Solution: Problems on Pigeonhole Principle

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1. **Prove that in a set containing n positive integers there must be a subset such that the sum of all numbers in it is divisible by n.**

[10 marks]

Answer:

Let the n positive integers be $a_1, a_2, \dots, a_n$. Consider 'n' new positive integers

$$b_1 = a_1,$$
$$b_2 = a_1 + a_2,$$
$$b_3 = a_1 + a_2 + a_3,$$
$$\dots \dots$$
$$b_n = a_1 + a_2 + \cdots + a_n.$$

Then all the 'n' values are distinct. When some of $b_1, b_2, \dots, b_n$ is divisible by n, the conclusion is proven. Otherwise if all b_i 's are not divisible by n, then their remainders are all not zero, i.e. at most they can take 'n-1' different values. By pigeonhole principle there must be b_i and b_j with $i < j$ such that $b_j - b_i \neq 0$ is divisible by 'n'.

Since $b_j - b_i = a_{i+1} + a_{i+2} + \cdots + a_j$, is a sum of some given numbers, the conclusion is proven.

2. **In a bag there are some balls of the same size that are coloured by 7 colours and for each colour the number of balls is 77. At least how many balls are needed to be picked out at random to ensure that one can obtain 7 groups of 7 balls each such that in each group the balls are homochromatic?**

[10 marks]

Answer:

For this problem, it is natural to let each colour be one pigeonhole and a ball drawn be a pigeon. At the first step, for getting a group of 7 balls with the same colour, at least 43 balls are needed to be picked out from the bag at random, since if only 42 balls are picked out, there may be exactly 6 for each colour.

By pigeonhole principle, there must be one colour such that at least $\left\lfloor \frac{43}{7} \right\rfloor + 1 = 7$ drawn balls have the same colour.

Next, after getting the first group, it is sufficient to pick out from the bag another 7 balls for getting 43 balls again. Then, by the same reason, the 2nd group of 7 homochromatic drawn balls can be obtained. Repeating this process for 6 times, the 7 groups of 7 homochromatic balls are obtained. Thus, the least number of drawn balls is $43 + 6 \times 7 = 85$.

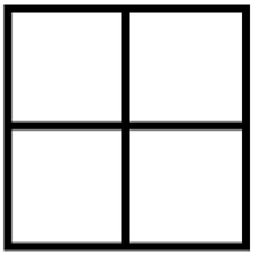
3. **There are 5 points in a square of side length 2. Prove that there exist 2 of them having a distance not more than $\sqrt{2}$.**

[5 marks]

Answer:

We can argue using the pigeon-hole principle.

Suppose we divide our square into 4 smaller square regions each of side length 1 unit, like so:



These smaller square regions will be our "pigeon-holes". Now, if one selects 5 points (our "pigeons") inside the larger square, and noting that 5 is one more than the number of smaller square regions -- the pigeon-hole principle requires that 2 of these points be in at least one of the smaller squares. Since the maximum distance between two points in one of these smaller squares is the length of the diagonal of that square, namely √2, we know there exists a pair of points out of the group of five originally selected that will be within √2 units of each other.

4. **There is a sequence of 100 integers. Prove that there is a sequence of consecutive terms such that the sum of these terms is divisible by 99.**

[5 marks]

Answer:

Note that a number divisible by 99 can be expressed as a difference of two numbers which have the same remainder when divided by 99. Therefore, this problem is equivalent to finding 2 sequences of consecutive terms where one of them is a subsequence of the other and they give the same sum of elements.

Having this concept, it is natural to think of the following construction: Let S_k be the subsequence formed by the first k terms of the original sequence and A_k be its sum of elements. Note that for any $1 \leq i < j \leq 100$, S_i is contained in S_j . Since the remainder when A_k is divided by 99 has at most 99 possibilities. By Pigeonhole Principle, among $A_1, A_2, \dots, A_{100}$ there are at least $\left\lceil \frac{100}{99} \right\rceil = 2$ of them having the same remainder when divided by 99, say A_i and A_j. Then the sequence obtained by removing the first ith term from S_j (i.e., the subsequence from the (i+1)th term to the jth term) has a sum of elements divisible by 99.

5. **76 points are aligned so that each row has 19 points and each column has 4 points. (A column is perpendicular to a row.) Each point is painted in red, blue or yellow. Prove that there exists a monochromatic rectangle. (i.e. 4 vertices are of same colour) with its sides parallel to the rows and columns.**

[10 marks]

Answer:

There are $3^4 = 81$ colouring schemes for a column. However, we see that it is not necessary to have 2 identical columns for the existence of a monochromatic rectangle. We only need two columns; each has 2 points in the same colour and in the same rows. Therefore, we can reduce our pigeonholes as follows:

Denote R, B, Y as red, blue and yellow respectively. Let the 19 columns be pigeons and $(X, \{a, b\})$ be pigeonholes where X = R, B, Y and $1 \leq a < b \leq 4.$ Note that there are 4 points in each column. By Pigeonhole Principle, at least $\left\lceil \frac{4}{3} \right\rceil = 2$ of them are in the same colour. For each column, if the points at column a and b are of color X, we put it into the pigeonhole $(X, \{a, b\})$. (If it has more than 1 pair of points in the same colour, just choose any one of the pigeonholes.) There are $3 \times 4C_2 = 18$ pigeonholes. By Pigeonhole Principle, $\left\lceil \frac{19}{18} \right\rceil = 2$ of them are in the same pigeonhole, say $(X, \{a, b\})$, then we have a rectangle in colour X_1 in columns $a_1,\ b_1$.