FIRST-ORDER LOGIC

SYNTAX

Term	Constants - a,b,c... Variables - x,y,z...
Function	Arity *n*: Takes *n* terms as input, and forms a term
Predicate	Arity *n*: Takes *n* terms as input, and forms an atom

SYNTAX

Atom	Predicate: p,q,r…
Logical Connectives	$\wedge$: and, $\vee$: or, $\neg$: not, $\rightarrow$: implies, $\leftrightarrow$: if and only if(iff)
Quantifier	$\forall$: Universal $\exists$: Existential
Literal	Atom or its negation
Formula	A literal or the application of logical connectives and quantifiers to formulae

EXAMPLE

$$\forall x \,.\, ((\exists y \,.\, p(f(x), y)) \to q(x))$$

EXAMPLE

$$\forall x\,.\,((\exists y\,.\,p(f(x),y)) \rightarrow q(x))$$

Variables

EXAMPLE

$$\forall x \,.\, ((\exists y \,.\, p(\boxed{f(x)}, y)) \rightarrow q(x))$$

Function

EXAMPLE

$$\forall x \,.\, ((\exists y \,.\, \boxed{p(f(x), y)}) \rightarrow \boxed{q(x)})$$

Predicate

EXAMPLE

$$\boxed{\forall x} . ((\boxed{\exists y} . p(f(x), y)) \rightarrow q(x))$$

Quantifier

EXAMPLE

$$\boxed{\forall x} . \underbrace{((\exists y . p(f(x), y)) \rightarrow q(x))}_{\text{Scope of x}}$$

EXAMPLE

$$\forall x \, . \, ((\boxed{\exists y} \, . \, \underbrace{p(f(x), y))}_{\text{Scope of y}} \rightarrow q(x))$$

EXAMPLE

$$\forall x\,.\,((\boxed{\exists y}\,.\,\underbrace{p(f(x), y))}_{\text{Scope of y}} \rightarrow q(x))$$

An occurrence of a variable is bound if it is in the scope of some quantifier

EXAMPLE

$$\forall x \,.\, ((\boxed{\exists y} \,.\, \underbrace{p(f(x), y)}_{\text{Scope of y}}) \rightarrow q(x))$$

An occurrence of a variable is bound if it is in the scope of some quantifier

An occurrence of a variable is free if it is not in the scope of some quantifier

SEMANTICS - EXAMPLES

- All Humans are mortal.
  - Assume unary predicates *human* and *mortal.*

$$\forall x \,.\, human(x) \rightarrow mortal(x)$$

SEMANTICS - EXAMPLES

- There always exists someone such that if (s)he laughs, then everyone laughs.
    - Assume unary predicate *laughs*.

$$\exists x \,.\, (\mathit{laughs}(x) \rightarrow \forall y \,.\, \mathit{laughs}(y))$$

SEMANTICS - EXAMPLES

- Every dog has its day.
  - $\forall x \,.\, dog(x) \rightarrow \exists y \,.\, day(y) \wedge itsDay(x, y)$
- Some dogs have more days than others.
  - $\exists x, y \,.\, dog(x) \wedge dog(y) \wedge \#days(x) > \#days(y)$
- All cats have more days than dogs.
  - $\forall x, y \,.\, (dog(x) \wedge cat(y)) \rightarrow \#days(y) > \#days(x)$

INTERPRETATIONS

- An interpretation I is an assignment from variables (in general, terms) to values in a specified domain.
- Domain, D_I
  - A nonempty set of values or objects. Also called universe of discourse.
  - Numbers, humans, students, courses, animals,…
- Assignment, α_I
  - Maps constants and variables to elements of the domain D_I (i.e. values)
  - Maps functions and predicate symbols to functions and predicates (of the same arity) over D_I

INTERPRETATIONS - EXAMPLE 1

- Suppose $D_I = \{A, B\}$
- Constants a and b are mapped to following elements in D_I
  - $\alpha_I(a) = B \quad \alpha_I(b) = A$
- A binary function symbol f is mapped to the following actual function on D_I:
  - $\alpha_I(f) = \{(A, A) \to B,\ (A, B) \to B,\ (B, A) \to A,\ (B, B) \to B\}$
- A unary predicate symbol p is mapped to the following actual predicate on D_I
  - $\alpha_I(p) = \{A \to True,\ B \to False\}$

INTERPRETATIONS - EXAMPLE 2

- Consider the formula: $x + y > z \rightarrow y > z - x$
  - Here, $+, -$ are functions and $>$ is a predicate.
  - Equivalent to $p(f(x, y), z) \rightarrow p(y, g(z, x))$.
- A standard interpretation for this formula would be:
  - Domain: $\mathbb{Z}$
  - $+, -$ would be mapped to the standard integer addition and subtraction functions.
  - $>$ would be mapped to the standard greater-than relation over integers.
  - x, y, z could be mapped to $5,10,9$ resp.

SEMANTICS: INDUCTIVE DEFINITION

Base Case:

$I \models \top$	
$I \not\models \bot$	
$I \models p$	iff I[p]=true
$I \not\models p$	iff I[p]=false

Inductive Case:

$I \models \neg F$	iff $I \not\models F$
$I \models F_1 \wedge F_2$	iff $I \models F_1$ and $I \models F_2$
$I \models F_1 \vee F_2$	iff $I \models F_1$ or $I \models F_2$
$I \models F_1 \rightarrow F_2$	iff $I \not\models F_1$ or $I \models F_2$
$I \models F_1 \leftrightarrow F_2$	iff $I \models F_1$ and $I \models F_2$, or $I \not\models F_1$ and $I \not\models F_2$

SEMANTICS: INDUCTIVE DEFINITION

Base Case:

What does this mean?

$I \models \top$	
$I \not\models \bot$	
$I \models p$	iff I[p]=true
$I \not\models p$	iff I[p]=false

Inductive Case:

$I \models \neg F$	iff $I \not\models F$
$I \models F_1 \wedge F_2$	iff $I \models F_1$ and $I \models F_2$
$I \models F_1 \vee F_2$	iff $I \models F_1$ or $I \models F_2$
$I \models F_1 \rightarrow F_2$	iff $I \not\models F_1$ or $I \models F_2$
$I \models F_1 \leftrightarrow F_2$	iff $I \models F_1$ and $I \models F_2$, or $I \not\models F_1$ and $I \not\models F_2$

SEMANTICS - CONTINUED...

$$I \models p(t_1, \ldots, t_n) \text{ iff } \alpha_I[p](\alpha_I[t_1], \ldots, \alpha_I[t_n]) = \top$$

$$\alpha_I[f(t_1, \ldots, t_n)] = \alpha_I[f](\alpha_I[t_1], \ldots, \alpha_I[t_n])$$

SEMANTICS - EXAMPLE

INTERPRETATION I

$D_I = \{A, B\}$

$\alpha_I(a) = B \quad \alpha_I(b) = A$

$\alpha_I(f) = \{(A, A) \to B,\ (A, B) \to B,\ (B, A) \to A,\ (B, B) \to B\}$

$\alpha_I(p) = \{A \to True,\ B \to False\}$

$$I \models p(b)$$

$$I \models p(f(a, b))$$

$$I \not\models p(f(b, a))$$

SEMANTICS - QUANTIFIERS

- An $x-$variant of interpretation $I = (D_I, \alpha_I)$ is an interpretation $J = (D_J, \alpha_J)$ such that

  - $D_I = D_J$;
  - and $\alpha_I[y] = \alpha_J[y]$ for all constant, free variable, function, and predicate symbols y, except possibly x.

- An $x-$variant of I, where x is mapped to some $v \in D_I$ is denoted by $I[x \mapsto v]$.

$$I \vDash \forall x \,.\, F \text{ iff for all } v \in D_I, I[x \mapsto v] \vDash F$$

$$I \vDash \exists x \,.\, F \text{ iff there exists } v \in D_I, I[x \mapsto v] \vDash F$$

SEMANTICS - QUANTIFIERS - EXAMPLE

INTERPRETATION I

$D_I = \{A, B\}$

$\alpha_I(a) = B \quad \alpha_I(b) = A$

$\alpha_I(f) = \{(A, A) \to B,\ (A, B) \to B,\ (B, A) \to A,\ (B, B) \to B\}$

$\alpha_I(p) = \{A \to True,\ B \to False\}$

$$I \models \exists x\,.\,p(x)$$

$$I \models \forall x\,.\,\neg p(f(b, x))$$

SATISFIABILITY AND VALIDITY

- A FOL formula F is satisfiable if there exists an interpretation I such that $I \vDash F$.
  - If no such interpretation exists, then it is unsatisfiable
- A FOL formula F is valid if for all interpretations I, $I \vDash F$
  - Technically, only for interpretations which assign to all the constants, variables, predicates, functions used in F.
- F is valid iff $\neg F$ is unsatisfiable.

FREE VARIABLES

- Given a FOL formula F, a variable x is free in F if there is a use of x in F which is not bound to any quantifier.
  - $free(F)$ denotes all variables free in F.
- A FOL formula F is closed if it does not contain any free variables.
- Technically, satisfiability and validity are only applicable for closed FOL formulae.
- However, we can extend these concepts to formulae with free variables by following the below convention:
  - For satisfiability, all free variables are implicitly existentially quantified.
  - For validity, all free variables are implicitly universally quantified.

SATISFIABILITY AND VALIDITY

EXAMPLES

- Is the formula $\forall x \,.\, \exists y \,.\, p(x, y)$ satisfiable?
  - Yes. A satisfying interpretation:
    $I = (\{A\}, < p \mapsto \{(A, A) \mapsto \top \} >)$
- Is the formula $\forall x \,.\, \exists y \,.\, p(x, y)$ valid?
  - No. A falsifying interpretation:
    $I = (\{A\}, < p \mapsto \{(A, A) \mapsto \bot \} >)$
- Is the formula $(\forall x \,.\, p(x)) \rightarrow (\exists y \,.\, p(y))$ valid?
- Is the formula $\forall x \,.\, (p(x) \rightarrow (\exists y \,.\, p(y)))$ valid?
  - What about $\forall x \,.\, (p(x) \rightarrow (\forall y \,.\, p(y)))$?

DECISION PROCEDURE FOR VALIDITY

- Semantic Argument Method
  - Deductive Approach
  - Proof by Contradiction
  - Assume that a falsifying interpretation exists.
  - Use proof rules to deduce more facts.
  - The goal is to find contradictory facts in each branch (also called closing the branch).
- Proof rules for negation, conjunction, disjunction, implication, iff carry over from Propositional logic

PROOF RULES

UNIVERSAL QUANTIFICATION

$$\frac{I \models \forall x\,.\,F}{I[x \mapsto v] \models F}$$ For any $v \in D_I$

PROOF RULES

UNIVERSAL QUANTIFICATION

$$\frac{I \models \forall x \,.\, F}{I[x \mapsto v] \models F}$$ For any $v \in D_I$

$$\frac{I \not\models \forall x \,.\, F}{I[x \mapsto v] \not\models F}$$ For a fresh $v \in D_I$

PROOF RULES

EXISTENTIAL QUANTIFICATION

$$\frac{I \vDash \exists x \,.\, F}{I[x \mapsto v] \vDash F}$$ For a fresh $v \in D_I$

PROOF RULES

EXISTENTIAL QUANTIFICATION

$$\frac{I \vDash \exists x \,.\, F}{I[x \mapsto v] \vDash F} \quad \text{For a fresh } v \in D_I$$

$$\frac{I \nvDash \exists x \,.\, F}{I[x \mapsto v] \nvDash F} \quad \text{For any } v \in D_I$$

PROOF RULES

CONTRADICTION

$$\frac{\begin{array}{c} J \vDash p(s_1, \ldots, s_n) \quad K \nvDash p(t_1, \ldots, t_n) \\ J = I[\ldots] \quad K = I[\ldots] \\ \alpha_J[s_i] = \alpha_K[t_i] \text{ for all } i = 1,\ldots,n \end{array}}{I \vDash \bot}$$

EXAMPLE

Prove that $(\forall x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))$ is valid

$$\frac{I \not\models (\forall x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))}{I \models (\forall x \,.\, p(x)) \qquad I \not\models (\forall y \,.\, p(y))}$$

EXAMPLE

Prove that $(\forall x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))$ is valid

$$\dfrac{I \not\models (\forall x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))}{I \models (\forall x \,.\, p(x)) \qquad \dfrac{I \not\models (\forall y \,.\, p(y))}{I[y \mapsto v] \not\models p(y)}\;\text{[for a fresh } v\text{]}}$$

EXAMPLE

Prove that $(\forall x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))$ is valid

$$\dfrac{I \not\models (\forall x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))}{\dfrac{I \models (\forall x \,.\, p(x))}{I[x \mapsto v] \models p(x)} \qquad \dfrac{I \not\models (\forall y \,.\, p(y))}{I[y \mapsto v] \not\models p(y)}\ \text{[for a fresh } v\text{]}}$$

EXAMPLE

Prove that $(\forall x\,.\,p(x)) \to (\forall y\,.\,p(y))$ is valid

$$\dfrac{I \not\models (\forall x\,.\,p(x)) \to (\forall y\,.\,p(y))}{\dfrac{I \models (\forall x\,.\,p(x))}{I[x \mapsto v] \models p(x)} \qquad \dfrac{I \not\models (\forall y\,.\,p(y))}{I[y \mapsto v] \not\models p(y)}}$$

[for a fresh v]

CONTRADICTION

EXAMPLE - 2

Prove that $\exists x \,.\, F \rightarrow G \leftrightarrow (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$ is valid

EXAMPLE - 2

Prove that $\exists x \,.\, F \rightarrow G \leftrightarrow (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$ is valid

$$I \not\models \exists x \,.\, F \rightarrow G \leftrightarrow (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$$

EXAMPLE - 2

Prove that $\exists x \,.\, F \rightarrow G \leftrightarrow (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$ is valid

$$\frac{I \not\models \exists x \,.\, F \rightarrow G \leftrightarrow (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)}{I \models \exists x \,.\, F \rightarrow G \quad I \not\models (\forall x \,.\, F) \rightarrow (\exists x \,.\, G) \quad \Big| \quad I \not\models \exists x \,.\, F \rightarrow G \quad I \models (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)}$$

EXAMPLE - 2

Prove that $\exists x . F \to G \leftrightarrow (\forall x . F) \to (\exists x . G)$ is valid

$$\frac{I \not\models \exists x . F \to G \leftrightarrow (\forall x . F) \to (\exists x . G)}{I \models \exists x . F \to G \qquad I \not\models (\forall x . F) \to (\exists x . G)}$$

EXAMPLE - 2

Prove that $\exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$ is valid

$$\dfrac{I \not\models \exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)}{\dfrac{I \models \exists x . F \rightarrow G}{I[x \mapsto v] \models F \rightarrow G} \qquad I \not\models (\forall x . F) \rightarrow (\exists x . G)}$$

EXAMPLE - 2

Prove that $\exists x \,.\, F \rightarrow G \leftrightarrow (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$ is valid

$$I \nvDash \exists x \,.\, F \rightarrow G \leftrightarrow (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$$

$$I \vDash \exists x \,.\, F \rightarrow G \qquad I \nvDash (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$$

$$I[x \mapsto v] \vDash F \rightarrow G$$

$$I[x \mapsto v] \nvDash F \quad | \quad I[x \mapsto v] \vDash G$$

EXAMPLE - 2

Prove that $\exists x . F \to G \leftrightarrow (\forall x . F) \to (\exists x . G)$ is valid

$$I \not\models \exists x . F \to G \leftrightarrow (\forall x . F) \to (\exists x . G)$$

$$I \models \exists x . F \to G \qquad I \not\models (\forall x . F) \to (\exists x . G)$$

$$I[x \mapsto v] \models F \to G \qquad I \models (\forall x . F) \quad I \not\models (\exists x . G)$$

$$I[x \mapsto v] \not\models F \quad | \quad I[x \mapsto v] \models G$$

EXAMPLE - 2

Prove that $\exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$ is valid

$$I \not\models \exists x . F \rightarrow G \leftrightarrow (\forall x . F) \rightarrow (\exists x . G)$$

$$I \models \exists x . F \rightarrow G \qquad I \not\models (\forall x . F) \rightarrow (\exists x . G)$$

$$I[x \mapsto v] \models F \rightarrow G \qquad I \models (\forall x . F) \quad I \not\models (\exists x . G)$$

$$I[x \mapsto v] \not\models F \mid I[x \mapsto v] \models G \qquad I[x \mapsto v] \models F$$

EXAMPLE - 2

Prove that $\exists x \,.\, F \rightarrow G \leftrightarrow (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$ is valid

$$I \not\models \exists x \,.\, F \rightarrow G \leftrightarrow (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$$

$$I \models \exists x \,.\, F \rightarrow G \qquad I \not\models (\forall x \,.\, F) \rightarrow (\exists x \,.\, G)$$

$$I[x \mapsto v] \models F \rightarrow G \qquad I \models (\forall x \,.\, F) \quad I \not\models (\exists x \,.\, G)$$

$$\boxed{I[x \mapsto v] \not\models F} \;\Big|\; I[x \mapsto v] \models G \qquad \boxed{I[x \mapsto v] \models F}$$

CONTRADICTION

EXAMPLE - 2

Prove that $\exists x\,.\,F \rightarrow G \leftrightarrow (\forall x\,.\,F) \rightarrow (\exists x\,.\,G)$ is valid

$$I \not\models \exists x\,.\,F \rightarrow G \leftrightarrow (\forall x\,.\,F) \rightarrow (\exists x\,.\,G)$$

$$I \models \exists x\,.\,F \rightarrow G \qquad I \not\models (\forall x\,.\,F) \rightarrow (\exists x\,.\,G)$$

$$I[x \mapsto v] \models F \rightarrow G \qquad I \models (\forall x\,.\,F) \quad I \not\models (\exists x\,.\,G)$$

$$I[x \mapsto v] \not\models F \;\Big|\; I[x \mapsto v] \models G \quad I[x \mapsto v] \models F \quad I[x \mapsto v] \not\models G$$

EXAMPLE - 2

Prove that $\exists x \,.\, F \to G \leftrightarrow (\forall x \,.\, F) \to (\exists x \,.\, G)$ is valid

$$\dfrac{I \not\models \exists x \,.\, F \to G \leftrightarrow (\forall x \,.\, F) \to (\exists x \,.\, G)}{\dfrac{\dfrac{I \models \exists x \,.\, F \to G}{I[x \mapsto v] \models F \to G}}{I[x \mapsto v] \not\models F \quad \Big| \quad \boxed{I[x \mapsto v] \models G}} \qquad \Big| \qquad \dfrac{I \not\models (\forall x \,.\, F) \to (\exists x \,.\, G)}{\dfrac{I \models (\forall x \,.\, F)}{I[x \mapsto v] \models F} \quad \dfrac{I \not\models (\exists x \,.\, G)}{\boxed{I[x \mapsto v] \not\models G}}}}$$

CONTRADICTION

Homework: Complete the proof in the other branch

MORE EXAMPLES

- Prove or disprove validity of following FOL formulae
    - $\forall x \,.\, F \to G \leftrightarrow (\exists x \,.\, F) \to (\forall x \,.\, G)$
    - $(\forall x \,.\, p(x)) \leftrightarrow \neg(\exists x \,.\, \neg p(x))$
    - $(\exists x \,.\, p(x)) \to (\forall y \,.\, p(y))$
    - $\exists x \,.\, (p(x) \to \forall y \,.\, p(y))$

DECIDABILITY OF VALIDITY OF FOL

- Church and Turing showed that it is undecidable to find whether a first-order formula is valid or not.
- But we have just seen the Semantic Argument-based decision procedure!
  - How to instantiate domain values in Proof rules for quantifiers?
  - What order should proof rules be applied in?
- The semantic argument-based method can be augmented to make the validity of FOL problem semi-decidable.
  - If the input formula is valid, then the method will halt and answer positive.
  - If the input formula is not valid, then the method may never halt.
  - More details in the BM book [Chapter 2, Section 2.7].

NORMAL FORMS OF FOL

- Negation Normal Form (NNF)
  - Should use only $\neg$, $\wedge$, $\vee$ as the logical connectives, and $\neg$ should only be applied to literals
  - $\neg(\forall x . F) \Leftrightarrow \exists x . \neg F$ and $\neg(\exists x . F) \Leftrightarrow \forall x . \neg F$

PRENEX NORMAL FORM

- A formula is in Prenex Normal Form (PNF) if all of its quantifiers appear at the beginning of the formula:
  - $Q_1 x_1 \ldots Q_n x_n \,.\, F[x_1, \ldots, x_n]$, where F is quantifier-free and may have $x_1, \ldots, x_n$ as free variables.
- How to convert an arbitrary formula F to PNF?

1. First, convert F to NNF (call it F_1).
2. If two quantified variables in F_1 have the same name, then rename them to fresh variables (obtaining the formula F_2).
3. Remove all quantifiers in F_2 to obtain F_3.
4. Add all the removed quantifiers at the beginning of F_3, ensuring that if Q_j was in the scope of Q_i in F_2, then Q_i occurs before Q_j

PRENEX NORMAL FORM

EXAMPLE

$$F: \ \forall x.\ \neg(\exists y.\ p(x,y)\ \wedge\ p(x,z))\ \vee\ \exists y.\ p(x,y)$$

STEP-1

$$F_1: \ \forall x.\ (\forall y.\ \neg p(x,y)\ \vee\ \neg p(x,z))\ \vee\ \exists y.\ p(x,y)$$

STEP-2

$$F_2: \ \forall x.\ (\forall y.\ \neg p(x,y)\ \vee\ \neg p(x,z))\ \vee\ \exists w.\ p(x,w)$$

STEP-3

$$F_3: \ \neg p(x,y)\ \vee\ \neg p(x,z)\ \vee\ p(x,w)$$

STEP-4

$$\forall x.\ \forall y.\ \exists w.\ \neg p(x,y)\ \vee\ \neg p(x,z)\ \vee\ p(x,w)$$