Discrete Mathematics
Types of Proofs – Predicate Logic

The most basic form of logic is propositional logic. Propositions, which have no variables, are the only assertions that are considered. Because there are no variables in propositions, they are either always true or always false.

Example –

1. **P:** 2 + 4 = 5. (Always false) is a proposition.
2. **Q:** y * 0 = 0. (Always true) is a proposition.

The majority of mathematical conclusions are expressed as implications: P and Q : P ⇒Q

We know that

P	Q	P ⇒ Q
T	T	T
T	F	F
F	T	T
F	F	T

Types Of Proofs

Let's say we want to prove the implication P ⇒ Q. Here are a few options for you to consider.

1. Trivial Proof –

If we know Q is true, then P ⇒ Q is true no matter what P's truth value is.

Example –

If there are 1000 employees in an organization, then $3^2= 9$.

Explanation –

Let P: There are 1000 employees in organization & Q: $3^2= 9$.

We know that Q is always true and in the truth table we can see that whenever Q is true, P ⇒ Q is true, whatever the true value of P is.

2. Vacuous Proof –

If P is a conjunction (example : P = A ^ B ^ C)of other hypotheses and we know one or more of these hypotheses is false, then P is false and so P → Q is vacuously true regardless of the truth value of Q.

Example –

If 5 ! = 100, then 3 ! = 6.

Explanation –

Let P : 5 ! = 100, & Q : 3 ! = 6.

We know that P is always false and in the truth table we can see that whenever P is False, $P \Rightarrow Q$ is true, whatever the truth value of Q is.

3. Direct Proof –

Assume P, and then prove Q using inference rules, axioms, definitions, and logical equivalences.

Example –

For all integers p and q, if p and q are odd integers, then p + q is an even integer. Let P denotes: p and q are odd integers

Q: p + q is an even integer To Prove: $P \Rightarrow Q$

Proof –

As p & q are odd integers, they can be represented as:

Assume: p = 2m + 1 and q = 2n + 1, where m & n are also some integers.

Then: p + q =

= (2m + 1) + (2n +1) (Substitution Law)

= am + 2n + 2 (associative and commutative law for addition)

= 2(m + n + 1) (distributive law)

= Number divisible by 2 & hence an even number.

4. Proof By Contradiction –

We start with the assumption that the hypotheses are correct and the conclusion is incorrect, and we try to find a contradiction.

Proof by contradiction is legitimate because: $\neg(P \wedge \neg Q)$ is equivalent to $P \Rightarrow Q$

If we can prove that $(P \wedge \neg Q)$ is false, then $\neg(P \wedge \neg Q)$ is true, and the equivalent statement $P \Rightarrow Q$ is likewise true.

Example –

Let x and y be real numbers. If 5a + 25b = 156, then a or b is not an integer.

Proof –

Let P: 5a + 25b = 156 & Q: a or b is not an integer

$\neg Q$: a or b is an integer

So, we assume that both a and b are integers ($\neg Q$) $\Rightarrow$ 5(a + 5b) = 156 (distributive law)

$\Rightarrow$ Since a and b are integers, this implies 156 is divisible by 5.

The integer 156, however, is anyway not divisible by 5. This contradiction gives the result.

It implies that $(P \wedge \neg Q)$ is false as P is false

then$\neg(P \wedge \neg Q)$ is true and the equivalent statement $P \Rightarrow Q$ is likewise true.

5. Proof by Contrapositive –

We can prove $P \Rightarrow Q$ indirectly by showing that $\neg Q \Rightarrow \neg P$. Assume $\neg Q$, and then prove $\neg P$ using inference rules, axioms, definitions, and logical equivalences.

Example: For all integers a and b, if a*b is even, then a is even or b is even.

Proof : We prove the contrapositive of the statement:

Let P: a*b is even & Q : a is an even integer or b is an even integer.

Then: $\neg P$: a*b is odd

$\neg Q$: a and b are odd integers

Say $\neg Q$ is true, i.e., a and b are both odd integers

a = 2m + 1 and b = 2n + 1 ; where m and n are integers.

Then: a*b= (2m + 1)(2n + 1) (by substitution)

= 4mn + 2m + 2n + 1 (by associative, commutative & distributive laws)

= 2(2mn + m + n) + 1 (by distributive law)

Since a*b is twice an integer (As: 2mn + m + n is also an integer) plus 1, a*b is odd. So it shows that $\neg Q \Rightarrow \neg P$. Hence $P \Rightarrow Q$

Q. Prove that: n can be odd if and only if n^2 is odd.

Solution:

We must prove two implications in order to prove this statement:

1. If n is odd, n^2 is odd
2. If n^2 is odd, n is odd

Assume –

P: n is odd & Q : n2 is odd.

1. $P \Rightarrow Q$: We are using direct proof to prove it. Assume n is an odd integer.

Then: n= 2p+ 1; for some integer p.

Then $n^2 = (2p+1)^2 = 4p^2 + 4p + 1 = 2(2p^2 + 2a) + 1$,; which is 2*(some integer) + 1. Thus, we can say that n^2 is odd. Thus $P \Rightarrow Q$

2. $Q \Rightarrow P$:

We are using contra-positive proof way here.

$\neg Q$: n^2 is even and $\neg P$: n is even.

We need to prove: $\neg P \Rightarrow \neg Q$ ($\neg P \Rightarrow \neg Q$ means that $Q \Rightarrow P$)

Assume n is an even integer, Then n= 2 ; for some integer p.

Then $n2 = (2p)^2 = 4p^2 = 2(2p^2)$, which is an even integer as it is divisible by 2. From(1.) $P \Rightarrow Q$ & from (2) $Q \Rightarrow P$, n can be odd if and only if n2 is odd.

Q. If a number is divisible by 4, then it is also divisible by 2.

Solution:

Using Direct Proof:

Assume: x is divisible by 4

Then: x = k * 4; where k is some integer (by definition of division) So, x = k * (2 * 2)

So, x = (k * 2)* 2 (Associative property of multiplication) so, x = P * 2 where P = k * 2; is an integer.

Thus, we can say that x is divisible by 2 also.

Q. By using contradiction, prove that: If y + y = y then y = 0.

Solution:

Let P: y +y = y & Q : y = 0

To prove: $(P \wedge \neg Q)$ is false as $(P \wedge \neg Q)$ is false, then $\neg(P \wedge \neg Q)$ is true, and the equivalent statement $P \Rightarrow Q$ is likewise true.

P: $y + y = y$ and $\neg Q$: $y \sim= 0$.

$(P \wedge \neg Q)$ means: Then $2y = y$ and as $y \sim= 0$ we can divide both sides by y. As a result, we get: $2 = 1$, which is a contradiction.

So, $(P \wedge \neg Q)$ is false and hence $P \Rightarrow Q$ is true.

Solved Examples on Types of Proofs – Predicate Logic

Example 1: Direct Proof

Prove: For all real numbers x and y, if $x > 0$ and $y > 0$, then $xy > 0$.

Proof: Let x and y be arbitrary real numbers such that $x > 0$ and $y > 0$.

Since $x > 0$ and $y > 0$, their product xy must be positive due to the properties of multiplication of positive numbers.

Therefore, $xy > 0$.

Thus, we have shown that for all real numbers x and y, if $x > 0$ and $y > 0$, then $xy> 0$.

Example 2: Proof by Contradiction

Prove: There is no largest integer.

Proof: Assume, for the sake of contradiction, that there is a largest integer n.

Consider the number $n + 1$.

Since n is an integer, $n + 1$ is also an integer.

By definition, $n + 1 > n$.

This contradicts our assumption that n is the largest integer.

Therefore, our assumption must be false, and there is no largest integer.

Example 3: Proof by Contraposition

Prove: For all integers n, if n^ 2 is even, then n is even.

Proof: We will prove the contrapositive: If n is odd, then n^2 is odd.

Let n be an odd integer. Then $n = 2k + 1$ for some integer k.

n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1

Let m = 2k^2 + 2k. Then n^2 = 2m + 1, which is an odd number.

Therefore, if n is odd, then n^2 is odd.

By contraposition, if n^2 is even, then n is even.

Example 4: Proof by Cases

Prove: For all integers n, n^2 + n is even.

Proof: We will consider two cases: when n is even and when n is odd.

Case 1: n is even

If n is even, then $n = 2k$ for some integer k.

n^2 + n = (2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k), which is even.

Case 2: n is odd

If n is odd, then $n = 2k + 1$ for some integer k.

n^2 + n = (2k + 1)^2 + (2k + 1) = 4k^2 + 4k + 1 + 2k + 1 = 4k^2 + 6k + 2 =2(2k^2 + 3k + 1), which is even.

In both cases, n^2 + n is even. Therefore, for all integers n, n^2 + n is even.

Example 5: Existence Proof

Prove: There exists a positive integer n such that n^ 2 – n – 12 = 0.

Proof: We can solve this equation:

n^2 – n – 12 = 0

$(n – 4)(n + 3) = 0$

$n = 4$ or $n = -3$

Since $n = 4$ is a positive integer, we have found a value that satisfies the equation.

Therefore, there exists a positive integer n (namely, 4) such that n^2 – n – 12 = 0.

Example 6: Uniqueness Proof

Prove: There is a unique real number x such that $x + 6 = 2x – 1$.

Proof: Existence: Let's solve the equation $x + 6 = 2x – 1$

$x + 6 = 2x – 1$

$6 = x - 1$
$7 = x$
So, $x = 7$ satisfies the equation.
Uniqueness: Assume y is another solution to the equation.
Then $y + 6 = 2y - 1$
Subtracting y from both sides: $6 = y - 1$
Adding 1 to both sides: $7 = y$
Therefore, $y = 7$
But we found earlier that $x = 7$. Since $y \neq x$, our assumption that there was another solution must be false.
Thus, $x = 7$ is the unique solution to the equation $x + 5 = 2x - 1$.

Example 7: Proof using Quantifiers

Prove: $\forall x \forall y(x < y \rightarrow \exists z(x < z \wedge z < y))$ for real numbers.

Proof: Let x and y be arbitrary real numbers such that $x < y$.
Consider $z = (x + y) / 2$.
We need to show that $x < z$ and $z < y$.
$x < z$:
$x < (x + y) / 2$
$2x < x + y$
$x < y$ (which is true by our assumption)
$z < y$:
$(x + y) / 2 < y$
$x + y < 2y$
$x < y$ (which is true by our assumption)
Therefore, for any x and y where $x < y$, we have found a z such that $x < z$ and $z < y$.
Thus, $\forall x \forall y(x < y \rightarrow \exists z(x < z \wedge z < y))$ is true for real numbers.

Example 8: Proof by Induction

Prove: For all positive integers n, $1 + 2 + 3 + \ldots + n = n(n+1)/2$.

Proof: Base case: When $n = 1$, LHS = 1, RHS = $1(1+1)/2 = 1$. The statement holds for $n = 1$.
Inductive step: Assume the statement is true for some positive integer k.
That is, $1 + 2 + 3 + \ldots + k = k(k+1)/2$.
We need to prove it's true for k+1:
$1 + 2 + 3 + \ldots + k + (k+1)$
$= [k(k+1)/2] + (k+1)$ (using the inductive hypothesis)
$= [k(k+1) + 2(k+1)] / 2$
$= [(k^2 + k) + (2k + 2)] / 2$
$= (k^2 + 3k + 2) / 2$
$= [(k+1)(k+2)] / 2$
This is exactly the formula for $n = k+1$.
Therefore, by the principle of mathematical induction, the statement is true for all positive integers n.

Example 9: Disproof by Counterexample

Disprove: For all real numbers x and y, if $xy = 0$, then $x = 0$ or $y = 0$.

Proof: Counterexample:
Let $x = 0$ and $y = 5$.
Then $xy = 0 * 5 = 0$.
However, while $x = 0$, $y \neq 0$.
This counterexample shows that the statement is false. The correct statement would be:
"For all real numbers x and y, if $xy = 0$, then $x = 0$ or $y = 0$ or both."

Example 10: Proof of an If and Only If Statement

Prove: For any integer n, n is even if and only if n^ 2 is even.

Proof: We need to prove both directions:

(→) If n is even, then n^2 is even:

Let n be an even integer. Then n = 2k for some integer k.

n^2 = (2k)^2 = 4k^2 = 2(2k^2)

Since 2k^2 is an integer, n^2 is even.

(←) If n^2 is even, then n is even:

We will prove the contrapositive: If n is odd, then n^2 is odd.

Let n be an odd integer. Then n = 2k + 1 for some integer k.

n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1

Since 2k^2 + 2k is an integer, n^2 is odd.

Practice Problems on Types of Proofs – Predicate Logic

- Prove that for all real numbers x, if $x^2 > 4$, then x > 2 or x < -2.
- Prove that for all integers n, if n is odd, then n^2 is odd.
- Disprove the statement: For all real numbers x and y, if $x^2 + y^2 = 1$, then x = y.
- Prove that for all positive integers n, if n is a multiple of 6, then n is even.
- Prove that for all real numbers x and y, if x < y, then $x^2 < y^2$.
- Disprove the statement: For all real numbers x, if $x^2 > x$, then x > 1.
- Prove that for all integers a and b, if a is even and b is odd, then a + b is odd.
- Prove that for all real numbers x, if x > 0, then 1/x > 0.
- Disprove the statement: For all real numbers x, if x2 ≥ 0, then x ≥ 0.
- Prove that for all positive integers n, if n is prime, then n is odd or n = 2.

Summary

Types of proofs in predicate logic include direct proofs, proof by contraposition, proof by contradiction, and proof by cases. These techniques are used to establish the truth or falsity of mathematical statements involving quantifiers and predicates. Understanding these proof methods is crucial for developing logical reasoning skills and solving complex mathematical problems.

- **What is predicate logic?**

Predicate logic is an extension of propositional logic that allows for more complex statements using quantifiers and predicates.

- **What are the main types of proofs in predicate logic?**

The main types are direct proofs, proof by contraposition, proof by contradiction, and proof by cases.

- **What is a direct proof?**

A direct proof starts with known facts and uses logical deductions to reach the desired conclusion.

- **What is proof by contraposition?**

Proof by contraposition involves proving the logically equivalent statement “if not Q, then not P” instead of “if P, then Q”.

- **What is proof by contradiction?**

Proof by contradiction assumes the negation of the statement to be proved and shows that this leads to a logical contradiction.