YALE UNIVERSITY

Circuit QED

by

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Project Report of Summer Internship

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Contents

Chapter 1

Black Box Quantization

Suppose we are given a non-dissipative linear circuit and access to one port(two distinct points on the circuit). By measuring the impedance(or admittance) at this port, we can characterize the circuit's quantum behaviour. We can determine the eigenmodes and eigenfrequencies of the given circuit, along with the quantum Hamiltonian. According to Foster theorem, any circuit consisting of only capacitors and inductors can be reduced to Foster's first form, as shown in the figure below.

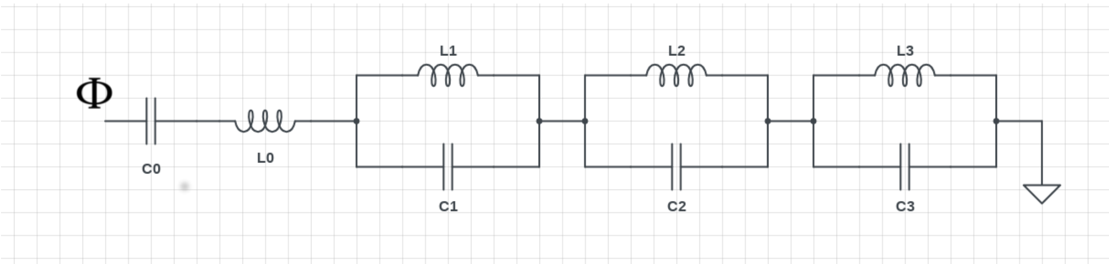


FIGURE 1.1: Foster first form

In the Foster first form, a pole in impedance (or zero in admittance) corresponds to a resonant frequency of the circuit, and the pole at zero (if present) corresponds to C_0, while the pole at infinity corresponds to L_0. Due to the fact that the impedance is an odd function of frequency, and is always increasing(Foster reactance theorem), there is either a pole or zero at $w = 0$ and at $w = \infty$.

1.1 Isolated Circuit

If the above circuit is isolated, we can ignore the inductance as no current flows through it and hence the flux across it is zero, and the capacitor has constant charge and can be dealt easily as it is a free particle state. Because of no current, the harmonic oscillators are decoupled, and hence each can be solved independently.

1.2 Non-Isolated Circuit

The main power of the Black body quantisation lies in its ability to predict the quantum Hamiltonian when another element is connected to the open port. In this case, the oscillators are not decoupled, and we need to diagonalise the obtained hamiltonian. If we add an element whose energy depends on $\dot{\phi}$ (ex: capacitor), then the definition of Q as obtained from Lagrangian will change, resulting in completely new operators and hence new commutation relationships. Hence, we will only consider the addition of a potential term to the Lagrangian, i.e the energy of the new element is $V(\phi)$ and has no dependence on $\dot{\phi}$. Let us consider 3 cases.

1.2.1 Case 1: $L_0 = 0, C_0 = 0$

This is the simplest case, as our new Hamiltonian is

$$H = H_0 + V(\phi)$$

where

$$H_0 = \sum_{p=1}^{n} \hbar w_p (a_p^\dagger a_p + 1/2)$$

and

$$\phi = \sum_{p=1}^{n} \sqrt{\frac{\hbar}{2 w_p C_p}} (a_p + a_p^\dagger)$$

Although all modes are coupled, the operators of different modes still commute, and hence the Hamiltonian can be diagonalised by usual methods.

1.2.2 Case 2: $L_0 = 0, C_0 \neq 0$

This case is the same as the previous one, except for the presence of the capacitance term in the Hamiltonian, and the corresponding modification of the total flux.
Also, we can take the term of order $\dot{\phi}^2$ into the Black box, and then after appropriate rearrangement, we will get a modified Foster form with $L_0' = 0, C_0' = 0$, where prime denotes values in the modified black box. The number of oscillators will get incremented

by one in some sense. The new oscillator has zero frequency, and can be seen as a free particle.

1.2.3 Case 3: $L_0 \neq 0, C_0 = 0$

This is the most interesting case, as the inductance and the new element $V(\phi)$ are both dependent only on ϕ, i.e they are potential terms in the Lagrangian. The current conservation results in a holonomic constraint as we shall see below.

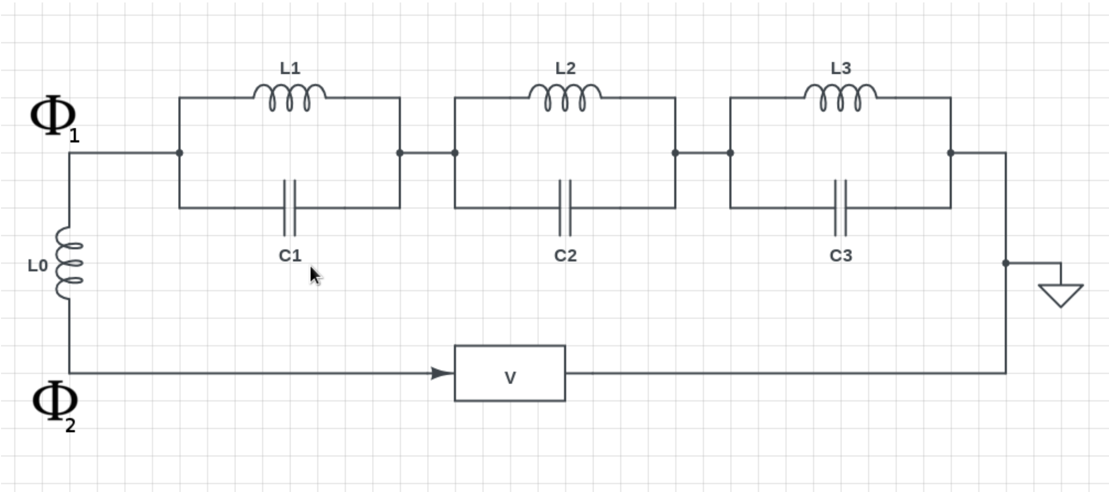


FIGURE 1.2: Circuit

Because of current conservation, we have

$$\frac{\phi_1 - \phi_2}{L_0} = \frac{\partial V(\phi_2)}{\partial \phi_2}$$

. Usually, we will be able to invert this relation to obtain $\phi_2 = f(\phi_1)$.

Now, we can use this relation before finding the quantum hamiltonian to eliminate ϕ_2 from the lagrangian, or we can use it after finding the Hamiltonian in terms of ϕ_1, ϕ_2 and see that it will become necessary to eliminate ϕ_2. Both of these methods are equivalent to putting the inductor L_0 outside the black box, and into the Potential term.

Eliminating in the lagrangian:
By using $\phi_2 = f(\phi_1)$, we get

$$\mathcal{L} = \mathcal{L}_0 - \frac{(f(\phi_1) - \phi_1)^2}{2L_0} - V(f(\phi_1))$$

and we can define $V'(\phi_1)$ to get

$$\mathcal{L} = \mathcal{L}_0 - V'(\phi_1)$$

or equivalently

$$H = H_0 + V'(\phi_1)$$

and we are done.

Eliminating in the Hamiltonian

We can write

$$H = H_0 + \frac{(\phi_2 - \phi_1)^2}{2L_0} + V(\phi_2)$$

Treating ϕ_1 and ϕ_2 as independent variables, we obtain the two hamilton's equations: $\dot{\phi}_2 = \frac{i}{\hbar}[H, \phi_2] = 0$ and $\dot{Q}_2 = \frac{i}{\hbar}[H, Q_2] = \frac{\phi_1 - \phi_2}{L_0} - \frac{\partial V(\phi_2)}{\partial \phi_2}$
Using the knowledge that Q_2 is zero from the lagrangian, we get the equation for current conservation.

Since ϕ_1 and ϕ_2 are not independent, we cannot choose a state with arbitrary ϕ_1 and ϕ_2. Hence it is useless to have both ϕ_1 and ϕ_2 in the hamiltonian, and we again end up putting L_0 into the potential.

Chapter 2

Uncoupled Modes

In black box quantization, we use equivalent Foster form of the given (often unimportant) circuit. In doing this, it it implicitly assumed that the equivalent circuits have the same hamiltonian. Here, two circuits are considered equivalent if they have same impedance function at the given port. As we shall see, the equivalent circuits can have very different Hamiltonians, and hence different energies.

Let us start with a simple example, as shown in the figure below.

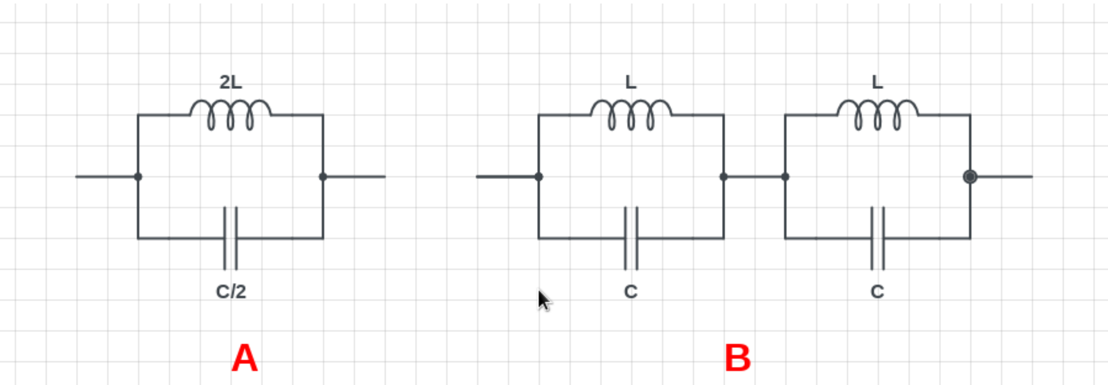


FIGURE 2.1: Two simple equaivalent circuits.

The Hamiltonians are

$$H_A = \hbar w(a^\dagger a + 1/2)$$

and

$$H_B = \hbar w(x^\dagger x + 1/2) + \hbar w(y^\dagger y + 1/2)$$

and we see that although both circuits are equivalent, the zero-point energy of B is twice of A, implying that they are not exactly the same. More importantly, the circuit A lacks degeneracy while degeneracy is present in the circuit B.

It will be easier to understand what is actually happening if we look at the circuit C.

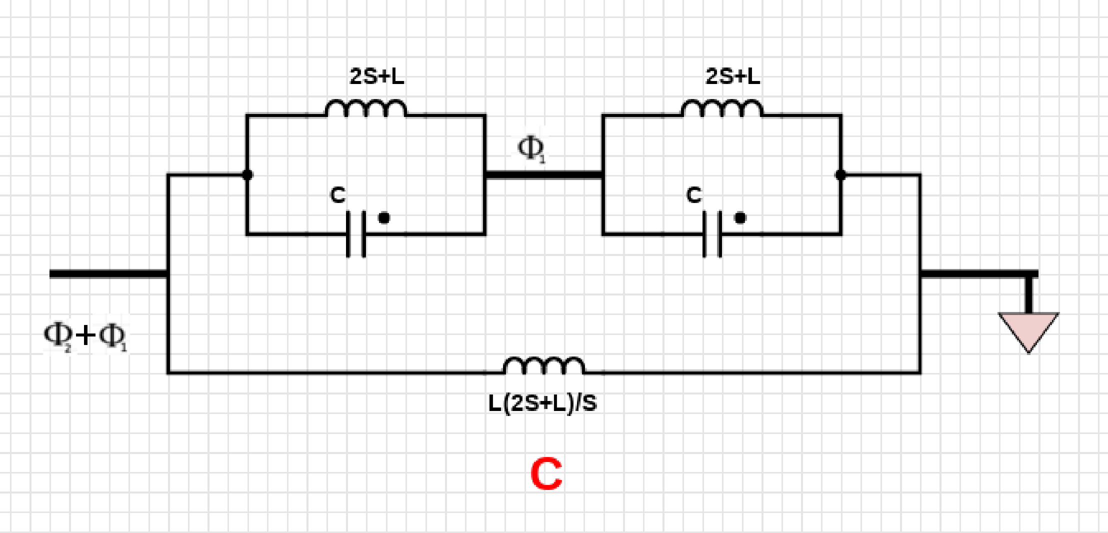


FIGURE 2.2: Another Equivalent Circuit

We can see that

$$Z_A = Z_B = Z_C = \frac{2j}{\frac{1}{wL} - wC}$$

In circuit C, if we assign ground to rightmost wire, ϕ_1 to the middle wire and $\phi_1 + \phi_2$ to the leftmost wire, we get

$$H_C = \frac{Q_1^2}{2C} + \frac{\phi_1^2}{2L}\left(\frac{S+L}{2S+L}\right) + \frac{Q_2^2}{2C} + \frac{\phi_2^2}{2L}\left(\frac{S+L}{2S+L}\right) + \frac{S\phi_1\phi_2}{L(2S+L)}$$

using the substitution $x_1 = (\phi_1 + \phi_2)/\sqrt{2}$, $x_2 = (\phi_1 - \phi_2)/\sqrt{2}$ and $p_1 = (Q_1 + Q_2)/\sqrt{2}$, $x_2 = (Q_1 - Q_2)/\sqrt{2}$, we get

$$H_C = \frac{p_1^2}{2C} + \frac{x_1^2}{2L} + \frac{p_2^2}{2C} + \frac{x_2^2}{2(2S+L)}$$

which tells us that $w' = 1/\sqrt{C(2S+L)}$ is also an eigenfrequency, which is invisible in the impedance.

It is obvious why this eigenfrequncy is invisible at output port, as at this frequency, the two oscillators cancel each other's effect, resulting in no flux difference across the outer inductor.

Hence, the black box quantization method fails when the circuit consists of two resonators capable of cancelling each other's effect at the output ports.

When measuring the classical impedance of circuit B, we can only see the combined oscillations of two resonators at the output port, and the information of each resonator remains hidden from us. Therefore, we arrive at the circuit A by measurement of the classical impedance of circuit B or circuit C.

We can look at another example showing similar phenomena.

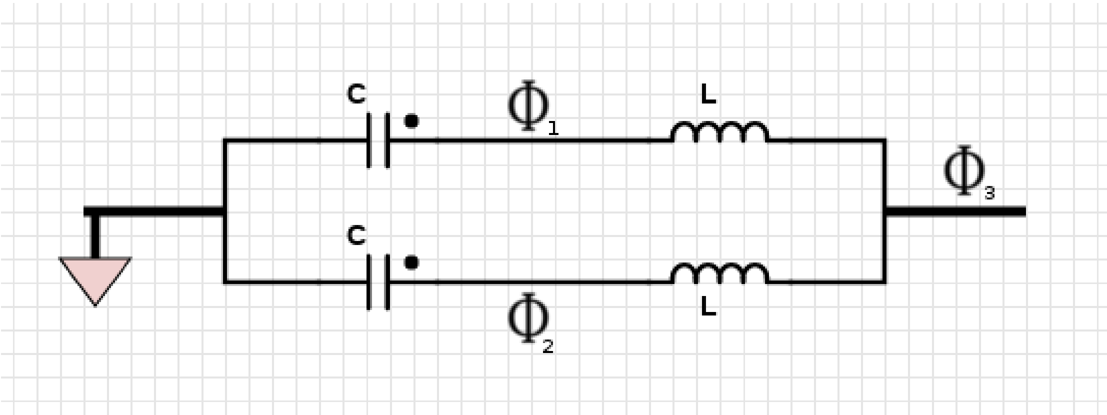


FIGURE 2.3: Another Peculiar Example

The Hamiltonian of the above circuit is

$$H = \frac{Q_1^2}{2C} + \frac{Q_2^2}{2C} + \frac{(\phi_1 - \phi_2)^2}{4L} = \hbar w(a^\dagger a + 1/2) + \frac{p^2}{2C}$$

where $w = 1/\sqrt{LC}$ and $p = (Q_1 + Q_2)/\sqrt{2}$

The Hamiltonian tells us that the circuit has oscillation frequency w, while the equivalent circuit as seen by output port simply consists of a capacitor $2C$ and an inductor $L/2$ in series. This is because the oscillation results in zero phase difference across the port.

In other words, the symmettry in the circuit allows us to have modes which cannot be coupled to at a particular output port, but can be coupled to if we select some other output port.

Chapter 3

JJ-L Problem

Let us look at the following element-

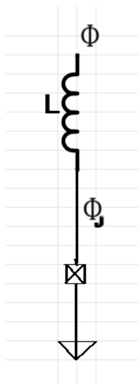


FIGURE 3.1: JJ and L in series

We have $V = \frac{(\phi-\phi_J)^2}{2L} - E_J \cos\varphi$ where $\varphi = \frac{\phi_J}{\phi_0}$ and $\phi_0 = \frac{\hbar}{2e}$ is the reduced flux quantum.
We need to express $\phi_J = f(\phi)$ to obtian V in terms of ϕ.
The current conservation equation is

$$\frac{\phi}{\phi_0} = \varphi + \lambda \sin\varphi$$

where $\lambda = \frac{LI_c}{\phi_0} = \frac{L}{L_J}$. also

$$V = -E_J \cos\varphi + \frac{\lambda}{2} E_J \sin\varphi$$

We can easily prove that $\lambda \leq 1$ implies that there exist a unique ϕ for every ϕ_J Unfortunately, a solution in closed form does not exist.

3.1 Small flux Approximation

From the equation of current conservation, we can see that $\phi \to 0 \implies \varphi \to 0$. Hence taking the approximation of small ϕ, we can write

$$\varphi = \frac{\phi}{\phi_0(\lambda+1)} + \frac{\lambda\phi^3}{6\phi_0^3(\lambda+1)^4} + \mathcal{O}(\phi^5)$$

Using this relation, we can write

$$V = E_J\left(-1 + \frac{\phi^2}{2\phi_0^2(\lambda+1)} - \frac{\phi^4}{24\phi_0^4(\lambda+1)^4} + \mathcal{O}(\phi^6)\right)$$

In the above expression, the second term can re-written as $\frac{\phi^2}{2(L+L_J)}$ and hence we can represent the whole element as a parallel combination of inductor $L+L_J$ and a non-linear device. From here, the rest of the theory is straightforward as done here.

3.2 Small flux Variation on a constant DC flux

Let us suppose that our element is driven by a superposition of a large DC flux ϕ_{DC} along with a small oscillating AC flux ϕ_s. Also, let $\varphi_{DC} = \varphi(\phi_{DC})$ which we need to find numerically.

Using the Fourier expansion, we can write $\varphi(\phi_{DC} + \phi_s)$ as function of φ_{DC} and ϕ_s

Define

$$V_0 = -E_J \cos\varphi_{DC} + \frac{\lambda}{2} E_J \sin\varphi_{DC}$$

We can easily find that

$$V = V_0 + E_J\left(\frac{\phi_s \sin\varphi_{DC}}{\phi_0} + \frac{\phi_s^2\cos\varphi_{DC}}{2\phi_0^2(1+\lambda\cos\varphi_{DC})} + \frac{\phi_s^3(-\sin\varphi_{DC})}{6\phi_0^3(1+\lambda\cos\varphi_{DC})^3} + \mathcal{O}(\phi^4)\right)$$

3.3 Junction Capacitance

If we consider the junction capacitance, which can be visualised as a capacitor C_J parallel to the Josephson Junction, then our work become much easier. We can then put the Inductor inside the black box, with the capacitor parallel to the complete circuit. Then we can include the capacitor in the black box, to get a modified black box. In this modified black box, Z has a zero at both $w = 0$ and $w = \infty$, and hence this black box can be reduced to the Foster first form with $L_0 = 0$ and $C_0 = 0$, which can be treated easily, as done here.

3.4 Fourier Series

We see that

$$\varphi(\phi + \phi_0) = \varphi(\phi) + 2\pi$$

This leads to some interesting results. From the equation, we note that $\varphi(\phi) - \frac{\phi}{\phi_0}$ is periodic in ϕ

Using $\varphi(\phi = \pi\phi_0) = \pi$ and the identity

$$(-1)^v \pi J_v(z) = \int_0^\pi d\varphi \cos\left(v\varphi + z\sin(\varphi)\right)$$

we obtain the following results ($\lambda \leq 1$ is assumed while integration):

$$\varphi = \frac{\phi}{\phi_0} + \sum_{k=1}^{\infty} \frac{-2(-1)^k}{k} J_k(k\lambda) \sin\left(k\frac{\phi}{\phi_0}\right)$$

$$\cos\varphi = \lambda/2 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} J_k'(k\lambda) \cos\left(k\frac{\phi}{\phi_0}\right)$$

$$\sin^2\varphi = 1/2 + \sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{k\lambda} J_k'(k\lambda) \cos\left(k\frac{\phi}{\phi_0}\right) + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2\lambda^2} J_k(k\lambda) \cos\left(k\frac{\phi}{\phi_0}\right)$$

where J_k is the Bessel function of order k. Combining the above terms, we get

$$V = E_J\left(-\lambda/4 + \sum_{k=1}^{\infty} \frac{2(-1)^k}{k^2\lambda} J_k(k\lambda) \cos\left(k\frac{\phi}{\phi_0}\right)\right)$$

It is obvious that V is even in ϕ as well as periodic with period ϕ_0.

However, as stated here, the function $J_v(x)$ decreases as $v^{-1/3}$. Hence the coefficients of φ goes like $k^{-4/3}$ while that of V goes like $k^{-7/3}$.

The absolute value of first 10 coefficients of fourier series of V for $\lambda = 0.3$ and $\lambda = 0.8$ is shown below, suggesting that the first two terms provide a good approximation to the function V.

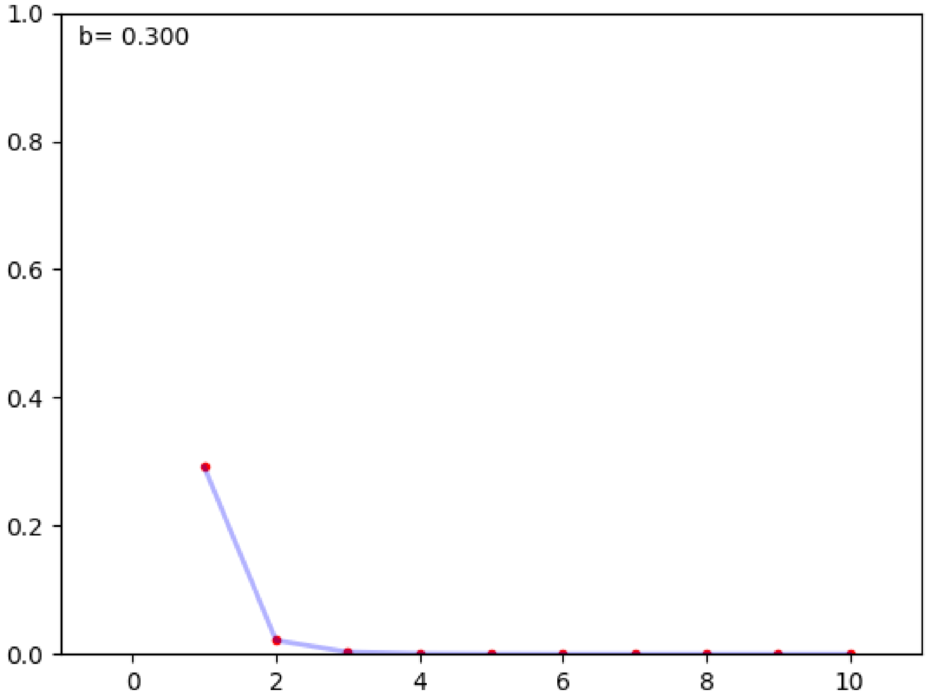


FIGURE 3.2: Absolute Value of Fourier Coefficients of V for lambda=0.3

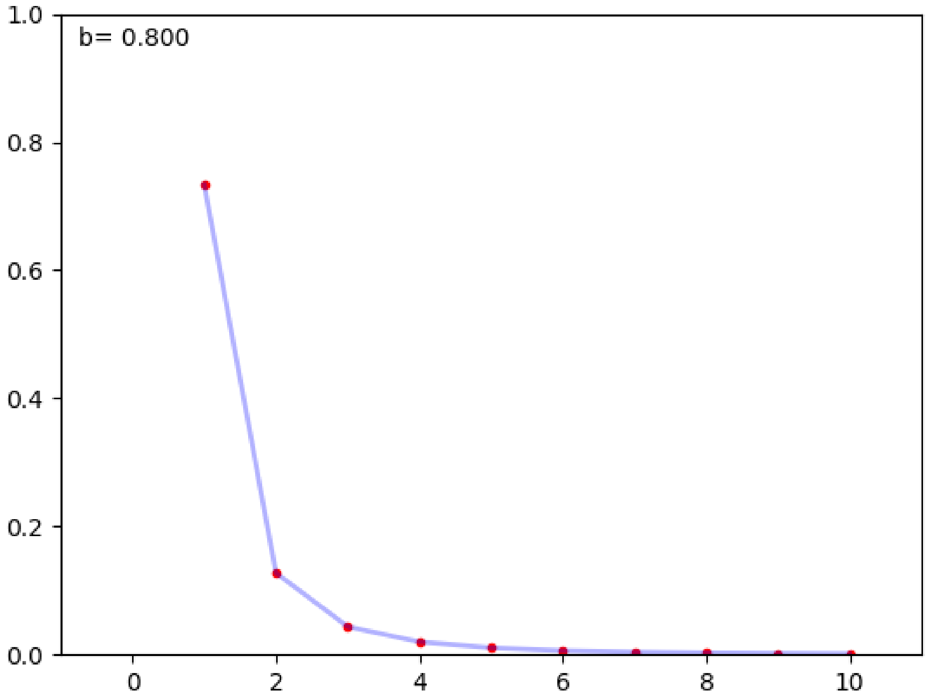


FIGURE 3.3: Absolute Value of Fourier Coefficients of V for lambda=0.8

Chapter 4

Cavity and Circuit QED

In QED, the interaction between a quantum electromagnetic field and an atom (artificial or natural) is given by

$$U = -\vec{p} \cdot \vec{E}$$

where the electric field inside the cavity $E \propto a + a^\dagger$. We will consider only the lowest 2 states, and assume that the cavity resonance frequency is far off from the rest of the transition frequencies. the two states results in different dipole of the atom, we can write (upto some constant)

$$V = H_{int} = \hbar g(a + a^\dagger)\sigma^x$$

the u$H_0 = \frac{1}{2}\hbar\sigma^z + \hbar w_c(a^\dagger a)$, we get the Rabi Hamiltonian

$$H = H_0 + H_{int}$$

Defining $\sigma^+ = \frac{1}{2}(\sigma^x + i\sigma^y)$ and $\sigma^+ = \frac{1}{2}(\sigma^x - i\sigma^y)$, we get

$$H_{int} = \hbar g(a\sigma^+ + a\sigma^- + a^\dagger\sigma^+ + a^\dagger\sigma^-) \approx \hbar g(a\sigma^+ + a^\dagger\sigma^-)$$

Where we have invoked the Rotating Wave Approximation (RWA) in the last step.

4.1 Rabi Oscillation

The detuning is given by $\Delta = w_a - w_c$, where w_a and w_c are the qubit frequency and the cavity frequency respectively. In the case when $\Delta = 0$, we have the stationary states

$$|\psi_{n\pm}\rangle = \frac{1}{\sqrt{2}}(|n+1,g\rangle \pm |n,e\rangle)$$

$$E_{n\pm} = \hbar w_a(n+1) \pm 0.5\hbar g\sqrt{n+1}$$

where the vector $|n,e\rangle(|n,g\rangle)$ denots the state with n photon and atom in excited(ground) state.

If $|\psi(0)\rangle = |n,e\rangle$, then after time t

$$|\psi(0)\rangle = \cos\bigl(0.5g\sqrt{n+1}t\bigr)\,|n,e\rangle - i\sin\bigl(0.5g\sqrt{n+1}t\bigr)\,|n+1,g\rangle$$

i.e. the state oscillates between $|n,e\rangle$ and $|n+1,g\rangle$

$$P_e = \cos^2(0.5g\sqrt{n+1}), P_g = \sin^2(0.5g\sqrt{n+1})$$

4.2 Dispersive regime

The dispersive regime is given by

$$\lambda = \frac{g}{\Delta} \ll 1$$

Defining, $X_\pm = a\sigma^+ \pm a^\dagger\sigma^-$, and rotating our frame by the unitary $D = e^{\lambda X_-}$ ($D^\dagger = e^{-\lambda X_-}$) to get (upto second order)

$$H^{'} = D^\dagger H_{RWA} D = H_{RWA} + \lambda[H_{RWA}, X_-] + \lambda^2[[H_{RWA}, X_-], X_-]$$

$$H^{'} = \hbar\left(w_c + \frac{g^2\sigma^z}{\Delta}\right)(a^\dagger a + 0.5) + \frac{\hbar}{2}\left(w_a + \frac{g^2}{\Delta}\right)\sigma^z$$

The frequency of the cavity excitation depends on state of atom $w_{cavity} = w_c \pm \frac{g^2}{\Delta}$ or equivalently,

$$H^{'} = \hbar w_c(a^\dagger a + \frac{1}{2}) + \frac{\hbar}{2}\left(w_a + \frac{2g^2}{\Delta}a^\dagger a + \frac{g^2}{\Delta}\right)\sigma^z$$

i.e. the frequency of the atomic transition depends on the state of the cavity. The Lamb shift is given by $\frac{g^2}{\Delta}$, while the Stark shift is $\frac{2g^2 n}{\Delta}$.

This means that we can use the cavity resonance to detect the state of the qubit, or we can use the qubit transition to detect the cavity state.

4.3 Exact Solution

Under RWA approximation, the Hamiltonian can be solved exactly. The Rabi Hamiltonian is given by

$$H_{RWA} = 0.5\hbar\sigma^z + \hbar w_c(a^\dagger a) + \hbar g(a\sigma^+ + a^\dagger\sigma^-)$$

The H_{RWA}, H_0 and H_{int} commute with the excitation state number $N_{ex} = a^\dagger a + \frac{1+\sigma^z}{2}$.

the degenerate eigenstates of N_{ex} are $|n,e\rangle$ and $|n+1,e\rangle$, therefore H_{RWA} is block diagonal with largest block of size 2×2.

U$\left|\psi^{n+1}\right\rangle = \alpha\,|n,e\rangle + \beta\,|n+1,g\rangle$ and solving, we get

$$E_{\pm}^{n+1} = (n+1/2)\hbar w_c \pm \frac{\hbar}{2}\sqrt{\Delta^2 + 4g^2(n+1)}$$

$$\left|\psi_{+}^{n+1}\right\rangle = \cos(\theta/2)\,|n,e\rangle + \sin(\theta/2)\,|n+1,g\rangle$$

$$\left|\psi_{-}^{n+1}\right\rangle = -\sin(\theta/2)\,|n,e\rangle + \cos(\theta/2)\,|n+1,g\rangle$$

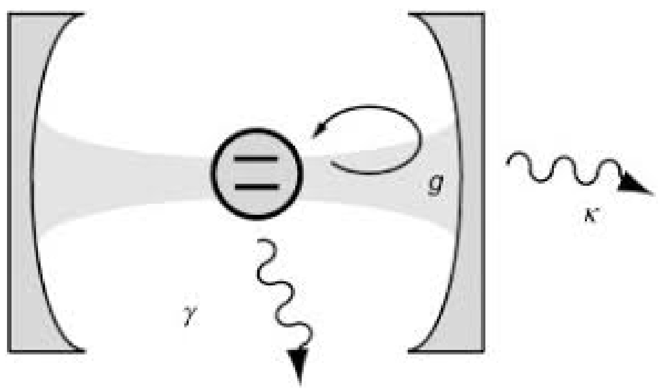
$$\tan\theta = \frac{2g\sqrt{n+1}}{\Delta}$$

From this, we can get both of the above limits (no detuning, dispersive).

4.4 Dissipation

Until now, we have ignored the dissipation in the Hamiltonian. The dissipation happens in 2 ways:

- Cavity loss: rate κ
- Non-resonant decay: rate γ



In the strong-coupling regime($g \gg \max \kappa, \gamma$) with $\Delta = 0$, we have $\theta/2 = 45°$ and hence each state is half qubit ($|n, e\rangle$) and half photon ($|n+1, g\rangle$). Therefore, the line width is

$$\tilde{\gamma} = \frac{\kappa + \gamma}{2}$$

For large detuning, to decay through the cavity, the qubit must emit a photon into the cavity which happens with low probability Δ is large and hence spontaneous emission through the cavity is weak. Equivalently, the cavity filters out the vacuum noise at the qubit frequency.

Chapter 5

3-wave Mixer

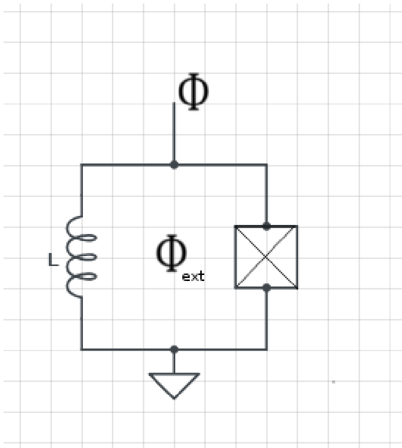


FIGURE 5.1: 3-wave mixer

Consider the circuit as shown above. ϕ_{ext} is the external flux through the loop. According to the Faraday's law,

$$\sum_a \phi_a = \phi_{ext}$$

where ϕ_a is the flux across element a of the loop. There will be an EMF generated such that the current tries to oppose the external flux.

Without Loss of Generality, we can put ϕ_{ext} across the josephson junction to get

$$V = \frac{\phi^2}{2L} - E_J \cos\left(\frac{\phi - \phi_{ext}}{\phi_0}\right)$$

Let ϕ_{min} be the value of ϕ at minimum of V. Then, at this point, the current conservation equation holds, and we get

$$\frac{dV}{d\phi} = \frac{\phi_{min}}{L} + I_c \sin\left(\frac{\phi_{min} - \phi_{ext}}{\phi_0}\right) = 0$$

and as in the JJ-L Problem, we have a unique minimum of V for every ϕ_{ext} if $\frac{L}{L_J} = \lambda < 1$. We will only consider cases of a minimum, we might not be able to consider Taylor expansion in case of multiple minima.

To get a three-wave mixer, the circuit should have a non-zero third-order term, and should not have a fourth order term in the Taylor expansion of V near the equilibrium. It can be easily seen that any circuit without an external field would be invariant under parity transformation, and hence can have a non-zero third-order term near equilibrium only if equilibrium is not at zero, which in turn imply that there will be multiple equilibriums of equal depth, disallowing us from uTaylor expansion. Hence to get a three-wave mixer, we definitely need an external magnetic field.

For zero fourth-order term at minima, we have

$$\frac{-E_J}{\phi_0^4}\cos\left(\frac{\phi_{min} - \phi_{ext}}{\phi_0}\right) = 0$$

Which gives the solutions

$$1 \rightarrow \phi_{min} = \phi_0\lambda; \phi_{ext} = \phi_0/4 + \phi_0\lambda$$

$$2 \rightarrow \phi_{min} = -\phi_0\lambda; \phi_{ext} = -\phi_0/4 + -\phi_0\lambda$$

We will be looking only at the first solution from now on, both of them are equivalent.

From the solution we observe that the current through the Josephson junction is same the as the current through Inductor, confirming that no current enters the device. Also, we see that for a three-wave mixer, the current is the maximum possible current, i.e. I_c,

5.1 Coupling to Harmonic Oscillators

The theory stated above is true only if the device is connected to a capacitor. However, in general, the 3 wave-mixer will be coupled to some circuit (which can be reduced to Foster first form) through a capacitor. Let us first analyze the situation when we directly couple the device to a harmonic oscillator without putting in a capacitor. In this case, we get the circuit shown in fig 5.2.

$$V = \sum_i \frac{\phi_i^2}{2L_i} + \frac{(\sum_i \phi_i)^2}{2L_0} - E_J \cos\left(\frac{\sum_i \phi_i - \phi_{ext}}{\phi_0}\right)$$

Here, the potential is multidimensional, and we need to minimize V in n dimensions. Let the flux across each oscillator be ϕ_i, and at minimum of V, $\phi_i = \alpha_i$. Then the

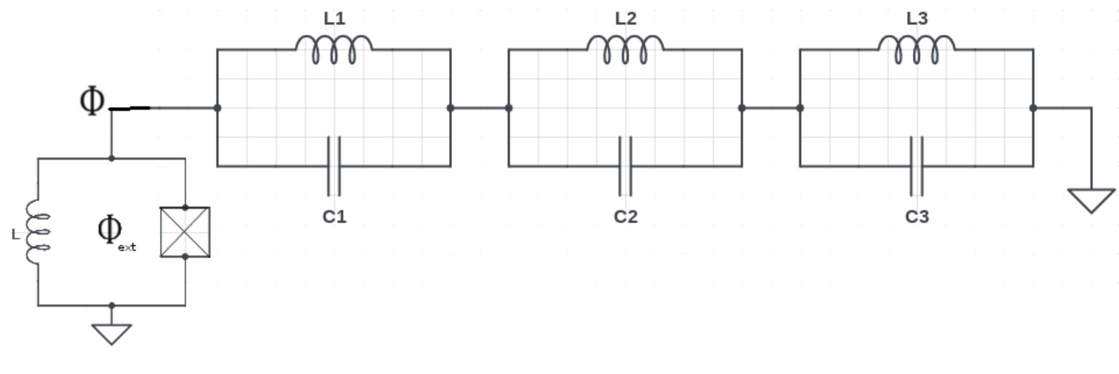


FIGURE 5.2: 3-wave mixer coupled to oscillators

equations at equilibrium and equation for the zero fourth-order term are

$$\frac{\alpha_j}{L_j} + \frac{(\sum_i \alpha_i)}{L_0} + I_c \sin\left(\frac{\sum_i \alpha_i - \phi_{ext}}{\phi_0}\right) = 0$$

$$-E_J \cos\left(\frac{\sum_i \alpha_i - \phi_{ext}}{\phi_0}\right) = 0$$

Where j can be any index. The above $n+1$ equations can be solved analytically to give

$$\alpha_j = L_j I_c \frac{L_{eff}}{\sum_i L_i}$$

$$\phi_{ext} = L_{eff} I_c + \phi_0/4$$

where we have defined $\frac{1}{L_{eff}} = \frac{1}{L_0} + \frac{1}{\sum_i L_i}$.

5.2 Coupling through a Capacitor

Now we finally look at the case in which the 3-wave mixing device is coupled to a microwave resonator through a Capacitor C_0. Let flux across each oscillator be ϕ_i and flux across the capacitor C_0 be $\widetilde{\phi}$.

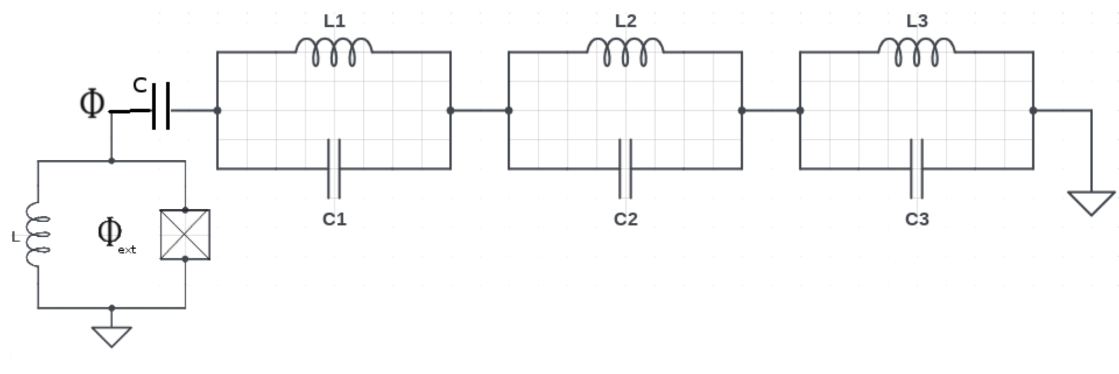


FIGURE 5.3: 3-wave mixer coupled capacitively

$$V = \sum_i \frac{\phi_i^2}{2L_i} + \frac{(\widetilde{\phi} + \sum_i \phi_i)^2}{2L_0} - E_J \cos\left(\frac{\widetilde{\phi} + \sum_i \phi_i - \phi_{ext}}{\phi_0}\right)$$

Again, taking $\phi_i = \alpha_i$ and $\widetilde{\phi} = \widetilde{\alpha}$ at equilibrium gives us the following $n+2$ equations-

$$\frac{\alpha_j}{L_j} + \frac{(\widetilde{\phi} + \sum_i \alpha_i)}{L_0} + I_c \sin\left(\frac{\widetilde{\alpha} + \sum_i \alpha_i - \phi_{ext}}{\phi_0}\right) = 0$$

$$\frac{(\widetilde{\alpha} + \sum_i \alpha_i)}{L_0} + I_c \sin\left(\frac{\widetilde{\alpha} + \sum_i \alpha_i - \phi_{ext}}{\phi_0}\right) = 0$$

$$-E_J \cos\left(\frac{\widetilde{\alpha} + \sum_i \alpha_i - \phi_{ext}}{\phi_0}\right) = 0$$

Which has the following solution- $\alpha_i = 0, \widetilde{\alpha} = L_0 I_c, \phi_{ext} = L_0 I_c + \phi_0/4$. Basically, by adding a capacitor, we are decoupling the three-wave mixing device from oscillator at the equilibrium point, a DC current cannot flow through the capacitor.

UTaylor expansion near equilibrium with $\phi_i = \alpha_i + \beta_i$ and $\widetilde{\phi} = \widetilde{\alpha} + \widetilde{\beta}$ where β is the small variation about the mean value, we get (upto some constants)

$$V = \sum_i \frac{\beta_i^2}{2L_i} + \frac{(\widetilde{\beta} + \sum_i \beta_i)^2}{2L_0} + \frac{E_J}{4\phi_0^3}(\widetilde{\beta} + \sum_i \beta_i)^3$$

For very large L_J, we can get a three-wave mixing device under the rotating term approximation.

$$V = \sum_i \frac{\beta_i^2}{2L_i} + \frac{\widetilde{\beta}^2}{2L_0} + \frac{(\sum_i \beta_i)^2}{2L_0} + \frac{\widetilde{\beta}(\sum_i \beta_i)}{L_0} + \frac{E_J}{4\phi_0^3}(\widetilde{\beta} + \sum_i \beta_i)^3$$

In the above expression, the first term corresponds to the potential energy of the oscillators, the second term is the new mode introduced by L_0 and C_0, the third term denotes the frequency shift of oscillators, the fourth term is the coupling between the new mode and the old modes while the last term shows the 3-wave mixing effect.

5.3 Comparision with a SQUID

We will now look at the comparision between our 3-wave mixer(A) and a SQUID(B) as shown in the figure.

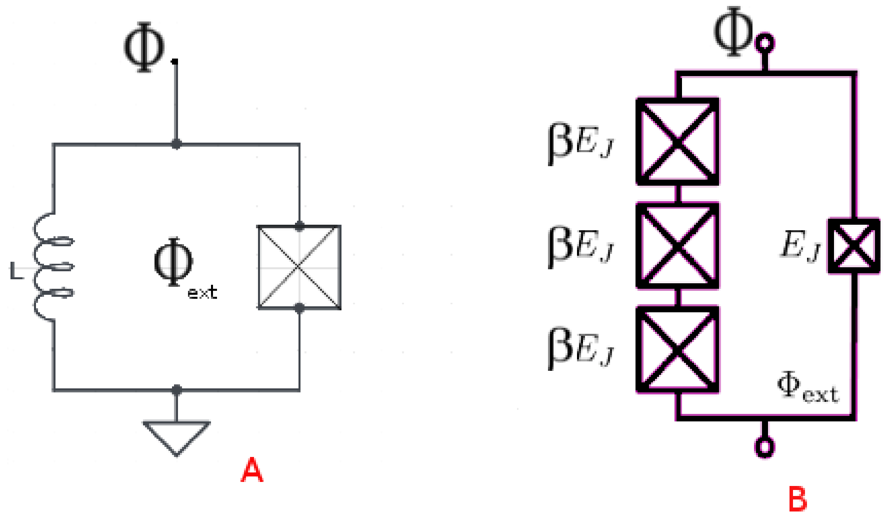


FIGURE 5.4: Comparision of 3-wave Mixers

In B, for large β the current through inductors in left will be small relative to the critical current, and therefore we expect the 3 junctions in left to behave roughly as an inductor, and hence $L = 3\frac{\phi_0^2}{\beta E_J} \implies \lambda_B = 3/\beta$.

$$V = -\frac{9E_J}{\lambda_B}\cos(\phi/3) - E_J\cos(\phi - \phi_{ext})$$

Writing the equation of equilibrium and equation for zero fourth-order terms,

$$3\sin(\phi_{min}/3) + \lambda_B\sin(\phi_{min} - \phi_{ext}) = 0$$

$$\frac{1}{9}\cos(\phi_{min}/3) + \lambda_B\cos(\phi_{min} - \phi_{ext})$$

Again, for V to have a minimum, $\lambda_B < 1$.

However, by squaring and adding the two equations, we get $\lambda > \frac{1}{9}$. This contrasts our expectation of approximating the junctions as an inductor for small λ_B (large β). It happens because at small λ_B, the fourth order terms generated by junctions in left are much larger than that in right(in B), and they cannot cancel each other out.