Matrix Algebra

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Introduction to matrix algebra

- A system of simultaneous linear models like $p = 4 - 2q$ and $p = 3.5 + q$ is easy to solve.
- You can use graphs, elimination, substitution, and do crammers rule all by hand.
- However, as the variables become numerous, it is becomes increasingly difficult to solve.
- Matrix algebra gives a short hand that allows us to manage these sets of equations.
- This chapter examines matrices and the way to use matrix algebra to solve common problems in economics, projects, and other areas.

Matrix algebra: Definitions and terminology

- A matrix (whose plural is matrices) is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns.
- A matrix with m rows and n columns is called an $m \times n$ matrix or m-by-n matrix, where m and n are called the matrix dimensions.
- Matrices can be used to compactly write and work with multiple linear equations, that is, a system of linear equations.
- Matrices and matrix multiplication reveal their essential features when related to linear transformations, also known as linear maps.

Matrix algebra: Definitions and terminology

- The element of a matrix is an individual item in a matrix.
- A row vector (matrix) is a matrix with a single row.
- A column vector (matrix) is a matrix with a single column.
- A square matrix is a matrix which has the same number of rows and columns.
- A 3 by 4 matrix (abbreviated 3 X 4) has three rows and 4 columns.
- The convention is to always start with rows (R) then columns (C)- RC.

Matrix algebra: An example

- Take a set of simultaneous equations as follows;
- $c_1 P_1 + c_2 P_2 = -c_0$, and;
- $\gamma_1 P_1 + \gamma_2 P_2 = -\gamma_0$, can be written in matrix form in three parts as;

$$M = \begin{bmatrix} c_0 \\ -\gamma_0 \end{bmatrix}$$

- Hence, $M * N = P$

$$N = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

Matrix algebra: An example

- The number of rows and columns of a matrix together define its **dimension**.
- M is a 2X2 matrix- it has 2 rows and 2 columns
- N is a 2X1 matrix. N is a special type of matrix called a column matrix as it has just one column. This is a vector, a special type of matrix.
- A matrix with one row and 2 or more columns is a row matrix. This is also a vector, a special type of matrix.
- A special matrix with one row and one column is a scalar, for instance number 4.

Matrix algebra: An example

- Our equation above them becomes

$$\underbrace{\begin{bmatrix} c_1 & c_2 \\ \gamma_1 & \gamma_2 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}}_{\mathbf{N}} = \underbrace{\begin{bmatrix} c_0 \\ -\gamma_0 \end{bmatrix}}_{\mathbf{P}}$$

- With $M * N = P$.
- Write the following set of equations in matrix format; Try solving it using crammers rule.
- $p + 4Q = 80$
- $2p - Q = 30$

Matrix algebra: The general form of linear equations using matrices

- We can generalize that any set of linear equations can be written as follows in linear algebra.

$$\underbrace{\begin{bmatrix} a & b & \dots & n \\ a & b & \dots & n \\ \vdots & \vdots & \vdots & \vdots \\ a & b & \dots & n \end{bmatrix}}_{\mathbf{a}} * \underbrace{\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}}_{\mathbf{b}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\mathbf{c}}$$

- With $a * b = c$
- a represents the coefficients. b stands for the variables.

Matrix algebra: An example

- Rewrite the following set of equations in matrix form as denoted in the previous page.
- $6x_1 + 3x_2 + x_3 = 22$
- $x_1 + 4x_2 - 2x_3 = 12$
- $4x_1 - x_2 + 5x_3 = 10$
- So far, we have NOT learnt enough to be able to solve this set of equations using crammers rule.

Matrix operations

- In this section, we cover matrix addition, subtraction, and multiplication. The matrices MUST have the same dimension though.
- Note that two matrices are equal if and only if (iff) they have the same dimensions and have identical elements in the corresponding locations in the array.

Matrix Operations: Adding/ subtracting two matrices

- The addition and subtraction of matrices is element-wise.
- For addition, it is as follows;

$$\underbrace{\begin{bmatrix} c_1 & c_2 \\ \gamma_1 & \gamma_2 \end{bmatrix}}_{\mathbf{A}} + \underbrace{\begin{bmatrix} d_1 & d_2 \\ \theta_1 & \theta_2 \end{bmatrix}}_{\mathbf{B}} = \underbrace{\begin{bmatrix} c_1 + d_1 & c_2 + d_2 \\ \gamma_1 + \theta_1 & \gamma_2 + \theta_2 \end{bmatrix}}_{\mathbf{P}}$$

- For subtraction, its similar;

$$\underbrace{\begin{bmatrix} a_1 & a_2 \\ \epsilon_1 & \epsilon_2 \end{bmatrix}}_{\mathbf{R}} - \underbrace{\begin{bmatrix} v_1 & v_2 \\ \alpha_1 & \alpha_2 \end{bmatrix}}_{\mathbf{S}} = \underbrace{\begin{bmatrix} a_1 - v_1 & a_2 - v_2 \\ \epsilon_1 - \alpha_1 & \epsilon_2 - \alpha_2 \end{bmatrix}}_{\mathbf{T}}$$

Matrix algebra: Multiplication/ division of a matrix with a scalar.

- Like addition and subtraction, multiplication and division of a matrix with a scalar is element-wise. For multiplication.

$$\underbrace{\begin{bmatrix} c_1 \end{bmatrix}}_{\mathbf{R}} * \underbrace{\begin{bmatrix} v_1 & v_2 \\ \alpha_1 & \alpha_2 \end{bmatrix}}_{\mathbf{S}} = \underbrace{\begin{bmatrix} c_1 * v_1 & c_1 * v_2 \\ c_1 * \alpha_1 & c_2 * \alpha_2 \end{bmatrix}}_{\mathbf{T}}$$

Matrix algebra: Multiplication/ division of a matrix with a scalar.

- For division, it is;

$$\underbrace{\begin{bmatrix} a_1 & a_2 \\ \epsilon_1 & \epsilon_2 \end{bmatrix}}_{\mathbf{K}} / \underbrace{\begin{bmatrix} d_1 \end{bmatrix}}_{\mathbf{L}} = \underbrace{\begin{bmatrix} a_1/d_1 & a_2/d_1 \\ \epsilon_1/d_1 & \epsilon_2/d_1 \end{bmatrix}}_{\mathbf{M}}$$

- Think of division of a matrix (A) with a scalar (K) as multiplying the matrix with $1/K$.

Matrix operations, exercises

- Given three matrices A, B and C as follows;

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 10 & 18 & 14 \\ 16 & 12 & 10 \\ 5 & 4 & -1 \end{bmatrix}$$

$$\begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} -12 & 4 & 3 \\ 1 & -12 & 25 \\ 15 & 12 & 10 \end{bmatrix}$$

$$\begin{bmatrix} C \end{bmatrix} = \begin{bmatrix} 100 & 37 & 4 \end{bmatrix}$$

Matrix operations, exercises

- Which 2 pairs of matrices are compatible for addition, subtraction.
- Compute 3A, -4B, and 6C.
- Compute $A + B$.
- Compute 3 * A.
- Compute $B/10$.

Matrix operations: Multiplication/ division of two matrices.

- We start with multiplication using a simplified illustration.
- Given two matrices A, B as follows;

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} \alpha_1 & \theta_1 \\ \theta_1 & \gamma_1 \end{bmatrix}$$

$$\begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} \rho_1 & \tau_1 \\ \psi_1 & \omega_1 \end{bmatrix}$$

- Then A X B will be;

Matrix operations: Multiplication/ division of two matrices.

$$\underbrace{\begin{bmatrix} \alpha_1 & \theta_1 \\ \lambda_1 & \gamma_1 \end{bmatrix}}_{\textbf{A}} * \underbrace{\begin{bmatrix} \rho_1 & \tau_1 \\ \psi_1 & \omega_1 \end{bmatrix}}_{\textbf{B}} = \underbrace{\begin{bmatrix} \alpha_1 * \rho_1 + \theta_1 * \psi_1 & \alpha_1 * \tau_1 + \theta_1 * \omega_1 \\ \lambda_1 * \rho_1 + \gamma_1 * \psi_1 & \lambda_1 * \tau_1 + \gamma_1 * \omega_1 \end{bmatrix}}_{\textbf{C}}$$

- Example

$$\underbrace{\begin{bmatrix} 2 & 5 \\ 7 & 1 \end{bmatrix}}_{\textbf{A}} * \underbrace{\begin{bmatrix} 3 & 0 \\ 8 & 11 \end{bmatrix}}_{\textbf{B}} = \underbrace{\begin{bmatrix} 2 * 3 + 5 * 8 & 2 * 0 + 5 * 11 \\ 7 * 3 + 1 * 8 & 7 * 0 + 1 * 11 \end{bmatrix}}_{\textbf{C}}$$

Matrix operations: Multiplication/ division of two matrices.

- Multiply the following matrices A and B.

$$\left[A\right] = \begin{bmatrix} 10 & 59 \\ -16 & 25 \end{bmatrix}$$

$$\left[B\right] = \begin{bmatrix} 1 & 95 \\ -61 & 52 \end{bmatrix}$$

Matrix operations: Multiplication/ division of two matrices.

- Multiply the following two matrices S and T.

$$\begin{bmatrix} S \end{bmatrix} = \begin{bmatrix} 10 & 59 & 70 \\ -16 & 25 & 12 \\ 39 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} T \end{bmatrix} = \begin{bmatrix} 1 & 95 & 7 \\ -61 & 52 & 21 \\ 93 & 0 & -10 \end{bmatrix}$$

Division of matrices

- While matrices can undergo addition, subtraction and multiplication, division is a bit tricky.
- Take two numbers 2 and 3. We can divide the two as $2/3$ which is also $2 * 1/3$.
- $2 * 1/3 = 2 * 3^{-1}$
- Similarly given two matrices A and B, we could then say $A/B = A * B^{-1}$.
- However, for matrices, this is not always the case.

Division of matrices

- For instance B^{-1} (called the inverse of B) may not exist.
- In other cases, even where B^{-1} is defined, $A * B^{-1}$ may not be defined.
- Still, even where B^{-1} is defined, $A * B^{-1}$ may not equal $B^{-1} * A$.
- We shall discuss this under matrix determinants and inverses.

Notes on the multiplication of vectors

- Remember vectors are special types of matrices.
- vectors are either row matrices or column matrices.
- When multiplying matrices, you first check for compatibility by ensuring the number of columns in the first matrix equals the number of rows in the second matrix.
- Only one exception;

Notes on the multiplication of vectors

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 10 \\ -16 \end{bmatrix}$$

$$\begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} 1 & 95 & 5 \end{bmatrix}$$

- Multiply the 10 in A with all elements in B.
- Multiply the -16 in A with all elements of B.

_ we get

$$\begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} 10 & 950 & 50 \\ -16 & -1520 & -80 \end{bmatrix}$$

The summation sign and its applications

- The Greek alphabet Σ is used to denote summations.
- For instance we may want to add $x_1 + x_2 + x_3 + x_4 + x_5 + x_6$. These are 6 numbers.
- We can let k demote the subscripts $1, 2, 3, ..., 6$.
- We can write this concisely as $\sum_{k=1}^{6} x_k$.
- We read this as summation for k $=$ 1 to 6 of x_k, and is equal to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6$
- Note that the k could be replaced by any other letter or symbol. k we have just chosen k for the example.

The summation sign and its applications: Example

- Write $ax_1 + ax_2 + ax_3 + ax_4$ concisely using the summation notation.
- We can rewrite this as $a(x_1 + x_2 + x_3 + x_4)$.
- Lets choose letter j this time to capture the subscripts. Note we have 4 summations, so $j = 1, \ldots.., 4$
- This becomes $a \sum_{j=1}^{4} x_j$

The summation sign and its applications: Example

- Write the following in concise summation notation.
- $ax_1^1 + ax_2^2 + ax_3^3 + ax_4^4 + ax_5^5 + ax_6^6$.
- we can rewrite this as $a(x_1^1 + x_2^2 + x_3^3 + x_4^4 + x_5^5 + x_6^6)$.
- Note that both the subscript and power move together in this case x_1^1, for example.
- Lets use the symbol w in this case which goes from 1 to 6 $x_1^1 to x_6^6$
- So we can say, $a \sum_{w=1}^{6} x_w^w$
- This will be a common occurrence in your economics and project management classes now and in the future. Please read more on this.

The concept of linear dependence

- A set of vectors $c_1, c_2, ..., c_n$ are said to be linearly dependent if we can write any one of them as a linear combination of the other vectors.
- Example.

The vector space

- Research on the idea of a vector space.
- How do we compute Euclidean distance?
- Give an example showing the computation of the Euclidean distance.
- Does the Pythagoras theorem represent Euclidean distance?

Square Matrices, Identity Matrices and Null Matrices

- The square matrix has the number of rows equal to the number of columns.
- Examples are matrices with dimensions, 2X2, 3X3, and so on.
- Null matrices of the other hand have all their elements equal to zero.
- The identity matrix (abbreviated as I) is a SQUARE matrix whose main diagonal elements are one (1), while all the other elements are zero.
- The main diagonal is the central diagonal in a matrix that slopes from left to right.

Examples of Identity Matrices

$$\begin{bmatrix} S \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Identity Matrices: Properties

- When you multiply a matrix B with the identity matrix I, the result is B.
- Think about when you multiply any number, say 549, by one. You get the same number 549.
- Similarly, the identity matrix behaves like the one in number multiplication.
- The NULL matrix is the zero. Multiply any matrix with the NULL matrix and you get the NULL matrix.
- $A * I = A$
- $A * NULL = NULL$

Identity Matrices: Properties

- Note that when you multiply any number, say 3, with its inverse, $\frac{1}{3}$, you always get one.
- Likewise, when you multiply a matrix with its inverse you get an identity matrix of the same dimension.
- $A * A^{-1} = I$, where I is the identity matrix.

Contradicitions in matrix algebra

- In matrix algebra, generally $A * B \neq B * A$. Order matters in multiplication.
- In ordinary maths, you cannot multiply two numbers both NOT EQUAL TO ZERO and get a zero.
- Remember a zero in numbers is equivalent to the NULL matrix in matrices.
- Consider this example;

$$\underbrace{\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}}_{\textbf{A}} * \underbrace{\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}}_{\textbf{B}} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\textbf{C}}$$

- Lesson: In Matrices, do not borrow from number operations blindly.

Transposes, Determinants, & Inverses

- When the rows and columns of a matrix are interchanged, such that the first row becomes the first column, and so on, we get the transpose of the matrix.
- Given a matrix A, the transpose is denoted A^T or A'.
- Example; Find the transpose B^T of;

$$\left[B\right] = \begin{bmatrix} 10 & 950 & 50 \\ -16 & -1520 & -80 \end{bmatrix}$$

Properties of Transposes

- $(A^T)^T = A$.
- The transpose of a transpose gives you the original matrix.
- $(A + B)^T = A^T + B^T$.
- $(AB)^T = B^T * A^T$.
- In this case note the order of A and B.

Properties of Transposes

- Example: Given the two matrices A and B below, show that these three properties hold.

$$\left[A\right] = \begin{bmatrix} 0 & 3 \\ 5 & 4 \end{bmatrix}$$

$$\left[B\right] = \begin{bmatrix} 5 & 8 \\ 1 & 4 \end{bmatrix}$$

The Determinant: simple case of a $2X2$ matrix.

- Given a $2X2$ matrix, the determinant is the difference between the products of the main diagonal and the secondary diagonal.
- Example: For the two matrices below compute the determinant.

The Determinant: simple case of a $2X2$ matrix.

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 5 & 4 \end{bmatrix}$$

$$\begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 1 & 4 \end{bmatrix}$$

- $|A| = (0 * 4) - (5 * 3) = -15$
- $|B| = (5 * 4) - (1 * 8) = 12$

The Determinant: Exercises

- Compute the determinant of the following matrices separately.

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 6 & 4 \end{bmatrix}$$

$$\begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 5 & 6 \end{bmatrix}$$

The inverse of a matrix: Simple case of a two by two matrix

- Some square matrices, say A, have an inverse, written as A^{-1} such that $A * A^{-1} = I$, where I is the identity matrix.
- For a $2X2$ matrix A, we get the inverse as follows;
  - Calculate the determinant (D) of the matrix.
  - Interchange the positions of the elements in the main diagonal, and then
  - Change the signs of the elements in the secondary diagonal so that $+$ becomes $-$ and vice versa.
  - You get a new matrix, call this X. The inverse of A, denoted A^{-1} is;
  - $A^{-1} = \frac{1}{D} * X$
- Calculate the inverses of the above two matrices in the previous page.

A new way to think about determinants

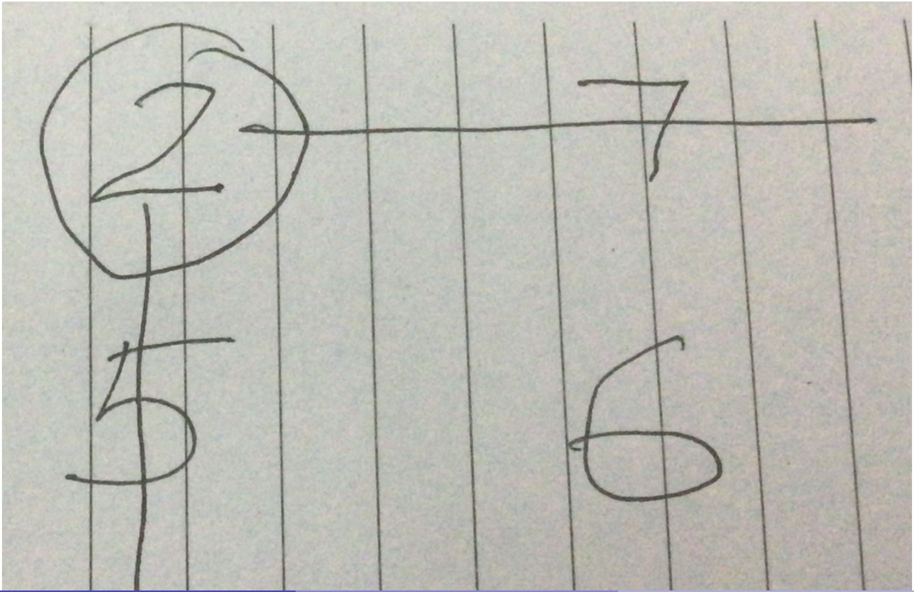
- Lets take an example of the matrix below.

$$\begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 5 & 6 \end{bmatrix}$$

A new way to think about determinants . . . cont'd.

- Starting with the top left corner, circle the element 2.
- Draw a vertical line starting from 2.
- Draw a horizontal line starting from 2. As follows;

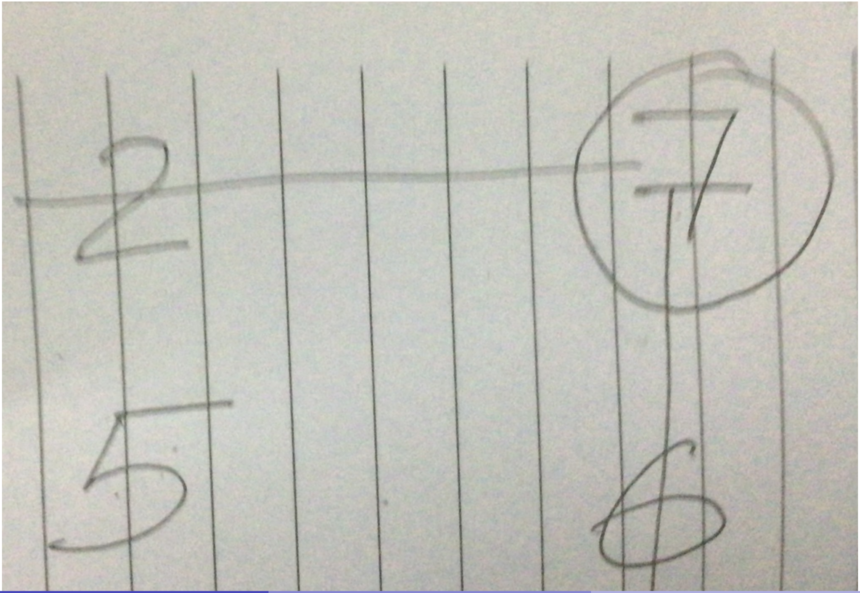
A new way to think about determinants . . . cont'd.



A new way to think about determinants . . . cont'd.

- We now take the number we have circled (2) and the number that none of the lines do not go through (6). i.e. 2 times $6 = 12$.
- Now we go to the number on the top right (7) and repeat the same procedure, as follows.
- Circle the 7 and draw a vertical and horizontal line accross the matrix. see below.

A new way to think about determinants . . . cont'd.



A new way to think about determinants . . . cont'd.

- Again, we take the circled number, 7 and the one that no line crosses, 5.
- We have 7 * 5 $=$ 35.
- Note now we have two products (2 * 6 $=$ 12) from the first and (7 * 5 $=$ 35) for the second.
- Now we assign a positive sign to the first product and a negative sign to the second.
- We have ($+$12) and (-35).
- Add the two to get your determinant. $12 - 35 = -23$
- is this the same figure you got using the straightforward method?

Higher order inverses and determinants.

- We shall next examine determinants and inverses of SQUARE matrices with dimensions higher than $2X2$.
- Let us start with the determinants.
- We shall use the technique just covered ... the new way to think about determinants.
- We shall just deal with 3 X 3 matrix although the technique extends to any size of matrix.

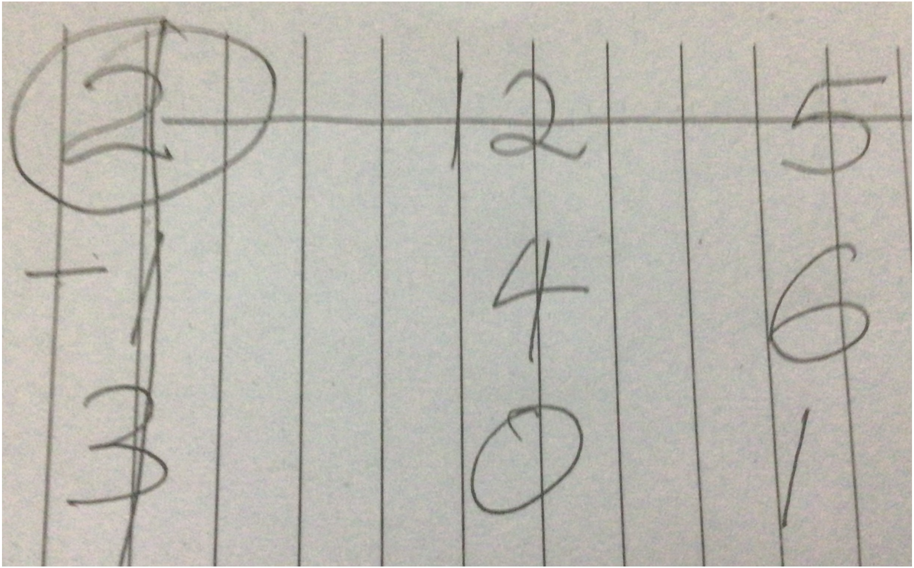
Higher order inverses and determinants.

- Consider a matrix;

$$\left[S\right] = \begin{bmatrix} 2 & 12 & 5 \\ -1 & 4 & 6 \\ 3 & 0 & 1 \end{bmatrix}$$

- We shall start with the top left and deal with the first row, number by number just like we did before.
- Lets begin by crossing out the top left number, then draw a vertical and horizontal straight line from this number to the rest of the matrix, as follows.

Higher order inverses and determinants.



Higher order inverses and determinants.

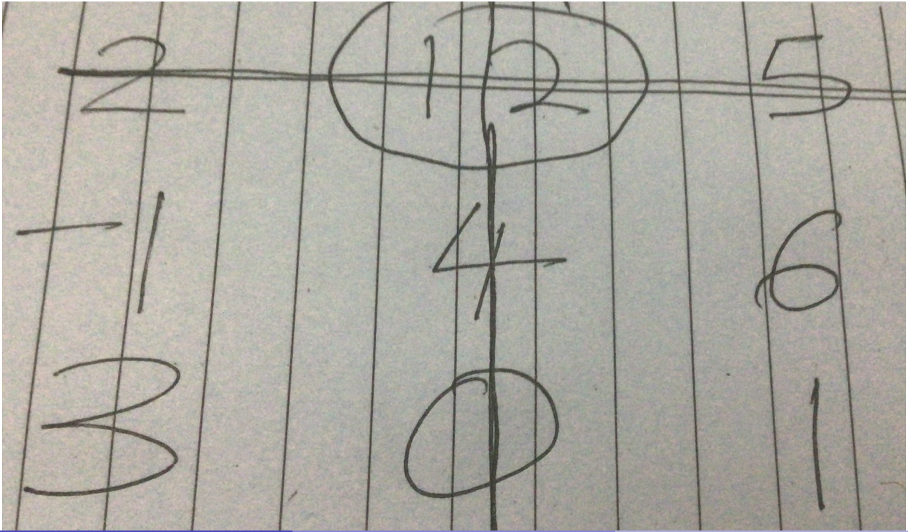
- like we did, take the circled number (2), and note this time we are left with a 2 X 2 matrix not touched by the vertical, horizontal lines.
- We have;
    - 2, and

$$\left[S\right] = \begin{bmatrix} 4 & 6 \\ 0 & 1 \end{bmatrix}$$

- Lets get the determinant of this 2 X 2 matrix = 4 - 0 = 4.
- Now we have 2 * 4.

Higher order inverses and determinants.

- Now lets do the same for the top row middle number. As follows.



Higher order inverses and determinants.

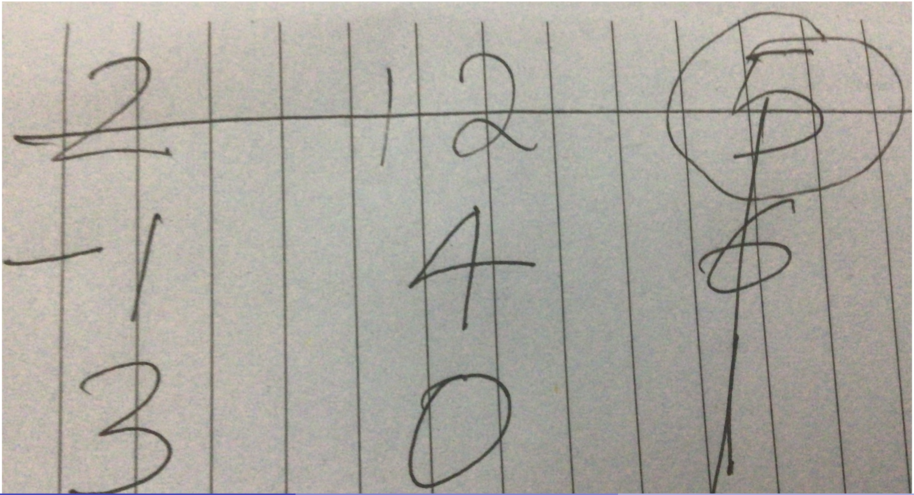
- Again, we are left with a circled number 12 and a matrix.

$$\begin{bmatrix} S \end{bmatrix} = \begin{bmatrix} -1 & 6 \\ 3 & 1 \end{bmatrix}$$

- The determinant of the matrix is -19.
- We now have another pair 12 * -19.

Higher order inverses and determinants.

- Lets repeat the same for the last figure in the first row of our 3X3 matrix. (5).



Higher order inverses and determinants.

- Again, we have a number 5 and a matrix whose determinant is -12.
- So in this case we have a product (5 * -12).
- We have three pairs of products $(2 * 4)$, $(12 * -19)$, $(5 * -12)$.
- We have 8, -228, and -60.
- We assign alternating $+$ and negative signs, starting with $+$
- The first product takes $+$, the second $-$, and the third $+$.
- Now we have 8, 228, and -60. Add up to get determinant of 176.

Exercises to ponder

- Get the determinant of the folowing matrices;
- Take your time to absorb this.

$$\begin{bmatrix} S \end{bmatrix} = \begin{bmatrix} 2 & 8 & 9 \\ -3 & 7 & 6 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} K \end{bmatrix} = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 4 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$