FIRST-ORDER LOGIC

SYNTAX

Term	Constants - a,b,c… Variables - x,y,z…
Function	Arity *n*: Takes *n* terms as input, and forms a term
Predicate	Arity *n*: Takes *n* terms as input, and forms an atom

SYNTAX

Atom	Predicate: p,q,r…
Logical Connectives	$\wedge$: and, $\vee$: or, $\neg$: not, $\rightarrow$: implies, $\leftrightarrow$: if and only if(iff)
Quantifier	$\forall$: Universal $\exists$: Existential
Literal	Atom or its negation
Formula	A literal or the application of logical connectives and quantifiers to formulae

EXAMPLE

$$\forall x\,.\,((\exists y\,.\,p(f(x), y)) \rightarrow q(x))$$

EXAMPLE

$$\forall \boxed{x} . ((\exists \boxed{y} . p(f(\boxed{x}), \boxed{y})) \rightarrow q(\boxed{x}))$$

Variables

EXAMPLE

$$\forall x \,.\, ((\exists y \,.\, p(\boxed{f(x)}, y)) \rightarrow q(x))$$

Function

EXAMPLE

$$\forall x \,.\, ((\exists y \,.\, \boxed{p(f(x), y)}) \rightarrow \boxed{q(x)})$$

Predicate

EXAMPLE

$$\boxed{\forall x}\,.\,((\boxed{\exists y}\,.\,p(f(x),y)) \to q(x))$$

Quantifier

EXAMPLE

$$\boxed{\forall x} \,.\, ((\exists y \,.\, p(f(x), y)) \rightarrow q(x))$$

Scope of x

EXAMPLE

$$\forall x\,.\,((\boxed{\exists y}\,.\,\underbrace{p(f(x), y))}_{\text{Scope of y}} \rightarrow q(x))$$

EXAMPLE

$$\forall x\,.\,((\boxed{\exists y}\,.\,\underbrace{p(f(x),y))}_{\text{Scope of y}} \rightarrow q(x))$$

An occurrence of a variable is bound if it is in the scope of some quantifier

EXAMPLE

$$\forall x \,.\, ((\boxed{\exists y} \,.\, \underbrace{p(f(x), y)}_{\text{Scope of y}}) \rightarrow q(x))$$

An occurrence of a variable is bound if it is in the scope of some quantifier

An occurrence of a variable is free if it is not in the scope of some quantifier

SEMANTICS - EXAMPLES

- All Humans are mortal.
  - Assume unary predicates *human* and *mortal.*

$$\forall x \,.\, human(x) \rightarrow mortal(x)$$

SEMANTICS - EXAMPLES

- There always exists someone such that if (s)he laughs, then everyone laughs.
    - Assume unary predicate *laughs*.

$$\exists x\,.\,(laughs(x) \rightarrow \forall y\,.\,laughs(y))$$

SEMANTICS - EXAMPLES

- Every dog has its day.
  - $\forall x \,.\, dog(x) \rightarrow \exists y \,.\, day(y) \land itsDay(x, y)$
- Some dogs have more days than others.
  - $\exists x, y \,.\, dog(x) \land dog(y) \land \#days(x) > \#days(y)$
- All cats have more days than dogs.
  - $\forall x, y \,.\, (dog(x) \land cat(y)) \rightarrow \#days(y) > \#days(x)$

INTERPRETATIONS

- An interpretation I is an assignment from variables (in general, terms) to values in a specified domain.
- Domain, D_I
    - A nonempty set of values or objects. Also called universe of discourse.
    - Numbers, humans, students, courses, animals,…
- Assignment, α_I
    - Maps constants and variables to elements of the domain D_I (i.e. values)
    - Maps functions and predicate symbols to functions and predicates (of the same arity) over D_I

INTERPRETATIONS - EXAMPLE 1

- Suppose $D_I = \{A, B\}$
- Constants a and b are mapped to following elements in D_I
  - $\alpha_I(a) = B \quad \alpha_I(b) = A$
- A binary function symbol f is mapped to the following actual function on D_I:
  - $\alpha_I(f) = \{(A, A) \to B,\ (A, B) \to B,\ (B, A) \to A,\ (B, B) \to B\}$
- A unary predicate symbol p is mapped to the following actual predicate on D_I
  - $\alpha_I(p) = \{A \to True,\ B \to False\}$

INTERPRETATIONS - EXAMPLE 2

DONEC QUIS NUNC

- Consider the formula: $x + y > z \rightarrow y > z - x$
  - Here, $+, -$ are functions and $>$ is a predicate.
  - Equivalent to $p(f(x, y), z) \rightarrow p(y, g(z, x))$.
- A standard interpretation for this formula would be:
  - Domain: $\mathbb{Z}$
  - $+, -$ would be mapped to the standard integer addition and subtraction functions.
  - $>$ would be mapped to the standard greater-than relation over integers.
  - x, y, z could be mapped to $5,10,9$ resp.

SEMANTICS: INDUCTIVE DEFINITION

Base Case:

$I \models \top$	
$I \not\models \bot$	
$I \models p$	iff I[p]=true
$I \not\models p$	iff I[p]=false

Inductive Case:

$I \models \neg F$	iff $I \not\models F$
$I \models F_1 \wedge F_2$	iff $I \models F_1$ and $I \models F_2$
$I \models F_1 \vee F_2$	iff $I \models F_1$ or $I \models F_2$
$I \models F_1 \rightarrow F_2$	iff $I \not\models F_1$ or $I \models F_2$
$I \models F_1 \leftrightarrow F_2$	iff $I \models F_1$ and $I \models F_2$, or $I \not\models F_1$ and $I \not\models F_2$

SEMANTICS: INDUCTIVE DEFINITION

Base Case:

What does this mean?

$I \models \top$	
$I \not\models \bot$	
$I \models p$	iff I[p]=true
$I \not\models p$	iff I[p]=false

Inductive Case:

$I \models \neg F$	iff $I \not\models F$
$I \models F_1 \wedge F_2$	iff $I \models F_1$ and $I \models F_2$
$I \models F_1 \vee F_2$	iff $I \models F_1$ or $I \models F_2$
$I \models F_1 \rightarrow F_2$	iff $I \not\models F_1$ or $I \models F_2$
$I \models F_1 \leftrightarrow F_2$	iff $I \models F_1$ and $I \models F_2$, or $I \not\models F_1$ and $I \not\models F_2$

SEMANTICS - CONTINUED...

$$I \models p(t_1, \ldots, t_n) \text{ iff } \alpha_I[p](\alpha_I[t_1], \ldots, \alpha_I[t_n]) = \top$$

$$\alpha_I[f(t_1, \ldots, t_n)] = \alpha_I[f](\alpha_I[t_1], \ldots, \alpha_I[t_n])$$

SEMANTICS - EXAMPLE

INTERPRETATION I

$D_I = \{A, B\}$

$\alpha_I(a) = B \quad \alpha_I(b) = A$

$\alpha_I(f) = \{(A, A) \to B,\ (A, B) \to B,\ (B, A) \to A,\ (B, B) \to B\}$

$\alpha_I(p) = \{A \to True,\ B \to False\}$

$$I \models p(b)$$

$$I \models p(f(a, b))$$

$$I \not\models p(f(b, a))$$

SEMANTICS - QUANTIFIERS

- An $x-$variant of interpretation $I = (D_I, \alpha_I)$ is an interpretation $J = (D_J, \alpha_J)$ such that

  - $D_I = D_J$;
  - and $\alpha_I[y] = \alpha_J[y]$ for all constant, free variable, function, and predicate symbols y, except possibly x.

- An $x-$variant of I, where x is mapped to some $v \in D_I$ is denoted by $I[x \mapsto v]$.

$I \vDash \forall x\,.\,F$ iff for all $v \in D_I, I[x \mapsto v] \vDash F$

$I \vDash \exists x\,.\,F$ iff there exists $v \in D_I, I[x \mapsto v] \vDash F$

SEMANTICS - QUANTIFIERS - EXAMPLE

INTERPRETATION I

$D_I = \{A, B\}$

$\alpha_I(a) = B \quad \alpha_I(b) = A$

$\alpha_I(f) = \{(A, A) \to B,\ (A, B) \to B,\ (B, A) \to A,\ (B, B) \to B\}$

$\alpha_I(p) = \{A \to True,\ B \to False\}$

$$I \models \exists x \,.\, p(x)$$

$$I \models \forall x \,.\, \neg p(f(b, x))$$

SATISFIABILITY AND VALIDITY

- A FOL formula F is satisfiable if there exists an interpretation I such that $I \models F$.
    - If no such interpretation exists, then it is unsatisfiable
- A FOL formula F is valid if for all interpretations I, $I \models F$
    - Technically, only for interpretations which assign to all the constants, variables, predicates, functions used in F.
- F is valid iff $\neg F$ is unsatisfiable.