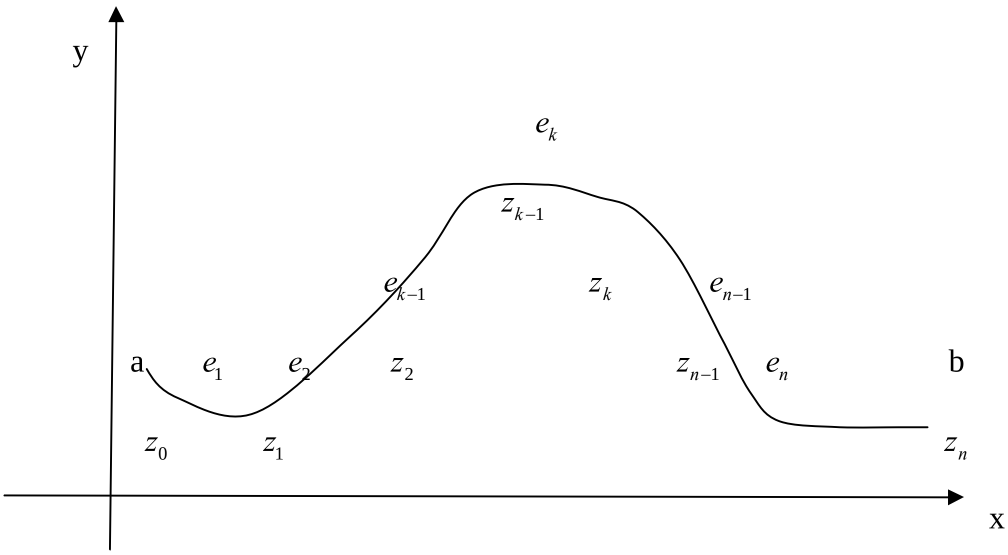
Complex line Integral

Let $f(z)$ be continous at all points of a curve c which we shall assume has a finite length.



Subdivide c into n parts by means of points $z_1, z_2, \dots z_{n-1}$,chosen arbitrarily and call $a = z_0, b = z_n$.

On each arc joining z_{k-1} to z_k ,choose a point $\frac{e}{k}$. Form the sum

$$S_n = f(e_1)(z_1 - a) + f\left(\frac{e}{2}\right)(z_2 - z_1) + \dots + f\left(\frac{e}{n}\right)(b - z_{n-1})$$

On writing $z_k - z_{k-1} = \triangle z_k$ é (1)

Equation (1) becomes $S_n = \sum_{k=1}^{n} f(e_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(e_k)\triangle z_k$ é (2)

Taking lim as $\triangle z_k \to 0$, $\int_a^b f(z)\,dz = \lim_{\triangle z \to 0} \sum_{k=1}^{n} f(e_k)\triangle z_k = \int_a^b f(z)\,dz$

$$\int_a^b f(z)\,dz = \int_c f(z)\,dz \text{ é } (3)$$

Equation (3) is called the complex line integral of $f(z)$ along curve c.

Note:The integral does not represent area but a complex number. If P(x, y) and Q(x, y) are real functions of x and y continous at all points of curve c ,the real line integral of $Pdx + Qdy$ along curve is defined by $\int_c \left[P(x,y)\,dx + Q(x,y)\,dy\right]$ é (4)

If c is a smooth curve and has parametric equations $x = f(t)$, $y = g(t)$ where $t_1 \le t \le t_2$, then the value of equation (4) is given by $\int_{t_1}^{t_2} P\left[f(t), g(t)\right]f'(t)\,dt + Q\left[f(t), g(t)\right]g'(t)\,dt$ é (5)

If $f(z) = u(x,y) + iv(x,y)$, the complex line integral (3) can be written in terms of the real line integral.

i.e. $\int_c f(z)\,dz = \int_c \left[u(x,y)+iv(x,y)\right]d(x+iy)$

$$= \int_c (u+iv)(dx+idy)$$

$$= \int_c udx + iudy + ivdx - vdy$$

$$= \int_c (udx - vdy) + i(udy + vdx)$$

$$= \int_c udx - vdy + i\int_c udy + vdx \text{ é } \quad (6)$$

Equation (5) is called the real line integral and equation (6) is called the complex line integral.

Example 40

Evaluate $= \int_{(0,3)}^{(2,4)} \left[2y + x^2\right)dx + (3x-y)dy$ along

(a) the parabola $x = 2t, y = t^2 + 3$.

(b) straight lines from (0, 3) to (2, 3) and then from (2, 3) to (2, 4).

(c) a straight line from (0, 3) to (2, 4).

Solution:

(a)along the parabola $x = 2t, y = t^2 + 3$ since 0r and y are in parametric form, we use the

$$\int_c P\left[f(t), g(t)\right]f^{'}(t)dt + Q\left[f(t), g(t)\right]g^{'}(t)dt$$

$$P(x,y) = 2y + x^2$$

$$Q(x,y) = 3x - y$$

But $x = 2t = f(t)\cdot y = t^2 + 3 = g(t)$

$$f(t) = 2t \Rightarrow f^{'}(t) = 2;\ g(t) = t^2 + 3 \Rightarrow g^{'}(t) = 2t$$

Limits (0, 3) and (2, 4), in terms of t ,are:

$(0,3): x = 0, \quad y = 3 \qquad (2,4): x = 2; y = 4$

$$\Rightarrow 2t = 0, \quad t^2 + 3 = 3, \quad 2t = t, \quad 3 + t^2 = 4$$

$$t = 0, \quad t = 0, \quad \Rightarrow t = 1, \quad t = 1$$

$\therefore (0,3) \to 0 \qquad \therefore (2,4) \to 1$

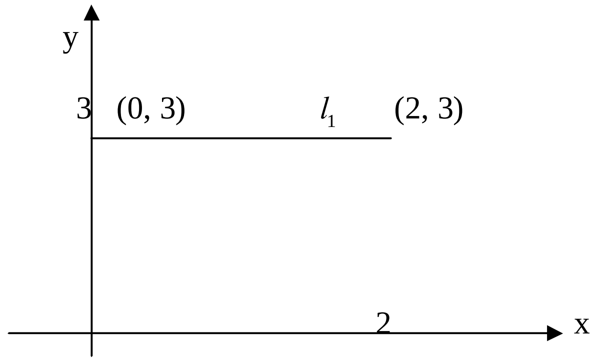
$$\therefore \int_0^1 \left[2(t^2+3) + (2t)^2\right]2dt + \left[3(2t) - (t^2+3)\right]2tdt$$

$$= \int_0^1 (12t^2 + 12)dt + \int_0^1 (12t^2 - 2t^3 - 6t)dt$$

$$= \left[4t^3 + 12t\right]_0^1 + \left[4t^3 - \frac{1}{2}t^4 - 3t^2\right]_0^1$$

$$= 4 + 12 + 4 - \frac{1}{2} - 3 = \frac{33}{2}$$

(b)(i) along the straight line from (0, 3) to (2, 3)



Along the straight line l_1, change in y is $0 \Rightarrow dy = 0, y = 3$

$P(x, y) = 2y + x^2 \Rightarrow$

$P(x,3) = 2(3) + x^2 = 6 + x^2$

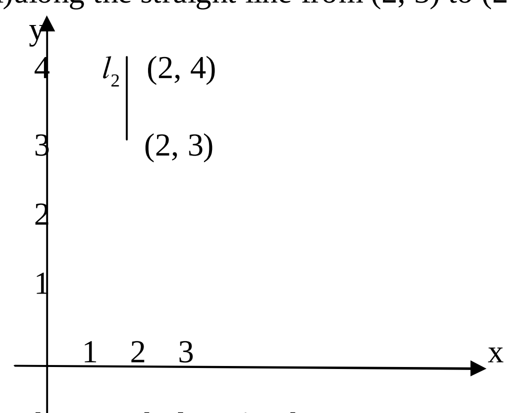
$Q(x, y) = 3x - y = 3x - 3$

Limits $\left.\begin{matrix}(0,3) \to 0 \\ (2,3) \to 2\end{matrix}\right\}$ since no change along y-axis

$: \int_{(0,3)}^{(2,3)} (2y + x^2)dx + (3x - y)dy$ becomes $\int_0^2 (6 + x^2)dx + (3x-3)dy$ but $dy = 0$

$= \int_0^2 (6 + x^2)dx = \left[6x + \frac{x^3}{3}\right] = 12 + \frac{8}{3} = \frac{44}{3}$

(b)(ii)along the straight line from (2, 3) to (2, 4)



Along the straight line l_2, change in x is $0 \Rightarrow dx = 0, x = 2$

$\therefore P(x, y) = 2y + x^2 \Rightarrow P(2, y) = 2y + 4$

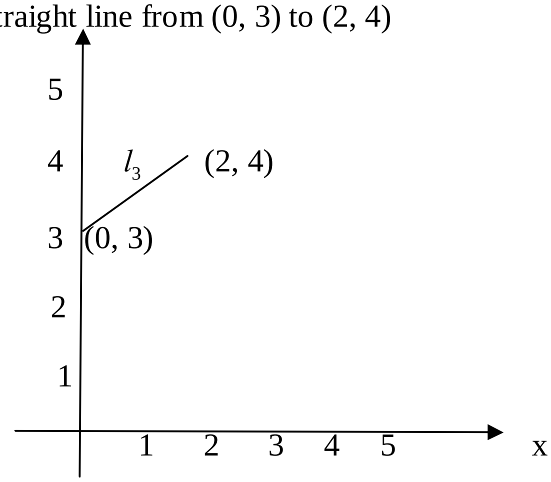
$Q(x, y) = 3x - y \Rightarrow P(2, y) = 6 - y$

Limits $\left.\begin{matrix}(2,3) \to 3 \\ (2,4) \to 4\end{matrix}\right\}$ since no change along x-axis.

$$\therefore \int_{(2,3)}^{(2,4)} \left(2y+x^2\right)dx+\left(3x-y\right)dy \text{ becomes } \int_3^4 \left(2y+4\right)dx+\left(6-y\right)dy; \text{ but } dx=0$$

$$=\int_3^4 \left(6-y\right)dy=\left[6y-\frac{y^2}{2}\right]_3^4=24-8-18+\frac{9}{2}=\frac{5}{2}$$

(c) a straight line from (0, 3) to (2, 4)



Get the equation of the line l_3

$$y=mx+c; \quad c=3; \quad m=\frac{4-3}{2-0}=\frac{1}{2}$$

$$\therefore y=\frac{1}{2}x+3 \Rightarrow 2y=x+6 \text{ or } 2y-x=6$$

$$x=2y-6 \Rightarrow dx=2dy$$

$$P(x,y)=2y+x^2 \Rightarrow P(x,y)=2y+(2y-6)^2=P(y)$$

$$Q(x,y)=3x-y \Rightarrow Q(x,y)=3(2y-6)-y=5y-18=Q(y)$$

Since $P(x,y)$ and $Q(x,y)$ reduce to *P(y)* and *Q(y)* respectively, we use the limits of *y* only, i.e. *y=3* to *y=4*.

$$\therefore \int_{(0,3)}^{(2,4)} \left(2y+x^2\right)dx+\left(3x-y\right)dy \text{ becomes } \int_3^4 \left[2y+\left(2y-6\right)^2\right]dx+\left(5y-18\right)dy \text{ but } dx=2dy$$

$$=\int_3^4 \left[2y+\left(2y-6\right)^2\right]2dy+\left(5y-18\right)dy$$

$$=\int_3^4 \left[\left(2y\right)+4y^2-24y+36\right]2dy+\left(5y-18\right)dy$$

$$=\int_3^4 \left[8y^2-44y+72+5y-18\right]dy$$

$$=\int_3^4 \left[8y^2-39y+54\right]dy$$

$$=\left[\frac{8y^3}{3}-\frac{39y^2}{2}+54y\right]_3^4=\frac{97}{6}$$

Note:the result can also be obtained using $y=\frac{1}{2}x+6$.

Example 40

Evaluate $\int_c \bar{z}$ from $z=0$ to $z=4+2i$ along the curve c given by

(a) $z=t^2+it$

(b)the line from $z=0$ to $z=2i$ and then the line from $z=2i$ to $z=4+2i$

Solution:

(a) $z=t^2+it$

Limits

$z=0 \Rightarrow 0=t^2+it \Rightarrow t=0$

$z=4+2i \Rightarrow 4+2i=t^2+it \Rightarrow t=2$

$z=0 \Rightarrow t=0$

$z=4+2i \to t=2$

$$\int_0^{4+2i} \bar{z}dz=\int_0^2 \left(t^2-it\right)dz=\int_0^2 \left(t^2-it\right)d\left(t^2+it\right)$$

$$=\int_0^2 \left(t^2-it\right)\left(2t+i\right)dt$$

$$=\int_0^2 \left(2t^3+it^2-2it^2+t\right)dt$$

$$=\int_0^2 \left(2t^3-it^2+t\right)dt$$

$$=10-\frac{8i}{3}$$

(b)(i)the given line integral is equal to

$$\int_c \bar{z}dz=\int_c \left(x-iy\right)d\left(x+iy\right)$$

$$=\int_c \left(x-iy\right)\left(dx+idy\right)$$

$$=\int_c xdx+ixdy-iydx+ydy$$

$$=\int_c \left(xdx+ydy\right)+i\left(xdy-ydx\right)$$

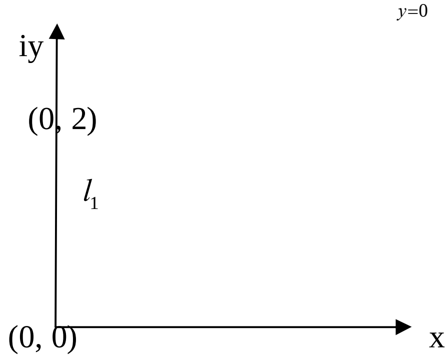
The line from $z=0$ to $z=2i$ is the same as the line from (0, 0) to (0, 2).

$z=0=0+oi \to x=0, y=0 \to \left(0,0\right)$

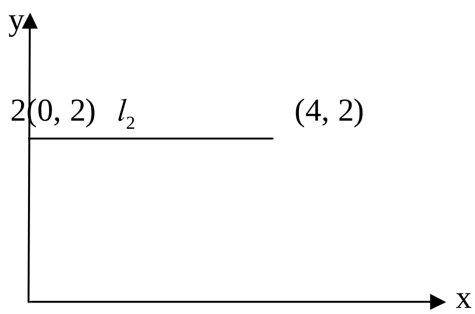
$z=2i=o+2i \to x=0, y=2 \to \left(0,2\right)$

$x=0 \Rightarrow dx=0$ (i.e. change in x is 0).

$$\therefore \int_c (xdx + ydy) + i(xdy - ydx) \text{ becomes } \int_{y=0}^{y=2} ydy = \left[\frac{y^2}{2}\right]_0^2 = 2$$



(b)(ii)the line from $z = 2i$ to $z = 4 + 2i$, this line is the same as the line l_2 from (0, 2) to (4, 2)



along line l_2, change on y is
$\Rightarrow dy = 0,$ and $y = 2$

limits $z = 2i$ to $z = 4 + 2i$ become $(0,2) \to (4,2) \to 0 \to 4$

$$\therefore \int_c (xdx + ydy) + i(xdy - ydx) \text{ becomes } \int_0^4 xdx + 2(0) + i\left[x(0) - 2dx\right]$$

$$= \int_0^4 xdx - i\int_0^4 2dx$$

$$= \left[\frac{x^2}{2}\right]_0^4 - i\left[2x\right]_0^4$$

$$= 8 - 8i$$

Properties of integrals

1. $\int_c \left(f(z) + g(z)\right]dz = \int_c f(z)dz + \int_c g(z)dz$
2. $\int_c Af(z)dz = A\int_c f(z)dz$ where A is any constant.
3. $\int_a^b f(z)dz = -\int_b^a f(z)dz$
4. $\int_a^b f(z)dz = \int_a^m f(z)dz + \int_m^b f(z)dz$

Where a, b, m are on the curve c.

5. $\left|\int_c f(z)dz\right| \le ml$, where $|f(z)| \le m$ i.e. m is an upper bound of $|f(z)|$ on c and l is the length of c.

Example 42:

Evaluate $\int_{c_1} z^2 dz$ where c_1 is the line segment OB from $z=0$ to $z=2+i$.

Method 1

Segment OB from $z=0$ to $z=2+i$

$$z=o=o+oi=(0,0)$$

$$z=2+i=(2,1)$$

Equation of a line joining (0, 0) and (2, 1) is given by $y=\frac{1}{2}x \Rightarrow x=2y$

$$\therefore \frac{dy}{dx}=\frac{1}{2} \Rightarrow dx=2dy$$

Now, $\int_{c_1} z^2 dz = \int (x+iy)^2 d(x+iy)$

$$=\int (x^2+2iyx-y^2)(dx+idy)$$

$$=\int \left[(x^2-y^2)+2iyx\right][dx+idy]$$

$$=\int (x^2-y^2)dx+i(x^2-y^2)dy+i2yxdx-2yxdy$$

$$=\int (x^2-y^2)\pm_{dx} 2yxdy+i\int (x^2-y^2)^{dy}+2yxdx$$

Now, $dx=2dy$ and $y=\frac{1}{2}x \Rightarrow x=2y$

$\therefore$ using limits of y and writing x in terms of y

$\int_{(0,0)}^{(2,1)} (x^2-y^2)dx-2yxdy+i\int_{(0,0)}^{(2,1)} (x^2-y^2)dy+2yxdx$ becomes

$$\int_0^1 \left[(2y)^2-y^2\right][2dy]-2y(2y)dy+i\int_0^1 \left[(2y)^2-y^2\right]dy+(2y)2y(2dy)$$

$$=\int_0^1 6y^2dy-4y^2dy+i\int 3y^2dy+8y^2dy$$

$$=\int_0^1 (6y^2-4y^2)+i\int_0^2 11y^2dy$$

$$=\left[\frac{2y^3}{3}\right]_0^1+i\left[\frac{11y^3}{3}\right]_0^1$$

$$=16/3+56/3\, i \quad \frac{2}{3}+\frac{11}{3}i$$

Method 2

$$\int_0^{2+i} z^2 dz = \left[\frac{z^3}{3}\right]_0^{2+i}$$

$$= \frac{(2+i)^3}{3}$$

$$= \left((1)8 + (3)4i - (3)2 - (1)i\right)/3$$

$$= \frac{1}{3}(8 + 12i - 6 - i)$$

$$= 2/3 + \frac{11}{3}i$$