## Singular Value Decomposition (SVD)

**Definition:** SVD decomposes any matrix $A$ (not necessarily square) into $A = U\Sigma V^T$, where:

- $U$: Matrix of left singular vectors.
- $\Sigma$: Diagonal matrix of singular values (representing the variance captured).
- $V^T$: Matrix of right singular vectors.

**Use in Dimensionality Reduction:** Select the top $k$ singular values and their corresponding vectors to approximate the original data in a lower-dimensional space.

**Applications:** Latent semantic analysis (LSA) in text processing, image compression, and collaborative filtering.

- Carry out SVD on $A = \begin{bmatrix} 1 & 1 \\ 7 & 7 \end{bmatrix}_{mxn}$

$$SVD\ of\ A = U_{mxm}\Sigma_{mxn}V_{nxn}{}^T$$

$$\text{Let A} = \begin{bmatrix} 1 & 1 \\ 7 & 7 \end{bmatrix}$$

$$\text{Transpose A to get } A^T = \begin{bmatrix} 1 & 7 \\ 1 & 7 \end{bmatrix}$$

Calculate $A \times A^T$

$$A \times A^T = \begin{bmatrix} 1 & 1 \\ 7 & 7 \end{bmatrix} \times \begin{bmatrix} 1 & 7 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} \overset{1+1}{2} & \overset{7+7}{14} \\ 14 & 98 \end{bmatrix}$$

Calculate $A^T \times A$

$$A^T \times A = \begin{bmatrix} 1 & 7 \\ 1 & 7 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 7 & 7 \end{bmatrix} = \begin{bmatrix} 50 & 50 \\ 50 & 50 \end{bmatrix}$$

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To find the value of left singular vector U,

let's find the eigen values and eigenvectors of $A \times A^T$

$$A \times A^T = \begin{bmatrix} 2 & 14 \\ 14 & 98 \end{bmatrix}$$

$$A^T \times A = \begin{bmatrix} 50 & 50 \\ 50 & 50 \end{bmatrix}$$

To find the value of left singular vector U,

let's find the eigen values and eigenvectors of $A \times A^T$

$AA^T - \lambda I = 0$

$$A \times A^T = \begin{bmatrix} 2 & 14 \\ 14 & 98 \end{bmatrix}$$

$$A^T \times A = \begin{bmatrix} 50 & 50 \\ 50 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & 14 \\ 14 & 98-\lambda \end{bmatrix} = 0$$

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$$A^T \times A = \begin{bmatrix} 50 & 50 \\ 50 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & 14 \\ 14 & 98-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)(98-\lambda) - 14 \times 14 = 0$$

$$\lambda^2 - 100\lambda + 196 - 196 = 0$$

$$\lambda^2 - 100\lambda = 0$$

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$$(2-\lambda)(98-\lambda) - 14 \times 14 = 0$$

$$\lambda^2 - 100\lambda + 196 - 196 = 0$$

$$\lambda^2 - 100\lambda = 0$$

$\lambda(\lambda - 100) = 0$

Hence $\lambda = 0$ or $\lambda = 100$

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$$\begin{bmatrix} 2-\lambda & 14 \\ 14 & 98-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)(98-\lambda) - 14 \times 14 = 0$$

$$\lambda^2 - 100\lambda + 196 - 196 = 0$$

$$\lambda^2 - 100\lambda = 0$$

$\lambda(\lambda - 100) = 0$

Hence $\lambda = 0$ or $\lambda = 100$

Let's find the 1st eigenvector corresponding to $\lambda = 0$. Let's call it $v = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} 2 & 14 \\ 14 & 98 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = 0 \times \begin{bmatrix} x \\ y \end{bmatrix}$$

$AA^T \cdot v = \lambda \cdot v$


To find the value of left singular vector U,

$AA^T - \lambda I = 0$

$$A \times A^T = \begin{bmatrix} 2 & 14 \\ 14 & 98 \end{bmatrix}$$

$$A^T \times A = \begin{bmatrix} 50 & 50 \\ 50 & 50 \end{bmatrix}$$

let's find the eigen values and eigenvectors of $A \times A^T$

$$\begin{bmatrix} 2-\lambda & 14 \\ 14 & 98-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)(98-\lambda) - 14 \times 14 = 0$$

$$\lambda^2 - 100\lambda + 196 - 196 = 0$$

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$\lambda(\lambda - 100) = 0$

Hence $\lambda = 0$ or $\lambda = 100$

Let's find the 1st eigenvector corresponding to $\lambda = 0$. Let's call it $v = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} 2 & 14 \\ 14 & 98 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = 0 \times \begin{bmatrix} x \\ y \end{bmatrix}$$

$AA^T \cdot v = \lambda \cdot v$


$$\begin{bmatrix} 2x + 14y \\ 14x + 98y \end{bmatrix} = 0$$

$x + 7y = 0$

Hence, $x = -7y$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ 1 \end{bmatrix}$$

To make the length of this eigenvector as 1, divide the eigenvector with its current length

$$\sqrt{(-7)^2 + (1)^2} = \sqrt{50} = 7.07$$

Hence, 1st eigenvector of unit length is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7/7.07 \\ 1/7.07 \end{bmatrix} = \begin{bmatrix} -0.99 \\ 0.14 \end{bmatrix}$$


To find the value of left singular vector U,

$$A \times A^T = \begin{bmatrix} 2 & 14 \\ 14 & 98 \end{bmatrix}$$

$$A^T \times A = \begin{bmatrix} 50 & 50 \\ 50 & 50 \end{bmatrix}$$

let's find the eigen values and eigenvectors of $A \times A^T$

$$\begin{bmatrix} 2-\lambda & 14 \\ 14 & 98-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)(98-\lambda) - 14 \times 14 = 0$$

$$\lambda^2 - 100\lambda + 196 - 196 = 0$$

$$\lambda^2 - 100\lambda = 0$$

Hence $\lambda = 0$ or $\lambda = 100$


Let's find the 2nd eigenvector corresponding to $\lambda = 100$. Let's call it $v = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} 2 & 14 \\ 14 & 98 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = 100 \times \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} -98x + 14y \\ 14x - 2y \end{bmatrix} = 0$$

$7x - y = 0$

$7x = y$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

To make the length of this eigenvector as 1, divide the eigenvector with its current length

$$\sqrt{(1)^2 + (7)^2} = \sqrt{50} = 7.07.$$

2nd eigenvector of unit length is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/7.07 \\ 7/7.07 \end{bmatrix} = \begin{bmatrix} 0.14 \\ 0.99 \end{bmatrix}$$

To find the value of left singular vector U,

$$A \times A^T = \begin{bmatrix} 2 & 14 \\ 14 & 98 \end{bmatrix}$$

$$A^T \times A = \begin{bmatrix} 50 & 50 \\ 50 & 50 \end{bmatrix}$$

let's find the eigen values and eigenvectors of $A \times A^T$

$$\begin{bmatrix} 2-\lambda & 14 \\ 14 & 98-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)(98-\lambda) - 14 \times 14 = 0$$

$$\lambda^2 - 100\lambda + 196 - 196 = 0$$

$$\lambda^2 - 100\lambda = 0$$

Hence $\lambda = 0$ or $\lambda = 100$

You sort the eigenvectors according to eigenvalues.

$$\begin{array}{ccc} & 100 & 0 \\ U = & \begin{bmatrix} 0.14 & -0.99 \\ 0.99 & 0.14 \end{bmatrix} \end{array}$$

To find the value of right singular vector $V^T$

$A^T A - \lambda I = 0$

$$A \times A^T = \begin{bmatrix} 2 & 14 \\ 14 & 98 \end{bmatrix}$$

$$A^T \times A = \begin{bmatrix} 50 & 50 \\ 50 & 50 \end{bmatrix}$$

let's find the eigenvalues and eigenvectors of $A^T \times A$

$$\begin{bmatrix} 50-\lambda & 50 \\ 50 & 50-\lambda \end{bmatrix} = 0$$

$$(50-\lambda)(50-\lambda) - 50 \times 50 = 0$$

$$\lambda^2 - 100\lambda + 2500 - 2500 = 0$$

$$\lambda^2 - 100\lambda = 0$$

Hence $\lambda = 0$ or $\lambda = 100$

Let's find the 1st eigenvector corresponding to $\lambda = 0$. Let's call it $v = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} 50 & 50 \\ 50 & 50 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = 0 \times \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 50x + 50y \\ 50x + 50y \end{bmatrix} = 0$$

Hence, $x = -y$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

To make the length of this eigenvector as 1, divide the eigenvector with its current length

$$\sqrt{(-1)^2 + (1)^2} = \sqrt{2} = 1.41$$

Hence, 1st eigenvector of unit length is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1/1.41 \\ 1/1.41 \end{bmatrix} = \begin{bmatrix} -0.70 \\ 0.70 \end{bmatrix}$$

To find the value of right singular vector $V^T$

$$A \times A^T = \begin{bmatrix} 2 & 14 \\ 14 & 98 \end{bmatrix}$$

$$A^T \times A = \begin{bmatrix} 50 & 50 \\ 50 & 50 \end{bmatrix}$$

let's find the eigenvalues and eigenvectors of $A^T \times A$

$$\begin{bmatrix} 50-\lambda & 50 \\ 50 & 50-\lambda \end{bmatrix} = 0$$

$$(50-\lambda)(50-\lambda) - 50 \times 50 = 0$$

$$\lambda^2 - 100\lambda + 2500 - 2500 = 0$$

$$\lambda^2 - 100\lambda = 0$$

Hence $\lambda = 0$ or $\lambda = 100$

Let's find the 2nd eigenvector corresponding to $\lambda = 100$. Let's call it $v = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} 50 & 50 \\ 50 & 50 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = 100 \times \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} -50x + 50y \\ 50x - 50y \end{bmatrix} = 0$$

Hence, $x = y$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To make the length of this eigenvector as 1, divide the eigenvector with its current length

$$\sqrt{(1)^2 + (1)^2} = \sqrt{2} = 1.41$$

2nd eigenvector of unit length is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/1.41 \\ 1/1.41 \end{bmatrix} = \begin{bmatrix} 0.70 \\ 0.70 \end{bmatrix}$$

To find the value of right singular vector $V^T$

let's find the eigenvalues and eigenvectors of $A^T \times A$

$$A \times A^T = \begin{bmatrix} 2 & 14 \\ 14 & 98 \end{bmatrix}$$

$$A^T \times A = \begin{bmatrix} 50 & 50 \\ 50 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 50-\lambda & 50 \\ 50 & 50-\lambda \end{bmatrix} = 0$$

$$(50-\lambda)(50-\lambda) - 50 \times 50 = 0$$

$$\lambda^2 - 100\lambda + 2500 - 2500 = 0$$

$$\lambda^2 - 100\lambda = 0$$

Hence $\lambda = 0$ or $\lambda = 100$

You sort the eigenvectors according to eigenvalues.

$$V^T = \begin{bmatrix} 0.70 & -0.70 \\ 0.70 & 0.70 \end{bmatrix}$$

Singular values are square root of eigenvalues and are sorted in descending order.

It is a matrix of $m \times n$ where all elements are 0 except the diagonal elements.

$$A \times A^T = \begin{bmatrix} 2 & 14 \\ 14 & 98 \end{bmatrix}$$

$$A^T \times A = \begin{bmatrix} 50 & 50 \\ 50 & 50 \end{bmatrix}$$

$$\varepsilon = \begin{bmatrix} \sqrt{100} & 0 \\ 0 & \sqrt{0} \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, SVD of $\begin{bmatrix} 1 & 1 \\ 7 & 7 \end{bmatrix}$ can be written as following.

$$\begin{bmatrix} 1 & 1 \\ 7 & 7 \end{bmatrix} = U \varepsilon V^T = \begin{bmatrix} 0.14 & -0.99 \\ 0.99 & 0.14 \end{bmatrix} \times \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0.70 & -0.70 \\ 0.70 & 0.70 \end{bmatrix}$$

$$= \begin{bmatrix} 0.98 & -0.98 \\ 6.93 & -6.93 \end{bmatrix}$$