Syntax and Semantics of Propositional Logic

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Syntax of Propositional Logic

Definition (Syntax of Propositional Logic)

Let A be a set of propositions. The set of propositional formulas (over A) is inductively defined as follows:

- Every $a \in A$ is a propositional formula over A.
- If ϕ is a propositional formula over A, then so is its negation $\neg\phi$.
- If ϕ and ψ are propositional formulas over A, then so is the conjunction $(\phi \wedge \psi)$.
- If ϕ and ψ are propositional formulas over A, then so is the disjunction $(\phi \vee \psi)$.

The implication $(\phi \rightarrow \psi)$ is an abbreviation for $(\neg\phi \vee \psi)$.
The biconditional $(\phi \leftrightarrow \psi)$ is an abbrev. for $((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi))$.

Syntax: Examples

Which of the following sequences of symbols are propositional formulas over the set of all possible letter sequences? Which kinds of formula are they (conjunction, disjunction, . . .)?

- $(A \land (B \lor C))$
- $((\text{EatFish} \land \text{Drink Water}) \rightarrow \neg\text{Eat Ice Cream})$
- $\neg(\land \text{Rain} \lor \text{StreetWet})$
- $\neg(\text{Rain} \lor \text{StreetWet})$
- $\neg(A = B)$
- $(A \land \neg(B \leftrightarrow)C)$
- $(A \lor \neg(B \leftrightarrow C))$
- $((A \leq B) \land C)$
- $((A_1 \land A_2) \lor \neg(A_3 \leftrightarrow A_2))$

Meaning of Propositional Formulas?

So far propositional formulas are only symbol sequences without any meaning.

For example, what does this mean:

$((\text{Eat Fish} \wedge \text{Drink Water}) \rightarrow \neg\text{Eat Ice Cream})$?

We need semantics!

Semantics of Propositional Logic

Definition (Semantics of Propositional Logic)

A truth assignment (or interpretation) for a set of propositions A is a function $\mathcal{I} : A \to \{0, 1\}$.

A propositional formula ϕ (over A) holds under $\mathcal{I}$ (written as $\mathcal{I} \models \phi$) according to the following definition:

$\mathcal{I} \models a$	iff	$\mathcal{I}(a) = 1$	(for $a \in A$)
$\mathcal{I} \models \neg\phi$	iff	not $\mathcal{I} \models \phi$	
$\mathcal{I} \models (\phi \wedge \psi)$	iff	$\mathcal{I} \models \phi$ and $\mathcal{I} \models \psi$	
$\mathcal{I} \models (\phi \vee \psi)$	iff	$\mathcal{I} \models \phi$ or $\mathcal{I} \models \psi$	

Question: should we define semantics of $(\phi \to \psi)$ and $(\phi \leftrightarrow \psi)$?

Semantics of Propositional Logic: Terminology

- For $I \models \phi$ we also say I is a model of ϕ
  - and that ϕ is true under I.
- If ϕ does not hold under I, we write this as $I \models \phi$
  - and say that I is no model of ϕ
  - and that ϕ is false under I.
- Note: $\models$ is not part of the formula
  - but part of the meta language (speaking about a formula).

Summary

- propositional logic based on propositions
- syntax defines what well-formed formulas are
- semantics defines when a formula is true
- interpretations are the basis of semantics