Kartikeye Chadda
3CO15
102103708

Parameter Estimation Assignment

1. Let $(x_1, x_2, ---)$ be random sample of size n taken from normal population with parameters : mean $= \theta_1$ & var $= \theta_2$. Find max likelihood estimates of 2 parameters.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$ (Pdf of normal dist.)

$\mu = \theta_1$, $\sigma^2 = \theta_2$

$$f(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$f(x_i) = \frac{1}{\sqrt{2\pi\theta_2}} e^{\frac{-(x-\theta_1)^2}{2\theta_2}}$$

Likelihood function

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

$$L(\theta_1, \theta_2) = (\theta_2)^{n/2} (2\pi)^{-n/2} e^{\prod_{i=1}^{n} \frac{(x_i-\theta_1)^2}{2\theta_2}}$$

$$L(\theta_1, \theta_2) = (\theta_2)^{-n/2} (2\pi)^{-n/2} e^{\left(\frac{-1}{2\theta_2} \prod_{i=1}^{n} (x_i-\theta_1)^2\right)} \quad \text{— (1)}$$

Taking log both sides

$$\ln(L(\theta_1, \theta_2)) = \ln\left[(\theta_2)^{-n/2} (2\pi)^{-n/2} e^{\frac{-1}{2\theta_2} \prod_{i=1}^{n} (x_i-\theta_1)^2}\right]$$

$$Z = \ln L(\theta_1, \theta_2) = -\frac{n}{2}\ln\theta_2 - \frac{n}{2}\ln(2\pi) - \frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2 \quad \text{— (2)}$$

Diff (2) wrt θ_1

$$\frac{dZ}{d\theta_1} = \frac{1}{\theta_2}\sum_{i=1}^{n}(x_i-\theta_i)$$

Now, $\frac{\partial Z}{\partial \theta_1} = 0$

$$\frac{1}{\theta_2}\left(\sum_{i=1}^{n} x_i - n\theta_1\right) = 0$$

$$\sum_{i=1}^{n} x_i = n\theta_1$$

$$\theta_1 = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\boxed{\theta_1 = \bar{x}_n} \quad — (3)$$

Diff (2) with θ_2

$$\frac{\partial Z}{\partial \theta_2} = \frac{-n}{2\theta_2} + \frac{1}{2(\theta_2)^2}\sum_{i=1}^{n}(x_i - \theta_1)^2$$

Now, $\frac{\partial Z}{\theta_2} = 0$

$$\frac{-n}{2\theta_2} + \frac{1}{2(\theta_2)^2}\sum_{i=1}^{n}(x_i - \theta_1)^2 = 0$$

$$\theta_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \theta_1)^2$$

Putting θ_1 from (3)

$$\boxed{\theta_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}_n)^2}$$

(2) Let $(x_1, x_2, \ldots x_n)$ be random sample from $B(m, \theta)$ dist. where $\theta \in \Theta = (0,1)$ is unknown & m is +ve integer. Compute value of θ with MLE.

$$f(x) = {}^{n}C_{x}\, p^{x}(1-p)^{n-x}$$

$$n = m, \; p = \theta$$

$$f(x) = {}^{m}C_{x}\, \theta^{x}(1-\theta)^{m-x}$$

Likelihood function

$$L(m,\theta) = \prod_{i=1}^{n} f(x_i)$$

$$L(m,\theta) = \prod_{i=1}^{n} {}^{m}C_{x_i}\, \theta^{x_i} (1-\theta)^{m-x_i}$$

$$L(m,\theta) = \prod_{i=1}^{n} {}^{m}C_{x_i} \prod_{i=1}^{n} \theta^{x_i} \prod_{i=1}^{n} (1-\theta)^{m-x_i}$$

$$L(m,\theta) = \prod_{i=1}^{n} {}^{m}C_{x_i}\, \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{nm-\sum_{i=1}^{n} x_i}$$

Taking log both sides

$$\ln(L(m,\theta)) = \ln\left(\prod_{i=1}^{n} {}^{m}C_{x_i}\, \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{mn-\sum_{i=1}^{n} x_i}\right)$$

$$z = \ln(L(m,\theta)) = \ln\left(\prod_{i=1}^{n} {}^{m}C_{x_i}\right) + \sum_{i=1}^{n} x_i \ln\theta + \left(mn-\sum_{i=1}^{n} x_i\right)(\ln(1-\theta)) \quad \text{— (1)}$$

Diff (1) wrt θ

$$\frac{\partial z}{\partial \theta} = \frac{1}{\theta}\sum_{i=1}^{n} x_i + \left(\frac{\sum_{i=1}^{n} x_i - mn}{1-\theta}\right)$$

$$\frac{\partial z}{\partial \theta} = 0$$

$$\frac{1}{\theta}\sum_{i=1}^{n} x_i + \left(\frac{\sum_{i=1}^{n} x_i - mn}{1-\theta}\right) = 0$$

$$1 - \frac{mn}{\sum_{i=1}^{n} x_i} = \frac{\theta-1}{\theta}$$

$$\boxed{\theta = \frac{\bar{x}_n}{m}}$$

$$\boxed{\theta_{MLE} \in (0,1) = \frac{\bar{x}_n}{m}}$$