IIT BOMBAY

Quantum Chaos and Many-Body systems

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Contents

Chapter 1

Quantum chaos in many-body systems

Quantum Chaos is simply the study of Chaotic Quantum systems. Quantum chaos reveals a significant amount of universality in the behavior of extraordinarily different physical systems.

There are three methods to characterize Quantum Chaos

- Semi-classical Methods
- Random Matrix Theory
- Out-of-Time-Order Correlators (OTOC)

Semi-classical methods involve solving the system in the classical or semi-classical limits. If this simplified system shows chaotic behavior, it is natural to expect that the original quantum system is chaotic as well. We will not proceed in this direction. We will look at Random Matrix Theory and it's importance in characterization of Chaos in the next chapter, and here instead focus on OTOCs, which have been motivated by the study of black holes and discovery of Gauge-Gravity correspondence.

In quantum systems, Chaos can be characterised by studying the growth of different time commutator of 2 rather general Hermitian operators given by $[W(t), V(0)]$. The commutator quantifies the effect of perturbation V on a later measurement of W. When be average over ensemble of initial states, it becomes necessary to use the squared commutator so that the phases do not cancel out, and hence we study the function

$$C(t) = \langle [W(t), V(0)]^{\dagger} [W(t), V(0)] \rangle = -\langle [W(t), V(0)]^2 \rangle \tag{1.1}$$

We will assume that V, W are simple Hermitian operators, i.e. they can be expressed as sum of terms with each term having $\mathcal{O}(1)$ degrees of freedom. Also, V, W have been shifted such that their thermal 1-point function is zero.

$C(t)$ can be easily connected to OTOC using the expansion of the commutator

$$C(t) = \langle W(t)VVW(t)\rangle + \langle VW(t)W(t)V\rangle - 2\,\mathrm{Re}\{\langle VW(t)VW(t)\rangle\} \tag{1.2}$$

where the last term is the OTOC while the first and second term are simply the thermal two point functions, which goes to $\langle V^2\rangle\langle W^2\rangle$ at large t.

If the system is chaotic, i.e. it exhibits *Butterfly effect*, then the effect of perturbation V will grow with time and the value of $C(t)$ saturates to $2\langle V^2\rangle\langle W^2\rangle$ [1]. On the other hand, for a non-chaotic systems, the operations commute at large time and the function $C(t)$ decays to zero.

One important subtlety regarding quantum chaos needs to be clarified. Since QM is a linear theory, if two states are close initially in the sense of having large inner product, they will remain close as time progresses. However, for systems having large number of degrees of freedom, orthogonal states can be physically similar. This physical similarity of the initial states can be destroyed by the time evolution in chaotic systems.

1.1 Brief overview of classical chaos

Classical chaos is a huge field in itself, and here we will only consider the properties relevant to our discussion in later sections.

We consider a classical thermal system with phase space denoted by $\boldsymbol{X} = (\boldsymbol{p}, \boldsymbol{q})$, where $\boldsymbol{p}$ and $\boldsymbol{q}$ are multidimensional momentum and coordinate vectors. In this system, we can determine the presence of Chaos through Sensitive Dependence on Initial Condition (SDIC).

Assume that we have a reference trajectory in phase space $\boldsymbol{X}(t)$ with initial conditions $\boldsymbol{X}(0) = \boldsymbol{X}_0$. If we make a small change in the initial conditions $\boldsymbol{X}_0 \to \boldsymbol{X}_0 + \delta\boldsymbol{X}_0$, the system follows a new trajectory given by $\boldsymbol{X}(t) \to \boldsymbol{X}(t) + \delta\boldsymbol{X}(t)$. For a chaotic system showing SDIC, the distance between the reference trajectory and the perturbed trajectory increases exponentially with time and this rate of increase is given by Lyapunov exponent λ_L

$$|\delta\boldsymbol{X}(t)| \sim |\delta\boldsymbol{X}_0| e^{\lambda_L t} \tag{1.3}$$

It must be noted that we actually obtain a spectrum of Lyapunov exponent depending upon the orientation of $\delta\boldsymbol{X}_0$. However, for most deviations, only the largest Lyapunov exponent is significant at large time, and unless otherwise stated, I will refer to the maximal Lyapunov exponent as the Lyapunov exponent λ_L.

The classical thermal systems exhibiting chaotic dynamics usually have two exponential behavior.

- Lyapunov behavior $\implies$ characterises SDIC
- Ruelle behavior $\implies$ characterises approach to thermal equilibrium

Since the Black holes can also be considered as thermal systems, the analogs of these two behaviors can be studied in Black hole physics.

When we consider the SDIC, we need to use squared values of deviation to avoid cancellation on averaging over initial systems and perturbations. We consider the function

$$F(t) = \left\langle \left(\frac{\partial \boldsymbol{X}(t)}{\partial \boldsymbol{X}_0} \right)^2 \right\rangle \sim \sum_k c_k e^{2\lambda_k t} \qquad \text{at large } t \tag{1.4}$$

Ruelle's behavior characterises how fast the system forgets the initial conditions and approach the thermal equilibrium. This approach is given by the 2-point function

$$G(t) = \langle X(t)X(0)\rangle_\beta - \langle X\rangle_\beta^2 \sim \sum_j b_j e^{-\mu_j t} \qquad \text{at large } t \tag{1.5}$$

At large time, the growth of function $F(t)$ is given by the maximum Lyapunov exponent λ_L and the decay of 2-point function is given by the minimum Ruelle resonance μ.

1.2 Quantum Analogs

In quantum systems, the analogs of F and G are given by the expectation value of the squared commutator and the and 2-point correlator respectively. To see the origin of the squared commutator, consider the measure of deviation given by

$$\frac{\partial q(t)}{\partial q(0)} \sim e^{\lambda_L t} \tag{1.6}$$

In quantum system, the analogous quantity to consider is the commutator obtained by replacing the classicla Poisson bracket with the commutator.

$$\frac{\partial q(t)}{\partial q(0)} = \{q(t), p(0)\} \rightarrow \frac{1}{i\hbar}[q(t), p(0)] \tag{1.7}$$

In the large N systems, we use general operators W, V instead of p, q to obtain the form of $C(t)$ as specified in equation 3.2.

$$F(t) = \left\langle \left(\frac{\partial q(t)}{\partial q(0)} \right)^2 \right\rangle \rightarrow C(t) = -\langle [W(t), V(0)]^2 \rangle \tag{1.8}$$

1.3 Behavior of $C(t)$

There are two important time scales which occurs in study of $C(t)$ for chaotic systems. [2]

- **Scrambling Time (t_*):** $C(t)$ usually starts from zero and grows. T_* is the time when $C(t)$ becomes significant.
- **Dissipation Time (t_d):** This scale characterize the exponential decay time of 2-point expectation values of type $\langle V(0)V(t) \rangle$ and also control the late time behaviour of $C(t)$.

For chaotic systems, the function $C(t)$ grows exponentially at time scales $t_d < t < t_*$ according to the largest Lyapunov exponent λ_L, i.e. $C(t) \sim e^{2\lambda_L t}$, as shown in Figure 1.1. We also have the following approximate relations [2]

$$t_* \sim \lambda_L^{-1} \log N_{dof} \quad , \quad t_d \sim \lambda_L^{-1} \tag{1.9}$$

As stated earlier, $C(t)$ saturates at large time at value $2\langle V^2 \rangle \langle W^2 \rangle$. For times smaller than t_d, $C(t)$ has a negligible value of order $\mathcal{O}(N_{dof}^{-1})$

1.4 Butterfly effect

In quantum system, the operator $W(t)$ is given by

$$W(t) = e^{iHt} W e^{-iHt} = \sum_{k=0}^{\infty} \frac{(-it)^k}{k!} [H, [H, ...[H, W]]] \tag{1.10}$$

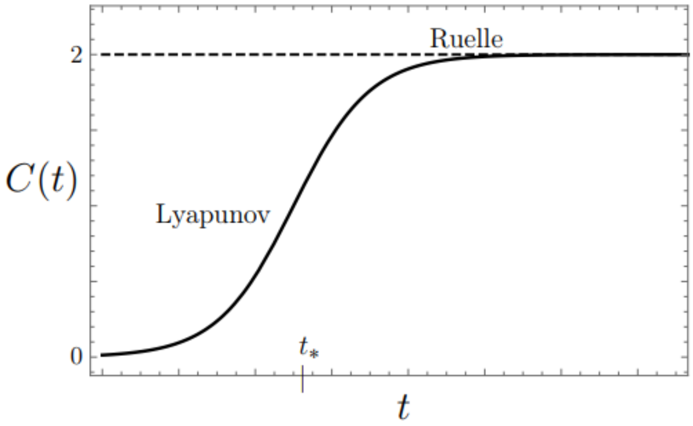

FIGURE 1.1: Typical Behavior of C(t) for chaotic systems [2]

In chaotic quantum system, operator $W(t)$ becomes more and more delocalised as time progresses. Therefore, the commutator grows exponentially in large N systems.

In we consider operators separated in space, i.e. $V(0, 0$ and $W(x, t)$, for time $t > t_*$, the $C(t, x)$ actually grows as $C(t) \sim \exp\left\{\lambda_L(t - t_* - \frac{|x|}{v_B})\right\}$, where v_B is called the *Butterfly velocity*. This velocity characterise the growth of operator W in physical space, and bound the rate of transfer of quantum information.

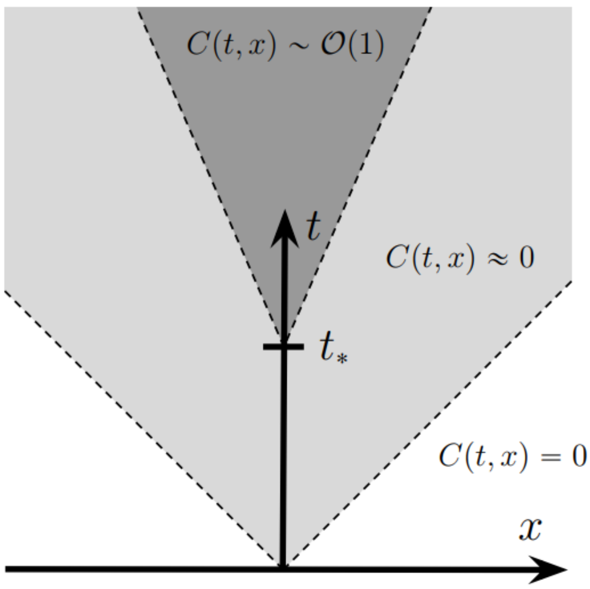

FIGURE 1.2: Butterfly effect cone [2]

For transfer of information about the perturbation, we obtain the above effective light cone where different regions are separated by different order of magnitude of $C(t)$

Chapter 2

Quantum Chaos and Random Matrix Theory

Random Matrix Theory (RMT) originated in nuclear physics as a statistical approach to understand the spectra of heavy atomic nuclei and it explained the distribution eigenvalue spacing of nuclear resonances. Motivated by the observation in SYK model [3], it is now believed that many chaotic quantum systems shares some similarities with Random Matrix Theory with regards to the Spectral statistics. Indeed, the comparison of spectral statistics of a system with that of RMT has been employed as a diagnostic of Chaos since a long time.

2.1 Introduction to RMT

For the purpose of this report, we will only consider the Gaussian Unitary Ensemble, a specific type of Random Matrix Theory. Other RMTs of interest include Gaussian Orthogonal Ensemble and Gaussian Symplectic Ensemble.

A Gaussian Unitary ensemble, specified by GUE(L, μ, σ) is an ensemble of $L \times L$ random Hermitian matrices, where each element is an independent random complex number chosen from a Gaussian distribution with mean μ and variance σ^2. Unless otherwise specified, we will be working with GUE$(L, 0, 1/\sqrt{L})$, where the variance has been chosen such that the eigenvalues do not scale with the system size. It can be shown that the energy spectrum of $GUE(L, 0, \sigma)$ lies in the range $(-2\sigma\sqrt{L}, 2\sigma\sqrt{L})$ [4].

The probability density function of matrix H in GUE is given by

$$P(H) = \exp\left\{-\frac{L}{2}\,\mathrm{Tr}\left\{H^2\right\}\right\} \tag{2.1}$$

The normalization factor has been absorbed in the measure $\mathcal{D}H$. It is obvious that GUE has $U(L)$ symmetry, and hence the integration measure is also $U(L)$ invariant, i.e. $\mathcal{D}H = \mathcal{D}(U^\dagger HU)$. Using this invariance, we can change the integral over H to integral over diagonal matrix $D = diag\{\lambda_i\}$ and unitary U according to

$$\mathcal{D}H = C|\Delta(\lambda)|^2\mathcal{D}U \quad \Pi_i \mathrm{d}\lambda_i \tag{2.2}$$

where the Vandermonde determinant $\Delta(\lambda)$ and the constant C are given by

$$\Delta(\lambda) = \Pi_{i<j}(\lambda_j - \lambda_i) \quad , \quad C = \frac{L^{L^2/2}}{(2\pi)^{L/2}\Pi_{p=1}^{L}(p!)} \tag{2.3}$$

We can absorb the integration over the Unitary group in the integration constant and define the joint probability distribution of eigenvalues as

$$P(\lambda_1, \lambda_1, ...\lambda_L) = C|\Delta(\lambda)|^2 \exp\left\{-\frac{L}{2}\sum_i \lambda_i^2\right\} \tag{2.4}$$

for simplicity, we define the measure over eigenvalues which absorbs their distribution as $\mathcal{D}\lambda = P(\lambda_1, ...\lambda_L)\ \Pi_i \mathrm{d}\lambda_i$. In this notation, the expectation value of the operator is given by

$$\langle O(\lambda)\rangle_{GUE} = \int \mathcal{D}\lambda O(\lambda) \tag{2.5}$$

Hence, we can obtain the spectral distribution by integrating over all but one eigenvalue

$$\rho(\lambda) = \int \mathrm{d}\lambda_1..\mathrm{d}\lambda_{L-1} P(\lambda_1, ..\lambda_{L-1}, \lambda) \tag{2.6}$$

In the large L limit, we obtain the Wigner semi-circle law for the distribution of eigenvalues [5]

$$\rho(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2} \quad \text{as} \quad L \to \infty \tag{2.7}$$

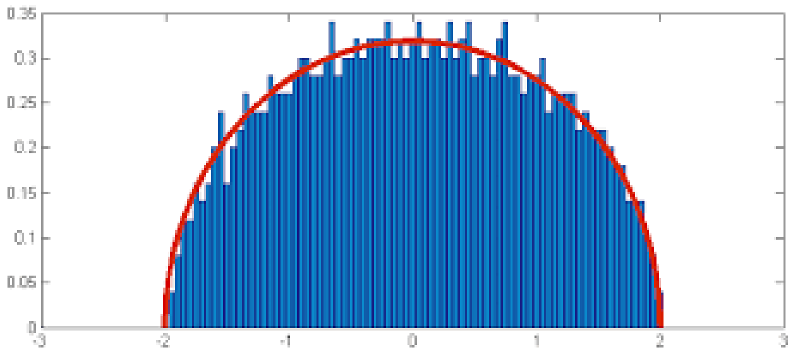

FIGURE 2.1: Wigner Semi-circle law distribution

Similarly, we can define the spectral n-point function as

$$\rho^{(n)}(\lambda_1, ...\lambda_n) = \int \mathrm{d}\lambda_{n+1}..\mathrm{d}\lambda_L P(\lambda_1, ..\lambda_{L-1}, \lambda_L) \tag{2.8}$$

According to this, the spectral 2-point function can be expressed as a sum of disconnected piece and a squared sine kernel [4] as

$$\rho^{(2)}(\lambda_1, \lambda_2) = \frac{L^2}{L(L-1)}\rho(\lambda_1)\rho(\lambda_2) - \frac{L^2}{L(L-1)}\frac{\sin^2 L(\lambda_1 - \lambda_2)}{[L\pi(\lambda_1 - \lambda_2)]^2} \tag{2.9}$$

2.2 Spectral Form Factor

For a given Hamiltonian H, the spectral form factor is defined using analytically continued partition function

$$\mathcal{R}_2^H(\beta, t) = Z(\beta, t)Z^*(\beta, t) = \mathrm{Tr}\Big(e^{-\beta H - iHt}\Big)\,\mathrm{Tr}\Big(e^{-\beta H + iHt}\Big) \tag{2.10}$$

In systems such as SYK model or RMT, we can average over all possible Hamiltonians to describe average spectral form factor

$$\mathcal{R}_2(\beta, t) = \langle Z(\beta, t)Z^*(\beta, t)\rangle_H \tag{2.11}$$

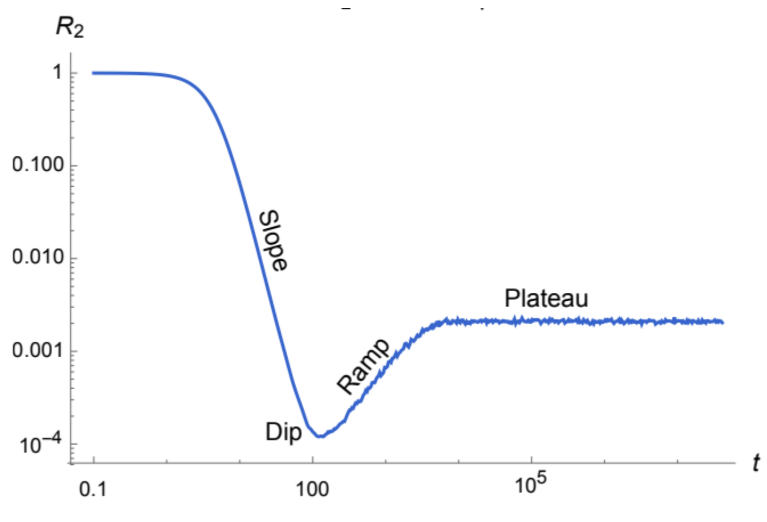

FIGURE 2.2: general behavior of spectral form factor for chaotic systems [4]

The definition of the above spectral form factor was motivated from consideration of the following 2-point correlation function

$$f_\beta(T) = \frac{1}{Z}\,\mathrm{Tr}\left(e^{-\beta H}O(T)e^{-\beta H}O(0)\right) = \frac{1}{Z}\sum_{n,m} e^{-(\beta+iT)E_n}e^{-(\beta-iT)E_m}|\langle n|O|m\rangle|^2 \tag{2.12}$$

Since, the squared part matrix elements vary smoothly on slightly changing the Hamiltonian, we are only interested in the oscillating part, which is

$$\mathcal{R}_2^H(\beta,t) = \sum_{n,m} e^{-(\beta+iT)E_n} e^{-(\beta-iT)E_m} \tag{2.13}$$

2.3 Spectral form factor in RMT

After taking the GUE average, the spectral form factor reduces to

$$\mathcal{R}_2(\beta,t) = \langle Z(\beta,t)Z^*(\beta,t)\rangle_{GUE} = \int \mathcal{D}\lambda \sum_{k,j} e^{-\beta(\lambda_k+\lambda_j)+i(\lambda_k-\lambda_j)t} \tag{2.14}$$

We separately study the behavior of spectral form factor in infinite T and finite T cases. Here, I will simply state the results and the detailed calculation can be found in reference [4].

2.3.1 Infinite Temperature Case

In this case, the spectral form factor reduces to

$$\mathcal{R}_2(t) = \int \mathcal{D}\lambda \sum_{k,j} e^{i(\lambda_k-\lambda_j)t} = L + L(L-1)\int \mathrm{d}\lambda_1 \mathrm{d}\lambda_2 \rho^{(2)}(\lambda_1,\lambda_2) e^{i(\lambda_1-\lambda_2)t} \tag{2.15}$$

where the first term arises from the case when $i = j$ and second term correspond to the case when $i \neq j$.

Separating $\rho^{(2)}(\lambda_1,\lambda_2)$ into connected and disconnected parts, we observe that the disconnected part is simply the Fourier transform of the spectral distribution function. The integration over the squared Sine kernel can be evaluated under the box approximation [5] to give a ramp function. The final result is

$$\mathcal{R}_2(t) = L + L^2 r_1^2(t) - L r_2(t) \tag{2.16}$$

The first contribution comes from the $i = j$ case which is constant at large t, the second contribution arises from the disconnected part and oscillates at all t while the last contribution arises from the Sine kernel and is responsible for the ramp behavior the the spectral form factor.

The functions r_1 and r_2 are given by

$$r_1(t) = \frac{J_1(2t)}{t} \tag{2.17}$$

$$r_2(t) = \begin{cases} 1 - \frac{t}{2L} & t < 2L \\ 0 & t \geq 2L \end{cases} \tag{2.18}$$

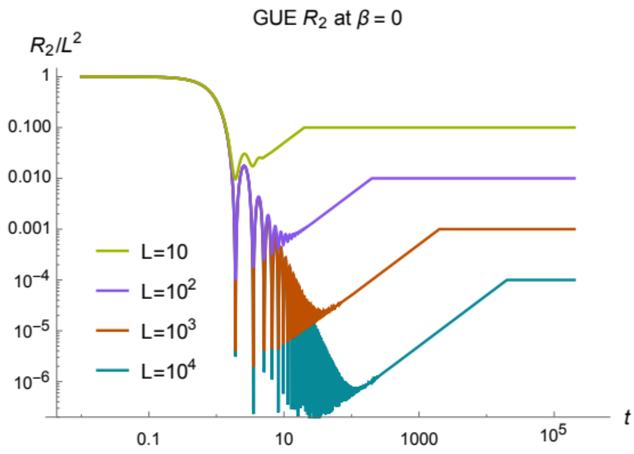

FIGURE 2.3: Spectral Form factor at infinite temperature [4]

The figure shows the results obtained by simulating the system at infinite temperature. We can distinctively identify the linear ramp, dip and plateau in the spectral form factor. The oscillations occurs due to oscillating nature of r_1 and gets smoothened out at finite temperature. The spectral form factor becomes almost constant after the plateau time t_p which can be found by function r_2 as $t_p = 2L$.

2.3.2 Finite temperature

The finite temperature spectral form factor can be written using analytical continuation of functions r_1 and r_2 as

$$\mathcal{R}_2(t) = Lr_1(2i\beta) + L^2 r_1(t + i\beta)r_1(-t + i\beta) - Lr_1(2i\beta)r_2(t) \tag{2.19}$$

This smoothens out the oscillations visible in infinite temperature case, as well as gives the correct slope of ramp as obtained in simulations.

2.4 Relation between OTOC and spectral form factor

In this section, we will relate 2-point OTOC to the spectral form factor, thus establishing the fact that both of them are more or less equivalent diagnostic of Chaos. We will be working with general hamiltonian H of dimensions $L = 2^n$. We will only consider the infinite temperature case here.

Let us consider 2-point correlation function between between a unitary operator A with time separation t. On averaging over all unitary operators

$$\frac{1}{L}\int \mathrm{d}A\langle A(0)A^{\dagger}(t)\rangle = \frac{1}{L}\int \mathrm{d}A\,\mathrm{Tr}\big(Ae^{-iHt}Ae^{+iHt}\big) \tag{2.20}$$

where $\mathrm{d}A$ is the Haar measure on $U(L)$. Using the mathematical identity $\int \mathrm{d}AA_{jk}A^{\dagger}_{lm} = \frac{1}{L}\delta_{jm}\delta_{lk}$, we obtain

$$\frac{1}{L}\int \mathrm{d}A\langle A(0)A^{\dagger}(t)\rangle = \frac{\mathcal{R}_2^H(t)}{L^2} \tag{2.21}$$

This definition not only provides a link between spectral statistics and 2-point functions, but also provide us a way to calculate the spectral form factor. In a system with large L, instead of integrating over all unitary matrices A, we only need to integrate over some randomly chosen unitary matrices.

Chapter 3

Many-body chaos at weak coupling

This chapter is primarily a reproduction of the paper "Many-body chaos at weak coupling" by Douglas Stanford [6]. The purpose of this reproduction was to get familiar with the methods of calculation in a Quantum Field theory with Matrix Field and apply similar tools in future work.

In the previous chapters, we have looked at some properties and diagnostics of a chaotic system. Here, we will study a large N chaotic system and analytically calculate the largest Lyapunov exponent of the model in the weak coupling regime. We will work with a matrix field of size N in four dimensions in the large N and small 't Hooft coupling($\lambda = g^2 N$) limit. The model is given by the ϕ^4 Lagrangian

$$\mathcal{L} = \frac{1}{2} Tr\left(\dot{\Phi}^2 - (\boldsymbol{\nabla}\Phi)^2 - m^2\Phi^2 - g^2\Phi^4\right) \tag{3.1}$$

We will calculate index averaged and space averaged square commutator given by equation 3.2. The aim of this calculation is to find out the Lyapunov exponent of the model.

$$C(t) = \frac{1}{N^4} \sum_{aba'b'} \int d^3\mathbf{x}\, \mathrm{Tr}\left(\sqrt{\rho}\, [\Phi_{ab}(t,\mathbf{x}), \Phi_{a'b'}]\, \sqrt{\rho}\, [\Phi_{ab}(t,\mathbf{x}), \Phi_{a'b'}]^\dagger\right) \tag{3.2}$$

where the trace is over the system states while the sum on the matrix elements are explicitly specified. In the above commutator, the thermal density matrix has been split into two factors of $\sqrt{\rho}$ to move the operators to opposite sides of thermal circle. This places all the four operators at different space-time points and hence regularizes the correlation function.

$$\mathrm{Tr}\left(\rho[A(t),B][A(t),B]^\dagger\right) \to \mathrm{Tr}\left(\rho[A(t-i\beta),B(-i\beta)][A(t),B]^\dagger\right) = \mathrm{Tr}\left(\sqrt{\rho}[A(t),B]\sqrt{\rho}[A(t),B]^\dagger\right) \tag{3.3}$$

This regularization should not affect the growth properties, and hence should not change λ_L. We expect that $C(t)$ grows as $e^{2\lambda_L t}$ at large time. To find out λ_L, we will expand the commutators and use Feynman diagram on each term. After expanding the commutator, we can write $C(t)$ diagrammatically as shown in figure 3.1. In the figure, the horizontal axis is the real time, while the verticle axis is the imaginary time with end points identified.

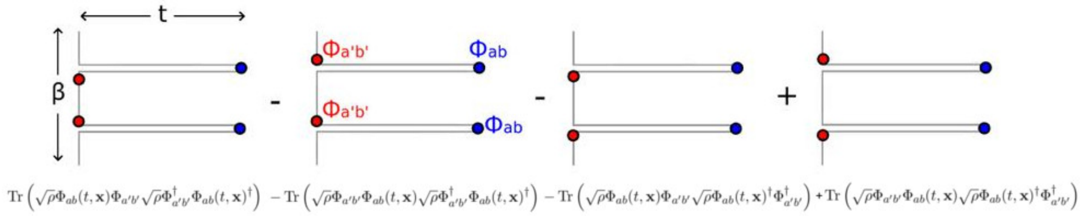

FIGURE 3.1: Diagrammatic representation of terms in $C(t)$

To calculate the Lyapunov exponent, we will employ the following simplifications

1. The N^{-2} term in expansion of $C(t)$ grows as $e^{2\lambda_L t}$, which is also the leading order term in the large N expansion. Therefore, we will be working in large N limit and will only consider planar diagrams.

2. We will restrict integration region only to the real time-fold, i.e. the interaction vertices will be placed only on real time-folds. The interaction over thermal circle should modify the thermal state of the system, but should not affect the growth exponents.

3. Since we want to compute the largest growth exponent, we will only keep the fastest growing terms at each power of λ. As will be shown later, this correspond to summing over only the powers of $\lambda^2 t$ and ignoring the other terms.

Because of these simplifications, the diagrams that must be summed over are the dressed ladder diagrams. There will arise two subtle points in the calculations

1. We are applying perturbation theory on the folded time manifolds, and hence must integrate on both sides of each fold. This summing over the integrals on both sides converts the side rails (horizontal propagator in diagram) into retarded

propagators, while the rungs (propagator between 2 folds) results in Wightman correlators.

2. In the retarded propagator, we need to include self energy correction arising because of propagators lying on the same fold.

The above points will be further clarified in the subsequent sections.

3.1 Free propagators

For the computation of $C(t)$, we will need the retarded propagator G_R and the Wightman correlator G_W defined as follows

$$\begin{aligned}\delta_{ab'}\delta_{ba'}G_R(\mathbf{x},t) &= \theta(t)\,\mathrm{tr}\left(\rho\left[\Phi_{ab}(\mathbf{x},t),\Phi_{a'b'}\right]\right)\\ \delta_{ab'}\delta_{ba'}G_W(\mathbf{x},t) &= \mathrm{tr}\left(\sqrt{\rho}\Phi_{ab}(\mathbf{x},t)\sqrt{\rho}\Phi_{a'b'}\right)\end{aligned} \tag{3.4}$$

At zeroth order in $\lambda = g^2N$, the theory is basically a free field complex Klein-Gordon theory, and the propagators can be calculated directly to give

$$\begin{aligned}G_R(\mathbf{x},t) &= \frac{i}{2E_{\boldsymbol{k}}}\left(\frac{1}{k^0 - E_{\boldsymbol{k}} + i\epsilon} - \frac{1}{k^0 + E_{\boldsymbol{k}} + i\epsilon}\right)\\ G_W(\mathbf{x},t) &= \frac{\pi}{2E_{\boldsymbol{k}}\sinh\frac{\beta E_{\boldsymbol{k}}}{2}}\left[\delta(k^0 - E_{\boldsymbol{k}}) + \delta(k^0 + E_{\boldsymbol{k}})\right]\end{aligned} \tag{3.5}$$

3.2 Order λ^0

At this order, we need to only sum over different ways of contracting the four operators appearing in the figure 3.1. There are three ways to contract the fields as shown in the figure 3.2.

Out of the above cases, the four terms in expansion of B and C adds up to zero, since the contraction does not care about the relative ordering of operators at same imaginary time. Only the case A gives non-zero contribution.

From the figure 3.3, the contribution of diagrams at order λ^0 can be calculated to be

$$\begin{aligned}C(t) &= \frac{1}{N^4}\sum_{a,b,a',b'}\int \mathrm{d}^3x\,\mathrm{Tr}\left(\rho[\Phi_{ab}(\boldsymbol{x},t),\Phi_{a'b'}]\right)\mathrm{Tr}\left(\rho[\Phi^\dagger_{a'b'},\Phi^\dagger_{ab}(\boldsymbol{x},t)]\right)\\ &= -\frac{1}{N^4}\sum_{a,b,a',b'}\int \mathrm{d}^3x\,\mathrm{Tr}\left(\rho[\Phi_{ab}(\boldsymbol{x},t),\Phi_{a'b'}]\right)\mathrm{Tr}\left(\rho[\Phi_{ba}(\boldsymbol{x},t),\Phi_{b'a'}]\right) = -\frac{1}{N^2}\int \mathrm{d}^3x G_R(x,t)^2\end{aligned} \tag{3.6}$$

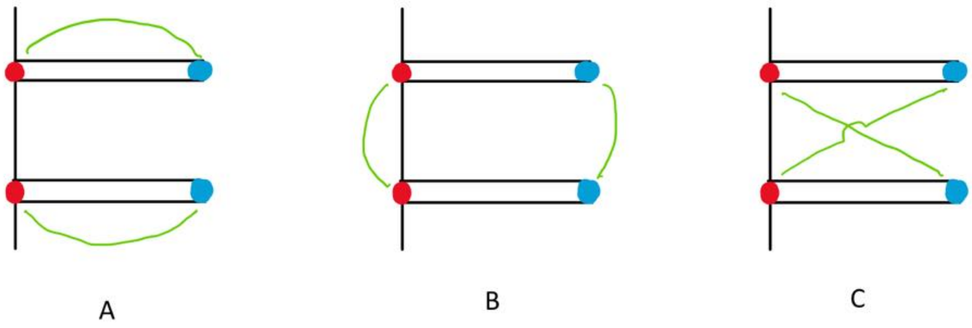

FIGURE 3.2: 3 ways to contract fields at order λ^0

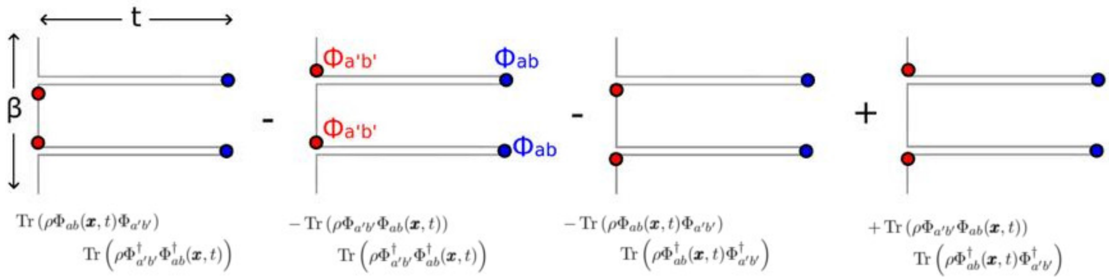

FIGURE 3.3: Terms in expansion of case A

this shows the emergence of the retarded propagator from simple correlators.

3.3 Order λ^1

At order λ, the only possible diagrams are one-loop correction on on the real time folds. These diagrams gives the thermal mass correction to the base mass. This contribution does not modify the Lyapunov at highest order exponent if $m \neq 0$, but contributes to λ_L when $m = 0$

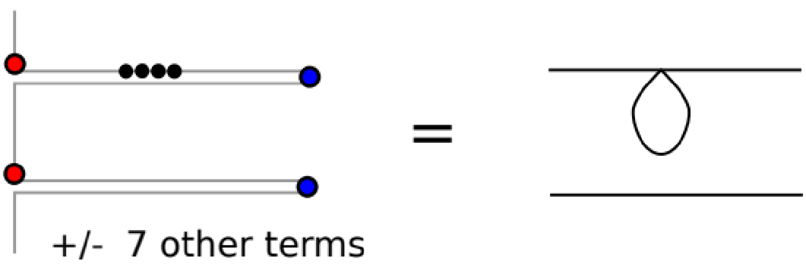

FIGURE 3.4: Diagrams at order λ [6]

We will show in section 3.5 that in the case when bare mass is zero, the one-loop correction generates thermal mass given by $m_{th}^2 = \frac{2\lambda}{3\beta^2}$

3.4 Order λ^2

When we have two integration vertices on the real time fold, we have the following three distinct types of diagrams

1. Two different one-loops (figure 3.5(a) : This correspond to the second term in geometric series generated by one-loop correction. We can ignore this correction in both cases when $m \neq 0$ or when $m = 0$ (relative to thermal mass).

2. Two-loop self energy (figure 3.5(b) : This can also be absorbed as a correction in the propagator according to the Swinger-Dyson equation

$$G(iw_n, (k))^{-1} = G_0(iw_n, (k))^{-1} + \Pi(iw_n, (k))$$

   The real part of the correction Π can be ignored relative to the bare mass or the thermal mass arising out of one-loop correction. The imaginary part of the self energy correction leads to the exponential decay of the correlation function due to scattering. This effect along with higher order dressings of the correlator function can be incorporated by including the imaginary part of the self energy correction in the retarded propagator

$$G_R(\mathbf{x}, t) = \frac{i}{2E_{\boldsymbol{k}}} \left(\frac{1}{k^0 - E_{\boldsymbol{k}} + i\Gamma_{\boldsymbol{k}}} - \frac{1}{k^0 + E_{\boldsymbol{k}} + i\Gamma_{\boldsymbol{k}}} \right) \tag{3.7}$$

   We will not include the effect of self-energy correction in the Wightman correlator because if affects λ_L at a sub-leading order.

3. Rung diagram (figure 3.5(b) : This is a qualitatively new type of diagram when each interaction vertex is attached to different time-fold. This diagram will be analyzed in the section 3.6 in detail.

3.5 Self energy correction

The Schwinger-Dyson equation for incorporating self energy in the propagator is given by

$$G(k)^{-1} = G_0(k)^{-1} + \Pi(k) \tag{3.8}$$

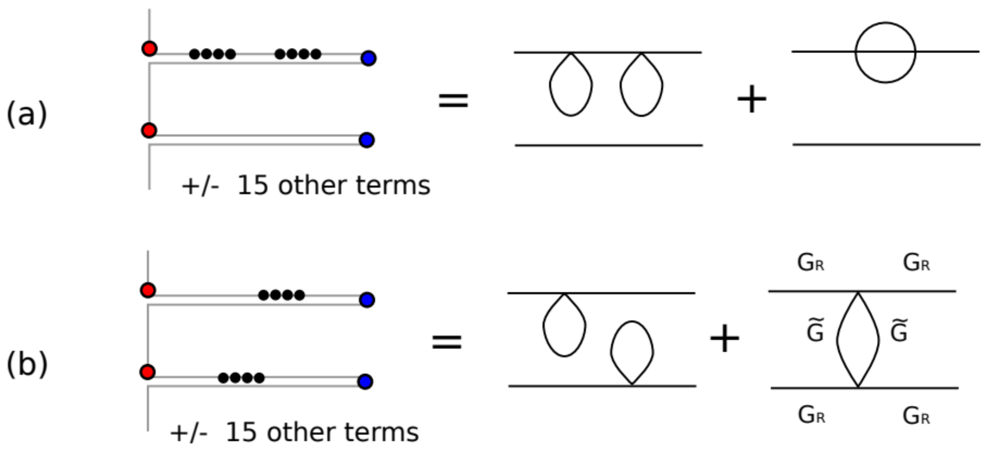

FIGURE 3.5: Diagrams at order λ^2 [6]

where G is the 2-point correlator. Using $G_R(k^0, \mathbf{k}) = -iG(k^0 + i\epsilon, \mathbf{k})$ and under the assumption that Π is small, we can write

$$G_R(k)^{-1} = -i(k^0 + i\Gamma_{\mathbf{k}})^2 + i(E_{\mathbf{k}}^2 + \Sigma_{\mathbf{k}}) \tag{3.9}$$

where we have defined

$$2k^0\Gamma_{\mathbf{k}} = -\operatorname{Im}\{\Pi(k)\} \qquad \Sigma_{\mathbf{k}} = \operatorname{Re}\{\Pi(k)\} \tag{3.10}$$

3.5.1 One-loop self energy

As obvious from the diagram below, the contribution of one-loop free energy does not depend upon the external momenta. The one loop free energy can be obtained by

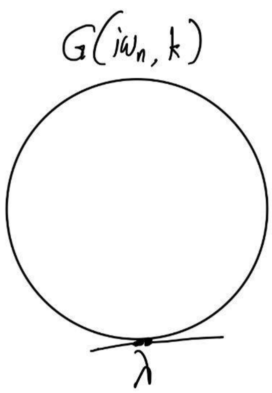

FIGURE 3.6: One loop diagram

summing over the internal momentum at finite temperature in the free propagator shown in the diagram 3.6. Since, this is only significant in the limit of zero bare mass, we write self energy at $m = 0$ as

$$\Pi = \frac{8\lambda}{\beta} \sum_n \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{w_n^2 + \mathbf{p}^2} \tag{3.11}$$

To look at self energy due to temperature T, we subtract the divergent zero temperature contribution from the above integral to get the thermal mass $m_{th}^2 = \frac{2\lambda}{3\beta^2}$.

3.5.2 Two-loop self energy

From the diagram 3.7, we get the two-loop self energy correction

$$\Pi(\tau) = -16\lambda^2 G^3(\tau) \tag{3.12}$$

We are only interested in Imaginary part of the self energy, at $p^0 = E_{\mathbf{p}}$.

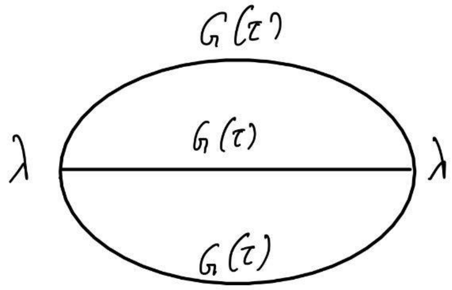

FIGURE 3.7: Two loop diagram

On straightforward evaluation, we can obtain the following expression for $\Gamma_{\mathbf{p}}$

$$\Gamma_{\mathrm{p}} = \frac{\sinh \frac{\beta E_{\mathrm{p}}}{2}}{6E_{\mathrm{p}}} \int \frac{d^4k}{(2\pi)^4} R(p-k)\tilde{G}(k) \tag{3.13}$$

where R is the rung function defined later in equation 3.15.

3.6 Rung Diagram

To evaluate the contribution of diagram 3.5(c), it is beneficial to go to the Fourier space. However, this is troublesome since we expect the function to be exponentially growing, for which the Fourier transform does not exist. Therefore, we will do a Laplace-like transform and define $C(w) = \int_0^\infty \mathrm{d}t e^{iwt} C(t)$. This transformation can be defines by the $i\epsilon$ prescription since the integrand will decay at long times. To recover back $C(t)$, we can simply use Mellin's inverse formula.

In the frequency space, we can directly write the contribution diagram 3.5(c) as

$$C_{rung}(w) = \frac{1}{N^2}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{d}^4q}{(2\pi)^4} G_R(w-p)G_R(p)R(p-q)G_R(w-q)G_R(q) \tag{3.14}$$

where $w-p$ refers to the four momentum $(w-p^0, -p^1, -p^2, -p^3)$. In the above equation, we have defined the Rung Propagator $R(p)$ for later use

$$R(p) = 48\lambda^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\widetilde{G}(p/2+k)\widetilde{G}(p/2-k) \tag{3.15}$$

The combinatorial factor of 48 arises since the one-rung diagram has three index structures, and each structure has 16 equivalent diagrams which must be summed over. The explicit evaluation of the Rung function gives the following form, which will be useful for later calculations

$$R(k) = \frac{6\lambda^2}{\pi\beta|\mathbf{k}|\sinh\frac{|k^0|\beta}{2}}\left[\theta\left((k^0)^2 - \mathbf{k}^2 - 4m^2\right)\log\frac{\sinh x_+}{\sinh x_-} + \theta\left(-(k^0)^2+\mathbf{k}^2\right)\log\frac{1-e^{-2x_+}}{1-e^{2x_-}}\right] \tag{3.16}$$

where the variables $x_\pm$ are defined by

$$x_\pm = \frac{\beta}{4}\left(|k^0| \pm |\mathbf{k}|\sqrt{1+\frac{4m^2}{\mathbf{k}^2 - (k^0)^2}}\right) \tag{3.17}$$

Since we are only interested in the leading time behavior of $C(t)$, we can further simplify the contribution of one rung diagram. This simplification will also make it evident why we are concerned only with powers of $\lambda^2 t$ in the expansion. In equation 3.14, two pairs of retarded propagators appear and each pair $G_R(w-p)G_R(p)$ is given by

$$-\frac{1}{4E_{\mathbf{p}}^2}\left(\frac{1}{p^0 - E_{\mathbf{p}} + i\epsilon} - \frac{1}{p^0 + E_{\mathbf{p}} + i\epsilon}\right)\left(\frac{1}{\omega - p^0 - E_{\mathbf{p}} + i\epsilon} - \frac{1}{\omega - p^0 + E_{\mathbf{p}} + i\epsilon}\right) \tag{3.18}$$

On integration on p^0, we will need to apply the Cauchy integral theorem and will have to take residues at the poles. These residues will generate terms proportional to w^{-1}, $(w+2E_{\mathbf{p}})^{-1}$ and $(w-2E_{\mathbf{p}})^{-1}$. Similar terms will arise when we integrate over the the other pair. Since we know that the fastest growing term in real time correspond to double poles of w in Laplace transform, we make the following change to retain only the terms which give the double pole on integration using Cauchy formula

$$G_R(w-p)G_R(p) \to -\frac{\pi i}{2E_{\mathbf{p}}^2}\frac{\delta(p^0 - E_{\mathbf{p}}) + \delta(p^0 + E_{\mathbf{p}})}{w + 2i\epsilon} \tag{3.19}$$

This was done for free propagator. Since the free energy correction occurs only at order λ, we can incorporate the effects of free energy by doing the same modification for dressed propagators.

$$G_R(w-p)G_R(p) \to -\frac{\pi i}{2E_{\mathbf{p}}^2}\frac{\delta(p^0-E_{\mathbf{p}})+\delta(p^0+E_{\mathbf{p}})}{w+2i\Gamma} \tag{3.20}$$

Because of this substitution, the calculation of $C_{rung}(w)$ will have a double pole at $w=0$. Hence, as specified earlier, the run diagram contributes at order $\lambda^2 t$.

It should be noted that all the Dirac functions appearing in the above expression and in the Wightman functions of the rung propagator forces the integration momenta to be on-shell.

3.7 Building the Ladder

The equation 3.14 can be visualised using the diagram 3.8.

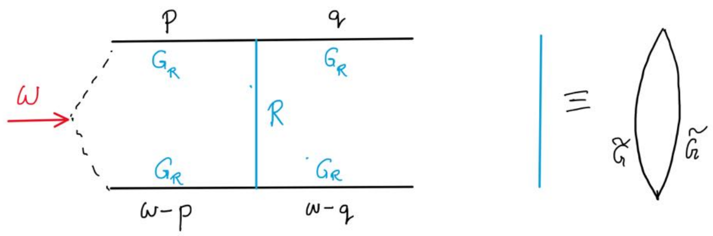

FIGURE 3.8: Diagram corresponding to equation 3.14

In the above diagram, four momentum $w=(w,0,0,0)$ is inserted from left, and we need to integrate over all internal momenta. We need to take care that the horizontal propagators in the diagram are retarded propagators.

At each higher order in λ, the fastest growing contribution will arise from the the additional rung diagram. Any other possible planar diagram at a given order of λ will have lower order of pole, and hence will not contribute to the fastest growing term. In the figure 3.9, the order of λ is 4 while the order of pole at $w=0$ is again 2. This will give subleading contribution of type $\lambda^4 t$.

Now it is evident that rung diagrams will give contribution $(\lambda^2 t)^n$, where n is the number of rungs. For further calculation, we define $f(w,p)$ by the equation

$$C(w) = \frac{1}{N^2}\int \frac{\mathrm{d}^4p}{(2\pi)^4} f(w,p) \tag{3.21}$$

where p is the four-momentum distributed initially between the two side rails.

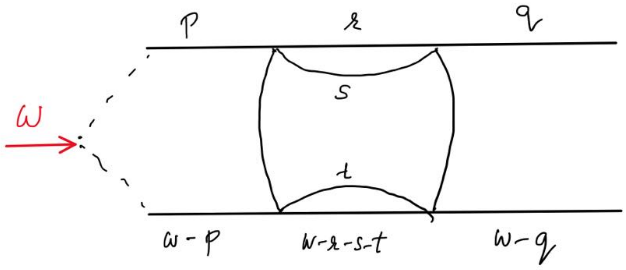

FIGURE 3.9: Diagram not contributing to leading term.

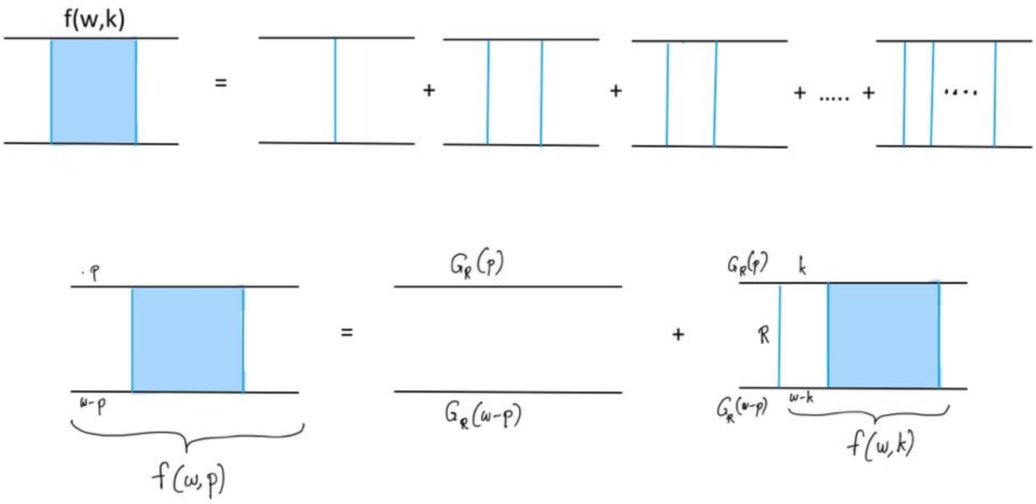

FIGURE 3.10: Ladder Diagrams for $f(w, p)$

As shown in figure 3.10, $f(w, p)$ is sum of infinite terms with increasing number of ladders. We can write the following equation by which must be satisfied by these diagrams

$$f(\omega, p) = -G_R(p)G_R(\omega - p)\left[1 + \int \frac{d^4k}{(2\pi)^4} R(k-p)f(\omega, k)\right] \tag{3.22}$$

This is an inhomogeneous equation, and we can find $C(t)$ by first doing an inverse Fourier transform (more accurately, inverse Laplace transform) on the w variable to get $f(t, p)$ and then integrating over the four momentum p. We expect that $f(t, p)$ is an exponentially increasing function at large time. However, the inverse Fourier transform of the first term give a decaying contribution, and hence we can ignore it when attempting to find the growth exponent.

We substitute the form of G_R obtained in equation 3.20 in the expression. This substitution forces the four momentum p to be on shell, which allows us to define h according to

$$f(w,p) = \delta((p^0)^2 - E_{\mathbf{p}}^2)h(w,\mathbf{p}) \tag{3.23}$$

After substituting approximate expression for G_R and this definition in equation 3.22, we get the following homogeneous equation on integration over k^0

$$-i\omega h(\omega,\mathbf{p}) = -2\Gamma_{\mathbf{p}} h(\omega,\mathbf{p}) + \int \frac{d^3k}{(2\pi)^3} m(\mathbf{k},\mathbf{p}) h(\omega,\mathbf{k}) \tag{3.24}$$

where the kernel $m(\mathbf{k},\mathbf{p})$ obtained by integrating over k^0 can be specified in terms of rung function as

$$m(\mathbf{k},\mathbf{p}) = \frac{R(k_+) + R(k_-)}{4E_{\mathbf{k}}E_{\mathbf{p}}} \qquad k_\pm = (E_{\mathbf{k}} \pm E_{\mathbf{p}}, \mathbf{k} - \mathbf{p}) \tag{3.25}$$

As claimed before, decay rate Γ influences the growth exponents. Writing the decay rate in terms of the kernel $m(\mathbf{k},\mathbf{p})$, we get the following equation

$$-i\omega h(\omega,\mathbf{p}) = \int \frac{d^3k}{(2\pi)^3} m(\mathbf{k},\mathbf{p}) \left(h(\omega,\mathbf{k}) - \frac{\sinh\frac{\beta E_{\mathbf{p}}}{2}}{3\sinh\frac{\beta E_{\mathbf{k}}}{2}} h(\omega,\mathbf{p}) \right) \tag{3.26}$$

In the real space, the above equation becomes

$$\frac{d}{dt}h(t,\mathbf{p}) = \sum_{\mathbf{k}} M_{\mathbf{pk}} h(t,\mathbf{k}) \tag{3.27}$$

where M is the integral operator specified above. We can now find λ_L by finding the largest eigenvalue of the matrix M, which can be done numerically.

3.8 Result and Discussion

By solving the equation 3.27, the following results can be obtained

Case	λ_L
$\beta m \ll 1$	$0.025\frac{\lambda^2}{\beta^2 m}$
$\beta m \gg 1$	Decays exponentially with βm
$m = 0$	$0.025\frac{\lambda^2}{\beta^2 m_{th}}$

In the last case when $m = 0$, we need to use the thermal mass arising because of the one-loop correction.

The computation for λ_L was done here only under the limits $\lambda = N^2 g \to 0$ and $N \to \infty$. Therefore, the correction to the λ_L are suppressed by either a factor of λ or by $\frac{1}{N}$.

[7]

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