KENYATTA UNIVERSITY

INSTITUTE OF OPEN LEARNING

SMA 305

COMPLEX ANALYSIS I

BY

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DEPARTMENT OF MATHEMATICS

PREFACE

This module is designed primarily to provide the readers with the best preparation possible for the Complex Analysis I (SMA 305 Complex Analysis I). In its present form this module has developed from the courses given by the author over the last **thirty two** years in various universities to the audience of Mathematicians, Physicists and Engineers in the university of Madras, Kenyatta University, University of Nairobi and Jomo Kenyatta University of Agriculture and Technology.

This module, Complex Analysis I is compiled from the Author's First course in Complex Variables, Oxford Publications, London and Nairobi. Most of the theory and problems are freely taken from the Author's Book for which the author has sole Copyright.

It is hoped that it will be of great interest to students of pure and Applied Mathematicians and Engineers following Complex Analysis I.

Each Lesson begins with a brief statement of definitions principles and important Theorems followed by a set of solved problems.

The author is pleased to acknowledge Dr. L. O. Odongo B.Ed (Hons) M.Sc (UON) and M.Sc (Canada), Ph.D (Canada), the Chairman of Department of Mathematics Kenyatta University who encouraged me to write this module.

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LESSON 1
Complex Numbers and their Representation on the Complex Plane

1.1 Introduction

We all know that $\sqrt{9}$ is either +3 or –3. Then what is the value of $\sqrt{-9}$? Mathematicians could not answer this question for a long time. The square root of –1, or in general the square root of negative numbers were considered to be meaningless and the term **imaginary** was applied to them. Now they are called **complex numbers** richly meaningful within the frame work of operations newly defined.

Complex numbers were used by Euler (1707 – 1788) to whom the symbol i for $\sqrt{-1}$ is due. The name complex numbers is due to Gauss who introduced the letter z to represent a complex number. Argand (1806) used the complex plane for the representation of complex numbers and hence the graph representing the complex numbers is called the Argand Diagram. It is Cauchy, the 'Intellectual Giant' who developed the new branch of mathematics called Complex Analysis.

1.2 Objectives of the Lesson

By the end of this Lesson you will be able to:

i) define a complex number

ii) workout arithmetic operations of complex numbers.

iii) define a complex plane (Argand Diagram) and to represent a complex numbers on the complex plane.

iv) understand the polar form (r, θ form) of complex numbers.

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

1.3 Definition of a Complex Number

A complex number can be defined as a number of the form z = x + iy where x and y are real numbers and i is such that $i^2 = -1$ or $i = \sqrt{-1}$

1.4 Arithmetic operations of complex numbers.

In performing operations with complex numbers we can proceed as in the algebra of real numbers keeping in mind that $i^2 = -1$.

Examples

i Addition of complex numbers

We define $(3 + 4i) + (6 + 2i) = 3 + 6 + 4i + 2i = 9 + 6i$

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

ii) **Subtraction of complex numbers.**

We define $(8 + 5i) – (3 - 2i) = 8 + 5i – 3 + 2i = 5 + 7i$

$$(a + bi) – (c + di) = (a – c) + (b – d)i$$

iii) Multiplication of a complex number by a scalar

we define $5(4 + 3i) = 20 + 15i$

In general $p(a + ib) = pa + ipb$ when p,a,b are real numbers.

iv) Multiplication of two complex numbers.

$(5 + 3i)(4 + 6i) = 5(4) + 5(6i) + 4(3i) + (3i)(6i)$
$= 20 + 30i + 12i + 18i^2$
$= 20 + 42i + 18(-1)$
$= 20 + 42i - 18$
$= 2 + 42i$

or $(a + bi)(c + di) = ac + adi + bci + bdi^2$
$= (ac - bd) + i(ad + bc)$

1.5 Conjugate of a complex number

If z = x + iy we define that the conjugate of the complex number z is $\bar{z} = x - iy$.

The conjugate of z is written as $\bar{z}$.

To get the conjugate of any complex number we just change i into –i. The product of a complex number and its conjugate is a real number.

For example if $z = 4 + 3i$, $\bar{z} = 4 - 3i$

$z\bar{z} = (4 + 3i)(4 - 3i) = 16 - 12i + 12i + 9$
$= 16 + 9 = 25.$

In general, if z = a + ib and $\bar{z}$ = a – ib

$z\bar{z} = (a + ib)(a - ib) = a^2 + b^2$

Example 1

Factorise: $16x^2 + 25y^2$

$16x^2 + 25y^2 = (4x + 5iy)(4x - 5iy)$

since $a^2 + b^2 = (a + ib)(a - ib)$

Example 2

Write down the conjugate of the following complex numbers:

$2 + 3i$, $2 - 3i$, $4 + 3i$, i, $6 - 5i$.

Solution. The conjugates of $2 + 3i$, $2 - 3i$, $4 + 3i$, i, and $6 - 5i$ are $2 - 3i$, $2 + 3i$, $4 - 3i$, $-i$ and $6 + 5i$ respectively

1.6 Division of one complex number by another.

We cannot divide one complex number by another complex number, however we define the division of a complex number as the elimination of the imaginary number i from the denominator using the conjugate of the denominator.

Examples 3

Simplify: $\dfrac{5+3i}{-i}$

To eliminate i in the denominator we multiply both numerator and the denominator by *i* to get

$$\frac{5+3i}{-i} = \frac{(5+3i)^i}{-i(i)} = \frac{5i+3i^2}{-i^2}$$

$$= \frac{5i-3}{+1} = 5i-3$$

Example 4

Simplify: $\dfrac{3+4i}{5+2i}$

The conjugate of 5 + 2i is 5 – 2i, multiply the numerator and the denominator by the conjugate of the denominator to get

$$= \frac{3+4i}{5+2i} = \frac{(3+4i)(5-2i)}{(5+2i)(5-2i)}$$

$$= \frac{15-6i+20i+8}{25+4}$$

$$= \frac{23+14i}{29}$$

1.7 Geometric Representation of Complex Numbers

Since a complex number z = x +iy can be considered as an ordered pair z = (x,y) of real numbers. We can represent such numbers by points on a graph or on the x y plane. The graph on which complex numbers are plotted is called the **complex plane** or **z plane** or an **Argand Diagram**. The x axis is called the Real axis and the y axis is called the imaginary axis of the complex plane.

To each complex number there corresponds only one point and conversely to each point in the plane there corresponds one and only one complex number

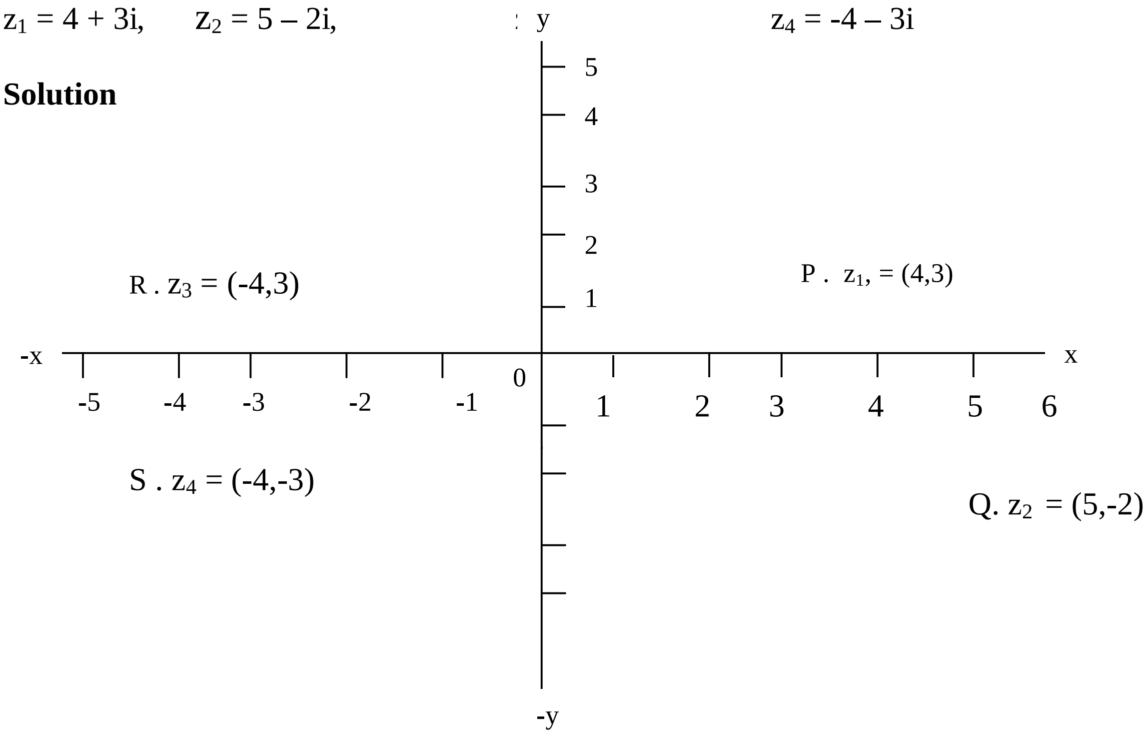
Representation of a complex number on the complex plane is illustrated in the following examples:

Example 5

Plot the complex numbers

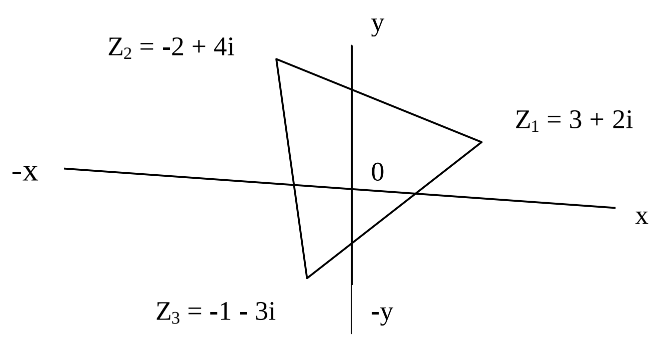
$z_1 = 4 + 3i$, $z_2 = 5 – 2i$, $z_4 = -4 – 3i$

Solution



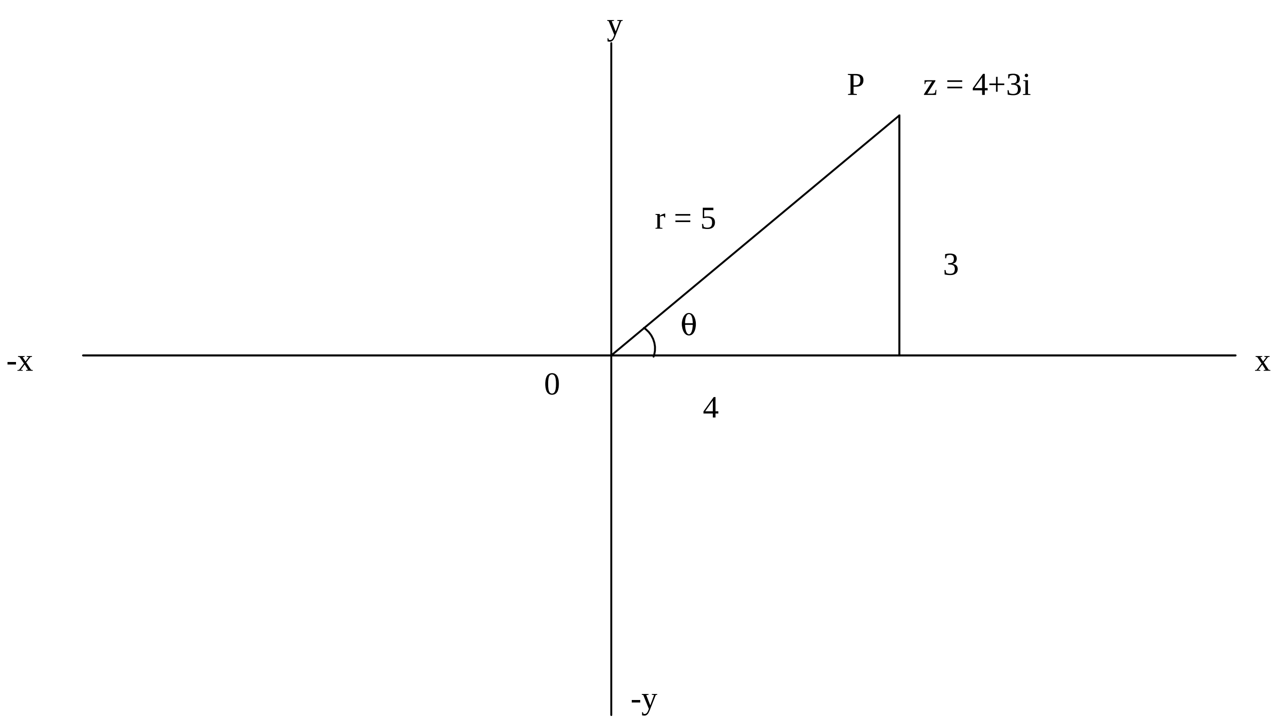
The point $z_1 = 4 + 3i = (4,3)$; P, represents z_1
The point $z_2 = 5-2i = (5,-2)$; Q, represents z_2
The point $z_3 = -4 + 3i = (-4,3)$; R represents z_3
The point $z_4 = -4 - 3i = (-4, -3)$; S represents z_4

Example 6
Plot the triangle on the complex plane given that the vertices of the triangle are
$z_1 = 3 + 2i$, $z_2 = -2 + 4i$ and $z_3 = -1 -3i$



The triangle is shown on the complex plane.

1.8. Polar form of complex numbers.



Consider the complex number $z = 4 + 3i$
Let r and θ be the polar coordinates of the Point p (4,3). Then

OP = r and angle Pox = θ

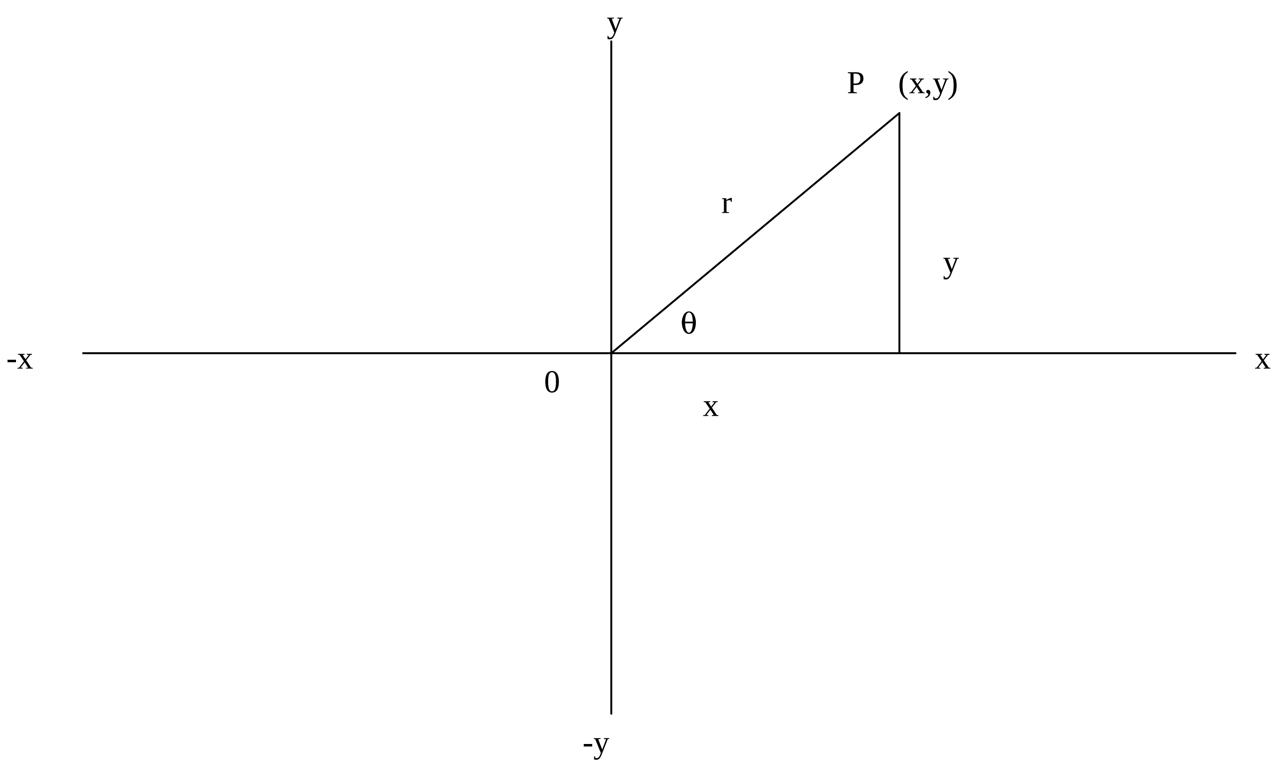
Now $r^2 = 4^2 + 3^2 = 25$ and r = 5

Now $\tan\theta = \dfrac{3}{4}$ or $\theta = \tan^{-1}\dfrac{3}{4} = 36.9^0$

$$\cos\theta = \frac{x}{r} \text{ and } \sin\theta = \frac{y}{r}$$

r is called the modulus of the complex number θ is called the argument of z

The formula connecting the polar form and Cartesian form of a complex number



In the figure $\cos\theta = \dfrac{x}{r}$ and $\sin\theta = \dfrac{y}{r}$ or $x = r\cos\theta$ and $y = r\sin\theta$

Hence the complex number Z is represented in polar form as,

$z = x + iy = r\cos\theta + i\,r\sin\theta$

or $z = r(\cos\theta + i\sin\theta)$

where $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$

$z = r(\cos\theta + i\sin\theta)$ is called the Polar Form of the complex number z.

It is represented by $z = (r, \theta)$

Example 7
Express z = 4 + 3i in polar form

In polar form $r = \sqrt{x^2 + y^2} = \sqrt{4^2 + 3^2} = 5$ and θ is given by $\tan\theta = \frac{y}{x} = \frac{3}{4}$ and the principal value of $\theta = 36.9^0$ approximately.
Hence Z = 4 + 3i
= r(cos θ + i sin θ)
= 5 (cos 36.9^0 + i sin 36.9^0)
Hence 4 + 3i = (5, 36.9^0)

1.9. Euler's representation of (cos θ + i sin θ)

Complex numbers were used by Euler to whom the symbol i for $\sqrt{-1}$ is due. He represented $\cos\theta + i\sin\theta = e^{i\theta}$ and $\cos\theta - i\sin\theta = e^{-i\theta}$
For example
Cos 10 θ + i sin 10 θ = $e^{i10\theta}$
Cos 10 θ - i sin 10 θ = $e^{-i10\theta}$

$$\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

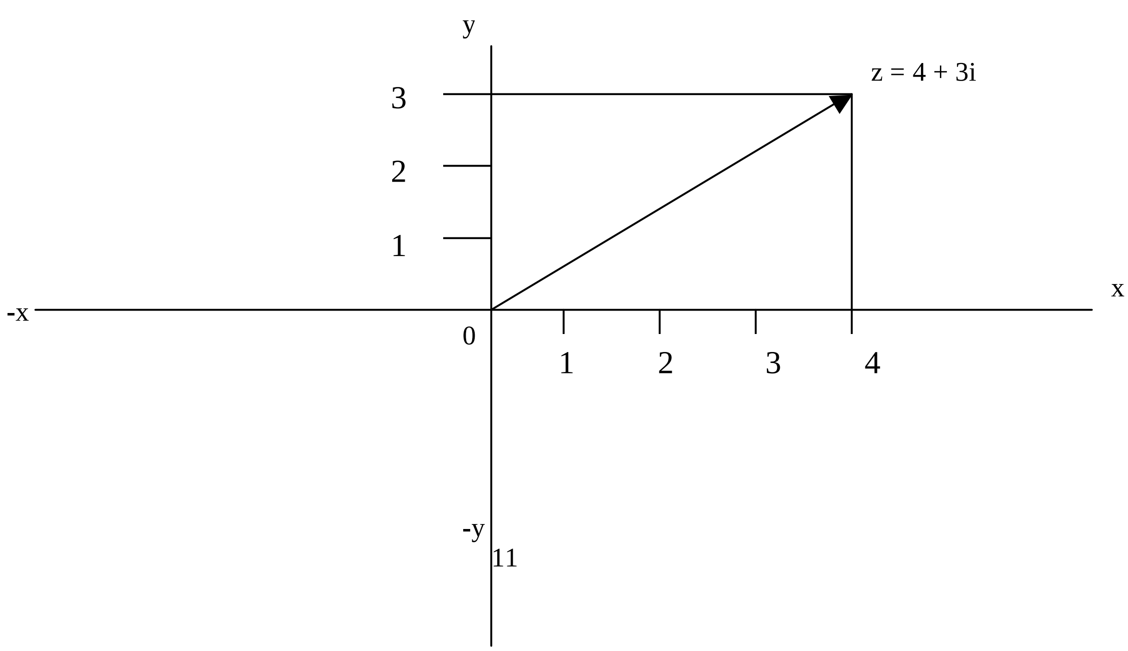
$$= e^{i\frac{\pi}{3}} . e^{i\frac{\pi}{2}}$$

$$= e^{i\left[\frac{\pi}{2} + \frac{\pi}{3}\right]}$$

$$= e^{i}\frac{5\pi}{6}$$

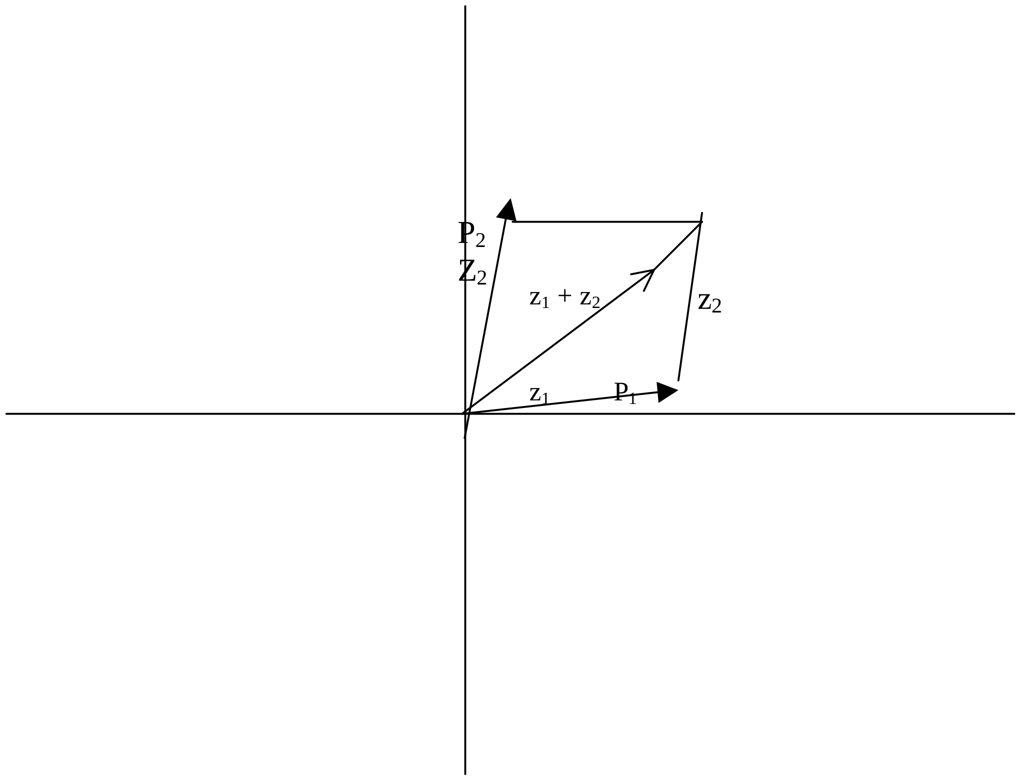
1.10. Vector representation of a complex number.

Consider a complex number 4 + 3i = (4, 3)
By taking 4 units on the x axis and 3 units on the y axis we can represent z = 4 + 3i as a vector shown in the diagram below.



In general z = a + ib can be considered as a vector whose x coordinate is a and the y coordinate is b.

1.11. The radius vector corresponding to the addition of two vectors Z_1 and Z_2



The radius vector corresponding to $z_1 + z_2$ is the diagonal of the parallelogram defined by the vectors OP_1, and OP_2.
In the same way we can represent $z_1 - z_2$ and z_1z_2.

Example 8
Simplify $(\cos 4\theta + i \sin 4\theta)(\cos 2\theta + i \sin 2\theta)$

Solution

$$(\cos 4\theta + i\sin 4\theta)(\cos 2\theta + i\sin 2\theta)$$

$$= e^{i4\theta}.e^{i2\theta}$$

$$= e^{i(6\theta)}$$

$$= (\cos 6\theta + i\sin 6\theta)$$

Example 9

Simplify: $$\frac{=(\cos 4\theta + i\sin 4\theta)(\cos 3\phi + i\sin 3\phi),}{(\cos\theta - i\sin\theta)(\cos 2\phi - i\sin 2\phi)}$$

Solution

$$\frac{=(\cos 4\theta + i\sin 4\theta)(\cos 3\phi + i\sin 3\phi)}{(\cos\theta - i\sin\theta)(\cos 2\phi - i\sin 2\phi)}$$

$$= \frac{e^{i4\theta} . e^{i3\phi}}{e^{-i\theta}\, e^{-i2\phi}}$$

$$= e^{i4\theta} . e^{i\theta} . e^{i3\phi} . e^{i2\phi}$$

$$= e^{i4\theta} . e^{i\theta} . e^{i3\phi} . e^{i2\phi}$$

$$= e^{i(5\theta + 5\phi)} = \cos(5\theta + 5\phi) + i\sin(5\theta + 5\phi)$$

Exercise I

Simplify (1 to 10)

1) i^3, i^4, i^5, i^6, i^{25}

2) $(3 + i) + (1 + 2i)$ 4) $(1 + i)(1 - i)$

3) $(5 - 3i) + (4 + 3i)$ 5) $(x + 2iy)(x - 2iy)$

6) $\frac{1-i}{1+i}$ 7) $\frac{1}{2-3i}$ 8) $\frac{3i-2}{1+2i}$

9) $(2+3i)^2$ 10) $\frac{(3-4i)}{5+12i}$

Write down the moduli of (11 and 12)

11) $3+4i$ 12) $\frac{1}{2} - \frac{1}{2}\sqrt{3i}$ i

Find the Principal Value of the arguments of

13) $1 - i$ 14) $\frac{1}{2} + \frac{1}{2}\sqrt{3i}$ 15 $\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$

16) Show that the equation $|z| = 2$ represents a circle with center the origin and radius 4 units.

If $z_1 = 3 + 2i$ and $z_2 = 5 + 8i$ determine the following

17) $z_1 + z_2$ 18) $2z_1 + 3z_2$ 19) $5z_1 - 3z_2$

20) $z_1 z_2$ 21) $3z_1 z_2$ 22) $\frac{z_1}{z_2}$

23) Show that $\left(\sqrt{2}-i\right)-i\left(1-\sqrt{2}i\right)=-2i$

24) Show that $(3+i)(3-i)\left(\frac{1}{5}+\frac{i}{10}\right)=2+i$

25) Solve the equation $z^2 + z + 1 = 0$ giving your answer in i.
26) Prove that $\overline{z_1 z_2} = \overline{z_1}\,\overline{z}$ (Hint: take $z_1 = x + iy$ and $z_2 = a + ib$).
27) Prove the associative law: $(z_1+z_2)+z_3 = z_1+(z_2+z_3)$ write down the following complex numbers in polar form
28) (i) -4 + 3i (ii) 12 – 5i (iii) -4 – 3i
(iv) 12 + 5i

29) Show that $\overline{z+3i} = \overline{z} - 3i$

30) Find one value of arg z when $z = \frac{-2}{1+i\sqrt{3}}$

31) Simplify: $\frac{(\cos 60 - i\sin 60)(\cos 120 + i\sin 120)}{(\cos 50 + i\sin 50)(\cos 40 + i\sin 40)}$

32) Simplify: $\frac{[\cos(-40)++i\sin(-40)][\cos 50 + i\sin 50]}{(\cos 60 - i\sin 60)}$

Summary of the Lesson
You have learnt the following from this Lesson:
i) Definition of a complex number: $z = x+iy$ where x and y are real numbers and $i=\sqrt{-1}$ or $i^2 = -1$

ii) To simplify $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$ and $\frac{z_1}{z_2}$

iii) To represent complex numbers x + iy or (x, y) on a complex plane
iv) Polar form of a complex number: x + iy = r(cos θ + i sin θ) where $r=\sqrt{x^2+y^2}$ and $\tan\theta = \frac{y}{x}$
v) Eulers representation: $\cos\theta + i\sin\theta = e^{i\theta}$

Further Reading.
1) Pure mathematics, A second course
By J.K Backhouse and others
Longman Group ltd
Longman's House, Harlow, Esssex. UK.

2) Theory and Problem of complex variables
By Murray R. Spiegel, Ph.D,

Sehaum' out line series
Mc Graw – Hill Book Company Singapore

3) Text Book of Complex Analysis
By Dr. D. Sengottaiyan, Ph.D.,
Oxford Publications
London Nairobi.

Lesson 2
Demoivre's Theorem and its Application to Roots of Real and Complex Numbers

2.1 Introduction

In the previous Lesson you have studied that a complex number z can be represented in polar form as $z = r\,(\cos\theta + i\sin\theta)$ where r is the **modulus** of the Complex number and θ is the **argument** of the number z. In this Lesson we shall see Euler's representation of $(\cos\theta + i\sin\theta)$ and Demoivre's theorem which is very useful to find roots of real and complex numbers.

2.2 Objectives of the Lesson

By the end of this Lesson you will be able to:

i). define Euler's representation of $(\cos\theta + i\sin\theta)$

ii). establish de Moivre's theorem

iii). and determine the roots of Real and Complex numbers using de Moivre's Theorem.

2.3 Euler's Representation of (cos θ + i sin θ)

In Lesson 1 you have seen that a complex number z is represented by

$z = x + iy = r\,(\cos\theta + i\sin\theta)$

where r is the modulus and θ is the argument of the complex number z. The famous Mathematician Euler represented,

$\cos\theta + i\sin\theta = e^{i\theta}$ and $\cos\theta - i\sin\theta = e^{-i\theta}$ and $e^{i(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2) + i\,\sin(\theta_1+\theta_2)$

Then the complex number

$z = x + iy = re^{i\theta}$

$(\cos\theta + i\sin\theta)$ is also represented as $cis\,\theta$, for convenience

Example 1

Let $z_1 = r_1 e^{i\theta_1}$

$z_2 = r_2 e^{i\theta_2}$

show that

i). $z_1 z_2 = r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

where $r_1 r_2$ is the modulus of $z_1 z_2$ and $(\theta_1+\theta_2)$ is the argument of $z_1 z_2$

ii). $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$

Solution

i). Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$

Then $z_1 z_2 = r_1 e^{i\theta_1} . r_2 e^{i\theta_2}$

$= r_1 r_2 e^{i(\theta_1+\theta_2)}$

$= R\ e^{i(\theta_1+\theta_2)}$

$$= R\ [\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)]$$

where R = r_1r_2 is the modules of z_1z_2 and $\theta_1+\theta_2$ is the argument of z_1z_2.

ii).

$$\frac{z_1}{z_2}=\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}=\frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

$$=\frac{r_1}{r_2}[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)]$$

2.4 De Moivre's Theorem

Let $z_1=r_1(\cos\theta_1+i\sin\theta_2)=r_1e^{i\theta_1}$

$z_2=r_2(\cos\theta_2+i\sin\theta_2)=r_2e^{i\theta_2}$

....

$z_n=r_n(\cos\theta_n+i\sin\theta_n)=r_ne^{i\theta_n}$

Then $z_1z_2...z_n=r_1r_2...r_ne^{i(\theta_1+\theta_2+...\theta_n)}$ (2)

If $z_1=z_2=...=z_n$ each equal to z = $re^{i\theta}$ in (2) we have $(z)^n=r^ne^{i(\theta+\theta+...)}=r^ne^{in\theta}$

or $r^n(\cos\theta+i\sin\theta)^n=r^n[\cos n\theta+i\sin n\theta]$

or $(\cos\theta+i\sin\theta)^n=(\cos n\theta+i\sin n\theta)$ (3)

Equation (3) is called **de Moivre's Theorem**. This theorem is true for all values of n. when n = 0, ±1, ±2... and also when n = positive or negative fractions.

de Moivre's theorem is very useful in computing roots of all the nonzero real and complex numbers.

Example 2

Simplify $\dfrac{(\cos3\theta+i\sin3\theta)(\cos5\theta-i\sin5\theta)}{(\cos2\theta-i\sin2\theta)(\cos7\theta+i\sin7\theta)}$

Solution

$$\frac{(\cos3\theta+i\sin3\theta)(\cos5\theta-i\sin5\theta)}{(\cos2\theta-i\sin2\theta)(\cos7\theta+i\sin7\theta)}=\frac{e^{i3\theta}.e^{-i5\theta}}{e^{-i2\theta}.e^{i7\theta}}$$

$$=e^{i3\theta}.e^{-i5\theta}.e^{i2\theta}.e^{-i7\theta}$$

$$=e^{i(3\theta-5\theta+2\theta-7\theta)}$$

$$=e^{-i7\theta}$$

$$=\cos7\theta-i\sin7\theta$$

2.5 The fundamental Theorem of Algebra

According to the fundamental Theorem of Algebra "an nth degree equation will have n roots and only n roots". This theorem is applicable for complex variable z. Thus an nth

degree equation in z will have n roots and only n roots. For example $z^3 + 1 = 0$, will have three roots one being real and two are complex roots.

In the following sections we shall find the roots of nonzero real and complex numbers using the fact

Cos 2 kπ + i sin 2 kπ = 1 **for k = 0, 1, 2, 3…**
Cos (2k + 1) π + i sin (2k + 1) π = -1 **for k = 0, 1, 2, 3…**

Example 3

Find the cube roots of –1 or Solve $z^{\frac{1}{3}} = -1$ or solve the equation $z^3 = -1^3 = -1$ giving three values of the roots.

Solution

Let $z^3 = -1$

or $z^3 = \cos(2k+1)\pi + i\sin(2k+1)\pi$

Taking cube roots on both sides

$$\left(z^3\right)^{\frac{1}{3}} = [\cos(2k+1)\pi + i\sin(2k+1)\pi]^{\frac{1}{3}}$$

or $z = \cos\dfrac{(2k+1)\pi}{3} + i\sin\dfrac{(2k+1)\pi}{3}$, k = 0, 1, 2…

using de Moivre's Theorem..

when k = 0, $z = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3} = (\cos 60 + i\sin 60) = \dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$

when k = 1, $z = \cos\dfrac{3\pi}{3} + i\sin\dfrac{2\pi}{3} = \cos 180 + i\sin 180 = -1$

when k = 2, $z = \cos\dfrac{5\pi}{3} + i\sin\dfrac{5\pi}{3} = \cos 300 + i\sin 300 = \dfrac{1}{2} - \dfrac{i\sqrt{3}}{2}$

When you put k = 3, 4, 5… we will get the same values repeated. Hence $z = \dfrac{1+i\sqrt{3}}{2}$, -1 and $\dfrac{1-i\sqrt{3}}{2}$ are the three values that satisfy the equation $z^3 = -1$. or the roots of – 1 .

You can verify that $\left(\dfrac{1+i\sqrt{3}}{2}\right)^3 = -1$

Example 4

Solve the equation $z^4 - 16 = 0$

(or find the fourth root of 16)

Solution

Let $16^{\frac{1}{4}} = z$ where the value of z is to be found.

or $z^4 = 16$ (1)

$z^4 = 16[\cos 2k\pi + i\sin(2k\pi)\pi]$

using de Moivere's Theoem we have

$z = 16^{\frac{1}{4}}[\cos 2k\pi + i\sin 2k\pi]^{\frac{1}{4}}$

$z = 2\left[\cos\dfrac{2k\pi}{4} + i\sin\dfrac{2k\pi}{4}\right]$ (1)

From (1),

If k = 0, $z = 2(\cos\theta + i\sin\theta) = 2$

If k = 1, $z = 2(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}) = 2(0+1) = 2i$

If k = 2, $z = 2(\cos\pi + i\sin\pi) = 2(-1+0) = -2$

If k = 3, $z = 2(\cos 270 + i\sin 270) = 0 - 2i = -2i$

If you put k = 4, 5, 6… we will get the same values repeated.

Hence $16^{\frac{1}{4}} = 2$, *2i, -2 and –2i*

Example 5

Find the square roots of $-15 - 8i$

Solution

Writing –15 – 8i in polar form, we have $r = \sqrt{(-15)^2 + (-8)^2} = \sqrt{225 + 64} = 17$

$r\cos\theta = -15$ and $r\sin\theta = -8$

$\cos\theta = \dfrac{-15}{17}$, $\sin\theta = \dfrac{-8}{17}$ or $\tan\theta = \dfrac{8}{15}$

Since cos θ and sin θ both negative θ is in the third quadrant from 180^0 to 270^0

Hence θ = 208.072^0

Let $-15 - 8i = 17[\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)]$ where k = 0, 1 and θ = 205.2^0

Then $(-15 - 8i)^{\frac{1}{2}} = 17^{\frac{1}{2}}[\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)]^{\frac{1}{2}}$

$= 17^{\frac{1}{2}}\left[\cos\dfrac{2k\pi + \theta}{2} + i\sin\dfrac{2k\pi + \theta}{2}\right]$

when k = 0, $\sqrt{-15 - 8i} = \sqrt{17}(\cos 104.036 + i\sin 104.036)$

when k = 1, $\sqrt{-15 - 8i} = \sqrt{17}(\cos 284.036 + i\sin 284.036)$

Example 6

Find all the 6^{th} roots of unity (or solve the equation $z^6 - 1 = 0$)

Solution

Let the 6th root of 1 be z

Then $1^{\frac{1}{6}} = z$ or $1 = z^6$

Hence $z^6 = 1 = (\cos 2k\pi + i\sin 2k\pi)$ k = 0, 1, 2…

or $z^6 = e^{2k\pi i}$

or $z = \left(e^{2k\pi i}\right)^{\frac{1}{6}}$

or $z = e^{\frac{k\pi i}{3}}$ (2)

From (2)

If k = 0, $z = e^0 = 1$

If k = 1, $z = e^{\frac{\pi i}{3}} = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

If k = 2, $z = e^{\frac{2\pi i}{3}} = (\cos\ 120 + i\sin\ 120) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

If k = 3, $z = e^{\pi i} = \cos\ 180 + i\sin\ 180 = -1$

If k = 4, $z = e^{\frac{4\pi i}{3}} = \cos\ 240 + i\sin\ 240 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

If k = 5, $z = e^{\frac{5\pi i}{3}} = \cos\ 300 + i\sin\ 300 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

[If k = 6, $z = e^{2\pi i} = \cos\ 360 + i\sin\ 360 = 1$ It is the same as in the case of k = 0]

2.6 The zeros of a polynomial with real coefficients always occur in conjugate pairs.

If a + ib is a root of a polynomial,

$a_0 z^n + a_1 z^{n-1} + \ldots + a_n = 0$ $a_0, a_1 \ldots a_n$ are real number.

show that its conjugate a – ib is also a root of the same equation or the zeros of a polynomial with real coefficients always occur in conjugate pairs.

Solution

Let $f(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n = 0$ (1)

Let $a + ib = re^{i\theta}$. Then

$f\left(re^{i\theta}\right) = a_0 r^n e^{in\theta} + a_1 r^{n-1} e^{i(n-1)\theta} + \ldots a_n = 0$ (2)

or $a_0 r^n(\cos n\theta + i\sin n\theta) + a_1 r^{n-1}[\cos(n-1)\theta + i\sin(n-1)\theta \ldots + a_n] = 0$

or $a_0 r^n(\cos n\theta) + a_1 r^{n-1}\cos(n-1)\theta + \ldots a_n = 0 + i\left[\left(a_0 r^n \sin n\theta\right) + a_1 \sin(n-1)\theta + \ldots\right] = 0$

Hence $a_0 r^n \cos n\theta + a_1 r^{n-1} \cos(n-1)\theta + \ldots a_n = 0$ (3)

and $a_0 r^n \sin n\theta + a_1 \sin(n-1)\theta + \ldots = 0$ (4)

(The real and imaginary part each must be zero)

or $a_0 r^n (\cos n\theta - \sin n\theta) + a_1 r^{n-1}[\cos(n-1)\theta - \sin(n-1)\theta] + \ldots a_n = 0$ from (3) and (4)

From (4) $a_0 \left(re^{-i\theta}\right)^n + a_1 \left(re^{-i\theta}\right)^{n-1} + \ldots a_n = 0$

Thus $re^{-i\theta}$ satisfies the polynomial

$a_0 z^n + a_1 z^{n-1} + \ldots a_n = 0$

where $z = a - ib = re^{-i\theta}$

Thus $a - ib$ is also a root of the same polynomial (1)

Exercise 2

1. Evaluate the following.
   a) $5(\cos 20 + i \sin 20)(\cos 40 + i \sin 40)$
   b) $\dfrac{(8cis40)^3}{(2cis60)^4}$ note that $cis\theta = \cos\theta + i\sin\theta$
   c) $\left(\dfrac{\sqrt{3}-1}{\sqrt{3}+1}\right)^4 \left(\dfrac{1+i}{1-i}\right)^5$
   d) $(2 \text{ cis } 50)^6$

2. Prove that the solution of $z^4 - 3z^2 + 1 = 0$ are given by z = 2 cos 36^0, 2 cos 72^0, 2 cos 216^0 and 2 cos 252^0

3. Prove de Moivre's Theorem for (a) negative integers and (b) rational numbers.
4. Find the roots of the following and locate them on a graph (Argand diagrams)

   a). $(-4+i)^{\frac{1}{5}}$ c). $64^{\frac{1}{6}}$

   b). $\sqrt[3]{4cis260^0}$

5. Solve the equation $z^4 + 81 = 0$
6. Solve $z^6 + 1 = \sqrt{3}i$
7. Find the square root 5 – 12i
8. Find the square root of 5 – 12i
9. Find the cube roots of –11 – 2i
10. Solve the following equations:

    a). $5z^2 + 2z + 10 = 0$

b). $z^5 + 2z^4 + z^3 + 6z - 4 = 0$

11. Find all the roots of $z^4 + z^2 + 1 = 0$

12. Find the two complex numbers whose sum is 4 and whose product is 8.

Summary of the Lesson

You have learnt the following from this Lesson

1. $(\cos\theta + i\sin\theta)$ is represented as $e^{i\theta}$ and
   $(\cos\theta - i\sin\theta)$ is represented as $e^{-i\theta}$

2. de Moivre's Theorem
   $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

3. We use $\cos 2k\pi + i\sin 2k\pi = 1$ and $\cos(2k+1)\pi + i\sin(2k+1)\pi = -1$ to determine the roots of functions.

Further Reading

1) Complex Variables and Applications
By R.V Churchill and others
Mc Graw-Hill, KOGAKUSHA Ltd
Tokyo Singapore

2) Theory and Problem of complex variables
By Murray R. Spiegel, Ph.D,
Sehaum' out line series
Mc Graw – Hill Book Company Singapore

3) Text Book of Complex Analysis
By Dr. D. Sengottaiyan, Ph.D.,
Oxford Publications
London Nairobi.

Lesson 3
Limits and Continuity of Functions of Complex variables.

3.1 Introduction

You have learnt the limits and continuity of functions of real variables. The definitions of limits and continuity of a function of complex variables is quite similar to those of functions of real variables.

3.2 Objectives of the Lesson

By the end of this Lesson you will be able to define

- i). single valued and many valued functions *f(z)*.
- ii). the limit of single valued functions *f(z)*
- iii). the continuity of a function *f(z)* at a point and the three conditions necessary for *f(z)* to be continuous.
- iv). The continuity of a function in a region.
- v). The uniform continuity of a function *f(z)*

3.3 Single valued and many valued functions of z

w = *f(z)* is said to be a single valued or many valued function of z according as for a given value of z in a domain D there corresponds only one, or more than one values of w.

For example ***w* = *f(z)* = *z*** is a single valued function but $w = f(z) = z^{\frac{1}{2}}$ is a many valued function.

3.4 Limit of a function *f(z)*

Let *f(z)* be any single valued function defined in the deleted neighbourhood of z = a.

We say that *f(z)* tends to the limit *l* as z tends to *a* **along any path** in a defined region, if to each positive number $\in$, however small there corresponds a positive number δ depending on $\in$ such that $|f(z) - l| < \in$ for all points of the region for which $0 < |z - a| < \delta$

In other words it means that there exists a deleted neighbourhood of the point z = a in which $|f(z) - l|$ can be made as small as we please

Symbolically we write $\lim\limits_{z \to a} f(z) = l$

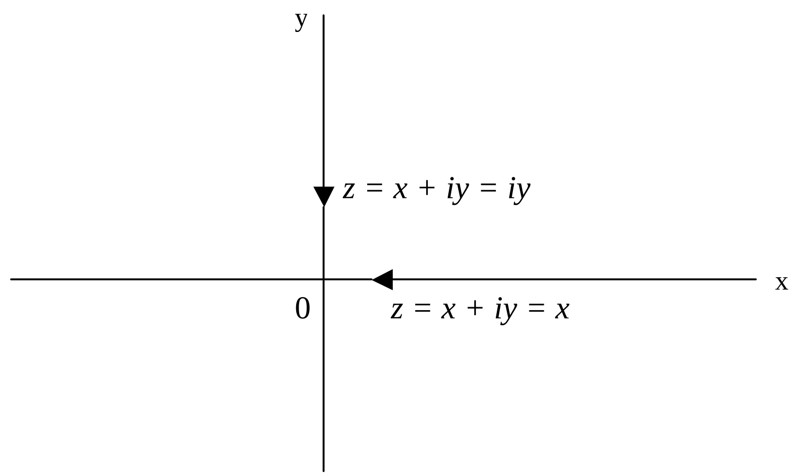
Example 1

Prove that $\lim\limits_{z \to 0} \dfrac{\bar{z}}{z}$ does not exist

Solution

Suppose the $\lim\limits_{z \to 0} \dfrac{\bar{z}}{z}$ exists

The limit must be independent of the manner in which z approaches the point 0



Let $z \to 0$ along the x axis. Then $y = 0$ and hence $z = x + iy$ becomes z = x and $\bar{z} = x - iy = x$.

Then the required limit is $\lim\limits_{z \to 0} \dfrac{\bar{z}}{z} = \lim\limits_{z \to 0} \dfrac{x}{x} = 1$.

Let $z \to 0$ along the y axis as shown in the figure. Then $x = 0$, $z = x + iy$ = iy and $\bar{z} = x - iy = -iy$.

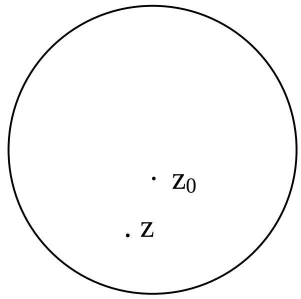
Then the required limit is $\lim\limits_{z \to 0} \dfrac{\bar{z}}{z} = \lim\limits_{y \to 0} \dfrac{-iy}{iy} = -1$

The limit of a function should be unique irrespective of the path. Here the limit shows +1 and –1 in two different paths.

Hence $\lim\limits_{z \to 0} \dfrac{\bar{z}}{z}$ does not exist.

3.5 Continuity of a function f(z).

Let $f(z)$ be defined and single valued in a neighbourhood of z_0 and also at z_0



Then *f(z)* is said to be continuous at $z = z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$

3.6 Three conditions for continuity of a function f(z) at z_0

The above definition implies three conditions which must be satisfied in order that *f(z)* be continuous at $z = z_0$.

1. $\lim_{z \to z_0} f(z) = l$
2. $f(z_0)$ must exist and finite
3. $f(z_0) = l$

3.7 Second definition of continuity of a function using $\in$ and δ

The function *f(z)* is said to be continuous at the point $z = z_0$ if for any given $\in > 0$ we can find $\delta > 0$ such that $|f(z) - f(z_0)| < \in$ whenever $|z - z_0| < \delta$

3.8 Continuity of *f(z)* in a region or domain

A function *f(z)* is said to be continuous in a region R if it is continuous at all points of the region R.

Example: All polynomials, e^z, sin z and cos z are continuous on the whole of the complex plane.

3.9 Properties of continuous functions

1. If *f(z)* and *g(z)* are continuous in a region then
   a). *f(z) + g(z)* b). *f(z) - g(z)*
   c). *f(z) . g(z)* are all continuous
   d). $\dfrac{f(z)}{g(z)}$ is also continuous if $g(z) \neq 0$

2. If *f(z)* is continuous in a closed region it is bounded in the region; i.e there exists a constant M such that $|f(z)| < M$ for all points z of the region.
3. If *f(z) = u(x, y) + iv(x, y)*, and if *f(z)* is continuous in a region then the functions *u(x, y)* and *v(x, y)* are also continuous in the region

4. A continuous function of a continuous function is continuous. For example e^z is a continuous function. Hence $\phi(e^z)$ is also continuous.

Example 2
Prove that $f(z) = 2z^2$ is continuous at $z = z_0$

Solution
If $f(x) = 2z^2$ is continuous it must satisfy three conditions namely

i). $\lim\limits_{z \to z_0} f(z) = l$

ii). $f(z_0)$ must exist

iii). $f(z_0) = l$

First we must show that $\lim\limits_{z \to z_0} f(z) = l$ i.e. we must show that given any $\in > 0$ we can find $\delta > 0$ (depending on $\in$) such that $|f(z) - f(z_0)| = |z^2 - z_0^2| < \in$ when $|z - z_0| < \delta$

If $\delta \le 1$ then $0 < |z - z_0| < \delta$ implies that

$$|z^2 - z_0^2| = |z - z_0|\ |z - z_0| < \delta|z - z_0 + 2z_0| < \delta[|z - z_0| + |2z_0|] < \delta[1 + 2|z_0|]$$

Taking δ as 1 or $\dfrac{\in}{1 + 2|z_0|}$

whichever is smaller we have $|z^2 - z_0^2| < \in$ where $|z - z_0| < \delta$.

Example 3

Show that $f(z) = \dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$ is not continuous at $z = i$

If $f(z)$ is continuous then the three fundamental conditions must be satisfied. They are:

i). $\lim\limits_{z \to i} f(z) = l$ must exist

ii). $f(z_0)$ or $f(i)$ must exist

iii). $f(z_0)$ or $f(i) = l$

Here $f(i)=\dfrac{3i^4-2i^3+8i^2-2i+5}{i-i}$

$=\dfrac{3-2i-8-2i+5}{0}$

$=\dfrac{0}{0}$

Thus *f(i)* does not exists even though the limit exists as shown below: -

$$\lim_{z\to i}=\frac{3z^4-2z^3+8z^2-2z+5}{z-i}=\lim_{z\to i}\frac{12z^3-6z^2+16z-2}{1}$$

$= 12i^3 – 6i^2 + 16i – 2$

$= -12i + 6 + 16i – 2$

$= 4i + 4$

Hence *f(z)* is not continuous

Example 4

For what values of z

$f(z)=\dfrac{3z^2+1}{z^2+1}$ is continuous.

Solution

$$f(z)=\frac{3z^2+1}{z^2+1}=\frac{3z^2}{(z+i)(z-i)}$$

When z = i and z = -i *f(z)* does not exist and it is not continuous. Thus *f(z)* is continuous at all points on the z plane except z = $\pm i$.

3.10 Uniform continuity of a function f(z)

Let *f(z)* be **continuous** in a region R. Then at each point z_0 of the region R and for any $\in > 0$ we can find $\delta > 0$ where δ depend on both $\in$ and the particular point z_0 such that $|f(x)-f(z_0)|<\in$ whenever $|z\text{-}z|<\delta$ suppose we can find δ depending on $\in$ **only but not on the particular point z_0** we say that *f(z)* is uniformly continuous in the region.

Note that of *f(z)* is continuous in a closed region R, it is uniformly continuous in R

Exercise

1. If $f(z) = z^2 + 2z$ prove that $\lim_{z\to i} f(x)=2i-1$
2. Prove that $\lim_{z\to 1+i}\dfrac{z^2-z+1+i}{z^2-2z+2}=1+\dfrac{1}{2}i$
3. Evaluate $\lim_{z\to 2i}(iz^4+3z^2-10i)$ using the theorem on limits.

4. Evaluate $\lim_{z \to e^{\frac{\pi i}{4}}} \frac{z^2}{z^4 + z + 1}$

5. Prove that if $f(z) = 3z^2 + 2z$ then $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = 6z_0^2 + 2$

6. Explain exactly what is meant by the statement $\lim_{z \to \infty} \frac{2z+1}{z^4+1} = 2$

7. Find all points of discontinuity for the functions

   i) $f(z)\frac{12z+1}{z^2+2z+2}$ ii) $f(z)$= cot z

   iii) $\frac{z^2+1}{z^4-16}$

8. Prove that $f(z) = z^2 - 2z + 3$ is continuous every where in the finite plane.

9. Show that $f(z) = \frac{z^2+1}{z^2-3z+2}$ is continuous for all z outside |z| = 2.

10. Prove that if $f(z)$ is continuous in a closed region, it is bounded in the region (hint: if $f(z)$ is continuous then it tends to a limit l which is finite)

11. Prove that $f(z) = 3z - 2$ is uniformly continuous in the region |z| ≤ 10. Prove that $f(z) = \frac{1}{z^2}$ is not uniformly continuous in the region |z| ≤ 1 0

    (Hint: since there is one point z = 0 where f(z) does not exist)

12. Prove that $f(z) = \frac{1}{z^2}$ is uniformly continuous in the region $\frac{1}{2} \le |z| \le 1$

13. Prove that f(z) = 3z-2 is uniformly continuous in the region |z| ≤ 10.

Summary of the Lesson

You have learnt the following from this Lesson:

i). Definition of single valued and many valued function *f(z)*

ii). Definition of limits of a function *f(z)*.

iii). Definition of continuity of a function *f(z)*

iv). Three conditions necessary and sufficient for a function to be continuous.

v). Definition of continuity of a function in a region.

vi). Four important properties of a continuous function *f(z)*

vii). Definition of uniform continuity of a function *f(z)*

Further Reading

1) Complex Variables and Applications
By R.V Churchill and others
Mc Graw-Hill, KOGAKUSHA Ltd
Tokyo Singapore

2) Theory and Problem of complex variables
By Murray R. Spiegel, Ph.D,
Sehaum' out line series
Mc Graw – Hill Book Company Singapore

3) Text Book of Complex Analysis
By Dr. D. Sengottaiyan, Ph.D.,
Oxford Publications
London Nairobi.

Lesson 4
Differentiability of Complex Functions, Analytic Functions and their Properties.

4.1 Introduction

You have studied the differentiability of real functions in Elementary Calculus. Differentiability of complex functions is nearly the same as that of real functions except **that Δz may tend to zero in any manner**. In this Lesson we shall study the definition of derivative of complex functions and that of analytic functions and their properties.

4.2 Objectives of the Lesson

By the end of this Lesson you will be able to

i) define the derivative $f'(z)$ of a function of complex variables and the differentiability of functions.
ii) define analytic function or holomorphic function at a point and in a region.
iii) define an entire function.
iv) state Cauchy –Reimann Equations for an analytic function.
v) define harmonic functions
vi) determine the analytic function f(z) = u + iv when u or v is given.

4.3 Derivative of a function of complex variables

If f(z) is a single valued function in some region R of the z plane, **the derivative** of f(z) is defined as

$$f'(z) \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

provided the limit exists **independent of the manner** in which $\Delta z \to 0$

If the derivative of f(z) exists at z_0 we say that f(z) is **differentiable at** z_0 and $f'(z)$ is called the **differential coefficient** of f(z) at z_0

4.4 Analytic function

A function f(z) is said to be analytic at a point z_0 if its derivative exists **not only at z_0** but also **at each point z in some neighborhood** of z_0.

4.5 Analytic function in a Region R

If(z) is said to be **analytic in a region R** if f(z)is analytic at each and every point of R. Analytic functions are usually defined on a certain domain.

Analytic functions at a point are also called **holomorphic functions** or **regular functions** at the point.

4.6 Entire function

An entire function is a function that is analytic at each point in the **entire complex plane.**

For example e^z, sin z, cos z are analytic at all points of the complex plane. Hence they are **entire functions.**

4.7 Differentiation of functions of complex variables

The formulae for differentiation of functions of complex variables are identical with those of functions of real variables in elementary calculus. Thus

1. $\frac{d}{dz}z^n = nz^{n-1}$ If $n \neq -1$

2. $\frac{d}{dz}\ln z = \frac{1}{z}$

3. $\frac{d}{dz}e^z = e^z$

4. $\frac{d}{dz}a^z = a^z \ln a$

5. $\frac{d}{dz}\sin z = \cos z$

6. $\frac{d}{dz}\cos z = -\sin z$

7. $\frac{d}{dz}\tan z = \sec^2 z$

8. $\frac{d}{dz}\cot z = -\cos ec^2 z$

9. $\frac{d}{dz}\sec z = \sec z \tan z$

10. $\frac{d}{dz}\cos ecz = -\cos ecz \cot z$

11. $\frac{d}{dz}\sinh z = \cosh z$

12. $\frac{d}{dz}\cosh z = \sinh z$

13. $\frac{d}{dz}\tanh z = \sec h^2 z$

14. $\frac{d}{dz}\coth z = -\cos ech^2 z$

15. $\frac{d}{dz}\sec hz = -\sec hz \tanh z$

16. $\frac{d}{dz}\cos echz = -\cos echz \coth z$

17. $\frac{d}{dz}\sinh^{-1} z = \frac{1}{\sqrt{1+z^2}}$

18. $\frac{d}{dz}\cosh^{-1} z = \frac{1}{\sqrt{z^2-1}}$

19. $\frac{d}{dz}\tanh^{-1} z = \frac{1}{1-z^2}$

20. $\frac{d}{dz}\coth^{-1} z = \frac{1}{1-z^2}$

21. $\frac{d}{dz}\sec h^{-1} z = \frac{-1}{z\sqrt{1-z^2}}$

22. $\frac{d}{dz}\cos ech^{-1} z = \frac{-1}{z\sqrt{z^2+1}}$

If f(z) and g(z) are analytic then

23. $\frac{d}{dz}[f(z) \pm g(z)] = \frac{d}{dz}f(z) \pm \frac{d}{dz}g(z)$

24. $\frac{d}{dz}(fg) = fg' + gf'$

25. $\frac{d}{dz}\left(\frac{f}{g}\right) = \frac{gf' - fg'}{g^2}$

$g \neq 0$

4.8 Cauchy – Reimann Equations of an Analytic Function

If a function f(z) = u(x,y)+ iv(x,y) is differentiable at a point z = x + iy then the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ should exist and satisfy the equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (or $u_x = v_y$ and $u_y = -v_x$)

Before we prove the above results let us define the meaning of the partial derivative $\frac{\partial u}{\partial x}$,

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} \text{ and } \frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

Proof of Cauchy – Reimann Equations

Since f(z) is analytic at z it is differentiable at z and $f^1(z)$ exists.

Now, $f^{'}(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ (1)

where $\Delta z \to 0$ in any manner near z.

since $\Delta z = \Delta x + i\Delta y$ (1) becomes

$$f^{'}(z) = \lim_{\substack{\nabla x \to 0 \\ \nabla y \to 0}} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)}{\Delta x + i\Delta y} - \frac{[u(x,y) + iv(x,y)]}{\Delta x + i\Delta y} \quad (2)$$

must exist independent of the manner in which Δz (or Δx and Δy) approaches zero.

Let us consider Δz = Δx + iΔy along real axis so that Δy = 0.

Grouping the result (2) in a different way we have (using Δy = 0) and Δz = Δx

$$f^{'}(z) = \lim_{< x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \frac{i[v(x+\Delta x, y)] - v(x,y)}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad (3)$$

Let Δz = Δx + iΔy tends to zero along the imaginary axis so that Δx = 0. Then

$$f^{'}(z) = \lim_{< y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \frac{i[v(x, y+\Delta y) - v(x,y)]}{i\Delta y} = \frac{\partial u}{i\Delta y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad (4)$$

But $f^{'}(z)$ should be identical irrespective of the manner in which $\Delta z \to 0$. Hence equating (3) and (4) we have

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

4.9 Finding the analytic function f(z) = u + iv when u is given

Theorem

If the real part of an analytic function f(z) = u + iv is given, then

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0)$$

Proof

Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ (1)

Then $\overline{f(z)} = \overline{f(x+iy)} = u(x,y) - iv(x,y)$ (2)

Adding $f(x+iy) + \overline{f(x+iy)} = 2u(x,y)$ using $\overline{f(x+iy)} = \overline{f}(x-iy)$ we have

$2u(x,y) = f(x+iy) + \overline{f}(x-iy)$ (3)

(3) is an identity for all x and y and we can substitute any value (including complex) for x and y.

substituting $x=\dfrac{z}{2}$ and $y=\dfrac{z}{2i}$ in (3) we have $2u\left(\dfrac{z}{2},\dfrac{z}{2i}\right)=f\left(\dfrac{z}{2}+i\dfrac{z}{2i}\right)+\overline{f}\left(\dfrac{z}{2}-i\dfrac{z}{2i}\right)$

$$=f(z)+\overline{f}(0)$$

Hence $f(z)=2u\left(\dfrac{z}{2},\dfrac{z}{2i}\right)-\overline{f}(0)$

Example 1
Let f(z) = u +iv be an analytic function.
If u = $u^3 - 3x^2y$ find f(z).

Solution
If u = u(x,y) then

$f(z)=2u\left(\dfrac{z}{2},\dfrac{z}{2i}\right)-\overline{f}(0)$ (by theorem)

since u = $x^3 - 3x^2y$ we have

$$f(z)=2u\left(\frac{z}{2},\frac{z}{2i}\right)-\overline{f}(0)$$

$$=2\left[\left(\frac{z}{2}\right)^3-3\left(\frac{z}{2}\right)^2\left(\frac{z}{2i}\right)\right]-\overline{f}(0)$$

$$=2\left[\frac{z^3}{8}-\frac{3z^2}{4}\left(\frac{z}{2i}\right)\right]-u(0,0)$$

$$=2\left(\frac{z^3}{8}-\frac{3z^3}{8i}\right)-0$$

$$f(z)=2\left(\frac{z^3}{8}+\frac{3iz^3}{8}\right)+ci$$

(by adding any imaginary constant at the end.)

Real and Imaginary parts of an analytic function satisfy Laplace Equation.

Let $f(z)=u(x,y)+iv(x,y)$. We must prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \text{ and } \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

Let $f(z)=u(x,y)+iv(x,y)$ be analytic

By Cauchy Reimann equations we have $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$.

Then $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 v}{\partial x\partial y}$ and $\dfrac{\partial^2 u}{\partial y^2}=-\dfrac{\partial^2 v}{\partial x\partial y}$

Adding $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\partial^2 v}{\partial x\partial y}-\dfrac{\partial^2 v}{\partial x\partial y}=0$

Similarly $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Thus Laplace differential equation is satisfied

Harmonic function

A function ϕ (x,y)which has continuous partial derivatives of the first and second orders and satisfies Laplace equation is called a **harmonic function.**

Hence u(x,y) and v(x,y) are harmonic functions.

Example 2

Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin, although the Cauchy – Reimann equations are satisfied at the origin.

Solution

Let $f(z) = u(x, y) + iv(x, y) = \sqrt{|xy|}$ so that $u = \sqrt{|xy|}$ and v = 0.

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0, y) - u(0,0)}{y} = \frac{0 - 0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(xo) - v(0,0)}{x} = \frac{0 - 0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0y) - v(0,0)}{y} = \frac{0 - 0}{y} = 0$$

$$\frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}$$

Hence Cauchy – Reimann equations are satisfied at the origin.

Now $f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{\sqrt{xy} - 0}{x + iy}$

If $f(z) = u + iv$ is analytic then u and v are harmonic functions.

Now suppose $z \to 0$ along the line y = mx

$f'(0) = \lim_{z \to 0} \frac{\sqrt{|mx^2|}}{x(1 + im)} = \frac{\sqrt{|m|}}{(1 + im)}$ since $\frac{\sqrt{|m|}}{(1 + im)}$ will have different values for different m, f'(0) is not unique at z =0 and f(z) is not analytic at the origin although Cauchy – Reimann equations are satisfied.

Exercise 4

1) Distinguish between an analytic function and an entire function. Give an example for an entire function .
2) Define a singular point using differentiability of a function.
3) Define f'(z) using limit.

4) Prove that f(z) = 3x + y + i(3y-x) is an entire function.

5) Prove that $f(z) = e^{-y}.e^{ix}$ is an entire function.

6) If u(x,y) = 2x(1-y) is harmonic find a harmonic conjugate v.

7) Let u(x,y) = sin x sin y find the function v(x,y) so that f(z) = u+iv is an analytic function.

8 a) State the condition for the two curves u(x,y) and v(x,y) cut orthogonally.

b) If f(z)=u(x,y) + iv(x,y) is an analytic function, show that the curves u(x,y) and v(x,y) cut orthogonally.

9 a) Prove that the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ satisfies Laplace equation.

b) Determine the analytic function $f(z) = u + iv$ where u is given above.

10) If $u = x^2 - y^2$ and $v = \dfrac{-y}{x^2 + y^2}$ prove both u and v satisfy Laplace equation but u + iv is not an analytic function [Hint: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ and $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$ but $\dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$].

11 a) If $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$ show that u satisfies Laplace equation.

b) Determine the function f(z) whose real part is u.

Summary of the Lesson

You have learnt the following from this Lesson.

1) Derivative of a function f(z) is defined as $f^{'}(z) = \lim\limits_{\nabla z \to 0} \dfrac{f(z + \Delta z) - f(z)}{\Delta z}$ when the limit exists and $\Delta z \to 0$ in any manner.

2) Definition of Analytic functions at a point. A function f(z) is said to be analytic at z_0 if its derivative exists not only at z_0 but also at each point in some neighborhood of z_0.

3) Definition of entire function.

4) Cauchy – Reimann equations for an analytic function.

$u_x = v_y$ and $u_y = -v_x$

or $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

5) Definition of harmonic functions: if $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ then u is called a harmonic function.

6) Determination of the Analytic function $f(z) = u + iv$ when u or v is given.

Further Reading

1) Complex Variables and Applications
By R.V Churchill and others
Mc Graw-Hill, KOGAKUSHA Ltd
Tokyo Singapore

2) Theory and Problem of complex variables
By Murray R. Spiegel, Ph.D,
Sehaum' out line series
Mc Graw – Hill Book Company Singapore

3) Text Book of Complex Analysis
By Dr. D. Sengottaiyan, Ph.D.,
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Lesson 5
Elementary Functions of Complex Variables

5.1 Introduction

You are already familiar with various elementary functions in the Calculus of Real Variables. In this Lesson we consider the elementary functions of a complex variable. We shall consider mainly the exponential, trigonometric and logarithmic functions of complex variables in this Lesson.

5.2 Objectives of the Lesson

By the end of this Lesson you will be able to

i) define e^z. sin z.cos z, other trigonometric functions of z and ln z.

ii) state the properties of exponential trigonometric and logarithmic functions.

iii) define hyperbolic functions in real and complex variables.

5.3 Definition of the Exponential Function

i) Exponential Functions are defined by

$$f(z) = e^z = e^{x+iy} = e^x.e^{iy}$$

$$e^z = e^x(\cos y + i\sin y) \qquad \textbf{(1) and}$$

$$e^{-z} = e^{-x-iy} = e^{-x}.e^{-iy} = e^{-x}(\cos y - i\sin y) \qquad \textbf{(2)}$$

Here e = 2.71825 … is an irrational number.

ii) **If a is real and positive we define** $a^z = e^{z\ln a} = e^x(\cos y + i\sin y)\ln a$ **(3)**

Hence ln a = $\ln_e$ a

If a =e then ln e =1 and (1) is a particular case of (3).

Complex exponential function have properties similar to those of real exponential functions e^x

For example

$$e^{z_1}.e^{z_2} = e^{z_1+z_2}$$

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$$

$$\left(e^{z_1}\right)^m = e^{mz_1}$$

Example 1

Prove that $e^{z_1}.e^{z_2} = e^{z_1+z_2}$ using the definition of e^z

Solution

By definition $e^z = e^x(\cos y + i\sin y)$

Then $e^{z_1}.e^{z_2} = e^{x_1}(\cos y_1 + i\sin y_1)e^{x_2}(\cos y_2 + i\sin y_2)$ where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

or $e^{z_1 z_2} = e^{x_1}.e^{x_2}(\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2)$

$e^{x_1+x_2}[\cos(y_1+y_2)+i\sin(y_1+y_2)]$ from Euler's theorem.

$e^{z_1+z_2}$ by definition of e^z

Example 2

Show that $f(z) = e^z$ is a periodic function with period 2k π i.

Solution

Let $f(z) = e^z = e^x(\cos iy + i\sin iy)$

Then f(z + 2k π i.) $= e^{z+2k\pi i}$

$e^{x+iy+2k\pi}$

$= e^x[\cos(y+2k\pi) + i\sin(y+2k\pi)]$

$= e^x[\cos y + i\sin y]$

$= f(z)$

since
$\cos(y+2k\pi) = \cos y$
$\sin(y+k\pi) = \sin y$

Since $f(z+2k\pi i) = f(z)$, for k =0,1,2,…..it is a periodic function with period $2k\pi i$

Example 3

Find all values of z such that $e^z = 2$

Solution

Let the value of $f(z) = e^z = \mathrm{Re}^{i\theta} = 2$

$\mathrm{Re}^{i\theta} = 2$ (1)

$= 2[\cos(2k\pi) + i\sin(2k\pi)]$

Hence R =2 and θ $= 2k\pi$

Then $e^z = \mathrm{Re}^{i\theta} = 2e^{2k\pi i}$

Taking logarithm on both sides of $e^z = 2e^{2k\pi i}$ we have $z = \ln 2 + 2k\pi i$

Example 4

Find all values of z such that $e^z = 1 + i\sqrt{3}$

Solution

First let $1+i\sqrt{3}=r(\cos\theta+i\sin\theta)$.

Then $r=\sqrt{1^2+3}=\sqrt{4}=2$

$\tan\theta=\frac{\sqrt{3}}{1}$ then $\theta=60^0$ or $\theta=2n\pi+\frac{\pi}{3}$ then $e^z=1+i\sqrt{3}$ becomes

$e^z=2e^{i\left(2n\pi+\frac{\pi}{3}\right)}$

Taking logarithm to base e we have, $z\ln e=\ln 2+i\left(2n\pi+\frac{\pi}{3}\right)\ln e$ or

$z=\ln 2+i\left(2n+\frac{1}{3}\right)\pi$

5.4 Definition of Trigonometric functions of Complex variable z in terms of e^z

From Euler's Formula we have, $e^{ix}=\cos x+i\sin x$ and $e^{-ix}=\cos x-i\sin x$.

It follows that $\cos x=\frac{e^{ix}+e^{-ix}}{2}$ and $\sin x=\frac{e^{ix}-e^{-ix}}{2i}$ for every real number x.

This is exactly applicable to the Complex variable z and we define

$$\boxed{\cos z=\frac{e^{iz}+e^{-iz}}{2} \quad \textbf{and} \quad \sin z=\frac{e^{iz}-e^{-iz}}{2i}}$$

using the definition of cos z and sin z we have the other trigonometric functions. Thus

$$\tan z = \frac{\sin z}{\cos z} = -i\left(\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right)$$

$$\cot z = \frac{\cos z}{\sin z} = i\left(\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}\right)$$

$$\sec z = \frac{1}{\cos z} = \frac{2}{e^{iz} + e^{-iz}}$$

$$\cos ecz = \frac{1}{\sin z} = \frac{2i}{e^{iz} - e^{-iz}}$$

5.5 Trigonometric identities in real variable x and complex variable z.

The following **ten** identities are still valid for complex variables.

	Trigonometric identities for real variable x	**Trigonometric identities for complex variables z**
1	$\sin^2 x + \cos^2 x = 1$	$\sin^2 z + \cos^2 z = 1$
2	$1 + \tan^2 x = \sec^2 x$	$1 + \tan^2 z = \sec^2 z$
3	$1 + \cot^2 x = \cos ec^2 x$	$1 + \cot^2 z = \cos ec^2 z$
4	$\sin(x_1 \pm x_2) = \sin x_1 \cos x_2 \pm \cos x_2 \sin x_1$	$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_2 \sin z_1$
5	$\cos(x_1 + x_2) = \cos x_1 \cos x_2 - \sin x_1 \sin x_2$	$\sin(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$
6	$\sin(-x) = -\sin x$	$\sin(-z) = -\sin z$
	$\cos(-x) = \cos x$	$\cos(-z) = \cos z$
7	$\sin(90 - x) = \cos x$	$\sin(90 - z) = \cos z$
8	$\sin 2x = 2\sin x \cos x$	$\sin 2z = 2\sin z \cos z$
9	$\cos 2x = \cos^2 x - \sin^2 x$	$\cos 2z = \cos^2 z - \sin^2 z$
10	$\sin x = 0$ **if and only if** $x = n\pi$	$\sin z = 0$ **if and only if** $z = n\pi$
11	$\cos x = 0$ **if and only if** $x = \left(n + \frac{1}{2}\right)\pi$	$\cos z = 0$ **if and only if** $z = \left(n + \frac{1}{2}\right)\pi$

5.6 Trigonometric functions of complex variables in terms of hyperbolic functions

You might have known the definition of hyperbolic functions:

$$\cosh y = \frac{e^y + e^{-y}}{2} \quad \sinh y = \frac{e^y - e^{-y}}{2}$$

Using these definitions we can express Trigonometric functions of z in terms of hyperbolic functions.
The following are important identities

- ***a) sin z = sin (x+iy) = sin x cos hy + i cos x sin hy***
- ***b) cos z = cos(x+iy) = cos x cos hy – i sin x sin hy***
- ***c) sin (iy) = i sin hy***
- ***d) cos (iy) = cos hy***

5.7 Prove that *sin z = sin x cos hy + i cos x sin hy*

By definition we have

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= \frac{e^{ix}.e^{-y}}{2i} - \frac{e^{-ix}.e^{y}}{2i}$$

$$= (\cos x + i \sin x).\frac{e^{-y}}{2i} - \frac{(\cos x - i \sin x)e^{y}}{2i}$$

$$= \cos x \frac{e^{-y}}{2i} + \sin x \frac{e^{-y}}{2} - \frac{\cos x e^{y}}{2i} + \frac{\sin x e^{y}}{2}$$

$$= \sin x \frac{(e^{y} + e^{-y})}{2} + i \cos x \frac{(e^{y} - e^{-y})}{2}$$

$$= \sin x \cos hy + i \cos x \sin hy \qquad (1)$$

Thus if $f(z) = \sin z = u + iv$ (u and v are real).

We have $u = \sin x \cosh y$ and $v = \cos x \sin hy$. In the same way you can prove that

$$\cos z = \cos x \cos hy - i \sin x \sin hy \qquad (2)$$

5.8 To show that *sin (iy) = i sin hy*

From (1) we have $\sin z = \sin x \cos hy + i \cos x \sin hy$

$$\sin(x + iy) = \sin x \cos hy + i \cos x \sin hy$$

Let $x = 0$, then $\sin (iy) = 0 + i\,(1) \sin hy$

$= i \sin hy$

In the same way from (2) you can show that $\cos (iy) = \cos hy$

Exercise

1. Define an exponential function e^z
2. Using the definition of exponential function prove that

   a). $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$ b). $\left| e^{iz} \right| = e^{-y}$

3. Prove that 2π is a period of e^{iz}
4. Prove that if $e^{3z} = 1$ then $z = \dfrac{2k\pi i}{3}$
5. Find all the values of z for $e^{4z} = i$.
6. Show that $e^{2 \pm 3\pi i} = e^{-2}$
7. Find all values of z such that $e^z = -2$
8. Find all values of z such that $e^{2z-1} = 1$
9. a). Define cos z and sin z using e^{iz} and e^{-iz}

   b). Hence express tan z, cot z, sec z and cosec z
10. a). Show that cos z = cos x cos hy – i sin x sin hy

    b). Deduce that cos iy = cos hy
11. What are the real and imaginary parts of cos z
12. Establish the formula 1 + $\tan^2$ z = $\sec^2$ z using the definition of sin z and cos z
13. Find all the roots of cos z = 2
14. Show that $\sin \bar{z}$ and $\cos \bar{z}$ are not analytic function of z any where on the complex plane.
15. Show that $\cos(i\bar{z}) = \overline{\cos(iz)}$ for all z and $\sin(i\bar{z}) = \overline{\sin iz}$ if and only if $z = n\pi i (n = 0, \ \pm 1, \ \pm 2..........)$

Summary of the Lesson

You have learnt the following from this Lesson.

i). Definition of e^z and a^z

$$e^z = e^x(\cos y + i\sin y)$$
$$a^z = e^x(\cos y + i\sin y)\ln a$$

ii). Definition of Trogonometric functions:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

From these definitions you can express the other trigonometric functions of z.

iii). Definition of logarithmetic function:

$\ln z = \ln r + i\theta$ …….principal value of lnz

$\ln z = \ln r + I(2n\pi + \theta)$ $n = 0, \pm 1, \pm 2$………. general value of ln z

iv). Hyperbolic functions in real variable:

$$\cos\ hx = \frac{e^{x} + e^{-x}}{2}; \quad \sin\ hx = \frac{e^{x} + e^{-x}}{2}$$

v). Definition of hyperbolic functions: in complex variable

$$\cos\ hz = \frac{e^{z} + e^{-z}}{2}; \quad \sin\ hz = \frac{e^{z} - e^{-z}}{2i}$$

Further Reading

1) Complex Variables and Applications
By R.V Churchill and others
Mc Graw-Hill, KOGAKUSHA Ltd
Tokyo Singapore

2) Theory and Problem of complex variables
By Murray R. Spiegel, Ph.D,
Sehaum' out line series
Mc Graw – Hill Book Company Singapore

3) Text Book of Complex Analysis
By Dr. D. Sengottaiyan, Ph.D.,
Oxford Publications
London Nairobi.

Lesson 6
Mapping by Elementary Functions

6.1 Introduction

Suppose w = f(z) = u (x,y) + iv(x,y) = u + iv
be a function of a complex variable z with

z = x + iy

To each pair of values (x, y) there correspond one value for u and another value for v, in

w = u + iv.

We utilize two separate complex planes for the representation of

z = (x, y) and w = (u, v).

The two planes are called **the z plane** and **the w plane** respectively.

The relationship, w = f(z) then establishes a connection between the points of a given region R in the z plane and the corresponding points of another region R' determined by

w = f(z) in the w plane

In this Lesson we shall study how various curves and regions in the z plane (with the x and the y axes) are mapped by elementary analytic functions on to the w plane (with the u and the v axes).

6.2 Objectives of the Lesson

By the end of this Lesson you will be able to:

i) define the mapping or transformation of points, curves and regions from z plane to w plane under a transformation function f(z).

ii) discuss the mapping by elementary functions such as polynomials, exponential, trigonometric and logarithmic functions.

6.3 Meaning of Mapping (or Transformation)

Consider

$$w = f(z) = 2z^2 + 3 \qquad (1)$$

where

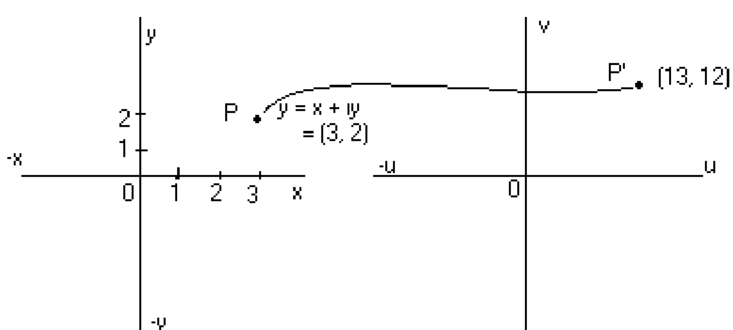
w = u + iv and z = x + iy

Then

$$\begin{aligned} u + iv &= 2(x+iy)^2 + 3 \\ &= 2(x^2 - y^2 + 2xiy) + 3 \\ &= 2x^2 - 2y^2 + 3 + 2xiy \qquad (2) \end{aligned}$$

Equating the real and the imaginary parts in (2) we have

$u = 2x^2 - 2y^2 + 3$ and v = 2xy



z plane w plane

Consider any point say

$z = 3 + 2i$ or $z = (3,2)$.

Then

$$u = 2x^2 - 2y^2 + 3 = 2(3)^2 - 2(2)^2 + 3 = 13$$

and

$$v = 2xy = 2(3)(2) = 12$$

Under the transformation,

$$w = 2z^2 + 3$$

where $z = (x, y) = (3, 2)$ and $w = (u, v) = (13, 12)$

We say that the point (3, 2) on the xy plane or z plane is mapped on to the point (13, 12) on the uv plane or w plane

The point (3, 2) is called the **object** on the z plane and the corresponding point P′ (13, 12) is called the **image** of P on the w plane.

$w = f(z)$ is called the **transformation function** or **mapping function.**

Example 1

Find the image of the point (4, 3) on z plane under the transformation $w = 2z^2 + 3$.

Solution

Since $u + iv = 2(x + iy)^2 + 3$

$$u = 2x^2 - 2y^2 + 3 = 2(16) - 2(9) + 3 = 17 \text{ and}$$
$$v = 2xy = 2(4)(3) = 24.$$

Hence the image of (4, 3) on z plane is (17, 24) on the w plane. We shall discuss some of the transformations in the following examples:

Example 2

Let $w = 3z + 4 - 5i = f(z)$

Find the values of w which corresponds to $z = -3 + i$ on the z plane.

Solution

Let $w = u + iv = 3z + 4 - 5i$ then,

Then

$$u + iv = 3(x + iy) + 4 - 5i$$

or

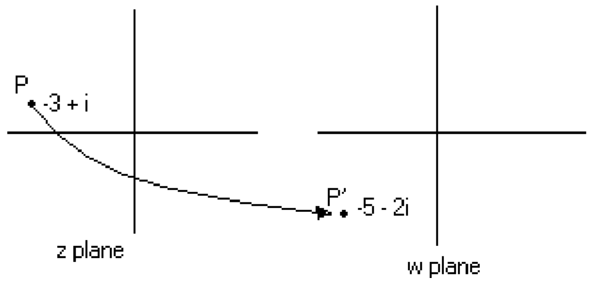
$$u + iv + (3x + 4) + i(3y - 5)$$

Then $u = 3x + 4$ and $v = 3y - 5$

When $x = -3$ and $y = 1$, $u = 3(-3) + 4 = -5$ and $v = 3(1) - 5 = -2$

thus the point $z = -3 + i$ is transformed (mapped) on to the point (-5, -2) on the w plane or $w = -5 - 2i$.

They are shown on the z plane and the w plane



Example 3
Explain the nature of the transformation w = z^2 considering the semi-circle with centre the origin and the radius r on the z –plane.

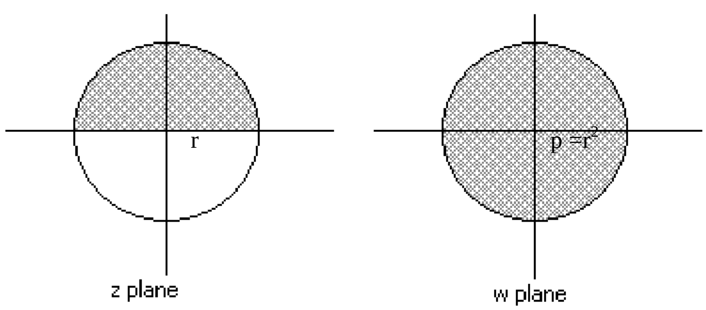
Solution
Let $z = re^{i\theta}$ and $w = \mathrm{R}e^{i\phi}$
Then w = z^2 becomes

$$\mathrm{R}e^{i\phi} = \left(re^{i\theta}\right)^2$$

$$\mathrm{R}e^{i\phi} = r^2 e^{i2\theta}$$

then R = r^2 and $\phi = 2\theta$
The range of variations of θ from $0 < \theta < \pi$ makes ϕ from $0 < \phi < 2\pi$
Let |z| = r be the circle with radius r.

Points on the upper half of the circle on z plane map into the entire circle $|w| = r^2$



Thus the semi-circle of radius r in the z plane is mapped into the full circle of radius R = r^2 in the w plane.

6.4 The linear Transformation
The transformation

$$w = az + b$$

where a and b are real or complex constants is called a **linear Transformation**

6.5 The Bilinear (or Fractional) Transformation
The transformation

$$w = \frac{az+b}{cz+d}\ , \quad \text{where} \qquad ad - bc \neq 0$$

is called a **bilinear Transformation**. It is also known as a **Fractional transformation** or **Mobius transformation**. Here, a, b, c, d are real or complex constants.

6.6 Cross Ratio of Four points z_1, z_2, z_3 and z_4

If z_1, z_2, z_3, z_4 are distinct then the ratio

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

is called the **cross ratio** of z_1, z_2, z_3, z_4.

The cross ratios of z_1, z_2, z_3, z_4 can be written in six different ways

For example $\frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)}$ is another way of writing the cross ratio of z_1, z_2, z_3, z_4. It is written as (z_1, z_2, z_3, z_4)

6.7 An important property of a Bilinear Transformation

If z_1, z_2, z_3, z_4 are four distinct points on the z plane and w_1, w_2, w_3, $w_4$are the images of z_1, z_2, z_3, z_4 respectively under a bilinear transformation

$$w = \frac{az+b}{cz+d}$$

then the cross ratio of z_1, z_2, z_3, z_4 is equal to that of w_1, w_2, w_3, w_4

or
$$\frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)} = \frac{(w_4 - w_1)(w_2 - w_3)}{(w_2 - w_1)(w_4 - w_3)}$$

Proof

This property can be proved by direct substitution for w_1, w_2, w_3, w_4.

Consider $w_2 - w_3 = \frac{az_2+b}{cz_2+d} - \frac{az_3+b}{cz_3+d}$

$$= \frac{(ad-bc)(z_2 - z_3)}{(cz_2+d)(cz_3+d)} \qquad (1)$$

Similarly
$$w_4 - w_1 = \frac{(ad-bc)(z_4 - z_1)}{(cz_4+d)(cz_1+d)} \qquad (2)$$

$$w_2 - w_1 = \frac{(ad-bc)(z_2 - z_1)}{(cz_2+d)(cz_1+d)} \qquad (3)$$

$$w_4 - w_3 = \frac{(ad-bc)(z_4 - z_3)}{(cz_4+d)(cz_3+d)} \qquad (4)$$

Substituting *(1), (2), (3), (4),* in $\frac{(w_4-w_1)(w_2-w_3)}{(w_2-w_1)(w_4-w_3)}$ we have

$$\frac{(w_4-w_1)(w_2-w_3)}{(w_2-w_1)(w_4-w_3)}=\frac{(z_4-z_1)(z_2-z_3)}{(z_2-z_1)(z_4-z_3)}$$

Hence the cross ratio of w_1, w_2, w_3, w_4 = cross ratio of z_1, z_2, z_3, z_4.

This is written as $(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$

Example 4

Find a bilinear transformation which maps the point $z = 0, -i, -1$ on the z plane into $w = i, 1, 0$ respectively on the w plane.

Solution

Let the points $0, -i, -1, z$ on the z plane be transformed into the points $i, 1, 0, w$ respectively on the w plane.

The cross ratio of w_1, w_2, w_3, w_4 is the same as the cross ratio of z_1, z_2, z_3, z_4

Then $\frac{(w_4-w_1)(w_2-w_3)}{(w_2-w_1)(w_4-w_3)}=\frac{(z_4-z_1)(z_2-z_3)}{(z_2-z_1)(z_4-z_3)}$ (1)

Substituting the values of z_1, z_2, z_3 and w_1, w_2, w_3 in the equation (1) and letting $w_4 = w$ and $z_4 = z$ we will get the solution (Try this yourself!)

Example 5

Find a bilinear transformation which maps the points $z_1 = 2, z_2 = i, z_3 = -2$ into the points $w_1 = 1, w_2 = i, w_3 = -1$ respectively.

Solution

Let a general point z be transformed into w under the same transformation.

Since the cross ratios of four points are preserved under a bilinear transformation

$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$

Then $\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$

$$\frac{w-1}{w+1}=\frac{(z-2)(3+4i)}{5(z+2)i}=\frac{(z-2)(3i-4)}{(z+2)(-5)}$$

Substituting the given points we have,

$$\frac{(w-1)(i+1)}{(1-i)(-1-w)} = \frac{(z-2)(i+2)}{(z-i)(-2-z)}$$

$$\frac{(w-1)(1+i)^2}{(1-i)(1+i)(-1-w)} = \frac{(z-2)(2+i)^2}{(z-i)(z+i)(-2-z)}$$

$$\frac{(w-1)(2i)}{w+1} = \frac{(z-2)(3+4i)}{(z+2)(5)}$$

or $\dfrac{w-1}{w+1} = \dfrac{(z-2)(3+-4i)}{5(z+2)i} = \dfrac{(z-2)(3i-4)}{(z+2)(-5)}$

or $\dfrac{w-1}{w+1} = \dfrac{3iz-4z-6i+8}{-5z-10}$ using componendo and dividendo

$\left(if\ \dfrac{a}{b} = \dfrac{c}{d} \text{ then } \dfrac{a+b}{b-a} = \dfrac{c+d}{d-c} \right)$

we have,

$$\frac{(w-1)+(w+1)}{(w+1)-(w-1)} = \frac{(3iz-4z-6i+\delta)+(-5z-10)}{(-5z-10)-(3iz-4z-6i+\delta)}$$

$$\frac{(w-1)+(w+1)}{(w+1)-(w-1)} = \frac{(3iz-4z-6i+\delta)+(-5z-10)}{(-5z-10)-(3iz-4z-6i+\delta)}$$

or $\dfrac{2w}{2} = \dfrac{-9z+3iz-6i-2}{-z-3iz+6i-18}$

or $w = \dfrac{3z(i-3)+2i(-3+i)}{iz(i-3)+6(i-3)}$

or $w = \dfrac{3z+2i}{(iz+6)}$ (2)

This is of the form $w = \dfrac{az+b}{cz+d}$. Hence the required transformation is given in 2

6.8 The transformation *w = lnz*

Let *w = lnz*

Then *u + iv = ln(x + iy).*

Raising both sides to the powers *e* we have $e^w = z$

$$e^{u+iv} = x+iy$$

$$e^u e^{iv} = x+iy$$

$$e^u(\cos v + i\sin v) = x+iy$$

Then $e^u \cos v = x$, $e^u \sin v = y$ (1)

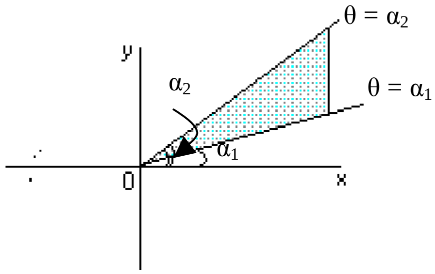
Thus (x,y) becomes (e^u cos v, e^u sin v)

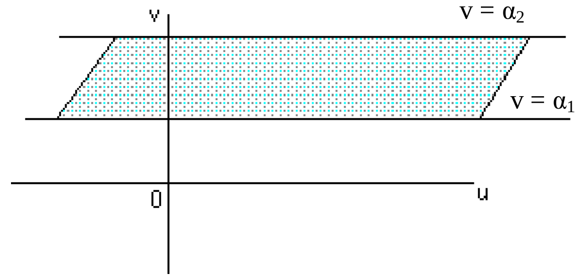
writing *w = lnz* in another way we have

$u+iv = \ln\left(re^{i\theta}\right)$ since $z = re^{i\theta}$, we have $u + iv = lnr + i\theta$

Hence $u = lnr$ and $v = \theta$ (2)

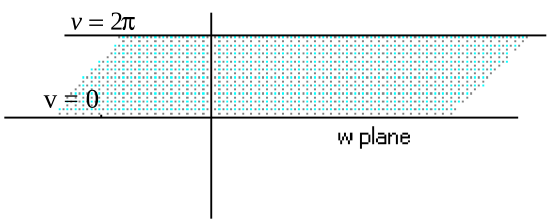
Consider the lines $\theta = \alpha_1$ and $\theta = \alpha_2$ on the z plane



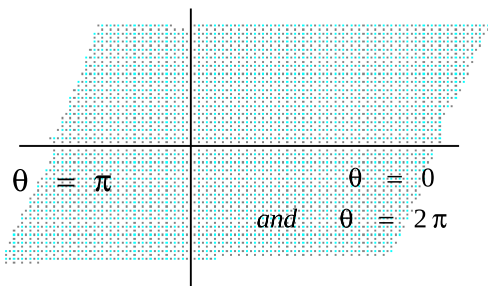


We see that the area between $\theta = \alpha_1$ and $\theta = \alpha_2$ on the z plane is mapped on to the finite strip between $v = \alpha_1$ and $v = \alpha_2$ on the w plane.

The infinite strip $0 \le v \le 2\pi$



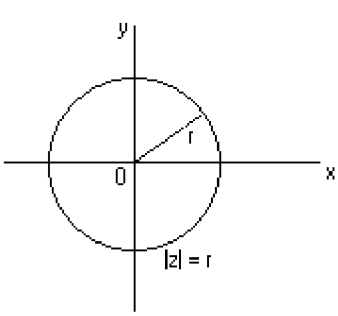
Let v = 0 on the w plane. Then θ = 0 on the z plane. If $v = 2\pi$ on the w plane then $\theta = 2\pi$ on the z plane.



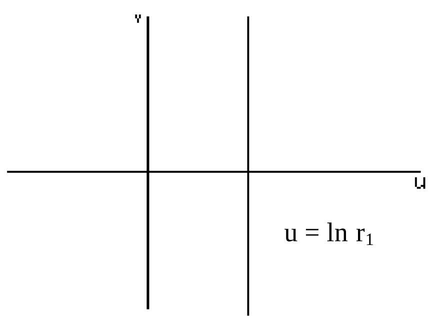
Hence the infinite strip v = 0 to $v = 2\pi$ is transformed into the whole of the z plane from $\theta = 0$ to $\theta = 2\pi$

The image of the circle with radius $r = r_1$

All the circles defined by $r = r_1$ in the z plane are mapped on to the straight lines $u = \ln r_1$



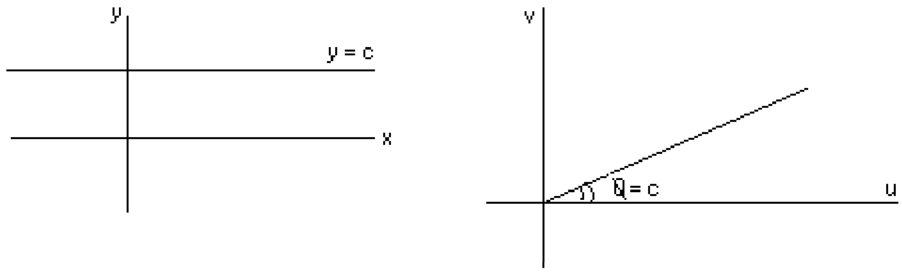
w = e



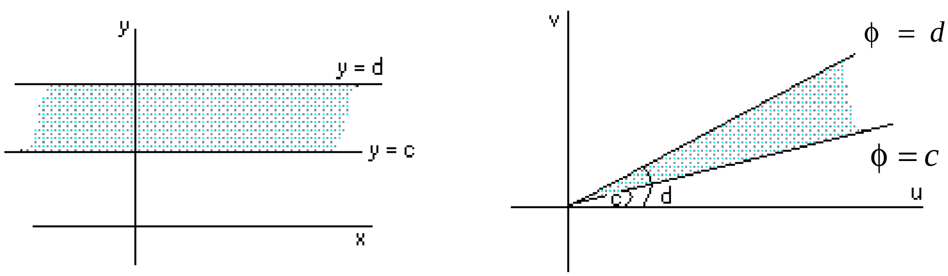
6.9 The transformation $w = e^z$

w = e^z is conformal since for every value of $\dfrac{dw}{dz} = e^z$ is analytic and not equal to zero anywhere on the w plane.

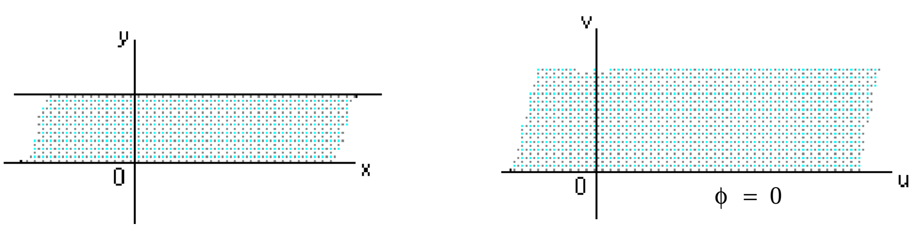
Consider y = c on the z plane. Then $\phi = c$ is a line making an angle ϕ on the w plane.



i) The infinite strip between y =c and y = d, (d > c) becomes $\phi = c$ and $\phi = d$

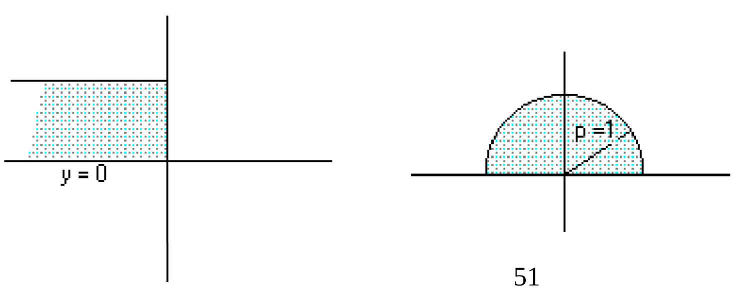


ii) The infinite strip between y = 0 and $y = \pi$ on the z plane becomes $\phi = 0$ and $\phi = \pi$ or the upper half of the w plane

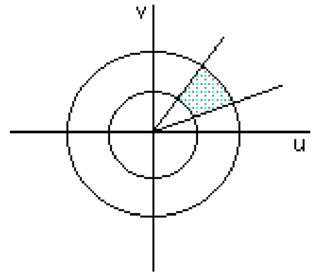


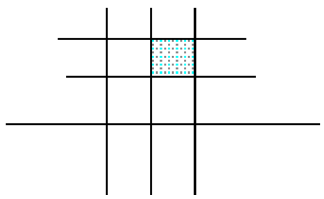
or

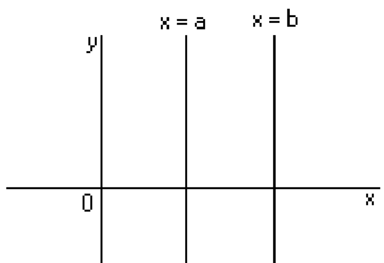
iii) When x varies from $-\infty$ to 0, R varies from 0 to 1. Thus the strip $0 \le y \le \pi$, $-\infty \le x \le 0$ is transformed into the unit semicircle R = 1, $0 \le \phi \le \pi$.

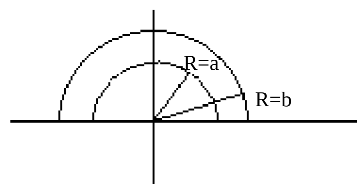


The rectangle bounded by x = a, x = b, y = c, y = d is transformed into the region $e^a \le R \le e^b,\ c \le \phi \le d$

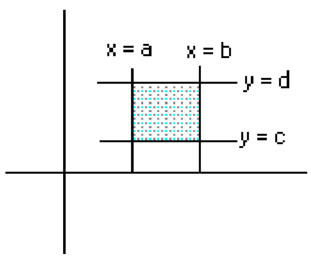


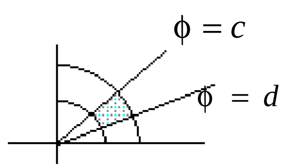




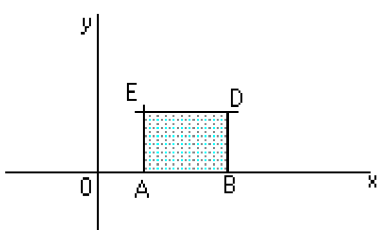


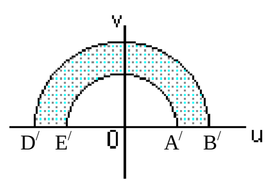
i) Thus the rectangle bounded by y = c, y = d and x = a, and x = b is mapped on to the region $e^a \le R \le e^b,\ c \le \phi \le d$. The mapping is shown in the following diagram.





ii) In particular the transformation w = ez when c = 0 and $d = \pi$ then $0 \le y \le \pi$, the corresponding rectangle is mapped onto half of a circular ring as shown below:



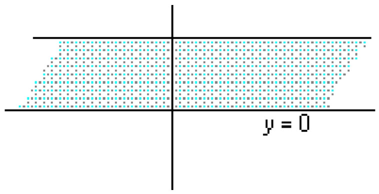


iii) What will happen if w =0?

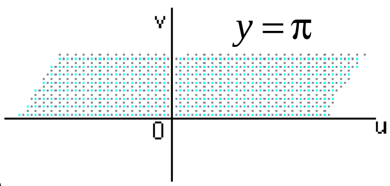
Since e^z is never zero w = 0 has no image on the z plane.

iv) What will be the image of $w = e^z$ if $0 < y < \pi$.

The strip $0 < y < \pi$ on the z plane



is a strip in the above figure then, taking logarithm on both sides of $e^z = w$ we have $z = \ln w + 2n\pi i$. Thus the infinite strip is mapped on the upper half R > 0:



Example 6

a) Find a bilinear transformation, which transforms the unit circle |z| = 1 into the real axis of the w plane in such a way that the points $z_1 = 1$, $z_2 = i$, $z_3 = -1$ are mapped onto $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$.

b) In what regions the interior and exterior of the circle are mapped.

Solution

Let a general point z be mapped onto w. Since bilinear transformations preserve cross ratio of four points we have

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

then

$$(w, 0, 1, \infty) = (z, 1, i, -1)$$

$$\frac{(w-0)(1-\infty)}{(0-1)(\infty - w)} = \frac{(z-1)(i+1)}{(1-i)(-1-z)}$$

$$\frac{(w)(1-x)}{(-1)(x-w)} = \frac{(z-1)}{-1-z}\frac{(i+1)^2}{(1-i)(1+i)}, \qquad \text{where } x \to \infty$$

$$\frac{w\left(\frac{1}{x}-1\right)}{\left(\frac{x}{x}-\frac{w}{x}\right)} = \frac{(z-1)}{1+z}\frac{(di)}{2}$$

$$\frac{w(-1)}{(1-0)} = \frac{-i(1-z)}{1+z}, \qquad \text{when } x \to \infty, \frac{1}{x} = \frac{w}{x} = 0$$

$$or \quad w = \frac{i(1-z)}{1+z}$$

is the required transformation.

6.9 Fixed points of a bilinear transformation

If a point $z_1 = x_1 + iy$, on the z plane may have the image w = u + iv. The points which coincide with their images under a bilinear transformation are called **Fixed points** of the transformation.

If P is a fixed point of the bilinear transformation $w = \dfrac{az+b}{cz+d}$ where $ad - bc \neq 0$ then w and z will be equal.

Hence $z = \dfrac{az+b}{cz+d}$.

or $cz^2 + dz = az + b$

or $cz^2 + (d - a)z - b = 0$

then $z = \dfrac{(a-d) \pm \sqrt{(a-d)^2 - 4(c)(-b)}}{2c}$

or

$$z = \frac{(a-d) \pm \sqrt{(a-d)^2 + 4(bc)}}{2c}$$

The two values of z are the **fixed points** of the bilinear transformation.

The nature of fixed points

i) If a = d, then the fixed points are $\pm \dfrac{2\sqrt{bc}}{2c} = \pm \dfrac{\sqrt{bc}}{c}$

ii) If c = 0 and $a \neq d$ we have one fixed point is finite and other is infinite.

iii) If $c \neq 0$ and $(a-d)^2 + 4bc$ is positive then there will be two finite fixed points.

Example 7

Find the fixed points of the bilinear transformation $w = \dfrac{3z-4}{3-1}$

Solution

At the fixed points z = w.

Hence $z = \dfrac{3z-4}{3-1}$

or

$$z^2 - 4z + 4 = 0$$

or

$$(z - 2)^2 = 0$$

then z = 2 is the only fixed point or we say that fixed points 2, 2 coincide.

Example 8

Find the fixed points of the bilinear transformation $w = \dfrac{z-1}{z+1}$

Solution

If the fixed points z = w

Hence

$$z = \frac{z-1}{z+1}$$

or

$$z^2 + 1 = 0 \qquad \text{or} \qquad z^2 = i^2$$

Hence $z = \pm i$ are the two distinct fixed points.

6.10 The transformation *z = c sin w, c* being real:

The image of the rectangle $u = \pm\dfrac{\pi}{2}$ and $v = \pm\alpha$ in *w* plane.

Let $z = x + iy$ and $w = u + iv$

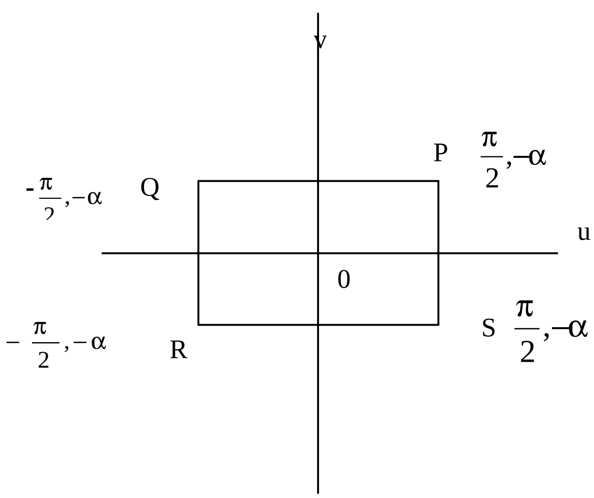
$z = c \sin w$ becomes,

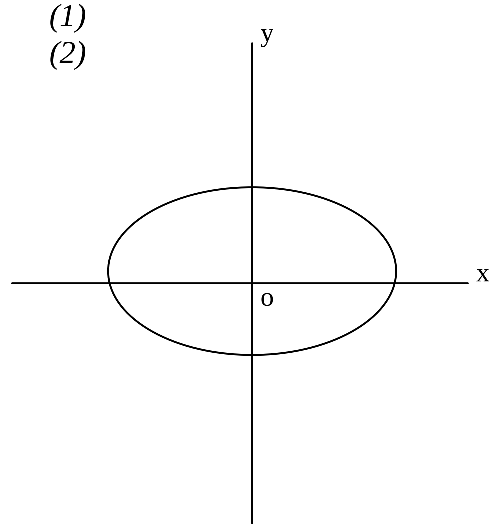
$x + iy = c \sin(u + iv)$

$\quad = c(\sin u \cos\ hv + i \cos u \sin v)$

Hence $x = c \sin u \cos hv$ *(1)*

and $y = c \cos u \sin hv$ *(2)*





Now when *v* is constant the corresponding curves in *z* plane are obtained by eliminating *u* from (1) and (2). Then

$$\frac{x^2}{c^2 \cosh^2 v} + \frac{y^2}{c^2 \sinh^2 v} = \sin^2 u + \cos^2 u$$

or $\dfrac{x^2}{c^2\cosh^2 v}+\dfrac{y^2}{c^2\sinh^2 v}=1$ **(3)**

If $v=\pm\alpha$ a constant, then

$\dfrac{x^2}{c^2\cosh^2\alpha}+\dfrac{y^2}{c^2\sinh^2\alpha}=1$ which represent an ellipse.

Also for $-\dfrac{\pi}{2}<u<\dfrac{\pi}{2}$ we have *cos u* is positive. For the line *PQ*, $v=\alpha$ and *u* varies from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$, x varies from $-c\cos h\alpha$, to $+c\cos h\,\alpha$

Thus we conclude that the side PQ of the rectangle corresponds to upper half of the ellipse in *z* plane.

The line RS

In the same way we conclude that the side RS of the rectangle corresponds to the lower half of the ellipse (or *y* negative)

The line PS

$u=\dfrac{\pi}{2}$ and v varies from $-\alpha$ to α so that from (2), v = 0 ($\cos\dfrac{\pi}{2}=0$) and x varies from c cos hα to c and then from c to c cos hλ according as v varies from $-\alpha$ to 0 and then from 0 to α Hence the rectangle enclosed by $u=\pm\dfrac{\pi}{2}$, $v=\pm\alpha$ in the w plane corresponds to the ellipse in the w plane corresponds to the ellipse in z plane with two slits.

Exercise 6

1. Define a mapping from *z* plane to *w* plane.
2. If the function is *f(z) = z +I* find the image of the point *p*, *4 + 5i* on the *w* plane.
3. A point *3 + bi* on the *z* plane is mapped on to the point (11, c) on the *w* plane by the mapping function $f(z) = 2z^2 + 1$. find the values of *b*and *c*.
4. Define a bilinear Transformation.
5. Find a bilinear Transformation which maps *z = 1, i, -1* respectively onto *w = i, 0, -1*.
6. Find a bilinear transformation which maps points *z = 0, -i, -1* onto *w = i, 1, 0* respectively
7. Find the invariant points of the transformation $w=\dfrac{2z-5}{z+4}$ (Hint: the fixed points are attained putting $z=\dfrac{2z-5}{z+4}$ and solving).
8. Define cross ratio of any four points z_1, z_2, z_3, z_4.

9. Consider the rectangle R formed by $x = 0, y = 0, x = 2, y = 1$ on the z lane. Determine the region R' of the w plane into which R is mapped under the transformation $w = z + (1 - 2i)$.

10. Determine the region of the w plane into which the first quadrant of the z plane is mapped by the transformation $w = z^2$.

11. Show that the Transformation $w = 2z - 3i\bar{z} + 5 - 4i$ is equivalent to $u = 2x + 3y + 5$, $v = 2y - 3x - 4$.

12. Find a bilinear transformation which maps the vertices $1 + i, -i, 2 - i$ of a triangle of the plane into the points $0, 1, i$ of the w plane.

13. Find a bilinear transformation which maps the points $i, -i, 1$ on the z plane into $0, 1, \infty$ of the w plane respectively.

14. Show that the transformation $w = \frac{1}{z}$ maps the inside of the unit circle on the z plane into the outside of the circle $|w| = 1$ on the w plane.

Summary

You have learnt the following from this Lesson

i) The meaning of mapping or transformation of pints, curves and regions from z plane to w plane under a function $f(z)$.

ii) The mapping of elementary functions such as $e^z, \ln z, \sin az$, polynomials, and $\frac{1}{z}$

iii) Bilinear transformation $w = \frac{az+b}{cz+d}$ and their properties.

iv) Definition of a Cross Ration of four points on the z plane.

v) If z_1, z_2, z_3, z_4 are four points on the z plane and w_1, w_2, w_3, w_4 are their images respectively under a bilinear transformation $w = \frac{az+b}{cz+d}$ then the cross ratio of z_1, z_2, z_3, z_4 = the cross Ratio of w_1, w_2, w_3, w_4.

vi) Determination of a bilinear transformation when three points on z plane and their images on w plane are given.

Further Reading

1) Complex Variables and Applications
By R.V Churchill and others
Mc Graw-Hill, KOGAKUSHA Ltd
Tokyo Singapore

2) Theory and Problem of complex variables
By Murray R. Spiegel, Ph.D,
Sehaum' out line series
Mc Graw – Hill Book Company Singapore

Lesson 7
Complex Integration and Cauchy's Theorem

7.1 Introduction

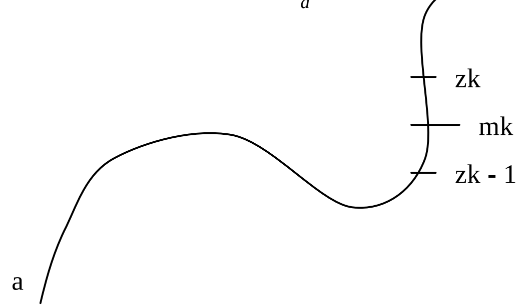
You have already studied the differentiation of the functions of a complex variables and analytic function at a point on the *z* plane. In this Lesson you will study the integration of functions of complex variables and Cauchy's fundamental Theorem in integration of *f(z)* over a simple closed curve when *f(z)* is analytic inside and on the closed curve.

7.2 Objectives of the Lesson

By the end of this Lesson you will be able to

i) state the meaning of complex integration along a curve or line.
ii) apply the meaning of complex integration along a line or curve.
iii) State Cauchy's fundamental theorem in complex analysis.
iv) Apply Cauchy's Theorem to evaluate integrals of functions of *z*.

7.3 Meaning of the complex integration, $\int_a^b f(z)dz$



let c be a curve of finite length and *f(z)* be continuous at all points on the curve c. let the curve be subdivided into n parts by means of points $z_1\ z_2\ \ldots\ldots z_{n-1}$ chosen arbitrarily .

Let us call the starting point a and the ending point b as z_0 and z_n respectively. Now the curve c is subdivided into *n* arcs from z_0 to z_n. Consider any one arc joining z_{k-1} to z_k. let m_k be a point on the arc $z_{k-1}\,z_k$ where *k* varies from *1* to *n*. let

Let $s_n = f(m_1)(z_1 - a) + f(m_2)(z_2 - z_1) + \ldots\ldots f(m_n)(b - z_{n-1})$

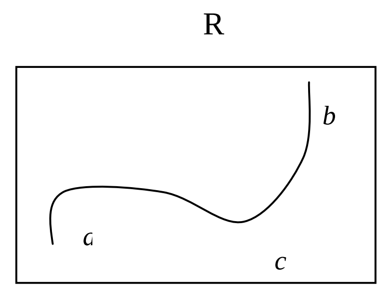
$$= \sum_{k-1}^{n} f(m_k)(z_k - z_{k-1})$$

$$= \sum_{k=1}^{n} f(m_k)\Delta z_k \ \text{ where } \ \Delta z_k = z_k - z_{k-1}$$

Let the number of subdivision, *n* increases and let the largest of the chord length $|\Delta z_k|$ tends to zero. Let the sum s_n approaches a limit *l*. We denote this limit *l* by

$$I = \int_a^b f(z)dz$$

I is called the **line integral of *f(z)* along curve c** or the definite integral of *f(z)* from *a* to *b* along the curve *c*.



7.4 Conditions for the limit *I* exists or $\int_c f(z)dz$ exists

If *f(z)* is analytic at all points of a Region *R* and if the curve *c* is lying in *R* then the limit *I* exists and *f(z)* is said to be integrable along *c*. The famous French Mathematician, Cauchy has discovered that *I = 0* if the curve *c* is closed or *a* and *b* coincide.

7.5 All the formulae for integration of functions of real variables hold good for integration of functions of complex variables.

For example

$$\int z^n dz = \frac{z^{n+1}}{n+1} + c, \quad \int \frac{dz}{z} = \ln z + c \quad \text{where } n \neq -1$$

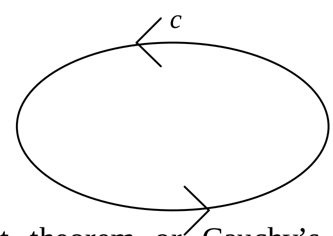
$$\int \cos z dz = \sin z + c \quad \text{and} \quad \int \sin z dz = -\cos z + c$$

Similarly all the other formulae for exponential, trigonometric and logarithmic functions hold good for the functions of complex variables.

7.6 Cauchy's Fundamental Theorem

If a function *f(z)* is analytic inside and on a simple closed curve *c*

then $\int_c f(z)dz = 0$



Cauchy's theorem is also called Cauchy – Goursat theorem or Cauchy's Integral Theorem.

7.7 Converse of Cauchy's Theorem or Morera's Theorem.

Let *f(z)* be continuous in a simply connected region *R* and suppose that

$$\int_c f(z)dz = 0$$

around every simple closed curve *c* in *R*. Then *f(z)* is analytic in *R*.

7.8 **Indefinite Integrals or antiderivative of *f(z)***

If *f(z)* and *F(z)* are analytic in a region *R* and $F^{'}(z) = f(z)$ then *F(z)* is called an indefinite integral or *F(z)* is called the anti-derivative of *f(z)* denoted by

$$F(z) = \int f(z)dz \text{ or sometimes we write } F(z) = \int f(z)dz + c$$

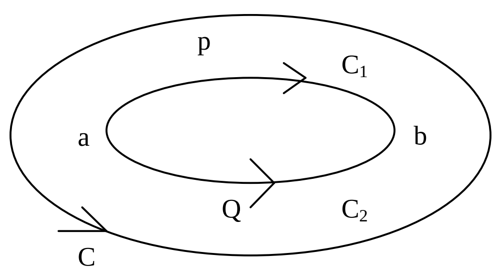
Example 1

Since we have $\frac{d}{dx}\left(5x^2 + e^{2x}\right) = 10x + 2e^{2x}$ we write $\int\left(10x + 2e^{2x}\right)dx = 5x + e^{2x} + c$

Here $5x^2 + e^{2x} + c$ is called the anti derivative of $10x + 2e^{2x}$.

Theorem

7.9 **If *f(z)* is analytic inside and on a the boundary *c* of *a* simply connected region *R*, then $\int_a^b f(z)dz$ is independent of the path in *R* joining the points *a* and *b* in *R*.**



Proof

Let *C* be any simple closed curve enclosing the region *R*. Let *a* and *b* be two points in *R*. Let *apb* and *aQb* be two paths connecting ***a*** and ***b*.**

By Cauchy's Theorem

$\int_{apbQa} f(z)dz = 0$ since *f(z)* is analytic inside and on *apbQa*

or $\int_{apb} f(z)dz + \int_{bQa} f(z)dz = 0$

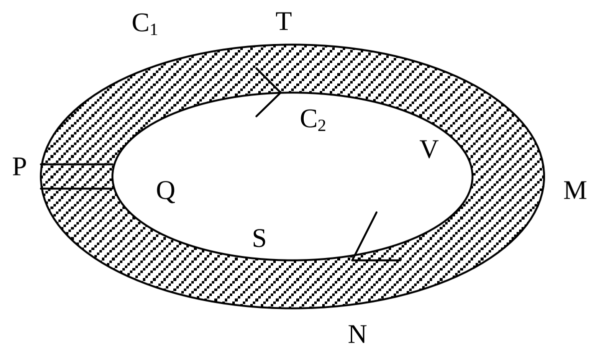
or $\int_{apb} f(z)dz = -\int_{bQa} f(z)dz$

or $\int_{apb} f(z)dz = -\int_{aQb} f(z)dz$

Hence the value of the integrals in the two paths *apb* and *aQb* are the same.

7.10 **If $f(z)$ is analytic in a region R bounced by two simple closed curves c_1 and c_2 and also on c_1 and c_2 then**

$$\int_{c_1} f(z)dz = \int_{c_2} f(z)dz$$



where c_1 and c_2 are both traversed in the anticlockwise direction relative to their interiors.

Proof

Construct a **cross – cut** PQ. Consider the curve $PQV\ SQPNMTP$ $f(z)$ is analytic inside the region R and on the curve $PQVS\ QPNMTP$.

By Cauchy's Theorem we have

$$\int_{PQVSQPNMTP} f(z)dz = 0 \int_{c_2} f(z)dz$$

$$\text{or } \int_{PQ} f(z)dz + \int_{QVSQ} f(z)dz + \int_{QP} f(z)dz + \int_{PNMTP} f(z)dz = 0$$

$$\text{or } \int_{QVSQ} + \int_{PNMTP} = 0$$

since the integrals along PQ and QP cancel.

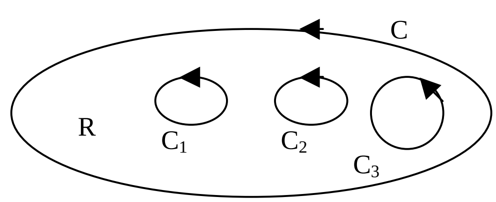
Then $$\int_{QVSQ} f(z)dz = -\int_{PNMTP} f(z)dz$$

$$or - \int_{QSVQ} f(z)dz = -\int_{PNMTP} f(z)dz$$

$$or \int_{QVSQ} f(z)dz = \int_{PNMTP}$$

$$or \int_{c_1} f(z)dz = \int_{c_2} f(z)dz$$

7.11 **The above result can be extended when there are more than one region bounded by c_1, c_2, …c_n**



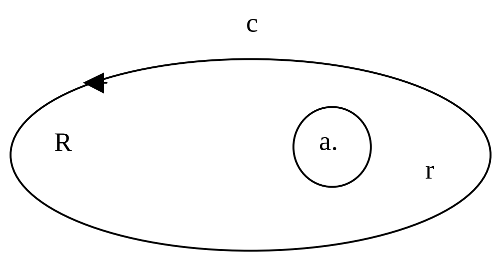
If $f(z)$ is analytic inside and on the curve c enclosing the region R and and $c_1, c_2 \ldots. c_n$ are closed curves in R. Then $\int_C f(z)dz = \int_{C_1} + \int_{C_2} + \ldots\ldots \int_{C_n}$

Theorem

7.12 **Cauchy's Integral Formula.**
If $f(z)$ is analytic inside and on the boundary c of a simply connected region R and a is any point inside the curve c then,

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}dz$$

Proof



Let c be any simple curve enclosing the Region R and a be any point inside c. Draw a circle r of center a and radius c.

Now $\int_c \frac{f(z)}{z-a}dz = \int_r \frac{f(z)}{.z-a}dz$ by Theorem.

Any point z on r is given by $z = a + \in e^{i\theta}$ where θ varies from 0 to 2π and $dz = \in e^{i\theta} i d\theta$

Hence $\int_r \frac{f(z)}{z-a}dz = \int_0^{2\pi} \frac{f\left(a+\in e^{i\theta}\right)\varepsilon e^{i\theta} i d\theta}{\in e^{i\theta}}$

$$= i\int_0^{2\pi} f\left(a+\in e^{i\theta}\right)d\theta$$

$$or \int_c \frac{f(z)}{z-a}dz \ = i\int_0^{2\pi} f\left(a+\in e^{i\theta}\right)d\theta \qquad (2)$$

Taking the limit $\in \to 0$, on both sides we have

$$\int_c \frac{f(z)}{z-a}dz = \lim_{\in \to 0} i\int_0^{2\pi} f\left(a+\in e^{i\theta}\right)d\theta$$

$$= i\int_0^{2\pi} f(a)d\theta$$

$$= if(a)\int_0^{2\pi} d\theta$$

$$= 2\pi i f(a)$$

Hence $f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}dz$

The above theorem can be extended by considering $\int_c \frac{f(z)}{(z-a)^2}$ within the curve c

Thus $f'(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^2}dz$

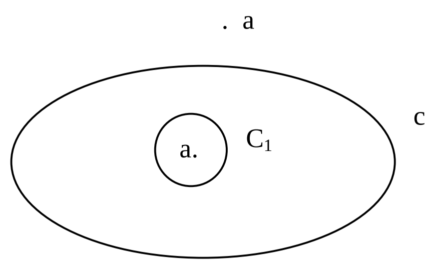
Also $f^{(n)}(a) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-a)^{n+1}}dz$

The theorems are also applicable for multiply connected Regions. We can prove this by making a cut.

Example 2

a) Evaluate $\int_c \frac{dz}{z-a}$ where c is any simple closed curve c and a is inside c,

b) What is the value of the integral if a is outside the closed curve?



Solution

Let c be any simple closed curve c and a be a point inside c

Let c_1 be a circle with center a and radius $\in$

Now $\int_c \frac{dz}{z-a} = \int_{c_1}\frac{dz}{z-a}$ By Cauchy's Theorem for multiply – connected region.

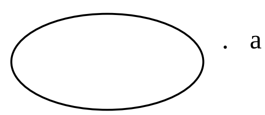
On c_1, $|z-a| = \in$ or $z-a = \in e^{i\theta}$

Hence $z = a + \in e^{i\theta}$ where $0 \le \theta \le 2\pi$ and $dz = i \in e^{i\theta} d\theta$

Then $\int_{c_1}\frac{dz}{z-a} = \int_{\theta=0}^{2\pi}\frac{i \in e^{i\theta} d\theta}{\in e^{i\theta}}$

$= i\int_a^{2\pi} d\theta = 2\pi i$

Hence $\int_c \frac{dz}{z-a} = 2\pi i$ if c is any simple closed curve and a is inside c.

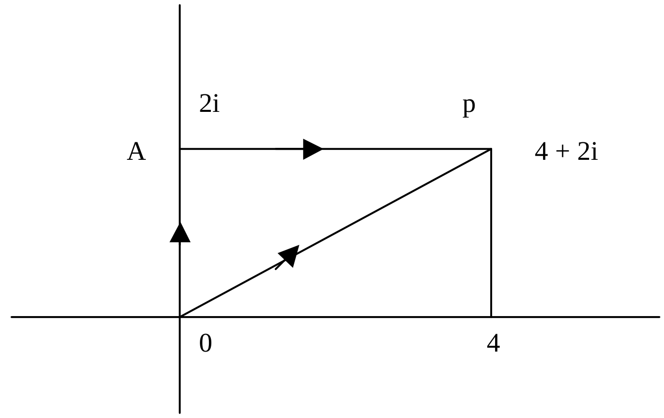


b) If a is outside the curve then $f(z) = \frac{1}{z-a}$ is analytic inside and on the closed curve c and hence by Cauchy's fundamental Theorem $\int_c \frac{dz}{z-a} = 0$

(see this problem after the Residue Theorem)

Example 3

Evaluate $\int \bar{z}dz$ from $z = 0$ to $z = 4 + 2i$ along the line joining $z = 0$ to $z = 2i$ and the line joining $z = 2i$ to $z = 4 + 2i$.



Solution

$$\int_c \bar{z}dz = \int_c (x-iy)d(x+iy) = \int_c (xdx + xidy - iydx + ydy)$$

$$= \int_c (xdx + ydy) + i\int (xdy - ydx)$$

The line from $z = 0$ to $z = 2i$ is OA. On OA $x = 0$, y is varying from 0 to 2.

Hence $\int \bar{z}dz = \int (0d(0) + ydy) + i\int 0dy - yd(0)$

$$= \int_0^2 ydy + i\int 0$$

$$= \int_0^2 ydy = \left.\frac{y^2}{2}\right|_0^2$$

$$= 2.$$

The line from $z = 2i$ to $z = 4 + 2i$ is the line AP on which $y = 2$ constant and x varies from 0 to 4

Hence $\int_{AP} \bar{z}dz = \int (x-iy)d(x+iy)$ becomes $\int_{AP} (x-iy)(dx+idy)$

$\int_{AP} (xdx + ixdy - iydx + ydy)$ on AP x is varying from 0 to 4 and $y = 2$ and $dy = 0.$

$$= \int (xdx + 0 - iydx + 0)$$

$$= \int_0^4 xdx - i\int_0^4 2dx$$

$$= \left.\frac{x^2}{2}\right|_0^4 - i\,2x\Big|_0^4$$

$$= 8 - i8$$

$$\int_c \bar{z}dz = \int_{OA} \bar{z}dz + \int_{AP} \bar{z}dz$$

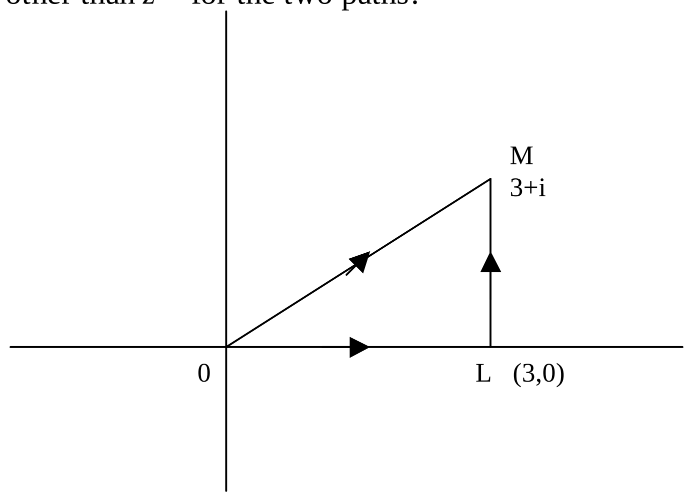
$$= 2 + 8 - 8i$$
$$= 10 - 8i$$

Example 4

a) Integrate z^2 along the straight line OM (direct) and also along the path OLM consisting of two straight line segments OL and LM. O is the origin and M is the point $z = 3 + i$.

b) Show that the integral of z^2 along the two different paths are equal.

c) Is the result true for any function other than z^2 for the two paths?



Let $I_1 = \int_{OM} z^2 dz = \left[\frac{z^3}{3}\right]_0^{3+i}$

$$= \frac{(3+i)^3}{3} - 0$$
$$= \frac{18+26i}{3}$$

Let $I_2 = \int_{OL} z^2 dz + \int_{LM} z^2 dz$

$= \int_{OL} x^2 dx + \int_{LM} (3+iy)^2 i dy$ Note that $z = x$ and $y = 0$ on LM and $x = 3$ and $y = 3 + i$ and dx = 0 *on* LM

hence $I_2 = \left[\frac{x^3}{3}\right]_{x=0}^{3} + i\left[\frac{(3+iy)^3}{3}\right]_{y=0}^{1}$

$$= 9 + \frac{1}{3}(26i - 9)$$
$$= \frac{18+26i}{3}$$

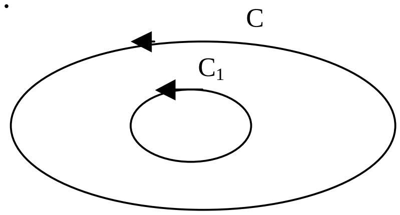
Thus $I_1 = I_2$

d) The result will be true for any function $f(z)$ other than z^2 provided $f(z)$ is analytic in the region inside OLM and on OLMO.

Exercise

1) State Cauchy's Integral Theorem.
2) State the converse of Cauchy's Theorem.
3) Give an example of $f(z)$ and the anti derivative of $f(z)$.
4) If $f(z)$ is analytic inside and on a closed contour c of a simply connected region R, prove that $\int_a^b f(z)dz$ is independent of the path in R joining the points a and b in R.

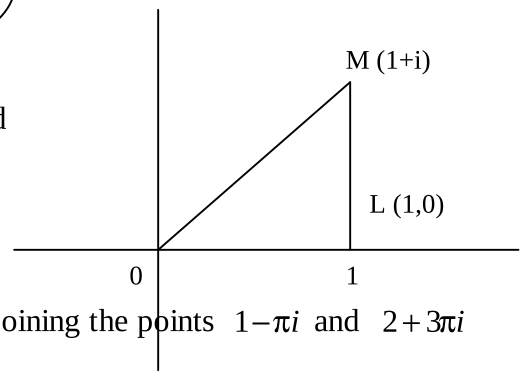
5) If *f(z)* is analytic inside and on a closed curve C (as in the figure) show that $\int_{c} f(z)dz = \int_{c_1} f(z)dz$.



6) Find the value of the integral $I_1 = \int_{OM} \left(z^2 + 4iz\right)dz$ and

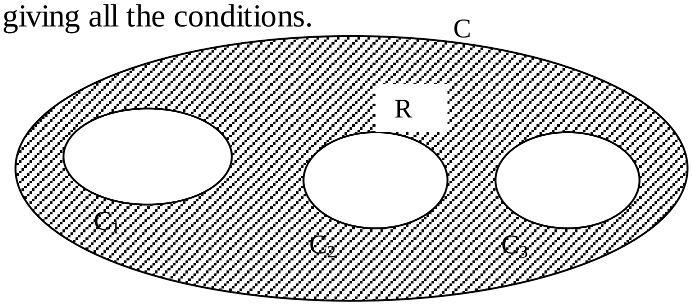
$$I_2 = \int_{OL} \left(z^2 + 4iz\right)dz + \int_{LM} \left(z^2 + 4iz\right)^2 dz$$

show that $I_1 = I_2$



7) Show that $\int_{c} e^{-2z} dz$ is independent of the path C joining the points $1-\pi i$ and $2+3\pi i$ and determine its value.

8) Prove Cauchy – Goursat – Theorem for the multiply connected region R shown in the figure shaded giving all the conditions.



9) Show that $\int_{3+4i}^{4-3i} \left(6z^2 + 8iz\right)dz$ has the same value along the following paths C joining the points 3 + 4i and 4 – 3i along a straight line and also along the straight line 3 + 4i to 4 + 4i and then from 4 + 4i to 4 – 3i.

10) Evaluate $\int_{\pi i}^{2\pi i} e^{3z} dz$

11) Show that $\int_0^{\frac{\pi}{2}} \sin^2 z dz = \int_0^{\frac{\pi}{2}} \cos^2 z dz = \frac{\pi}{4}$

12) Show that $\int \frac{dz}{z^2 - a^2} = \frac{1}{2a} \ln\left(\frac{z-a}{z+a}\right) + c_1 \frac{1}{3} \ln\left(z^3 + 3z + 2\right) + c$

13) Evaluate $\int \frac{z^2 + 1}{z^3 + 3z + 2} dz$.

Summary of the Lesson

You have learnt the following from this Lesson

i) the meaning of complex Integration or the line integral of *f(z)*.

ii) Cauchy's fundamental Theorem: If *f(z)* is analytic inside and on a simple closed curve C then $\int_c f(z)dz = 0$.

iii) Converse of Cauchy's Theorem (Morera's Theorem).

iv) Three important Theorems derived from Cauchy's Theorem.

v) Cauchy's Integral Formula.

If *f(z)* is analytic inside and on the boundary of a simply connected region and a is any point inside c then $f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}dz$ and its extensions.

Further reading

1. Complex variables and Applications
   By R.V Churchill and others
   Mc Graw – Hill, Kogakusha Ltd
   Tokyo Singapore.

2. Complex variables
   By Murray R. Spiegel, Ph.D
   Schaum outline series.
   Mc Graw – Hill Book Company
   Singapore.

3. Text Book of Complex Analysis
   By Dr. D. Sengottaiyan Ph. D
   Oxford Publications
   London Nairobi.

Lesson 8
Laurent series and Singularities of Functions

8.1 Introduction

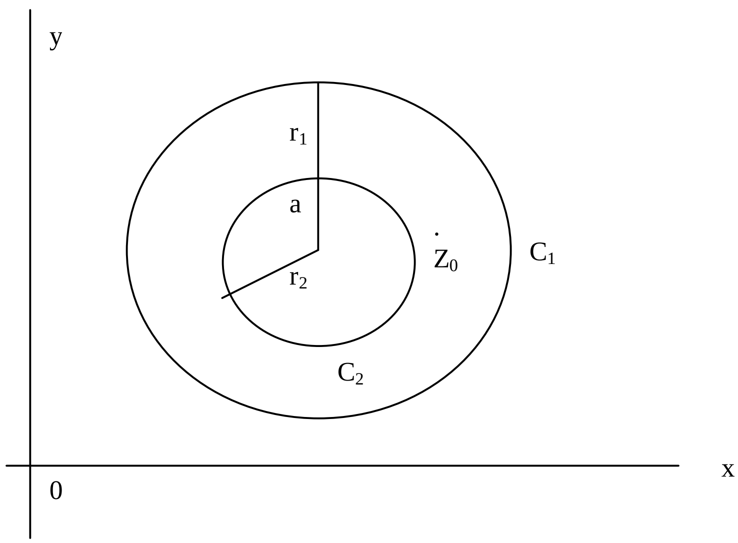
You have studied analytic functions in the previous Lessons. In this Lesson you will study the series representation of analytic functions. You shall study mainly Laurent series expansion in a ring shaped region between two concentric circles.

8.2 Objectives of the Lesson.

By the end of this Lesson you should be able to

i) state Laurent's series
ii) deduce Taylor's series from Laurent Series.
iii) define Singularities of functions
iv) classify the singularities of functions.
v) expand a function *f(z)* at its singularities.

8.3 Laurent series



Let c_1 and c_2 be two concentric circles of radii r_1 and r_2 respectively and cetre at a. suppose that a function $f(z)$ is single valued and analytic on c_1 and c_2 and in the ring shaped region between c_1 and c_2 shown in the figure. If z_0 is any point in R, then we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad (1)$$

where $a_n = \dfrac{1}{2\pi i} \displaystyle\int_{c_1} \frac{f(z)}{(z - z_0)^{n+1}}$ n = 0, 1, 2 ….

$$b_n = \frac{1}{2\pi} \int_{c_2} \frac{f(z)}{(z - z_0)^{-n+1}} \qquad n=1, 2 …..$$

The path of integration is taken counter clockwise.

The series (1) is called Laurent Series.

The part of Laurent Series $\sum_{n=0}^{\infty} an(z-z_0)^n$ is called the **analytic** part of Laurent Series and $\sum_{n=1}^{\infty} \frac{bn}{(z-z_0)^n}$ is called the **principal** part of Laurent Series.

8.4 Taylor's Series from Laurent Series

If $f(z)$ is analytic at all points inside and on c_1 the function $\frac{f(z)}{(z-z_0)^{-n+1}}$ is analytic inside and on c_2 since $-n+1 \leq 0$. Hence by Cauchy's fundamental Theorem b_n becomes zero. In this case Laurent Series reduces to Taylors series. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

where $a_n = \frac{1}{2\pi i} \int_{c_1} \frac{f(z)}{(z-z_0)^{n+1}}$ n=0, 1, 2, …..

8.5 Singular points or Singularities.

A point on the z plane at which $f(z)$ fails to be analytic is called **a singular point** or singularity of $f(z)$. There are various types of singularities:

i) Isolated Singular points

A point $z = z_0$ is called an isolated singular point of $f(z)$ if we can find $\delta > 0$ such that the circle $|z-z_0| = \delta$ encloses no singular point other than z_0.

In other words there exists a deleted neighborhood of z_0 containing no singularity. If no such δ can be found we call z_0 a non – isolated singularity.

ii) Poles

Consider a point z_0 on the z plane at which $f(z)$ becomes infinite. If we can find a positive integer n such that $\lim_{z \to z_0} (z-z_0)^n f(z) = c \neq 0$. Then the point z_0 is called a pole of order n

Example 1

$f(z)=\dfrac{2z}{z-3}$ has a pole of order 1. Here $\dfrac{2z}{z-3}$ becomes ∞ at $z = 3$, but

$\lim_{z\to 3}(z-3)\dfrac{2z}{z-3}$ = 6 which is not zero

Hence $f(z)=\dfrac{2z}{z-3}$ has a pole of order one.

Example 2

$\dfrac{2z+4}{(z-2)^3(z-1)^4(z+4)}$ has

a pole of order 3 at $z = 2$
a pole of order 4 at $z = 1$ and
a pole of order one at $z = -4$.
A pole of order one is called a simple pole.

For a multiple valued function all the Branch points are called singular points.

Examples: $f(z)=z^{\frac{1}{2}}$ has a branch point at $z = 0$

$f(z)=(z-5)^{\frac{1}{2}}$ has a branch point at $z = 5$

$f(z)=\ln(z^2+3z-10)$ or $ln(z-2)(z+5)$ has branch points at $z = 2$ and $z = -5$

iii) Removable singularities

If z_0 is a singular point of $f(z)$ but $\lim_{z\to z_0} f(z)$ exists, then the singular point z_0 is called a removable singularity of $f(z)$.

Examples

Consider

$f(z)=\dfrac{\sin z}{z}$. $z=0$ is not an ordinary point of $f(z)$ since it takes the form $\dfrac{0}{0}$. But

$\lim_{z\to 0}\dfrac{\sin z}{z}=1$ by L Hospital Rule.

Hence $z=0$ is a removable singularity for $f(z)=\dfrac{\sin z}{z}$

iv) Essential Singularity

A function $f(z)$ may have a singular point $z = z_0$, but if this singular point is neither an isolated singularity, nor a pole or branch point or removable singularity then the singularity z_0 is called an essential singularity of $f(z)$.

Example 4

$f(z)=\dfrac{1}{e^{3-z}}$ has an essential singularity at $z = 3$.

Suppose $f(z)$ becomes infinity but we cannot find any positive integral n such that $\lim_{z\to z_0}(z-z_0)^n f(z)=R\neq 0$ then $z = z_0$ is an essential singularity.

v) Singularities at Infinity

A function $f(z)$ may not have any pole at $z = z_0$, but $f(\frac{1}{z})$ may have poles. Such poles are called singularities at infinity.

vi) Branch point

Example 5

Consider $f(z)=z^4$. $f(z)$ has actually no poles but $f(\frac{1}{z})=z^{\frac{1}{4}}$ has a pole at $z = 0$. Hence $f(z)$ has a pole of order 4 at $z=\infty$

8.6 Laurent series about the singularities.

If z_0 is any kind of a singularity for $f(z)$, we can expand the function $f(z)$ in an infinite series about the Singularity such series are called **Laurent Series**.
We shall consider some functions having singularities at $z = z_0$ and expand the functions at z_0 in the following examples.

Example 6

Consider the function $f(z)=\dfrac{e^{3z}}{(z-2)^3}$

a) state the singularity of $f(z)$.
b) what is the kind of the singularity of $f(z)$.
c) expand $f(z)$ in a Laurent series.
d) state the region of convergence of the series.

Solution

a) when $z=2$. $f(z)$ becomes infinite and it is not analytic Hence $z = 2$ is a singularity of $f(z$.

b) $\lim_{z\to 2}\dfrac{(z-2)^3 e^{3z}}{(z-3)^3}=e^{6z}$ which is not equal to zero Then $z = 2$ is a pole of order 3.

Let $f(z)=\dfrac{e^{3z}}{(z-2)^3}$ and let $z - 2 = u$ so that $z = u + 2$.

Then $\dfrac{e^{3z}}{(z-2)^3} = \dfrac{e^{3(u+2)}}{u^3}$

$$= \frac{e^{6+3u}}{u^3}$$

$$= \frac{e^6.e^{3u}}{u^3}$$

$$= \frac{e^6}{u^3}\left[1 + \frac{3u}{1!} + \frac{(3u)^2}{2!} + \frac{(3u)^3}{3!} + \frac{(3u)^4}{4!} + ...\right]$$

$$= \frac{e^6}{u^3} + \frac{3e^6}{u^2} + \frac{9e^6}{u.2!} + \frac{27e^6}{3!} + \frac{81u}{4!} + ...$$

$$= \frac{e^6}{(z-2)^3} + \frac{3e^6}{(z-2)^2} + \frac{9e^6}{2!(z-2)} + \frac{27e^6}{3!} + \frac{81u}{4!} + ...$$

The series converges at all points except at $z = 2$.

Example 7

Consider the function $f(z) = \dfrac{z - \sin z}{z^3}$

a) state the singularity of *f(z)*.
b) what is the kind of singularity of *f(z)*.
c) find the Laurent series of *f(z)*.
d) what is the region of convergence of *f(z)?*.

Solution

Let $f(z) = \dfrac{z - \sin z}{z^3}$

a) $z = 0$ is a singularity since at $z = 0$ the function is not defined and hence not analytic.

b) $\lim\limits_{z \to 0} \dfrac{z - \sin z}{z^2} = \lim\limits_{z \to 0} \dfrac{1 - \cos z}{2z} = \lim\limits_{z \to 0} \dfrac{1 + \sin z}{2} = \dfrac{1}{2}$ (By L' Hospital Rule the limit exists). Hence $z = 0$ is a removable singularity.

c) Let $z = 0$

$$\frac{z - \sin z}{z^2} = \frac{1}{z^2}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ...\right)\right]$$

$$= \frac{1}{z^2}\left[\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - .....\right]$$

$$= \frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!} + ...$$

d) The series converges for all values of z i.e. the series converges on the whole of z plane.

Example 8

Consider the function $f(z) = (z-4)\sin\frac{1}{z+2}$

a) state the singularity of $f(z)$
b) what is the kind of singularity of $f(z)$.
c) expand $f(z)$ in a Laurent series.
d) state the region of convergence of the series.

Solution

a) Let $f(z) = (z-4)\sin\frac{1}{z+2}$ when $z = -2$ $f(z)$ is not defined and hence it is not analytic. Hence $z = -2$ is a singularity of $f(z)$.
b) $z = -2$ is neither a pole nor Branch point. It is an essential singularity.
c) Let $z + 2 = u$ so that $z = u - 2$

Then $(z-4)\sin\frac{1}{z+2} = (u-6)\sin\frac{1}{u}$

$$= (u-6)\left[\frac{1}{u} - \frac{1}{3!u^3} + \frac{1}{5!u^5} + ...\right]$$

$$= 1 - \frac{1}{3!u^2} + \frac{1}{5!u^4} + ... - \frac{6}{u} + \frac{6}{3!u^3} - \frac{6}{5!u^5} + ...$$

$$= 1 - \frac{6}{u} - \frac{1}{3!u^2} + \frac{6}{3!u^3} + \frac{1}{5!u^4} - \frac{6}{5!u^5} + ...$$

$$= 1 - \frac{6}{z+2} - \frac{1}{3!(z+2)^2} + \frac{6}{3!(z+2)^3} + \frac{1}{5!(z+2)^4} - \frac{6}{5!(z+2)^5}$$

The series converges at all points except at $z = -2$.

Example 9

Expand $f(z) = \frac{3}{z^2(z-3)^2}$ in a Laurent Series at $z = 3$.

Solution

We can expand $f(z)$ using Binomial theorem.

$$f(z) = \frac{3}{z^2(z-3)^2}$$

$z = 0$ and $z = 3$ are poles of $f(z)$.
Let $z - 3 = u$ or $z = u + 3$.

$$f(z) = \frac{3}{z^2 (z-3)^2} = \frac{3}{(u+3)^2 (u^2)}$$

$$= \frac{3}{\left[3\left(1+\frac{u}{3}\right)\right]^2 u^2}$$

$$= \frac{3\left(1+\frac{u}{3}\right)^{-2}}{9u^2}$$

$$= \frac{1}{3u^2}\left[1+(-2)\frac{u}{3}+\frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{u}{3}\right)^3+.....\right]$$

$$= \frac{1}{3u^2} - \frac{2}{9u} + \frac{1}{9} - \frac{4}{81}u + ...$$

$$f(z) = \frac{1}{3(z-3)^2} - \frac{2}{9(z-3)} + \frac{1}{9} - \frac{4}{81}(z-3) - .....$$

This series converges for all values of z such that $0 < |z-3| < 3$.

Example 10

Expand $f(z) = \frac{1}{(z+3)(z+1)}$ is a Laurent series valid for $0 < |z+1| < 2$.

Solution

Let $f(z) = \frac{1}{(z+3)(z+1)}$ (1)

Let $(z + 1) = u$ then

$$f(z) = \frac{1}{u(u+2)} = \frac{1}{2u\left(1+\frac{u}{2}\right)}$$

$$= \frac{1}{2u}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + ...\right) \qquad (2)$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + ...$$

The expansion of Binomial theorem is valid when $\frac{u}{2} < 1$ or $\frac{z+1}{2} < 1$ or $z + 1 < 2$ or $|z+1| < 2$.

Example 11

Expand $f(z) = \frac{1}{(z+1)(z+3)}$

Solution

Let $f(z)=\dfrac{1}{(z+1)(z+3)}$

$=\dfrac{1}{2}\left(\dfrac{1}{z+1}\right)-\dfrac{1}{2}\dfrac{1}{(z+3)}$ Resolving into partial fraction

$$\frac{1}{2(z+1)}=\frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}\right)=\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+.... \qquad (2)$$

$$\frac{1}{2(z+3)}=\frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+....\right) \qquad (3)$$

The two expansions are valid if $\left|\dfrac{1}{z}\right|$ < 1 and $\left|1+\dfrac{z}{3}\right|$ < 1 or if $|z|$ > 1 and $|z|$ < 3 or

$1 < |z| < 3$. The required Laurent series is the sum of the series in (2) and (3). Hence

$$f(z)=\frac{1}{(z+1)(z+3)}=....-\frac{1}{2z^4}+\frac{1}{2z^3}-\frac{1}{2z^2}+\frac{1}{2z}-\frac{1}{6}+\frac{z}{18}-\frac{z^2}{54}+\frac{z^3}{162}........$$

Exercise 8

1) State Laurent Series for a function *f(z)*.
2) Derive Taylor's series from Laurent series.
3) Define singularity of a function *f(z)*.
4) Give one example for each of the following
    - i) Isolated singularity
    - ii) Poles
    - iii) Branch point
    - iv) Removable singularity
    - v) Essential singularity
    - vi) Singularity at infinity.
5) Expand $f(z)=\dfrac{1}{z-3}$ in a Laurent series valid for i) $|z| < 3$ and ii) $|z| > 3$.
6) Expand $f(z)=\dfrac{1}{z(z-2)}$ in a Laurent series valid for a) $0 < |z| < 2$ b) $|z| > 2$.
7) Find the singularities of the functions $\dfrac{z}{e^z-1}$ and classify the singularity.
8) a) Expand $f(z)=\dfrac{z}{e^{z-2}}$ in a Laurent series about $z = 2$ and determine the region of convergence of this series .

b) Classify the singularities of *f(z)*.

Summary of the Lesson

You have learnt the following from this Lesson.

i) Laurent series expansion of an analytic function in a ring shaped region.
ii) Taylor's series derived from Laurent's series.
iii) Singularities of *f(z)* and their classification.
iv) Expansion of functions at the singularities.

Further Reading

1. Complex Variables and Applications
By R.V Churchill and others
Mc Graw – Hill, KOGAKUSHA Ltd
Tokyo Singapore.

2. Complex Variables
By Murray R. Spiegel, Ph.D
Schaum outline series.
Mc Graw – Hill Book Company
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3. Text Book Complex Analysis
By Dr. D. Sengottaiyan Ph. D
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Lesson 9
Poles and Residues of a Function

9.1 Introduction

You have studied Cauchy's fundamental theorem which states that if a function is analytic every where inside and on a simple closed contour (curve) c, then the integral of a function around that contour is zero. If however the functions fails to be analytic at a **finite number** of points inside C those points may be called poles of the function. Each of these points contributes to the value of the integral. These contributions are called the Residues of the function. You will learn, in this Lesson to determine the poles and the residues at the poles of a function of complex variables.

9.2 Objectives of the Lesson

By the end of this Lesson you will be able to: -

i).define the pole of a function *f(z)*
ii).determine the pole of *f(z)*
iii).define the Residues of a function at its poles.
iv).determine the residue of a function *f(z)* at its poles of order 1, 2…n.

9.3 Definition of poles of a function *f(z)*

Let *f(x)* be any function of z. Generally $\lim_{z\to a}(z-a)f(z)=0$

Suppose $\lim_{z\to a}(z-a)f(z)=A$ which is not zero. Then z = a is called a pole of order one of *f(z)*.

Similarly if $\lim_{z\to a}(z-a)^m f(z)=A,\neq 0$ then z = a is called a pole of order m of *f(z)*.

9.4 Determination of the Poles of f(z) at its poles.

At the pole the function *f(z)* becomes infinite. Hence to find the poles of *f(z)* we put *f(z)* = ∞ and find z which are poles. If *f(z)* = $\frac{\phi(z)}{g(z)}$ we solve *g(z)* = 0 and the roots are poles of *f(z)*.

Example 1

Find the poles of $\frac{z^3}{(z-1)(z+3)^2(z-8)^5}$ and state the order of each pole.

Solution

At the pole *f(z)* becomes infinite. If $(z-1)(z+3)^2(z-8)^5=0$, *f(z)* becomes infinite.

Hence z = 1 is pole of order 1
z = -3 is pole of order 2
z = 8 is pole of order 5

Generally to find the pole of *f(z)* put the denominator of *f(z)* to zero and solve for z.

Example 2

Determine the poles of $\dfrac{e^{iz}}{z^2(z^2+2z+2)}$

Solution

Let $f(z)\dfrac{e^{iz}}{z^2(z^2+2z+2)}$

The poles of *f(z)* are obtained by solving $z^2(z^2+2z+2)=0$
One pole is z = 0 of order 2.

Solving $z^2+2z+2=0$ we get $z=\dfrac{-2+\sqrt{4-8}}{2}$ or z = -1 + i and – 1 – i both of them are simple poles.

Thus z = 0, 0, -1 + i, -1 – i are the four poles.

Example 3

Find the poles of $\dfrac{2z^2+5}{z^4+16}$

Solution
The poles of *f(z)* are obtained by solving the equation $z^4 + 16 = 0$

or $z^4 = -16$
or $z^4 = 16\ (-1)$

Hence $z=2(-1)^{\frac{1}{4}}$

$$=2\left[\cos(2n+1)\pi+i\sin(2n+1)\pi\right]^{\frac{1}{4}}$$
$$=2\left[\cos\frac{(2n+1)\pi}{4}+i\sin\frac{(2n+1)\pi}{4}\right]$$

n = 0, 1, 2, 3 (by Demoivre's Theorem).

If n = 0, $z=2\left(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right)=2\left(\dfrac{1}{\sqrt{2}}+\dfrac{i}{\sqrt{2}}\right)$

If n = 1, $z=2\left(\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)=2\left(-\dfrac{1}{\sqrt{2}}+\dfrac{i}{\sqrt{2}}\right)$

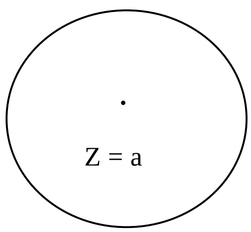
If n = 2, $z = 2\left(\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 2\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$

If n = 3, $z = 2\left(\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$

Thus the poles of $\frac{2z^2+5}{z^4+16}$ are given by $z = \sqrt{2} \pm i, \ \sqrt{2} \pm i$

9.5 Residue of *f(z)* at its pole

Let *f(z)* be single valued and analytic inside and on a circle c whose center is a *f(z)* is not analytic at the point z = a (center of the circle)



Then *f(z)* has a Laurent series about z = a given by: -

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n \qquad (1)$$

where $a_n = \frac{1}{2\pi i}\int \frac{f(z)}{(z-a)^{n+1}}dz$ n = 0, ± 1, ± 2, ……….. (2)

suppose n = -1 we have from (2) $\int f(z)dz = 2\pi i \quad a_{-1}$ (3)

(3) involves only the coefficient a_{-1} in (1). We call a_{-1} **the residue** of *f(z)* at z = a. It is denoted by R.

Useful formula for the residue of *f(z)* at the pole *z = a*

9.6 Determination of Residues of *f(z)* at its poles

i). If z =a is a simple pole for *f(z)* then the residue of *f(z)* at *a* is given by

$$R = \lim_{z \to a}(z-a)f(z)$$

ii). If z =a is a pole of order two then

$$R = \lim_{z \to a}\frac{1}{1!}\frac{d}{dz}\left[(z-a)^2 f(z)\right]$$

iii). If z = a is a pole of order n then

$$R = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right]$$

The following examples illustrate the method of finding the Residues of *f(z)* at its poles.

Example 4

Let $f(x) = \dfrac{z^3 + 5z + 1}{z - 2}$

a) Determine the pole of *f(z)*
b) Calculate the residue of *f(z)* at its pole.

Solution

a) The pole *f(z)* is obtained by solving the denominator z – 2 = 0
   Hence z = 2 is a simple pole of *f(z)*

b) The Residue of *f(z)* at z =a is given by

$$R = \lim_{z \to 2} (z-2) f(z)$$

$$= \lim_{z \to 2} (z-2) \frac{(z^3 + 5z + 1)}{z - 2}$$

$$= \lim_{z \to 2} (z^3 + 5z + 1)$$

$$= 8 + 10 + 1$$

$$= 19$$

Then the Residue of *f(x)* at z = 2 is 19

Example 5

Let $f(z) = \dfrac{z^2 - 7z + 10}{(z-3)^2}$

Determine the pole of *f(z)*. Calculate the residue of *f(z)* at its poles.

Solution

a) The pole of *f(z)* is obtained by equating *f(z)* = ∞ or by solving the denominator
$(z – 3)^2$ = 0
$(z – 3)^2$ = 0 gives z = 3, 3 Hence z = 3 is a pole of order 2.

If z = a is a pole of order n, then $R = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right]$

Hence if z = 3 is a pole of order 2, we have

$$R = \lim_{z \to 3} \frac{1}{(2-1)!} \frac{d}{dz}\left[(z-3)^2 \frac{(z^2-7z+10)}{(z-3)^2}\right]$$

$$= \lim_{z \to 3} \frac{d}{dz}(z^2 - 7z + 10)$$

$$= \lim_{z \to 3}(2z - 7)$$

R = - 1

Exercise 9

1. Define the pole of a function *f(z)*
2. State the formula for finding the pole of *f(z)* = $\frac{\phi(z)}{g(z)}$
3. Let z = a be a simple pole for *f(z)* state the formula for finding the Residue of *f(z)* at the pole z = a.
4. Let z =a be a pole of order 4 for the function *f(z)*. Write down the formula for finding the Residue of *f(z)* at z = *a*.

For each of the following functions determine the pole and the Residues at the poles (5 to 10).

5. $\frac{2z+1}{z^2-z-2}$
6. $\frac{(z+1)^2}{(z-1)^2}$
7. $\frac{1}{z^4+1}$
8. $\frac{3z^3+2}{(z-1)(z^2+9)}$
9. $\frac{1}{z^2(z+4)}$
10. $\frac{1}{z^4+81}$

11. a). Find the 6 poles of the function $\frac{1}{z^6+1}$

    b). Determine the residues at each pole of *f(z)*

12. a). Find the three poles of $f(z) = \frac{1}{z^3-1}$

    b). Determine the residues of *f(z)* at its 3 poles.

Summary of the Lesson

You have learnt the following from this Lesson.

1. Definition of the poles of a function *f(z)*:
   If $\lim_{z\to a}(z-a)\,f(z) = A$ which is not zero, then z = a is called a pole of *f(z)*.

2. Method of finding the poles of *f(z)* = $\frac{\phi(z)}{g(z)}$: i.e. we put *g(z)* = 0 and solve; the values of z are poles of the function.

3. Definition of Residue of *f(z)* at its poles: if z = a is a simple pole of *f(z)* then $\lim_{z\to a}(z-a)\,f(z)$ = Residue of *f(z)* at z =a.

   If z = a is a pole of order n then $\lim_{z\to a}\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]$ = Residue of *f(z)* at z =a.

Further Reading

1) Complex Variables and Applications
By R.V Churchill and others
Mc Graw-Hill, KOGAKUSHA Ltd
Tokyo Singapore

2) Theory and Problem of complex variables
By Murray R. Spiegel, Ph.D,
Sehaum' out line series
Mc Graw – Hill Book Company Singapore

3) Text Book of Complex Analysis
By Dr. D. Sengottaiyan, Ph.D.,
Oxford Publications
London Nairobi.

LESSON 10
Residue Theorem and its Applications to Integration

10.1 Introduction

You have already studied Cauchy's fundamental Theorem when a function is analytic inside and on a simple closed contour C then the integral of the function along the curve is zero. What will be the value of the integral if the function has some **finite** number of poles inside the closed curve? In this Lesson we shall find the answer which is called the Residue Theorem.

10.2 Objectives of the Lesson

By the end of this Lesson you will be able to: -

i). state Residue Theorem

ii). apply Residue Theorem for the evaluation of three types of integrals:

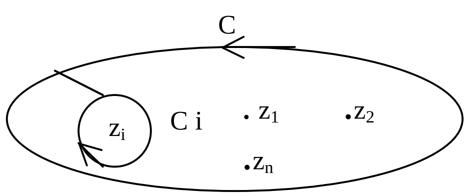
a) improper integrals of the type $\int_{-\infty}^{\infty} f(x)dx$

b) definite integrals of the Trigonometric type $\int f(\sin\theta,\ \cos\theta)d\theta$ and

c) integration round a Branch point

iii). state some theorems useful for integration.

10.3 Residue Theorem



If $f(z)$ is analytic inside and on a closed curve C except at a finite number of poles a_1, a_2 ……a_n inside C at which the residues are R_1, R_2…R_n respectively, then

$$\int_c f(z)dz = 2\pi i(R_1 + R_2 + \ldots R_n)$$

$$= 2\pi i \sum \text{ Residues}$$

Proof

Let z_i be one pole inside C. Draw a small circle C_i such that C_1 is inside C and no other poles inside the circle C_i.

According to extension of Cauchy's fundamental theorem for the multiply connected region

$$\int_c f(z)dz = \int_{c_i} f(z)dz$$

$$\int_c f(z)dz = \int_{c_i} f(z)dz = 2\pi i R_1$$

In the same way taking $i = 1, 2, \ldots n$ we have

$$\int_c f(z)dz = 2\pi i(R_1 + R_2 + \ldots R_n)$$

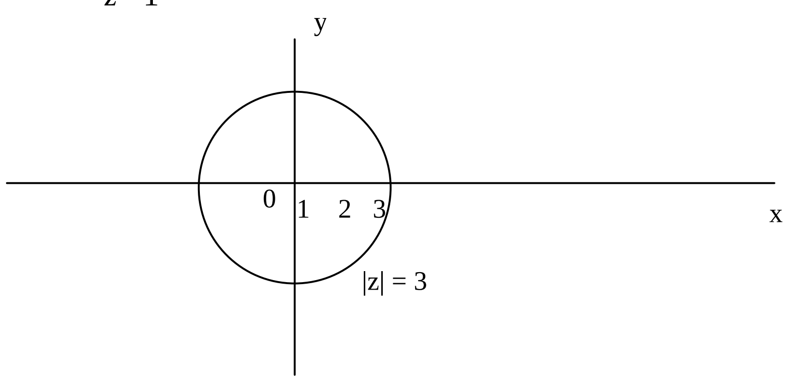
$$= 2\pi i \text{ (sum of Residues)}$$

Example 1

Evaluate $\int_c \frac{2z+3}{z-1} dz$ around the circle $|z| = 3$

Solution

The integrand $\frac{2z+3}{z-1}$ has a pole at z = 1. (put the denominator = 0 and solve for z).



This pole z = 1 + 0i is inside the circle with center origin and radius 3 units. The Residue at z = 1 is given by

$$\lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{(z-1)(2z+3)}{z-1}$$

$$= \lim_{z\to 1}(2z+3) = 5$$

There is only one pole and one Residue = 5. Hence by Residue Theorem,

$$\int_c \frac{(2z+3)}{z-1} dz = 2\pi i \text{ (sum of residues) where C is } |z| = 3$$

$$= 2\pi i(5)$$

$$= 10\pi i$$

Example 2

Evaluate $\int_c \frac{e^z}{(z-2)(z-4)} dz$

when

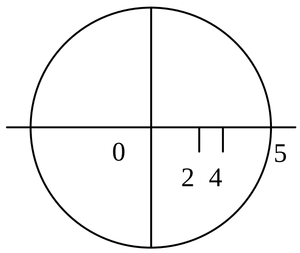
i). c is the circle |z| = 5

ii). c is the circle |z| = 3

iii). c is the circle |z| = 1

Solution

i). The integrand $\dfrac{e^z}{(z-2)(z-4)}$ has two simple poles at z = 2 and z = 4 inside |z| = 5 (since the denominator (z – 2) (z – 4) = 0 gives z = 2 and z = 4).



Residue at z = 2 is

$$\lim_{z\to 2}\frac{(z-2)e^z}{(z-2)(z-4)} = \lim_{z\to 2}\frac{e^z}{(z-4)}$$

$$= \frac{e^2}{2-4} = \frac{e^2}{-2}$$

Residue at z = 4 is

$$\lim_{z\to 4}\frac{(z-4)e^z}{(z-2)(z-4)} = \lim_{z\to 4}\frac{e^z}{z-2} = \frac{e^4}{2}$$

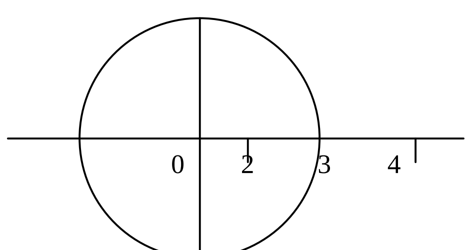
Sum of Residues = $\dfrac{e^2}{2} + \dfrac{e^4}{2} = \dfrac{e^4 - e^2}{2}$

Hence $\displaystyle\int_c \frac{e^z}{(z-2)(z-4)}dz$ = 2πi (sum of residues)

$$= 2\pi i\frac{(e^4 - e^2)}{2}$$

(if c is |z| = 5) $= \pi i\,(e^4 - e^2)$

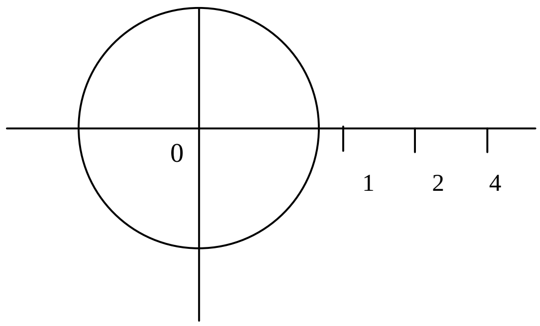
ii). If |z| = 3, i.e. c is the circle with center origin and radius 3 units



there is only one pole z = 2 inside |z| = 3 and the corresponding Residue at z = 2 is $\frac{e^2}{-2}$.

Hence $\int_c \frac{e^z\ dz}{(z-2)(z-4)} = \frac{2\pi i e^2}{-2} = -\pi i e^2$ if c is |z| = 3

iii). If |z| = 1 or c is the circle with center origin and radius 1 unit,



there is no pole inside the circle |z| = 1. Since both the poles z =2 and z =4 are outside the circle.

Hence by Cauchy's fundamental Theorem

$$\int_c \frac{e^z\ dz}{(z-2)(z-4)} = 0$$; (since no pole no Residue)

10.4 Application of Residue theorem for various Types of Integrals

Using Cauchy's Residue Theorem we can evaluate the following types of Integrals.

i). Improper Real integrals of the type $\int_{-\infty}^{\infty} f(x)dx$**, provided the integral is convergent.**

ii). Definite integrals of the Trigonometric functions of the type $\int f(\sin\theta,\ \cos\theta)d\theta$

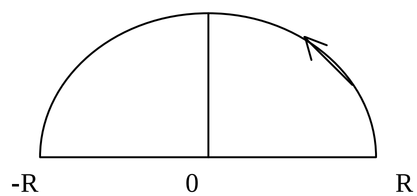
iii). Integration round a branch point.

We shall consider some examples in each of the three types of Integrals.

10.5 Improper integrals of the type

$$\int_{-\infty}^{\infty} f(x)dx \quad \text{and} \quad \int_{0}^{\infty} g(x)dx$$

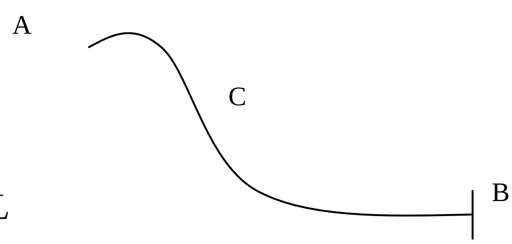
Some of the integrals of the above type can be evaluated without the help of complex integration, but the harder type of such integrals can be evaluated using Cauchy's Residue Theorem. We consider the curve C, in this case, as a semi-circle with center origin and radius R units and finally we take the limit R → ∞



10.6 Some important Theorems

Theorem 1
If *f(z)* is analytic then it is bounded. Hence | *f(z)* | ≤ M where M is an upper bound of | *f(z)* |

Theorem 2
If *f(z)* is analytic

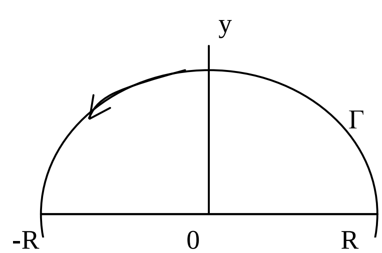


then $\left|\int_c f(z)dz\right| \leq ML$

where M is the upper bound of *f(z)* on the curve c and L is the length of the curve c from A to B.

Theorem 3

If $\left|f(z)dz\right| \leq \dfrac{M}{R^k}$ then



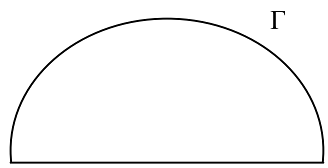
$$\lim_{R\to\infty}\int_{\Gamma} f(z) = 0 \text{ if k > 1}$$

where Γ is a semi circular arc whose center is origin and radius R units.

Proof

From Theorem (2) $\left|\int_{c} f(z)dz\right| \leq ML$

For a semicircle



The length of the arc $= \dfrac{2\pi R}{2} = \pi R$

Hence $\left|\int_{\Gamma} f(z)dz\right| \leq \dfrac{M}{R^k}.\pi R$

$$\leq \frac{M\pi}{R^{k-1}}$$

if k – 1 > 0 or k > 1 then Mπ is finite and $R^{k-1} \to \infty$

Hence $\dfrac{M\pi}{R^{k-1}} \to 0$ and it cannot be negative.

Then $\left|\int_{\Gamma} f(z)dz\right| = 0$

The following examples illustrate the evaluation of the improper integrals of real variables.

Example 3

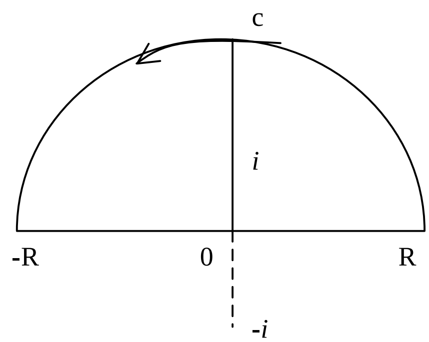
Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$

Solution

i). We consider the integral

$$\int_c \frac{dz}{z^2+1}$$ and

ii). The curve c is a semicircle with center at origin and Radius R (where R $\rightarrow \infty$)



iii). The poles of $f(z)=\dfrac{1}{z^2+1}$ are obtained by solving $z^2 + 1 = 0$ or $z^2 = -1$. Hence $z=\sqrt{-1}=\pm i$ are the two simple poles inside the circle |z| = R within the semicircle there is only one pole at z = i, the other being outside the semi-circle.

The residue at z =i for $f(z)=\dfrac{1}{z^2+1}$ is given by $\lim\limits_{z\to i}\dfrac{(z-i)}{z^2+1}=\lim\limits_{z\to i}\dfrac{(z-i)}{(z+i)(z-i)}=\dfrac{1}{2i}$

Now $\int_c f(z)dz=\int_{AB} f(z)dz+\int_{\Gamma} f(z)dz=2\pi iR_1$ where AB is on the x axis and Γ is the semicircular arc.

On the line AB, z = x+ iy becomes z = x since y = 0 on the x axis.
On the semicircle $z=R\,e^{i\theta}$, $dz=R\,e^{i\theta}id\theta$

$$\text{Hence } \int_c f(z)dz=\int_{-R}^{R} f(x)dx+\int_c \frac{1}{z^2+1}dz$$

$$=\int_{-R}^{R}\frac{1}{x^2+1}dx+\int_{\theta=0}^{\pi}\frac{R\;e^{i\theta}id\theta}{\left(R\,e^{i\theta}\right)^2+1}$$

$$=\int_{-\infty}^{\infty}\frac{1}{x^2+1}dx+0 \text{ when } R\rightarrow\infty$$

Since $\int \frac{R\,e^{i\theta}\,id\theta}{R^2\,e^{2i\theta}+1} = \int \frac{e^{i\theta}\,id\theta}{R\,e^{2i\theta}+0}$ when R $\to \infty$
= 0 when R $\to \infty$.

$$\int_C f(z)dz = \int_{-\infty}^{\infty} \frac{1}{x^2+1}dx = 2\pi i \text{ (sum of Residues)}$$

or $\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = 2\pi i\left(\frac{1}{2i}\right) = \pi$

Since the integrand $\frac{1}{x^2+1}$ is even function

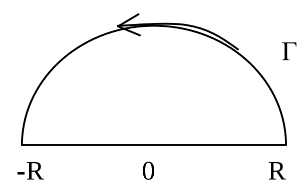
$$2\int_0^{\infty} \frac{dx}{x^2+1} = \pi$$

or $\int_0^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{2}$.

Example 4

Prove that $\int_0^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$

Solution



Consider $\int_c \frac{dz}{z^4+1}$ where c is the closed contour, consisting of the x axis from – R to R and the semi circle Γ traversed in the anticlockwise direction.

The poles of the integrand is obtained by solving
$z^4 + 1 = 0$ or $z^4 = -1$
or $z^4 = \cos(2r+1)\pi + i\sin(2r+1)\pi$ r = 0, 1, 2, 3
or $z^4 = e^{(2r+1)\pi i}$

Hence $z = e^{\frac{(2r+1)\pi i}{4}}$ r = 0, 1, 2, 3

or $z = e^{\frac{\pi i}{4}}$, $e^{\frac{3\pi i}{4}}$, $e^{\frac{5\pi i}{4}}$, $e^{\frac{7\pi i}{4}}$ only the poles $e^{\frac{\pi i}{4}}$ and $e^{\frac{3\pi i}{4}}$, lie within c, other two are below the x axis.

Residue at $z = e^{\frac{\pi i}{4}}$ $= \lim\limits_{z \to e^{\frac{\pi i}{4}}} \left[\dfrac{\left(z - e^{\frac{\pi i}{4}} \right)}{z^4 + 1} \right]$

$$= \lim_{z \to e^{\frac{\pi i}{4}}} \left[\frac{1}{4z^3} \right] \text{ using L' Hospital Rule}$$

$$= \frac{1}{4} e^{\frac{-3\pi i}{4}}$$

Residue at $z = e^{\frac{3\pi i}{4}}$ $= \lim\limits_{z \to e^{\frac{3\pi i}{4}}} \left[\dfrac{z - e^{\frac{3\pi i}{4}}}{z^4 + 1} \right]$

$$= \lim_{z \to e^{\frac{3\pi i}{4}}} \left[\frac{1}{4z^3} \right] \text{ using L' Hospital Rule}$$

$$= \frac{1}{4} e^{\frac{-9\pi i}{4}}$$

Thus $\displaystyle\int_c \frac{dz}{z^4 + 1} = 2\pi i \left[\frac{1}{4} e^{\frac{-3\pi i}{4}} + \frac{1}{4} e^{\frac{-9\pi i}{4}} \right]$

$$= \frac{\pi i}{2} \left[\cos\frac{3\pi}{4} - i \sin\frac{3\pi}{4} + \cos\frac{9\pi}{4} + i \sin\frac{9\pi}{4} \right]$$

$$= \frac{\pi i}{2} [\cos 135 - i \sin 135 + \cos 405 + i \sin 405]$$

$$= \frac{\pi i}{2} \left[-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right]$$

$$= \frac{\pi i}{2} \left(\frac{-2i}{\sqrt{2}} \right)$$

$$= \frac{\pi}{\sqrt{2}}$$

10.7 Definite Integrals of the Trigonometric functions of the type $\int f(\sin\theta, \cos\theta)\,d\theta$

Consider a real integral $\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta$ (1)

The evaluation of such integrals as (1) can be reduced to the calculation of a rational function of z along the **Unit Circle** |z| = 1.

Since rational functions have no singularities other than poles, the Residue theorem provides a simple means for evaluating integrals of the form (1).

We set $z = e^{i\theta}$ so that $dz = e^{i\theta} i\,d\theta$

If $z = e^{i\theta} = \cos\theta + i\sin\theta$

then $\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$

Hence $z + \frac{1}{z} = 2\cos\theta$

and $z - \frac{1}{z} = 2i\sin\theta$

Thus we substitute

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ and

$dz = e^{i\theta} i\,d\theta$.

Thus we have

$$\boxed{\begin{aligned} \cos\theta &= \frac{z + \frac{1}{z}}{2} \\ \sin\theta &= \frac{z - \frac{1}{z}}{2i} \\ d\theta &= \frac{dz}{iz} \end{aligned}}$$

in (1) we have

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta = \int_c \phi(z)dz = 2\pi i \sum R$$

where c is the unit circle.

The method of evaluating the integrals of the type (1) is illustrate in the following examples.

Example 5
Use residue theorem to how that

$$\int \frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{3}$$

Solution
We put

$$z = \cos\theta + i\sin\theta = e^{i\theta}$$

$z^{-1} = \cos\theta - i\sin\theta = e^{-i\theta}$ and take the contour |z| = 1

$$dz = ie^{i\theta}d\theta \quad \text{or} \quad d\theta = \frac{dz}{iz}$$

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_c \frac{dz}{iz\left[5+4(z-z^{-1}).\frac{1}{2i}\right]}$$

$$= \int_c \frac{2idz}{iz(10i+4z-4z^{-1})}$$

$$= \int_c \frac{2dz}{4z^2+10iz-4}$$

where c is a circle with center origin and radius 1 unit.

The poles of $= \int_c \frac{2}{4z^2+10iz-4}$ are obtained by solving $4z^2+10iz-4=0$ and are given by

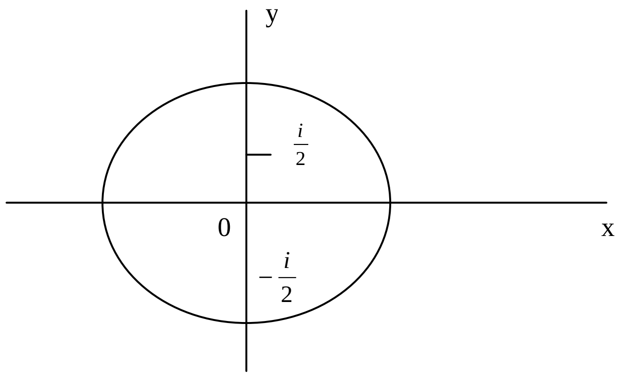
$$z = \frac{-10i \pm \sqrt{-100+64}}{8}$$

$$= \frac{-10i \pm \sqrt{-36}}{8}$$

$$= \frac{-10i \pm 6i}{8} = \frac{-5i \pm 3i}{4}$$

$$= -\frac{i}{2} \text{ and } -2i$$

only $-\frac{i}{2}$ lies inside the unit circle |z| = 1



Residue at $z = -\frac{i}{2}$ is $\lim_{z \to \frac{-i}{2}} \frac{\left(z + \frac{i}{2}\right)2}{4z^2 + 10iz - 4}$

$$= \lim_{z \to -\frac{i}{2}} \frac{2}{(8z + 10i)} \text{ Using L' Hospital Rule}$$

$$= \frac{2}{8\left(-\frac{i}{2}\right) + 10i} = \frac{2}{-4i + 10i} = \frac{1}{3i}$$

Thus $\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} = \int_c \frac{2dz}{4z^2 + 10iz - 4} = 2\pi i \sum R$

$$= 2\pi i \left(\frac{1}{3i}\right)$$

$$= \frac{2\pi}{3}$$

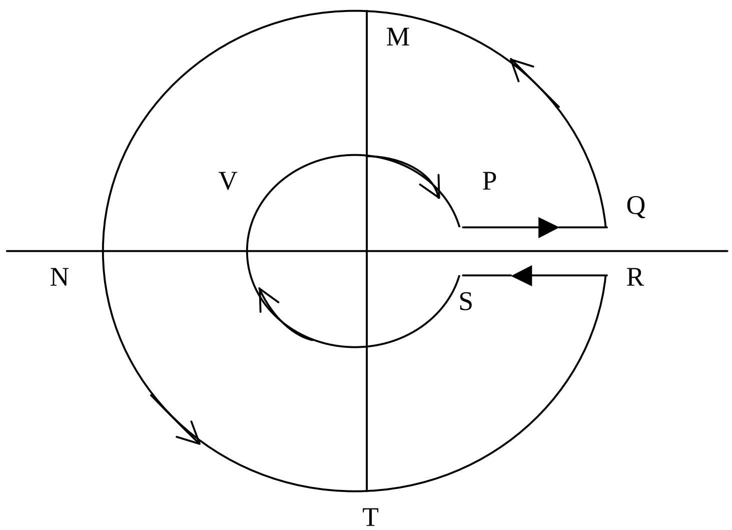
10.8 Integration round a Branch point

We have already studied the definition of the branch points and we now consider an example involving Branch points and Branch cuts. We shall evaluate the improper real integral,

$$\int_0^\infty \frac{x^{p-1}}{1+x}dx \qquad (1)$$

where p is a positive proper fraction or 0 < p < 1.

To evaluate the integral (1) we consider $\int \frac{z^{p-1}}{1+z}dz$ since z = 0 is a branch point, we can choose c as the contour where the positive x axis is the branch line PQ and RS coincident with x axis but shown separated



The integrand $\frac{z^{p-1}}{1+z}dz$ has a simple pole at $z = -1 = \cos\pi i + i\sin\pi i = e^{\pi i}$

Residue R at z = -1 is given by

$$R = \lim_{z \to e^{\pi i}} \frac{(z+1)z^{p-1}}{z+1} = \left(e^{\pi i}\right)^{p-1}$$

Then $\int_c \frac{z^{p-1}}{1+z} = 2\pi i$ (sum of Residues)

$$= 2\pi i\left(e^{\pi i}\right)^{p-1}$$

or $\int_c \frac{z^{p-1}}{1+z}dz = \int_{PQ} + \int_{QMNTR} + \int_{RS} + \int_{SVP} \qquad (3)$

Any point z, on PQ is given by z = x + iy = x

Any point z, on the circle QMNTR is given by $z = R\,e^{i\theta}$ where R is the radius of the outer circle.

Any point on RS is given by z = x + iy = x and the points on the circle SVP is given by $z = \in e^{i\theta}$ where $\in$ is the radius of the inner circle.

Hence (3) becomes

$$\int_c \frac{z^{p-1}}{1+z}dz = \int_{\in}^{R} \frac{x^{p-1}}{1+x}dx + \int_0^{2\pi} \frac{\left(R\,e^{i\theta}\right)^{p-1} R\,e^{i\theta}\,id\theta}{1+R\,e^{i\theta}} + \int_R^{\in} \frac{\left(xe^{2\pi i}\right)^{p-1}}{1+xe^{2\pi i}}dx + \int_{2\pi}^{0} \frac{\left(\in e^{i\theta}\right)^{p-1} \in e^{i\theta}\,id\theta}{1+\in e^{i\theta}}d\theta$$

Taking the limit $\in \to 0$ and $R \to \infty$ we have the second and the fourth integral tend to zero.

Hence using Residue Theorem;

$$\int_c \frac{z^{p-1}}{1+zdz} = \int_0^{\infty} \frac{x^{p-1}}{1+x}dx + \int_{\infty}^{0} \frac{e^{2\pi i(p-1)}}{1+x}dx = 2\pi e^{(p-1)\pi i}$$

or $$\int_0^{\infty} \frac{x^{p-1}}{1+x}dx - \int_0^{\infty} \frac{e^{2\pi i(p-1)}x^{p-1}}{1+x}dx = 2\pi e^{(p-1)\pi i}$$

or $$\int_0^{\infty} \frac{x^{p-1}}{1+x}dx - e^{2\pi i(p-1)}\int_0^{\infty} \frac{x^{p-1}}{1+x}dx = 2\pi e^{(p-1)\pi i}$$

or $$\int_0^{\infty} \frac{x^{p-1}}{1+x}dx\left[1-e^{2\pi i(p-1)}\right] = 2\pi\,e^{(p-1)\pi i}$$

or $$\int_0^{\infty} \frac{x^{p-1}}{1+x}dx = \frac{2\pi i\,e^{(p-1)\pi i}}{1 - e^{2\pi i(p-1)}}$$

Dividing the Numerator and denominator of the Right Hand side by $e^{(p-1)\pi i}$ we have

$$\int_0^{\infty} \frac{x^{p-1}dx}{1+x} = \frac{2\pi\,i}{e^{-(p-1)\pi i} - e^{(p-1)\pi i}}$$

$$= \frac{2\pi i}{e^{-p\pi i}.e^{\pi i} - e^{p\pi i}.e^{-\pi i}}$$

$$= \frac{2\pi i}{e^{-p\pi i}(-1) - e^{p\pi i}(-1)}$$

$$= \frac{2\pi i}{e^{p\pi i} - e^{-p\pi i}}$$

$$= \frac{2\pi i}{2i \sin p\pi}$$

$$= \frac{\pi}{\sin p\pi}$$

Summary

You have learnt the following from this Lesson: -

i) Statement of Cauchy'c Residue Theorem namely:
If f(z) is analytic inside and on a closed curve C except at a finite number of poles at a_1, a_2, … a_n inside C at which the Residues are R_1, R_2,… R_n respectively, then,

$$\int_C f(z)dz = 2\pi i(R_1 + R_2 + \ldots R_n)$$

ii) To apply Residue Theorem for the evaluation of

a). Improper integrals of the type

$$\int_{-\infty}^{\infty} f(x)dx$$

b). Definite integrals of the Trigonometric type $\int f(\sin\theta, \cos\theta)d\theta$

c). Integration round a branch point of the type $\int_0^{\infty} \frac{x^{p-1}}{1+x}dx$ when 0 < p <1

Exercise 10

1. State the Residue Theorem
2. a). Find the poles of $f(z) = \dfrac{z+2}{(z-1)(z-3)}$

   b). Determine the Residues of *f(z)* at its poles.

   c). What is the sum of the Residues of *f(z)*?

   d). Evaluate $\int_c \frac{z+2}{(z-1)(z-3)}dz$ where c is the circle |z| = 4

3. a). Find the poles of the function

   $$f(z) = \frac{2z+3}{z^2+25}$$

b). Determine the Residues of *f(z)* at its poles
c). State the sum of the residues at its poles
d). Evaluate $\int_c \frac{2z+3}{z^2+25} dz$ where c is the circle |z| = 6

4. a). Find the poles $f(z) = \frac{1}{z^4+16}$
b). Determine the poles of *f(z)*
c). Calculate the residues of *f(z)* at its poles.
d). Evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^4+16}$

5. a). Find the poles and the residues at the poles for the function $f(z) = \frac{z}{(z+1)(z+2)^2}$

b). Evaluate $\int_c \frac{z}{(z+1)(z+2)^2} dz$ when c is the circle |z| = 2.

6. Evaluate $\int_0^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2}$

7. Evaluate $\int_0^{\infty} \frac{dx}{x^6+1}$

8. Show that $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}$

(Hint: The two poles are z = i of orders 2 and z = -1 + i of order I. The respective residues are $\frac{9i-12}{100}$ and $\frac{3-4i}{25}$).

9. Show that $\int \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$

10. Show that $\int \frac{dx}{(1+x^2)^3} = \frac{3\pi}{8}$

11. Show that $\int_0^{\infty} \frac{x^6}{(x^4+1)^2} = \frac{3\sqrt{2}}{16}\pi$

12. Show that $\int_0^{\infty} \frac{dx}{x^3+1} = \frac{\pi}{3}$

13. a). Show that $I = \int_0^{2\pi} \frac{d\theta}{13+5\cos\theta}$ is reduced to $I = \frac{2}{i}\int_c \frac{dz}{5z^2+26z+5}$ (where c is the unit circle |z| = 1). Using the substitution $z = e^{i\theta}$.

b). Hence evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\cos\theta}$

14. Evaluate $\int_0^{2\pi} \frac{d\theta}{13+12\cos\theta}$

(Hint: Put $z = e^{i\theta}$, the integral becomes $I = \frac{2}{i}\int \frac{dz}{12z^2+26z+12}$)

15. Show that $\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta} = \frac{\pi}{2}$

16. Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$

17. Prove that $\int \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$

18. Prove that $\int \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$, $a > b$.

19. Show that $\int \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$, $0 < p < 1$.

20. Show that $\int_0^{\infty} \frac{\ln(x^2+1)}{x^2+1} \pi \ln 2$

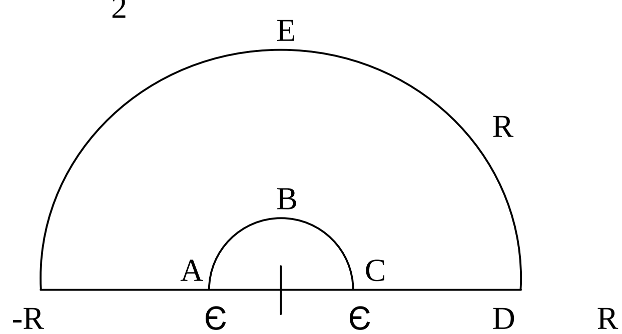
(Hint: Take the contour as the semi circle with center origin and radius $R \to \infty$. The pole is z = i and the residue is $\frac{\ln 2i}{(2i)} = \ln 2 + \ln i = \ln 2 + \ln e^{\frac{\pi i}{2}} = \ln 2 + \frac{\pi i}{2}$).

21. Prove that $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta = \frac{\pi}{2}$

(Hint: Put $z = e^{i\theta}$, $\cos 3\theta = \frac{e^{i3\theta}+e^{-i3\theta}}{2} = \frac{z^3+z^{-3}}{2}$ and $I\ z\ d\ \theta$).

22. Show that $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$

(Hint: Consider the contour FABCDEF



$$\int_{-R}^{-\epsilon}\frac{e^{ix}}{x}dx+\int_{ABC}\frac{e^{iz}}{z}dz+\int_{\epsilon}^{R}\frac{e^{ix}}{x}dx+\int_{DEF}\frac{e^{iz}}{z}dz=0$$

23. Show that $\int_{0}^{\infty}\frac{1}{\sqrt{x}(1+x)}dx=\pi$

(Hint: Take $p=\frac{1}{2}$ in 10.8)

24. Show that $\int_{0}^{\infty}\frac{x^{\frac{-3}{4}}}{(1+x)}dx=\sqrt{2}\pi$

Further Reading

1) Complex Variables and Applications
By R.V Churchill and others
Mc Graw-Hill, KOGAKUSHA Ltd
Tokyo Singapore

2) Theory and Problem of complex variables
By Murray R. Spiegel, Ph.D,
Sehaum' out line series
Mc Graw – Hill Book Company Singapore

3) Text Bokk of Complex Analysis
By Dr. D. Sengottaiyan, Ph.D.,
Oxford Publications
London Nairobi.

COMPLEX ANALYSIS ANSWERS

Exercise one

1. -i, , i, -1, i	2. $4 + 3i$	3. 9	4. $1^2 - i^2 = 2$
5. x2 + 4y2	6. -i	7. $\frac{2+3i}{13}$	8. $\frac{4+7i}{5}$
9. - 5 +12i	10. -33 – 56i	11. 5	12. 1
13. 3150	14. 600	15. 3000	16. 8 + 10i
17. 21 + 28i	18. -14i	19. -1 + 34i	20. -3 + 102i
21. $z = \frac{-1 \pm i\sqrt{3}}{2}$	22.	23.	24.
25.	26.	27.	
28. 5 cis 143. 1^0	13 cis	5 cis 216.9^0	13 (cos + i sin)
120^0	$\frac{i\left(1+\sqrt{3}\right)}{2}$	Cos 70 + i sin 70	

Exercise 2

$5(\cos 60 + i\sin 60) = \frac{5}{2}\left(1 + i\sqrt{3}\right)$ $\quad$ $32(\cos 120 - i\sin 120) = \left(16 - 6\sqrt{3i}\right)$

$\frac{-\sqrt{3}}{2} - \frac{i}{2}$ $\quad$ $2^6\left(\cos 300 + i\sin 300\right) = 32\left(\frac{1}{2} - i\sqrt{3}\right)$

$\sqrt{2}$ cis 27, $\sqrt{2}$ cis 99, $\sqrt{2}$ cis 171, $\sqrt{2}$ cis 143

cis 60, cis 180, cis 300,

2 cis 0, $\sqrt{2}$ cis 315, 2 cis 60, 2 cis 120, 2 cis 180, 2 cis 240, 2 cis 300

3 cis 45, 3 cis 135, 3 cis 225, 3 cis 315

$\sqrt[6]{2}$ cis 40, $\sqrt[6]{2}$ cis 100, $\sqrt[6]{2}$ cis 160, $\sqrt[6]{2}$ cis 220, $\sqrt[6]{2}$ cis 280, $\sqrt[6]{2}$ cis 340

3 – 2i, -3 + 2i

$1+2i,\ \frac{1}{2}-\sqrt{3}+\left(1+\frac{\sqrt{3}}{2}\right)i, \quad -\frac{1}{2}-\sqrt{3}+\left(\frac{1}{2}\sqrt{3}-1\right)i$

$\frac{-1\pm 7i}{5}$

$1,\ 1,\ 2, \quad -1\pm i$

$\frac{-1\pm i\sqrt{3}}{2}, \qquad \frac{-1\pm\sqrt{3}}{2}$

$2+2i, \qquad 2-2i$

Exercise 3

$-12+6i \qquad \frac{\sqrt{2}}{2}(1+i)$

$-1\pm i \qquad k\pi,\ k=0,\ 1,\ 2\ \ldots$

$\pm 2, \quad \pm 2i$

Exercise 4

$v(x,y)=x^2-y^2+2y$

$v=e^x(x\sin y+y\cos y)+c$

$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \qquad z^3+3z^2+c$

$(1-2i)(\sin z+z^2)+c$

$v=\tan^{-1}\left(\frac{-x}{y}\right)+c$

Exercise 5

$e^z=e^x(\cos y+i\sin y)$

$\frac{1}{8}\pi i+\frac{1}{2}k\pi i$ where $k=0, \quad \pm 1, \quad \pm 2\ldots\ldots\ldots\ldots$

$\ln 2+(2n+1)\pi i,\ n=0,, \quad \pm 1, \quad \pm 2$

$z = \frac{2k\pi i}{3}$ $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

$si\ \ z = \frac{e^{iz} - e^{-iz}}{2i}$; $\tan z = \frac{\sin z}{\cos z}$ etc

real part: cos x cos hy, Imaginary part: - sin x sin hy

7 and 26

Exercise 6

1. See the book
2. Image is (8,8)
3. b= 2, c = 24
4. See the book
5. w = $\frac{i(1-z)}{1+z}$
6. w = $\frac{i(z+1)}{z-1}$
7. $z = -1 \pm 2i$
8. See the book
9. u = 1, 3, v = -1 - 2
10. θ becomes 2θ, the upper half of w plane
11. w = $\frac{2z-2-2i}{(i-1)z-3-5i}$
12. w = $\frac{(1-i)(z-i)}{2(z-1)}$

Exercise 7

See the text

$f(z) = z^2$, antiderivative $= \int z^2 dz = 2z$

see the text

I, $\mathrm{I}_2 = -\frac{1}{3}(5+4i)$

$\frac{1}{4}\left(e^2+1\right)$ $\frac{1}{4}\ln\left(z^3+3z^2+2\right)$

Exercise 8

1 – 4 see text

$-\frac{1}{3} - \frac{1}{9}z - \frac{x^2}{27} - \frac{7}{8}x^3 - \frac{15}{16}x^4 + \ldots$

and $z^{-1} + 3z^{-2} + 9z^{-3} + 27z^{-4} + \ldots$

$z = 2n\pi i, \quad n = \pm 1, \quad \pm 2 \ldots$ Simple poles

z = $z = \infty$ essential singularity

$$e\left[1 + 2(z-2)^{-1} + \frac{2^2(z-2)^{-2}}{2!} + \frac{2^3(z-2)^{-3}}{3!}\ldots\right]$$

$|z-2| > 0$, z = 2 essential singularity.

Exercise 9

See the text 2). Put g(z) = 0 and solve for z

3). $R = \lim_{z \to a} (z-a) f(z)$

4). $R = \lim_{z \to a} \frac{1}{3!} \frac{d^3}{dz^3} \left[(z-a)^4 f(z)\right]$ 5). z = - 1, z = 2, $R_1 = \frac{1}{3}$, $R_2 = \frac{5}{3}$

6). Pole is z = 1, Residues is 4

7). Z^4 = -1, Solving

9). Poles are – 1 and ± 2. Residues are $\frac{-14}{25}$, $\frac{7 \pm i}{25}$

10). Poles are: z = 1, z = -1 Residues $\frac{1}{4}$ and $-\frac{1}{4}$ respectively.

11). (a). $e^{\frac{\pi i}{6}}$, $e^{\frac{3\pi i}{6}}$, $e^{\frac{5\pi i}{6}}$, $e^{\frac{7\pi i}{6}}$, $e^{\frac{9\pi i}{6}}$, $e^{\frac{11\pi i}{6}}$

(b). $\frac{1}{6} e^{\frac{-5\pi i}{6}}$, $\frac{1}{6} e^{\frac{-5\pi i}{2}}$, $\frac{1}{6} e^{\frac{-25\pi i}{6}}$,...

12). $z = e^{\frac{\pi i}{2}}$, $e^{\frac{3\pi i}{2}}$, $e^{\frac{5\pi i}{2}}$

Exercise 10

1. See the book
2. a)Poles are z = 1 and z = 3.

   b). Residue at z = 1 is $\frac{3}{-2}$

   c). Residue at z = 3 is $\frac{5}{2}$

   d). Sum of Residues = 1

The value of the integral is 1

3. a). Poles are z = 5i and z = -5i

   b). Residue are $\frac{10-3i}{10}$ and $\frac{3i-10}{10}$ respectively.

   c). Sum of residues = 4 πi

   d). $-8\pi^2$.

4. a). Pole are given by

   $z_1 = 2e^{i\frac{\pi}{4}}, \qquad z_2 = 2e^{\frac{3\pi i}{4}}$

   $z_3 = 2e^{\frac{5\pi i}{4}}, \qquad z_4 = 2e^{\frac{7\pi i}{4}}$

   b). z_1 and z_2 lie in the semicircle

   Sum of Residues at z_1 and $z_2 = \frac{1}{32}\left(2i\sin\frac{\pi}{4}\right)$

   $$\int_{-\infty}^{\infty}\frac{1}{z^4+16}dz = 2\pi i\left(\frac{-\sqrt{2}i}{32}\right) = \frac{\sqrt{2}\pi}{16}$$

5. a). z= 1 and z = - 1 are the poles

   Residue at z = 1 is $\frac{1}{4}$

   Residue at z = - 1 is $-\frac{1}{4}$

   b). The value of the integration $2\pi i(0) = 0$

6. $\frac{5\pi}{288}$

7. $\frac{\pi}{3}$

13 b). $\frac{\pi}{6}$

14. $\frac{2\pi}{5}$

16. $\frac{\pi}{6}$