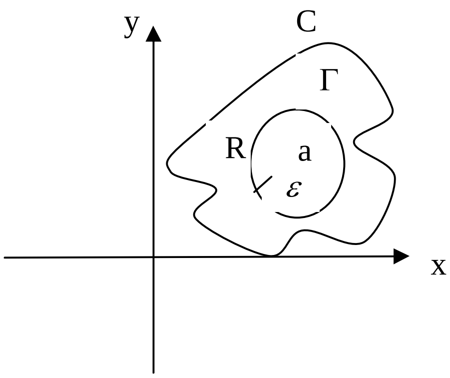
Cauchy's Integral formula

If $f(z)$ is analytic inside and on a simple closed curve *c* and *z=a* is any point inside *c*, then

$f(a)=\frac{1}{2\pi i}\oint\frac{f(z)}{z-a}dz$ where *c* is transverse in the positive direction (counter-clockwise form).

Cauchy's Integral formula for 1st derivative of an analytic function.

If $f(z)$ is analytic inside and on the boundary *c* of a simply connected region R, and if *a* is any point inside the curve, then $f'(a)=\frac{1}{2\pi i}\oint_c\frac{f(z)}{(z-a)^2}dz$.



In the case of the second derivative, we have the above formula becoming

$f''(a)=\frac{2!}{2\pi i}\oint_c\frac{f(z)}{(z-a)^3}dz$ .The same conditions apply.

Example 47

Evaluate $\oint\frac{\cos z}{z(z^2+8)}dz$

Solution: Singular points to occur at *z=0*, $\frac{\cos z}{z^2+8}=\frac{\cos 0}{0+8}=\frac{1}{8}$

$\oint\frac{\cos z}{z(z^2+8)}dz=2\pi i f(a) \quad =2\pi i f(0) \quad =2\pi i\left(\frac{1}{8}\right) \quad =\frac{\pi i}{4}$

Example 48

Evaluate $\int_c\frac{z}{2z+1}dz$

Solution:

$\int_c\frac{z}{2\left(z+\frac{1}{2}\right)}dz$;singular point is $z=-\frac{1}{2}$

$$f(z)=\frac{z}{2}$$

$$\therefore f\left(-\frac{1}{2}\right)=-\frac{1}{2}\div 2=-\frac{1}{4}$$

$$\int_c \frac{zdz}{2z+1}=2\pi i f(a)=2\pi i f\left(-\frac{1}{2}\right)=2\pi i\left(-\frac{1}{4}\right)=-\frac{\pi}{2}i$$

Exercise 22

1.Evaluate $\int_c \frac{z+1}{z^2+2z+4}dz$ along $c:|z+1+i|=1$.

2.Evaluate $\int_c \frac{z+3}{(z+1)(z-i)}dz$ along $c:|z+1|=\frac{1}{2}$.

3.Let c be a simple closed contour and write $g(z)=\int \frac{s^3+2s}{(s-z)^3}ds$,show that $g(z)=6\pi iz$.

Singular points

A point at which $f(z)$ fails to be analytic is called a singular point or singularity of $f(z)$.

Types of singularities

1. Isolated singularities

The point $z=z_0$ is called an isolated singularity or isolated singular point of $f(z)$ If, in addition, there is some neighbourhood of z_0 throughout which $f(z)$ is analytic except at the point itself.

e.g. (1) $f(z)=\frac{1}{z}\to z=0$ is isolated, $\therefore f(z)$ is analytic anywhere except at *z=0*.Hence the origin is an isolated singular point of that function.

e.g. (2) $f(z)=\frac{(z+1)}{z^3(z^2+1)}$ singular points occur at $z^3(z^2+1)=0\Rightarrow z=0, z=\pm i$

$\therefore f(z)$ has 3 isolated points, $z=0, z=\pm i$.

2. Poles

If we can find a positive integer *n* such that $\lim_{z\to z0}(z-z_0)^n f(z)=A\neq 0$, then $z=z_0$ is called a pole of order *n*.If *n=1*, z_0 is called a simple pole.

e.g.(1) $f(z)=\frac{1}{(z-2)^3};(z-2)^3=0\Rightarrow z=2$

$f(z)$ has a pole of order 3 at *z=2*

$$\lim_{z\to 2}(z-2)^3\cdot\frac{1}{(z-2)^3}=1$$

e.g.(2) $f(z)=\frac{3z-2}{(z-1)^2(z+1)(z-4)}$

Poles occur at $(z-1)^2(z+1)(z-4)=0$

$$\Rightarrow z=1, z=-1, z=4$$

$z=1$ has order 2

$\left.\begin{array}{l} z=-1 \\ z=4 \end{array}\right\}$ have order 1, thus they are simple poles.

$$\lim_{z\to 1}(z-1)^2\frac{3z-2}{(z-1)^2(z+1)(z-4)}=-\tfrac{1}{6}$$

3. Branch Points

A multiple-valued function has a branch point and are singular points.

e.g.(1) $f(z)=(z-3)^{\frac{1}{3}}$ has a branch point at *z=3*.

e.g.(2) $f(z)=\ln\left(z^2+z-2\right)$ is multiple-valued and has branch points at *z=1* and *z=-2*

or

If a function $f(z)$ is many-valued e.g. $f(z)=z^{\frac{1}{n}}, n=1,2,3,\ldots$ then the point where its not analytic is called a branch point.

e.g.(3) $f(z)=(z-2)^{-\frac{1}{8}}$ has a branch point at *z=2*.

4. Removable singularities

The singular point z_0 is called a removable singularity of $f(z)$ if $\lim_{z\to z_0} f(z)$ exists or if $\lim_{z\to z_0} f(z)=\frac{0}{0}$ or $\frac{\infty}{\infty}$ but the limit at óz_0 ôexists and its finite, then $f(z)$ is said to have a removable singularity.

e.g. $f(z)=\dfrac{z^2-9}{z-3}$

$$\lim_{z\to 3}\frac{z^2-9}{z-3}=\frac{0}{0}=\lim_{z\to 3}2z=6$$

Hence the limit exists.

$\therefore z=3$ is a removable singularity.

e.g.(2) $f(z)=\dfrac{\sin z}{z}; \lim_{z\to 0} f(z)=\lim_{z\to 0}\dfrac{\sin z}{z}=\dfrac{0}{0}$ but $\lim_{z\to 0}\dfrac{\sin z}{z}=\lim_{z\to 0}\dfrac{\cos z}{1}=1$ hence $z=0$ is a removable singularity.

5. Essential singularities

A singularity that is not removable, not a pole or a branch point is an essential singularity.

e.g.(1) $f(z)=e^{\frac{1}{z-2}}$ has an essential singularity at *z=2* since $e^{\frac{1}{z-2}}=\left(1+\dfrac{1}{z-2}+\dfrac{1}{2!(z-2)}+\ldots\right)$

$\therefore$ point $(z-2)$ is essential.

e.g.(2) $(z-3)\sin\dfrac{1}{z-2}, z=2$ is essential

since $f(z)=(z-3)\left\{\dfrac{1}{z-2}-\dfrac{1}{3!(z-2)^3}+\dfrac{1}{5!(z-2)^5}+\ldots\right\}$ $\therefore z=2$ is essential.

Exercise 23

1. Locate and name all the singularities of

(a) $f(z)=\dfrac{z^8+z^4+2}{(z-1)^3(3z+2)^2}$

(b) $f(z)=\dfrac{z^2-3z}{z^2+2z+2}$

(c) $f(z)=\dfrac{\ln(z+3i)}{z^2}$

(d) $\sqrt{z(z^2+1)}$

Taylor Series

Taylor theorem

Let f be analytic everywhere inside a circle c with centre at z_0 and radius ε .Then at each point z inside c, $f(z)=f(z_0)+f'(z_0)(z-z_0)+f''\dfrac{(z_0)}{2!}(z-z_0)^2+\ldots+\dfrac{f^n}{n!}(z_0)(z-z_0)^n+\ldots$

This is the expansion of $f(z)$ into a Taylor series about the point z_0 .If $z_0=0$,we obtain the Maclaurin series.

Some special series.

$$e^z=1+\frac{z}{2}+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots+\frac{z^n}{n!}$$

$$\sin z=z-\frac{z^3}{2!}+\frac{z^5}{5!}-\ldots(-1)^{n-1}\frac{z^{2n-1}}{(2n-1)!}+\ldots$$

$$\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\ldots(-1)^{n-1}\frac{z^{2n-2}}{(2n-1)!}+\ldots$$

$$\ln(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}-\ldots(-1)^{n-1}\frac{z^n}{n}+\ldots$$

$$\tan^{-1}z=z-\frac{z^3}{3}+\frac{z^5}{5}-\ldots(-1)^{n-1}\frac{z^{2n-1}}{2n-1}$$

$$\frac{1}{1-z}=1+z+z^2+\ldots+z^n;\frac{1}{1+z}=1-z+z^2-z^3+\ldots$$

Laurent Series

$$f(z)\sum_{-\infty}^{\infty}b_n(z-z_0)^n$$

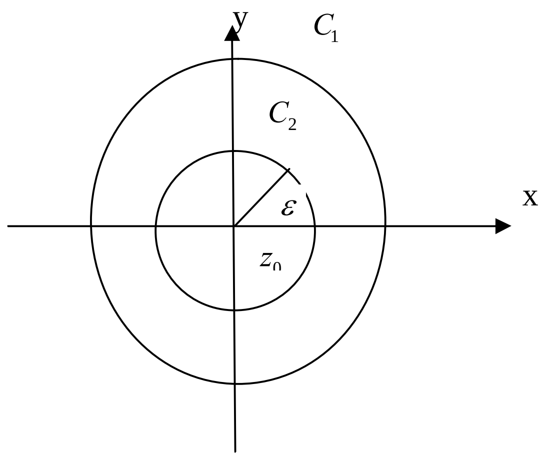
$$=\ldots+b_{-3}(z-z_0)^{-3}+b_{-2}(z-z_0)^{-2}+b_{-1}(z-z_0)^{-1}+b_0+b_1(z-z_0)^1+b_2(z-z_0)^2+\ldots$$

$$=b_0+b_1(z-z_0)^1+b_2(z-z_0)^2+\ldots+b_{-1}(z-z_0)^{-1}+b_{-2}(z-z_0)^{-2}+b_{-3}(z-z_0)^{-3}+\ldots$$

$$= b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \ldots + \frac{b-1}{z-z_0} + \frac{b-2}{(z-z_0)^2} + \frac{b-3}{(z-z_0)^3} + \ldots \text{ where}$$

$$b_n = \frac{1}{2\pi i}\oint_{c_1} \frac{f(z)}{(z-z_0)^{n+1}} dz \quad n = 0,1,2,\ldots \text{ and}$$

$$b_{-n} = \frac{1}{2\pi i}\oint_{c_2} (z-z_0)^{n-1} dz \quad n = 1,2,3,\ldots$$



$f(z)$ is analytic inside and on the boundary of the shaded region.

Example 49

(a) Find the poles of $f(z) = \dfrac{1}{(z+2)(z-3)}$.

(b) Expand into a Laurentôs series at the point $z=-2$.

Solution:

(a) Poles occur at $(z+2)(z-3) = 0 \Rightarrow z = -2, z = 3$-simple poles.

(b) Let $z+2 = u$; then $z = u-2 \Rightarrow z-3 = u-2-3 = u-5$

$$\therefore f(z) = \frac{1}{u(u-5)} \quad ,u = \text{small}$$

$$\therefore f(z) = \frac{1}{5u\left(\frac{u}{5}-1\right)} = \frac{-1}{5u\left(1-\frac{u}{5}\right)}$$

$$\therefore f(z) = \frac{-1}{5u}\left(1-u/5\right)^{-1} \text{ where } \left|\frac{u}{5}\right| < 1 \text{ for it to converge.}$$

By binomial theorem,

$$\left(1-u/5\right)^{-1} = 1 + (-1)(1)\left(\frac{-u}{5}\right) + \frac{(-1)(-2)}{2!}\left(\frac{-u}{5}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{-u}{3}\right)^3 + \ldots$$

$$= 1 + \frac{u}{5} + \frac{u^2}{25} + \frac{u^3}{125} + \ldots$$

$$f(z)=\frac{-1}{5u}\left[1+\frac{u}{5}+\frac{u^2}{25}+\frac{u^3}{125}+\dots\right]$$

$$=\frac{-1}{5}\left[\frac{1}{u}+\frac{1}{5}+\frac{u}{25}+\frac{u^2}{125}+\dots\right.$$

But $u=z+2$

$$\therefore f(z)=\frac{-1}{5}\left[\frac{1}{z+2}+\frac{1}{5}+\frac{z+2}{25}+\frac{(z+2)^2}{125}+\dots\right]$$

Example 50

Expand $f(z)=(z-3)\sin\frac{1}{(z+5)}$ into a Laurent series at z=-5.

Solution:

Let $u=z+5\Rightarrow z=u-5\Rightarrow z-3=u-5-3=u-8$

$\therefore f(z)=(u-8)\sin\frac{1}{u}$ but $\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}+\dots$

$$\therefore \sin\left(\frac{1}{u}\right)=\frac{1}{u}-\frac{1}{3!u^3}+\frac{1}{5!u^5}+\dots$$

$$f(z)=(u-8)\left\{\frac{1}{u}-\frac{1}{3!u^3}+\frac{1}{5!u^5}+\dots\right\}$$

$$=\frac{u-8}{u}-\frac{u-8}{3!u^3}+\frac{u-8}{5!u^5}+\dots \quad \text{but } u=z+5$$

$$=\frac{z-3}{z+5}-\frac{z+3}{3!(z+5)^3}+\frac{z-3}{5!(z+5)^5}+\dots$$

$$=(z-3)\left\{\frac{1}{(z+5)}-\frac{1}{3!(z+5)^3}+\frac{1}{5!(z+5)^5}+\dots\right\}$$

Example 51

Find the Laurent series about $=(z-3)\left\{\frac{1}{(z+5)}-\frac{1}{3!(z+5)^3}+\frac{1}{5!(z+5)^5}+\dots\right\}$

Solution:

Let $u=z-1\Rightarrow z=u+1$

t $\therefore f(z)=\frac{e^{2(u+1)}}{u^3}=\frac{e^{2u}e^2}{u^3}=\frac{e^2}{u^3}\left\{e^{2u}\right\}$

but $e^z=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots$

$$\therefore e^{2u}=1+2u+\frac{(2u)^2}{2!}+\frac{(2u)^3}{3!}+\dots$$

$$=1+2u+2u^2+\frac{4}{3}u^3+\dots$$

$$\therefore f(z)=\frac{e^2}{u^3}\left\{1+2u+2u^2+\frac{4}{3}u^3+\dots\right\} \text{ but } u=z-1$$

$$=\frac{e^2}{(z-1)^3}\left\{1+2(z-1)+2(z-1)^2+\frac{4}{3}(z-1)^3+\cdots\right\}$$

$$=\frac{e^2}{(z-1)^3}+\frac{2e^2}{(z-1)^2}+\frac{2e^2}{(z-1)}+\frac{4}{3}e^2+\dots$$

Example 52

Expand $f(z)=\frac{1}{z^2(z-3)^2}$ about $z=3$.

Solution:

Let $z-3=u \Rightarrow z=+3$

$$\therefore f(z)=\frac{1}{(u+3)^2(u)^2}=\frac{1}{u^2(u+3)^2}$$

$$=\frac{1}{u^2}(u+3)^{-2}$$

$$=\frac{1}{u^2}\left\{3\left(1+\frac{u}{3}\right)\right\}^{-2}$$

$$=\frac{1}{9u^2}\left\{1+\frac{u}{3}\right\}^{-2}$$

By binomial expansion,

$$=\frac{1}{9u^2}\left\{1-(2)\frac{u}{3}+\frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2+\dots\right\}$$

$$=\frac{1}{9u^2}\left\{1-\frac{2u}{3}+\frac{1u^2}{3}-\frac{4}{27}u^3+\dots\right\}$$

$$=\frac{1}{9u^2}-\frac{2}{27u}+\frac{1}{27}-\frac{4}{243}u+\dots \text{ but } u=z-3$$

$$=\frac{1}{9(z-3)^2}-\frac{2}{27(z-3)}+\frac{1}{27}-\frac{4}{23}(z-3)+\cdots$$

Example 53

Expand $f(z)=\frac{1}{(z+1)(z+3)}$ in a Laurent series valid for

(a) $1<|z|<3$

(b) $|z|<3$

(c) $0<|z+1|<2$

(d) $|z|<1$

Solution:

(a)Resolve $\frac{1}{(z+1)(z+3)}$ into partial fractions

$$\frac{1}{(z+1)(z+3)}=\frac{A}{z+1}+\frac{B}{z+3}$$

$$1=A(z+3)+B(z+1)$$

$$1=Az+3A+Bz+B$$

$$0Z+1=(A+B)z+3A+B$$

$$\therefore A+B=0 \Rightarrow A=-B$$

$$3A+B=1 \Rightarrow 3A-A=11 \Rightarrow A=\frac{1}{2} \Rightarrow B=\frac{-1}{2}$$

$$\therefore f(z)=\frac{1}{2(z+1)}-\frac{1}{2(z+3)}$$

If $|z|>1, \frac{1}{2(z+1)}=\frac{1}{2z\left(1+\frac{1}{z}\right)}=\frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}$

$$=\frac{1}{2z}\left\{1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\dots\right\}$$

$$=\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+\dots$$

$$\left[|z|<1 \Rightarrow 1>\frac{1}{|z|} \Rightarrow \frac{1}{|z|}<1\right]$$

If $|z|<3, \frac{|z|}{3}<1$

$$\frac{1}{2(z+3)}=\frac{1}{2\cdot 3\left(1+\frac{z}{3}\right)}=\frac{1}{6\left(1+\frac{z}{3}\right)}=\frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$=\frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}=\frac{1}{6}\left\{1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\dots\right\}$$

$$\frac{1}{2(z+3)}=\frac{1}{6}-\frac{z}{18}+\frac{z^2}{54}-\frac{z^3}{162}+\dots$$

Then the required Laurent expansion valid for $|z|>1$ and $|z|<3$, i.e. $1<|z|<3$ is

$$\dots\frac{-1}{24z^4}+\frac{1}{2z^3}-\frac{1}{2z^2}+\frac{1}{2z}-\frac{1}{6}+\frac{z}{18}-\frac{z^2}{54}+\frac{z^3}{162}+\dots$$

(b)if $|z|>1$, we have $\frac{1}{2(z+3)}=\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{24z^4}+\dots$ (1)

If $|z|>3, 1>\frac{3}{|z|}, \frac{3}{|z|}<1$

$$\frac{1}{2(z+3)}=\frac{1}{2z\left(1+\frac{3}{z}\right)}=\frac{1}{2z}\left(1-\frac{3}{z}+\frac{9}{z^2}-\frac{27}{z^3}+\ldots\right)$$

$$=\frac{1}{2z}-\frac{3}{27z^2}+\frac{9}{2z^3}-\frac{27}{27z^4}+\ldots(2)$$

Combining (1) and (2), we have

$$\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+\ldots-\frac{1}{2z}+\frac{3}{2z^2}-\frac{9}{2z^3}+\frac{27}{2z^4}\ldots$$

$$=\frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-\frac{40}{z^5}+\ldots$$

(NB $z>1$ includes even $z>3$)

(c) $0<|z+1|<2$

$|z+1|$ must be greater than zero.

$$f(z)=\frac{1}{(z+1)(z+3)}$$

Let $z+1=u \Rightarrow z+3=u+2$

$$|u|<2$$

$$\Rightarrow \frac{|u|}{2}<1$$

$$\therefore f(z)=\frac{1}{u(u+2)}=\frac{1}{u\cdot 2\cdot\left(1+\frac{u}{2}\right)}=\frac{1}{2u\left(1+\frac{u}{2}\right)}=\left(\frac{1}{2u}+u/2\right)^{-1}$$

$$=\frac{1}{2u}\left\{\frac{1}{u}-u/2+\frac{u^2}{4}-u^3/8+\ldots\right\}$$

$$=\frac{1}{2}\left\{\frac{1}{u}-\frac{1}{2}+\frac{u}{4}-\frac{u^2}{8}+\ldots\right\};\text{ but } u=z+1$$

$$=\frac{1}{2}\left\{\frac{1}{z+1}-\frac{1}{z}+\frac{z+1}{4}-\frac{(z+1)^2}{8}+\ldots\right\}$$

$$=\frac{1}{2(z+1)}-\frac{1}{2}+\frac{z+1}{8}-\frac{(z+1)^2}{16}+\ldots$$

(d)If $|z|<1$, $f(z)=\frac{1}{2(z+1)}-\frac{1}{2(z+3)}$

$z<1\Rightarrow$

$$\frac{1}{2(z+1)}=\frac{1}{2}(1+z)^{-1}$$

$$= \frac{1}{2}(1 - z + z^2 - z^3 + \dots\}$$

$$= \tfrac{1}{2} - \frac{1}{2}z + \frac{z^2}{2} - \frac{z^3}{2} + \dots (1)$$

If $|z| < 3, \frac{|z|}{3} < 1$

$$\Rightarrow \frac{1}{2(z+3)} = \frac{1}{2(3)\left(1+\frac{z}{3}\right)} = \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \dots (2)$$

$\therefore |z| < 1$

Subtract (2) from (1) to get

$$\frac{1}{2} - \frac{1}{2z} + \frac{z^2}{2} + \frac{z^3}{2} + \dots - \left(\frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \dots\right)$$

$$= \tfrac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \dots$$

Exercise 24

(a)Expand $f(z) = \dfrac{z}{(z-1)(2-z)}$ in a Laurent series valid for

(a) $|z| < 1$ Ans. $-\frac{1}{2}z - \frac{3}{4}z^2 - \frac{7}{8}z^3 + \frac{15}{16}z^4 + \dots$

(b) $1 < |z| < 2$ Ans. $\frac{1}{z^2} + \frac{1}{z} + 1 + \frac{1}{2}z + \frac{1}{4}z^2 + \dots$

(c) $|z| > 2$ Ans. $\frac{1}{z^2} + \frac{1}{z} + 1 + \frac{1}{2}z + \frac{1}{4}z^2 + \dots$

(d) $|z-1| > 1$ Ans. $-(z-1)^{-1} - 2(z-1)^{-2} - 2(z-1)^{-3} + \dots$

(e) $0 < |z-2| < 1$ Ans.