Interest Rate Risk

A. Defining the Problem

What is interest rate risk (IRR)? It is the risk that a financial institution (or firm more generally) will lose money if interest rates change. FSFs are often exposed naturally to this risk. For example, the traditional business model of a savings institution is to hold mainly long-term fixed rate mortgages as assets funded by relatively short term deposits. Thus, if interest rates rise, the S&L loses because it must pay more for its funding, but the yield on its mortgage portfolio does not change much. A more sophisticated analysis would say that the market value of the S&L's assets falls by more than its liabilities, thus the S&L suffers a decline in its net worth.

Let's look analytically at the IRR on a simple fixed income security, a T-year coupon bond.

Here is the bond price (for bond with $100 face value):

$$P = \Sigma\, C / (1+r)^t + 100 / (1+r)^T, \text{ where } t = 1, 2, ..., T$$

where C is the annual coupon payment on $100 of principal paid at maturity, and r is the yield-to-maturity.

NB: If C is paid semi-annually, then the formula would be:

$$P = \Sigma\, C / (1+r/2)^{2t} + 100 / (1+r/2)^{2T}, \text{ where } t = 1/2, 1, 1.5, ..., T$$

I will analyze the problem in terms of the yield-to-maturity (r). This is a little bit over simplified because it assumes that we are dealing with a flat term structure of interest rates; alternatively, you could think of this analysis as pertaining to cases in which we are considering only *parallel shifts in the yield curve*. If you take a course on fixed income securities, you can analyze the problem from the primitives securities–that is, from yields on zero coupon bonds. This approach allows you to be completely flexible in modeling the yield curve.

The key issue is, how sensitive is P to changes in r. That is, we want to know what is the following derivative:

$$\frac{\Delta P}{\Delta r}$$

Examples: Suppose the current interest rate is 8% and the yield curve is flat. Compute the change in the price of a 3-year, 6% coupon bond if interest rates rise to 9%. Do the same for a 10-year, 6% coupon bond.

If you did things correctly, you will find the following:

(1) Both bond prices <u>fall</u> as the yield <u>rises</u>;
(2) The longer term bond falls more than the shorter term bond;

Now, see what happens when you increase rates further, from 9% to 10%. If you did it right, you should find that the changes in the prices of the two bonds are smaller than they were in the first part of the example. This illustrates the third fact about interest rate risk:

(3) The effect of changes in interest rates <u>diminishes</u> as interest rates increase.

These are the three principles of interest rate risk. The first two facts are related to the idea of <u>duration.</u> Specifically, bonds with longer duration are more sensitive to interest rate changes than bonds of shorter duration. The second idea is related to the notion of <u>convexity</u>, which says that interest rate sensitivity (duration) declines at the level of interest rates rises.

B. Market Value Approach #1: The Duration Model

Go back to the first example with the 6% coupon bond. I had you compute the effect on the market value of this bond to a 1 percentage point change in interest rates. We concluded that the sensitivity of the bond's market price to interest rate changes depended on the maturity of the bond. This idea is formalized with the notion of duration.

Here is a definition of <u>Macaulay Duration</u> (D):

$$D = \sum_{t=1}^{T} w_t t$$

D is a weighted average of the time that you wait to receive each payment on the bond, where the weights are equal to the percentage of value coming from each payment. So, if most of the value of the bond comes from payments that happen way in the future, then the bond has a high duration (you've got to wait a long time to get your $). Here is the definition of the weights:

$$w_t = \frac{Payment_t}{P \times (1+r)^t} = \frac{PV(Payment_t)}{P}$$

Example: Compute Macaulay Duration for the 6% coupon bond (assume annual payments) with 3 years until maturity when interest rates are 8% (use a spreadsheet).

Now, what we are after is the sensitivity of the bond price to interest rates. To get the exact derivative we are after, we need to modify the Macaulay duration slightly. Define Modified Duration (D*) as follows:

$$D* = \frac{D}{1 + r}$$

NB: If you are dealing with semi-annual coupons, then:

$$D^* = D / (1+r/2)$$

D* is a useful analytical tool. With some tedious algebra, one can show the following:

$$\frac{\partial P}{\partial r} = -D^* P$$

Why is this helpful? Because we can use D* to construct an estimate of the change in the bond price with respect to a change in r. For those interested in technical details, we are appealing to the Taylor Series Approximation, which states the following:

$$\Delta P = \frac{\partial P}{\partial r}\Delta r + ... \Rightarrow \ \Delta P \approx -D^* P \Delta r$$

So, we now have an approximate relationship between D* and the sensitivity to the bond price to a change in interest rates.

Example: Compute D* for the 6% coupon bond that matures in 3-years when interest rates are 8%. Now, compute the change in the value of this bond when rates move to 9%. How does this approximate change compare with the actual change computed earlier?

If you did it correctly, you should find that the actual change in the bond price (when rates rise from 8% to 9%), is smaller than the change predicted by the duration approach. The reason is that, as we said before, the sensitivity to rate changes diminishes as rates rise. Thus, the linear approximation overstates the change. In technical terms, bonds exhibit <u>positive convexity.</u>

D. Market Value Approach #2: Convexity

We can improve on our approximation by taking account of the convexity of the asset. In general terms, <u>convexity</u> is defined as the second derivative of the asset price (as a percentage of the price) with respect to the interest rate. Convexity is important because it allows us to get a better approximation of the change in the price of the bond. Here is the Second-Order Taylor Series:

$$\Delta P = \frac{\partial P}{\partial r}\Delta r + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}(\Delta r)^2 = -D^* P\Delta r + \frac{1}{2}\Gamma P(\Delta r)^2$$

Where the Greek letter gamma (upper case) is traditionally used to measure convexity. For coupon bonds with annual payments, convexity equals the following expression:
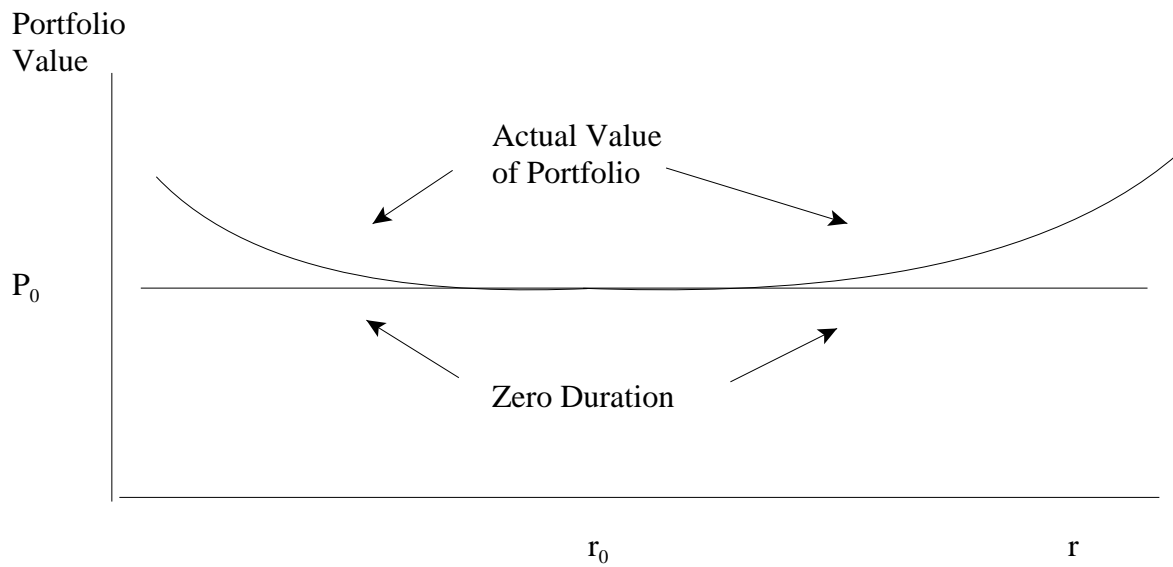
$$\Gamma = \frac{1}{(1+r)^2}\sum_{t=1}^{T} t(t+1)w_t$$
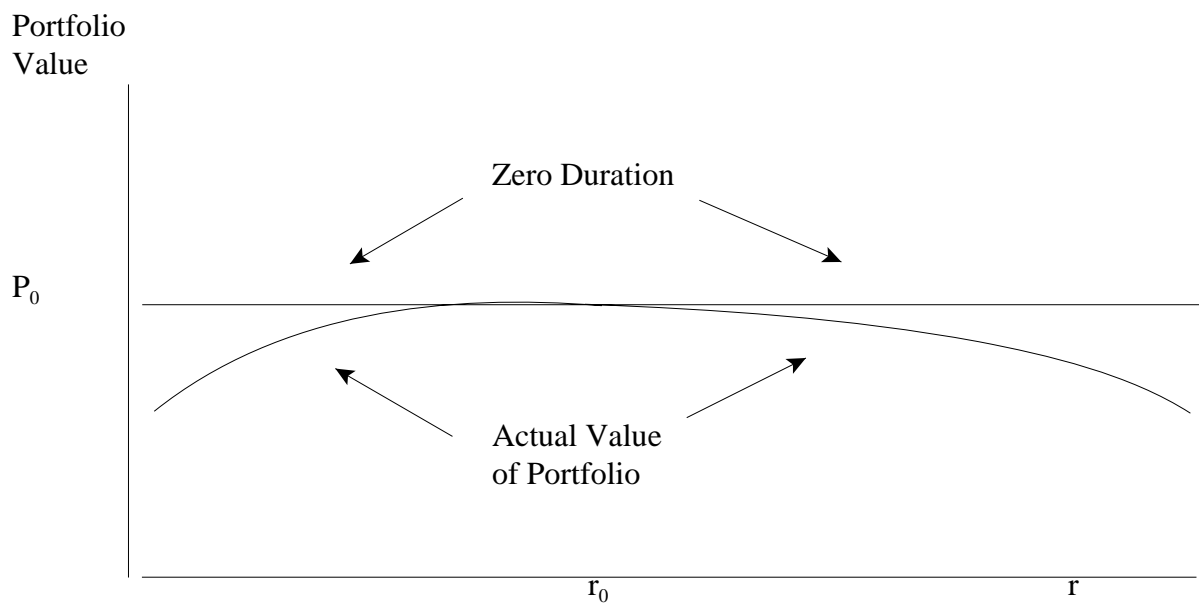
where the weights in the summation are the same as before.

Example: Go back an re-compute the effect of a change in rates from 8% to 9% for the 6% coupon bond with 3-years to maturity using the convexity term. Convince yourself that you get a better approximation this way. (Hint: I would suggest using a spreadsheet for this.)

Who cares about convexity? The risk manager, that's who. Often what FSFs will do is hedge the risk by structuring their portfolio so that the duration of net worth equals zero (more below). For such a scheme, however, the risk manager wants to know what the portfolio's convexity is. If convexity is positive, the risk manager can sleep at night confident in the knowledge that the portfolio will make money regardless of how rates change (up or down). Conversely, if convexity is negative, the risk manager will worry because $ will be lost regardless of how rates change.

Here is the graph of a zero-duration portfolio with <u>positive convexity</u> (r_0 is the current interest rate, P_0 the current market value of the portfolio):

Portfolio
Value

Actual Value
of Portfolio

P_0

Zero Duration

r_0 r

And, <u>negative convexity:</u>

Portfolio
Value

Zero Duration

P_0

Actual Value
of Portfolio

r_0 r

Page 5 of 11

D. Hedging

How do FSFs hedge their interest rate risk? The approach we will take is based on the idea that the FSF wants the <u>market value</u> of its net worth to be insensitive to (small) changes in interest rates. That is, it wants its net worth to have a zero duration. How do we do it? Start with the balance sheet equation:

$$NW = A_1 + \ldots + A_k - L_1 - \ldots - L_m$$

Where NW means market value of net worth; A means market value of assets; L means market value of liabilities; there are k different assets and m different liabilities. Now, take a derivative of this with respect to r (again, we are doing this assuming a flat yield curve; r is the yield to maturity):

$$\delta NW/\delta r = \delta A_1/\delta r + \ldots + \delta A_k/\delta r - \delta L_1/\delta r - \ldots - \delta L_m/\delta r$$

So, we can substitute in the modified duration for each asset and liability class to get:

$$\frac{\delta NW}{\delta r} = -D^*_{A1}A_1 - \ldots - D^*_{Ak}A_k + D^*_{L1}L_1 + D^*_{Lm}L_m$$

Since the modified duration equals the (negative of) the derivative with respect to interest rates, divided by the market value of the asset:

$$D^*_{NW} = -\frac{\delta NW}{\delta r} \times \frac{1}{NW}$$

So:

$$D^*_{NW} = (D^*_{A1}A_1 + \ldots + D^*_{Ak}A_k - D^*_{L1}L_1 - D^*_{Lm}L_m) \times \frac{1}{NW}$$

From before, we know that:

$$\Delta NW \approx - D^*{}_{NW} \times NW \times \Delta r$$

So,

$$\Delta NW \approx (- D^*{}_{A1} A_1 - ... - D^*{}_{Ak} A_k + D^*{}_{L1} L_1 + D^*{}_{Lm} L_m) \times \Delta r$$

If there is just one asset category and one liability category, this boils down to:

$$\Delta NW \approx - (D^*{}_A A - D^*{}_L L) \times \Delta r$$

Another way to write the last equation is:

$$\Delta NW \approx - (D^*{}_A - D^*{}_L k) \times A \times \Delta r$$

where k=L/A, which is a measure of leverage. So, to eliminate interest rate risk from net worth, we want a weighted average of the modified durations of the individual assets and liabilities to be equal to zero. This would eliminate interest rate risk, at least for small changes in rates.

<u>Key issue</u>: The last equation illustrates something VERY IMPORTANT about leverage: *leverage affects the way that duration of assets and liabilities should be weighted to eliminate interest rate risk!*

Example. Let's see how this works. Look at a simple bank. Here is its balance sheet:

Assets
6% coupon bond with maturity of 3 years   1000
Liabilities
1-year CDs   900
Net Worth   100

Now, we are assuming that this balance sheet is based on market value! Also, to make the math easy, assume that the 1-year CD operates like a zero-coupon bond. Interest rates are 8%, so this CD will pay $900x1.08 in 1 year.

First, compute the modified durations of the assets and liabilities. Next, compute the exact change in net worth for a 1% increase in interest rates. Then, using the duration of net worth, compute the approximate change stemming from a 1% increase in interest rates.

If you did it correctly, you will find that there a lot of interest rate risk in the portfolio. There are 2 reasons for this; first, because the assets have a longer duration than the liabilities, net worth falls when interest rates rise. The percentage change in the value of net worth is very large because of <u>leverage</u>. This creates a very strong motivation for FSFs to hedge their interest rate risk (after all, a 1% change in rates is not very unusual).

Question: Without doing the calculus, what is the convexity of this portfolio?

Now, we have shown there is significant risk. The challenge is to figure out how to hedge the risk. One way to go would be to buy or sell assets until you eliminate the duration of net worth. But this is usually inefficient because banks earn profits from their core business of making loans and taking deposits. It makes more sense for them to alter the duration of net worth by doing a side transaction in the derivatives markets, and leave their core business alone. This is what most FSFs do today. Before the rise of these markets in the 1980s, most FSFs (except insurance companies) did little to hedge their exposure to interest rates.

Using derivatives to hedge the interest rate risk is very simple. All you need to do is add a derivative to your portfolio with the same duration as your net worth, but with opposite sign. I will show you how to compute the duration for two commonly used instruments, forwards/futures on Treasury securities, and interest rate swaps.

Hedging with futures

A little background for those who are not comfortable with financial forwards and futures. A forward contract is an agreement between two parties to make a transaction at a future date at a pre-specified price. That price is called the forward price.

A futures contract is almost the same; the only difference is that futures contracts trade on organized exchanges and the parties mark to market at the end of each day. Thus, if the futures price rises by $1 during the day, the short (future seller) pays $1 to the long (future buyer). The other difference is that futures contracts are standardized. This is an advantage for FSFs looking to hedge because there is a lot of liquidity in the market. Financial futures and forwards are contracts where the thing being transacted in the future is a financial asset. Common underlying assets are Treasury securities. If you are long in a futures contract with a Treasury bond as the underlying, it means that you are obligated to buy that bond in the future at today's futures price; thus you make a profit if the price of the underlying rises. Hence, in terms of changes in your wealth, being long the futures is just like owning the bond outright! The reverse is true if you are short the futures. So, we can analyze the change in the value of a futures contract in the same way that we did for the bonds themselves:

$$\Delta F = -D^*_F \ \text{x} \ (N_F \text{x} P_F) \ \text{x} \ \Delta r$$

In the equation:

ΔF is the change in your position in the futures

D^*_F is the modified duration of the futures contract, <u>which equals the modified duration of the underlying</u>.

$(N_F \text{x} P_F)$ is the <u>position size</u>, where N_F indicates the number of contracts and P_F indicates the futures price for each contract. If the position is long, this product will be positive; if short, it will be negative. For example, you might have one long futures contract with a $100,000 Treasury Bond as the underlying, and the futures price is $97,000. In the example, $P_F = \$97,000$, $N_F = +1$ (+ because you are long; it would be -1 if you are short), and D_F = modified duration of the Treasury Bond. (I am not going to go thru this, but it turns out that the futures price generally is less than the spot price; in the case, 97,000 < 100,000. The reason is that you save some interest using futures relative to buying the underlying outright.)

Example: Let's continue the example from above. Suppose you want to eliminate the interest rate risk of net worth using a 3-month futures contract on Treasury bond with 10 years to maturity and annual coupon rate of 6%. What is the position size that you need to open to duration hedge your net worth? Are you long or short in the futures contract?

Hedging with interest rate swaps

Interest rate swaps are the other derivative most commonly used to hedge interest rate risk. An interest rate swap is an agreement by two parties to exchange fixed for floating rate interest payments on an agreed upon amount of principal, called the notional amount. The term notional refers to the fact that the two parties never swap the notional itself, only the interest on that amount.

Example. Notional principal equals $1 million. Counterparty A pays the fixed rate (8%) to counterparty B, in exchange for 1-year LIBOR. Interest payments are swapped annually, and the contract lasts for 5 years.

Here is one possible set of cash flows for each side (note that the example depends on the path of interest rates that occurs over the life of the swap; this is something that is not known at the outset of the deal):

Year	1-Year LIBOR	Net Payment to A	
0	5%		
1	8%	$1mx(0.05-0.08)=	-$30,000
2	6%	$1mx(0.08-0.08)=	$0
3	7%	$1mx(0.06-0.08)=	-$20,000
4	8.6%	$1mx(0.07-0.08)=	-$10,000
5	-	$1mx(0.086-0.08)=	+$6,000

The swap provides the same cash flow that Counterparty A would receive if it had borrowed $1 million at a fixed rate for 5 years, and invested the proceeds in 1-year notes. The difference, however, is that there is netting in a swap contract, which lowers the credit risk.

Question: What is the market value of a swap contract when it is first initiated?

FSFs often prefer swaps to hedge rather than futures because they have a cash flow pattern that more nearly replicates their assets and liabilities. For example, a swap can be put into place to offset a specific transaction and then left alone for the life of the transaction.

Example. Suppose a bank has no interest rate risk. A borrower then comes along wanting to borrow $1 million for 5 years on a fixed rate basis. For simplicity, assume that the borrower has no credit risk and interest rates are currently 8%, so the annual coupon rate is set at 8% per year. The bank can raise funds most efficiently, however, by borrowing in the short-term interbank market. It decides to fund the loan by borrowing $1 million at the 1-year LIBOR. How can the bank use an interest rate swap to offset interest rate risk that this transaction adds to its balance sheet? (Aside: the deal in this example has no profit for the bank; in reality, the bank would be able to charge slightly more than the market interest rate to its borrower to realize a profit. The hedging would be slightly more complicated in this case.)

Notice that if a bank manages risk on an incremental basis (i.e. matching a swap to individual deals as the come along), it will develop a very large set of swap positions over time. However, many of these swap positions will be offsetting. Keep this in mind when you see hard-to-fathom statistics about how big the derivatives markets are. These statistics are usually based on notional amounts.

Swaps can also be used to hedge the overall interest rate risk of net worth in the same way that we showed you could use futures. The only thing you need to understand is how to compute how the value of a swap changes with interest rates. It is straightforward once you recognize that a swap is just a long position in the received-rate side funded by a short position in a pay-rate side.

So, for someone who receives fixed (i.e. the fixed side is an asset), and pays floating (i.e. the floating rate side is a liability):

$$\Delta S \approx -(D^*_{Fixed} - D^*_{Variable}) \times NP \times \Delta r$$

Where NP equals the notional principal on the swap. Notice the symmetry with what we derived about for net work. Think of a swap as a balance sheet that is 100% levered. There is not leverage term here (k=L/A) because both the fixed rate and floating rate sides have a market value equal to the notional principal amount of the swap at the outset of the contract.

Example: How would you hedge the risk in the net worth position from above using the five-year interest rate swap described above? (That is, how big does the position need to be and which side do you want to be on?)