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Quantum Measurement at Variable Strength

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Project Report for
Japan-India International Linkage Degree Program

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December 2018

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Chapter 1

Quantum Measurement

1.1 Introduction

Quantum measurement forms the bridge between the quantum theory and our observable reality which makes it one of the most important aspect of quantum theory. In quantum measurement, a quantum system interacts with a macroscopic detector and the interaction leaves the detector in a classical state which provides information about some observable of the quantum system. However, unlike classical measurement, the quantum state is reduced to one of the eigenstate of the measured observable. The above description is formalised as one of the postulate of quantum theory-

When a measurement of an observable $A = \sum a_i |a_i\rangle \langle a_i|$ *is made on a generic state* $|\psi\rangle$*, then the state is reduced to one of the eigenstates* $|a_i\rangle$ *with the probability given by* $| \langle a_i|\psi\rangle |^2$.

Such a measurement is termed as "Projective Measurement". Fortunately, it is possible to have more general measurements of quantum system by indirect method comprising of two steps:

1. *Interaction*: Entangle a meter M with the system S.

2. *Reduction*: Subsequant projective measurement on the meter.

Such a measurement is called "Variable Strength Measurements", as it is equivalent to measurement using a detector whose pointer variable is only partially coupled to the system. It was shown by Von-Neumann[1] that any general measurement can be enacted in such a way. In this report, it will be assumed that the interaction duration is sufficiently small and the reduction is carried out quickly after interaction so that the free evolution hamiltonian of both the system and the meter can be neglected during the measurement process. We now proceed to the description of the measurement process.

Suppose system state lies in a N dimensional hilbert space and our aim is to measure an observable A of the system. The initial state of system $|\psi_0\rangle$ and operator A corresponding to the observable are given by

$$|\psi_0\rangle = \sum_{i=1}^{N} c_i |a_i\rangle \tag{1.1}$$

$$A = \sum_{i=1}^{N} a_i |a_i\rangle \langle a_i| \tag{1.2}$$

The hamiltonian during the interaction is H and the unitary acting on $S+M$ during interaction is given by $U = exp(-itH)$. The combined state of system and meter evolves as

$$|\chi_0\rangle = |\psi_0\rangle_S |\phi_0\rangle_M \rightarrow |\chi(t)\rangle = U |\psi_0\rangle_S |\phi_0\rangle_M \tag{1.3}$$

The interaction between system and meter is intended for measurement of observable A and therefore should not perturb the eigenstates of A. In other words, the Hamiltonian and time evolution operator must satisfy

$$H = \sum |a_i\rangle \langle a_i| R_i \tag{1.4}$$

$$U |a_i\rangle = |a_i\rangle U_i \tag{1.5}$$

Where U_i are operators acting only on meter state. This ensures that if the initial state of the system is one of the eigenstates of A, then there is no change in the system due to interaction. The states after the interaction are given given by

$$|\chi(t)\rangle = \sum_{i=1}^{N} c_i |a_i\rangle U_i |\phi_0\rangle \tag{1.6}$$

$$\rho_S = Tr_M\{|\chi(t)\rangle \langle\chi(t)|\} = \sum_{i,j} c_i c_j^* |a_i\rangle \langle a_j| \langle\phi_0| U_j^\dagger U_i |\phi_0\rangle_M \tag{1.7}$$

$$\rho_M = Tr_S\{|\chi(t)\rangle \langle\chi(t)|\} = \sum_{i} |c_i|^2 U_i |\phi_0\rangle \langle\phi_0| U_j^\dagger \tag{1.8}$$

The expression for $|\chi(t)\rangle$ convey the manifestation of entanglement between system and meter due to the interaction. For obtaining the information about the system, a projective measurement on the meter can be performed in an arbitrary basis $\{|m\rangle_M\}$. The reduction collapses the state of the $S+M$ into product state depending upon the outcome of the meter measurement. The probability of outcome $|m\rangle$ and the system state after obtaining outcome $|m\rangle$ is given by

$$p_M(m) = \sum_i |c_i|^2 |\langle m| U_i |\phi_0\rangle|^2 \tag{1.9}$$

$$|\psi^m\rangle = \frac{1}{\sqrt{p(m)}} \sum_i c_i |a_i\rangle \langle m| U_i |\phi_0\rangle \tag{1.10}$$

In the strong measurement limit, i.e. complete entanglement case, for the correct meter basis we have $|\langle m| U_i |\phi_0\rangle|^2 = \delta_{mi}$ and therefore $p(m) = |c_m|^2$.

As one would expect, the system desity matrix will not be affected by the projective measurement performed on the meter because of causality, but the density matrix of the meter is decohered because of the projective measurement.

It is obvious that the entanglment between system and meter increases during the interaction till the limit of strong measurement is reached. In this regard, we have the following theorem-

Theorem 1.1. *The entropy of entanglement is maximum when the eigenstate of system present in Smidth Decomposition are the eigenstates of the measurement observable A.*

The above theorem implies that the entropy of entanglement is greatest in the strong measurement limit.

Proof. The maximum possible entanglement under this unitary is assumed when $|\chi_{max}\rangle = \sum_i c_i |a_i\rangle |i\rangle$. Let $U_i(t)|\phi_0\rangle = |\tilde{i}\rangle$. Then the set $\{|i\rangle\}$ is orthonormal, while $\{|\tilde{i}\rangle\}$ might not be orthogonal. Since the combined state of system and meter is pure, $S(\rho^S) = S(\rho^M)$ in both cases. We need to show that

$$S(\rho^S_{max}) \geq S(\rho^S(t)) \text{ or equivalently } S(\rho^M_{max}) \geq S(\rho^M(t))$$

Consider an ancilla A such that combined state of $A+M$ is given by

$$\rho^{AM}_{max} = \sum_i |c_i|^2 |i\rangle_A \langle i|_A \otimes |i\rangle_M \langle i|_M$$

$$\rho^{AM}(t) = \sum_i |c_i|^2 |i\rangle_A \langle i|_A \otimes |\tilde{i}\rangle_M \langle\tilde{i}|_M$$

Consider a operation E on ρ^{AM}_{max} given by kraus operators $\hat{M}^j = I_A \otimes |\tilde{i}\rangle \langle i|$. Then,

$$E(\rho^{AM}_{max}) = \rho^{AM}(t)$$

Since then channel acts only on meter, the mutual information can only decrease.

$$S(\rho^A_{max}) + S(\rho^M_{max}) - S(\rho^{AM}_{max}) \geq S(\rho^A(t)) + S(\rho^M(t)) - S(\rho^{AM}(t))$$

But we know that $S(\rho^A_{max}) = S(\rho^A(t)) = S(\rho^{AM}_{max}) = S(\rho^{AM}(t)) = S(\rho^M_{max})$, and therefore,

$$S(\rho^M_{max}) \geq S(\rho^M(t))$$

□

1.2 Joint Measurement

The uncertainity principle is one of the most fundamental principles of quantum mechanics. It states that a quantum state cannot have definite value of two non-commuting observables and there exsits some tradeoff between the uncertainity of those observables. This princicle, as will be shown later, lead to the property of complementarity in quantum measurement which states that it is impossible to precisely measure values of two non-commuting observables of an arbitrary state in a single measurement scheme. This means that any attempt to measure a property of a system will disturb the state and will therefore limit the accuracy of the subsequent measurement on the system.

Fortunately, it is possible to jointly measure two (or more) non-commuting observables by allowing some degree of imprecision in each measurement. One way to implement a joint measurement is to first perform a variable strength measurement of observable A followed by a precise measurement of observable B. In the remainder of this report, "Joint Measurement" will be used to refer to such a measurement. It must be noted that there exist joint measurement schemes different from the one stated above such as "Arthurs-Kelly Joint Measurement Scheme"[2].

The observable B is given by $B = \sum_{i=1}^N b_i |b_i\rangle \langle b_i|$.

The measurement of A disturbs the state which limits the precision of measurement of B. The disturbance caused is dependent of the strength of interaction between system and meter, which also decides the precision of measurement of A. For higher precision of B, the disturbance must be lower which implies that the interaction should be weaker which in turn result in lower precision of A.

The probabilities of obtaining result $|b_i\rangle$ on subsequent projective measurement of B after measurement of A is given by

$$p(m, b_i) = |\langle b_i, m| U |\psi_0, \phi_0\rangle|^2 \tag{1.11}$$

$$p_S(b_i) = \sum_m p(m, b_i) = \sum_m p(m) |\langle b_i|\psi^M\rangle|^2 = ||\langle b_i| U |\psi_0, \phi_0\rangle||^2 \tag{1.12}$$

which is different from the probability $p_S^0(b_i) = |\langle b_i|\psi_0\rangle|^2$ of obtaining $|b_i\rangle$ without measurement of observable A because of disturbance caused by the interaction.

A joint measurement should be such that the probability of outcome $|m\rangle$ of meter is close to probability of obtaining $|a_m\rangle$ in the projective measurement of A and the probability of obtaining outcome $|b_i\rangle$ in the susbequent measurement is close to probabiltiy of obtaining $|b_i\rangle$ in absence of coupling with meter. Therfore, we aim for the following approxiamte statement.

$$p_0(a_m, b_i) \approx p(m, b_i)$$

The probability of outcome (m, b_i) in the joint measurement is same as the probability of projecting the system state $|\psi_0\rangle$ on the (non-normalised) vector $\langle\phi_0| U^\dagger |b_i, m\rangle$. This justifies the earlier assertion that the complimentarity follows from the Uncertianity principle, and tells us the maximum precision of measurement of two non-commuting observables is same as the minimum possible spread in observables A and B.

1.3 Weak Measurement

The weak measurement introduced by Aharanov et al.[3] are a subclass of general measurement in which the interaction between the system and the meter is very weak. In this way, it is possible to obtain some information about the system without causing an appreciable disturbance to the system. In such a measurement, most of the outcomes obtained by projective measurement of the meter do not give much information about the system and do not cause any disturbance. However, some outcomes which occur with very low probability gives information about the system and cause large disturbance.

Let us assume that the interaction hamiltonian is given by $H = A \otimes R$ where R is an hermitian operator acting on meter states. Then in the weak limit, we can write

$$U \approx 1 - itH = 1 - itA \otimes R \tag{1.13}$$

In this case, t acts as the strength parameter. Since $A\left|\alpha\right\rangle = \langle A\rangle\left|\alpha\right\rangle + \Delta A\left|\alpha^{\perp}\right\rangle$, we can write

$$|\chi(t)\rangle = U\,|\chi_0\rangle = |\psi_0,\phi_0\rangle - itAR\,|\psi_0,\phi_0\rangle = \\ |\psi_0,\phi_0\rangle - it(\langle A\rangle\,|\psi_0\rangle + \Delta A\left|\psi^{\perp}\right\rangle)(\langle R\rangle\,|\phi_0\rangle + \Delta R\left|\phi^{\perp}\right\rangle) \quad (1.14)$$

where $\left|\psi^{\perp}\right\rangle$ and $\left|\phi^{\perp}\right\rangle$ are orthogonal to $|\psi_0\rangle$ and $|\phi_0\rangle$ respectively. If the initial state is chosen such that $\langle R\rangle = 0$, we can write $\chi(t)$ in the smidth form by defining $|\phi\rangle = (1 - it\langle A\rangle\Delta R\,|\phi_0\rangle$

$$|\chi(t)\rangle = |\psi_0,\phi\rangle - it\Delta A\Delta R\left|\psi^{\perp},\phi^{\perp}\right\rangle \quad (1.15)$$

which shows that the state is perturbed by a small factor. It must be noted that $\left\langle \phi_0 \middle| \phi^{\perp}\right\rangle$ is of order t^2.

Let $p^0_M(m)$ and $p^0_S(b_i)$ denote the probabilities of obtaining outcome $|m\rangle$ and $|b_i\rangle$ respectively on measurement before the interaction. Then after interaction, the probabilities are given by (upto first order in t)-

$$p^M(m) = p^0_M(m)(1 + 2t\langle A\rangle\,\mathrm{Im}\{R_w\}) \quad (1.16)$$

$$p^S(b_i) = p^0_S(b_i)(1 + 2t\langle R\rangle\,\mathrm{Im}\{A_w\}) \quad (1.17)$$

$$p(b_i|m) = p^0_S(b_i)(1 + 2t\langle R\rangle\,\mathrm{Im}\{A_w\} - 2t\,\mathrm{Im}\{A_w R_w\}) \quad (1.18)$$

Where $R_w = \frac{\langle m|R|\phi_0\rangle}{\langle m|\phi_0\rangle}$ and $A_w = \frac{\langle b_i|A|\psi_0\rangle}{\langle b_i|\psi_0\rangle}$ are the weak values.

The above probabilities are symmetric due to obvious symmetry in the measurement scheme. If the initial meter state is chosen such that $\langle R\rangle = 0$, then the probability of measurement of B on the system (before reduction of meter) is not affected even though information about $\langle A\rangle$ is transferred to the meter. However, the bayesian update on system caused by reduction of meter can still have considerable effect on system.

From equation (1.16), we can see that the imaginary part of the weak value R_w causes the shift in pointer position. When $\langle R\rangle = 0$, the conditional probability of the system is affected because of the terms $\mathrm{Re}\{A_w\}\,\mathrm{Im}\{R_w\}$ and $\mathrm{Im}\{A_w\}\,\mathrm{Re}\{R_w\}$. The first term denotes the change in the pointer caused by the meter, and the second term contains the change in the system caused because of the pointer. This implies that during the interaction, both system and pointer applies "force" on each other and change each other's properties.

Chapter 2

Measurement Interaction

As stated earlier, the interaction between system and meter causes transfer of information about observable A of system to meter and the disturbance caused by meter on the system. The focus of this chapter would be the entanglement caused by this interaction. The term "back action" refers to the change of the system caused due to this interaction.

$$\text{back action: } \rho_S^0 = |\psi_0\rangle \langle\psi_0| \rightarrow \rho_S = Tr_M\{U |\psi_0, \phi_0\rangle \langle\psi_0, \phi_0| U^\dagger\}$$

2.1 Measurement Operator

Althrough the readout process do not change the density matrix, it updates our knowledge about the system. From equation (1.9) it is easy to figure out the measurement opertors which are obviously diagonal in $\{|a_i\rangle\}$ basis.

$$\hat{M}^m = \langle m| U |\phi_0\rangle \tag{2.1}$$

$$\hat{M}^m_{ik} = \delta_{ik} \langle m| U_i |\phi_0\rangle \tag{2.2}$$

According to the polar decomposition, each $\hat{M}^m$ can be expressed as a product of a unitary and a positive semi-definite operator given by $\hat{M}^m = V^m P^m$ where the elements of V and P are given by

$$P^m_{ik} = \delta_{ik} | \langle m| U_i |\phi_0\rangle | \tag{2.3}$$

$$V^m_{ik} = \delta_{ik} \frac{\langle m| U_i |\phi_0\rangle}{| \langle m| U_i |\phi_0\rangle |} \tag{2.4}$$

The operator P^m results in change of probability of outcome m depending on the state $|\psi_0\rangle$ of the system while V^m is just a unitary conditioned on the measurement output.

2.2 Dephasing Channel

The quantum measurement is a quantum channel which is specified by the initial meter state and the interaction hamiltonian. Also, since the measurement process does not perturb the eigenstates of A, it is same as dephasing channel between $\{|a_i\rangle\}$ basis, i.e. it decreases the coherence between eigenstates of A. This can be seen from the structure of $\rho_S(t)$

$$\rho_{ij}^S = c_i c_j^* \langle\phi_0| U_j^\dagger U_i |\phi_0\rangle \tag{2.5}$$

The decoherence is caused by the time dependent term $\langle\phi_0| U_j^\dagger U_i |\phi_0\rangle$ in the density matrix. This term has two parts - The absolute value denotes the coherence between eigenstates of A while the phase denotes results only in change of phase of each state among $\{|a_i\rangle\}$.

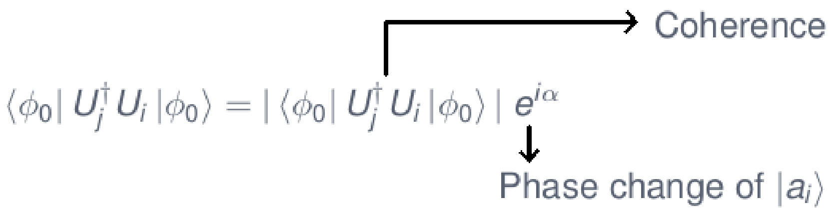

FIGURE 2.1: Change in ρ_{ij}^S

At $t = 0$ all $U_i = I$ and hence there is no dephasing while the coherence between any two distinct states $|a_i\rangle$ and $|a_j\rangle$ is completely lost in the limit of strong measurement. Therefore, we can define the dephasing between states $|a_i\rangle$ and $|a_j\rangle$ caused due to interaction by

$$D(i,j) = 1 - |\langle\phi_0| U_j^\dagger U_i |\phi_0\rangle| \tag{2.6}$$

2.3 Resolution

Resoultion refer to the ability of a measurement to distinguish between two states $|a_i\rangle$ and $|a_j\rangle$ on the basis of the measurement outcome. Different initial states results in different probability distributions in the outcome. Hellinger distance is a robust distance measure between two probability distributions in statistics and can be easily extended for quantum measurement. Therefore, the square of the Hellinger distance will be used to quantify the resolution between states $|a_i\rangle$ and $|a_j\rangle$ according to

$$R(i,j) = H^2(\vec{p}_M(a_i), \vec{p}_M(a_j)) = \frac{1}{2}\sum_m(\sqrt{p(m|a_i)} - \sqrt{p(m|a_j)})^2 \tag{2.7}$$

$$R(i,j) = 1 - \sum_m \sqrt{p(m|a_i)p(m|a_j)} = 1 - \sum_m |\langle\phi_0| U_i^\dagger |m\rangle \langle m| U_j |\phi_0\rangle| \tag{2.8}$$

where the second relation is obtained by utilising the relation between the Hellinger distance and the Bhattacharya coefficient.

It is not surprising that the resolution depends on the readout basis. After the entanglement between the meter and the system, the choice of the readout basis determines the amount of knowledge obtained about the system. The back action caused by the interaction can be partially reversed by applying unitaries on the system conditioned on measurement outcome m, and the irreversible part provides us the information about the system.

The division of back action between irreversible and reversible parts can be seen through the Measurement operator $\vec{M}^m = V^m P^m$. Since P^m is responsible for transfer of information about system to meter, the change caused due to P^m is irreversible while the change caused due to V^m can easily be reversed. By removing the reversible part, we get

$$\rho^S_{irr} = \sum_m (V^m)^\dagger \hat{M}^m \rho^S_0 (\hat{M}^m)^\dagger V^m = \sum_{i,j} c_i c_j^* |a_i\rangle |a_j\rangle (\sum_m |\langle\phi_0| U_i^\dagger |m\rangle \langle m| U_j |\phi_0\rangle|) \tag{2.9}$$

$$\rho^S_{irr} = \sum_{i,j} c_i c_j^* |a_i\rangle |a_j\rangle (1 - R(i,j)) \tag{2.10}$$

While the ρ^S obtained after the interaction was

$$\rho^S = \sum_{i,j} c_i c_j^* |a_i\rangle |a_j\rangle (1 - D(i,j))e^{i\alpha_{ij}} \tag{2.11}$$

where α_{ij} are constants depending on the interaction and $|\phi_0\rangle$.

Equation (2.10) and (2.11) clearly show that the total back action is caused due to the decoherence out of which the irreversible part is a result of the resolution of the readout. As intuition suggests, the following inequality is true-

$$R(i,j) \leq D(i,j) \tag{2.12}$$

And for a given interaction, the best resoultion is obtained by choosing a basis where the terms $\langle m| U_i |\phi_0\rangle$ with different m have the same phase so that the unitary conditioned by measurement evolution is given by a global phase factor.

The significance of the choice of readout basis can be seen by taking two cases is the strong measurement limit, i.e. $|\chi_f\rangle = \sum c_i |a_i\rangle_S |i\rangle_M$ or equivalently $D(i,j) = 1 - \delta_{ij}$. The measurement in the basis $\{|i\rangle\}$ gives complete information about the system and have $R(i,j) = D(i,j)$ while the measurement in the mutually-uncorrelated basis $\{|\tilde{i}\rangle\}$ with $\left|\tilde{k}\right\rangle = \sum_j e^{2i\pi jk/N} |j\rangle$ gives no information about the system and hence $R(i,j) = 0$. In the latter case, the state of the system is changed by a unitary conditioned on measurement outcome, and therefore the state change is completely reversible if the measurement outcome is known.

2.4 Disturbance-Resolution Inequality

The back-action due to interaction causes the change in system $|\psi_0\rangle\langle\psi_0| \to \rho^S$. To quantify this disturbance the quantity $1 - F^2$ can be used, where F is the statistical fidelity between states ρ_0^S and ρ^S given by $F = \sqrt{\langle\psi_0|\rho^S|\psi_0\rangle}$. Of course, the back action depends on the spread of initial state in the eigenbasis of A since very narrow spread will not suffer much disturbance. Therefore it makes sense to look at the following quantity

$$\frac{\text{Back Action}}{\text{Spread}} = \frac{1 - F^2}{1 - ||\vec{p_S}||^2} = \frac{\sum_{i\neq j} |c_i|^2 |c_j|^2 (1 - \mathrm{Re}\left\{\langle\phi_0| U_j^\dagger U_i |\phi_0\rangle\right\})}{\sum_{i\neq j} |c_i|^2 |c_j|^2} \tag{2.13}$$

which is the weighted sum of the distance between conditional states of the meter, i.e. $0.5||U_i|\phi_0\rangle - U_i|\phi_0\rangle||^2$. The vector $\vec{p_S} = (|c_0|^2, |c_1|^2, ...|c_{N-1}|^2)$ is the probability vector of the system in basis $|a_i\rangle$. Using the inequality $R(i,j) \leq 0.5||U_i|\phi_0\rangle - U_i|\phi_0\rangle||^2$, we get

$$\min_{i\neq j} R(i,j) \leq \frac{1 - F^2}{1 - ||p_S||^2} \leq \max_{i\neq j} \frac{||U_i|\phi_0\rangle - U_j|\phi_0\rangle||^2}{2} \tag{2.14}$$

The inequality (2.14) says that the back action caused by the interaction is always greater than the minimum resolution in optimum basis times the spread of the initial state. For the case of strong measurement, we have $LHS = 1$ which forces $F = ||p_S||$, while for the weak measurement, $RHS = O(t^2)$ which forces $1 - F^2 = O(t^2)$ (given $||p_S|| < 1$), i.e. there is almost no back action on the system.

Chapter 3

Isotropic Measurement Model

In this chapter, we will consider a simple measurement model which contains all the important features which were discussed in the previous chapter. The model is isotropic in the sense that all states $|a_i\rangle$ are on equal footing and all the states $|a_i\rangle$ $(i \neq j)$ "behave" similarly with respect to $|a_j\rangle$.

Consider the system with hilbert space of dimension N and the meter with hilbert space of $N+1$ dimensions. We aim to find the distribution of orthonormal basis $\{|a_1\rangle, \{|a_2\rangle, ...\{|a_N\rangle\}$ by measuring meter in basis $\{|0\rangle, |1\rangle, |2\rangle, ... |N\rangle\}$. The operators L_i $(i = 1, 2, 3...N)$ are generators of transition between states $|0\rangle_M$ and $|i\rangle_M$ with their action given by $e^{\alpha L_i} |0\rangle = \cos\alpha |0\rangle + \sin\alpha |i\rangle$. The interaction hamiltonian of the system is given by

$$H = \sum_{i=1}^{N} \omega |a_i\rangle \langle a_i| \otimes L_i \tag{3.1}$$

and the initial meter state is $|0\rangle_M$. The combined state of system and meter at minimum and maximum entanglement are given by $|\chi_0\rangle = \sum_i c_i |a_i\rangle |0\rangle_M$ and $|\chi_f\rangle = \sum_i c_i |a_i\rangle |i\rangle_M$. Using these conventions, the combined state at an arbitrary time t is given by

$$|\chi(t)\rangle = \cos\omega t |\chi_0\rangle - \sin\omega t |\chi_f\rangle \tag{3.2}$$

For this model, the results of readout in basis $\{|0\rangle, |1\rangle, |2\rangle, ... |N\rangle\}$ is tabulated in table (3.1).

Output	Probability	$\|\psi\rangle$ after readout
$\|0\rangle_M$	$\cos^2 \omega t$	$\|\psi_0\rangle$
$\|i\rangle_M$	$\|c_i\|^2 \sin^2 \omega t$	$\|a_i\rangle$

Table 3.1: Result of Readout

From the table, it can be seen that this measurement is equivalent to performing a projective measurement with probability $\sin^2(wt)$ and can be called a stochastic measurement.

In the limit of weak measurement, i.e. $t << 1$, the outcome $|i\rangle$ $(i = 1, 2, 3...N)$ are rare, but change the state completely and gives information about the system. On the other hand, the outcome $|0\rangle$ appears with high probability and do not cause any change in the state of the system but do not give any information about the system.

As expected, the entropy of entanglement increases till time $t = \pi/(2\omega)$

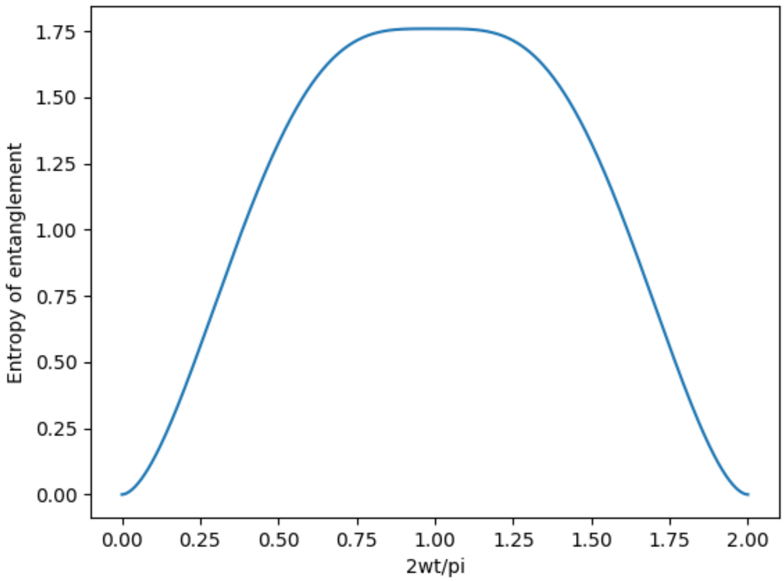

Figure 3.1: Entropy of entanglement.

All the off-diagonal entries of the density matrix of the system are reduced by a factor of $\sec^2(wt)$ which means that all pairs of states decohere at equal rate. The resolution $R(i,j)$ has the clear meaning as the seperation of the states $U_i |\phi_0\rangle$ and $U_j |\phi_0\rangle$. Since the seperation between any two states is the same, we have the equality condition in (2.14)

$$\min_{i \neq j} R(i,j) \leq \frac{1 - F^2}{1 - ||p_S||^2} \leq \max_{i \neq j} \frac{||U_i |\phi_0\rangle - U_j |\phi_0\rangle ||^2}{2} = \sin^2(\omega t) \tag{3.3}$$

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