5 Color Theorm Implementation

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Summary: In this project we are implementing 5 color theorm which is a result from graph theory.It is used for coloring of planar graphs (that is special labelling of graphs). According to this theorm we can color a given planar graph such that no two adjacent vertices has same color with 5 colors .We generally apply this in coloring geographical maps. Implementing this theorm help in clearing basics of graphs, backtracking and efficiently using programming language C syntax. This project is useful in learning debugging, use of git and git bash .

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1. Introduction

5 Color Theorm states that given a plane separated into regions, such as a political map of the countries or a state, the regions may be colored using no more than five colors in such a way that no two adjacent regions receive the same color. It was based on a failed attempt at the four color proof by Alfred Kempe in 1879. Percy John Heawood found an error 11 years later, and proved the five color theorem based on Kempe's work.
5 Color Theorm has application in different fields.Some real life applications are:

- Coloring maps (reduce cost and save time);
- Security camera placement optimization in a large building with many corners such that you minimize overlap;
- construction of a wildlife reserve (applying knowledge of food chains to see what combination of animals can live together and not completely wipe each other out).

`Planarity of graph`

Since 5 Color Theorm is only applicable on planar graphs ,it is important to understand planarity of graphs.Basically a planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other.

`Coloring of Graph`

Graph coloring is a special case of graph labeling. There is a rule for coloring graphs which states that no two adjacent vertices may have the same color.Generally graph coloring is used to solve problems where you have a limited amount of resources or other restrictions. The colors are just an abstraction for whatever resource you're trying to optimize, and the planar graph is an abstraction of your problem for example geographical maps. The Figure 1 gives us an idea how to color any graph.

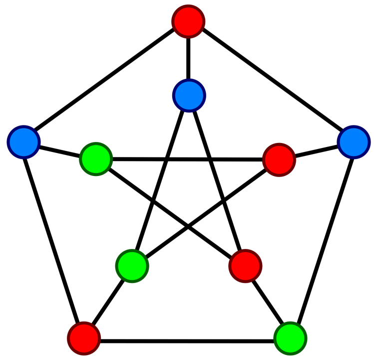

Figure 1: Coloring of Petersen Graph.

2. Proof of 5 Color Theorm

Because G is a simple planar, i.e. it may be embedded in the plane without intersecting edges,and it does not have two vertices sharing more than one edge, and it doesn't have loops, then it can be shown (using the Euler characteristic of the plane) that it must have a vertex shared by at most five edges. (Note: This is the only place where the five-color condition is used in the proof. If this technique is used to prove the four-color theorem, it will fail on this step. In fact, an icosahedral graph is 5-regular and planar, and thus does not have a vertex shared by at most four edges.) Find such a vertex, and call it v.
Now remove v from G. The graph G' obtained this way has one fewer vertex than G, so we can assume by induction that it can be colored with only five colors. If the coloring did not use all five colors on the five neighboring vertices of v, it can be colored in G with a color not used by the neighbors. So now look at those five vertices v1,v2,v3,v4,v5 that were adjacent to v in cyclic order (which depends on how we write G). So we can assume that v1,v2,v3,v4,v5 are colored with colors 1, 2, 3, 4, 5 respectively.
Now consider the subgraph G(1,3) of G' consisting of the vertices that are colored with colors 1 and 3 only and the edges connecting them. To be clear, each edge connects a color 1 vertex to a color 3 vertex (this is called a Kempe chain). If v1 and v3 lie in different connected components of G(1,3), we can swap the 1 and 3 colors on the component containing v1 without affecting the coloring of the rest of G'. This frees color 1 forv completing the task. If on the contrary v1 and v3 lie in the same connected component of G(1,3) , we can find a path in G(1,3) joining them that consists of only color 1 and 3 vertices.
Now turn to the subgraph G(2,4) of G' consisting of the vertices that are colored with colors 2 and 4 only and the edges connecting them, and apply the same arguments as before. Then either we are able to reverse the 2-4 coloration on the subgraph of G(2,4) containing v2 and paint v color 2, or we can connect v2 and v4 with a path that consists of only color 2 and 4 vertices. Such a path would intersect the 1-3 colored path we constructed before since v1 through v5 were in cyclic order. This is clearly absurd as it contradicts the planarity of the graph.
So G can in fact be five-colored, contrary to the initial presumption.

3. Algorithms

We scan ionformation of planar graph from the user and build a graph using adjacency list ,to color the vertices with 5 colors we use backtracking to avoid giving same color to two adjacent vertices. The algorithm used to color vertices is shown in Algorithm 1.

Algorithm 1 5 Coloring

```
given an undirected graph G:(V,E)
for every vertex v in G do
  if vertex v is not colored then
    Int i->0
    while check for color of adjacent nodes using backtrack do
      if i-> 4 then
        i->0
      end if
    else
      i++
    end while
    color of vertex v become i
  end if
end for
```

4. Some useful Theorms

Theorems that help in understanding 5 Color Theorm are as follows:

Theorem 4.1. *Given a planar map, a planar graph can be constructed such for each coloring of the regions of the map, there is a coloring of the vertices of the graph, and conversely.*

Proof. Construct one vertex for each region and construct an edge between two vertices iff the regions share a boundary. As shown in left side of Figure 2

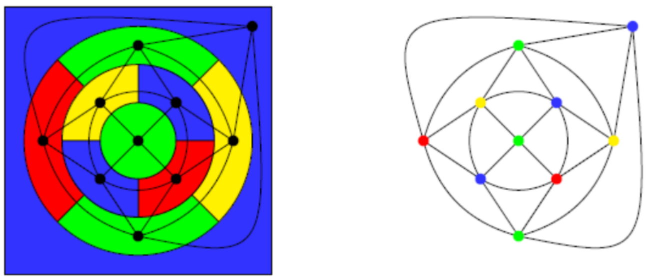

Figure 2: Converting into graph with vertices.

Theorem 4.2. *A planar map can be colored with four colors so that no adjacent regions share the same color.*

Proof. The proof of this theorem is extremely difficult; here we prove the much simpler fivecolor theorem, originally proved in the nineteenth century.

5. Conclusions

Any planar undirected graph can be colored with 5 different colors such that no two adjacent vertices are of same color. It helps in understanding applying the knowledge of graphs to simplify some real life problems.

6. Bibliography and citations

Stating sources here which helped us to implement 5 color theorm.

- lecture notes [3] gave us a basic understanding of graphs and other concepts used.
- the book [2] CLRS helped to understand the concepts better.
- the site [4] helped us in knowing about 5 color theorm
- the site [5] helped us in knowing about graph coloring

- the site [1] helped us with the figures used and some important information.

Acknowledgements

We want to thank our TA Simran Setia who helped us in picking this topic and monitered us through the process.

References

[1] Moti Ben-Ari. The five-color theorem.

[2] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. Introduction to algorithms -3rd edition.

[3] Dr.Anil Shukla. Graphs. Master's thesis, IIT Ropar, punjab, 11 2012. classroom.

[4] Wikipedia. 5 color theorm.

[5] Wikipedia. Graph coloring, 1999.

A. Appendix A

Some useful properties of planar graph

- If a connected planar graph G has e edges and r regions, then r is less than or equal to (2/3)e;
- If a connected planar graph G has e edges, v vertices, and r regions, then (v-e+r) is equal to 2;
- If a connected planar graph G has e edges and v vertices, then (3v-e) is greater than or equal to 6.
- A complete graph Kn is a planar if and only if nis less than 5.