\{H\})\rangle = \frac{\int[\mathcal{D}H]f(\{H\})e^{-S_{eff}}}{\int[\mathcal{D}H]e^{-S_{eff}}} \approx \frac{1}{N_c}\sum_C f(\{H\}_C) \tag{3.1}$$

where the set of configurations $\{C\}$ is generated using the Monte Carlo method.

3.1 Integration Variables

To calculate the expectation value in equation (3.1), we can choose to integrate in terms of the Higgs field. However, since both effective action and the function f are gauge invariant, we can change our integration variables to Gauge Invariant Order parameter Q_{lm} (different from [1]) given by

$$Q_{lm} = H_l^a H_m^a \tag{3.2}$$

To make this change of variables, we need to introduce the identity

$$1 = \int [\mathcal{D}Q_{lm}]\delta(Q_{lm} - H_l^a H_m^a) \tag{3.3}$$

and integrate over the Higgs field. This reduces the number of integration variables. However, the integration of the delta functions introduces a jacobian factor which must also be calculated numerically due to large number of fields. Because of this additional complexity, we prefer to perform the integration in terms of the Higgs field and not the order parameters Q_{lm}.

3.2 Metropolis Algorithm

The most direct way to study the action is to update Higgs field using the Metropolis Algorithm. However, due to a large number of terms involved in the action, the direct implementation of the Metropolis algorithm turned out to be quite slow for effective numerical computations. It is therefore necessary to implement more clever algorithms to study the model. It must be pointed out that the speed of the Metropolis algorithm can be improved to some extent by using the gauge invariance property of the effective action, as outlined in appendix A.

In the next three sections, I will describe several update algorithms for the direction of the various flavors of Higgs field. In these algorithms, the magnitude of the Higgs field will be kept constant. The magnitude update can be implemented separately using Metropolis algorithm.

3.3 Heat Bath algorithm

Suppose we want to update the H_l at site i. The rest of the configuration is known, and we can write down the action as

$$S = \vec{B} \cdot H_l + H_l^T \mathcal{M} H_l \tag{3.4}$$

Because of it's complicated structure, heat bath update cannot be directly applied to the above action. However, we can generate the new H_l according to the probability distribution $e^{\vec{B} \cdot H_l}$ and then accept it with the probability $e^{-H_l^T \mathcal{M} H_l}$. This combination of the Heat Bath and the Metropolis update satisfy the detailed balanced condition. This scheme does not provide any improvement in the speed, but provide smarter guess for H_l and allows us to explore the phase space more efficiently.

3.4 Wolff Algorithm

The gauge invariance of the theory prevents us from directly applying the Wolff algorithm, since reflection of the Higgs field does not change the action. Fortunately, it is easy to apply the Wolff algorithm in the flavor space because of the $O(4)$ symmetry of the model. Inverting the role of flavors and components of the Higgs field, we can use the following representation.

$$\underline{H}^a = \begin{bmatrix} H_1^a \\ H_2^a \\ H_3^a \\ H_4^a \end{bmatrix} \tag{3.5}$$

In this representation, the action given by equation (2.3) can be written as

$$S_{eff} = \sum_r \Big(-\frac{a^2}{2}\kappa(1/g^2)\sum_\mu [H^a(r+\mu)\cdot H^b(r+\mu)][H^c(r)\cdot H^d(r)] \\ + (6a + s\frac{a^3}{2})[H^a \cdot H^a] + a^3(u_0 - \frac{u_1}{4})[H^a \cdot H^a]^2 + a^3 u_1 [H^a \cdot H^b][H^a \cdot H^b] \Big) \tag{3.6}$$

From the above expression, we can see that on rotating or reflecting all the Higgs "components" at a particular site, the action only changes because of the link term. Hence, we can implement the Wolff algorithm as follows.

1. Choose a random 4D axis $\vec{n}$ and a lattice site. Reflect all Higgs "components" about 3D hyper plane perpendicular to this axis.
2. For all points j not in the cluster which are neighbours of points i in the cluster, add them to the cluster with probability $P(i,j) = max0, 1 - e^{-S'-S}$ where S is the value of link term if the Fields on site i is reflected and fields on site j are not reflected while S' is the value of the link term in fields on both sites i, j are reflected.
3. Repeat the last step for new points added to the cluster

It is easy to show that above algorithm satisfy the detailed balance condition. The cluster algorithm can modify the configuration non-locally and therefore we expect the cluster algorithm to solve the issue of critical slowing down near the critical point. However, since we necessarily have to reflect all the "components" of the Higgs field, the update method is non-ergodic as evident from the fact that not all the terms in the action play role in the update method. This shortcoming will be resolved later by combining multiple update methods.

3.5 Over-relaxation

Over relaxation can be implemented trivially in the 3D space by randomly choosing any two flavors and then reflecting the other two flavors about the plane formed by the chosen flavors. It has been observed in previous studies that the over-relaxation update stir-up the configurations and the equilibrium is quickly achieved [2]. Like the Wolff update, the over-relaxation update algorithm is non-ergodic and must be combined with Heat Bath algorithm to to satisfy the ergodicity requirement.

3.6 Combining multiple update algorithms

As pointed out, each update algorithm has it's own benefits and downfall. Although we expect the Wolff and the over-relaxation updates to be speed up the simulation, they cannot be employed independently because of their non-ergodic nature. Therefore, the best possible strategy seems to combine the different kinds of updates so that we can make use of benefits of each update algorithm.

Chapter 4

SYK-like Pairing Model

Unlike the rest of this report, this chapter is on the SYK-like model in which I attempted to obtain symmetry-breaking transition in which two different species of complex fermions can pair together on reducing the temperature of the system. The two species of fermions were coupled only by the disorder and there was no direct interaction between the species. The chapter is not related to the rest of the report describing the Gauge theory of the doped cuprates.

4.1 Model Description

Lets considered a lattice model of two species of fermions $\{\hat{\psi}_i\}$ and $\{\hat{\phi}_i\}$ with disordered local on-site Sachdev-Ye-Kitaev interactions of 4th order.

$$H = \sum_{\substack{i<j \\ k<l}} K_{ijkl}\hat{\psi}_i{}^\dagger\hat{\psi}_j{}^\dagger\hat{\psi}_k\hat{\psi}_l + L_{ijkl}\hat{\phi}_i{}^\dagger\hat{\phi}_j{}^\dagger\hat{\phi}_k\hat{\phi}_l \tag{4.1}$$

We impose the following constraint on disordered coupling constants K_{ijkl},L_{ijkl}-

$$K_{ijkl} = L^*_{ijkl} = J_{ijkl} \tag{4.2}$$

where the complex coupling constants $\{J_{ijkl}\}$ are chosen randomly from the Gaussian probability distribution $\exp\left\{-\frac{|J_{ijkl}|^2N^3}{J^2}\right\}$. The Hamiltonian reduces to

$$H = \sum_{\substack{i<j \\ k<l}} J_{ijkl}\left(\hat{\psi}_i{}^\dagger\hat{\psi}_j{}^\dagger\hat{\psi}_k\hat{\psi}_l + \hat{\phi}_i\hat{\phi}_j\hat{\phi}_k{}^\dagger\hat{\phi}_l{}^\dagger\right) \tag{4.3}$$

This model can also be thought as describing particles $\{\hat{\psi}_i\}$ and holes $\{\hat{\phi}_i\}$ with the same on-site SYK_4 interaction. Let μ_a and μ_b be the chemical potential of the of fermions $\{\hat{\psi}_i\}$ and $\{\hat{\phi}_i\}$ respectively. Then the action is given by

$$\mathrm{S}{=}\int \mathrm{d}\tau \left(\sum_i(\psi_i^\dagger(\partial_\tau-\mu_a)\psi_i+\phi_i^\dagger(\partial_\tau-\mu_b)\phi_i)+\sum_{\substack{i<j\\k<l}} J_{ijkl}\left(\hat{\psi}_i{}^\dagger\hat{\psi}_j{}^\dagger\hat{\psi}_k\hat{\psi}_l+\hat{\phi}_i\hat{\phi}_j\hat{\phi}_k{}^\dagger\hat{\phi}_l{}^\dagger\right)\right)$$

On taking the disorder average,

$$\begin{aligned} S' = N\Bigg(\int \mathrm{d}\tau(\overline{\psi^\dagger(\partial_\tau-\mu_a)\psi}+\overline{\phi^\dagger(\partial_\tau-\mu_b)\phi})-J^2\int \mathrm{d}\tau_1\int \mathrm{d}\tau_2[(\overline{\psi^\dagger(1)\psi(2)})^2(\overline{\psi^\dagger(2)\psi(1)})^2 \\ +(\overline{\phi^\dagger(1)\phi(2)})^2(\overline{\phi^\dagger(2)\phi(1)})^2+2(\overline{\psi^\dagger(1)\phi^\dagger(2)})^2(\overline{\psi(1)\phi(2)})^2]\Bigg) \end{aligned} \tag{4.4}$$

In the above expression, the bar signifies the average over all indices, for example, $\overline{\phi^\dagger(2)\phi(1)} = \frac{1}{N}\sum_i \phi_i^\dagger(2)\phi_i(1)$.

4.2 Effective Action

To simplify the disorder-averaged partition function, let us include the following identity

$$\begin{aligned} 1 = \int \mathcal{D}(\mathcal{AF})exp\{N\int \mathrm{d}\tau_1\int \mathrm{d}\tau_2\Big(\Xi_1(1,2)(G_1(2,1)+\overline{\psi^\dagger(1)\psi(2)}) \\ +\Xi_2(2,1)(G_2(1,2)+\overline{\phi^\dagger(2)\phi(1)})+\Lambda_1(2,1)(F_1(2,1)+\overline{\phi(1)\psi(2)}) \\ +\Lambda_2(1,2)(F_2(1,2)+\overline{\psi^\dagger(1)\phi^\dagger(2)})\Big)\} \end{aligned} \tag{4.5}$$

where $\mathcal{D}(\mathcal{AF})$ is the integration measure over auxiliary fields. The aim is to integrate out the fermionic fields. In the limit of large N, the integral for each index i is given by

$$\begin{aligned} &\int \mathcal{D}\psi^\dagger\mathcal{D}\psi\mathcal{D}\phi^\dagger\mathcal{D}\phi\ exp\Big(-\int \mathrm{d}\tau[\psi^\dagger(\partial_\tau-\mu_a)\psi+\phi(\partial_\tau+\mu_b)\phi^\dagger] \\ &+\int \mathrm{d}\tau_1\mathrm{d}\tau_2[\Xi_1(1,2)\psi^\dagger(1)\psi(2)-\Xi_2(2,1)\phi(1)\psi^\dagger(2)+\Lambda_1(2,1))\phi(1)\psi(2) \\ &+\Lambda_2(1,2))\psi^\dagger(1)\phi^\dagger(2)]\Big) \end{aligned} \tag{4.6}$$

In writing the above equation, the fact that $\phi^\dagger \partial_\tau \phi = \phi \partial_\tau \phi^\dagger$ has been used.

Also, since Hamiltonian does not explicitly depends on time, it can be assumed that for any non-local function $\xi(\tau_1, \tau_2) = \xi(\tau_1 - \tau_2)$. After fourier transform, the above integral becomes

$$\int \mathcal{D}\psi^\dagger \mathcal{D}\psi \mathcal{D}\phi^\dagger \mathcal{D}\phi \; exp\Big(-\frac{1}{\beta}\sum_m \Big[[iw_m - \mu_a - \Xi_1(w_m)]\psi^\dagger(w_m)\psi(w_m) \\ + [iw_m + \mu_b + \Xi_2(-w_m)]\phi(-w_m)\psi^\dagger(-w_m) \\ - \Lambda_1(-w_m)\phi(-w_m)\psi(w_m) - \Lambda_2(w_m)\psi(w_m)\phi(-w_m)\Big]\Big) \tag{4.7}$$

Introduce the nambu spinors

$$\eta(w_m) = \begin{bmatrix} \psi(w_m) \\ \phi^\dagger(-w_m) \end{bmatrix} \quad , \quad \bar{\eta}(w_m) = \begin{bmatrix} \psi^\dagger(w_m) & \phi(-w_m) \end{bmatrix}$$

The above integral becomes

$$\prod_m \int -\mathrm{d}\eta(w_m)\mathrm{d}\bar{\eta}(w_m) \exp\{-\bar{\eta}(w_m) athcalM(w_m)\eta(w_m)\} \\ = \prod_m (-det(\mathcal{M}(w_m))) = \exp\Big\{\sum_m \ln(-|\mathcal{M}(w_m)|)\Big\} \tag{4.8}$$

Where the matrix $\mathcal{M}(w_m)$ is given by

$$\mathcal{M}(w_m) = \frac{1}{\beta}\begin{bmatrix} [iw_m - \mu_a - \Xi_1(w_m)] & -\Lambda_2(w_m) \\ -\Lambda_1(-w_m) & [iw_m + \mu_b + \Xi_2(-w_m)] \end{bmatrix}$$

Also, define

$$\mathcal{P}(w_m) = \beta^2 det(\mathcal{M}(w_m)) = \{iw_m - \mu_a - \Xi_1(w_m)\}\{iw_m + \mu_b + \Xi_2(-w_m)\} - \Lambda_1(-w_m)\Lambda_2(w_m) \tag{4.9}$$

In terms of Matsubara frequencies, the effective action is

$$\frac{S_{eff}}{N} = \sum_m \Bigg(-\ln\Bigg(-\frac{\mathcal{P}(w_m)}{\beta^2} \Bigg) - \Xi_1(w_m)G_1(-w_m) - \Xi_2(w_m)G_2(-w_m) - \Lambda_1(w_m)F_1(-w_m) - \Lambda_2(w_m)F_2(-w_m) \Bigg)$$
$$+\frac{J^2}{\beta^2}\sum_{m,n,p}\Bigg(G_1(w_m)G_1(w_n)G_1(w_p)G_1(w_m+w_n-w_p)+G_2(w_m)G_2(w_n)G_2(w_p)G_2(w_m+w_n-w_p) + 2F_1(w_m)F_1(w_n)F_2(w_p)F_2(-w_m-w_n-w_p)\Bigg) \quad (4.10)$$

4.3 Dyson Equation

Using the above effective action, we obtain the following 8 Dyson equations by varying the indicated field

$$\Xi_1 \implies \frac{1}{\mathcal{P}(-w_q)}[iw_q - \mu_b - \Xi_2(-w_q)] + G_1(-w_q) = 0 \quad (4.11)$$

$$\Xi_2 \implies \frac{1}{\mathcal{P}(w_q)}[iw_q - \mu_a - \Xi_1(-w_q)] + G_2(-w_q) = 0 \quad (4.12)$$

$$\Lambda_1 \implies \frac{-1}{\mathcal{P}(-w_q)}\Lambda_2(-w_q) + F_1(-w_q) = 0 \quad (4.13)$$

$$\Lambda_2 \implies \frac{-1}{\mathcal{P}(w_q)}\Lambda_1(-w_q) + F_2(-w_q) = 0 \quad (4.14)$$

$$G_1 \implies 4J^2G_1^2(\beta-\tau)G_1(\tau) - \Xi_1(\tau) = 0 \quad (4.15)$$

$$G_2 \implies 4J^2G_2^2(\beta-\tau)G_2(\tau) - \Xi_2(\tau) = 0 \quad (4.16)$$

$$F_1 \implies 4J^2F_1(\tau)F_2^2(\tau) - \Lambda_1(\tau) = 0 \quad (4.17)$$

$$F_2 \implies 4J^2F_2(\tau)F_1^2(\tau) - \Lambda_2(\tau) = 0 \quad (4.18)$$

The last four equations in the frequency representation are given by-

$$4\frac{J^2}{\beta^2}\sum_{n,p} G_i(w_n)G_i(w_p)G_i(w_q+w_n-w_p) - \Xi_i(-w_q) = 0 \quad (4.19)$$

$$4\frac{J^2}{\beta^2}\sum_{n,p} F_i(w_n)F_j(w_p)F_j(-w_q-w_n-w_p) - \Lambda_i(-w_q) = 0 \quad (4.20)$$

where $i, j \in \{1, 2\}$ and $i \neq j$.

From equations, it can be seen that by assuming $F_1 = F_2 = 0$ identically, the equations decouples into Dyson equations of two independent SYK_4 models. However, to possibly obtain an alternate solution, we should assume that F_1, F_2 are non-zero obtain their values using self consistency.

4.4 Fourier Transform Formulae

The following formulae for Fourier transforms were used in the above derivation

$$\begin{aligned} f(\tau) &= \sum_n e^{iw_n\tau} f(w_n) \\ f(w_n) &= \int \frac{\mathrm{d}\tau}{\beta} e^{-iw_n\tau} f(\tau) \end{aligned} \tag{4.21}$$

The following identities for obtaining the δ functions were employed. Here $\tau_1, \tau_2 \in (0, \beta)$

$$\begin{aligned} \sum_n e^{iw_n(\tau_1-\tau_2)} &= \beta\delta(\tau_1 - \tau_2) \\ \sum_n e^{iw_n(\tau_1+\tau_2)} &= -\beta\delta(\tau_1 + \tau_2 - \beta) \\ \int \mathrm{d}\tau e^{i\tau(w_n-w_p)} &= \beta\delta_{m,n} \end{aligned} \tag{4.22}$$

4.5 Results

On performing the numerical simulations, it was found the transition does not occur at any temperature and the two species of fermions are not correlated.

Chapter 5

Appendix A: Integration Measure

I will first sketch down the proof for $N_h = 1$ flavor. We start with the integration measure given by

$$\prod_r \left(\int \mathrm{d}H_1(r)\mathrm{d}H_2(r)\mathrm{d}H_3(r) \prod_\mu \int \mathrm{d}U_\mu(r) \right) \tag{5.1}$$

Represent the Higgs Field as $\tilde{H}(r) = \rho(r)\tilde{X}^\dagger(r)\sigma_z\tilde{X}(r)$, where ρ is the magnitude of the Higgs Field while $\tilde{X}(r) \in SU(2)$

Let us change the integration on Gauge fields $U_\mu(r)$ to $V_\mu(r) = \tilde{X}(r+\mu)U_\mu(r)\tilde{X}^\dagger(r)$. The action is thus only a function of $V_\mu(r)$ and $\rho(r)$. Since we are integrating over the gauge fields before integrating over the Higgs field, $\tilde{X}(r)$ is constant element of $SU(2)$ in each integral of the gauge field $U_\mu(r)$.

Therefore, as $U_\mu(r)$ goes over the SU(2) manifold, $V_\mu(r) = \tilde{X}(r+\mu)U_\mu(r)\tilde{X}^\dagger(r)$ must also cover the entire SU(2) manifold. Since Haar measure is invariant under the left and right multiplication,

$$\int_{SU(2)} \mathrm{d}U_\mu(r) = \int_{SU(2)} \mathrm{d}\Big(\tilde{X}(r+\mu)U_\mu(r)\tilde{X}^\dagger(r)\Big) = \int_{SU(2)} \mathrm{d}V_\mu(r) \tag{5.2}$$

Now, the partition function is given by

$$Z = \prod_r \left(\int \mathrm{d}H_1(r)\mathrm{d}H_2(r)\mathrm{d}H_3(r) \prod_\mu \int \mathrm{d}V_\mu(r) \right) e^{-S} \tag{5.3}$$

and the action is a function only of $\rho(r)$ and $V_\mu(r)$. Hence, we can equivalently write the action as

$$Z = \prod_r \left(4\pi \int \rho^2 \mathrm{d}\rho(r) \prod_\mu \int \mathrm{d}V_\mu(r) \right) e^{-S} \tag{5.4}$$

thus reducing the integration variables by 2.

The above proof can be easily modified to include more flavors. In this case, we choose one flavor, say first, and then apply the above transformation. From now, denote the Higgs field by H_{la} where l is the flavor index and a is the color index. After above transformation, the action still depends on $\tilde{X}$. To remove this dependence, we rotate the other flavors by $\tilde{X}$ according to $\tilde{H}_l(r) = \tilde{X}^\dagger(r) Y_l \tilde{X}(r)$. Now, the action is a function only of ρ, V_μ, Y_2, Y_3 and Y_4. Since, the above transformation of H to Y is actually an SO(3) rotation in the adjoint representation, the integral measure of Higgs field changes as

$$\int \mathrm{d}H_{l1} \mathrm{d}H_{l2} \mathrm{d}H_{l3} \to \int \mathrm{d}Y_{l1} \mathrm{d}Y_{l2} \mathrm{d}Y_{l3} \tag{5.5}$$

Hence, we have again reduced the integration variable by 2 by replacing the components of H_1 by its magnitude.

This simplification seems reasonable because the effective action obtained after integrating out the gauge fields is invariant under SU(2) transformation, which allows us to choose an orientation of the coordinate system at each site. The action depends only on the relative orientation of Higgs field, and hence on each site we can choose local coordinate system such that the first flavor of Higgs field always point in the $z-$direction.

Extending the above argument, after fixing the $\vec{H}_1$ field in the $z-$direction, we can rotate our coordinate frame such that the field $\vec{H}_2$ lies in the xy plane. Hence, we can substitute the following representation of the fields $\vec{H}_1$ and $\vec{H}_2$

$$\vec{H}_1 = \rho \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{H}_2 = \begin{bmatrix} \alpha \\ 0 \\ '23 \end{bmatrix} \tag{5.6}$$

while changing the integration measure as follows

$$\int \mathrm{d}^3\vec{H}_1 \int \mathrm{d}^3\vec{H}_2 \int \mathrm{d}^3\vec{H}_3 \int \mathrm{d}^3\vec{H}_4 \to 8\pi^2 \int \rho^2 \mathrm{d}\rho \int \alpha \mathrm{d}\alpha \int \mathrm{d}'23 \int \mathrm{d}^3\vec{H}_3 \int \mathrm{d}^3\vec{H}_4 \tag{5.7}$$

This reduced the Integration variable to 9. However, now we need to cleverly generate the random numbers ρ, α as they appear in the integration measure. This can be done simply by Generating random numbers $H_{11}, H_{12}, H_{13}, H_{11}, H_{12}$ and then using $\rho = \sqrt{H_{11}^2 + H_{12}^2 + H_{13}^2}$ and $\alpha = \sqrt{H_{21}^2 + H_{22}^2}$.

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