LU Decomposition of a matrix to solve systems of linear equations

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LU Decomposition defined

- LU decomposition of a matrix is the factorization of a given square matrix $A_{(nxn)}$ into **two triangular matrices, one upper triangular matrix (U) and one lower triangular matrix (L).**
- The product of these two matrices (**L and U**) gives the original matrix. **A=LU**
- Matrix decomposition was introduced by Alan Turing in 1948, who also created the Turing machine.

Applications of LU

- ✓This method of factorizing a matrix as a product of two triangular matrices has various applications such as:
- ✓ a solution of a system of equations, which itself is an integral part of many applications such as
  - ***finding current in a circuit*** and
  - ***solution of discrete dynamical system problems***;
  - ***finding the inverse of a matrix*** and
  - ***finding the determinant of the matrix.***

- LU decomposition is helpful whenever it is possible to model the problem to be solved into matrix form.
- Conversion to the matrix form and solving with triangular matrices makes it easy to do calculations in the process of finding the solution.

- A square matrix A can be decomposed into **two square matrices L and U** such that: **<u>A = LU</u>** where **U** is an upper triangular matrix formed as a result of applying the Gauss Elimination Method on A, and **L** is a lower triangular matrix with diagonal elements being equal to 1.

For A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, we have L = $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and U = $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$; such that A = L U.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 0 \end{bmatrix} * \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Here value of l_{21} , u_{11} etc can be compared and found.

Gauss Elimination Method

According to the Gauss Elimination method:

- Any zero row should be at the bottom of the matrix.
- The first non zero entry of each row should be on the right-hand side of the first non zero entry of the preceding row.
- This method reduces the matrix to row echelon form.

Steps to follow in LU Decomposition:

1. Given a set of linear equations, first convert them into matrix form A X = C where **A is the coefficient matrix**, **X is the variable matrix** and **C is the matrix of numbers on the right-hand side of the equations**.
2. Now, reduce the coefficient matrix A to row echelon form using Gauss Elimination Method. The matrix so obtained is **U**.
3. To find L, we have two methods.
   a) The first one is to assume the remaining elements as some artificial variables, make equations using A = L U and solve them to find those artificial variables.

Steps for LU Decomposition…cont

b) The other method is that the remaining elements **are the multiplier coefficients because of which the respective positions became zero in the U matrix.** (This method is a little tricky to understand by words but would gets clear in the example that follows)

4. Now, we have A (the *nxn* coefficient matrix), L (the *nxn* lower triangular matrix), U (the *nxn* upper triangular matrix), X (the nx1 matrix of variables) and C (the nx1 matrix of numbers on the right-hand side of the equations).

5. The given system of equations is A X = C. We substitute A = L U. Thus, we have L U X = C.
We put Z = U X, where Z is a matrix or artificial variables and solve for L Z = C first and then solve for U X = Z to find X or the values of the variables, which was required.

LU Decomposition Method or Factorisation

Consider the system of equations in three variables:

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

These can be written in the form of AX = B as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Here,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \; B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Now follow the steps given below to solve the above system of linear equations by LU Decomposition method.

Step 1: Generate a matrix A = LU such that L is the lower triangular matrix with principal diagonal elements being equal to 1 and U is the upper triangular matrix.

That means,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

and

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Step 2: Now, we can write AX = B as:

LUX = B....(1)

Step 3: Let us assume UX = Y....(2)

$$\text{Where } Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Step 4: From equations (1) and (2), we have;

LY = B

On solving this equation, we get y_1, y_2, y_3.

Step 5: Substituting Y in equation (2), we get UX = Y

By solving equation, we get X, x_1, x_2, x_3.

The above process is also called the Method of Triangularisation.

Example:

Solve the system of equations $x_1 + x_2 + x_3 = 1$, $3x_1 + x_2 - 3x_3 = 5$ and $x_1 - 2x_2 - 5x_3 = 10$ by LU decomposition method.

Solution:

Given system of equations are:

$x_1 + x_2 + x_3 = 1$

$3x_1 + x_2 - 3x_3 = 5$

$x_1 - 2x_2 - 5x_3 = 10$

These equations are written in the form of AX = B as:

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$$

Step 1: Let us write the above matrix as LU = A.

That means,

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix}$$

By expanding the left side matrices, we get;

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix}$$

Thus, by equating the corresponding elements, we get;

$u_{11} = 1$, $u_{12} = 1$, $u_{13} = 1$

$l_{21}u_{11} = 3$,

$l_{21}u_{12} + u_{22} = 1$,

$u_{21}u_{13} + u_{23} = -3$

$l_{31}u_{11} = 1$,

$l_{31}u_{12} + l_{32}u_{22} = -2$,

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = -5$

Solving these equations, we get;

$u_{22} = -2$, $u_{23} = -6$, $u_{33} = 3$

$l_{21} = 3$, $l_{31} = 1$, $l_{32} = 3/2$

Step 2: LUX = B

Step 3: Let UX = Y

Step 4: From the previous two steps, we have LY = B

Thus,

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$$

So,

$y_1 = 1$

$3y_1 + y_2 = 5$

$y_1 + (3/2)y_2 + y_3 = 10$

Solving these equations, we get;

$y_1 = 1$, $y_2 = 2$, $y_3 = 6$

Step 5: Now, consider UX = Y. So,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

By expanding this equation, we get;

$x_1 + x_2 + x_3 = 1$

$-2x_2 - 6x_3 = 2$

$3x_3 = 6$

Solving these equations, we can get;

$x_3 = 2$, $x_2 = -7$ and $x_1 = 6$

Therefore, the solution of the given system of equations is (6, -7, 2).

Example: Solve the following system of equations using LU Decomposition method:

$$x_1 + x_2 + x_3 = 1$$
$$4x_1 + 3x_2 - x_3 = 6$$
$$3x_1 + 5x_2 + 3x_3 = 4$$

Solution: Here, we have

A = $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$ $, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$ such that A X = C.

Now, we first consider $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$ and convert it to row echelon form using Gauss Elimination Method.

So, by doing

$$R_2 \to R_2 - 4R_1 \quad (1)$$

$$R_3 \to R_3 - 3R_1 \quad (2)$$

we get

$$\begin{bmatrix}1 & 1 & 1\\4 & 3 & -1\\3 & 5 & 3\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 1\\0 & -1 & -5\\0 & 2 & 0\end{bmatrix}$$

Now, by doing

$$R_3 \to R_3 - (-2)R_2 \qquad (3)$$

we get

$$\sim \begin{bmatrix}1 & 1 & 1\\0 & -1 & -5\\0 & 0 & -10\end{bmatrix}$$

(Remember to always keep ' − ' sign in between, replace ' + ' sign by two ' − ' signs)

Hence, we get L = $\begin{bmatrix}1 & 0 & 0\\4 & 1 & 0\\3 & -2 & 1\end{bmatrix}$ and U = $\begin{bmatrix}1 & 1 & 1\\0 & -1 & -5\\0 & 0 & -10\end{bmatrix}$

(Remember to always keep ' − ' sign in between, replace ' + ' sign by two ' − ' signs)

Hence, we get L = $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$ and U = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$

(notice that in L matrix, $l_{21} = 4$ is from (1), $l_{31} = 3$ is from (2) and $l_{32} = -2$ is from (3))

Now, we assume Z $= \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ and solve L Z = C.

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

So, we have $z_1 = 1,\ 4z_1 + z_2 = 6,\ 3z_1 - 2z_2 + z_3 = 4.$

Solving, we get $z_1 = 1$, $z_2 = 2$ and $z_3 = 5$.

Now, we solve U X = Z

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Therefore, we get $x_1 + x_2 + x_3 = 1,$ $-x_2 - 5x_3 = 2$, $-10x_3 = 5.$

Thus, the solution to the given system of linear equations is $x_1 = 1$, $x_2 = 0.5$, $x_3 = -0.5$

and hence the matrix X = $\begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix}$

Question..........

Write on both sides of the paper

in either margin

$$\begin{bmatrix} 1- & 1 & 1 \\ -6 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

A   X   B

$A = LU$

$L\underline{U}X = B$

let $UX = Y$ to solve for X

$LY = B$ to solve for ~~X~~ Y

$$\begin{bmatrix} 1 & -1 & 1 \\ -6 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$$ $R_2 + 6R_1$
$R_3 - 3R_1$

$1 - 3(-1) = 1 + 3 = 4$

$1 - 3(1) = 1 - 3 = -2$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -6 & 1 & 0 \\ 3 & -\frac{4}{5} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -5 & 5 \\ 0 & 4 & -2 \end{bmatrix}$$

$4 -$

$R_3 + \frac{4}{5}R_2$

$4 + \frac{4}{5}(-5) = 4 - 4 = 0$

$-2 + \frac{4}{5}(5) = -2 + 4 = 2$

$4 + x(-5) = 0$

~~$4 + 5x = 0$~~

$4 + x(-5) = 0$

$4 - 0 = 5x$

$4 = 5x$

$x = \frac{4}{5}$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -5 & 5 \\ 0 & 0 & 2 \end{bmatrix} = U \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ -6 & 1 & 0 \\ 3 & -4/5 & 1 \end{bmatrix}$$

$UX = Y$ $\qquad$ $LY = B$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -5 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -6 & 1 & 0 \\ 3 & -4/5 & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -5 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \\ 10 \end{bmatrix}$$

$2x_3 = 10$

$x_3 = 5$

$-5x_2 + 5x_3 = 15$

$-5x_2 + 5(5) = 15$

$-5x_2 + 25 = 15$

$-5x_2 = 15 - 25 = -10$

$-5x_2 = -10$

$x_2 = 2$

$x_1 - x_2 + x_3 = 2$

$x_1 - 2 + 5 = 2$

$x_1 + 3 = 2$

$x_1 = 2 - 3 = -1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

$Y_1 = 2$

$-6Y_1 + Y_2 = 3$

$-6(2) + Y_2 = 3$

$-12 + Y_2 = 3$

$Y_2 = 3 + 12$

$Y_2 = 15$

$3Y_1 - \frac{4}{5}Y_2 + Y_3 = 4$

$3(2) - \frac{4}{5}(15) + Y_3 = 4$

$6 - 12 + Y_3 = 4$

$Y_3 = 4 + 12 - 6 = 10$

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \\ 10 \end{bmatrix}$$

LU Decomposition Method Problems

1. Solve the following equations by LU decomposition method.

   $6x_1 + 18x_2 + 3x_3 = 3, 2x_1 + 12x_2 + x_3 = 19, 4x_1 + 15x_2 + 3x_3 = 0$

2. Solve the below given system of equations by LU decomposition.

   $x + y + z = 1, 4x + 3y - z = 6, 3x + 5y + 3z = 4$

3. Find the solution of the system of equations by LU decomposition.

   $x + 2y + 3z = 9, 4x + 5y + 6z = 24, 3x + y - 2z = 4$

Questions & Comments