YALE UNIVERSITY

Circuit QED

by

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Contents

Chapter 1

Black Body Quantization

Suppose we are given a non-dissipative linear circuit, and access to one port(two distinct points on the circuit). By measuring the impedance(or admittance) at this port, we can characterize the circuit's quantum behaviour. We can determine the eigenmodes and eigenfrequencies of the given circuit, along with the quantum hamiltonian. According to foster theorem, any circuit consisting of only capacitors and inductors can be reduced to the foster's first form, as shown in the figure below.

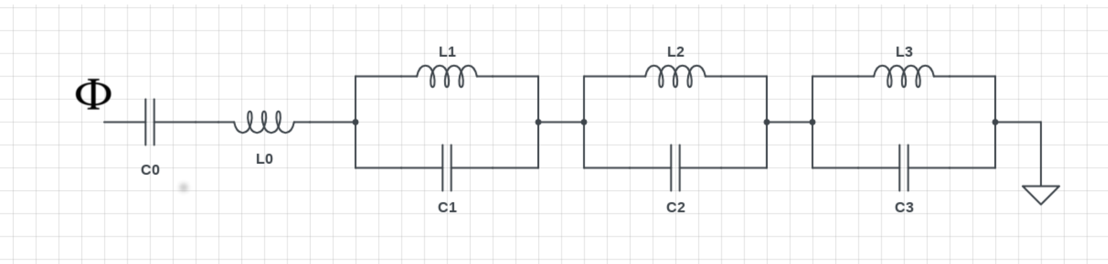

FIGURE 1.1: Foster first form

In the foster first form, the pole in Impedance(or zero in admittance) correspond to frequencis of the oscillators, and the pole at zero(if present) correspond to C_0, while the pole at infinity correspond to L_0. Due to the fact that Impedance is an odd function of frequency, and is always increasing(foster reactance theorem), there is either a pole or zero at $w = 0$ and at $w = \infty$.

1.1 Isolated Circuit

If the above circuit is isolated, we can ignore the inductance as no current flows through it and hence the flux across it is zero, and the capacitor has constant charge, and can be dealt easily as it is a free particle state. Because of no current, the harmonic oscillators are decoupled, and hence each can be solved independently.

1.2 Non-Isolated Circuit

The main power of the Black body quantisation lies in its ability to predict the quantum hamiltonian when another element is connected to the open port. In this case, the oscillators are not decoupled, and we need to diagonalise the obtained hamiltonian. If we add an element whose energy depends on $\dot{\phi}$ (ex:capacitor), then the definition of Q as obtained from Lagrangian will change, resulting in completely new operators and hence new commutation relationships. Hence, we will only consider addition of a potential term to the lagrangian, i.e the energy of the new element is $V(\phi)$ and has no dependence on $\dot{\phi}$. Let us consider 3 cases.

1.2.1 Case 1: $L_0 = 0, C_0 = 0$

This is the simplest case, as our new Hamiltonian is

$$H = H_0 + V(\phi)$$

where

$$H_0 = \sum_{p=1}^{n} \hbar w_p (a_p^\dagger a_p + 1/2)$$

and

$$\phi = \sum_{p=1}^{n} \sqrt{\frac{\hbar}{2w_p C_p}} (a_p + a_p^\dagger)$$

Although all modes are coupled, but the operators of different modes still commute, and hence the hamiltonian can be diagonalised by usual methods.

1.2.2 Case 2: $L_0 = 0, C_0 \neq 0$

This case is the same as the previous one, except for the presence of the capacitance term in the hamiltonian, and the corresponding modification of total flux.
Also, we can take the term of order ϕ^2 into the Black box, and then are appropriate rearrangement, we will get a modified foster form with $L_0' = 0, C_0' = 0$, where prime denotes values in modified black box. The number of oscillators will get incremented by one.

1.2.3 Case 3: $L_0 \neq 0, C_0 = 0$

This is the most interesting case, as the inductance and the new element are both dependent only on ϕ, i.e they are potential terms in the lagrangian. The current conservation results in a holonomic constraints, as we shall see below.

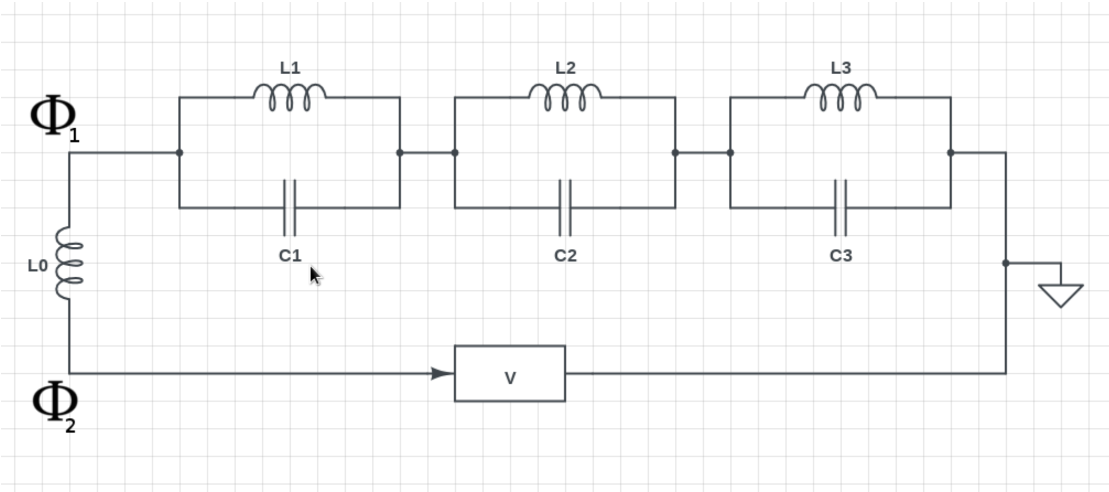

FIGURE 1.2: Circuit

Because of current conservation, we have

$$\frac{\phi_1 - \phi_2}{L_0} = \frac{\partial V(\phi_2)}{\partial \phi_2}$$

. Usually, we will be able to invert this relation to obtain $\phi_2 = f(\phi_1)$.

Now, we can use this relation before finding the quantum hamiltonian to eliminate ϕ_2 from the lagrangian, or we can use it after finding the Hamiltonian in terms of ϕ_1, ϕ_2 and see that it will become necessary to eliminate ϕ_2. Both of this is equivalent to putting the inductor L_0 outside the black box, and into the Potential term.

Eliminating in lagrangian:
By using $\phi_2 = f(\phi_1)$, we get

$$\mathcal{L} = \mathcal{L}_0 - \frac{(f(\phi_1) - \phi_1)^2}{2L_0} - V(f(\phi_1))$$

and we can define $V'(\phi_1)$ to get

$$\mathcal{L} = \mathcal{L}_0 - V'(\phi_1)$$

or equivalently

$$H = H_0 + V'(\phi_1)$$

and we are done.

Eliminating in Hamiltonian

We can write

$$H = H_0 + \frac{(\phi_2 - \phi_1)^2}{2L_0} + V(\phi_2)$$

Treating ϕ_1 and ϕ_2 independently, we get the two hamiltonaian equations:
$\dot{\phi}_2 = \frac{i}{\hbar}[H, \phi_2] = 0$ and $\dot{Q}_2 = \frac{i}{\hbar}[H, Q_2] = \frac{\phi_1 - \phi_2}{L_0} - \frac{\partial V(\phi_2)}{\partial \phi_2}$
Using the knowledge that Q_2 is zero from the lagrangian, we get the equation for current conservation.

Since ϕ_1 and ϕ_2 are not independent, we cannot choose a state with arbitrary ϕ_1 and ϕ_2. Hence it is useless to have both ϕ_1 and ϕ_2 in the hamiltonian, and we again end up putting L_0 into the potential.

Chapter 2

Equivalent Circuit Problem

In the black body quantisation, we use equivalent foster form of the given(often unimportant) circuit. In doing this, it it implicitly assumed that the equivalent circuits have the same hamiltonian. Here, two circuits are considered equivalent if they have same impedance function at the given port. As we shall see, the equivalent circuits can have very different hamiltoanian, and hence different energies.

Let us start with a simple circuit, as shown in the figure below.

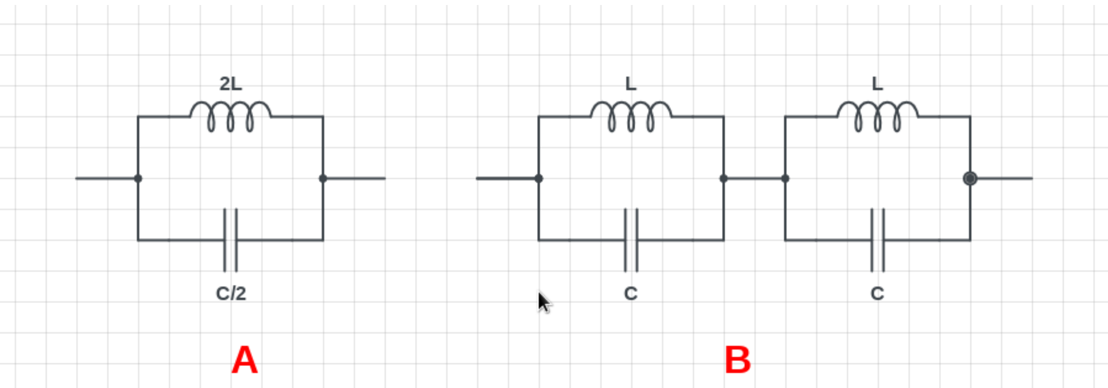

FIGURE 2.1: Two simple equaivalent circuits.

The hamiltonains are

$$H_A = \hbar w(a^\dagger a + 1/2)$$

and

$$H_B = \hbar w(x^\dagger x + 1/2) + \hbar w(y^\dagger y + 1/2)$$

and we see that although both circuits are equivalent, the zero-point energy of B is twice of A, implying that they are not exactly the same.

It will be easier to understand what is actually happening if we look at the circuit C.

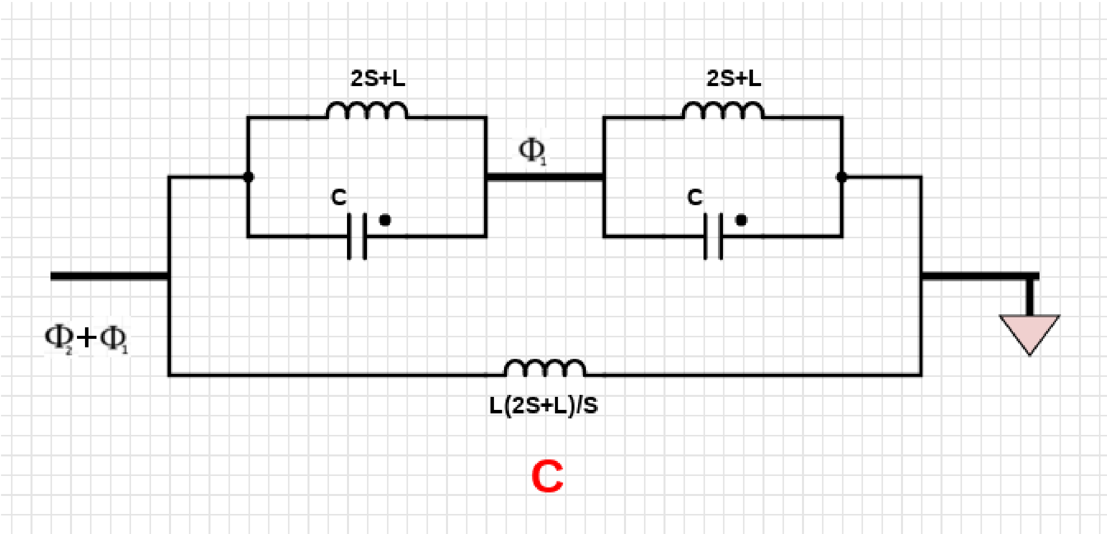

FIGURE 2.2: Another Equivalent Circuit

We can see that

$$Z_A = Z_B = Z_C = \frac{2j}{\frac{1}{wL} - wC}$$

In circuit C, if we assign ground to rightmost wire, ϕ_1 to the middle wire and $\phi_1 + \phi_2$ to the leftmost wire, we get

$$H_C = \frac{Q_1^2}{2C} + \frac{\phi_1^2}{2L}\left(\frac{S+L}{2S+L}\right) + \frac{Q_2^2}{2C} + \frac{\phi_2^2}{2L}\left(\frac{S+L}{2S+L}\right) + \frac{S\phi_1\phi_2}{L(2S+L)}$$

using the substitution $x_1 = (\phi_1 + \phi_2)/\sqrt{2}$, $x_2 = (\phi_1 - \phi_2)/\sqrt{2}$ and $p_1 = (Q_1 + Q_2)/\sqrt{2}$, $x_2 = (Q_1 - Q_2)/\sqrt{2}$, we get

$$H_C = \frac{p_1^2}{2C} + \frac{x_1^2}{2L} + \frac{p_2^2}{2C} + \frac{x_2^2}{2(2S+L)}$$

which tells us that $w' = 1/\sqrt{C(2S+L)}$ is also an eigenfrequency, which is invisible in the impedance.

It is obvious why this eigenfrequncy is invisible at output port, as at this frequency, the two oscillators cancel each other's effect, resulting in no flux difference across the outer inductor.

Hence, the black body quantisation fails when the circuit consists of two resonators capable of cancelling each other's effect at the output ports.

When measuring classical impedance of circuit B, we can only see the combined oscillations of two resonators at the output port, and the information of each resonator remains hidden from us. Therefore, we arrive at the circuit A by measurement of classical impedance of circuit B or circuit C.

We can look at another example showing similar phenomena.

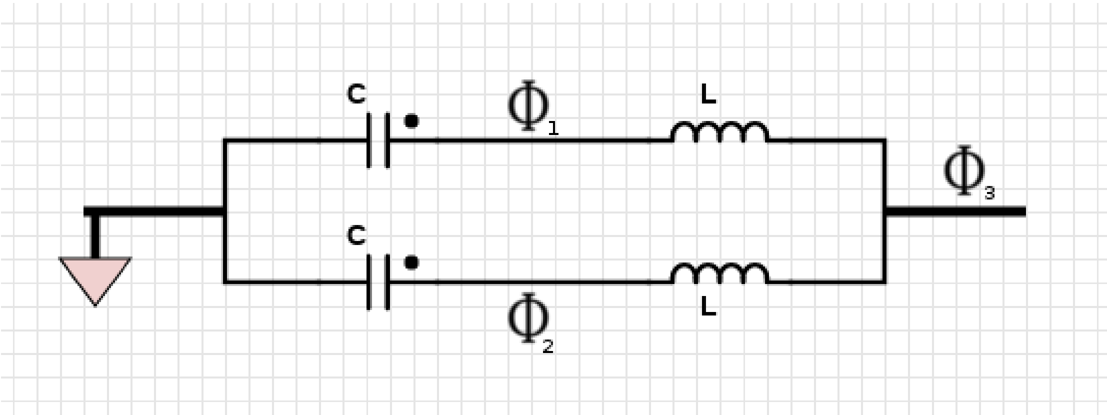

FIGURE 2.3: Another Peculiar Example

The Hamiltonian of the above circuit is

$$H = \frac{Q_1^2}{2C} + \frac{Q_2^2}{2C} + \frac{(\phi_1 - \phi_2)^2}{4L} = \hbar w(a^\dagger a + 1/2) + \frac{p^2}{2C}$$

where $w = 1/\sqrt{LC}$ and $p = (Q_1 + Q_2)/\sqrt{2}$

The hamiltonian tells us that the circuit has oscillation frequency w, while the equivalent circuit is simply a capacitor $2C$ and an inductor $L/2$ in in series. This is because the oscillation results in zero phase difference across the port.

Chapter 3

JJ-L Problem

Let us look at the following element-

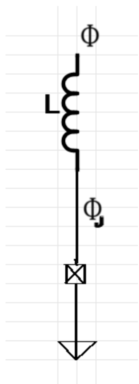

FIGURE 3.1: JJ and L in series

We have $V = \frac{(\phi - \phi_J)^2}{2L} - E_J Cos\varphi$ where $\varphi = \frac{\phi_J}{\phi_0}$ and ϕ_0 is the reduced flux quantum. We need to express $\phi_J = f(\phi)$ to obtaian V in terms of ϕ.
The current conservation equation is

$$\frac{\phi}{\phi_0} = \varphi + \lambda Sin\varphi$$

where $\lambda = \frac{LI_c}{\phi_0} = \frac{L}{L_J}$. also

$$V = -E_J Cos\varphi + \frac{\lambda}{2} E_J Sin\varphi$$

We can easily prove that $\lambda \leq 1$ implies that there exist unique ϕ for every ϕ_J Unfortunately, a solution in close form does not exist.

3.1 Small flux Approximation

From the equation of current conservation, we can see that $\phi \to 0 \implies \varphi \to 0$. Hence taking the approximation of small ϕ, we can write

$$\varphi = \frac{\phi}{\phi_0(\lambda+1)} + \frac{\lambda\phi^3}{6phi_0^3(\lambda+1)^4} + \mathcal{O}(\phi^5)$$

Using this relation, we can write

$$V = E_J\left(-1 + \frac{\phi^2}{2\phi_0^2(\lambda+1)} - \frac{\phi^4}{24\phi_0^4(\lambda+1)^4} + \mathcal{O}(\phi^6)\right)$$

In the above expression, the second term can re-written as $\frac{\phi^2}{2(L+L_J)}$ and hence we can represent the whole element as a parallel combination of inductor $L + L_J$ and a non linear device. From here, the rest of the theory is straightforward as done here.

3.2 Small flux Variation on a constant DC flux

Let us suppose that our element is driven by a superposition of a large DC flux ϕ_{DC} along with a small oscillating AC flux ϕ_s. Also, let $\varphi_{DC} = \varphi(\phi_{DC})$ which we need to find numerically.

Using the fourier expansion, we can write $\varphi(\phi_{DC} + \phi_s)$ as function of φ_{DC} and ϕ_s

Define

$$V_0 = -E_J Cos\varphi_{DC} + \frac{\lambda}{2} E_J Sin\varphi_{DC}$$

We can easily find that

$$V = V_0 + E_J\left(\frac{\phi_s Sin\varphi_{DC}}{\phi_0} + \frac{\phi_s^2 Cos\varphi_{DC}}{2\phi_0^2(1+\lambda Cos\varphi_{DC})} + \frac{\phi_s^3(-Sin\varphi_{DC})}{6\phi_0^3(1+\lambda Cos\varphi_{DC})^3} + \mathcal{O}(\phi^4)\right)$$

3.3 Junction Capacitance

If we consider the junction capacitance, which can be visualised as a capacitor C_J parallel to the Josephson Junction, then our work become much easier. We can then put the Inductor inside the black box, with the capacitor parallel to the complete circuit. Then we can include the capacitor in the black box, to get a modified black box. In this modified black box, Z has a zero at both $w = 0$ and $w = \infty$, and hence this black box can be reduced to the foster first form with $L_0 = 0$ and $C_0 = 0$, which can be treated easily, as done here.

3.4 Fourier Series

We see that

$$\varphi(\phi + \phi_0) = \varphi(\phi) + 2\pi$$

This leads to some interesting results. From the equation, we note that $\varphi(\phi) - \frac{\phi}{\phi_0}$ is periodic in ϕ

Using $\varphi(\phi = \pi\phi_0) = \pi$ and the identity

$$(-1)^v \pi J_v(z) = \int_0^\pi d\varphi Cos(v\varphi + zSin(\varphi))$$

we obtain the following results($\lambda \leq 1$ is assumed while integration):

$$\varphi = \frac{\phi}{\phi_0} + \sum_{k=1}^{\infty} \frac{-2(-1)^k}{k} J_k(k\lambda) Sin(k\frac{\phi}{\phi_0})$$

$$Cos\varphi = \lambda/2 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} J'_k(k\lambda) Cos(k\frac{\phi}{\phi_0})$$

$$Sin^2\varphi = 1/2 + \sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{k\lambda} J'_k(k\lambda) Cos(k\frac{\phi}{\phi_0}) + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2\lambda^2} J_k(k\lambda) Cos(k\frac{\phi}{\phi_0})$$

Combining the above terms, we get

$$V = E_J\left(-\lambda/4 + \sum_{k=1}^{\infty} \frac{2(-1)^k}{k^2\lambda} J_k(k\lambda) Cos(k\frac{\phi}{\phi_0})\right)$$

It is obvious that V is even in ϕ as well as periodic with period ϕ_0.

However, as stated here, the function $J_v(x)$ decreases as $v^{-1/3}$. Hence the coefficients of φ goes like $k^{-4/3}$ while that of V goes like $k^{-7/3}$.

The absolute value of first 10 coefficients of fourier series of V for $\lambda = 0.3$ and $\lambda = 0.8$ is shown below, suggesting that the first two terms provide a good approximation to the function V.

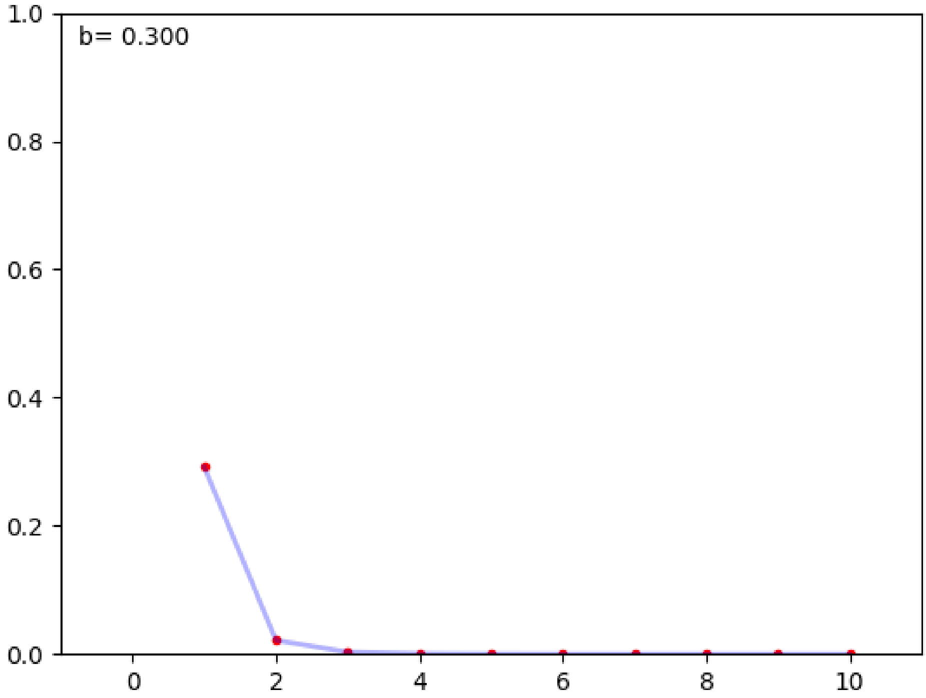

Figure 3.2: Absolute Value of Fourier Coefficients of V for lambda=0.3

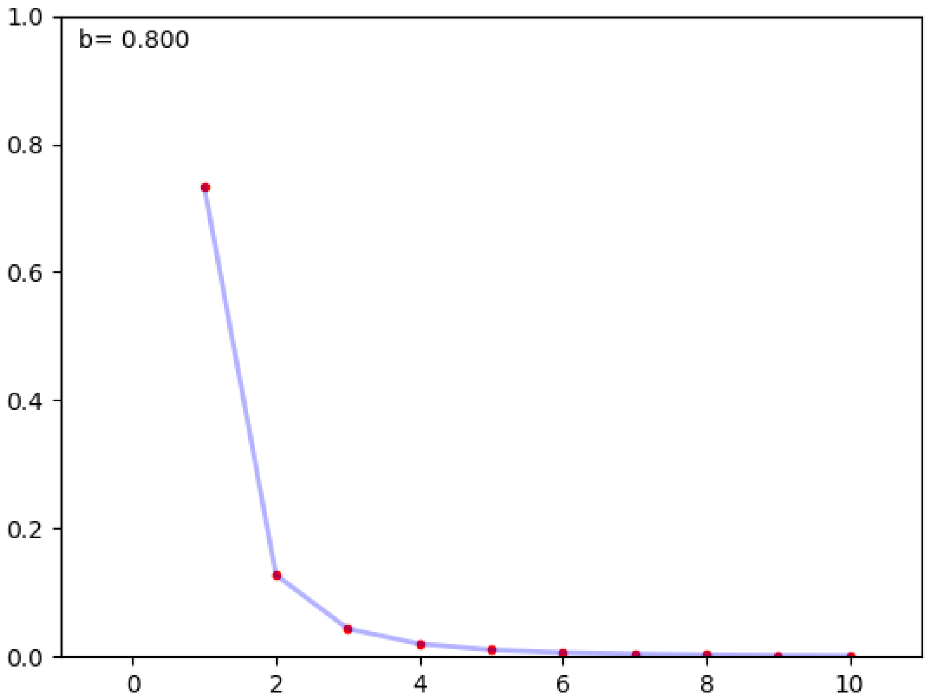

Figure 3.3: Absolute Value of Fourier Coefficients of V for lambda=0.8