RESIDUES

Let $f(z)$ be single-valued and analytic inside and on c except at the point $z=a$ chosen as the centre of c.Then the Laurent series about $z=a$ is given by

$$f(z)=\sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

$$=a_0+a_1(z-a)+a_2(z-a)^2+\ldots+\frac{a-1}{(z-a)}+\frac{a-2}{(z-a)^2}+\ldots(1)$$

Where $a_n=\frac{1}{2\pi i}\oint_6 \frac{f(z)}{(z-a)^{n+1}}dz, n=0,\pm1,\pm2$ é (2)

In the special case $n=-1$,we have from equation (2) $a-1=\frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-a)^0}dz$ and so

$$a-1=\frac{1}{2\pi i}\oint_c f(z)dz$$

$$\Rightarrow \oint_c f(z)dz=2\pi i a_{-1}$$ é (3)

Calculation of residues

To obtain the residue of a function $f(z)$ at $z=a$,it may appear from equation (1)that the Laurent expansion of $f(z)$ about $z=a$ must be obtained. However, in the case where $z=a$ is a pole of order *k*,there is a simple formular for a_{-1} given by $a_{-1}=\lim_{z\to a}\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left\{(z-a)^k f(z)\right\}$

é (4)

If *k=1*(Simple pole)

$a_{-1}=\lim_{z\to a}(z-a)f(z)$where $0!=1$

Example 54

Find the poles of $f(z)=\frac{z^3+5z+1}{z-2}$ and the corresponding residue.

Solution:

Pole: $z-2=0\Rightarrow z=2$

Residue at *z=2* is given by $\lim_{z\to 2}(z-2)\frac{z^3+5z+1}{(z-2)}=8+10+1=19$

Example 55

Find the residue of $f(z)=\frac{z}{(z-1)(z+1)^2}$ at the poles.

Solution:

Poles occur at $(z-1)(z+1)^2 = 0 \Rightarrow z=1$ (order 1) and $(z=-1)$ order 2 at

$$z=1, \lim_{z\to1}(z-1)\frac{z}{(z-1)(z+1)^2} = \frac{1}{4}$$

At $z=-1, \lim_{z\to-1}\frac{1}{1!}\frac{d}{dz}\left\{(z+1)^2\frac{z}{(z-1)(z+1)^2}\right\}$

$$= \lim_{z\to-1}\frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= \lim_{z\to-1}\frac{z-(z-1)}{(z-1)^2} = \lim_{z\to-1}\frac{1}{(z-1)^2} = \frac{1}{4}$$

Example 56

Confirm the answer to example 55 by using Laurent Series.

Solution:

$$f(z) = \frac{z^3+5z+1}{z-2}$$

Pole occurs at $z=2$

Laurent expansion around $z=2$

Let $z-2=u \Rightarrow z=u+2$

$$f(z) = \frac{(u+2)^3+5(u+2)+1}{u}$$

$$= \frac{u^3+3u^2(2)+3(2)^2u+8+5u+10+1}{u}$$

$$= \frac{u^3+6u^2+12u+8+5u+11}{u}$$

$$= \frac{u^3+6u^2+17u+19}{u}$$

$$= u^2+\frac{19}{u}+6u+17$$

$$\therefore f(z) = (z-2)^2+\frac{19}{(z-2)}+6(z-2)+17$$

$\therefore$ co-efficient of $\frac{1}{z-2}$ is the residue at $z=2$,which is 19.

Example 57

Find the residues of $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)}$ at all its poles.

Solution:

Poles: $z=-1-$ order 2
$z=\pm 2i-$ order 1

At $z=-1$, $\lim\limits_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left\{\frac{(z+1)^2(z^2-2z)}{(z+1)^2(z^2+4)}\right\}$

$$=\lim_{z\to -1}\frac{z^2-2z}{z^2+4}$$

$$=\lim_{z\to -1}\frac{(z^2+4)(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2}$$

$$=\frac{(5)(-4)-(3)(-2)}{25}=-\frac{14}{25}$$

At $z=2i$,

$$\lim_{z\to 2i}\frac{(z-2i)(z^2-2z)}{(z+1)^2(z-2i)(z+2i)}=\lim_{z\to 2i}\frac{(z^2-2z)}{(z+1)^2(z+2i)}$$

$$=\frac{(-4-4i)}{(2i+1)^2(4i)}$$

$$=\frac{(-4-4i)}{(-4+4i+1)(4i)}$$

$$=\frac{-4-4i}{(-3+4i)(4i)}$$

$$=\frac{4(-1-i)}{4i(-3+4i)}=\frac{1(-1-i)}{i(-3+4i)}$$

$$=\frac{-(1+i)}{-3i-4}=\frac{-(1+i)}{-(3i+4)}=\frac{1+i}{3i+4}$$

$$=\frac{(1+i)(4-3i)}{(4+3i)(4-3i)}=\frac{4+i+3}{16+9}$$

$$=\frac{7+i}{25}$$

At *z=-2i*, (order 1)

$$\lim_{z\to -2i}\left\{\frac{(z+2i)(z^2-2z)}{(z+1)^2(z+2i)(z-2i)}\right\}$$

$$\text{Res }=\lim_{z\to -2i}\frac{z^2-2z}{(z+1)^2(z-2i)}$$

$$=\frac{(-2i)^2-2(-2i)}{(-2i+1)^2(-4i)}$$

$$=\frac{-4+4i}{(-4-4i+1)(-4i)}=\frac{-4+4i}{(-3-4i)(-4i)}$$

$$=\frac{4(-1+i)}{(-3-4i)(-4i)}=\frac{(-1+i)}{(3i-4)}$$

$$=\frac{(-1+i)(-4-3i)}{(3i-4)(-4-3i)}=\frac{4+3i-4i+3}{25} \quad =\frac{7-i}{25}$$

At $z=-2i \quad (\text{order } 1, \Rightarrow k=1)$

$$\text{Res}=\lim_{z\to -2i}\frac{(z+2i)(z^2-2z)}{(z+1)^2(z+2i)(z-2i)}$$

$$=\lim_{z\to -2i}\frac{(z^2-2z)}{(z+1)^2(z-2i)}$$

$$=\frac{(-2i)^2-2(-2i)}{(-2i+1)^2(-2i-2i)}$$

$$=\frac{-4+4i}{(-4-4i+1)(-4i)}$$

$$=\frac{4i-4}{(-4i-3)(-4i)}=\frac{4(i-1)}{-4i(-4i-3)}$$

$$=\frac{(i-1)}{(-4+3i)}\times\frac{(-4-3i)}{(-4-3i)}$$

$$=\frac{-4i+3+4+3i}{(-4)^2+(3)^2}$$

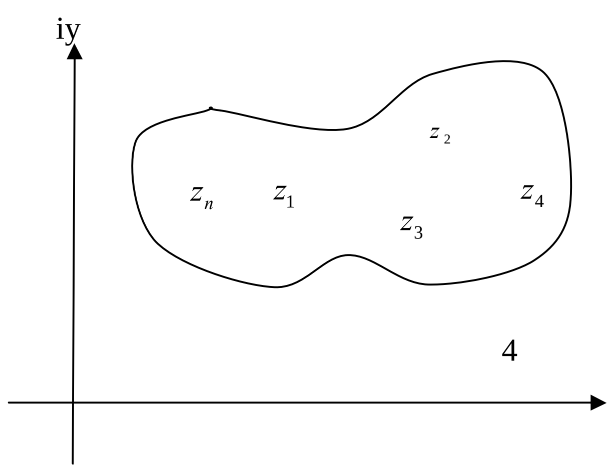
$$=\frac{i+7}{25}$$

THE RESIDUE THEOREM

Let $f(z)$ be a simple closed contour within and on which $f(z)$ is analytic except for a finite number of singular points $z_1, z_2, \ldots z_n$ interior to c.

If $\beta_1, \beta_2, \ldots, \beta_n$ denote the residues of $f(z)$ at these points, then

$\oint_c f(z)\,dz = 2\pi i(\beta_1+\beta_2+\ldots+\beta_n)$ where c is described in the positive sense.



4

Examples

1. Evaluate $\int \frac{2z+3}{z-1} dz$ around the circle $|z|=3$.

Solution:Pole: $z=1$-order 1

Residue at $z=1$ is given by $\lim_{z\to 1} \frac{(z-1)(2z+3)}{(z-1)}$

$$= \lim_{z\to 1}(2z+3) = 5$$

$$\therefore \oint_c \frac{2z+3}{z-1} = 2\pi i(5) = 10\pi i$$

OR

By Cauchy,s integral formula

$$f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}; z=1, f(z) = 2z+3$$

$$\Rightarrow f(1) = 5$$

$$\Rightarrow f(1) = \frac{1}{2\pi i}\oint_c \frac{2z+3}{z-1} dz$$

$$5 = \frac{1}{2\pi i}\oint_c \frac{2z+3}{z-1} dz \Rightarrow \oint_c \frac{2z+3}{z-1} dz = 10\pi i$$

2. Evaluate $\oint \frac{e^z}{(z-2)(z-4)}$ when

(a) $|z|=5$ (b) $|z|=3$ (c) $|z|=1$

Solution:

(a) $|z|=5$

Poles are at $z=5$ and $z=4$,therefore they are inside c.

At $z=2$,residue $= \lim_{z\to 2} \frac{(z-2)e^z}{(z-2)(z-4)} = \frac{e^z}{(z-4)} = \frac{e^2}{-2} = B_1$

At $z=4$,residue $= \lim_{z\to 4} = \frac{(z+4)e^z}{(z-2)(z+4)} = \frac{e^z}{z-2} = \frac{e^4}{2} = B_2$

$$\int_c \frac{e^z}{(z-2)(z-4)} dz = 2\pi i\left(-\frac{e^2}{2} + \frac{e^4}{2}\right) = \pi i\left(e^4 - e^2\right)$$

(b) $|z|=3$

Pole $z=2$ is inside while pole $z=4$ is outside.

Pole $z=4$ is outside $\Rightarrow$ residue at $z=4$ is 0.

Pole $z=2$ is inside $\Rightarrow$ residue at $z=2$ is $-\frac{e^2}{2}$ (see (a) above)

$$\therefore \oint_c \frac{e^z}{(z-2)(z-4)}dz = 2\pi i\left(0+-\frac{e^2}{2}\right) = 2\pi i\left(-\frac{e^2}{2}\right) = -\pi i e^2$$

(c) $|z|=1$

$\therefore$ Both poles $z=2$ and $z=4$ are outside c.

$\therefore$ By Cauchyôs Theorem, $\int \frac{e^z}{(z-2)(z-4)}dz = 0$

3. Show that $\oint_c \frac{3z^2+2}{(z-1)(-z^2+9)}dz = \pi i$,where $c:|z-2|=2$

4. Evaluate $\oint \frac{dz}{z^2-iz+6}$,where $c:|z-2i|=1$

Solution:

Poles are at $z^2-iz+6=0$

$$\Rightarrow z = \frac{i \pm \sqrt{(-i)^2-4(1)(6)}}{2}$$

$$= \frac{i \pm \sqrt{-1-24}}{2}$$

$$= \frac{i \pm \sqrt{-25}}{2}$$

$$= \frac{i \pm 5i}{2}$$

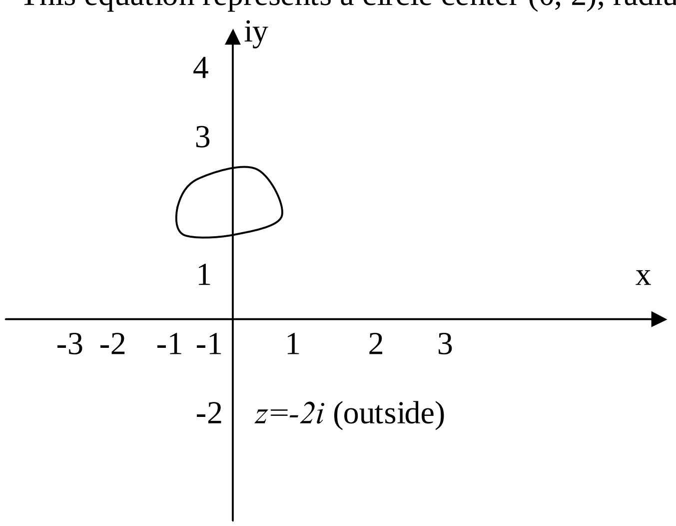
$$z_1 = \frac{6i}{2} = 3i \text{ or } z_2 = \frac{i-5i}{2} = -2i$$

$$c:|z-2i|=1 \Rightarrow |(x+iy)-2i|=1 \Rightarrow |x+i(y-2)|=1$$

$$\Rightarrow \sqrt{x^2+(y-2)^2}=1$$

$$\Rightarrow x^2+(y-2)^2=1$$

This equation represents a circle center (0, 2), radius 1.



$\therefore\ z = 3i$ is on the curve while $z = -2i$ is outside.
Residue at $z = 3i$

$k = 1$

$$\therefore \operatorname{Re} s = \lim_{z \to 3i} (z - 3i) \frac{1}{z^2 - iz + 6}$$

$$= \lim_{z \to 3i} \frac{(z-3i)}{(z-3i)(z+2i)} = \lim_{z \to 3i} \frac{1}{z+2i} = \frac{1}{5i}$$

$$\therefore \oint \frac{dz}{z^2 - iz + 6} = 2\pi i \left\{ \frac{1}{5i} \right\} = \frac{2\pi}{5}$$

Exercise:

Evaluate

1. $\int_c \frac{2z^2}{z^3 \left(z^2 + 2z + 7\right)} dz, \quad c : |z| = 2$

2. $\int_c \frac{e^{2z}}{z \left(z^2 + z + 3\right)^2} dz, \quad c : |2z + 3i| = 1$