Curves

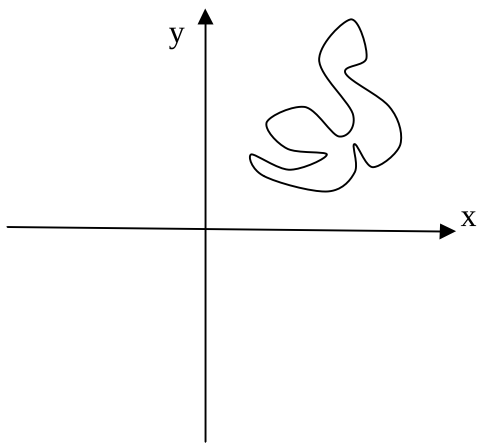
If $f(t)$ and $g(t)$ are real functions of the real variable t assumed continous in $t_1 \le t \le t_2$, the parametric equations $z = x + iy = f(t) + ig(t) = z(t), t_1 \le t \le t_2$ define a continous curve or arc in the z-plane joining $a = z(t_1)$ and $b = z(t_2)$.

If $t_1 \ne t_2$ while $z(t_1) = z(t_2)$ i.e. $a = b$, the end points coincide and the curve is said to be closed. A closed curve which does not intersect itself anywhere is called a simple closed curve.

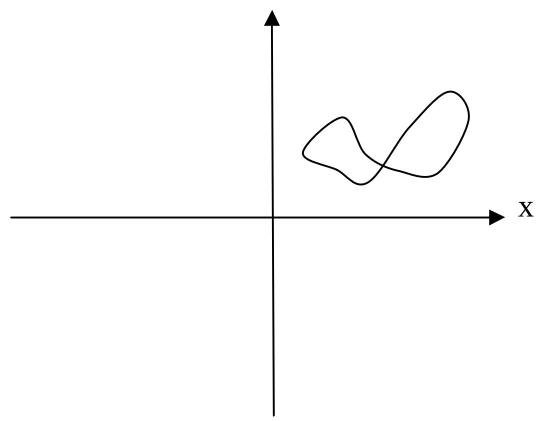
If $f(t)$ and $g(t)$ have continous derivatives in $t_1 \le t \le t_2$, the curve is often called a smooth curve or arc.

A curve which is composed of a finite number of smooth arcs is called a piece-wise or sectionally smooth curve or a contour.

e.g



simple closed curve



not simple

SIMPLY AND MULTIPLY CONNECTED REGIONS

A region R is called simply connected if any simple closed curve which lies in R can be shrunk to a point without leaving R. A region R which is not simply-connected is called multiply-connected.

Example 44

Suppose R is the region defined by $|z| < 2$ in the figure below.

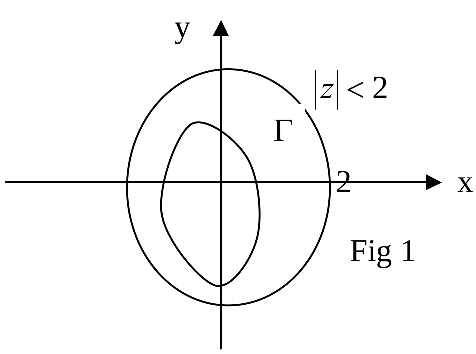


Fig 1

$$|z| < 2$$

$$\sqrt{x^2 + y^2} < 2$$

$$x^2 + y^2 < 4$$

If Γ is any simple closed curve lying in R, we see that it can be shrunk to a point which lies in R so that R is simply connected.

On the other hand, if R is the region defined by $1<|z|<2$ shown in figure 2 below, then there is a simple closed curve Γ lying in R which cannot possibly be shrunk to a point without leaving R, so that R is multiply connected.

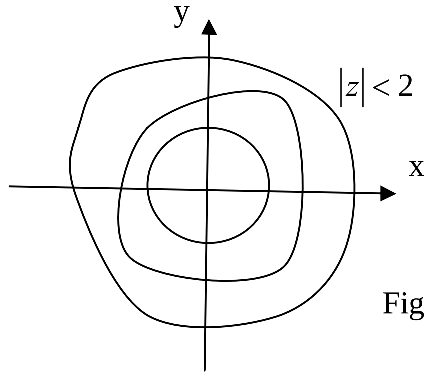
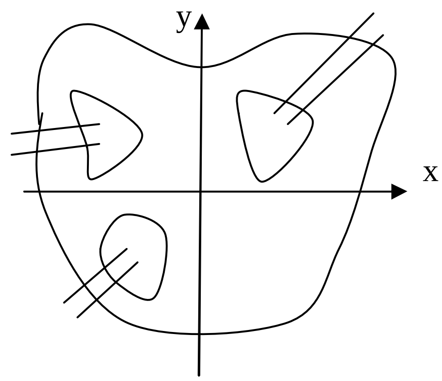


Fig 2



This can be changed to simple closed by introducing a cross cut.

Intuitively, a simply-connected region is one which does not have any "hole" in it while a multiply-connected region is one which does.

Thus the multiply connected regions of fig 2 and 3 have respectively one and three holes in them.

Note: The symbol $\oint f(x)\,dz$ is used to denote integration $f(z)$ around the boundary c in the positive sense or direction i.e. it is always the counter-clockwise direction.

CAUCHY'S THEOREM

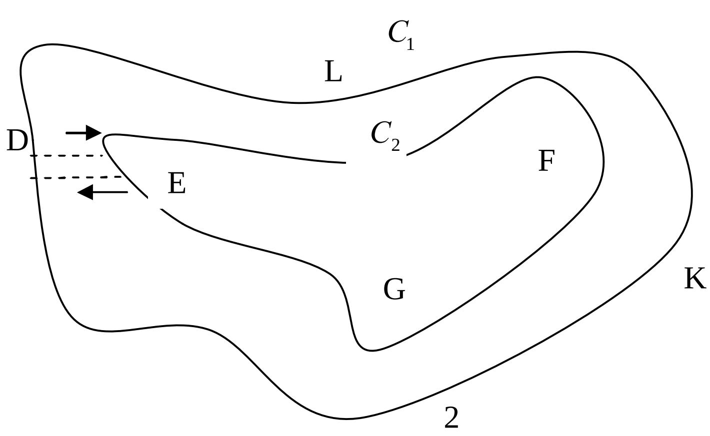
Cauchy's Fundamental theorem

Let $f(z)$ be analytic in the region R and on the boundary c. Then, $\oint f(z)\,dz = 0$. This is often called Cauchy's Integral theorem. It is valid for both simply and multiply connected regions. It is also known as Cauchy-Gourstat theorem.

e.g. If c is a simple closed contour, then $\int_c dz = 0, \int_c z\,dz = 0, \int_c z^2 dz = 0$.

Extension of Cauchy's theorem

Theorem 1: Let $f(z)$ be analytic in the region R bounded by the simple closed curves c_1 and c_2. Then, $\oint_{c_1} f(z)\,dz = \oint_{c_2} f(z)\,dz$ where c_1 and c_2 are both transveresed in the positive sense.



H J

To connect it to a simple closed curve, cross-cut DE. Then since $f(z)$ is analytic in the region R. we have by Caunchyôs theorem $\int\limits_{DEFGEDHJKLD} f(z)\,dz = 0$

Example 44

Evaluate $\oint \frac{dz}{z-a}$ where c is any simple closed curve and z=a is

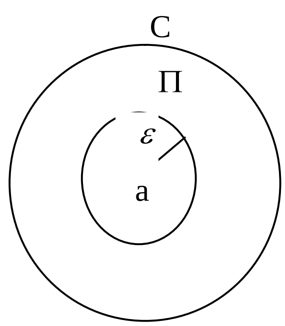
(a)outside c

(b)inside c

Solution:

(a)If a is outside c, then $f(z) = \frac{1}{z-a}$ is analytic anywhere inside and on c. Hence by Caunchyôs theorem, $\oint \frac{dz}{z-a} = 0$.

(b)Suppose a is inside c and let Γ be a circle of radius ε with centre z=a so that Γ is inside c (Caunchyôs theorem is not applicable since $f(z)$ is discontinuous at z=a)



By the above theorem;(theorem 1) $\oint \frac{dz}{z-a} = \int\limits_{\Gamma} \frac{dz}{z-a}$

On $\Gamma, |z-a| = \varepsilon$

$$\therefore z-a = \varepsilon e^{i\theta} \quad o < \theta \le 2\pi$$

$$\Rightarrow z = \varepsilon i^{i\theta} + a; \frac{dz}{d\theta} = i\varepsilon e^{i\theta}$$

$$dz = i\varepsilon e^{i\theta} d\theta$$

$$\therefore \oint\limits_{\Gamma} \frac{dz}{z-a} = \int\limits_{0}^{2\pi} \frac{i\varepsilon e^{i\theta} d\theta}{\varepsilon e^{i\theta} + a - a}$$

$$= \int\limits_{0}^{2\pi} \frac{i\varepsilon e^{i\theta} d\theta}{\varepsilon e^{i\theta}} = \int\limits_{0}^{2\pi} i\,d\theta = [i\theta]_0^{2\pi} = 2\pi i$$

$$\therefore \oint_c \frac{dz}{z-a} = \oint_\Pi \frac{dz}{z-a} = 2\pi i$$

Example 45

Find $\oint_c \frac{dz}{z^2+4}$ along the circle $c:|z|=1$

Solution:

Singular points of $f(z)=\frac{1}{z^2+4}$ occur when $z^2+4=0 \Rightarrow z=\pm 2i$

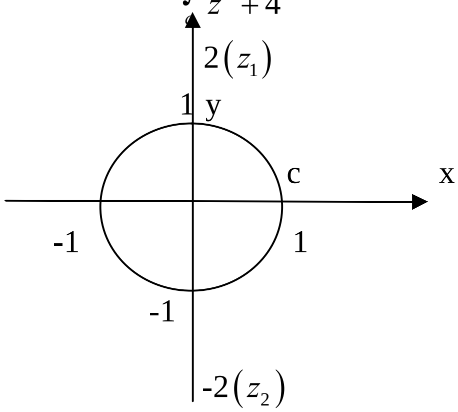
Plot the points to determine whether they are inside or outside the circle $|z|=1$.

$z_1 = 2i = o + 2i$

$z_2 = -2i = 0 - 2i$

Circle $|z|=1$ has radius 1

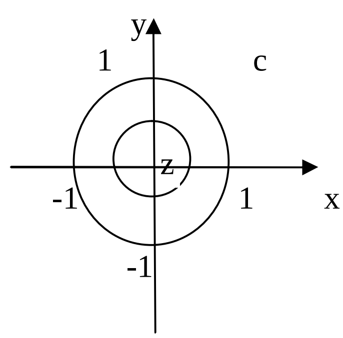
Both points are outside the circle, hence $f(z)$ is analytic on and inside c. Thus by Caunchyӧs theorem, $\oint_c \frac{dz}{z^2+4} = 0$.



Example 46

Evaluate $\oint_c \frac{dz}{z}$ along the circle $c:|z|=1$.

Solution:the singular point of $f(z)=\frac{1}{z}$ occurs at $z=0$, which is inside the circle $c:|z|=1$ of radius 1



$\therefore f(z)=\dfrac{1}{z}$ is discontinuous and therefore not inside. Cauchyôs formula therefore does not apply since at $z=0$ $f(z)$ is not analytic.

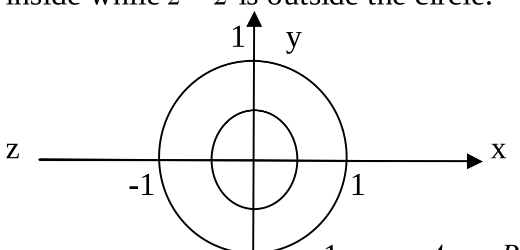
Using theorem1, $\oint \dfrac{dz}{z}=2\pi i$

(see example 45)

Example 47

Evaluate $\oint\limits_c \dfrac{dz}{z(z+2)}$ along the circle $c:|z|=1$.

Solution: Singular points of $f(z)=\dfrac{dz}{z(z+2)}$ occur when $z(z+2)=0 \Rightarrow z=0$ and $z=-2$,*z=0* is inside while *z=-2* is outside the circle.



By partial fractions, $\dfrac{1}{z(Z+2)}=\dfrac{A}{Z}+\dfrac{B}{z+2}$

$$\frac{1}{Z(z+2)}=A(z+2)+Bz$$
$$1=Az+2A+Bz$$
$$0z+1=(A+B)z+2A$$
$$A+B=0 \Rightarrow A=-B; 2A=1 \Rightarrow A=\frac{1}{2} \Rightarrow B=-\frac{1}{2}$$
$$\therefore \frac{1}{z(z+2)}=\frac{1}{2z}-\frac{1}{2(z+2)}$$
$$\therefore \oint\limits_c \frac{dz}{z(z+2)}=\int\left[\frac{1}{2z}-\frac{1}{2(z+2)}\right]^{dz}$$
$$=\int \frac{1}{2z}dz-\int\frac{1}{2(z+2)}dz$$
$$=\frac{1}{2}\int\frac{1dz}{z}-\frac{1}{2}\int\frac{1}{z+2}dz$$

$\oint\limits_c \dfrac{1}{z}dz=2\pi i$ since the singular point *z=0* is inside the circle.

$\oint \dfrac{1}{z+2}dz=0$ since the singular point *z=-2* is not inside *c*.

$$\therefore \oint_c \frac{dz}{z(z+2)} = \frac{1}{2}(2\pi i) - \frac{1}{2}(0) = \pi i$$

Exercise 21

1. Find $\oint \frac{dz}{z^2+1}$ along $c: |z| = \frac{1}{2}$.
2. Find $\oint \frac{dz}{(z+1)(z+2)}$ along $c: |z| = 1$.