FUZZY LOGIC

What is Fuzzy?

- Fuzzy means not clear, distinct or precise;
- not crisp (well defined);
- blurred (with unclear outline).

FUZZY LOGIC

- Introduced by **Lofti Zadeh** (1965)
- It is a powerful problem-solving methodology
  - Builds on a set of user-supplied human language rules

- It deals with **uncertainty** and ambiguous criteria or values
  - Example: “the weather outside is cold”
    - but, how cold is actually the coldness you described?
    - What do you mean by ‘cold’ here?
  - As you can see a particular temperature is cold to one person but it is not to another
  - It depends on one’s relative definition of the said term.

- Well known paradoxes can not be solved using classical logic.

- **Russell’s paradox**

  “All of the men in this town either shaved themselves or were shaved by the barber. And the barber only shaved the men who did not shave themselves“

  - Answer to question: “ **<u>Who shaves the barber ?</u>** ” is contradictory
  - Assume that **<u>he did shave himself</u>**. But we see from the story that he shaved only those men who did not shave themselves. Therefore, **<u>he did not shave himself</u>**.
  - But we notice that every man either shaved himself or was shaved by the barber. So **<u>he did shave himself</u>**. We have a contradiction.

- “ All Cretans are liars”, said the Cretan
  - If the Cretan is liar then his claim can not be believed and so is not a liar.
  - If he is not liar then he is telling truth. But because he is Cretan, he must therefore a liar.

- **Main idea behind Fuzzy systems**
  - Truth values (in fuzzy logic) or membership values are indicated by a value in the range [0,1] with 0 for absolute falsity and 1 for absolute truth.

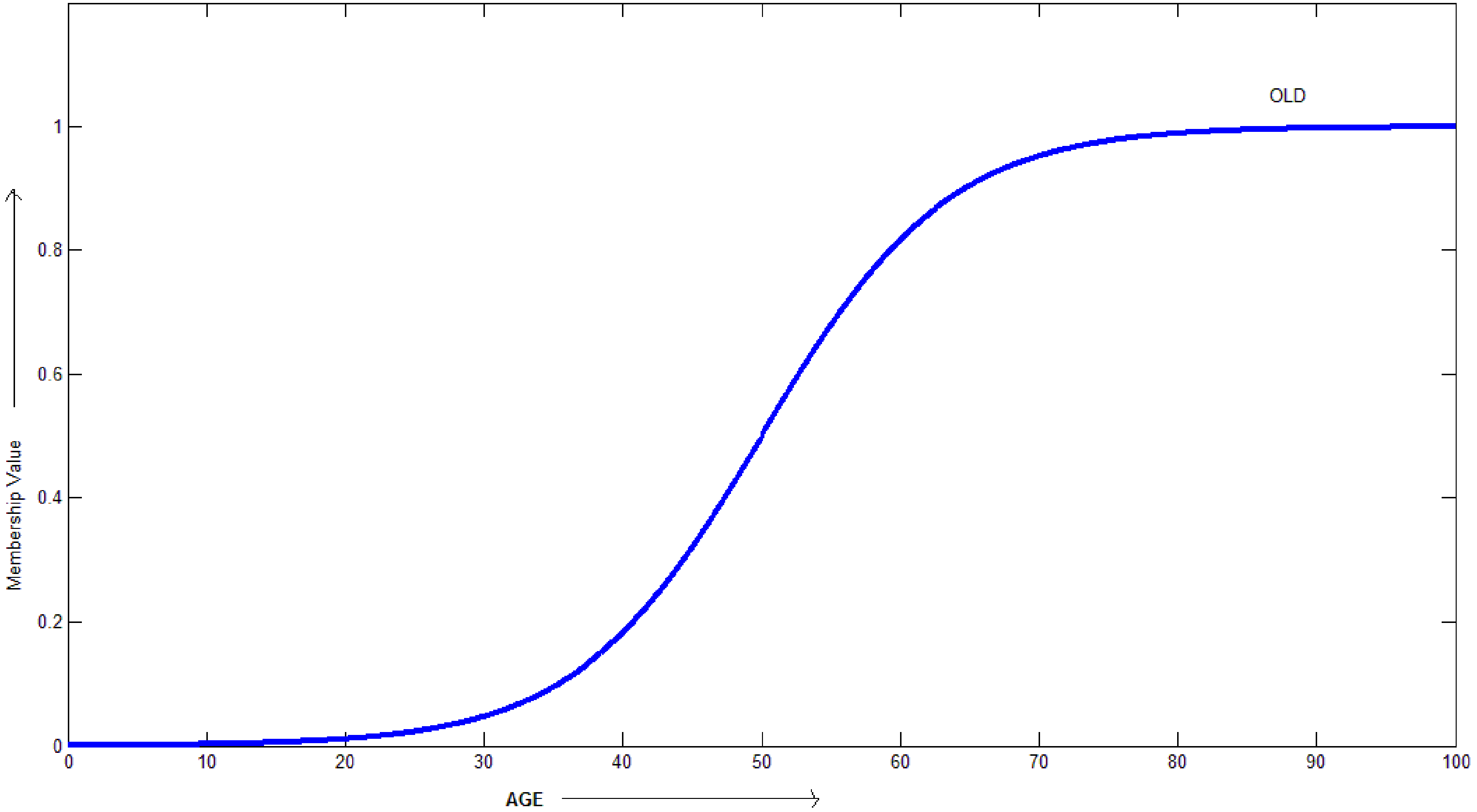
Comparing with Conventional Set Theory

Conventional Set Theory	Fuzzy Logic
1. Value of an element can be either True or False/ 1 and 0.	1. Elements in a fuzzy set X posses membership values between 0 and 1
2. Each element either fully belongs to the set or is completely excluded from the set	2. Allows each element of a given set to belong to that set to some degree
3. Represents a special case of the more general fuzzy set theory	3. General representation.

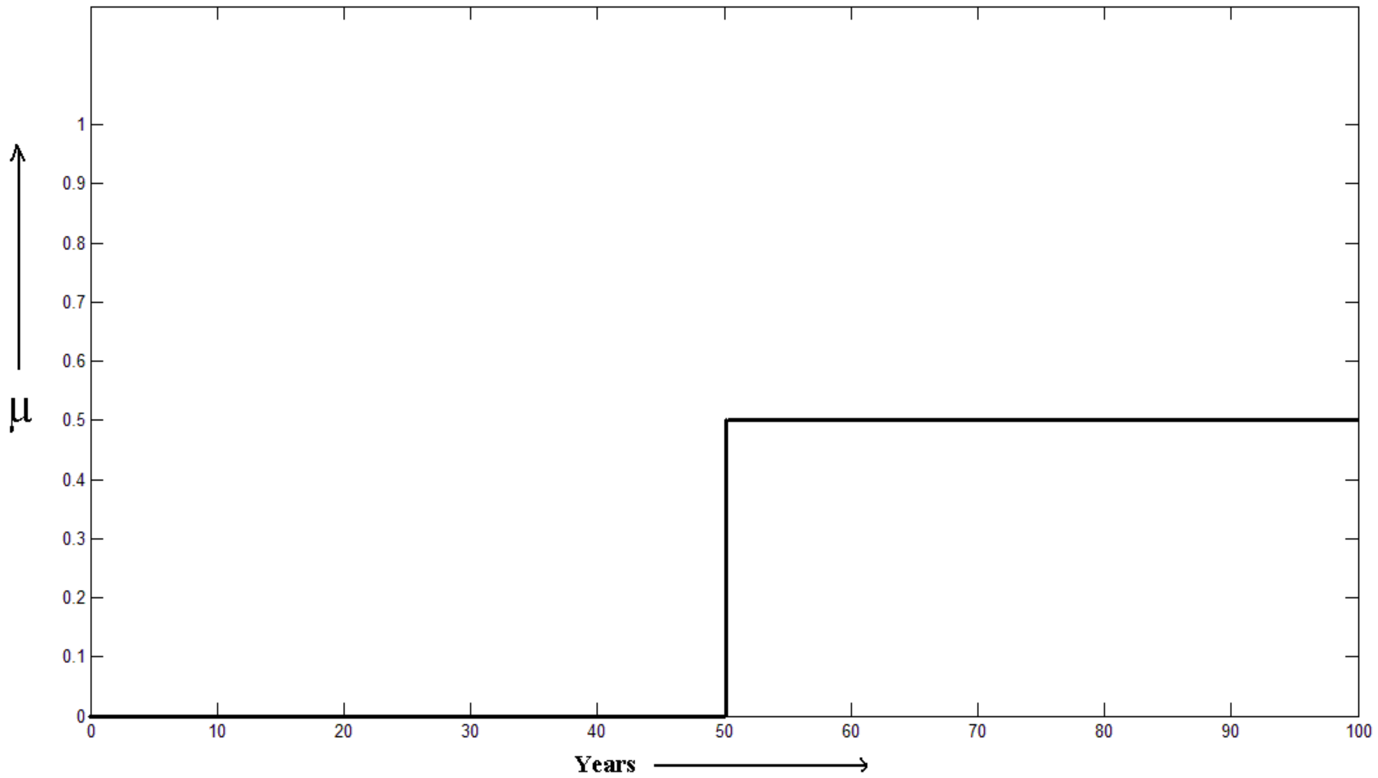
Comparing with Probability Theory

Probability approach	Fuzzy approach
1.Likelyhood of an event or a chance that a particular event will occur	1. A statement can be both true or false and also can be neither true nor false
2. P(A)=$\frac{\text{(Number of outcomes of A)}}{\text{(Total number of events)}}$	2. The membership grades are not probabilities
Representing "Helen is old"	
"Helen is old" the truth value of 0.95. The interpretation is that there is 95% chance of Helen is old	Helen is a member of the set of old people. It could be expressed in symbolic notation of fuzzy set as λ_{OLD}(Helen) = 0.95 i.e., Helen's degree of membership within the set of old people = 0.95
Distinction in two views	
There are 5% chances that Helen may not old	There is no chance of Helen being young and she is more or less old

Membership function μ_{OLD} for the fuzzy set OLD is represented as



Membership function for crisp (conventional) set older than 50 years is represented as:



How does Fuzzy Logic resembles Human intelligence?

- It can handle at certain level of imprecision and uncertainty.
- It reflects some forms of the human reasoning process by
  - Setting hypothetical rules
  - Performing inferencing
  - Performing logic reasoning on the rules

DEFINITION

- If X (Universal set) is a collection (set) of objects denoted by x, then a fuzzy set F in X is a set of ordered pairs

  $F = \{ (x, \mu_{F(x)})) \mid x \in X\}$

  where

  - $\mu_{F(x)}$ is the **membership function** of x in F which maps x to the membership space [0,1].

    F – Fuzzy Set

    x – Elements of X

    X – Universe of Discourse

- Grade of membership 1 is assigned to those objects that fully and completely belong to F and 0 to those who do not belong to F at all.

MEMBERSHIP FUNCTION

- A membership function is defined as a curve that indicates how each point in an input space is mapped to a membership value (or degree of membership) between 0 and 1.
- A membership function can be represented by an arbitrary curve whose shape defines a function that is
  - Convenient
  - Efficient
  - Varies between 0 and 1
- Generally represented by $\mu_{F(x)}$ for fuzzy set F(x)
- Features
  - $\mu_{F(x)}$ (a) = 1.0, where 'a' is a real number close to F(x)
  - $\mu_{F(x)}$ is symmetric w.r.t z, i.e. $\mu_{F(x)}(a+z)=\mu_{F(x)}(a-z)$
  - $\mu_{F(x)}$ decreases monotonically

EXAMPLE

Example: Consider the outside ambient temperature. Classical set theory can only classify the temperature as hot or cold (*i.e.*, either 1 or 0). It cannot interpret the temperature between 20°F and 100°F. In other words, the characteristic function for the classical logic for the above example is given by

$$\mu_{HOT}(x) = \begin{cases} 1 \text{ iff } x \geq 50^\circ F & \text{Classifies as hot} \\ 0 \text{ iff } x < 50^\circ F & \text{Classifies as cold} \end{cases}$$

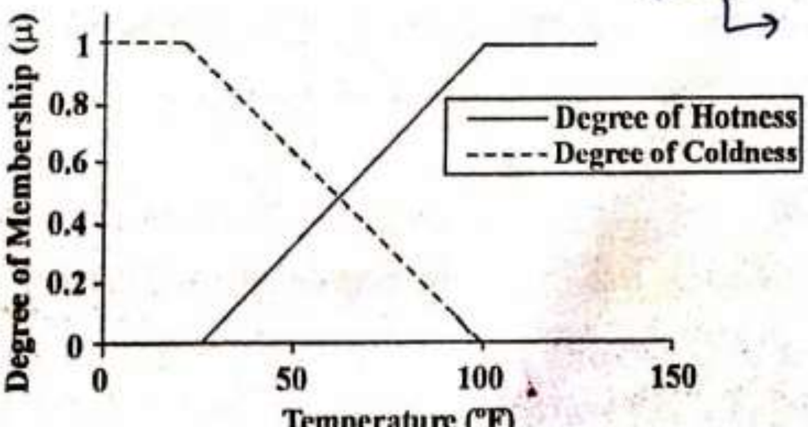
The boundary 50°F is taken because classical logic cannot interpret intermediate values.

On the other hand, fuzzy logic solves the above problem with a membership function as given by

$$\mu_{HOT}(x) = \begin{cases} 0. & \text{if } x \leq 20^0 F \\ \dfrac{x-20}{80} & \text{if } 20^0 F \leq x \leq 100^0 F \\ 1 & \text{if } x \geq 100^0 F \end{cases}$$

Temperature (°F)	Degree of Hotness	Degree of Coldness
20	0	1
30	0.13	0.87
40	0.25	0.75
50	0.375	0.625
60	0.5	0.5
70	0.625	0.375
80	0.75	0.25
90	0.875	0.125
100	1	0

Table 7.1 Membership function of temperature



Fuzzy Logic Operations

Fuzzy Logic Operators are used to write logic combinations

1) **Intersection**: The logic operator corresponding to the intersection of sets is AND.

$$\mu_{A \cap B} = \min(\mu_A, \mu_B)$$

2) **Union**: The logic operator corresponding to the union of sets is OR.

$$\mu_{A \cup B} = \max(\mu_A, \mu_B)$$

3) **Negation:** The logic operator corresponding to the complement of a set is the negation.

$$\mu_{\overline{A}} = 1 - \mu_A$$

EXAMPLE

Let $U = \{ 1,2,3,4,5,6,7\}$

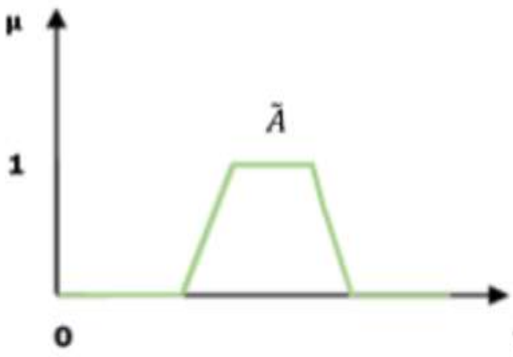
$A = \{ (3, 0.7), (5, 1), (6, 0.8) \}$ and

$B = \{(3, 0.9), (4, 1), (6, 0.6) \}$

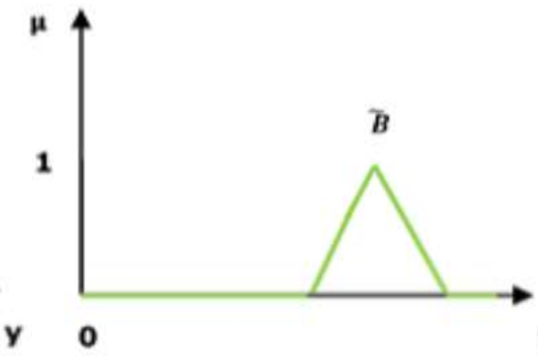
$A \cap B = \{ (3, 0.7), (6, 0.6) \}$

$A \cup B = \{ (3, 0.9), (4, 1), (5, 1), (6, 0.8) \}$

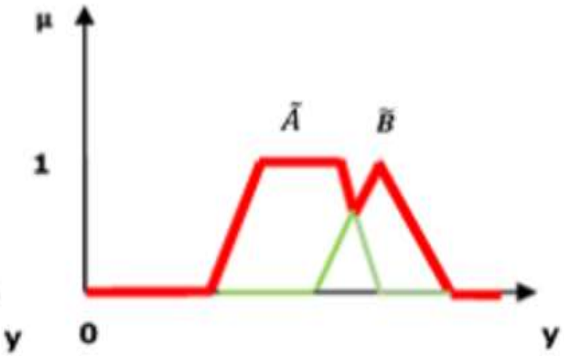
$A' = \{ (1, 1), (2, 1), (3, 0.3), (4, 1), (6, 0.2), (7, 1) \}$



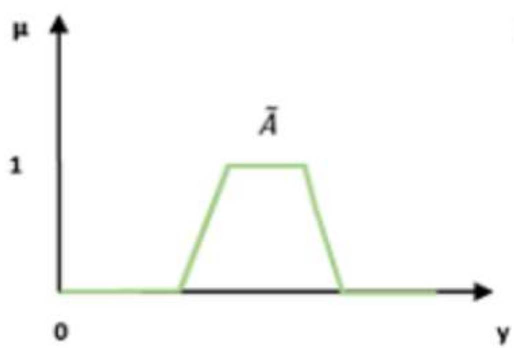
Fuzzy set $\tilde{A}$



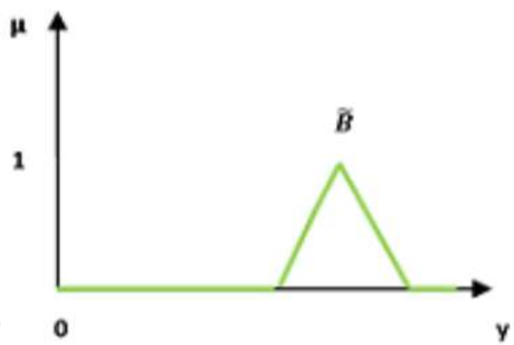
Fuzzy set $\tilde{B}$



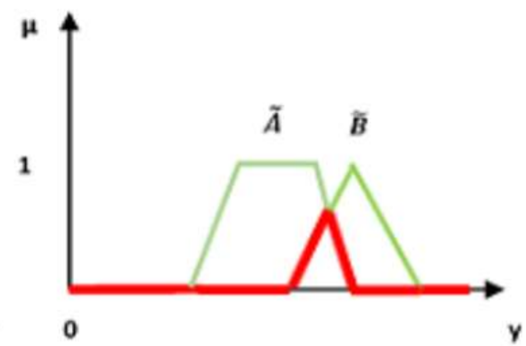
Union of two Fuzzy sets



Fuzzy set $\tilde{A}$



Fuzzy set $\tilde{B}$



Intersection of two Fuzzy sets

ADDITIONAL OPERATIONS

1. Equality: $A = B$, if $\mu_{A(x)} = \mu_{B(x)}$, $\forall x \in X$

2. Not equal: $A \neq B$, if $\mu_{A(x)} \neq \mu_{B(x)}$ for at least one $x \in X$

3. Containment: $A \subseteq B$ if and only if $\mu_{A(x)} \leq \mu_{B(x)}$, $\forall x \in X$

4. Proper subset: If $A \subseteq B$ and $A \neq B$

5. Product: A.B is defined as $\mu_{A.B(x)} = \mu_{A(x)} \cdot \mu_{B(x)}$

6. Power : A^N is defined as: $\mu_A^{N(x)} = (\mu_{A(x)})^N$

Various Types of Membership Functions

- S-shaped function
- Z-shaped function
- Triangular Membership Function
- Trapezoidal Membership Function
- Gaussian Distribution Function

S-shaped function

- S-membership function may be defined as follows:

$$\mu_S(x, a, b, c) = \begin{cases} 0\,, & \text{for } x \le a \\ 2[(x-a)/(c-a)]^2\,, & \text{for } a \le x \le b \\ 1-2[(x-c)/(c-a)]^2\,, & \text{for } b < x \le c \\ 1\,, & \text{for } x \ge c \end{cases}$$

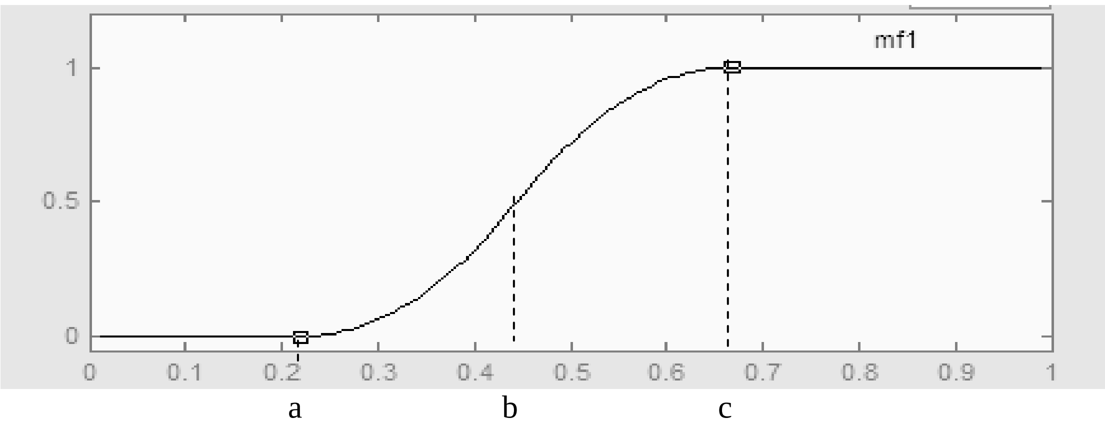
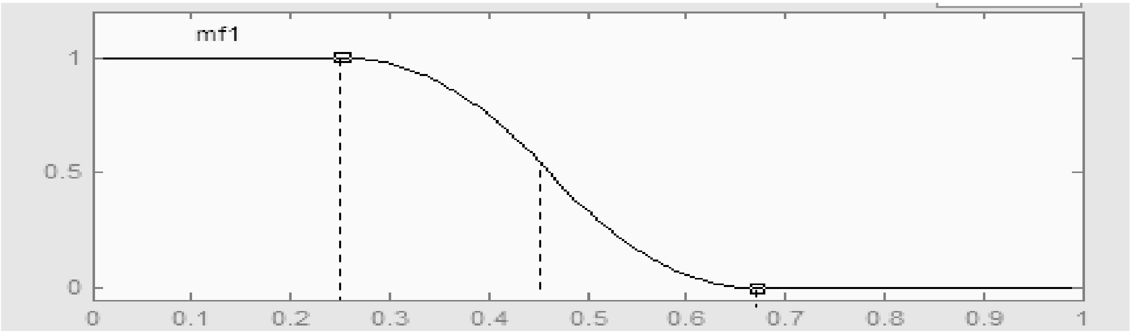


Figure S-shaped Membership Function

Z-SHAPED FUNCTION

- It represents an asymmetrical polynomial curve open to the left.
- Z-membership function may be defined as follows:

$$\mu_Z(x, a, b, c) = \begin{cases} 1, & \text{for } x \le a \\ 1- 2[(x-a) / (c-a)]^2, & \text{for } a \le x \le b \\ 2[(x-c) / (c-a)]^2, & \text{for } b < x \le c \\ 0, & \text{for } x \ge c \end{cases}$$



a b c

Figure Z membership function

TRIANGULAR MEMBERSHIP FUNCTIONS

- Triangular membership function may be defined as follows.

$$\mu_F(x, a, b, c) = \begin{cases} 0, & \text{if } x < a \\ (x - a) / (b - a), & \text{if } a \leq x \leq b \\ (c - x) / (c - b), & \text{if } b \leq x \leq c \\ 0, & \text{if } c < x \end{cases}$$

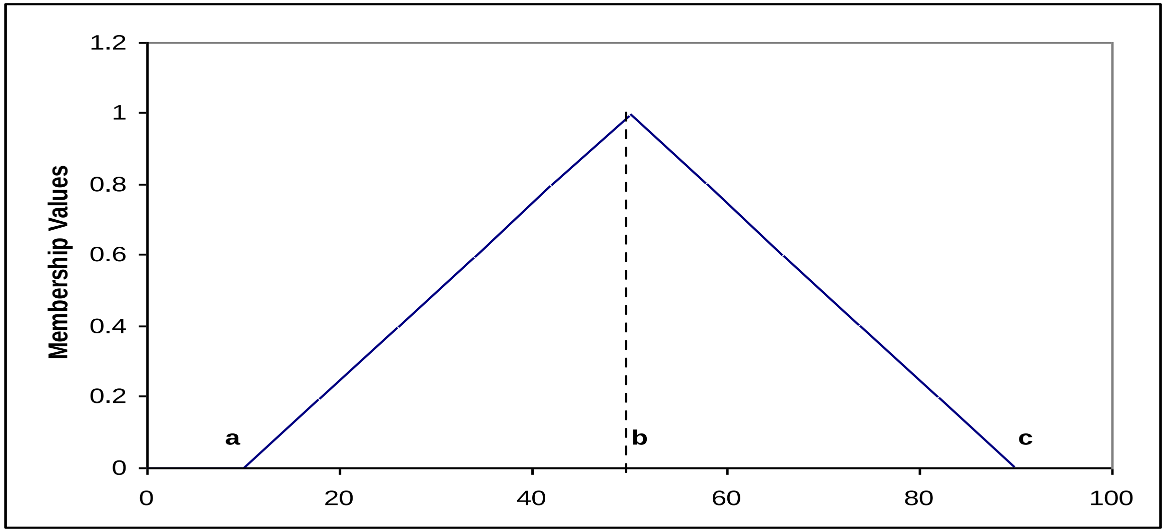
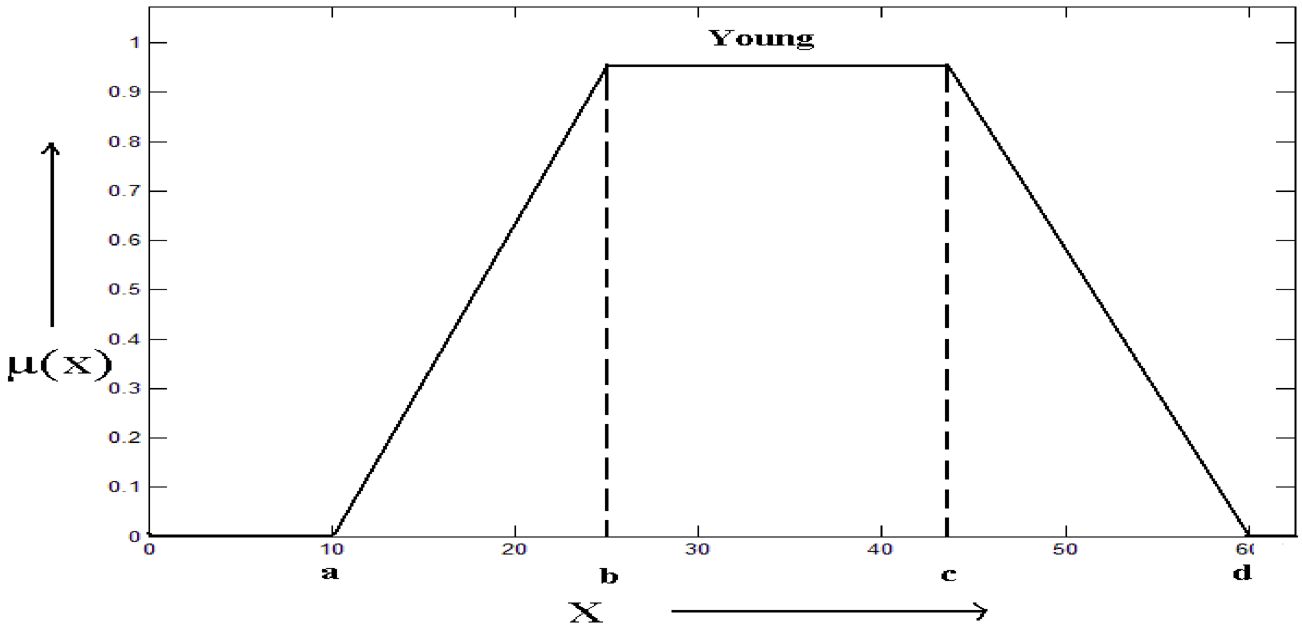


Figure Triangular Function

TRAPEZOIDAL MEMBERSHIP FUNCTION

- Trapezoidal membership function may be defined as follows.

$$\mu_F(x, a, b, c, d) = \begin{cases} 0, & \text{if } x < a \\ (x - a) / (b - a), & \text{if } a \le x \le b \\ 1, & \text{if } b < x < c \\ (d - x) / (d - c), & \text{if } c \le x \le d \\ 0, & \text{if } d < x \end{cases}$$

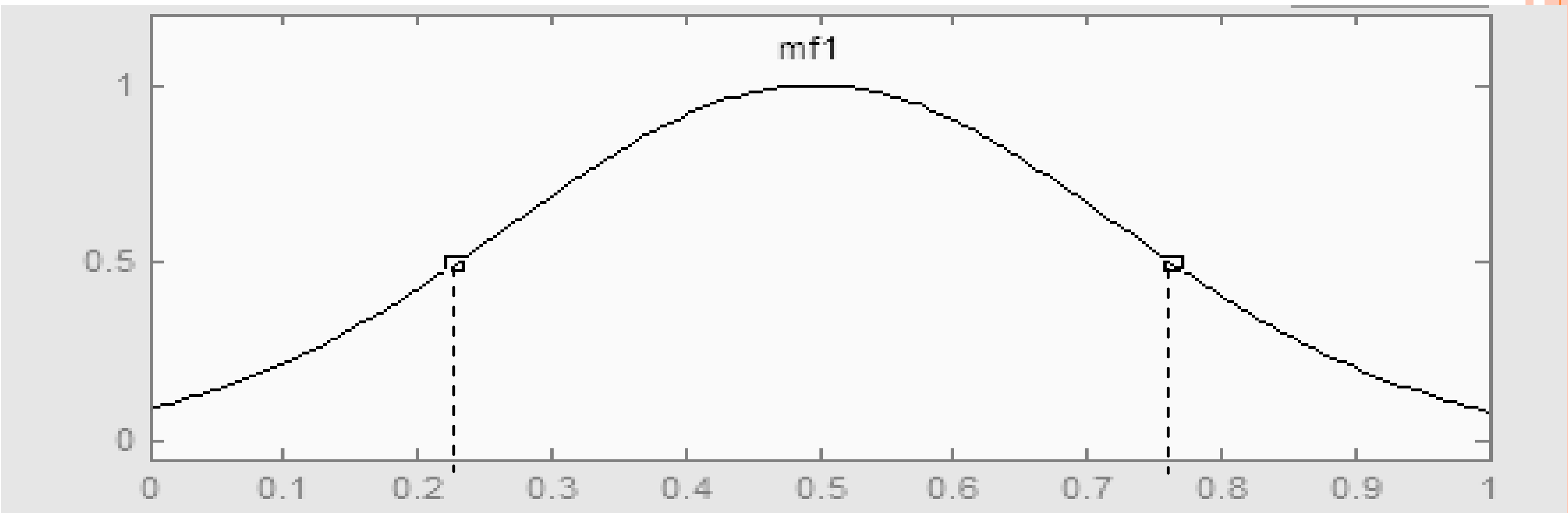


GAUSSIAN MEMBERSHIP FUNCTION

The Gaussian membership function can be defined as.

$$\mu(x,a,b) = e^{\frac{-(x-b)}{2a^2}}$$

The graph given in below fig. is for parameters *a* = 0.22, *b* = 0.78



a b

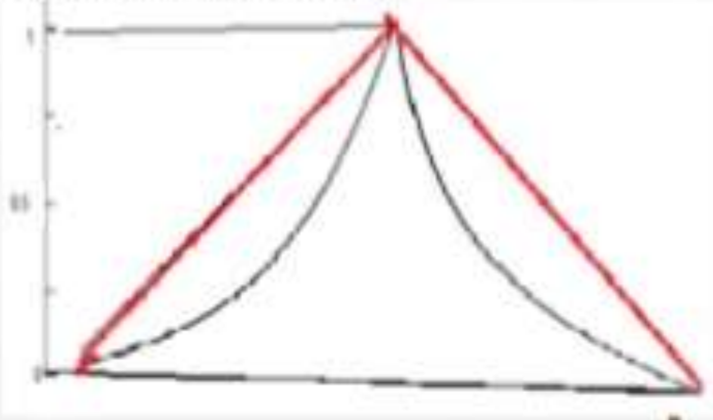
Figure Gaussian Membership Function

Basic Operations

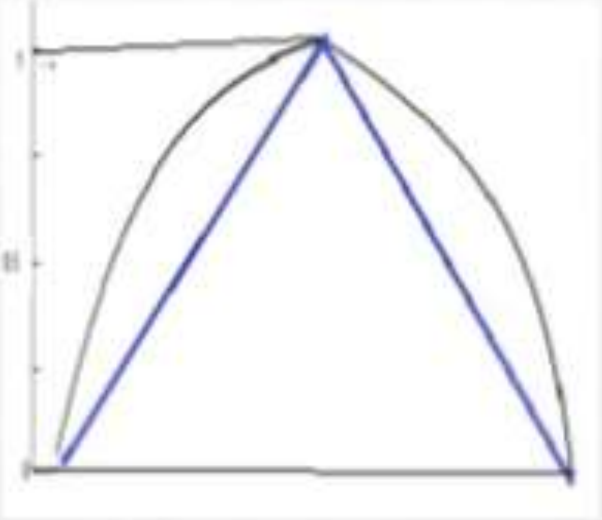
- For reshaping the membership functions, following three operations can be used.
  - **Dilation (DIL) :** It increases the degree of membership of all members by spreading out the curve.
  - **Concentration (CON):** It decreases the degree of membership of all members.
  - **Normalization (NORM)** : It discriminates all membership degree in the same order unless maximum value of any member is 1.

A fuzzy set is called normalized when at least one of its elements attains the maximum possible membership grade i.e., 1.

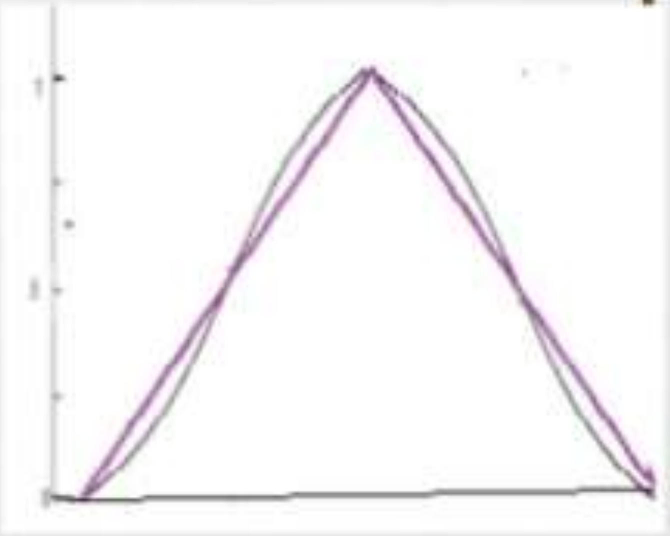
Graphical representation



- Concentration



- Dilation
- Intensification



Fuzzy Logic Systems Architecture

It has four main parts as shown –

- **Fuzzification Module** – It transforms the system inputs, which are crisp numbers, into fuzzy sets. It splits the input signal into five steps such as –

LP	x is Large Positive
MP	x is Medium Positive
S	x is Small
MN	x is Medium Negative
LN	x is Large Negative

- **Knowledge Base** − It stores IF-THEN rules provided by experts.

  Statements used to formulate the conditional statements that comprise fuzzy logic

  Example:

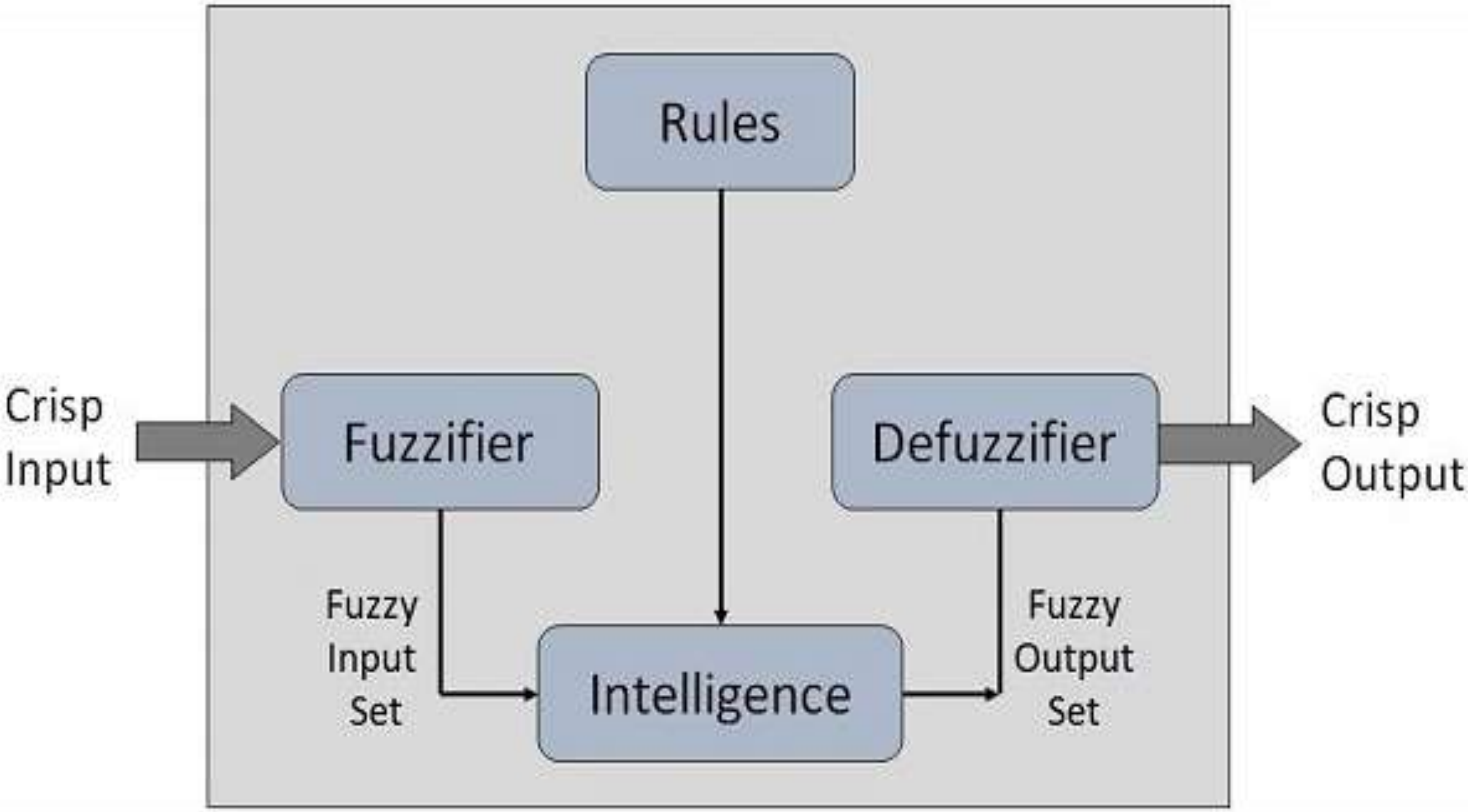
  if x is A then y is B where,

  A & B – Linguistic values

  x – Element of Fuzzy set X

  y – Element of Fuzzy set Y

- **Inference Engine** − It simulates the human reasoning process by making fuzzy inference on the inputs and IF-THEN rules.
- **Defuzzification Module** − It transforms the fuzzy set obtained by the inference engine into a crisp value.


Rules
Crisp
Input
Fuzzifier
Defuzzifier
Crisp
Output
Fuzzy
Input
Set
Intelligence
Fuzzy
Output
Set

Application of Fuzzy Methods

Fuzzy Logic success is mainly due to its introduction into consumer products such as:

- air conditioner
- washing machines
- refrigerators
- television
- rice cooker
- Etc.

Drawbacks to Fuzzy logic

- Requires tuning of membership functions.
- Fuzzy Logic control may not scale well to large or complex problems.
- Deals with imprecision, and vagueness, but not uncertainty.