ABSTRACT FORWARD PROPAGATE

KILDALL'S ALGORITHM

```
AbstractForwardPropagate(Γ_c,P)
  S := {l_0};
  μ̂(l_0) := α(P);
  μ̂(l) := ⊥, for l ∈ L\{l_0};
  while S ≠ ∅ do{
     l := Choose S;
     S := S \ {l};
     foreach (l,c,l') ∈ T do{
        F := ŝp(μ̂(l),c);
        if ¬(F ≤ μ̂(l')) then{
           μ̂(l') := μ̂(l') ⊔ F;
           S := S ∪ {l'};
        }
     }
  }
```

- Does this algorithm actually calculate the abstract JOP?
- What are the conditions under which it is guaranteed to terminate?

A Unified Approach to Global Program Optimization. Gary Kildall. POPL 1973.

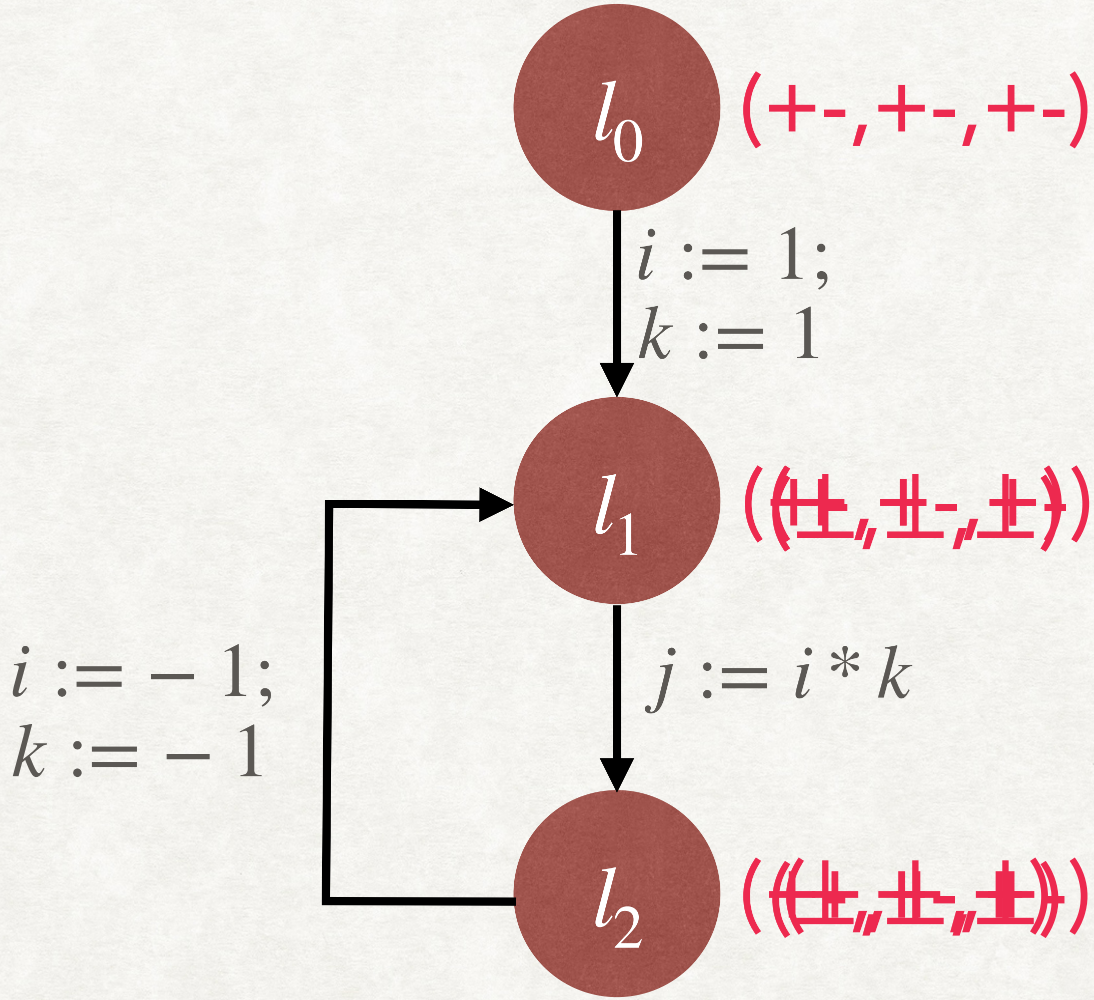
ABSTRACT FORWARD PROPAGATE

KILDALL'S ALGORITHM

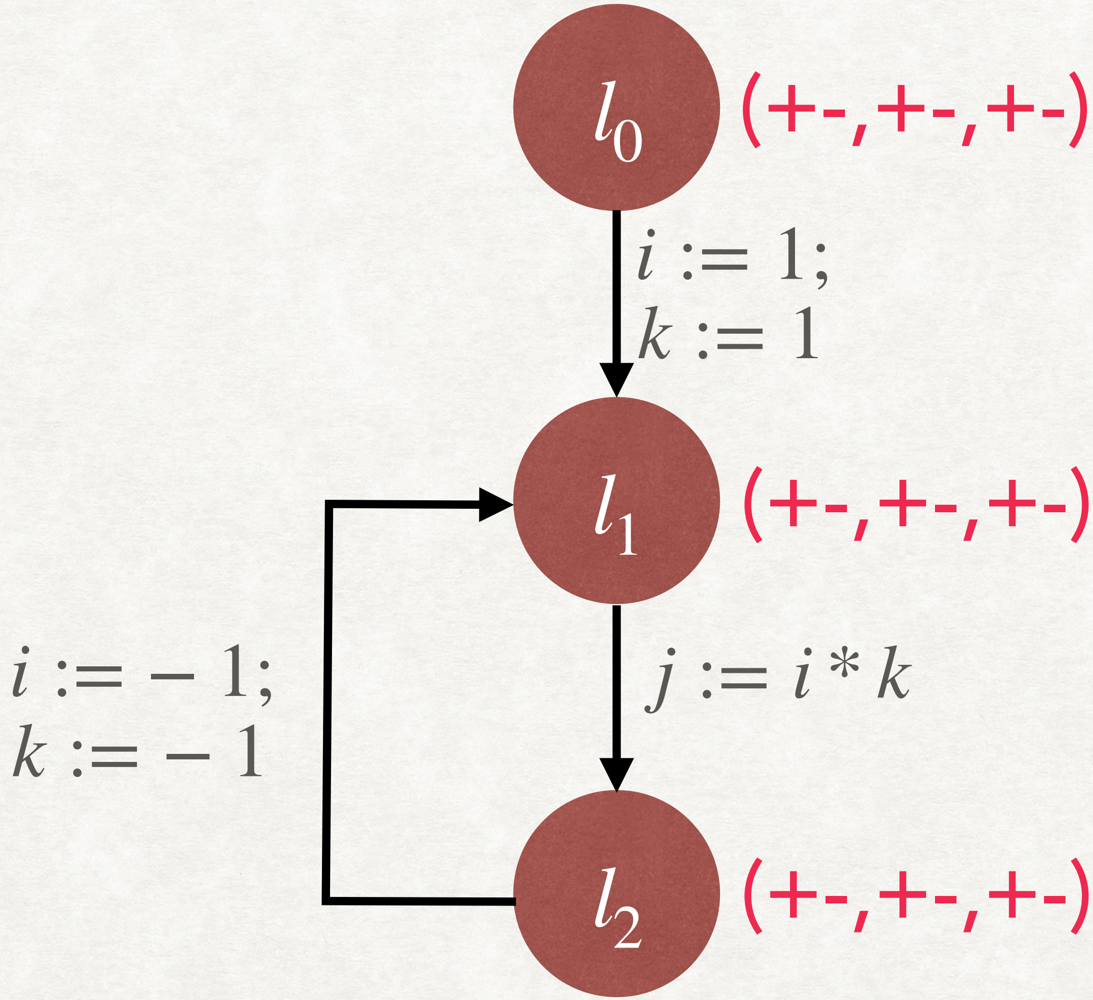
AbstractForwardPropagate(Γ_c,P)
  S := $\{l_0\}$;
  $\hat{\mu}_K(l_0)$:= $\alpha(\mathsf{P})$;
  $\hat{\mu}_K(l)$:= $\perp$, for $l \in L \backslash \{l_0\}$;
  while S $\neq \varnothing$ do{
    l := Choose S;
    S := S \ $\{l\}$;
    foreach $(l, c, l') \in T$ do{
      F := $\hat{f}_c(\hat{\mu}_K(l))$;
      if $\neg(\mathsf{F} \leq \hat{\mu}_K(l'))$ then{
        $\hat{\mu}_K(l') := \hat{\mu}_K(l') \sqcup F$;
        S := S $\cup \{l'\}$;
      }
    }
  }

A Unified Approach to Global Program Optimization. Gary Kildall. POPL 1973.

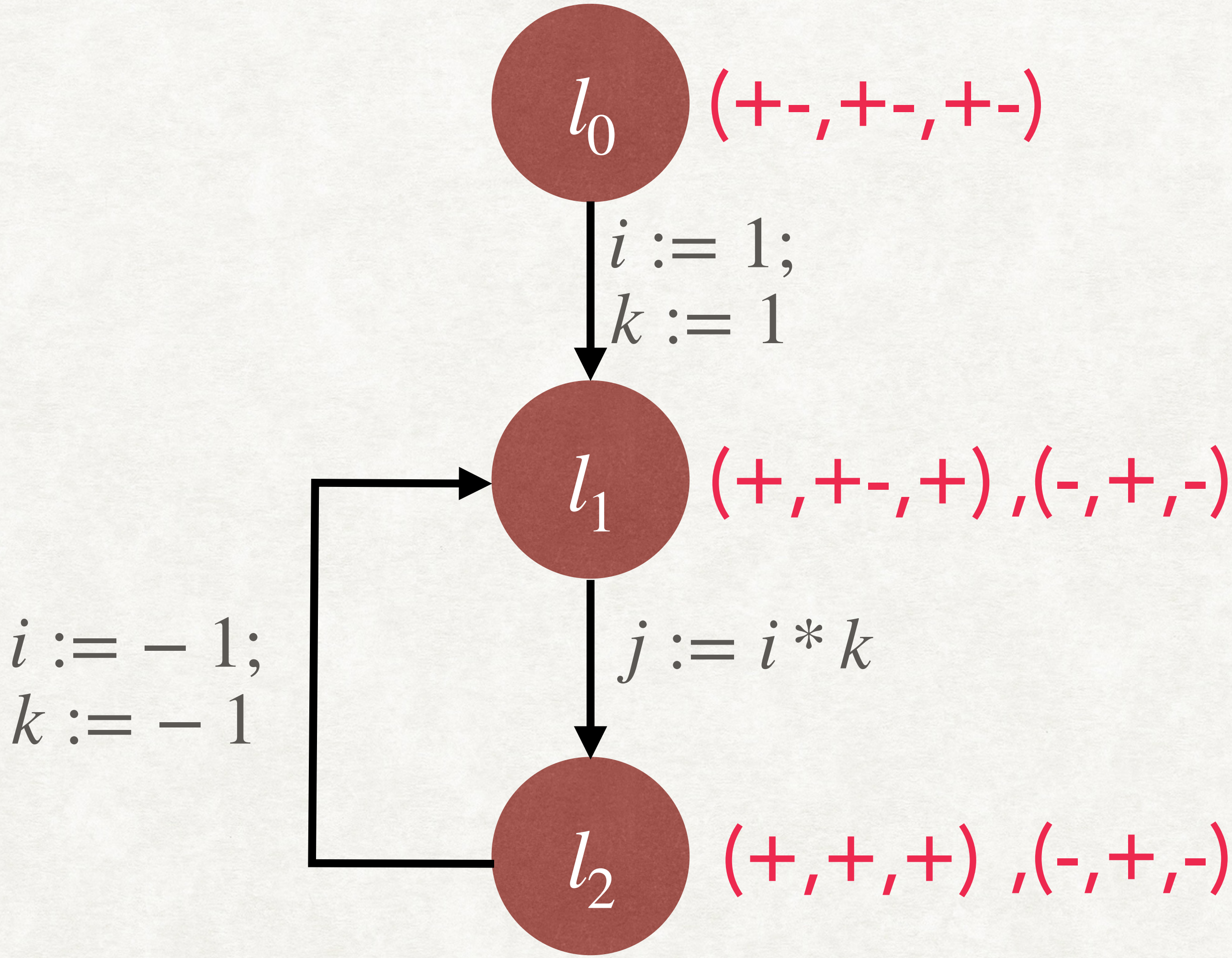
EXAMPLE - KILDALL'S ALGORITHM



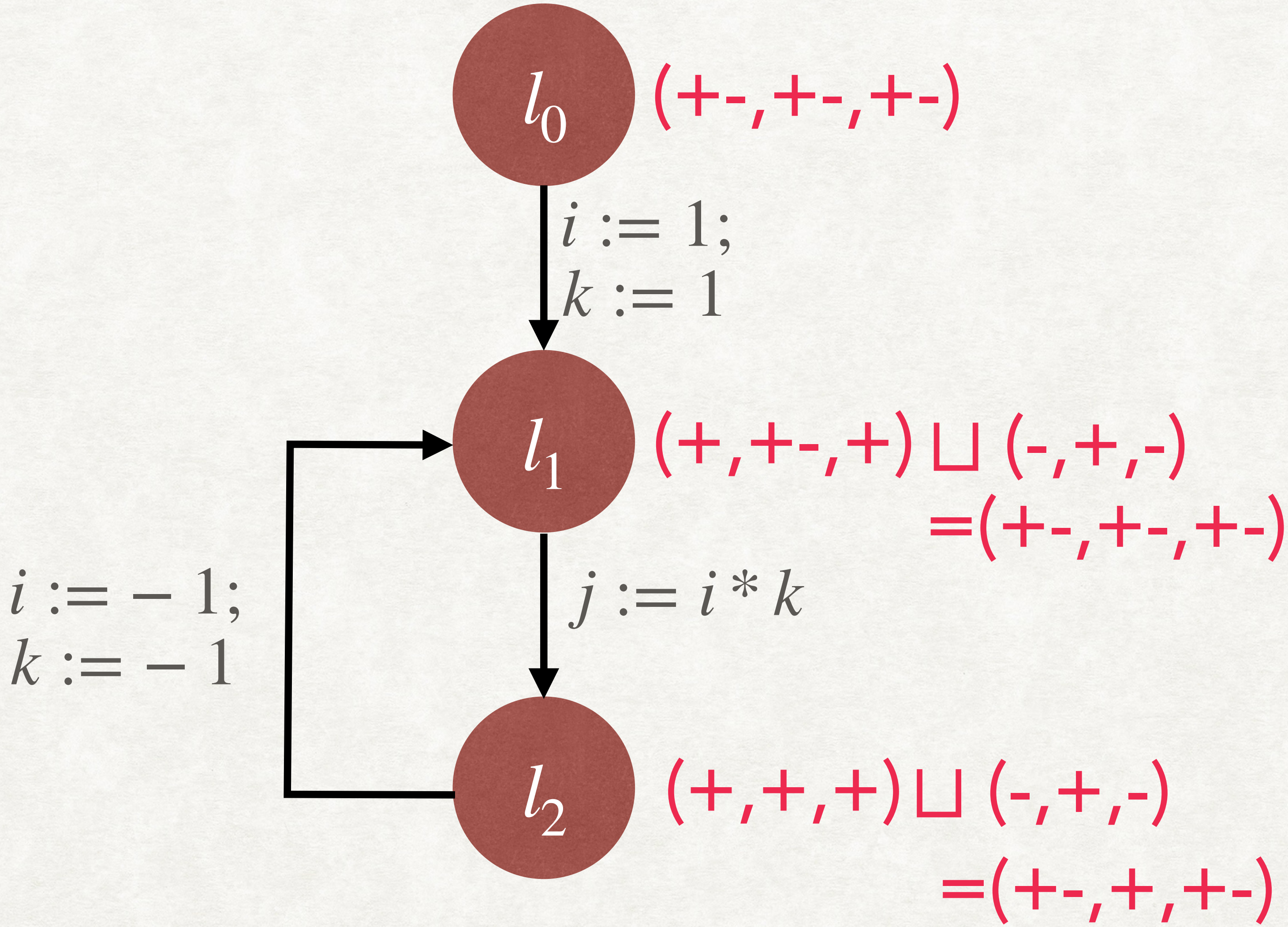
EXAMPLE - KILDALL'S ALGORITHM



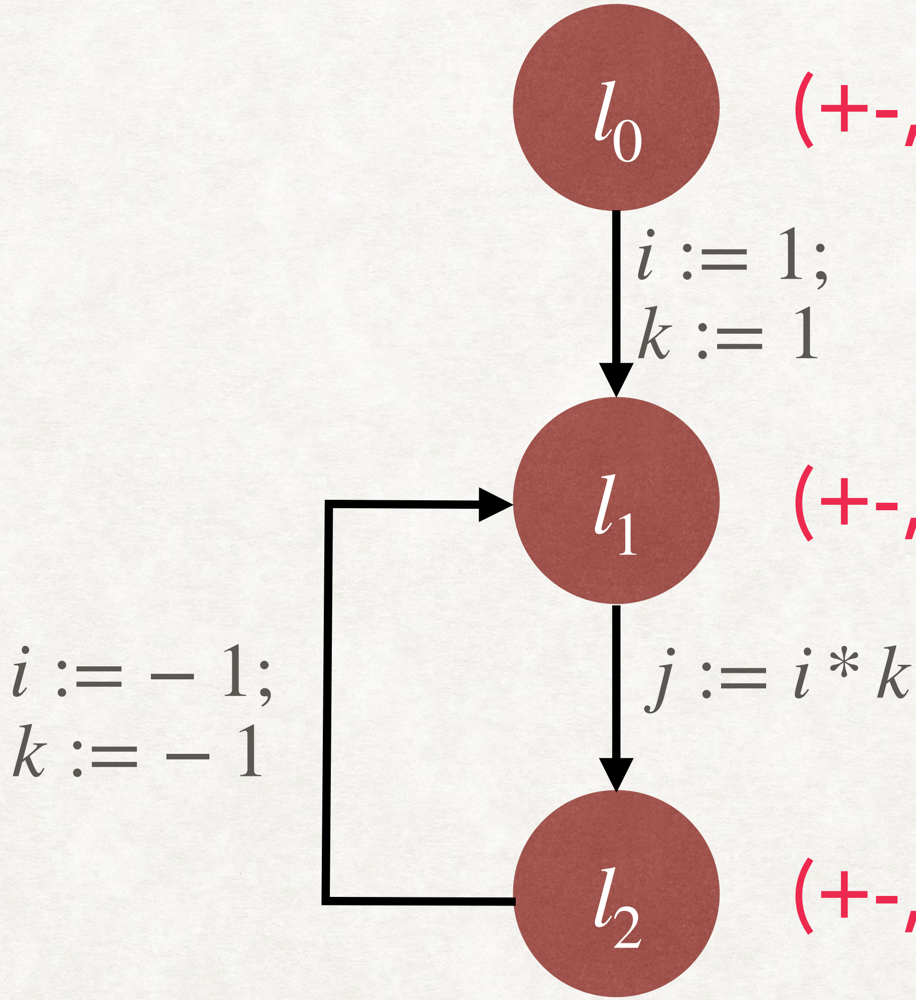
EXAMPLE - ABSTRACT JOP



EXAMPLE - ABSTRACT JOP



EXAMPLE - KILDALL VS ABSTRACT JOP



	$\hat{\mu}_K$	$\hat{\mu}$
l_0	(+-,+-,+-)	(+-,+-,+-)
l_1	(+-,+-,+-)	(+-,+-,+-)
l_2	(+-,+-,+-)	(+-,+,+-)

$$\hat{\mu}_K \neq \hat{\mu}$$

We will prove that $\hat{\mu}_K \geq \hat{\mu}$

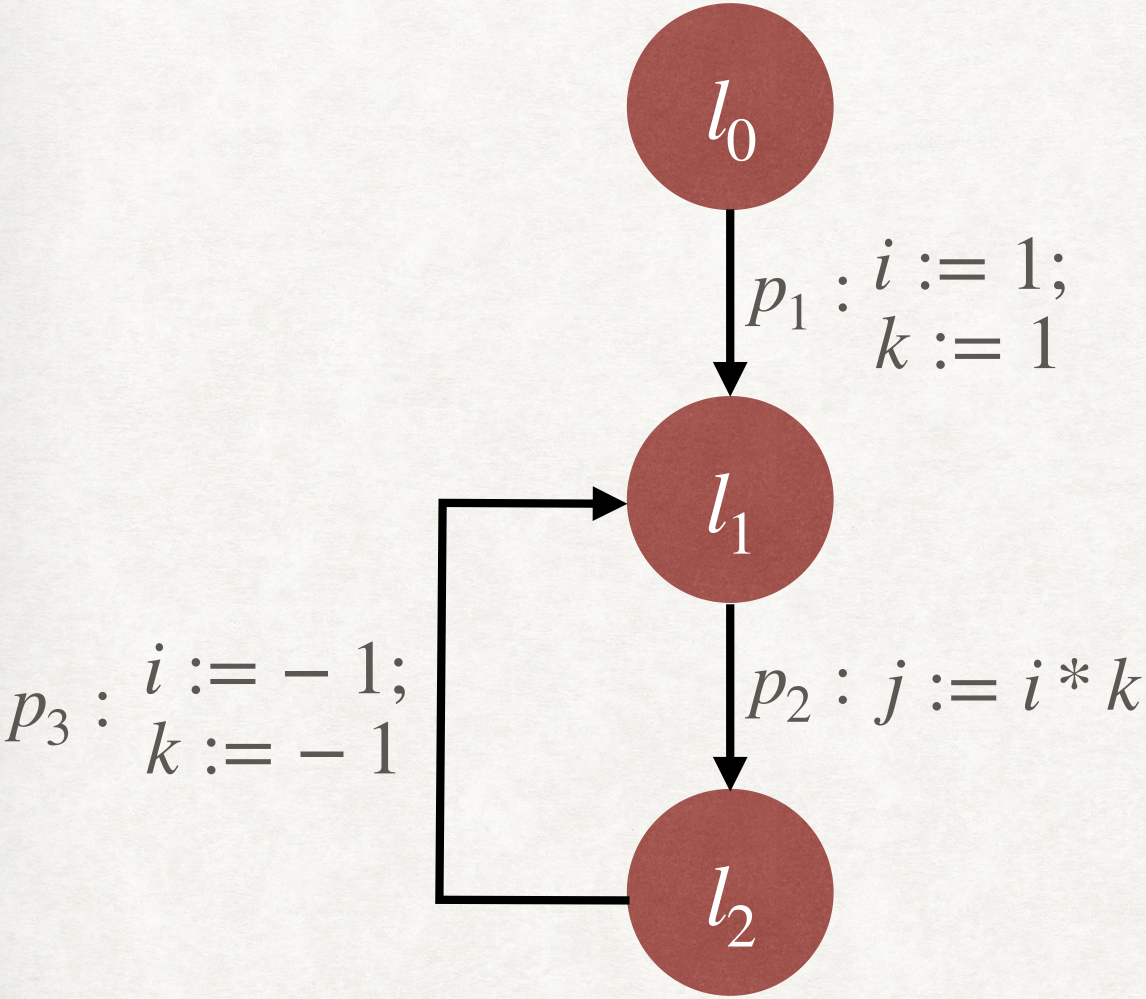
PROPERTIES OF KILDALL'S ALGORITHM

1. The values computed using Kildall's algorithm are an over-approximation of the abstract JOP, if the underlying AI framework is monotonic.
2. In general, Kildall's algorithm computes the least solution to a system of equations.
3. If the abstract domain satisfies the ascending chain condition, then Kildall's algorithm is guaranteed to terminate.

DATAFLOW EQUATIONS

- Program $\Gamma_c = (V, L, l_0, l_e, T)$ induces a system of data flow equations:
  - $X_{l_0} = d_0$
  - For all other locations $l \in L \backslash \{l_0\},\ X_l = \bigsqcup_{(l',c,l) \in T} \hat{f}_c(X_{l'})$
- For collecting semantics, replace d_0 with c_0, $\sqcup$ with $\cup$ and $\hat{f}_c$ with f_c.

EXAMPLE - DATAFLOW EQUATIONS



$$X_{l_0} = d_0$$

$$X_{l_1} = \hat{f}_{p_1}(X_{l_0}) \sqcup \hat{f}_{p_3}(X_{l_2})$$

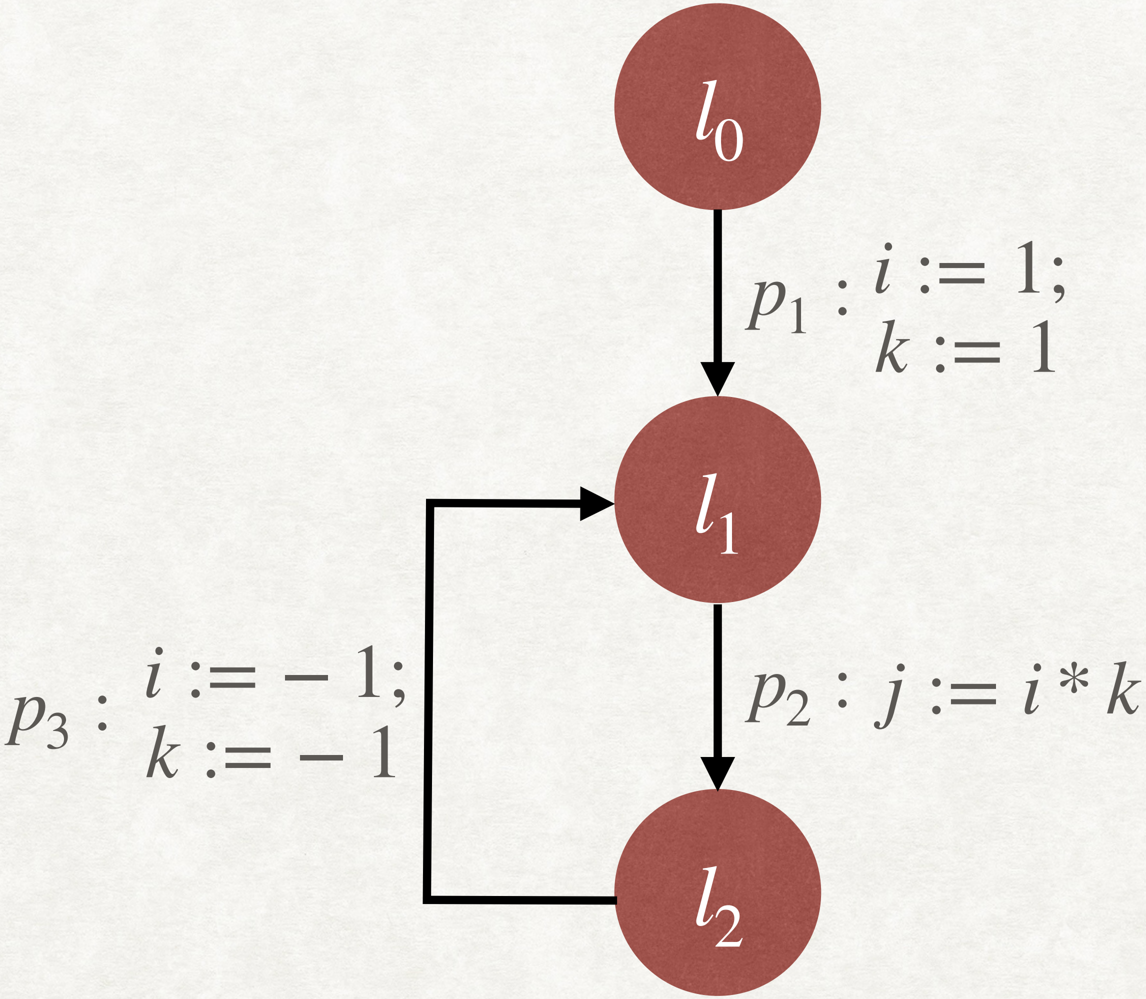
$$X_{l_2} = \hat{f}_{p_2}(X_{l_1})$$

DATAFLOW EQUATIONS AS FUNCTION

- Consider the 'vectorised' lattice $(\bar{D}, \bar{\leq})$, where $\bar{D} = D^{|L|}$.
  - $\bar{d} \mathrel{\bar{\leq}} \bar{d}' \Leftrightarrow \forall l \in L \,.\, \bar{d}(l) \leq \bar{d}'(l)$
  - Homework: Prove that if $(D, \leq)$ is a complete lattice, then $(\bar{D}, \bar{\leq})$ is also a complete lattice.
- We can view the data flow equations as a function $\bar{f} : \bar{D} \to \bar{D}$:
  - $(\bar{f}(\bar{d}))(l_0) = d_0$
  - $(\bar{f}(\bar{d}))(l) = \bigsqcup_{(l', c, l) \in T} \hat{f}_c(\bar{d}(l'))$

DATAFLOW EQUATIONS AS FUNCTION

EXAMPLE



Notice that a fixpoint of $\bar{f}$ is a solution to the dataflow equations

$$\bar{f}(d_{l_0}, d_{l_1}, d_{l_2}) = (d_0, \hat{f}_{p_1}(d_{l_0}) \sqcup \hat{f}_{p_3}(d_{l_2}), \hat{f}_{p_2}(d_{l_1}))$$

DATAFLOW EQUATIONS AS FUNCTION

- If every abstract transfer function $\hat{f} : D \to D$ is monotonic, then the function $\bar{f} : \bar{D} \to \bar{D}$ is also monotonic.
  - Homework: Prove this.
- We have a monotonic function $\bar{f}$ on a complete lattice $\bar{D}$. Hence, we can apply Knaster-Tarski theorem.
- The least fixpoint $lfp(\bar{f})$ exists, and is in fact the least solution to the dataflow equations.
- We will show that Kildall's algorithm actually computes $lfp(\bar{f})$.
- Note that we can also use the sequence $\perp, \bar{f}(\perp), \bar{f}^2(\perp), \ldots$ to compute $lfp(\bar{f})$.
  - This method is also called Kleene Iteration.

ABSTRACT JOP $\leq lfp(\bar{f})$ FOR MONOTONIC AI FRAMEWORK

PROOF

- Given AI $(D, \leq, \alpha, \gamma, \hat{F}_D)$, if all functions in $\hat{F}_D$ are monotonic, then Abstract JOP $\leq lfp(\bar{f})$.

Proof: Abstract JOP $\hat{\mu}(l) = \bigsqcup_{\pi \in \Pi_l} \hat{f}_\pi(d_0)$

Let $lfp(\bar{f}) = \bar{d}$. We have to show that $\forall l \in L \,.\, \hat{\mu}(l) \leq \bar{d}(l)$.

We will show that for all locations l, all paths $\pi \in \Pi_l$, $\hat{f}_\pi(d_0) \leq \bar{d}(l)$.

This proves the required result. Why?

We will use induction on length of the paths.

Base Case: Paths π of length 0 are empty and end at l_0. Hence, $\hat{f}_\pi(d_0) = d_0$.

Since $\bar{f}(\bar{d}) = \bar{d}$ and $(\bar{f}(\bar{d}))(l_0) = d_0$, we have $\bar{d}(l_0) = d_0$.

Thus, $\hat{f}_\pi(d_0) \leq \bar{d}(l_0)$

ABSTRACT JOP $\leq lfp(\bar{f})$ FOR MONOTONIC AI FRAMEWORK

PROOF

Inductive Case: Assume that the claim holds for all paths of length n.

Consider a path π of length $n+1$ ending at location l.

Let π' be the prefix of the path of length n, ending at location l'.

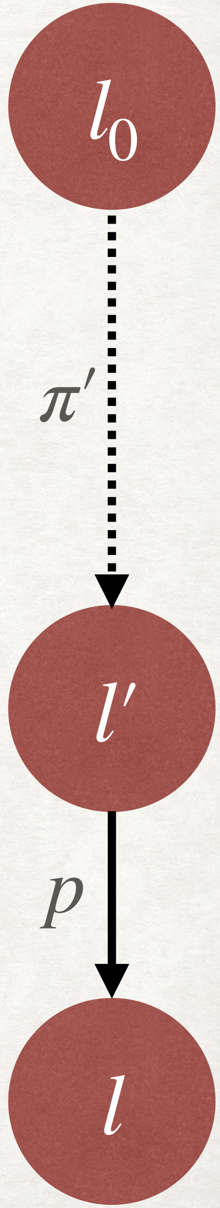
By Inductive Hypothesis, $\hat{f}_{\pi'}(d_0) \leq \bar{d}(l')$.

Since $\hat{f}_p$ is monotonic, $\hat{f}_p(\hat{f}_{\pi'}(d_0)) \leq \hat{f}_p(\bar{d}(l'))$.

Hence, $\hat{f}_\pi(d_0) \leq \hat{f}_p(\bar{d}(l'))$.

Now $\bar{f}(\bar{d}) = \bar{d}$. Hence, $\bar{d}(l) = \bigsqcup_{(l',p,l)\in T} \hat{f}_p(\bar{d}(l'))$.

Hence, $\hat{f}_p(\bar{d}(l')) \leq \bar{d}(l)$. Thus, $\hat{f}_\pi(d_0) \leq \bar{d}(l)$.



ABSTRACT JOP $= lfp(\bar{f})$ FOR INFINITELY DISTRIBUTIVE AI FRAMEWORK

PROOF

- Given AI $(D, \leq, \alpha, \gamma, \hat{F}_D)$, if all functions in $\hat{F}_D$ are infinitely distributive, then Abstract JOP $= lfp(\bar{f})$.

Proof: We will show that Abstract JOP $(\hat{\mu})$ is a fixpoint of $\bar{f}$. This is sufficient to prove the result. Why?

$$
\begin{aligned}
(\bar{f}(\hat{\mu}))(l) &= \bigsqcup_{(l',p,l)\in T} \hat{f}_p(\hat{\mu}(l')) \\
&= \bigsqcup_{(l',p,l)\in T} \hat{f}_p\Big(\bigsqcup_{\pi\in\Pi_{l'}} \hat{f}_\pi(d_0)\Big) \\
&= \bigsqcup_{(l',p,l)\in T} \bigsqcup_{\pi\in\Pi_{l'}} \hat{f}_p \circ \hat{f}_\pi(d_0)
\end{aligned}
$$

ABSTRACT JOP = $lfp(\bar{f})$ FOR INFINITELY DISTRIBUTIVE AI FRAMEWORK

PROOF

$$(\bar{f}(\hat{\mu}))(l) = \bigsqcup_{(l',p,l)\in T} \bigsqcup_{\pi\in\Pi_{l'}} \hat{f}_p \circ \hat{f}_\pi(d_0)$$

And we know that $\hat{\mu}(l) = \bigsqcup_{\pi'\in\Pi_l} \hat{f}_{\pi'}(d_0)$.

Then, $(\bar{f}(\hat{\mu}))(l) = \hat{\mu}(l)$. Why?

Thus, $\hat{\mu}$ is a fixpoint of $\bar{f}$. We know from previous result that $\hat{\mu} \leq lfp(\bar{f})$. Thus, $\hat{\mu} = lfp(\bar{f})$.

RECALL: SIGN ABSTRACT DOMAIN

- Sign abstract domain is not infinitely distributive.

Consider p : j := i*k and $d_1 = (+\,, +-\,, +\,)$, $d_2 = (-\,, +-\,, -\,)$.

Then, $\hat{f}_p(d_1 \sqcup d_2) = ???$

RECALL: SIGN ABSTRACT DOMAIN

- Sign abstract domain is not infinitely distributive.

Consider p : j := i*k and $d_1 = (\,+\,,+-\,,+\,)$, $d_2 = (\,-\,,+-\,,-\,)$.

Then, $\hat{f}_p(d_1 \sqcup d_2) = (\,+-\,,+-\,,+-\,)$.

RECALL: SIGN ABSTRACT DOMAIN

- Sign abstract domain is not infinitely distributive.

Consider p : j := i*k and $d_1 = (\, + \,,\, + - \,,\, + \,)$, $d_2 = (\, - \,,\, + - \,,\, - \,)$.

Then, $\hat{f}_p(d_1 \sqcup d_2) = (\, + - \,,\, + - \,,\, + - \,)$.

$\hat{f}_p(d_1) \sqcup \hat{f}_p(d_2) = ???$

RECALL: SIGN ABSTRACT DOMAIN

- Sign abstract domain is not infinitely distributive.

Consider p : j := i*k and $d_1 = (+\,,+-\,,+)$, $d_2 = (-\,,+-\,,-)$.

Then, $\hat{f}_p(d_1 \sqcup d_2) = (+-\,,+-\,,+-)$.

$\hat{f}_p(d_1) \sqcup \hat{f}_p(d_2) = (+\,,+\,,+) \sqcup (-\,,+\,,-) = (+-\,,+\,,+-)$

RECALL: SIGN ABSTRACT DOMAIN

- Sign abstract domain is not infinitely distributive.

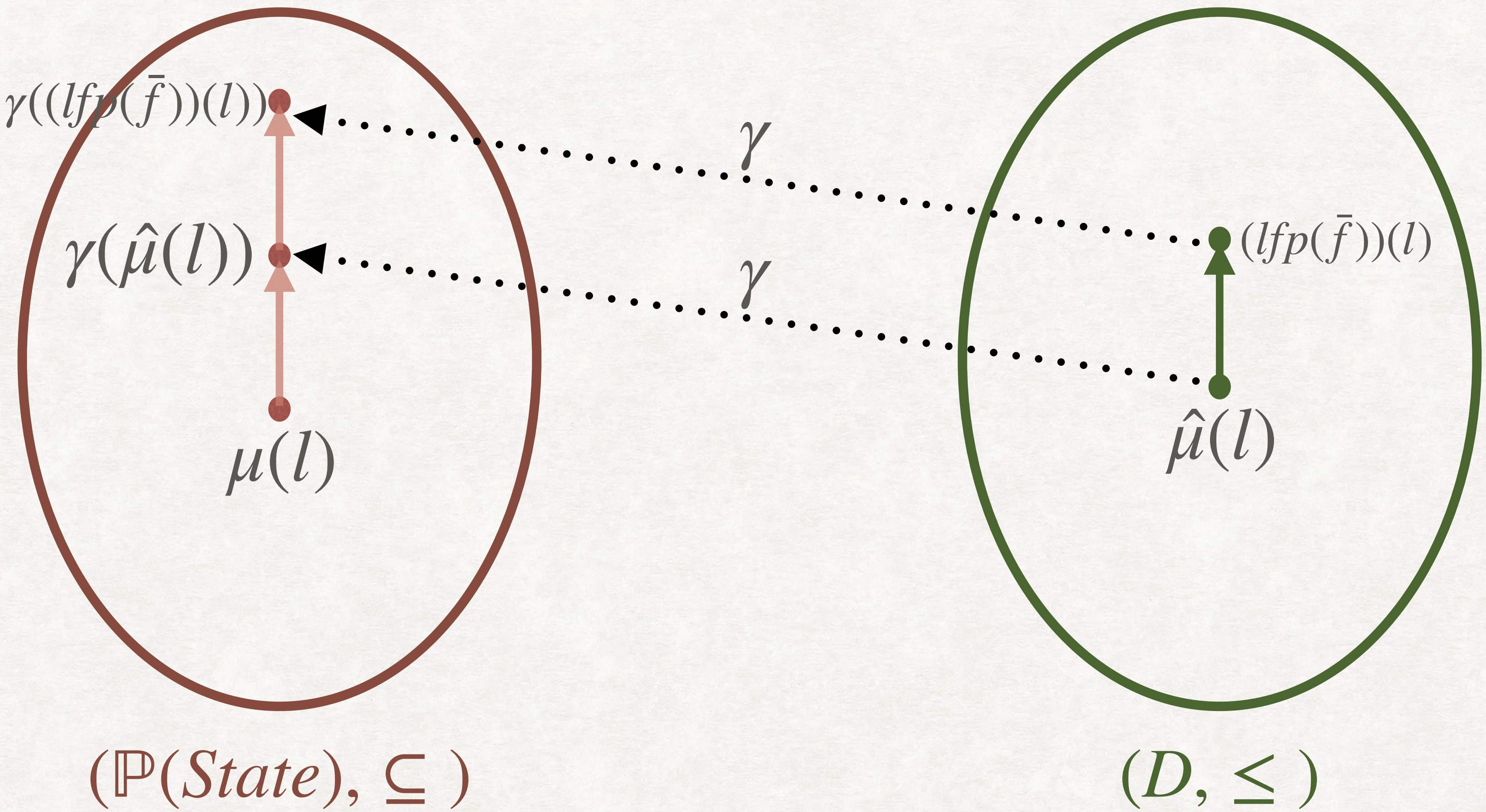
Consider p : j := i*k and $d_1 = (+\,,+-\,,+)$, $d_2 = (-\,,+-\,,-)$.

Then, $\hat{f}_p(d_1 \sqcup d_2) = (+-\,,+-\,,+-)$.

$\hat{f}_p(d_1) \sqcup \hat{f}_p(d_2) = (+\,,+\,,+) \sqcup (-\,,+\,,-) = (+-\,,+\,,+-)$

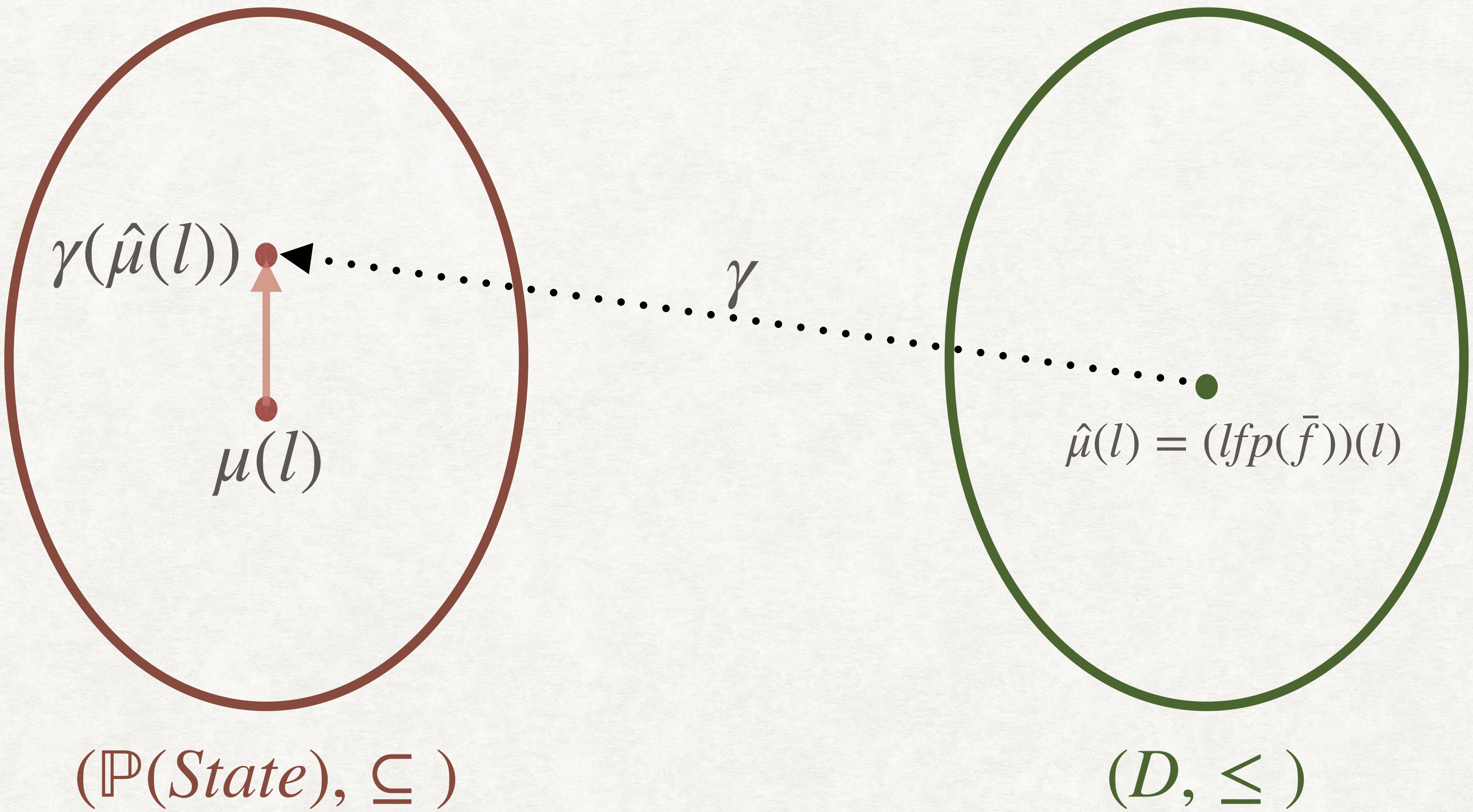
- The concrete transfer functions are all infinitely distributive. Hence, the concrete JOP is the least solution of the data-flow equations.

BIG PICTURE



For Monotonic AI Framework

BIG PICTURE



For Infinitely Distributive AI Framework

KILDALL'S ALGORITHM COMPUTES $lfp(\bar{f})$

PROOF

- First, we will show that $\hat{\mu}_K \leq lfp(\bar{f})$

We will show that $\hat{\mu}_K \leq lfp(\bar{f})$ is a loop invariant of the outer while loop.

At the beginning, $\hat{\mu}_K(l_0) = \alpha(P) \leq d_0$.

Hence, $\forall l \,.\, \hat{\mu}_K(l) \leq (lfp(\bar{f}))(l)$.

Assuming that the claim holds at the beginning of some iteration, let $\hat{\mu}_K = \bar{d}$, $lfp(\bar{f}) = \bar{g}$. We have $\bar{d} \leq \bar{g}$.

For some successor l' of l, $\hat{\mu}_K(l') = d(l') \sqcup \hat{f}_c(d(l))$.

Now, $\bar{d}(l) \leq \bar{g}(l) \Rightarrow \hat{f}_c(\bar{d}(l)) \leq \hat{f}_c(\bar{g}(l))$

```
AbstractForwardPropagate(Γ_c,P)
  S := {l_0};
  μ̂_K(l_0) := α(P);
  μ̂_K(l) := ⊥, for l ∈ L\{l_0};
  while S ≠ ∅ do{
      l := Choose S;
      S := S \ {l};
      foreach (l,c,l') ∈ T do{
          F := f̂_c(μ̂_K(l));
          if ¬(F ≤ μ̂_K(l')) then{
              μ̂_K(l') := μ̂_K(l') ⊔ F;
              S := S ∪ {l'};
          }
      }
  }
```

KILDALL'S ALGORITHM COMPUTES $lfp(\bar{f})$

PROOF

Now, $\bar{g}(l') = \bigsqcup_{(l,c,l') \in T} \hat{f}_c(\bar{g}(l))$

Hence, $\bar{g}(l') \geq \hat{f}_c(\bar{g}(l)) \geq \hat{f}_c(\bar{d}(l))$

We also know that $\bar{g}(l') \geq \bar{d}(l')$.

Thus, $\bar{g}(l') \geq \bar{d}(l') \sqcup \hat{f}_c(\bar{d}(l))$.

Hence, $\bar{g}(l') \geq \hat{\mu}_K(l')$.

```
AbstractForwardPropagate(Γ_c,P)
  S := {l_0};
  μ̂_K(l_0) := α(P);
  μ̂_K(l) := ⊥, for l ∈ L\{l_0};
  while S ≠ ∅ do{
     l := Choose S;
     S := S \ {l};
     foreach (l,c,l') ∈ T do{
        F := f̂_c(μ̂_K(l));
        if ¬(F ≤ μ̂_K(l')) then{
           μ̂_K(l') := μ̂_K(l') ⊔ F;
           S := S ∪ {l'};
        }
     }
  }
```

KILDALL'S ALGORITHM COMPUTES $lfp(\bar{f})$

PROOF

Next, we will show that $\hat{\mu}_K \geq lfp(\bar{f})$.

To prove this, we will show that when the algorithm terminates, the final $\hat{\mu}_K$ is a post-fixpoint of $\bar{f}$, i.e. $\bar{f}(\hat{\mu}_K) \leq \hat{\mu}_K$.

Then, by Knaster-Tarski theorem, $lfp(\bar{f})$ is the glb of all post-fixpoints, and hence the claim follows.

We will prove that following is a loop invariant of the outer while-loop:

$$\forall l \in L \backslash S \,.\, \forall l' \in L \,.\, (l, c, l') \in T \Rightarrow \hat{\mu}_K(l') \geq \hat{f}_c(\hat{\mu}_K(l))$$

```
AbstractForwardPropagate(Γ_c,P)
  S := {l_0};
  μ̂_K(l_0) := α(P);
  μ̂_K(l) := ⊥, for l ∈ L\{l_0};
  while S ≠ ∅ do{
      l := Choose S;
      S := S \ {l};
      foreach (l,c,l') ∈ T do{
          F := f̂_c(μ̂_K(l));
          if ¬(F ≤ μ̂_K(l')) then{
              μ̂_K(l') := μ̂_K(l') ⊔ F;
              S := S ∪ {l'};
          }
      }
  }
```

KILDALL'S ALGORITHM COMPUTES $lfp(\bar{f})$

PROOF

$\forall l \in L \backslash S \,.\, \forall l' \in L \,.\, (l, c, l') \in T$

$\Rightarrow \hat{\mu}_K(l') \geq \hat{f}_c(\hat{\mu}_K(l))$

On exiting the loop, we will have

$\forall l, l' \in L \,.\, (l, c, l') \in T \Rightarrow \hat{\mu}_K(l') \geq \hat{f}_c(\hat{\mu}_K(l))$

$$\Rightarrow \forall l' \in L \,.\, \hat{\mu}_K(l') \geq \bigsqcup_{(l,c,l') \in T} \hat{f}_c(\hat{\mu}_K(l))$$

$$\Rightarrow \forall l' \in L \,.\, \hat{\mu}_K(l') \geq (\bar{f}(\hat{\mu}_K))(l')$$

At the beginning, the invariant holds, assuming that $\hat{f}_c(\perp) = \perp$.

Note that if $\hat{f}_c(\perp) \neq \perp$, we can initialise S with L.

```
AbstractForwardPropagate(Γ_c,P)
  S := {l_0};
  μ̂_K(l_0) := α(P);
  μ̂_K(l) := ⊥, for l ∈ L\{l_0};
  while S ≠ ∅ do{
      l := Choose S;
      S := S \ {l};
      foreach (l,c,l') ∈ T do{
          F := f̂_c(μ̂_K(l));
          if ¬(F ≤ μ̂_K(l')) then{
              μ̂_K(l') := μ̂_K(l') ⊔ F;
              S := S ∪ {l'};
          }
      }
  }
```

KILDALL'S ALGORITHM COMPUTES $lfp(\bar{f})$

PROOF

$$\forall l \in L \backslash S \,.\, \forall l' \in L \,.\, (l, c, l') \in T \Rightarrow \hat{\mu}_K(l') \geq \hat{f}_c(\hat{\mu}_K(l))$$

Assume that the claim holds at the beginning of some iteration.

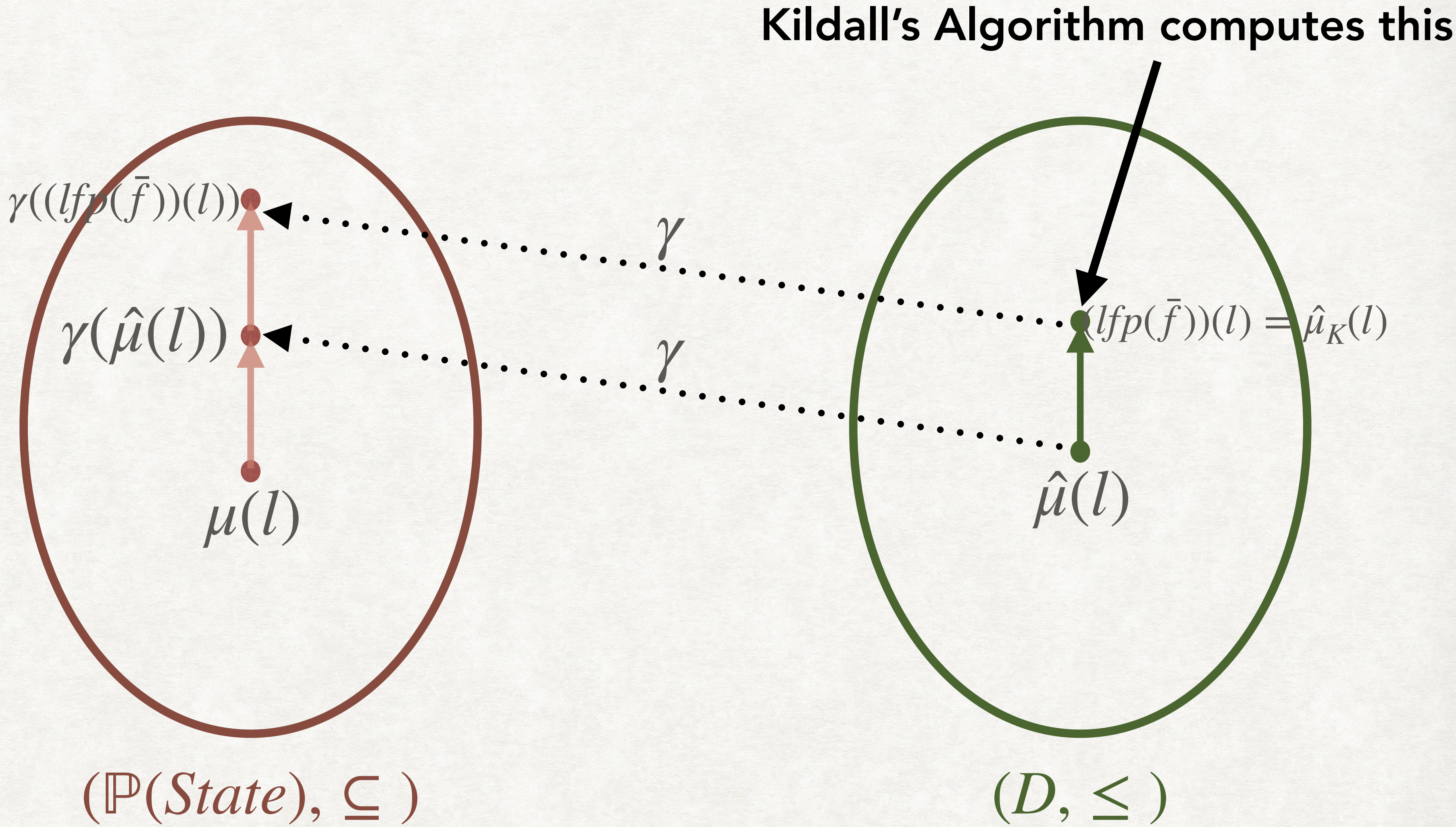
For each successor l' of l, either $\hat{\mu}_K(l') \geq \hat{f}_c(\hat{\mu}_K(l))$, or we enter the if-body and re-assign $\hat{\mu}_K(l')$ to ensure that $\hat{\mu}_K(l') \geq \hat{f}_c(\hat{\mu}_K(l))$.

Thus, the loop invariant continues to hold.

This concludes the proof that the final $\hat{\mu}_K = lfp(\bar{f})$.

```
AbstractForwardPropagate(Γ_c,P)
  S := {l_0};
  μ̂_K(l_0) := α(P);
  μ̂_K(l) := ⊥, for l ∈ L\{l_0};
  while S ≠ ∅ do{
    l := Choose S;
    S := S \ {l};
    foreach (l,c,l') ∈ T do{
      F := f̂_c(μ̂_K(l));
      if ¬(F ≤ μ̂_K(l')) then{
        μ̂_K(l') := μ̂_K(l') ⊔ F;
        S := S ∪ {l'};
      }
    }
  }
```

BIG PICTURE



For Monotonic AI Framework

KILDALL'S ALGORITHM: TERMINATION

- Consider the vector of values maintained by the algorithm across locations.
- After each iteration of the outer loop, either this vector increases or it stays the same and S decreases.
- If $(D, \leq)$ satisfies the ascending chain condition, then so does $(\bar{D}, \bar{\leq})$.
  - In this case, the loop is guaranteed to terminate.

```
AbstractForwardPropagate(Γ_c,P)
  S := {l_0};
  μ̂_K(l_0) := α(P);
  μ̂_K(l) := ⊥, for l ∈ L\{l_0};
  while S ≠ ∅ do{
      l := Choose S;
      S := S \ {l};
      foreach (l,c,l') ∈ T do{
          F := f̂_c(μ̂_K(l));
          if ¬(F ≤ μ̂_K(l')) then{
              μ̂_K(l') := μ̂_K(l') ⊔ F;
              S := S ∪ {l'};
          }
      }
  }
```

KILDALL'S ALGORITHM

SUFFICIENT CONDITIONS

- Kildall's Algorithm can be used with an abstract domain $(D, \leq, \alpha, \gamma, \hat{F}_D)$ if:
  - $(D, \leq)$ is a complete lattice.
  - $(\mathbb{P}(State), \subseteq) \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} (D, \leq)$
  - Every abstract transfer function in $\hat{F}_D$ is a consistent abstraction of the corresponding concrete transfer function.
  - Every abstract transfer function in $\hat{F}_D$ is monotonic.
  - $(D, \leq)$ satisfies the ascending chain condition.