Solving Systems of Equations Using Matrix inverse method

Solving the system of equations using an inverse matrix

$x + 2y + 2z = 5$

$3x - 2y + z = -6$

$2x + y - z = -1$

$$\underbrace{\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & -1 \end{bmatrix}}_{A \text{ (coefficients)}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X \text{ (variables)}} = \underbrace{\begin{bmatrix} 5 \\ -6 \\ -1 \end{bmatrix}}_{B}$$

$$\text{Det } A = 1\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} - 2\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} + 2\begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}$$

$$= 1(2-1) - 2(-3-2) + 2(3 - -4)$$

$$1(1) - 2(-5) + 2(7)$$

$$1 - -10 + 14 = 25$$

A^{-1}

$$\tfrac{1}{25}(\text{adjoint } A) = \begin{bmatrix} +\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \\ +\begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (2-1) & -(-3-2) & +(3 - -4) \\ -(2-2) & +(-1-4) & -(1-4) \\ (2 - -4) & -(1-6) & +(-2-6) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 & 7 \\ 4 & -5 & 3 \\ 6 & 5 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 7 \\ 4 & -5 & 3 \\ 6 & 5 & -8 \end{bmatrix}$$

$$\frac{1}{25}\begin{bmatrix} 1 & 4 & 6 \\ 5 & -5 & 5 \\ 7 & 3 & -8 \end{bmatrix} = \begin{bmatrix} 1/25 & 4/25 & 6/25 \\ 5/25 & -5/25 & 5/25 \\ 7/25 & 3/25 & -8/25 \end{bmatrix} \times \begin{bmatrix} 5 \\ -6 \\ -1 \end{bmatrix} = \begin{bmatrix} 5/25 - 24/25 - 6/25 \\ 25/25 - 30/25 - 5/25 \\ 35/25 - 18/25 + 8/25 \end{bmatrix} = \begin{bmatrix} \frac{5-24-6}{25} = \frac{-25}{25} = -1 \\ \frac{25-30-5}{25} \\ \frac{35-18+8}{25} = \frac{25}{25} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{5} + -\frac{24}{25} - \frac{6}{25} = -\frac{25}{25} \\ \frac{1}{5} + \frac{6}{5} - \frac{1}{5} = 1 + \frac{5}{5} \\ \frac{35-10}{25} = \frac{25}{25} \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$x = -1$

$y = 2$

$z = 1$

in eq 2

$3(-1) - 2(2) + (1) = -6$

$-3 - 4 + 1 = -6$

$-7 + 1 = -6$

$-6 = -6$

Question 1: Find the following of the given matrix $A = \begin{bmatrix} 3 & 2 & -1 \\ -5 & 0 & -2 \\ 3 & 4 & -1 \end{bmatrix}$

- **determinant of matrix A**
- **cofactor matrix A**
- **adjoint of matrix A**
- **inverse of matrix A**

Solution:

The given matrix is $A = \begin{bmatrix} 3 & 2 & -1 \\ -5 & 0 & -2 \\ 3 & 4 & -1 \end{bmatrix}$

- *Determinant of the A =*

$$3(0 \times (-1) - 4 \times (-2)) + 5(2 \times (-1) - 4 \times (-1)) + 3(2 \times (-2) - 0 \times (-1))$$

$= 3(0+8)+5(-2+4)+3(-4)$

$= 3 \times 8 + 5 \times 2 + 3 \times (-4)$

$= 24 + 10 - 12$ *units*

- *Cofactor of matrix A =*

$C_{11} = 0 \times (-1) - 4 \times (-2) = 0 + 8 = 8$

$C_{12} = -((-5) \times (-1) - 3 \times (-2)) = -(5 + 6) = -11$

$C_{13} = (-5) \times 4 - 3 \times 0 = -20$

$C_{21} = -(2 \times (-1) - 4 \times (-1)) = -(-2 + 4) = -2$

$C_{22} = 3 \times (-1) - 3 \times (-1) = -3 + 3 = 0$

$C_{23} = -(3 \times 4 - 3 \times 2) = -(12 - 6) = -6$

$C_{31} = 2 \times (-2) - 0 \times (-1) = -4$

$C_{32} = -(3 \times (-2) - (-5) \times (-1)) = -(-6 - 5) = 11$

$C_{33} = 3 \times 0 - (-5) \times 2 = 10$

Cofactor matrix of A = $C = \begin{bmatrix} 8 & -11 & -20 \\ -2 & 0 & -6 \\ -4 & 11 & 10 \end{bmatrix}$

- *AdjoinT of matrix A = transpose of cofactor matrix C =*

$$C = \begin{bmatrix} 8 & -11 & -20 \\ -2 & 0 & -6 \\ -4 & 11 & 10 \end{bmatrix}'$$

$$C = \begin{bmatrix} 8 & -2 & -4 \\ -11 & 0 & 11 \\ -20 & -6 & 10 \end{bmatrix}$$

- *Inverse of matrix A =* $A^{-1} = \frac{1}{|A|} adj.A$

$$= \frac{1}{22} \begin{bmatrix} 8 & -2 & -4 \\ -11 & 0 & 11 \\ -20 & -6 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{11} & \frac{-1}{11} & \frac{-2}{11} \\ \frac{-1}{2} & 0 & \frac{1}{2} \\ \frac{-10}{11} & \frac{-3}{11} & \frac{5}{11} \end{bmatrix}$$

- Question 2: Ram is hired for a job with a monthly payment of a specific amount and an annual increase of a predetermined amount. Find his beginning pay and yearly increase if his salary was \$300 per month at the end of the first month after 1 year of service and \$600 per month at the end of the first month after 3 years of service.

- Solution: Let "x" and "y" represent the monthly salary and a yearly increase of a certain amount, respectively.

- According to the question;

- $x + y = 300 \dashrightarrow$ (i)

- $x + 3y = 600 \dashrightarrow$ (ii)

- This can be written as AX = B, where

$A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 300 \\ 600 \end{bmatrix}$

Determinant of A = 1 × 3 - 1 × 1 = 3 - 1 = 2

Adjoin of A = $\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$

Thus, $A^{-1} = \frac{1}{|A|} adj.A$

$A^{-1} = \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$

Using Matrix Inverse,

$X = A^{-1}B$

$X = \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 300 \\ 600 \end{bmatrix}$

$X = \frac{1}{2} \begin{bmatrix} 3 \times 300 + (-1) \times 600 \\ (-1) \times 300 + 1 \times 600 \end{bmatrix}$

$X = \frac{1}{2} \begin{bmatrix} 300 \\ 300 \end{bmatrix}$

$X = \begin{bmatrix} 150 \\ 150 \end{bmatrix}$

Therefore; x = $150, y = $150

So, the monthly salary is $150 and the annual increment is $150.

- Question 3: The sum of three numbers is 3. If we multiple the second number by 2 and add the first number to it, we get 6. If we multiply the third number by 4 and add the second number to it, we get 10. Represent it algebraically and find the numbers using the matrix method.
- Solution: Let x, y, and z represent the first, second, and third numbers, respectively. Then, according to the question, we have
- $x + y + z = 3$
- $x + 2y = 6$
- $y + 4z = 10$

This can be written as AX = B, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 3 \\ 10 \\ 6 \end{bmatrix}$

Here, $|A| = 1(8 - 0) - 1(4 - 0) + 1(1 - 0) = 8 - 4 + 1 = 5 \neq 0$. Now, find adj A.

$A_{11} = 8 - 0 = 8, A_{12} = -(4 - 0) = -4, A_{13} = 1 - 0 = 1$

$A_{21} = -(4 - 1) = -3, A_{22} = 4 - 0 = 4, A_{23} = -(1 - 0) = -1$

$A_{31} = 0 - 2 = -2, A_{32} = -(0 - 1) = 1, A_{33} = 2 - 1 = 1$

Adj. $A = \begin{bmatrix} 8 & -3 & -2 \\ -4 & 4 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

Thus, $A^{-1} = \frac{1}{|A|} Adj.of A = \frac{1}{5} \begin{bmatrix} 8 & -3 & -2 \\ -4 & 4 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

$X = A^{-1}B$

$$X = \frac{1}{5} \begin{bmatrix} 8 & -3 & -2 \\ -4 & 4 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 10 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 24 - 30 - 12 \\ -12 + 40 + 6 \\ 3 - 10 + 6 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -18 \\ 34 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{-18}{5} \\ \frac{34}{5} \\ \frac{-1}{5} \end{bmatrix}$$

Therefore;

$$X = \frac{-18}{5}, Y = \frac{34}{5}, Z = \frac{-1}{5}$$

- Question 4: Assume Joe, Max, and Polly went shopping at the mall. Joe pays 45/- for 4 kg of apples, 7 kg of bananas, and 6 kg of guavas, Max pays 30/- for 2 kg of apples and 5 kg of guavas, and Polly pays 35/- for 3 kg of apples, 1 kg of bananas, and 4 kg of guavas. How much do apples, bananas, and guavas cost per kilogram?

- Solution: Let x, y, and z represent the number of apples, bananas, and guavas, respectively. In accordance to the question:
- $4x + 7y + 6z = 45$
- $2x + 5z = 30$
- $3x + y + 4z = 35$

Matrix A contains the kg of apples, bananas, and guavas bought by Joe, Max, and Polly. Matrix B contains the prices paid by the three and matrix X contains the variables.

$$A = \begin{bmatrix} 4 & 7 & 6 \\ 2 & 0 & 5 \\ 3 & 1 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 45 \\ 30 \\ 35 \end{bmatrix}$$

The solution of the given system of equations be $X = A^{-1} B$.

In order to find the inverse of A, we will first find the determinant of A.

$$A = \begin{bmatrix} 4 & 7 & 6 \\ 2 & 0 & 5 \\ 3 & 1 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 45 \\ 30 \\ 35 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj.A$$

Determinant of A = $|A| = 4(0 \times 4 - 1 \times 5) - 7(2 \times 4 - 5 \times 3) + 6(2 \times 1 - 3 \times 0)$

$= 4(0 - 5) - 7(8 - 15) + 6(2 - 0)$

$= -20 - 7(-7) + 12$

$= -20 + 49 + 12 = 41$

Adj. of A = $\begin{bmatrix} -5 & -22 & 35 \\ 7 & -2 & -6 \\ 2 & 17 & -14 \end{bmatrix}$

$$A^{-1} = \frac{1}{41} adj.A$$

$$X = A^{-1}B = \frac{1}{41} \begin{bmatrix} -5 & -22 & 35 \\ 7 & -2 & -6 \\ 2 & 17 & -14 \end{bmatrix} \times \begin{bmatrix} 45 \\ 30 \\ 35 \end{bmatrix}$$

$$X = \frac{1}{41} \begin{bmatrix} 340 \\ 45 \\ 110 \end{bmatrix} = \begin{bmatrix} 8.3 \\ 1.1 \\ 2.7 \end{bmatrix}$$

The cost of apples per kg = 8.3/-

The cost of bananas per kg = 1.1/-

The cost of guavas per kg = 2.7/-

- Question 5: The cost of 2 kg potatoes, 3 kg tomatoes, and 2 kg flour is 50. The cost of 5 kg potatoes, 1 kg tomatoes and 6 kg flour is 40. The cost of 4 kg potatoes, 6 kg tomatoes and 3 kg flour is 60. Find the cost of each item per kg by the inverse of a matrix.

- Solution: Let x, y, and z represent the kg of potatoes, tomatoes, and flour, respectively. In accordance to the question:

- $2x + 3y + 2z = 50$
- $5x + 1y + 6z = 40$
- $4x + 6y + 3z = 60$

Matrix A contains the kg of potatoes, tomatoes and flour. Matrix B contains the prices paid and matrix X contains the variables. This can be written as AX = B, where

$$A = \begin{bmatrix} 2 & 3 & 2 \\ 5 & 1 & 6 \\ 4 & 6 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 50 \\ 40 \\ 60 \end{bmatrix}$$

The solution of the given system of equations is $X = A^{-1}B$. In order to find the inverse of A, we will first find the determinant of A.

Determinant of A $|A| = 2(3 - 36) - 3(15 - 24) + 2(30 - 4) = 2 \times (-33) - 3(-9) + 2(26) = -66 + 27 + 52 = 13$

Now, find the adjoint of A to get the inverse of A.

$A_{11} = 3 - 36 = -33$, $A_{12} = -(15 - 24) = 9$, $A_{13} = 30 - 4 = 26$

$A_{21} = -(9 - 12) = 3$, $A_{22} = 6 - 8 = -2$, $A_{23} = -(12 - 12) = 0$

$A^{\{-1\}} =$ $$A^{-1} = \frac{1}{|A|} adj.A = \frac{1}{13}\begin{bmatrix} -33 & 3 & 16 \\ 9 & -2 & -2 \\ 26 & 0 & -13 \end{bmatrix}$$

$A_{31} = 18 - 2 = 16$, $A_{32} = -(12 - 10) = -2$, $A_{33} = 2 - 15 = -13$

Thus, $$Adj.A = \begin{bmatrix} -33 & 3 & 16 \\ 9 & -2 & -2 \\ 26 & 0 & -13 \end{bmatrix}$$

$X = A^{-1}B$

$$X = \frac{1}{13}\begin{bmatrix} -33 & 3 & 16 \\ 9 & -2 & -2 \\ 26 & 0 & -13 \end{bmatrix}\begin{bmatrix} 50 \\ 40 \\ 60 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -33 \times 50 + 3 \times 40 + 16 \times 60 \\ 9 \times 50 - 2 \times 40 - 2 \times 60 \\ 26 \times 50 + 0 \times 40 - 13 \times 60 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -570 \\ 250 \\ 520 \end{bmatrix} = \begin{bmatrix} -43.8 \\ 19.2 \\ 40 \end{bmatrix}$$

$x = 43.8$, $y = 19.2$, $z = 40$

Questions?
Comments?