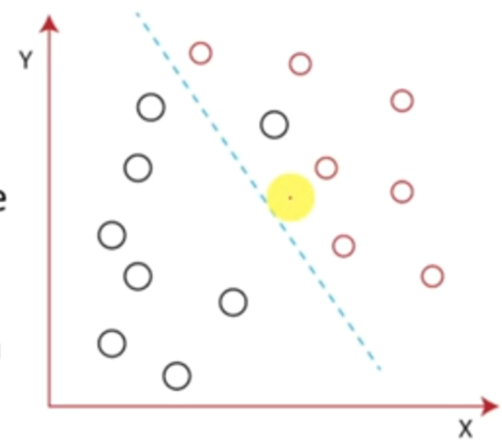
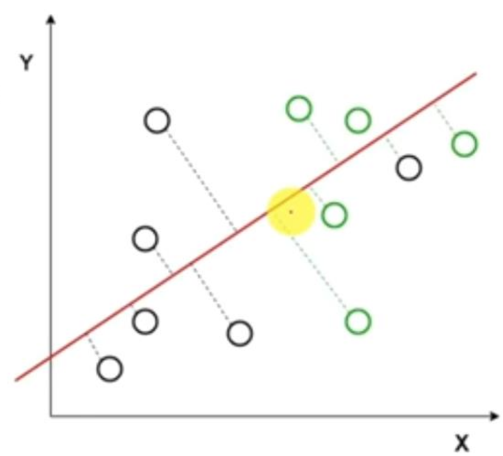
Linear Discriminant Analysis - Introduction

- Linear Discriminant Analysis or Normal Discriminant Analysis or Discriminant Function Analysis is a dimensionality reduction technique that is commonly used for supervised classification problems.
- It is used to project the features in higher dimension space into a lower dimension space.
- Suppose we have two sets of data points belonging to two different classes that we want to classify.
- When the data points are plotted on the 2D plane, there's no straight line that can separate the two classes of the data points completely.



- Here, Linear Discriminant Analysis uses both the axes (X and Y) to create a new axis and projects data onto a new axis in a way to maximize the separation of the two categories and hence, reducing the 2D graph into a 1D graph.



Linear Discriminant Analysis - Introduction

- Two criteria are used by LDA to create a new axis:
  - Maximize the distance between means of the two classes.
  - Minimize the variation within each class.

1. Compute the class means of dependent variable
2. Derive the covariance matrix of the class variable
3. Compute the within class — scatter matrix (S1+S2)
4. Compute the between class scatter matrix
5. Compute the Eigen values and eigen vectors from the within class and between class scatter matrix
6. Sort the values of eigen values and select the top k values
7. Find the eigen vectors corresponds to the top k eigen vectors
8. Obtain the LDA by taking the dot product of eigen vectors and original data

$$\mu_1 = \frac{1}{N_1}\sum_{x\in\omega_1} x$$

$$S_1 = \sum_{x\in\omega_1}(x-\mu_1)(x-\mu_1)^T$$

$$S_w = S_1 + S_2$$

$$S_B = (\mu_1-\mu_2)(\mu_1-\mu_2)^T$$

$$S_W^{-1}S_B w = \lambda w$$

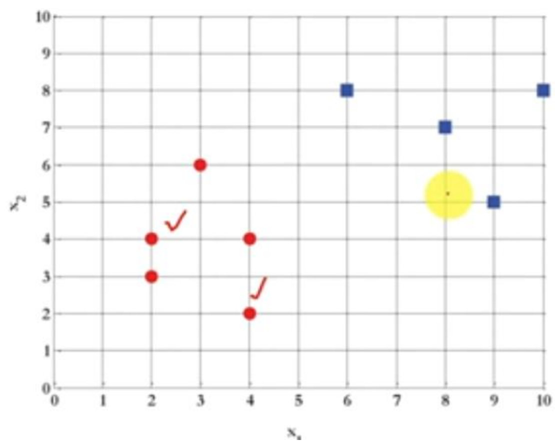
$$\left(S_W^{-1}S_B - \lambda I\right)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0$$

Linear Discriminant Analysis - Solved Example

- Compute the Linear Discriminant projection for the following two dimensional dataset.

Samples for class $\boldsymbol{\omega_1}$: $\mathbf{X_1}=(x_1,x_2)=\{(4,2),(2,4),(2,3),(3,6),(4,4)\}$

Sample for class $\boldsymbol{\omega_2}$: $\mathbf{X_2}=(x_1,x_2)=\{(9,10),(6,8),(9,5),(8,7),(10,8)\}$



Samples for class $\boldsymbol{\omega_1}$: $\mathbf{X_1}=(x_1,x_2)=\{(4,2),(2,4),(2,3),(3,6),(4,4)\}$

Sample for class $\boldsymbol{\omega_2}$: $\mathbf{X_2}=(x_1,x_2)=\{(9,10),(6,8),(9,5),(8,7),(10,8)\}$

The classes mean are :

$$\mu_1 = \frac{1}{N_1}\sum_{x\in\omega_1} x = \frac{1}{5}\left[\begin{pmatrix}4\\2\end{pmatrix}+\begin{pmatrix}2\\4\end{pmatrix}+\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}3\\6\end{pmatrix}+\begin{pmatrix}4\\4\end{pmatrix}\right]=\begin{pmatrix}3\\3.8\end{pmatrix}$$

$$\mu_2 = \frac{1}{N_2}\sum_{x\in\omega_2} x = \frac{1}{5}\left[\begin{pmatrix}9\\10\end{pmatrix}+\begin{pmatrix}6\\8\end{pmatrix}+\begin{pmatrix}9\\5\end{pmatrix}+\begin{pmatrix}8\\7\end{pmatrix}+\begin{pmatrix}10\\8\end{pmatrix}\right]=\begin{pmatrix}8.4\\7.6\end{pmatrix}$$

Linear Discriminant Analysis - Solved Example

Samples for class $\boldsymbol{\omega_1}$: $\mathbf{X_1}=(x_1,x_2)=\{(4,2),(2,4),(2,3),(3,6),(4,4)\}$

Sample for class $\boldsymbol{\omega_2}$: $\mathbf{X_2}=(x_1,x_2)=\{(9,10),(6,8),(9,5),(8,7),(10,8)\}$

Covariance matrix of the first class:

$$S_1 = \sum_{x\in\omega_1} \frac{(x-\mu_1)(x-\mu_1)^T}{N-1} = \left[\begin{pmatrix}4\\2\end{pmatrix}-\begin{pmatrix}3\\3.8\end{pmatrix}\right]\left[\begin{pmatrix}4\\2\end{pmatrix}-\begin{pmatrix}3\\3.8\end{pmatrix}\right]^T+\left[\begin{pmatrix}2\\4\end{pmatrix}-\begin{pmatrix}3\\3.8\end{pmatrix}\right]\left[\begin{pmatrix}2\\4\end{pmatrix}-\begin{pmatrix}3\\3.8\end{pmatrix}\right]^T$$

$$+\left[\begin{pmatrix}2\\3\end{pmatrix}-\begin{pmatrix}3\\3.8\end{pmatrix}\right]\left[\begin{pmatrix}2\\3\end{pmatrix}-\begin{pmatrix}3\\3.8\end{pmatrix}\right]^T+\left[\begin{pmatrix}3\\6\end{pmatrix}-\begin{pmatrix}3\\3.8\end{pmatrix}\right]\left[\begin{pmatrix}3\\6\end{pmatrix}-\begin{pmatrix}3\\3.8\end{pmatrix}\right]^T+\left[\begin{pmatrix}4\\4\end{pmatrix}-\begin{pmatrix}3\\3.8\end{pmatrix}\right]\left[\begin{pmatrix}4\\4\end{pmatrix}-\begin{pmatrix}3\\3.8\end{pmatrix}\right]^T \Big/ N-1$$

$$=\begin{pmatrix}1 & -0.25\\ -0.25 & 2.2\end{pmatrix}$$

Samples for class $\boldsymbol{\omega_1}$: $\mathbf{X_1}=(x_1,x_2)=\{(4,2),(2,4),(2,3),(3,6),(4,4)\}$

Sample for class $\boldsymbol{\omega_2}$: $\mathbf{X_2}=(x_1,x_2)=\{(9,10),(6,8),(9,5),(8,7),(10,8)\}$

Covariance matrix of the second class:

$$S_2 = \sum_{x\in\omega_2} \frac{(x-\mu_2)(x-\mu_2)^T}{N-1} = \left[\begin{pmatrix}9\\10\end{pmatrix}-\begin{pmatrix}8.4\\7.6\end{pmatrix}\right]\left[\begin{pmatrix}9\\10\end{pmatrix}-\begin{pmatrix}8.4\\7.6\end{pmatrix}\right]^T+\left[\begin{pmatrix}6\\8\end{pmatrix}-\begin{pmatrix}8.4\\7.6\end{pmatrix}\right]\left[\begin{pmatrix}6\\8\end{pmatrix}-\begin{pmatrix}8.4\\7.6\end{pmatrix}\right]^T$$

$$+\left[\begin{pmatrix}9\\5\end{pmatrix}-\begin{pmatrix}8.4\\7.6\end{pmatrix}\right]\left[\begin{pmatrix}9\\5\end{pmatrix}-\begin{pmatrix}8.4\\7.6\end{pmatrix}\right]^T+\left[\begin{pmatrix}8\\7\end{pmatrix}-\begin{pmatrix}8.4\\7.6\end{pmatrix}\right]\left[\begin{pmatrix}8\\7\end{pmatrix}-\begin{pmatrix}8.4\\7.6\end{pmatrix}\right]^T+\left[\begin{pmatrix}10\\8\end{pmatrix}-\begin{pmatrix}8.4\\7.6\end{pmatrix}\right]\left[\begin{pmatrix}10\\8\end{pmatrix}-\begin{pmatrix}8.4\\7.6\end{pmatrix}\right]^T \Big/ N-1$$

$$=\begin{pmatrix}2.3 & -0.05\\ -0.05 & 3.3\end{pmatrix}$$

Samples for class $\boldsymbol{\omega_1}$: $\mathbf{X_1}=(x_1,x_2)=\{(4,2),(2,4),(2,3),(3,6),(4,4)\}$

Sample for class $\boldsymbol{\omega_2}$: $\mathbf{X_2}=(x_1,x_2)=\{(9,10),(6,8),(9,5),(8,7),(10,8)\}$

Within-class scatter matrix:

$$S_w = S_1 + S_2 = \begin{pmatrix}1 & -0.25\\ -0.25 & 2.2\end{pmatrix}+\begin{pmatrix}2.3 & -0.05\\ -0.05 & 3.3\end{pmatrix}$$

$$=\begin{pmatrix}3.3 & -0.3\\ -0.3 & 5.5\end{pmatrix}$$

Linear Discriminant Analysis - Solved Example

Samples for class ω_1 : $\mathbf{X}_1=(x_1,x_2)=\{(4,2),(2,4),(2,3),(3,6),(4,4)\}$

Sample for class ω_2 : $\mathbf{X}_2=(x_1,x_2)=\{(9,10),(6,8),(9,5),(8,7),(10,8)\}$

$$S_B = (\mu_1 - \mu_2)(\mu_1 - \mu_2)^T$$

$$= \left[\begin{pmatrix} 3 \\ 3.8 \end{pmatrix} - \begin{pmatrix} 8.4 \\ 7.6 \end{pmatrix}\right]\left[\begin{pmatrix} 3 \\ 3.8 \end{pmatrix} - \begin{pmatrix} 8.4 \\ 7.6 \end{pmatrix}\right]^T$$

$$= \begin{pmatrix} -5.4 \\ -3.8 \end{pmatrix}\begin{pmatrix} -5.4 & -3.8 \end{pmatrix}$$

$$= \begin{pmatrix} 29.16 & 20.52 \\ 20.52 & 14.44 \end{pmatrix}$$

- Find Eigen Values

$$S_W^{-1} S_B w = \lambda w$$

$$\Rightarrow \left| S_W^{-1} S_B - \lambda I \right| = 0$$

$$\Rightarrow \left| \begin{pmatrix} 3.3 & -0.3 \\ -0.3 & 5.5 \end{pmatrix}^{-1} \begin{pmatrix} 29.16 & 20.52 \\ 20.52 & 14.44 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{pmatrix} 0.3045 & 0.0166 \\ 0.0166 & 0.1827 \end{pmatrix} \begin{pmatrix} 29.16 & 20.52 \\ 20.52 & 14.44 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{pmatrix} 9.2213 - \lambda & 6.489 \\ 4.2339 & 2.9794 - \lambda \end{pmatrix} \right|$$

$$= (9.2213 - \lambda)(2.9794 - \lambda) - 6.489 \times 4.2339 = 0$$

$$\Rightarrow \lambda^2 - 12.2007\lambda = 0 \Rightarrow \lambda(\lambda - 12.2007) = 0$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = 12.2007$$

- Find Eigen Vector

$$\left(S_W^{-1} S_B - \lambda I \right)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0$$

$$w_1 = \begin{pmatrix} -0.5755 \\ 0.8178 \end{pmatrix} \qquad w_2 = \begin{pmatrix} 0.9088 \\ 0.4173 \end{pmatrix} = w^*$$

- Find Eigen Vector

$$\left(S_W^{-1} S_B - \lambda I\right)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0$$

$$w_1 = \begin{pmatrix} -0.5755 \\ 0.8178 \end{pmatrix} \qquad w_2 = \begin{pmatrix} 0.9088 \\ 0.4173 \end{pmatrix} = w^*$$

Or directly;

$$w^* = S_W^{-1}(\mu_1 - \mu_2) = \begin{pmatrix} 3.3 & -0.3 \\ -0.3 & 5.5 \end{pmatrix}^{-1}\left[\begin{pmatrix} 3 \\ 3.8 \end{pmatrix} - \begin{pmatrix} 8.4 \\ 7.6 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0.3045 & 0.0166 \\ 0.0166 & 0.1827 \end{pmatrix}\begin{pmatrix} -5.4 \\ -3.8 \end{pmatrix}$$

$$= \begin{pmatrix} 0.9088 \\ 0.4173 \end{pmatrix}$$

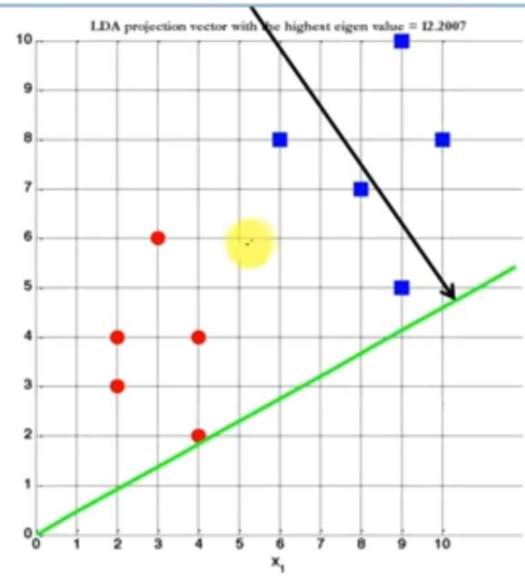
- Obtain the LDA by taking the dot product of eigen vectors and original data

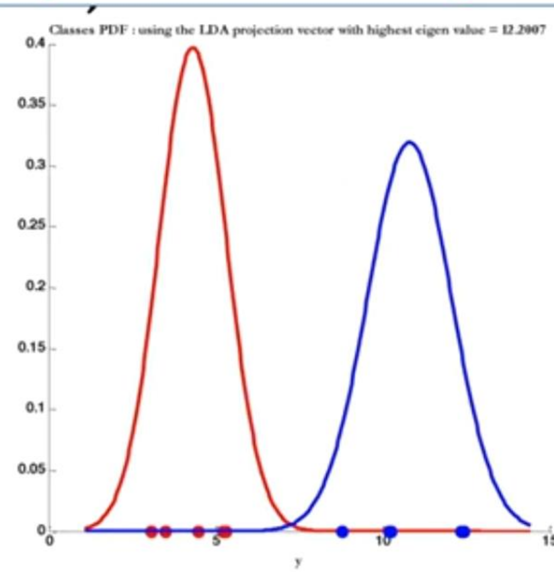
$$w_2 = \begin{pmatrix} 0.9088 \\ 0.4173 \end{pmatrix} = w^*$$

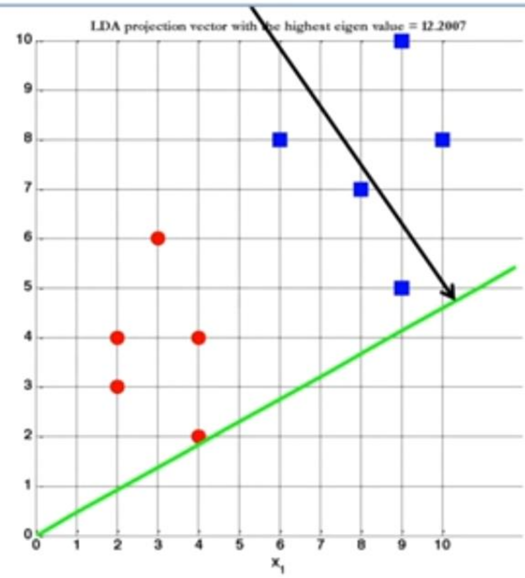
Samples for class $\boldsymbol{\omega_1}$: $\mathbf{X}_1=(x_1,x_2)=\{(4,2),(2,4),(2,3),(3,6),(4,4)\}$

Sample for class $\boldsymbol{\omega_2}$: $\mathbf{X}_2=(x_1,x_2)=\{(9,10),(6,8),(9,5),(8,7),(10,8)\}$

X1	**4**	**2**	**2**	**3**	**4**	**9**	**6**	**9**	**8**	**10**
X2	**2**	**4**	**3**	**6**	**4**	**10**	**8**	**5**	**7**	**8**
1st LD	**4.46**	**3.48**	**3.06**	**5.2**	**5.3**	**12.35**	**8.8**	**10.2**	**10.19**	**12.42**


LDA projection vector with the highest eigen value = 12.2007
x_1


Classes PDF : using the LDA projection vector with highest eigen value = 12.2007
y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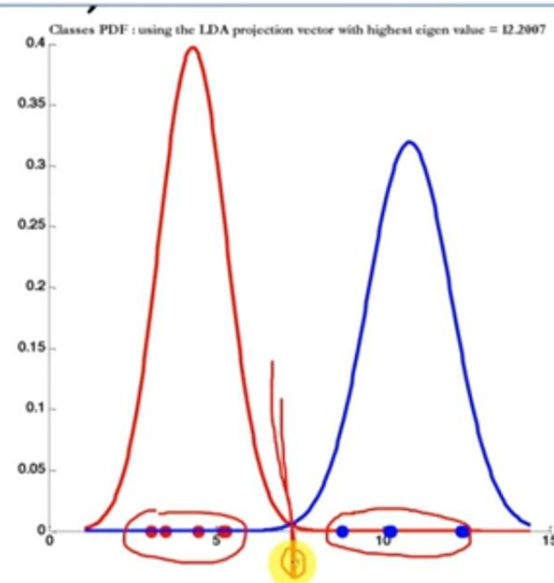
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