KENYATTA UNIVERSITY

DIGITAL SCHOOL OF VIRTUAL AND OPEN LEARNING

DEPARTMENT OF MATHEMATICS AND ACTUARIAL SCIENCE

SMA 202: LINEAR ALGEBRA I

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INTRODUCTION

Linear Algebra is an important part of Mathematical background required not only for Mathematicians but also for other Scientists.

This module introduces the learner to the foundations of Linear Algebra. It begins with the background information on matrices and their basic operations, determinants and inverses. Matrices are later used in solving systems of linear equations. The module also covers vectors, their basic operations and their application to equations of lines and planes in three dimensions. Other topics include Vector spaces, linear independence and Independence.

The content is divided into thirteen short lessons. Each lecture begins with a brief introduction and lesson learning outcomes before discussing the main content. Each concept is followed by an e-tivity which is intended to help you to test your understanding. In addition, answers to selected self help questions are given at the ended of the module. Further reading is suggested at the end of each lecture. This is intended to help the learner get exposed to other approaches to concepts and hopefully to more challenging exercises.

The learner is strongly advised to do all the e-tivities in each lesson before proceeding to the next lesson.

WEEK	TOPIC
WEEK 0:	INTRODUCTION
WEEK 1:	MATRICES
WEEK 2:	PROPERTIES OF MATRIX OPERATIONS
WEEK 3:	DETERMINANTS OF 2×2 AND 3×3 MATRICES
WEEK 4:	PROPERTIES OF DETERMINANTS
WEEK 5:	INVERSES OF 2 x 2 AND 3 x 3 MATRICES
WEEK 6:	ROW-ECHELON FORM
WEEK 7:	SOLUTION OF SYSTEMS OF LINEAR EQUATIONS-(GAUSS-JORDAN METHOD)
WEEK 8:	CRAMER'S RULE AND INVERSE MATRIX METHOD
WEEK 9:	VECTORS
WEEK 10:	VECTOR SPACES
WEEK 11:	LINEAR COMBINATIONS
WEEK 12:	LINEAR DEPENDENCE AND INDEPENDENCE, BASIS AND DIMENSION
WEEK 13:	PLANES AND LINES IN $\Re^3$
WEEK 14 &15	EXAMINATION

OVERVIEW OF THE COURSE

Week 0: Introduction (Your Context, Your Goals)

This lesson is intended to help you acclimatize to blended learning and to create a community of learners who will motivate each other during the course. You will be required to introduce yourself to your lecturer and colleagues either physically during a face to face session or even online before other academic interactions start. This will be at the discretion of individual universities and lecturers. It will be important to also state your context and goals as well as what you think about businesses. You can also share any experience that you may have regarding businesses.

Week 1:

This first lesson is divided into two subtopics.

In the first subtopic, you will be introduced to the concept of matrices. Basic definitions are given including the order of a matrix, equal matrices, row matrix, column matrix e.t.c.

The second subtopic deals with operations on matrices. These include addition, subtraction, scalar multiplication, dot product and matrix multiplication.

Each concept is illustrated by several examples.

Week 2:

In this lesson, we deal with the transpose of a matrix and also properties of matrix operations.

The subtopic introduces you to the transpose of a matrix. The definition is given, followed by a number of properties.

We also deal with properties of matrix operations. These include additive commutativity, of additive and multiplicative associativity, distributive laws etc.

Each concept is illustrated by several examples

Week 3:

This lesson is divided into two subtopics.

The first subtopic covers determinants of 2×2 and 3×3 matrices. In each case, several worked out examples are given. The second subtopic deals with a general formula for finding the determinant of $n\times n$ matrices.

Week 4:

In this lesson you will be introduced to the concept of properties of determinants of matrices, which can be used to find the determinant of a matrix from the determinant of another matrix.

The concept is illustrated by several examples.

Week 5:

This lesson This lecture is divided into two subtopics.

The first subtopic deals with inverses of 2 x 2 of matrices while the second one covers the inverses of 3 x 3 matrices.

However this lecture restricts itself to the method of using the adjoint of a matrix to find the inverses. Row reduction method is covered in the following lecture.

Each concept is illustrated by several examples

Week 6:

This lesson is divided into three sections, each section dealing with a specific subtopic.

The first subtopic covers the inverse of a matrix using row reduction method, while the second one covers the general concept of reducing a matrix to echelon form.

Finally, the learner is introduced to the canonical form of a matrix.

Each concept is illustrated by several examples.

Week 7:

This lesson starts with a general introduction to the solution of a system of linear equations, followed by a more detailed section on the solution of equations using Gauss Jordan method.

Other methods are covered in lesson 8

Week 8:

The content of this lesson is divided into two subtopics.

In the first subtopic, you will be introduced to the solution of a system of linear equations using Cramer's rule, or the method of determinants.

The second subtopic deals with the inverse matrix method of solving linear equations. Each method is illustrated using several examples

Week 9:

This lesson covers the concept of vectors, their dot product and cross product. The application of dot product in looking for the angle between vectors is included. Application of cross product in the equations of lines and planes is covered in the last lecture in the module. Each concept is illustrated by several examples.

Week 10:

This lesson is divided into two subtopics.

In the first subtopic, you will be introduced to the concept of a vector space.

The second subtopic deals with subspaces of vector spaces.

Each concept is illustrated by several examples.

Week 11:

This lesson covers linear combinations and linear spans. Several examples have been given on how to write a given vector as a linear combination of other vectors. The relationship between linear spans, vector spaces and subspaces is also included.Each concept is illustrated by several examples

Week 12:

The content of this lesson is divided into two subtopics.

In the first subtopic, the learner is introduced to the concept of linear dependence and independence.

The second subtopic deals with the application of linear dependence and independence in finding the basis and dimension of a given vector space.

Each concept is illustrated by several worked out examples.

Week 13:

This lesson is divided into two subtopics.

In the first subtopic, you will be introduced to the vector equation of a line, while the second subtopic covers the vector equation of a plane. Each concept is illustrated by several examples.

Week 14 &15: Examination

These two weeks bring together the work you have been doing to an end. This course unit will be examined and will partially contribute to the award of the degree in the programme that you are undertaking. We acknowledge that different universities across East Africa may have different Semester dates. It is however anticipated that most Universities will have a minimum of 13 weeks' semester. We have therefore placed examinations in the last two weeks but Universities are allowed to go with their schedules. Your university examinations regulations will apply.

MODULE LEARNING OUTCOMES

By the end of this module, you will be able to:

1. Perform basic operations on matrices
2. Find the determinants of 2 X 2 and 3 X 3 matrices
3. Find the inverses of 2 X 2 and 3 X 3 matrices
4. Solve systems of linear equations using Gauss Jordan method, Cramer's rule and Inverse matrix method
5. Determine whether or not a given set of vectors is linearly dependent or linearly independent
6. Perform basic operations on vectors
7. Find the equations of planes and lines in R^3
8. Determine the basis and dimension of given of given vector spaces

COURSE DESCRIPTION

This is a common course unit meant for all students who are not taking a Bachelor of Business Administration degree. The rationale of offering the course is to equip students whose degree programmes are not business- related with knowledge and skills that can assist them to undertake entrepreneurial activities should the need arise. This common unit or course is usually taken in the first year of study and has no prerequisite unit. The general purpose of the course therefore is to introduce you to various concepts and theories in entrepreneurship. Emphasis will be placed on the motivation behind entrepreneurship, types of entrepreneurships and various business models that you can adapt as you plan to start businesses. Also covered will be Intellectual property rights and how you can integrate technology in business. The course will take you 39 instructional hours some of which will be covered face to face and others in online activities. You will therefore be required to set aside about 5 hours per lesson to complete this course successfully.

COURSE REQUIREMENTS

This is a blended learning course that will utilize the flex model. This means that learning materials and instructions will be given online and the lessons will be self-guided with the lecturer being available briefly for face to face sessions and support and also on-site (online) most of the time. Your lecturer will be meeting you face to face to introduce a lesson and put it into perspective and you will actively participate in your search for knowledge by undertaking several online activities. This means that some of the 39 instructional hours of the course will be delivered face to face while other lessons will be taught online through various learner and lecturer activities. It is important for you to note that one instructional hour is equivalent to two online hours. Three instructional hours will be needed per week. Out of these, one will be used for face to face contact with your lecturer (also referred as e-moderator in the online activities) while the other two instructional hours (translating to four online hours) will be used for online activities otherwise referred to as e-tivities in the lessons. This will add up to the 5 hours requirement per lesson earlier mentioned. There are 27 online activities each taking at least two hours and totaling to 54 online hours. You are advised to follow the topic flow-chart given so that you cover at least a lesson every week.

You will be required to participate and interact online with your peers and the e-moderator who in this case is your lecturer. Guidelines for the online activities (which we shall keep referring to as e-tivities) will be provided whenever there is an e-tivity. Please note that since the online e-tivities are part of the learning process, they may be graded at the discretion of your e-moderator. Such grading will however be communicated in the e-tivity guidelines and feedback given as soon as possible after the e-tivity. The e-tivities will include but will not be limited to online assessment quizzes, assignments and discussions. There are also assessment questions that you can attempt at the end of every lesson to test your understanding of the lesson. The answers to all the assessment questions are at the end of the module after lesson 10. All the resource that have been used in this module in form of books are available under the resources section after the answers to the questions.

ASSESSMENT

It is important to note that the module has embedded certain learner formative assessment feedback tools that will enable you gauge your own learning progress. The tools include online collaborative discussions forums that focus on team learning and personal mastery and will therefore provide you with peer feedback, lecturer assessment and self- reflection. You will also be required to do one major assignment/project that is meant to assess the application of the skills and knowledge gained during the course. The project score in combination with scores for e-tivities (where graded) will account for 30% of your final examination score with the remaining 70% coming from a face to face sit-in final written examination that will be guided by your university examination policy and procedures.

We wish you the very best of experiences in this course.

LESSON 1

MATRICES

1.1 Introduction

This first lesson is, you were introduced to the concept of matrices. Basic definitions were given including the order of a matrix, equal matrices, row matrix, column matrix etc. You also studied operations on matrices including addition, subtraction, scalar multiplication, dot product and matrix multiplication.

1.2 Lesson Learning Outcomes

By the end of this lesson, you will be able to:

1.2.1 Perform basic operations on matrices including addition, subtraction, scalar multiplication and matrix multiplication.

1.2.1 Definitions

Matrices are used as a shorthand for keeping essential data arranged in rows and columns i.e matrices are used to summarize data in tabular form.

Definition: A matrix is an ordered rectangular array of numbers, usually enclosed in parenthesis or square brackets. Capital (Upper – case) letters are used to denote matrices.

Order of a Matrix

The size of a Matrix is specified by the number of rows (horizontal) and the number of columns (vertical).

A general matrix of order m $\times$n is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{22} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{mn} \end{bmatrix} \quad \begin{array}{l} = \lfloor a_{ij} \rfloor \\ i \to ith\ \ row \\ j \to jth\ column \end{array}$$

A Square Matrix is a one with the same number of rows and columns i.e
m $\times$ m matrix. Two matrices are of the same size if they have the same order.

A vector is a matrix with one row (1 $\times$ n) or one column (n $\times$ 1). A row vector is of the form 1 $\times$ n, and a column vector is of the form m $\times$ 1.

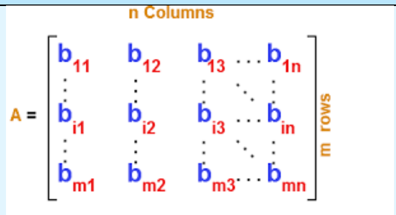
A zero matrix of order m $\times$ n is the matrix with $a_{ij} = 0 \quad i = 1, \cdots m, \quad j = 1 \ldots n$.

Similarly we talk of zero rows and column vector.

$$0 = (0,0,\cdots 0) \text{ or } \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Equal Matrices:
Two matrices A & B are said to be equal if they have the same order (size) m $\times$ n and $a_{ij} = b_{ij} \ \forall i \quad \& \forall j$

E-tivity 1.2.1 - Definitions

Numbering, pacing and sequencing	1.2.1
Title	**Definitions**
Purpose	To identify matrices of different orders
Brief summary of overall task	Watch the video on definition of matrices
Spark	
Individual task	After watching the video, give examples of 1. 3X3 2. 2X2 3. 2X3 4. 3X2 Matrices
Interaction begins	View your colleagues responses in Discussion forum 1.2.1 and summarize them in your note book
E-moderator interventions	Note that 1. The order of a matrix is important as it determines what operations can be performed on the matrices 2. A vector can be considered to be a matrix, with either one row or one column
Schedule and time	Week 1. The activity will take 1 hour
Next	**Operations on matrices**

1.2.2 Operations on matrices

Addition and Subtraction of matrices
This is performed on matrices of the same order (size). Let A and B be m $\times$ n matrices.

A + B = $\left[a_{ij}\right] \pm \left[b_{ij}\right] = \left[a_{ij} \pm b_{ij}\right] = \left[c_{ij}\right]$ $(m \times n)$

Scalar Multiplication

This is performed on any matrix and the resulting matrix is the same size . $c\,A = c\left[a_{ij}\right] = \left[ca_{ij}\right]$. Each entry is multiplied by same number (scalar).

Dot product: let $\vec{a}$ and $\vec{b}$ be any two vector of size n (matrices with a single column or row).

$\vec{a} = [a_1, a_2, \cdots a_n], \vec{b} = (b_1, b_2, \cdots b_n)$. The dot product of $\vec{a}$ & $\vec{b}$ denoted $\vec{a} \cdot \vec{b}$, is given by

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i$ .The dot product is also scalar product.

Dot product of a row & column vector of order *n*.

$$(a_1, a_2 \cdots a_n)\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i = \vec{a} \cdot \vec{b}$$

Matrix Multiplication:

Let $A = [a_{ik}]$ be an m × n matrix, and $B = \left[b_{kj}\right]$ an n × s matrix. The matrix product AB is the m × s matrix $C = \left[c_{ij}\right]$ where c_{ij} the dot product of the i^{th} row of A and the j^{th} column of B.

i.e. $AB = C$, $[a_{ik}]\left[b_{kj}\right] = \left[c_{ij}\right]$; $C_{ij} = A_i \cdot B_j = \sum_{k=1}^{n} a_{ik} b_{kj}$

Remark:

1.Let $A(m \times n)$, $B(s \times r)$ be two matrices .

$C = AB$ exists iff $n = s$ & C is $m \times n$

$C = BA$ exists iff $r = m$ & C is $s \times n$

2. It's possible for AB to be defined while BA is not defined. i.e. matrix multiplication is not commutative.

Examples

1. Let $A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & 2 \\ -7 & 1 & 8 \end{pmatrix}$. Then

$$A + B = \begin{pmatrix} 1+3 & -2+0 & 3+2 \\ 4-7 & 5+1 & -6+8 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 5 \\ -3 & 6 & 2 \end{pmatrix}$$

2. $3A = \begin{pmatrix} 3\cdot 1 & 3\cdot(-2) & 3\cdot 3 \\ 3\cdot 4 & 3\cdot 5 & 3\cdot(-6) \end{pmatrix} = \begin{pmatrix} 3 & -6 & 9 \\ 12 & 15 & -18 \end{pmatrix}$

3. $2A-3B=\begin{pmatrix}2 & -4 & 6\\ 8 & 10 & -12\end{pmatrix}+\begin{pmatrix}-9 & 0 & -6\\ 21 & -3 & -24\end{pmatrix}=\begin{pmatrix}-7 & -4 & 0\\ 29 & 7 & -36\end{pmatrix}$

4. $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}=\begin{pmatrix}1\cdot1+2\cdot0 & 1\cdot1+2\cdot2\\ 3\cdot1+4\cdot0 & 3\cdot1+4\cdot2\end{pmatrix}=\begin{pmatrix}1 & 5\\ 3 & 11\end{pmatrix}$

$$\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}=\begin{pmatrix}1\cdot1+1\cdot3 & 1\cdot2+1\cdot4\\ 0\cdot1+2\cdot3 & 0\cdot2+2\cdot4\end{pmatrix}=\begin{pmatrix}4 & 6\\ 6 & 8\end{pmatrix}$$

The above example shows that matrix multiplication is not commutative, i.e.the products AB and BA of matrices need not be equal.

E-tivity 1.2.2 _ Operations on matrices

Numbering, pacing and sequencing	1.2.1
Title	**Operations on matrices**
Purpose	To perform basic operations on matrices
Brief summary of overall task	Watch the videos on matrix addition, subtraction and scalar multiplication and matrix multiplication
Spark	
Individual task	After watching the videos, answer the following questions 1. Does the order of a matrix determine the operations that can be performed on it? 2. Why is it possible to get the product AB but not BA? 3. Can matrices of different orders be added or subtracted
Interaction begins	In discussion forum 1.2.2, comment on the answers posted by two of your colleagues
E-moderator interventions	Note that 1. Matrices can only be added or subtracted if they are of the same order. 2. Matrices of different orders can be multiplied, but the number of columns of the first matrix has to be equal to the number of rows of the second matrix
Schedule and time	Week 1. The activity will take 2 hours
Next	**Properties of matrix operations**

1.3 Assessment Questions

1. Let $A=\begin{pmatrix}1 & 3\\ 2 & -1\end{pmatrix}$ and $B=\begin{pmatrix}2 & 0 & -4\\ 3 & -2 & 6\end{pmatrix}$. Find (a) AB, (b) BA

2. Given $A=(2,-1)$ and $B=\begin{pmatrix}1 & -2 & 0\\ 4 & 5 & -3\end{pmatrix}$, find (a) AB, (b) BA

3. Given $A=\begin{pmatrix}2 & -1\\ 1 & 0\\ -3 & 4\end{pmatrix}$ and $B=\begin{pmatrix}1 & -2 & -5\\ 3 & 4 & 0\end{pmatrix}$, find (a) AB, (b) BA

1.4 References

1. Linear Algebra: Schaum's Outline Series
2. Linear Algebra by J. N. Sharma, A.R. Vasishta
3. Linear Algebra by Michael O'nan, Herbert Enderton
4. https://youtu.be/JhikgDtwpLM?list=TLPQMDIwODIwMjAGDiXU4camOg
5. https://youtu.be/QXUbFzEd3Ww
6. https://youtu.be/iJERwUVuwtY
7. https://youtu.be/vzt9c7iWPxs

LESSON 2

PROPERTIES OF MATRIX OPERATIONS

2.1 Introduction

In the first lesson, you were introduced to the concept of matrices. Basic definitions were given including the order of a matrix, equal matrices, row matrix, column matrix e.t.c. You also studied operations on matrices including addition, subtraction, scalar multiplication, dot product and matrix multiplication.

In this lesson, we deal with the transpose of a matrix and also properties of matrix operations.

The first subtopic introduces you to the transpose of a matrix. The definition is given, followed by a number of properties.

We also deal with properties of matrix operations. These include additive commutativity, of additive and multiplicative associativity, distributive laws etc.

Each concept is illustrated by several examples

2.2 Lesson Learning Outcomes

By the end of this lesson, you will be able to:

2.2.1 Identify and verify properties of the transpose of a matrix.

2.2.2 Identify and verify properties of matrix operations like additive commutativity,

additive and multiplicative associativity, distributive laws etc.

2.2.1 Transpose of a matrix

The transpose of the matrix A is matrix $B = A^T$, such that $b_{ij} = a_{ji}$ i.e. the rows become columns and vice versa.

1. If $A = A^T$, we say A is symmetric. (only for a square matrix). If A is symmetric,

a) $A\,A^T$ and $A + A^T$ are symmetric.

b) A^k symmetric $\forall_k$

c) If A, B are symmetric, $\alpha\,A + BA$ is symmetric.

2. If $A^T = -A$, we say A is skew symmetric. In that case,

(a) $A - A^T$ is skew symmetric

(b) If A, B are skew symmetric, $\alpha 4 + B\,A$ is skew symmetric.

3. If A, S are $n \times n$ (square) matrices and A is symmetric, then $S^T A S$, $S A S^T$ are symmetric

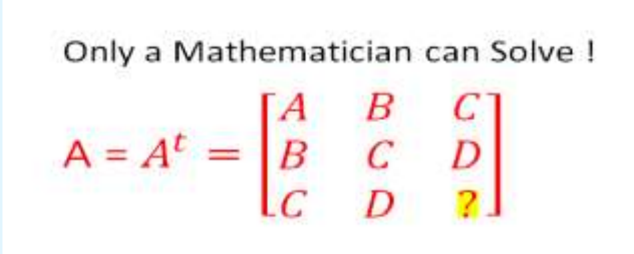
4. Every square matrix can be expressed as the sum of a symmetric & skew-symmetric matrix. i.e. $A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right)$

$S(A) = \frac{1}{2}\left(A + A^T\right)$ is the symmetric part; $K(A) = \frac{1}{2}\left(A - A^T\right)$ is the skew-symmetric part.

5. If A is $n \times n$ and $f(x)$ any polynomial, then $f\left(A^T\right) = [f(A)]^T$

E-tivity 2.2.1 Transpose of a matrix

Numbering, pacing and sequencing	**2.2.1**
Title	**Transpose of a matrix**
Purpose	To identify symmetric matrices
Brief summary of overall task	Watch the video on symmetric matrices

Spark	
Individual task	Give two examples of symmetric matrices Is it possible to get a 2X3 symmetric matrix?
Interaction begins	Follow the responses by your colleagues (at least 2) and
E-moderator interventions	Comment in discussion forum 2.2.1. Are their answers correct? Note that symmetric matrices have to be square matrices since
Schedule and time	Week 2. The activity will take 1 hour
Next	**Properties of Matrix operations**

2.2.2 Properties of Matrix operations

1. $A+B=B+A$ - Addition is commutative
2. $(A+B)+C=A+(B+C)$ - Addition is associative
3. $A+0=0+A=A, 0$ is the identity for addition
4. $\alpha(A+B)=\alpha A+\alpha B$, left distributive law.
5. $(\alpha+\beta)A=\alpha A+\beta A$, right distributive law
6. $(\alpha\,\beta)A=\alpha(\beta A)$, associativity of scalar multiplication
7. $(\alpha\,A)B=A(\alpha\beta)-$ scalar pull through
8. $(A\,B)C=A(BC)$ - associativity of matrix multiplication
9. $1_n A=A$, $AI_m=A$, $A(m\times m)$ - identity for matrix multiplication
10. $A(B+C)=AB+AC$, left distributive law
11. $(A+B)C=AC+BC$, right distributive law
12. $(A^T)^T=A$
13. $(A+B)^T=A^T+B^T$ - transpose of the sum=sum of transpose
14. $(AB)^T=B^T\,A^T$ - transpose of product = product of transpose
    $(AB)^*=B^*A^*$.

Proof of most of these properties involve routine computations.

Show that $(A-B)(A+B)=A^2-B^2$ iff $AB=BA$ i.e A & B commute
Solution:
$\Rightarrow$Suppose $(A-B)(A+B)=A^2-B^2$ and show that $A\,\&\,B$ commute.
$A^2-B^2=(A-B)(A+B)$

$$= A(A+B) - B(A+B)$$
$$= A^2 + AB - BA - B^2$$

Hence $AB - BA = 0$; $AB = BA$ and so $A \& B$ commute.

$\Rightarrow$ Assume A & B commute & show that $(A-B)(A+B) = A^2 - B^2$

Since $A \& B$ commute $AB = BA$

$$(A-B)(A+B) = A(A+B) - B(A+B)$$
$$= A^2 + AB - BA - B^2$$

but $AB = BA$; $= A^2 + AB - AB - B^2 = A^2 - B^2$

2.2.3 E-tivities 2.2.2 Properties of Matrix operations

Numbering, pacing and sequencing	2.2.1
Title	**Properties of Matrix operations**
Purpose	To identify properties of matrix operations
Brief summary of overall task	Watch the videos properties of matrix multiplication and matrix addition
Spark	
Individual task	After watching the videos, what do you call the properties that involve both multiplication and addition? Using suitable 3 X3 matrices, verify the property A(B+C)=AB+AC
Interaction begins	Follow the responses by your colleagues (at least 2) and see what you can learn from them
E-moderator interventions	The property is called distributive property, which can be either left or right. (A+B)C=AC+BC is also a distributive property
Schedule and time	Week 2. The activity will take 2 hours
Next	Determinants of 2×2 and 3×3 matrices

2.3 Assessment Questions

1. Find the transpose A^t of the matrix $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 5 \\ 4 & 4 & 4 & 4 \end{pmatrix}$

2. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}$. Find (a) AA^t , (b) A^tA

2.4 References

3 Linear Algebra by Michael O'nan, Herbert Enderton
4 A First Course in Linear Algebra by Daniel Zelisky
5 Elementary Linear Algebra by Bennard Kolman
6 Elementary Linear Algebra by Howard Anton
7 https://youtu.be/wwXCDY9-bAA
8 https://youtu.be/vEjnB3jZ7kA
9 https://youtu.be/NEWk5WsoXE4

LESSON 3

DETERMINANTS OF 2×2 AND 3×3 MATRICES

3.1 Introduction

In lesson 2, we dealt with the transpose of a matrix and also properties of matrix operations.

You were introduce to the transpose of a matrix. The definition was given, followed by a number of properties.

You also dealt with properties of matrix operations. These included additive commutativity, of additive and multiplicative associativity, distributive laws etc.

In this lesson we will cover determinants of 2×2 and 3×3 matrices. In each case, several worked out examples will be given. We will also deal with a general formula for finding the determinant of $n\times n$ matrices.

3.2 Lesson learning outcomes

By the end of the lesson, you will be able to;

3.2.1 Find the determinants of a 2×2 matrix

3.2.2 Find the determinants of a 3×3 matrix

3.2.3 Use the general formula for finding the determinant of $n\times n$ matrices to find the determinants of a 4 X 4 and 5 X 5 matrix

3.2.1 Determinants of 2×2 and 3×3 matrices

Let $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Det $A=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}\,a_{22}-a_{12}\,a_{21}$. The determinant is a scalar.

Examples:

1. Find the value of λ such that $\begin{vmatrix} \lambda & \lambda \\ 3 & \lambda-2 \end{vmatrix}=0$.

2. Let $\begin{vmatrix} 1+x & 1 \\ 2+2x & 2 \end{vmatrix}=$ find *x*.

3. Let $\begin{vmatrix} x & 3 \\ 2 & 2x+1 \end{vmatrix}=4$. Find *x*.

3×3 **Matrices**

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

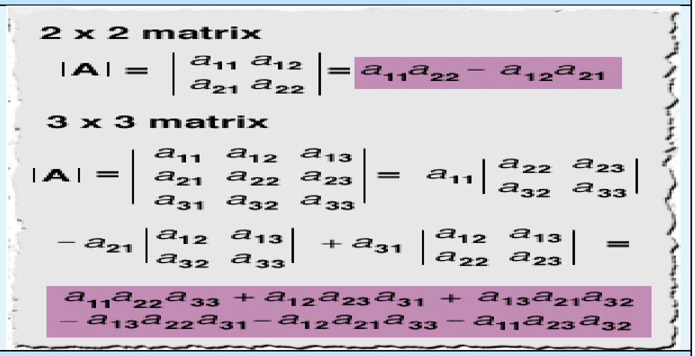
$$= a_{11}\left(a_{22}\,a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}\,a_{31}\right) + a_{13}\left(a_{21}a_{32}\right)$$

$$= a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{23}\,a_{32} - a_{12}\,a_{21}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32} - a_{13}\,a_{22}\,a_{31}$$

$$= a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32} - a_{11}\,a_{23}\,a_{32} - a_{12}\,a_{21}\,a_{33} - a_{13}\,a_{22}\,a_{31}$$

Note: This is a sum of **6 products**, *3 positive* and *3 negative.* Each product has exactly one factor from each row and column.

E-tivity 3.2.1 Determinants of 2×2 and 3×3 matrices

Numbering, pacing and sequencing	3.2.1
Title	Determinants of 2×2 and 3×3 matrices
Purpose	To calculate the determinants of 2×2 and 3×3 matrices
Brief summary of overall task	Watch the videos on Determinants of 2 by 2 and 3 by 3 matrices
Spark	
Individual task	From the two videos, how many determinants of 2X 2 matrices will be calculated in the process of finding the determinants of four 3 X 3 matrices?
Interaction begins	In discussion forum 3.2.1, follow the solutions posted by your colleagues and comment on their answers
E-moderator interventions	In calculating the determinant of one 3X3 matrix, one ends up calculating the determinants of three 2X2 matrices. This means that if the 3X3 matrices are four, the one will end up calculating the determinants of twelve 2 X 2

	matrices
Schedule and time	Week 3. The activity will take 2 hours
Next	**Determinant of an $n \times n$ matrix**

3.2.2 Determinant of an $n \times n$ matrix

For a large square matrix the determinant is a sum of products, half of which have minus signs added. Each product will have exactly one factor from each row and one factor from each column.

There are $n!$ summations.

1. Evaluate $\begin{vmatrix} 3 & 1 & 2 \\ -1 & -1 & 4 \\ -2 & -1 & 1 \end{vmatrix}$

Solution: $\begin{vmatrix} 3 & 1 & 2 \\ -1 & -1 & 4 \\ -2 & -1 & 1 \end{vmatrix} = 3[(-1)1+1(4)]-1[-1(1)+8]+2[1-2]=0$

2. Evaluate $\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 2 \\ 1 & -1 & 2 \end{vmatrix}$

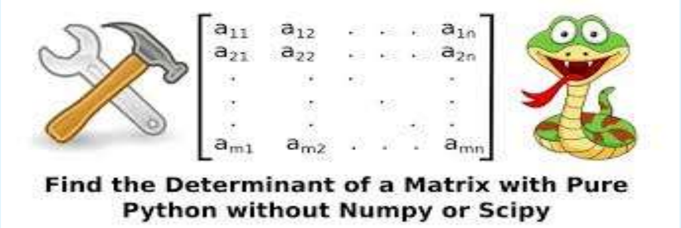
Solution: $\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 2 \\ 1 & -1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = 1(4)-3(2)+1(-3)=-5$

3. Evaluate $\begin{vmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 2 & 0 \\ -1 & -2 & -1 & 1 \\ 1 & -1 & 3 & 3 \end{vmatrix}$

Solution: $\begin{vmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 2 & 0 \\ -1 & -2 & -1 & 1 \\ 1 & -1 & 3 & 3 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 & 0 \\ -2 & -1 & 1 \\ -1 & 3 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 & 0 \\ -1 & -1 & 1 \\ 1 & 3 & 3 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 & 0 \\ -1 & -2 & 1 \\ 1 & -1 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 & 2 \\ -1 & -2 & -1 \\ 1 & -1 & 3 \end{vmatrix}$

$= 1(4)-2(-4)+3(-6)-1(-6)=0$

E-tivity 3.2.2 Determinant of an $n \times n$ matrix

Numbering, pacing and sequencing	3.2.2
Title	**Determinant of an $n \times n$ matrix**
Purpose	To find the determinants of matrices of order higher than 3.
Brief summary of overall task	Watch the video on finding the determinant of [n X n]() matrices
Spark	
Individual task	From the video, how many determinants of 3X 3 matrices will be calculated in the process of finding the determinants of five 4 X 4 matrices?
Interaction begins	In discussion forum 3.2.2, follow the solutions posted by your colleagues and comment on their answers
E-moderator interventions	In calculating the determinant of one 4X4 matrix, one ends up calculating the determinants of four 3X3 matrices. This means that if the 4X4 matrices are five, the one will end up calculating the determinants of twenty 3 X 3 matrices
Schedule and time	Week 3. The activity will take 2 hours
Next	**Properties of determinants**

3.3 Assessment Questions

1. Evaluate the determinant of each matrix:

(a) $\begin{pmatrix} 3 & -2 \\ 4 & 5 \end{pmatrix}$, (b) $\begin{pmatrix} a-b & a \\ b & a+b \end{pmatrix}$

(i) $\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 3 \cdot 5 - (-2) \cdot 4 = 23$ (ii) $\begin{vmatrix} a-b & a \\ b & a+b \end{vmatrix} = (a-b)(a+b) - a \cdot a = -b^2$

2. Determine those values of k for which $\begin{vmatrix} k & k \\ 4 & 2k \end{vmatrix} = 0$.

$\begin{vmatrix} k & k \\ 4 & 2k \end{vmatrix} = 2k^2 - 4k = 0$, or $2k(k-2) = 0$.Hence $k = 0$;and $k = 2$.That is, if $k = 0$ or $k = 2$, the determinant is zero.

3. Compute the determinant of each matrix:

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 2 & 5 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 & 1 \\ 4 & 2 & -3 \\ 5 & 3 & 1 \end{pmatrix}$ (c)

3.4 References

1. Linear Algebra by Fraleigh & Beauregard
2. Linear Algebra: Schaum's Outline Series
3. Elementary Linear Algebra by Howard Anton
4. https://youtu.be/OU9sWHk_dlw
5. https://youtu.be/3ROzG6n4yMc
6. https://youtu.be/H9BWRYJNIv4

LESSON 4

PROPERTIES OF DETERMINANTS

4.1 Introduction

In the previous lesson we covered determinants of 2×2 and 3×3 matrices. In each case, several worked out examples were given. We also dealt with a general formula for finding the determinant of $n\times n$ matrices.

In this lesson you will be introduced to the concept of properties of determinants of matrices, which can be used to find the determinant of a matrix from the determinant of another matrix. The concept is illustrated by several examples.

4.2 Learning Outcomes

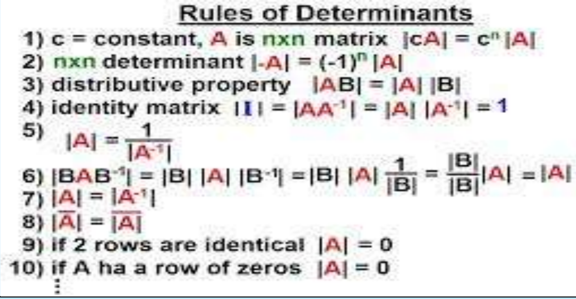
By the end of the lesson, you will be able to;

4.2.1 Use properties of determinants of matrices to find the determinant of a given matrix from the determinant of another matrix.

4.2.1 Properties of determinants

1. Det (AB) = Det A . Det B
2. If one row or column of A is multiplied by scalar r to get B, det (B) = r det A
3. If row i = 0 or column j = 0, det A =0.
4. For an n $\times$n matrix, det (r A) = r^n det A
5. If two rows or columns are identical det A = 0
6. If one row or column is a scalar multiple of another, det A = 0.
7. Adding a scalar multiple of a row or a column to another row or column respectively leaves the determinant unchanged.

E-tivity 4.2.1 Properties of determinants

Numbering, pacing and sequencing	4.2.1
Title	Properties of determinants
Purpose	To find the determinant of a given matrix from the determinant of another matrix.
Brief summary of overall task	Watch the video on properties of determinants
Spark	 **Rules of Determinants** 1) c = constant, A is nxn matrix $\lvert cA\rvert = c^n \lvert A\rvert$ 2) nxn determinant $\lvert -A\rvert = (-1)^n \lvert A\rvert$ 3) distributive property $\lvert AB\rvert = \lvert A\rvert \lvert B\rvert$ 4) identity matrix $\lvert I\rvert = \lvert AA^{-1}\rvert = \lvert A\rvert \lvert A^{-1}\rvert = 1$ 5) $\lvert A\rvert = \frac{1}{\lvert A^{-1}\rvert}$ 6) $\lvert BAB^{-1}\rvert = \lvert B\rvert \lvert A\rvert \lvert B^{-1}\rvert = \lvert B\rvert \lvert A\rvert \frac{1}{\lvert B\rvert} = \frac{\lvert B\rvert}{\lvert B\rvert}\lvert A\rvert = \lvert A\rvert$ 7) $\lvert A\rvert = \lvert A^{-1}\rvert$ 8) $\lvert \overline{A}\rvert = \overline{\lvert A\rvert}$ 9) if 2 rows are identical $\lvert A\rvert = 0$ 10) if A ha a row of zeros $\lvert A\rvert = 0$ ⋮
Individual task	After watching the video, illustrate property number 3 using an 3 X 3 matrix of your choice
Interaction begins	In discussion forum 4.2.1, go through the illustrations of three of your colleagues and comment on whether their answers are correct
E-moderator interventions	Note that the answer to this question lies in the definition of the determinant of a 3X 3 matrix
Schedule and time	Week 4. The activity will take one hour
Next	**Inverse of a 2 X 2 matrix**

4.3 Assessment Questions

Evaluate

1. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix}$

2. $\begin{vmatrix} 5 & 0 & 9 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix}$

3. $\begin{vmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & 4 & 6 \end{vmatrix}$

4. $\begin{vmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 1 & 2 & 3 \end{vmatrix}$

1.4 References

2. Linear Algebra by J. N. Sharma, A.R. Vasishta
3. Linear Algebra by Michael O’nan, Herbert Enderton
4. A First Course in Linear Algebra by Daniel Zelisky
5. https://youtu.be/0OJGV1zlnXY

LESSON 5

INVERSES OF 2 x 2 AND 3 x 3 MATRICES

5.1 Introduction

In the previous lesson you were introduced to the concept of properties of determinants of matrices, which were used to find the determinant of a matrix from the determinant of another matrix. This lesson deals with inverses of 2 x 2 matrices and the inverses of 3 x 3 matrices. However this lesson restricts itself to the method of using the adjoint of a matrix to find the inverses. Row reduction method is covered in lesson 6. Each concept is illustrated by several examples.

5.2 Learning Outcomes

By the end of the lesson, you will be able to:

5.2.1 Find the inverses of 2 x 2 and 3 x 3 matrices using the method of the adjoint of a matrix.

5.2.1 Inverse of a 2 X 2 matrix

Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ det $A = ad-bc \neq 0$; $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -c \\ -c & a \end{bmatrix}$

Note:

$$A A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & -ab+ab \\ cd-cd & sd-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ad-bc \neq 0$$

Let $A=\left[a_{ij}\right]$ be a square matrix. The classical adjoint of A is the matrix

$adj\,(A) = \left[a_{ij}\right]^T$, $a_{ij}' = (-1)^{i+j}\det(A_{ij})$ where a_{ij} is the (i, j) cofactor of A.

$$A^{-1} = \frac{1}{\det(A)} \text{ adj } (A)$$

For a 2 x 2 matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Adj(A) = $\begin{bmatrix} a_{11}' & a_{12} \\ a_{21}' & a_{22}' \end{bmatrix}^T = \begin{bmatrix} d & -c \\ -b & d \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$; $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

E-tivity 5.2.1 Inverse of a 2 X 2 matrix

Numbering, pacing and sequencing	5.2.1
Title	Inverse of a **2 X 2** matrix
Purpose	To find the inverse of a 2 X 2 matrix
Brief summary of overall task	Watch the video on the inverse of a 2 X 2 matrix
Spark	$[A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $[A]^{-1} = \frac{1}{det[A]} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
Individual task	After watching the videos, name two instances when a given matrix may not have on inverse
Interaction begins	Follow the answers by your colleagues in discussion forum 5.2.1 and compare with yours
E-moderator interventions	Note that the two instances a matrix may not have an inverse are when the matrix is not a square matrix, or then the determinant of the matrix is zero
Schedule and time	Week 5. The activity will take 2 hours
Next	**5.2.2 Inverse of a** 3 X 3 **matrix**

5.2.3 Inverse of a 3 X 3 matrix

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{32} & a_{33} \end{bmatrix}; \text{ Adj (A)} = \begin{bmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31}' & a_{32}' & a_{33}' \end{bmatrix}^T = \begin{bmatrix} |A_{11}| & -|A_{12}| & |A_{13}| \\ -|A_{21}| & |A_{22}| & -|A_{23}| \\ |A_{31}| & -|A_{32}| & |A_{33}| \end{bmatrix}^T$$

$$|A_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\ a_{33} - a_{23}\ a_{32}\ ; \quad |A_{12}| = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}\ a_{33} - a_{23}\ a_{31}$$

$$|A_{13}| = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\ a_{32} - a_{22}\ a_{31}; \quad |A_{21}| = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}\ a_{33} - a_{13}a_{32}$$

$$|A_{22}| = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11}\ a_{33} - a_{31}\ a_{13}\ ; \quad |A_{23}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}\ a_{32} - a_{12}\ a_{31}$$

$$|A_{31}| = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = a_{12}\ a_{23} - a_{13}\ a_{22}\ ; \quad |A_{32}| = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = a_{11}\ a_{23} - a_{13}\ a_{21}$$

$|A_{33}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\ a_{22} - a_{12}\ a_{21}$; Det $A = a_{11}\ |A_{11}| - a_{12}\ |A_{12}| + a_{13}\ |A_{13}|$; $A^{-1} = \dfrac{1}{\det\ A}$ *adj(A)*

Examples: Find the inverse of $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}$

Solution:

Det A = $4\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} = 4(2) - 0 + 1(2-6) = 8 - 4 = 4$; Therefore, A^{-1} exists.

$adj(A) = \begin{bmatrix} |A_{11}| & -|A_{12}| & |A_{13}| \\ -|A_{21}| & |A_{22}| & -|A_{23}| \\ |A_{31}| & -|A_{32}| & |A_{33}| \end{bmatrix}^T$; $|A_{11}| = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2$; $|A_{12}| = \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = 2$; $|A_{13}| = \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} = -4$;

$|A_{21}| = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$; $|A_{22}| = \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} = 1$; $|A_{23}| = \begin{vmatrix} 4 & 0 \\ 3 & 1 \end{vmatrix} = 4$; $|A_{31}| = \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} = -2$

$|A_{32}| = \begin{vmatrix} 4 & 1 \\ 2 & 0 \end{vmatrix} = -2$ $|A_{33}| = \begin{vmatrix} 4 & 0 \\ 2 & 2 \end{vmatrix} = 8 = 8\ adj(A) = \begin{bmatrix} 2 & -2 & -4 \\ 1 & 1 & -4 \\ -2 & 2 & 8 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 & -2 \\ -2 & 1 & 2 \\ -4 & -4 & 8 \end{bmatrix}$;

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ -1 & -1 & 2 \end{bmatrix}$$

E-tivity 5.2.2 Inverse of a 3 X 3 **matrix**

Numbering, pacing and sequencing	5.2.2

Title	Inverse of a **3 X 3** matrix
Purpose	To find the inverse of a 3 X3 matrix
Brief summary of overall task	Watch the video on the inverse of a 3X3 matrix
6.2.1 Spark **Inverse of a matrix (row reduction method)**	
Individual task	After watching the video, confirm if the inverse of matrix A is correct
Interaction begins	Go to discussion forum 5.2.2, and compare the answers given by three of your classmates with yours
E-moderator interventions	Note that it is always possible to confirm the accuracy when looking for the inverse of any invertible matrix, since $AA^{-1} = A^{-1}A = I$
Schedule and time	Week 5. The activity will take 1 hour
Next	**Inverse of a matrix (row reduction method)**

5.3 Assessment Questions

Find A^{-1} if

1. A = $\begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$

5.4 References

1. Linear Algebra by Fraleigh & Beauregard
2. Elementary Linear Algebra by Bennard Kolman
3. Elementary Linear Algebra by Howard Anton
4. https://youtu.be/7PrzCQSjE2g
5. https://youtu.be/xfhzwNkMNg4

LESSON 6
ROW-ECHELON FORM

6.1 Introduction

The previous lesson dealt with inverses of 2 x 2 matrices and the inverses of 3 x 3 matrices.The lesson restricted itself to the method of using the adjoint of a matrix to find the inverses.

This lesson covers the inverse of a matrix using row reduction method, the general concept of reducing a matrix to echelon form and finally you will be introduced to the concept of the canonical form of a matrix.

Each concept is illustrated by several examples.

1.2 Learning outcomes

By the end of the lesson, you will be able to;

1.2.1 Find the inverse of a matrix using row reduction method

1.2.2 Reduce a given matrix to echelon form.

1.2.3 Reduce given matrix to canonical form.

6.2.1 Inverse of a matrix (row reduction method)

The inverse of a matrix a can be found using row reduction to echelon form of the augmented matrix (A/I) to get $\left(I / A^{-1}\right)$.

Example:

Find the inverse of $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 4 & 0 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ 2R_2 - R_1 \\ 4R_3 - 3R_1 \end{array} \quad \rightarrow \quad \left[\begin{array}{ccc|ccc} 4 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & -1 & -1 & 2 & 0 \\ 0 & 4 & 1 & -3 & 0 & 4 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 4 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & -1 & -1 & 2 & 0 \\ 0 & 0 & 2 & -2 & -2 & 4 \end{array}\right]\begin{array}{l} 2R_1 - R_3 \\ 2R_2 + R_3 \\ \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 8 & 0 & 0 & 4 & 2 & -4 \\ 0 & 8 & 0 & -4 & 2 & 4 \\ 0 & 0 & 2 & -2 & -2 & 4 \end{array}\right]\begin{array}{l} R1/8 \\ R2/8 \\ R3/2 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right]; \quad \text{Inverse} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ -1 & -1 & 2 \end{bmatrix}$$

E-tivity 6.2.1 Inverse of a matrix (row reduction method)

Numbering, pacing and sequencing	6.2.1
Title	Inverse of a matrix (row reduction method)
Purpose	To find the inverse of a matrix using_ method
Brief summary of overall task	Watch the video on row reduction method of finding the inverse of a 3X3 matrix
Spark	
Individual task	After watching the video, find the inverse of $\begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$ using row-reduction method
Interaction begins	Confirm the accuracy of your answer by using the fact that $AA^{-1} = A^{-1}A = I$
E-moderator interventions	Go to discussion forum 6.2.1 and comment on the answers posted by at least 3 of your colleagues. Point out any mistakes that they may have made and correct the same
Schedule and time	Week 5. This activity will take 1 hour
Next	**Echelon form of a matrix**

6.2.1 Echelon form of a matrix

A $m \times n$ matrix B is said to be in row echelon form if it is of the form.

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} & \\ 0 & b_{22} & b_{23} & \cdots & b_{xn} & \\ 0 & 0 & b_{33} & \cdots & b_{3n} & \\ \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & \dots & b_{ii} & \cdots & b_{in} \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & \cdots & & b_{mm} & b_{mn} \end{bmatrix} if\ m > n$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{in} \\ 0 & b_{22} & b_{23} & \dots & b_{2n} \\ 0 & 0 & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & 0 & \cdots & b_{nn} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \dots & \dots & 0 \end{bmatrix} if\ n > m$$

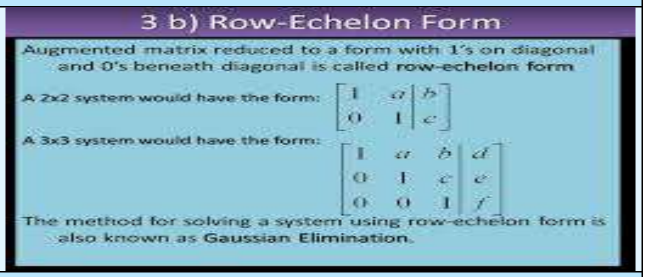
$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & & b_{in} \\ 0 & b_{22} & \cdots & & b_{2n} \\ & & & & \vdots \\ & \ddots & & & \vdots \\ & & & & \dots \\ & & & \ddots & b_{nn} \end{bmatrix} m = n$$

Note:
For a matrix in echelon form, for subsequent rows, the non-zero entries occur in later and later columns.

For a matrix in echelon form, all entries below the main diagonal are zero. Given any matrix B (not in echelon form) we perform the following **elementary row operations** to change it to echelon form:

1. Change the order of the rows (interchange some rows)
2. Multiply one row by a nonzero constant.
3. Add a multiple of one row to a nonzero multiple of another row.

E-tivity 6.2.2 Echelon form of a matrix

Numbering, pacing and sequencing	6.2.2
Title	Echelon form of a matrix
Purpose	To reduce a matrix to echelon form
Brief summary of overall task	Watch the video on reducing a matrix to echelon form
Spark	
Individual task	After watching the video, reduce $\begin{bmatrix} 3 & 4 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ 4 & -3 & 11 & 2 \end{bmatrix}$ to rechelon form.
Interaction begins	Compare your answers with those of three of your classmates and comment on their method
E-moderator interventions	Note that sometimes the sequence of steps followed by your classmates could be different even though the answer could still be correct
Schedule and time	Week 6. This activity will take 1 hour
Next	Reduced row-echelon form

6.2.3 Reduced row-echelon form

Definition: A matrix is said to be in reduced row echelon form (canonical form) if:
1. Each nonzero row begins with a pivot entry 1. (Leading 1 of the row)
2. The rest of the columns containing the pivot entry 1 consists of 0s.
3. In subsequent rows, the pivot entries occur in later and later columns.
4. The all-zero rows are at the bottom (they are the unused rows).

Examples

1. Reduce $\begin{bmatrix} 3 & 4 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ 4 & -3 & 11 & 2 \end{bmatrix}$ to reduced row-echelon (canonical) form.

Solution: $\begin{bmatrix} 3 & 4 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ 4 & -3 & 11 & 2 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & -1 & 3 & 1 \\ 3 & 4 & -1 & 1 \\ 4 & -3 & 11 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 4R_1 \end{matrix}$

$$\rightarrow \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 7 & -10 & -2 \\ 0 & -1 & -1 & -2 \end{bmatrix} R_2/7 \rightarrow \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & -\dfrac{10}{7} & -\dfrac{2}{7} \\ 0 & -1 & -1 & -2 \end{bmatrix} \begin{matrix} R_1 + R_2 \\ \\ R_3 + R_2 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & \frac{11}{7} & \frac{5}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{2}{7} \\ 0 & 0 & -\frac{17}{7} & -\frac{16}{7} \end{bmatrix} \begin{matrix} \\ \\ -\frac{7}{17} R_3 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{11}{7} & \frac{5}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{2}{7} \\ 0 & 0 & 1 & \frac{16}{17} \end{bmatrix} \begin{matrix} R_1 - \frac{11}{7} R_3 \\ R_2 + \frac{10}{7} R_3 \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{13}{17} \\ 0 & 1 & 0 & \frac{18}{17} \\ 0 & 0 & 1 & \frac{16}{17} \end{bmatrix}$$

2. Reduce to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 2 & 0 \\ -1 & -2 & -1 & 1 \\ 1 & -1 & 3 & 4 \end{bmatrix} \begin{matrix} \\ R_2 - 2R1 \\ R_3 + R_1 \\ R_4 - R_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -4 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & -3 & 0 & 2 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - R_2 \end{matrix} \rightarrow$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -4 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 4 & 4 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - 2R_3 \end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -4 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

E-tivity 6.2.3 Reduced row-echelon form

Numbering, pacing and sequencing	6.2.3
Title	Reduced row-echelon form
Purpose	To reduce a matrix to reduced row echelon form
Brief summary of overall task	Watch the video on reduced row-echelon form
Spark	

Individual task	After watching the video, reduce matrix $A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}$ to row-reduced echelon form.
Interaction begins	Compare your solution to that of your colleagues posted in discussion forum 6.2.3. What can you learn from their solutions?
E-moderator interventions	Note that a matrix in echelon form takes only a few more steps for it to be reduced to reduced row-echelon (Canonical) form
Schedule and time	Week 6. This activity takes one hour
Next	**Solution of systems of linear equations**

6.3 Assessment Questions

1. Reduce the matrix $\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 7 & -4 & 1 \end{pmatrix}$ to echelon form

2. Determine if the following matrices are in echelon canonical form or not.

(a) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & 5 & -2 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 8 & 7 & -3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

6.4 References

1. Linear Algebra: Schaum's Outline Series
2. Linear Algebra by J. N. Sharma, A.R. Vasishta
3. Linear Algebra by Michael O'nan, Herbert Enderton
4. https://youtu.be/G1_8E4oEVII
5. https://youtu.be/l69YjkuUym0
6. https://youtu.be/1rBU0yIyQQ8

LESSON 7

SOLUTION OF SYSTEMS OF LINEAR EQUATIONS -(GAUSS-JORDAN METHOD)

7.1 Introduction

During the previous lesson you were introduced to finding inverse of a matrix using row reduction method, the general concept of reducing a matrix to echelon form and finally you were introduced to the concept of the canonical form of a matrix.

This lesson starts with a general introduction to the solution of a system of linear equations, followed by a more detailed section on the solution of equations using Gauss Jordan method. Other methods will be covered in lesson 8

7.2 Learning Outcomes

By the end of this lesson, you will be able to;

7.2.1 Solve a system of linear equations using Gauss Jordan method.

7.2.1 Solution of systems of linear equations

A Linear equation is an equation of the form

$a_1\ x_1 + a_2\ x_2 + \cdots + a_n\ x_n = b$

$a_1, a_2, \cdots, a_n, b$ are constants while $x_1, x_2, \cdots x_n$ are the variables to be determined. A solution to the equation is an *n*-tuple $(c_1, c_2, \cdots, c_n)$ such that

$a_1\ c_1 + a_2\ c_2 + \cdots + a_n\ c_n = b$

Example: $5x_1 - 3x_2 + 6x_3 - 4x_2 = 10$; $x_1 = 3$; $x_2 = 5$; $x_3 = 3$; $x_4 = 2$. i.e. (3,5,3,2) is a solution.

Two systems of equations are equivalent if they have the same solution set. We get equivalent system if we perform any of the following:

1. Change the order of listing the equations
2. Multiply one or more equations by a non-zero constant.
3. Add a multiple of one equation to a multiple of another equation.

There are various methods of solving a system of linear equations

Elimination Method:

The strategy is to eliminate one variable at a time:

Example:

Solve by elimination method

$x_1 + x_2 + x_3 = 5 \quad \cdots \qquad (1)$

$x_1 + 2x_2 + 3x_3 = 10 \cdots \qquad (2)$

$2x_1 + x_2 + x_3 = 0 \cdots \qquad (3)$

Solution:

Eliminate x_1

(2) – (1) $x_2 + 2x_3 = 5 \cdots$ (4)

(3)-2(1) $-x_2 - x_3 = -4 \cdots$ (5)

Eliminate x_2

(4) +(5) $x_3 = 1$

Substitute in equation 5

$-x_2 - 1 = -4$, $x_2 = 3$

Substitute in equation 1

$x_1 + 3 + 1 = 5$, $x_1 = 1$

The elimination method is very tedious when there are many variables. We use a much organized elimination method in matrix form called Gauss-Jordan method.

7.2.2 Gauss-Jordan elimination method

Given a linear system

$$a_{11}\ x_1 + a_{12}\ x_2 + \cdots + a_{1n} x_n = b_1$$

$$ax_1\ x_1 + a_{22}\ x_2 + \cdots + a_{2n} x_n = b_2$$

$$\vdots$$

$$a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n = b_n$$

The system can be represented in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Or simply $Ax = b$

Where A is the n$\times$ n is the coefficient matrix

$x = (x_1, x_2 \cdots x_n)^T$, $b = [b_1, b_2 \dots b_n]^T$

An augmented matrix is the matrix $[A/b]$

The Gauss-Jordan method involves reducing the augmented matrix to reduced row echelon form.

Example:

Solve using Gauss-Jordan method

$$2x_1 - 4x_2 + 6x_3 = 20$$

$$3x_1 - 6x_2 + x_3 = 22$$

$$-2x_1 + 5x_2 - 2x_3 = -18$$

Solution:

Matrix form

$$\begin{bmatrix} 2 & -4 & 6 \\ 3 & -6 & 1 \\ -2 & 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 22 \\ -18 \end{bmatrix}$$

Augmented matrix

$$\left[\begin{array}{ccc|c} 2 & -4 & 6 & 20 \\ 3 & -6 & 1 & 22 \\ -2 & 5 & -2 & -18 \end{array}\right]$$

Divide row 1 by 2 i.e. $\frac{1}{2}R_1$ and use it to reduce a_{21} and a_{31} to zeros

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 10 \\ 3 & -6 & 1 & 22 \\ -2 & 5 & -2 & -18 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 10 \\ 0 & 0 & -8 & -8 \\ 0 & 1 & 4 & 2 \end{array}\right] R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 10 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & -8 & -8 \end{array}\right] R_1 + 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 11 & 14 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & -8 & -8 \end{array}\right] \begin{array}{l} 8R_1 + 11R_3 \\ 2R_2 + R_3 \\ \end{array} \qquad \left[\begin{array}{ccc|c} 8 & 0 & 0 & 24 \\ 0 & 2 & 0 & -4 \\ 0 & 0 & -8 & -8 \end{array}\right] \begin{array}{l} R_1/8 \\ R_2/2 \\ R_3/-8 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \text{Solution} \quad x_1 = 3, \; x_2 = -2, \; x_3 = 1$$

The general procedure to solve $Ax = b$ by Gauss-Jordan method is as follows:

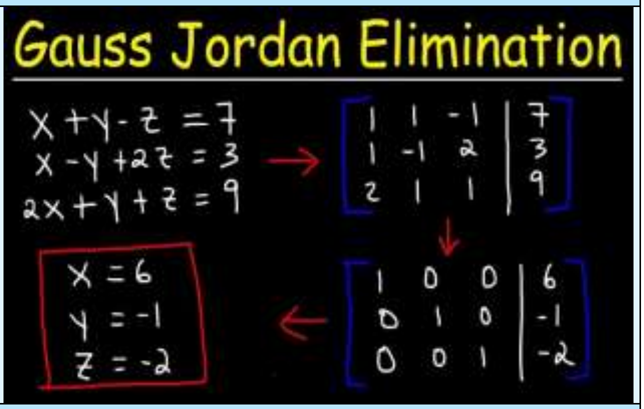
1. Get the augmented matrix $[A|b]$
2. Get 1 in (1,1) position of matrix by
   a) rearranging the rows
   b) dividing row 1 by $a_{11} \neq 0$
3. Get zeros in all other positions of column 1
4. Get 1 in (2,2) position by rearranging rows or dividing all of row 2 by $a_{22} \neq 0$
5. Get zeros in all other positions of column 2.
6. Get 1 in (3,3), (4,4)… and in each case get zeros in other positions of that column.
7. Each row gives the solution

$$\left[\begin{array}{c|c} & c_i \\ I & c_2 \\ & \vdots \\ & c_n \end{array}\right] x_1 = c_1\,; x_2 = c_2, \ldots x_n = c_n$$

A system of linear equations may have a unique solution, many solutions or no solution.
Unique solution: $[A|b]$ reduces to $(I|c)$
No solution: The last row is a form $00\ldots a \neq 0$
Many solutions: Some variables can be written in terms of others

Previous examples have unique solutions

E-tivity 7.2.1 Gauss-Jordan elimination method

Numbering, pacing and sequencing	7/2.1
Title	**Gauss-Jordan elimination method**
Purpose	Solve a system of linear equations using Gauss Jordan method.
Brief summary of overall task	Watch the video on Gauss-Jordan elimination method of solving a system of equations
Spark	
Individual task	After watching the video, Solve the following system of linear equations using Gauss Jordan method. $2x-5y+2z=7$ $x+2y-4z=3$ $3x-4y-6z=5$
Interaction begins	Compare your answers to those posted by two of your colleagues in Discussion forum 7.2.1. How does your answer compare with other answers?
E-moderator interventions	Observe that in order to be sure whether the answers are correct, all you need to do is to substitute the values of the answers back into the original equation. If the answers are correct, the RHS and LHS of the equations will be equal
Schedule and time	Week 7. This activity will take 2 hours
Next	**Cramer's Rule**

7.3 Assessment Questions

Solve the following system by Gauss Jordan (elimination) method.

1. $$\begin{aligned} x_1 - x_2 + x_3 + 2x_4 &= 1 \\ 2x_1 - x_2 + 3x_4 &= 0 \\ -x_1 + x_2 + x_3 + x_4 &= -1 \\ x_2 + x_4 &= 1 \end{aligned}$$

2. $$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 4 \\ x_1 + 2x_2 - x_3 - x_4 &= 7 \\ 2x_1 - x_2 - x_3 - x_4 &= 8 \\ x_1 - x_2 + 2x_3 - 2x_4 &= -7 \end{aligned}$$

3. $$\begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 3 \\ x_1 + 2x_2 + x_3 + x_4 &= 2 \\ x_1 + x_2 + x_3 + 2x_4 &= 1 \\ 2x_1 + x_2 + x_3 + x_4 &= 4 \end{aligned}$$

4. $$\begin{aligned} X_1 - X_2 + 2X_3 &= 2 \\ 2X_1 + 3X_2 - X_3 &= 14 \\ 3X_1 + 2X_2 + X_3 &= 16 \\ X_1 + 4X_2 - 3X_3 &= 12 \end{aligned}$$

7.4 References

1. Linear Algebra: Schaum's Outline Series
2. Linear Algebra by J. N. Sharma, A.R. Vasishta
3. Linear Algebra by Michael O'nan, Herbert Enderton
4. https://youtu.be/eYSASx8_nyg

LESSON 8

CRAMER'S RULE AND INVERSE MATRIX METHOD

8.1 Introduction

This previous gave a general introduction to the solution of a system of linear equations, followed by a more detailed section on the solution of equations using Gauss Jordan method.

In this lesson you will be introduced to the solution of a system of linear equations using Cramer's rule, or the method of determinants. You will also learn how to use the inverse matrix method to solve as system of linear equations. Each method is illustrated using several examples

8.2 Lesson Learning Outcomes

By the end of the lesson, you will be able to;

8.2.1 Solve a system of linear equations using Cramer's rule, or the method of determinants.

8.2.2. Solve a system of linear equations using inverse matrix method

8.2.1 Cramer's Rule

This method uses determinants to solve a linear system $Ax = b$ provided det A (coefficient matrix) is nonzero.

Suppose $\det A \neq 0$, A square matrix $x = A^{-1}b$, $A^{-1} = \frac{1}{\det A} adj(A)$

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} |A_{11}| & -|A_{21}| \\ -|A_{12}| & |A_{22}| \\ \vdots & \vdots \\ & \end{bmatrix}^T \text{; Let } \overline{a_{ij}} = (-1)^{i+j} \quad |A_{ij}|\text{; } A^{-1} = \frac{1}{\det A}\begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots & \overline{a_{n1}} \\ \overline{a_{12}} & \overline{a_{22}} & & \overline{a_{n2}} \\ \overline{a_{1n}} & \overline{a_{2n}} & & \overline{a_{nn}} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} \overline{a_{11}} & a_{21} & \cdots & a_{n1} \\ \overline{a_{12}} & a_{22} & & \overline{a_{n2}} \\ \vdots & \vdots & & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & & \overline{a_{nn}} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$x_j = \frac{1}{\det A}\left[b_1\,\overline{a_{1j}} + b_2\,\overline{a_{2j}} + \ldots + b_n\,\overline{a_{nj}}\right]$$

But $b_1\,\overline{a_{ij}} + b_2\,\overline{a_{2j}} + \ldots + b_n\,\overline{a_{nj}}$ is the determinant of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{ij-1} & b_1 & a_{ij+1} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{1j-1} & b_2 & a_{2j+1} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj-1} & b_n & b_{nj+1} & \dots & a_{nn} \end{bmatrix}$$

i.e. matrix A with column j replaced with $(b_1, b_2 \dots, b_n)$. The expansion is done along column j. Note that the determinant of A expanded along column j is

$a_{1j}\ \overline{a_{ij}} + a_{2j}\ \overline{a_{2j}} + \dots + a_{nj}\ \overline{a_{nj}}$

Thus by replacing columns j by $[b_1, b_2 \cdots b_n]$ we have the determinants of the new matrix.

For 3×3 matrix $Ax = b$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$X_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \qquad X_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & b_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \qquad X_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

Example

Solve the linear system using Crammer's rule

$$\begin{aligned} -2x_1 + 3x_2 - x_3 &= 1 \\ x_1 + 2x_2 - x_3 &= 4 \\ -2x_1 - x_2 + x_3 &= -3 \end{aligned}$$

Solution

$$|A| = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2 \text{ .Then} \qquad x_1 = \frac{\begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix}}{|A|} = \frac{-4}{-2} = 2,$$

$$x_2 = \frac{\begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix}}{|A|} = \frac{-6}{-2} = 3,$$

And $x_3 = \dfrac{\begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{vmatrix}}{|A|} = \dfrac{-8}{-2} = 4$

E-tivity 8.2.1 Cramer's rule

Numbering, pacing and sequencing	8.2.1
Title	Cramer's rule
Purpose	To solve a system of linear equations using Cramer's rule,
Brief summary of overall task	Watch the video on Cramer's rule
Spark	
Individual task	After watching the video, follow the steps illustrated and solve $\begin{array}{l} x_1 + 3x_2 + 2x_3 = 3 \\ 2x_1 + 4x_2 + 2x_3 = 8 \\ x_1 + 2x_2 - x_3 = 10 \end{array}$ using Cramer's rule
Interaction begins	Go to discussion forum 8.2.2 and compare your solution to that of three of your colleagues. If they have made mistakes correct them and if you have done a mistake learn from them
E-moderator interventions	Observe that the reason this method is also called the method of determinant is because it involves finding the determinants of four 3X3 matrices. The first determinant to find should be that of the coefficient matrix. This is because if the determinant is zero, then the system had no solution and hence there is no need to proceed to the other determinants
Schedule and time	Week 8. This activity will take 2 hours
Next	**Inverse matrix method**

8.2.2 Inverse matrix method

Consider the system of linear equations represented in matrix form $A\,x = b$. Where A is an m × n matrix.

If det $A \neq 0$, then A^{-1} exists and $A^{-1}(A\,x) = A^{-1}b$; $Ix = x = A^{-1}b$. This method works only

if det $A \neq 0$, and there is a unique solution $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A^{-}b = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

Examples:

1. Solve the linear system using the inverse matrix

$x + 2y = 6$

$4x + 3y = 3$

Solution:

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}; \quad A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \quad \det\ A = 3 - 8 = -5; \quad A^{-1} = -\frac{1}{5}\begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\,b = -\frac{1}{5}\begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 12 \\ -21 \end{bmatrix} = \begin{bmatrix} -12/5 \\ 21/5 \end{bmatrix};$$ **Solution**: $x = -12/5$, $y = 21/5$

2. Solve the linear system using the inverse matrix

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 3 \\ 2x_1 + 4x_2 + 2x_3 &= 8 \\ x_1 + 2x_2 - x_3 &= 10 \end{aligned}$$

Solution:

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 10 \end{bmatrix}; \ A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 2 \\ 1 & 2 & -1 \end{bmatrix}; \ \det \mathrm{A} = 4; \ A^{-1} = \begin{bmatrix} -2 & 7/4 & -1/2 \\ 1 & -3/4 & 1/2 \\ 0 & 1/4 & -1/2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 & 7/4 & -1/2 \\ 1 & -3/4 & 1/2 \\ 0 & 1/4 & -1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}; \ x_1 = 3, \ x_2 = 2, \ x_3 = -3$$

E-tivities 8.2.2 Inverse matrix method

Numbering, pacing and sequencing	**8.2.1**
Title	Solution of a system of linear equations in three unknowns using inverse Matrix method
Purpose	The purpose of the task is to enable you obtain the solution of three equations in three unknowns using inverses of matrices

Brief summary of overall task	Watch videos on these links
Spark	
Individual task	After watching the videos, solve the following equations using inverse matrix method. 1. $\begin{aligned} 3x+2y-z&=4 \\ -x+4y-2z&=1 \\ 2x-5y+z&=-5 \end{aligned}$ 2. $\begin{aligned} 5b+2a+4c&=12 \\ a-2b+2c&=-3 \\ 3a+b+16&=c \end{aligned}$
Interaction begins	(a) Follow the solutions posted by at least 2 of your colleagues (b) In discussion forum 3.2.1 state what you have learnt from their method
E-moderator interventions	(a) How did the rearrangement of equations in question 2 affect the solution (b) Do all systems of equations in three unknowns have solutions (c) It is important to note that (i) When using matrices to solve equations, all the unknowns in the equation have need to be aligned in order to obtain the correct coefficient matrix (ii) If the coefficient matrix has a zero determinant, the equations have no solution
Schedule and time	This activity should take one hour
Next	**Vectors**

8.3 Assessment Questions

Solve using Crammer's rule.

1. $$\begin{aligned} 2x-5y+2z&=7 \\ x+2y-4z&=3 \\ 3x-4y-6z&=5 \end{aligned}$$

$$x_1 + 3x_2 + 2x_3 = 3$$

2. $2x_1 + 4x_2 + 2x_3 = 8$

$$x_1 + 2x_2 - x_3 = 10$$

3. Solve the following three equations using inverse matrix method

$$\begin{aligned} x + y + 2z &= 1 \\ x + 2y - z &= -2 \\ x + 3y + z &= 5 \end{aligned}$$

4.
$$\begin{aligned} x + 2y + z &= 4 \\ 3x - 4y - 2z &= 2 \\ 5x + 3y + 5z &= -1 \end{aligned}$$

8.4 References

1. A First Course in Linear Algebra by Daniel Zelisky
2. Elementary Linear Algebra by Bennard Kolman
3. Elementary Linear Algebra by Howard Anton
4. https://youtu.be/qmjapjGxf2s
5. https://youtu.be/JyISylNXGzE-INVERSE

LESSON 9

VECTORS

9.1 Introduction

In the previous lesson you were taught how to find the solution of a system of linear equations using Cramer's rule, (the method of determinants) and also using the inverse matrix method. In this lesson you will revise the topic of vectors. You will cover the topic of dot product and cross product of vectors. The application of dot product in looking for the angle between vectors is also included. Application of cross product in the equations of lines and planes is covered in the last lecture in the module.

9.2 Learning Outcomes

By the end of the lesson, you will be able to;

9.2.1 Perform addition subtraction and scalar multiplication of vectors

9.2.2 Find the dot product and cross product of vectors

9.2.3 Use the dot product to find the angle between given vectors

9.2.1. Vectors (Co-Ordinate Systems)

Any point on the Cartesian *x-y* plane can be represented as a pair (x_0, y_0) where x_0 is the *x* coordinate and y_0 is *y*-coordinate. The Cartesian plane is simply $\Re^2$. $\Re^3$ consists of three planes, *x-y* plane, *x-z* plane & *y-z* planes. The *x*-axis, *y*-axis and *z* – axes all meet at 90° (are perpendicular) at the origin (0,0,0). The *x*-axis consists of points $(x_0, 0, 0)$, *y*-axis $(0, y_0, 0)$ and *z*-axis $(0, 0, z_0)$. The *x-y* plane $(x_0, y_0, 0)$ the *x-z* plane $(x_0, 0, z_0)$ and the *y-z* plane $(0, y_0, z_0)$. $\Re^n$ consists of all n-tuples consisting of real entries i.e. $\Re^n = \{(a_1, a_2, \ldots, a_n) : a_i \in \Re \; \forall_i = 1, 2, \ldots n\}$

Vectors: (Quantities defined by magnitude & direction)

A **column vector is** an n-tuple of numbers written vertically $\begin{bmatrix} a_i \\ \vdots \\ a_n \end{bmatrix}$ (or n × 1 matrix)

If the numbers $a_1, a_2 \ldots, a_n$ are real we have a real column vector. If they are complex we have a complex column vector. The number a_i in the i^{th} (row) slot is the i^{th} component.
A row *n*-vector is an *n*-tuple $[a_1, a_2, \ldots, a_n]$ of numbers written horizontally.

Equality of vectors

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \Rightarrow m = n \quad a_i = b_i \quad \forall_i = 1, \ldots n \qquad i.e. \; a_1 = b_1, \; a_2 = b_2, \ldots, a_n = b_n$$

Addition & Subtraction

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \pm \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ \vdots \\ a_n \pm b_n \end{bmatrix}$$

Scalar multiplication

$$\alpha \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \alpha a_1 \\ \alpha a_2 \\ \vdots \\ \alpha a_n \end{bmatrix}$$ Where α is real scalar or complex scalar.

Properties:

Let a, b, c be n-column vectors , λ, μ scalars

1. $a+b=b+a$ - **Commutative law**
2. $\left(a+b\right)+c=a+\left(b+c\right)$ - **Associative law**
3. $0+a=a+0=a$ - **Additive identity**
4. $a+(-a)=(-a)+a=0$ **Additive inverse**
5. $\lambda\left(a+b\right)=\lambda a+\lambda b$
6. $(\lambda+\mu)a=\lambda a+\mu a$; $\quad (\lambda\mu)a=\lambda\left(\mu a\right)$

Length/Magnitude of Vector:

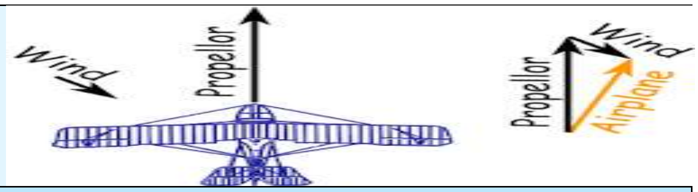
Let $a=[a_1, a_2, \ldots, a_n]$

$$|a|=\sqrt{a_1^2+a_2^2+\ldots+a_n{}^2}$$

$$|a+b|\le|a|+|b| \quad \left|r\,a\right|=|r|\left|a\right|$$

E-tivity 9.2.1- Vectors (Co-Ordinate Systems)

Numbering, pacing and sequencing	9.2.1
Title	Vectors (Co-Ordinate Systems)
Purpose	To perform operations of vector addition, subtraction and multiplication
Brief summary of overall task	Watch the video on operations of vectors

Spark	
Individual task	After watching the video find ***5u-6v*** where $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and $v = -\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$
Interaction begins	Compare your answers with those of your classmates (At least three of them) in Discussion forum 9.2.1
E-moderator interventions	Note that these operations are similar to those performed on algebraic expressions
Schedule and time	Week 9. The activity will take 30 minutes
Next	**Dot Product**

9.2.2 Dot Product

Let $\vec{a} = [a_1, a_2, \ldots, a_n]$, $\vec{b} = [b_1, b_2, \ldots, b_n]$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$

$\vec{a} \cdot \vec{b} = a^T\, b$ if a & b are column vectors

$= a\, b^T$ if a & b are row vectors.

Properties of the dot product:

1. $u \cdot v = v \cdot u$ Commutative property
2. $u \cdot (v + w) = u \cdot v + u \cdot w$ Distributive Property
3. $(ru) \cdot v = u \cdot rv = r(u \cdot v)$ Homogeneous property
4. $u \cdot u \geq 0$ and $u \cdot u = 0$ iff $u = 0$

Note: $|a| = \sqrt{a \cdot a}$

Definition: A unit vector is a vector of length (magnitude) 1 i.e $|a| = \sqrt{a \cdot a}$ =1

Let *u* and *v* be nonzero vectors and let θ be the angle between them.

Then

$$u \cdot v = |u|\ |v| \cos\ \theta$$

$$\cos\ \theta = \frac{u \cdot v}{|u|\ \ |v|}$$

$$\theta = \cos^{-1} \frac{u \cdot v}{|u|\ |v|}$$

Example:

Find the angle between $u = i + j + k$ and $v = i + j - k$

$(i+j+k)\cdot(i+j-k)=|i+j+k|\ |i+j-k|\cos\theta$

$1+1-1=(\sqrt{3})(\sqrt{3})\ \cos\ \theta$

$\text{Cos}\ \theta = {}^{1}\!/_{3},\quad \theta=\cos^{-1}\left(\frac{1}{3}\right)=70.5^{0}$

Orthogonal /Perpendicular vectors:

The vectors u and v are orthogonal/perpendicular if the angle between then is 90^{0}.
$\theta=90^{0}$, $\cos\ 90^{0}\ =0$
and so $u\cdot v=|u|\ |v|\ \cos\ 90^{\circ}$ and therefore $u\cdot v=|u|\ |v|\ (0)=0$ $\Rightarrow u.v=0$ (where $u\neq 0, v\neq 0$).

Exercise: Show that the vectors $u=[\sin\theta, \cos\theta]$ and $v=[\cos\theta-\sin\theta]$ are orthogonal

E-tivity 9.2.2- Dot Product

Numbering, pacing and sequencing	9.2.2
Title	Dot product
Purpose	To find the dot product of vectors and use it to calculate the angle between vectors
Brief summary of overall task	Watch the video on dot product
Spark	
Individual task	After watching the video, use dot product to find the angle between the vectors $\underset{\sim}{u}=-\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k}$ and $\underset{\sim}{v}=2\underset{\sim}{i}-\underset{\sim}{j}+2\underset{\sim}{k}$
Interaction begins	Compare your answer to at least two of your classmates in discussion forum 9.2.2 and see what you can learn from them.
E-moderator interventions	Note that the angle between the vectors is 90, if and only if the dot product is zero
Schedule and time	Week 9. This activity will take 30 minutes
Next	Cross Product

9.2.3 Cross Product

Cross/vector product:

The cross product of $\vec{a} = a_1 i + a_2 j + a_3 k$ and $\vec{b} = b_1 i + b_2 j + b_3 k$, denoted by $\vec{a}\times\vec{b}$ is a vector orthogonal to both $\vec{a}$ and $\vec{b}$ defined by

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix} + k\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Example: Let $\overrightarrow{a} = \left(3i - j + k\right)$; $\overrightarrow{b} = \left(i + 2j - k\right)$

(a) Find $\vec{a}\times\vec{b}$
(b) Show that $\vec{a}\times\vec{b}$ is orthogonal to $\vec{a}$
(c) Show that $\vec{a}\times\vec{b}$ is orthogonal to $\vec{b}$

Solution:

(a) $\left(3i - j + k\right)\times\left(i + 2j - k\right) = \begin{vmatrix} i & j & k \\ 3 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = i(1-2) - j(-3-1) + k(6--1) = -i + 4j + 7k$

(b) $\overrightarrow{a}\cdot(\vec{a}\times\vec{b}) = \left(3i - j + k\right)\cdot\left(-i + 4j + 7k\right) = -3 - 4 + 7 = 0$ and therefore $\vec{a}\times\vec{b}$ is orthogonal to $\vec{a}$

(c) $\overrightarrow{b}\cdot(\vec{a}\times\vec{b})\left(i + 2j - k\right)\cdot\left(-i + 4j + 7k\right) = -1 + 8 - 7 = 0$ and therefore $\vec{a}\times\vec{b}$ is orthogonal to $\vec{b}$

$$\vec{a}\cdot(\vec{b}\times\overrightarrow{c}) = (a_1 i + a_2 j + a_3 k)\cdot\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (a_1 i + a_2 j + a_3 k)\cdot\left[i\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - j\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + k\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right]$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Properties of the cross product:
Let a, b, c be vectors, α, β and y scalars

1. $a\times b = -b\times a$
2. $a\times\left(b + c\right) = \left(a\times b\right) + \left(a\times c\right)$
   $\left(a + b\right)\times c = a\times c + b\times c$
3. $\alpha\left(a\times b\right) = \left(\alpha a\times b\right) = a\times\alpha b$

4. $a \times a = -(a \times a)$

Two vectors $\vec{a}$ and $\vec{b}$ are parallel if $\vec{a} = k\vec{b}$ for some scalar *k*.

E.tivity 9.2.3 –Cross Product

Numbering, pacing and sequencing	9.2.3
Title	Cross Product
Purpose	To calculate the vector product of given vectors
Brief summary of overall task	Watch the video on cross product of vectors
Spark	
Individual task	After watching the video, 1. prove that $\underset{\sim}{a} \times (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{a} \times \underset{\sim}{c}$ given $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}, \underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{c} = 3\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$ to 2. Show that for any three vectors $\underset{\sim}{a} = a_1 i + a_2 j + a_3 k, \underset{\sim}{b} = b_1 i + b_2 j + b_3 k$ and $\underset{\sim}{c} = c_1 i + c_2 j + c_3 k$ $\underset{\sim}{a} \times (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{a} \times \underset{\sim}{c}$
Interaction begins	Go to discussion forum 9.2.3 and compare your answers to those of at least three classmates. See what you can learn from them
E-moderator interventions	Observe that: 1. These operations are similar to those performed on algebraic expressions 2. The cross product can be used to evaluate the determinant of a matrix
Schedule and time	Week 9. This activity will take two hours
Next	Vector spaces

9.3 Assessment Questions

1. Compute the length of the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

2. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.Find the angle between ***u*** and ***v***.

3. Show that the vectors $\mathbf{u} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ are orthogonal

9.4 References

1. A First Course in Linear Algebra by Daniel Zelisky
2. Elementary Linear Algebra by Bennard Kolman
3. Elementary Linear Algebra by Howard Anton
4. https://youtu.be/gCWiw5ZqjnA
5. https://youtu.be/Te8eL5r7aJs
6. https://youtu.be/gPnWm-IXoAY

LESSON 10

VECTOR SPACES

10.1Introduction

The previous lesson revised the topic of vectors. You covered the topic of dot product and cross product of vectors and the application of dot product in finding the angle between vectors was also included.

10.2 Learning Outcomes

By the end of the lesson, you will be able to:

10.1.1 Show that a given set is a vector space

10.1.2 Show that a subset of a given vector space is a subspace of the vector space

10.2.1 Vector space

Definition: Let K be a given field and let V be a non empty set with rules of addition and scalar multiplication which assigns to any $u, v \in V$ a *sum* $u + v \in V$ and to any $u \in V, k \in K$ a *product* $ku \in V$. Then V is called a vector space over K (and the elements of V are called vectors) if the following axioms hold:

$[A_1]$: For any vectors $u, v, w \in V$, $(u+v)+w = u+(v+w)$.

$[A_2]$: There is a vector in V, denoted by 0 and called the *zero vector*, for which $u + 0 = u$ for any vector $u \in V$

$[A_3]$: For each vector $u \in V$ there is a vector in V, denoted by $-u$, for which $u + (-u) = 0$.

$[A_4]$: For any vectors $u, v \in V$, $u + v = v + u$.

$[M_1]$: For any scalar $k \in K$ and any vectors $u, v \in V$, $k(u+v) = ku + kv$.

$[M_2]$: For any scalars $a, b \in K$ and any vector $u \in V$, $(a+b)u = au + bu$.

$[M_3]$: For any scalars, $a, b \in K$ and any vector $u \in V$, $(ab)u = a(bu)$.

$[M_4]$: For the unit scalar $1 \in K$, $1u = u$ for any vector $u \in V$.

Examples:

1. Show that the set $\Re^n = \{(a_1, a_2, \ldots, a_n) | a_n \in \Re\}$ is a vector space.
2. Show that $M_{n,m}(\Re)$, the set of all $n \times m$ matrices is a vector space.
3. Show that $p_{\Re}(x)$, the set of all polynomials in x with real coefficients is a vector space.
4. Show that $F(\Re)$, the set of all functions $f : \Re \to \Re$, is a vector space.
5. Define $\Re^{n+}$ to be the set of n-tuples $(x_1, \ldots, x_n)$ such that $x_i > 0 \quad \forall_i$
   Define addition and scalar multiplication by:
   $(x_1, x_2, \ldots, x_n) \oplus (y_1, y_2, \ldots, y_n) = (x_1 y_1, \ldots, x_n y_n)$
   $\lambda \otimes (x_1, \ldots x_n) = (x_1^{\lambda}, \ldots, x_n^{\lambda})$. Show that $\Re^{n+}$ is a vector space.

Solution:

Condition 1

$$(x_1, x_2, \ldots, x_n) \oplus (y_1, y_2, \ldots, y_n) = (x_1 y_1, \ldots, x_n y_n) \in \Re^{n+}$$

Condition 2

$$\lambda \otimes (x_1, \ldots x_n) = (x_1^{\lambda}, \ldots, x_n^{\lambda}) \in \Re^{n+}$$

Condition 3

$$0 \otimes (x_1, \ldots, x_n) = (x_1^0, \ldots, x_n^0) = (1, \ldots, 1)$$

$(x_1, x_2, \ldots, x_n) \oplus (1, \ldots, 1) = (x_1, \ldots, x_2) \Rightarrow (1, \ldots, 1)$ is the zero vector

Condition 4

$$(x_1, \ldots, x_n) \oplus \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right) = \left(x_1\left(\frac{1}{x_1}\right), x_2\left(\frac{1}{x_2}\right), \ldots, x_n\left(\frac{1}{x_n}\right)\right) = (1, 1, \ldots, 1)$$

Then additive inverse of $(x_1,\dots,x_n)$ is $\left(\frac{1}{x_1},\dots,\frac{1}{x_n}\right)$. Therefore $\Re^{n+}$ is a vector space.

E-tivity 10.2.1 – Vector space

Numbering, pacing and sequencing	10.2.1
Title	Vector space
Purpose	To show whether a given set is a vector space
Brief summary of overall task	Watch the video on the definition of a vector space
Spark	
Individual task	Watch the video on vector space and summarize the properties of vector spaces
Interaction begins	Follow the answers given by at least three of your colleagues in Discussion forum 10.2.1 and compare their solutions with yours
E-moderator interventions	Observe that these properties have mainly to do with addition, scalar multiplication and distribution
Schedule and time	Week 10. This activity will take one hour
Next	**Subspace of a subspace**

10.2.2 Subspace of a subspace

Definition: Let V be a vector space. A subset W of V is a subspace of V if W fulfils the requirements of a vector space, where addition and scalar multiplication in W produce the same vectors as these operations did in V.

Lemma: (Test for subspace)
A non-empty subset W of a vector space V is a subspace of V if and only if
(i) $\forall\, u, w \in W,\quad u + w \in W$ **-Closure under addition:**
(ii) $\forall\, r \in \Re, w \in W,\quad rw \in W$ **-Closure under scalar multiplication.**
i.e. W is a subspace of a vector space V *iff* its closed under vector addition and scalar multiplication.

Examples:
1. The set of diagonal n $\times$ n matrices is a subspace of M_n, the set of (n $\times$ n matrices).
2. Let $W \subset \Re^n$ with $w = (w_1 \dots w_n)/W_i \in$ □ with

a) $w_1 = 0$

b) w_1 even

c) w_1 is divisible by k

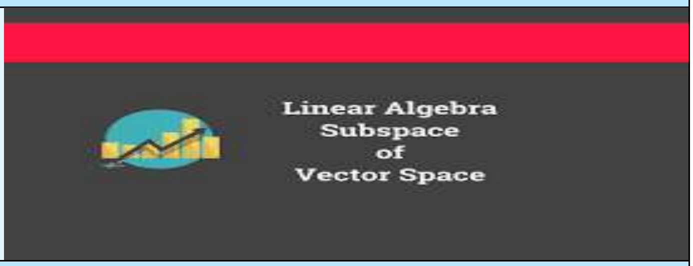
Show that each of them is a vector space.

3. The set D of all differentiable functions from $f:\Re \to \Re$ is a subspace of F, the set of all functions $f:\Re \to \Re$

4. (a)The set of all functions $f:\Re \to R$ such that $f(0)=1$ is not a subspace of the set $f:\Re \to \Re$.

(b) The set of all functions $f:\Re \to R$ such that $f(1)=0$ is a subspace of the set $f:\Re \to \Re$.

E-tivity 10.2.2- Subspace of a subspace

Numbering, pacing and sequencing	10.2.2
Title	Subspace of a subspace
Purpose	To show that a subset of a given vector space is a subspace of the vector space
Brief summary of overall task	Read the definition of a subspace of a vector space And watch a video on the same
Spark	
Individual task	Follow the definition and illustration in the video and give at least three examples of vector spaces
Interaction begins	In discussion forum 10.2.2, read the examples given by four of your classmates (at least) and comment on them. Get their feedback also and see how to improve on your answer
E-moderator interventions	The most important point to note here is that when proving that a given subset of a vector space is a subspace, only two properties are needed; the rest of them are inherited from the '*parent set*'
Schedule and time	Week 10. This activity will take two hours
Next	Linear Combinations

10.3 Assessment Questions

1. Show that the line $y = 2x$ is a subspace of $\Re^2$

2. Show that the line $y = x + 1$ is not a subspace of $\Re$ .i.e. $W = \{(x, y) / y = x + 1\}$
$= \{(x, x+1) / x \in \Re\}$

3. Show that the set of all **invertible** n × n matrices is not a subspace of the set $M_n(R)$ of all n × n matrices.

10.4 References

1. Linear Algebra by Fraleigh & Beauregard
2. Linear Algebra: Schaum's Outline Series
3. Linear Algebra by J. N. Sharma, A.R. Vasishta
4. https://youtu.be/EP2ghkO0lSk
5. https://www.researchgate.net/profile/Sepideh_Stewart/publication/37986944/figure/fig37/AS:650477213597712@1532097261446/Definition-of-a-subspace-in-students-course-manual.png
6. https://youtu.be/Eawc_ZuQI_8

LESSON 11

LINEAR COMBINATIONS

11.1Introduction

In lesson 10, you were introduced to the concept of a vector space and a subspace of a vector space.
This lesson covers linear combinations and linear spans. You will learn how to write a given vector as a linear combination of other vectors. The relationship between linear spans, vector spaces and subspaces will also be discussed.

11.2 Learning Outcomes

By the end of the lesson, you will be able to;

11.2.1 Write a given vector as a linear combination of a given set of vectors

11.2.2 Verify relationships between a linear span, a vector space and a subspace of a vector space

11.2.1 Linear Combinations

Let $v_1, v_2, \ldots, v_n$ be vectors in a vector space V over scalar field K, and $a_1, a_2, \ldots a_n \in K$. A linear combination of the vectors $v_1, v_2, \ldots, v_n$ is any expression of the form
$v = a_1v_1 + a_2 v_2 + \ldots + a_nv_n$

Example 1 $2(3-1, 4)+5(1,0,-1)-6(0,0,1)$ is a linear combination of (3, -1, 4), $(1,0,-1)$ and $(0,0,1)$.

Example 2 Write the vector $v = (1,-2,5)$,as a linear combination of the vectors $e_1 = (1,1,1), e_2 = (1,2,3)$ and $e_3 = (2,-1,1)$.
Solution: We wish to express v as $v = xe_1 + ye_2 + ze_3$, with x, y and z as yet unknown scalars. Thus we require

$$
\begin{aligned}
(1,-2,5) &= x(1,1,1) + y(1,2,3) + z(2,-1,1) \\
&= (x,x,x) + (y,2y,3y) + (2z,-z,z) \\
&= (x+y+2z, x+2y-z, x+3y+z)
\end{aligned}
$$

Form the equivalent system of equations by setting corresponding components equal to each other, and then reduce to echelon form:

$$
\begin{matrix}
x+y+2z=1 & & x+y+2z=1 & & x+y+2z=1 \\
x+2y-z=-2 & \text{or} & y-3z=-3 & \text{or} & y-3z=-3 \\
x+3y+z=5 & & 2y-z=4 & & 5z=10
\end{matrix}
$$

Note that the above system is consistent and so has a solution. Solve for the unknowns to obtain $x=-6, y=3, z=2$ .Hence $v = -6e_1 + 3e_2 + 2e_3$.

The span of $\{v_1, v_2, \ldots, v_n\}$ over K is the set of all linear combinations of $v_1, v_2, \ldots, v_n$.

i.e. Span $\left(v_1, v_2, \ldots, v_n\right) = \left\{a_1 v_1 + a_2 v_2 + \ldots + a_n\ v_n \,/\, a_1, a_2 \ldots a_n \in K\right\}$

For real vector spaces, Span $\left(v_1, v_2, \ldots, v_n\right) = \left\{a_1 v_1 + a_2 v_2 + \ldots + a_n v_n \,/\, a_i \in \Re\right\}$

Theorem: Span $\left(v_1, v_2, \ldots, v_n\right)$ is a subspace of the vector space V.

Proof: Show closure under addition & scalar multiplication.

Let $u, w \in$ span $\left(v_1, v_2, \ldots, v_n\right)$

Condition 1 of a subspace-**Closure under addition:**

Let $u = a_1\ v_1 + a_2\ v_2 + \ldots + a_n v_n$ for some $a_i \in \Re$ and $w = b_1 v_1 + b_2 v_2 + \ldots + b_n v_n$ for some $b \in \Re$

$u + w = \left(a_1 + b_1\right) v_1 + \left(a_2 + b_2\right) v_2 + \ldots + \left(a_n + b_n\right) v_n \in$ span $\left(v_1, v_2, \ldots, v_n\right)$.

Since $a_i + b_i \in \Re$ for $i = 1, 2, \ldots, n$

i.e. a linear combination of $\left(v_1, v_2, \ldots, v_n\right)$ + a linear combination of $\left(v_1, v_2, \ldots, v_n\right)$

= a linear combination of $\left(v_1, v_2, \ldots, v_n\right)$

Condition 2 of a subspace-**Closure under scalar multiplication:**

$\alpha u = \alpha (a_1\ v_1 + a_2\ v_2 + \ldots + a_n v_n) = \left(\alpha\ a_1\right) v_1 + \left(\alpha\ a_2\right) v_2 + \ldots + \left(\alpha\, a_n\right) v_n \in$ span $\left(v_1, v_2, \ldots, v_n\right)$.

since $\alpha\, a_i \in \Re\ \forall_i = 1, 2, \ldots, n$

i.e. a scalar multiple of a linear combination of $\left(v_1, v_2, \ldots, v_n\right)$ is also a linear combination of $\left(v_1, v_2, \ldots, v_n\right)$

$\Rightarrow$ span $\left(v_1, v_2, \ldots, v_n\right)$ is a subspace of *V*.

Example:

1. Let $P_{\Re}(x)$ be the set of all polynomials in *x* with real coefficients over $\Re$.

Span $\left(1, x, x^2\right) = \left\{a + bx + cx^2 \,/\, a, b, c \in \Re\right\}$ = $P_2\left(x\right)$, set of all polynomials in $P_{\Re}\left(x\right)$ of degree ≤ 2.

Observe that

1. Span $\left(1, x, x^2, 3 + 5x\right) \subset$ Span $\left(1\, x, x^2\right)$
2. Span $\left(1, 3 + 5x, x^2\right) \subset$ Span $\left(1, x, x^2\right)$

Theorem: Let V be a vector space and W_1 and W_2 be subspaces of V with W_1 = Span $\left(v_1, v_2, \ldots, v_n\right)$, W_2 = Span $\left(u_1, u_2, \ldots, u_m\right)$. If each u_i is a linear combination of the v_i's,

Then Span $\left(u_1, u_2, \ldots, u_m\right) \subset$ Span $\left(v_1, v_2, \ldots, v_n\right)$. $W_2 \subset W_1$

Proof: Since each u_i is a linear combination of v_i's, $u_i \in W_i\ \forall_i = 1, 2, \ldots, n$.

By definition of subspace, any linear combination in W_2 is in W_2

$\Rightarrow$ Span $\left(u_1, \ldots, u_n\right) = W_2 \subset W_1$

Corollary: Let W_1 = Span $(v_1, v_2, \ldots, v_n)$ and W_2 = Span $(u_1, u_2, \ldots, u_n)$. If each v_i is a linear combination of u_i's and each u_i is a linear combination of v_i's. Then $W_1 = W_2$

Proof: $W_1 \subset W_2$ and $W_2 \subset W_1$ implies that $W_1 = W_2$

If W = Span$(v_1, v_2, \ldots, v_n)$, we say W is a subspace **generated** or **spanned** by $v_1, v_2, \ldots, v_n$ and that $\{v_1, v_2, \ldots, v_n\}$ is a generating/spanning set for W. If a vector space V has a finite generating set, we say V is **finitely generated.**

Claim:

1. Span $(v_1, v_2, \ldots v_n)$ = span $(v_1, v_2, \ldots v_{i-1}, \lambda v_i, v_{i+1}, \ldots, v_n)$ for any $\lambda \neq 0$.Thus, replacing a generator by a nonzero scalar multiple of itself leaves the subspaces unchanged.
2. Span $(v_1, v_2, \ldots, v_i, \ldots, v_j + \lambda v_i, \ldots, v_n)$= Span $(v_1, \ldots, v_n)$ $i \neq j$.

Replacing a generator by the sum of itself and a scalar multiple of another generator leaves the space unchanged.

Definition: Let $W_1, W_2, \ldots, W_n$ be subspaces of a vector space V. The set spanned by $W_1, W_2, \ldots, W_n$

is the sum of W_1, $W_2, \ldots, W_n$, denoted $W_1 + W_2 + \ldots + W_2$ and defined by

$W_1 + W_2 + \ldots + W_2 = \{u_1 + u_2 + \ldots u_n / u_i \in W_i\}$

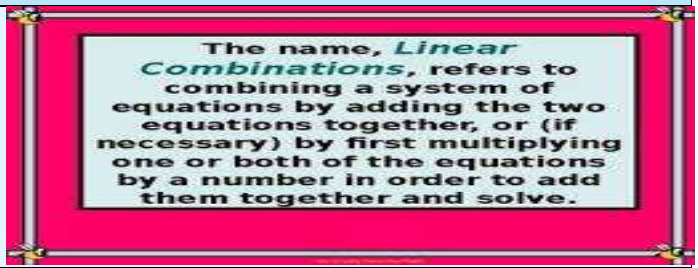
Lemma: If $W_1, W_2, \ldots, W_n$ are subspaces of V, then $W_1 + W_2 + \ldots + W_n$ is a subspace of V

Examples:

1. Let $V = P_{\Re}^5(x)$, the set of all polynomials with real coefficients with degree ≤ 5. Then, $W_2 = P_{\Re}^2(x)$, the set of all polynomials with real coefficients with degree ≤ 2 is a subspace of V.
$W_2 = P_{\Re}^3(x)$, the set of all polynomials with real coefficients with degree $\leq$ 3 is a subspace of V. $W_4 = P_{\Re}^4(x)$, the set of all polynomials with real coefficients with degree $\leq$ 4 is a subspace of V.
Now, $W_2 + W_3 = W_3$ and $W_2 + W_1 + W_4 = W_4$
2. Let $V = \Re^3$, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$
Let also W_1 = Span (e_1), W_2= Span(e_2), W_3 = Span (e_3) and W_4= Span (e_2, e_3)
Then $V = W_1 + W_2 + W_3$; $= W_1 + W_4$ $= W_1 + W_2 + W_4$ Also, $W_4 = W_2 + W_3$

E-tivity 11.2.1 - Linear Combinations

Numbering, pacing and sequencing	11.2.1
Title	Linear Combinations
Purpose	Write a given vector as a linear combination of a given set of vectors

Brief summary of overall task	Watch the video on [linear combinations]
Spark	
Individual task	After reading the notes in section 10.2.1 and watching the video above, show that $(11,3,-8)$ is a linear combination of $(1,1,0)$ and $(2,1,-1)$
Interaction begins	Go to discussion forum 10.2.1 and compare your solution with that of three of your classmates
E-moderator interventions	Note that Linear combinations is what will be used to define linear dependence and independence in Lesson 12
Schedule and time	Week 11. This activity will take one hour
Next	Linear dependence and independence

11.3 Assessment Questions

1. Write the vector $v = (2,-5,3)$ in $\Re^3$ as a linear combination of the vectors $e_1 = (1,-3,2), e_2 = (2,-4,-1)$ and $e_3 = (1,-5,7)$.

2. For which value of k will the vector $u = (1,-2,k)$ in $\Re^3$ be a linear combination of the vectors $u = (3,0,-2)$ and $w = (2,-1,-5)$?

3. Show that the vectors $u = (1,2,3), v = (0,1,2)$ and $w = (0,0,1)$ generate $\Re^3$.

4. Find conditions on a, b and c so that $(a,b,c) \in \Re^3$ belongs to the space generated by $u = (2,1,0), v = (1,-1,2)$ and $w = (0,3,-4)$

11.4 References

1. Linear Algebra: Schaum's Outline Series
2. Linear Algebra by J. N. Sharma, A.R. Vasishta
3. Linear Algebra by Michael O'nan, Herbert Enderton
4. Elementary Linear Algebra by Bennard Kolman
5. Elementary Linear Algebra by Howard Anton

6. https://youtu.be/PJEjKztLOvI

LESSON 12

LINEAR DEPENDENCE AND INDEPENDENCE, BASIS AND DIMENSION

12.1Introduction

During the previous lesson you covered linear combinations and linear spans. You learnt how to write a given vector as a linear combination of other vectors. The relationship between linear spans, vector spaces and subspaces was also discussed.

In this lesson, you will be introduced to the concept of linear dependence and independence.

You will also apply the knowledge of linear dependence and independence in finding the basis and dimension of a given vector space.

12.2 Learning Outcomes

By the end of the lesson, you will be able to;

12.2.1 Determine whether a given set of vectors is linearly dependent and independent.

12.2.2 Use the concept of linear dependence and independence to find the basis and dimension of a given vector space

12.2.1 Linear Dependence and independence

Definition: The vectors $v_1, v_2, \ldots, v_n$ of a vector space V are ***linearly dependent*** if there exists real numbers $a_1, a_2, \ldots, a_n$ with at least one $a_i \neq 0$ such that $a_i v_i + a_2 v_2 + \ldots + a_n v_n = 0$.The vectors are ***linearly independent*** if they are not dependent i.e. for every linear combination $a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$ then $a_i = 0 \; \forall \quad i = 1, 2, \ldots, n$.

Note:

1. Any set of vectors including the zero vector is linearly dependent.

2. If none of the vectors is zero and $a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$ with v_i's dependent, then at least two of the a_i's are non-zero.

Criterion for dependence

A finite list of nonzero vectors $v_1, \ldots, v_n$ in a vector space V is *linearly dependent* iff some vector is a linear combination of its predecessors. (OR $v_1, \ldots, v_n$ are *linearly dependent* iff one v_i is a linear combination of others)

Proof:

$\Rightarrow$ Suppose a vector v_k is a linear combination of $v_1, v_2, \ldots, v_{k-1}$, say

$v_k = b_1 v_1 + \ldots + b_{k-1} v_{k-1}, bi \neq 0$ for some i. Then

$b_1 v_1 + b_2 v_2 + \ldots + b_{k-1} v_{k-1} - 1v_k + 0v_{k+1} + \ldots + 0v_n = 0$.

$\Leftarrow$ Suppose $a_1 v_1 + a_2 v_2 + a_n v_n = 0$ where $v_1, \ldots, v_n$ is a *linearly dependent* set.

Let $a_k \neq 0$ where $a_{k+1} = 0,\ a_{k+2} = 0, \ldots, a_n = 0$

Then $a_1 v_1 + \ldots + a_{k-1} v_{k-1} + a_k v_k = 0$ i.e.

$v_k = \left(-\dfrac{a_1}{a_k}\right) v_1 + \left(\dfrac{-a_2}{a_k}\right) v_2 + \ldots + \left(\dfrac{-a_{k-1}}{a_k}\right) v_{k-1}$ i.e. v_k is a linear combination of its predecessors.

Criterion for independence

A finite list of vector $v_1, v_2, \ldots, v_n$ in a vector space V is *linearly independent* iff no vector is a linear combination of the others.

Claim: Let $\{v_1, v_2, \ldots, v_n\}$ be a *linearly independent* set in V, and $u \in V$. Then $v_1, v_2, \ldots, v_n, u$ are *linearly dependent* iff u is a linear combination of $v_1, v_2, \ldots, v_n$.

Examples:

1. The vectors $e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, 0, \ldots, 0, 1)$ in $\Re^n$ are linearly independent.
2. The set $\{1, x, x^2, \ldots, x^n\}$ in $P^{n+1}(x)$ is linearly independent.
3. The set $\{1, x, 3-2x\}$ is linearly dependent since $3 - 2x = 3(1) - 2(x)$
4. Determine whether or not $\{(1,2,3,1), (2,2,1,3), (-1,2,7-3)\}$ in $\Re^4$ is linearly dependent.

Method 1:

Find a_1, a_2, a_3 such that $a_1 (1,2,3,1) + a_2 (2,2,1,3) + a_3 (-1,2,7,-3) = (0,0,0,0)$

Method 2: Simply reduce the matrix $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 3 \\ -1 & 2 & 7 & -3 \end{pmatrix}$ to echelon form.

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 3 \\ -1 & 2 & 7 & -3 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_1 + R_3 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 2 & 5 & -1 \\ 0 & 4 & 10 & -2 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 2R_2 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 2 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

OR

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 3 \\ -1 & 2 & 7 & -3 \end{pmatrix}\begin{matrix} V_1 \\ V_2 \\ V_3 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 2 & 5 & -1 \\ 0 & 4 & 10 & -2 \end{bmatrix}\begin{matrix} V_1 \\ V_2 - 2V_1 \\ V_1 + V_3 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 2 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{matrix} V_1 \\ V_2 - 2V_1 \\ (V_1 + V_3) - 2(V_2 - 2V_1) \end{matrix}$$

$\Rightarrow (V_1 + V_3) - 2(V_2 - 2V_1) = 0 \Rightarrow -3V_1 + 2V_2 + V_3 = 0$ but the coefficients of V_1, V_2, V_3 are not zero $\Rightarrow$ V_1, V_2, V_3 are *linearly dependent.*

5. Show that the set $\{1, \sin^2 x, \cos^2 x\}$ is a *linearly dependent* set of functions in the vector space F of all functions mapping $\Re$ to $\Re$.

Solution: $\cos^2 x = 1 - \sin^2 x$,i.e. $\cos^2 x$ is a linear combination of $\sin^2 x$ and 1.
OR $\sin^2 x = 1 - \cos^2 x$ i.e. $\sin^2 x$ is a linear combination of $\cos^2 x$ and 1.
Since one vector can be written as a linear combination of the others, we conclude that $\{1, \sin^2 x, \cos^2 x\}$ is *linearly dependent*.

6. Any set of more than n vectors in $\Re^n$ is *linearly dependent*. In particular, $(1,-2,1)$, $(3,-5,2)$, $(2,-3,6)$ and $(1,2,1)$ in $\Re^3$ are *linearly dependent*.

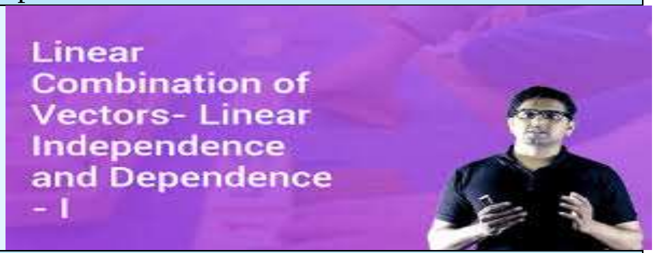
Theorem: Let $v_1, v_2, \ldots, v_n$ be n vectors in $\Re^n$ .The following conditions are equivalent

1. The vectors are *linearly independent*.
2. The vectors generate all of $\Re^n$
3. The matrix A having these vectors as columns (or rows) is invertible.

An infinite set S of vectors in a vector space V is *linearly independent* if there is no dependence relation involving a finite number of vectors in S.
e.g. The set $\{1, x, x^2, x^3 \ldots\}$ is *linearly independent* in the vector space $P(x)$ of all polynomials with real coefficients.

E-tivity 12.2.1 - Linear Dependence and independence

Numbering, pacing and sequencing	12.2.1
Title	Linear Dependence and independence
Purpose	To determine whether a given set of vectors is linearly dependent and independent.

Brief summary of overall task	Watch the video on linearly dependent and independent vectors
Spark	
Individual task	After watching the video and reading notes in section 12.2.1, determine whether or not the vectors in $\Re^4$ are linearly dependent : {(1,3,-4,2),(3,8,-5,7),(2,9,4,23)}
Interaction begins	Follow the solutions posted by three of your colleagues and comment on their method. What can you learn from them
E-moderator interventions	Observe that there are two ways of working out the question, and they are applied depending on the question
Schedule and time	Week 12. This activity will take one hour
Next	**Basis and dimension**

12.2.2 Basis and Dimension

Consider a vector space V generated/spanned by $v_1, v_2, \ldots, v_n$. If this set of vector is *linearly dependent* , then one vector, say v_n ,is a linear combination of $v_1, v_2, \ldots, v_{n-1}$ and hence Span $(v_1, v_2, \ldots, v_n) = Span\,(v_1, v_2, \ldots, v_{n-1})$.

We may delete superfluous generators until we have a *linearly independent*, set say $v_1, v_2, \ldots, v_k$

Span $v_1, v_2, \ldots, v_n$ = Span $v_1, v_2, \ldots, v_k$, $k \le n$.

If we delete any vector in the *linearly independent* set, we would no longer generate *V*. A set of generators is minimal if every proper subset of them fails to span/generate *V*. Minimality and Independence are equivalent for a set of generators.

Definition: Let *V* be a vector space. A set of vectors in *V* is a **basis** for *V* if

(1) The vectors generate/span *V*

(2) The vectors are *linearly independent*

Examples:

1. The set $\{e_1, e_2, \ldots, e_n\}$ where $e_1 = (1,0,0,\ldots,0), e_2 = (0,1,0,\ldots,0),\ldots, e_n = (0,0,0,\ldots,0,1)$ form a basis for $\Re^n$

2. The set $\{1, x, x^2, \ldots, x^{n-1}\}$ form a basis for $p_{\Re}(x)$, the set of all polynomials in *x* of degree $\le n$ with real coefficients.

3. Vectors $e_1 = (1,0)$ and $e_2 =$ (1,1) from a basis for $\Re^2$

Theorem: Let *V* be a vector space with basis $B=\{b_1,b_2,\ldots,b_n\}$.Each vector $v\in V$ can be uniquely expressed in the form $v=\alpha_1b_1+\alpha_2b_2+\ldots+\alpha_nb_n$, $\alpha_i\in\Re$ (*i.e.* there is exactly one choice for each α_i)

Proof: Suppose $v=\alpha_1b_1+\alpha_2b_2+\ldots+\alpha_n\ b_n \quad \alpha_i\in\Re$

$=\beta_1b_1+\beta_2b_2+\ldots+\beta_n\ b_n \quad \beta_i\in\Re$ with $\alpha_i\neq\beta_i$ for some *i*. Then

$(\alpha_1-\beta_1)b_1+(\alpha_2-\beta_2)b_2+\ldots+(\alpha_i-\beta_i)b_i+\ldots+(\alpha_n-\beta_n)b_n=0$ with $\alpha_i-\beta_i\neq 0$. This contradictions the fact that $b_1,b_2,\ldots,b_n$ are linearly independent by definition of a basis. Hence $\alpha_i=\beta_i \quad \forall i$

Example

1.Show that the vectors $(1,2,-1,0)$ (0,1,0,1), (-1,-5,2,0) and (2,3,-2,7) form a basis for $\Re^4$.

Solution: Show that the vectors are linearly independent.

Theorem: Let *V* be a vector space and $\{v_1,\ldots,v_n\}$ a basis of *V*.

(a) If $m>n$, then any set of *m* vectors of *V* is linearly dependent.
(b) Any other basis contains precisely *n* elements.
(c) *n* can be characterized as either the minimum number of generators of *V* or the maximum number of linearly independent vectors in *V*.

Definition: The dimension of a finitely generated vector space *V* is the number of elements in any basis of *V*, denoted *dim(V)*. We say *V* is an *n*-dimensional vector space.

Example: Find dimension of the subspace $W=S\text{pan}\,(1,-3,1)$, $(-2,6,-2)$, $(2,1,-4)$, $(-1,10,-7)$ of $\Re^3$.

Solution:

$$\begin{bmatrix}1&-3&1\\-2&6&-2\\2&1&-4\\-1&10&-7\end{bmatrix}\to\begin{bmatrix}1&-3&1\\0&0&0\\0&-7&-6\\0&7&-6\end{bmatrix}\to\begin{bmatrix}1&-3&1\\0&7&-6\\0&0&0\\0&0&0\end{bmatrix};\ W=\text{span}\ \{(1,-3,1),(2,1,-4)\}\ \Rightarrow$$

Dim(W)=2.

Theorem: Let *V* be an *n*-dimensional vector space and $\{v_1,\ldots,v_m\}$ a linearly independent set of vectors in *V*.

(i) The set $\{v_1,\ldots,v_m\}$ is a basis for *V* iff $m=n$.

(ii) Any linearly independent set of vectors of a finite dimensional space can be enlarged to a basis.

Proof:

(i) $\Rightarrow$ Suppose $\{v_1,v_2,\ldots,v_m\}$ is a basis for *V*. Any two basis for a finitely generated vector space *V* have same number of elements called dim *V*. Hence $m=n$.

$\Leftarrow$ Suppose $m=n$. If $u\in V$ is not a linear combination of $v_1,v_2,\ldots,v_m$. Then $\{v_1,\ldots,v_m,k\}=S$ is a linearly independent set with $|S|=n+1$. But dim *V* = *n* is the maximum number of linearly independent vectors in *V*. This is a contradiction and hence $u\in$ Span $(v_1,v_2,\ldots v_m)$ and $(v_1,v_2,\ldots,v_m)$ is a basis of *V*.

(ii) Let $\{v_1,v_2,\ldots,v_k\}$ be a linearly independent set. If *k*=*n*, the proof follows from (i).

If $k < n$ take $V_{k+1} \notin \text{Span}\left(v_1,\ldots,v_k\right)$.Then $\left\{v_1, v_2,\ldots,v_k, v_{k+1}\right\}$ is a linearly independent set. If $k+1=n$, we have a basis from (i). If $k+1<n$ we can repeat the process. The process stops when the number of elements in the enlarged set is *n*.

If *V* is an *n*-dimensional vector space and *W* a subspace of *V*, then *W* is finite dimensional, dim $W \subseteq$ dim *V* and any basis of *W* can be extended to a basis of *V*.

$W = V$ iff dim $W = \dim V$.

Example: Let $V = \Re^4$, W = Span {(1,0,0,0), (1,0,1,0)}. $\{(1,0,0,0), (1,0,1,0)\}$ is a basis for *W*. To enlarge this basis to a basis of $\Re^4$, we start with $U = \left\{(1,0,0,0), (1,0,1,0), e_1, e_2, e_3, e_4\right\}$ where { e_1, e_2, e_3, e_4 } is the usual basis for $\Re^4$. $\Re^4$ = span *(U)*

We delete vectors in *U* that are a linear combination of (1,0,0,0) and (1,0,1,0).

$e_1 = (1,0,0,0)$, $e_3 = -(1,0,0,0) + (1,0,1,0)$

The set $\left\{(1,0,0,0), (1,0,1,0), e_2\right\}$ is linearly independent.

The set $\left\{(1,0,0,0), (1,0,1,0), e_2, e_3\right\}$ is linearly dependent since $e_3 = -(1,0,0,0) + 1(1,0,1,0)$; Delete e_3

The set $\left\{(1,0,0,0), (1,0,1,0)\, e_2, e_4\right\}$ is linearly independent hence a basis of $\Re^4$.

E-tivity 12.2.2 Basis and Dimension

Numbering, pacing and sequencing	12.2.2
Title	**Basis and Dimension**
Purpose	
Brief summary of overall task	Watch the video on basis and dimension
Spark	
Individual task	After watching the video above and reading notes in section 20.2.2, find a basis and the dimension of the subspace **W** of $\Re^4$ spanned by $(1,-4,-2,1)(1,-3,-1,2)$ and $(3,-8,-2,7)$
Interaction begins	Follow the posts by three of your classmates in discussion forum 10.2.2 and see what you can learn from them. Offer constructive criticism and also be ready to receive the same
E-moderator interventions	Note that the concept of Linear dependence and independence is the one used to find the basis and dimension. A set of vectors need to be linearly independent in order to qualify as a basis.
Schedule and time	Week 12. This activity will take one hour 30 minutes
Next	**The vector equation of a line**

12.3 Assessment Questions

1. Determine whether or not u and v .are linearly dependent if:

(a) $u=(3,4), v=(1,-3)$ (b) $u=(2,-3), v=(6,-9)$ (c) $u=(4,3,-2), v=(2,-6,7)$

(d) $u=(-4,6,-2), v=(2,-3,1)$ (e) $u=\begin{pmatrix}1 & -2 & 4\\ 3 & 0 & -1\end{pmatrix}, v=\begin{pmatrix}2 & -4 & 8\\ 6 & 0 & -2\end{pmatrix}$

(f) $u=\begin{pmatrix}1 & 2 & -3\\ 6 & -5 & 4\end{pmatrix}, v=\begin{pmatrix}6 & -5 & 4\\ 1 & 2 & -3\end{pmatrix}$ (g) $u=2-5t+6t^2-t^3, v=3+2t-4t^2+5t^3$

2. Determine whether or not the following vectors in $\Re^3$ are linearly dependent:

(a) $(1,-2,1),(2,1,-1),(7,-4,1)$

(b) $(1,-3,7),(2,0,-6),(3,-1,-1),(2,4,-5)$

(c) $(1,2,-3),(1,-3,2),(2,-1,5)$

(d) $(2,-3,7),(0,0,0),(3,-1,-4)$

3. Determine whether or not the following form a basis for the vector space $\Re^3$:

(a) $(1,1,1)$ and $(1,-1,5)$

(b) $(1,2,3),(1,0,-1),(3,-1,0)$ and $(2,1,-2)$

(c) $(1,1,1),(1,2,3)$ and $(2,-1,1)$

(d) $(1,1,2),(1,2,5)$ and $(5,3,4)$

12.4 References

1. Linear Algebra by Michael O'nan, Herbert Enderton
2. A First Course in Linear Algebra by Daniel Zelisky
3. Elementary Linear Algebra by Bennard Kolman
4. Elementary Linear Algebra by Howard Anton
5. https://youtu.be/32fqO07p0Y8
6. https://youtu.be/aAvTFc2gfhw

LESSON 13

PLANES AND LINES IN $\Re^3$

13.1 Introduction

In lesson 12, you learnt about linear dependence and independence. You applied this knowledge in finding the basis and dimension of a given vector space.

In this lesson, you will learn about the vector equation of a line and the vector equation of a plane.

13.2 Learning Outcomes

By the end of the lesson, you will be able to;

13.1.1 Find the parametric and symmetric equation of a line.

13.1.2 Find the equation of a plane

13.1.3 Find the line of intersection of two planes

13.2.1 The vector equation of a line

Example: Find the equation of the line through the point A(1,2,3) and B(4,4,4), and find the co-ordinates of the point where the line meets the plane z = 0.

Solution:

$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 3i + 2j + k$$

Let R be any point on line AB, then

$\overrightarrow{OR} = \overrightarrow{OA} + t\ \overrightarrow{AB}$, where t is a scalar.

$$\vec{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad \text{OR}$$

$$\overrightarrow{OR} = \overrightarrow{OR} + (1-t)\ \overrightarrow{BA} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} + (1-t) \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} + \begin{pmatrix} -3+3t \\ -2+2t \\ -1+t \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

The line meets Z = 0 where the z-coordinate is 0. i.e. where 3 +t =0 $\Rightarrow$ t = -3.

At this point, $\vec{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 3 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ -4 \\ 0 \end{pmatrix}$. Point R has coordinates $(-8, -4, 0)$

Example:

1. (a) Write the line $\dfrac{x-2}{3} = \dfrac{y-4}{5} = \dfrac{z-7}{2}$ in form $\vec{r} = \vec{a} + t\vec{u}$

   (b)Show that the line passes through *(8,14,11)*

(c) Find the unit vector parallel to this line

Solution:

(a) $t = \frac{x-2}{3} = \frac{y-4}{5} = \frac{z-7}{2} \Rightarrow \quad x = 2+3t,\ y = 4+5t,\ z = 7+2t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2+3t \\ 4+5t \\ 7+2t \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$$

(b) Showing that it passes through *(8,14,11)*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2+3t \\ 4+5t \\ 7+2t \end{pmatrix} = \begin{pmatrix} 8 \\ 14 \\ 11 \end{pmatrix} \Rightarrow \begin{pmatrix} 2+3t = 8 \\ 4+5t = 14 \\ 7+2t = 11 \end{pmatrix} \Rightarrow t = 2$$

$$\begin{pmatrix} 8 \\ 14 \\ 11 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$$

(c)The unit vector parallel to this line is

$$u = \frac{1}{\sqrt{3^2+5^2+2^2}}\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{39}}\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$$

2. Show that the equations $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + m\begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 15 \\ -3 \end{pmatrix} + n\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$ represent the same line.

Solution: $\begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix}$ is parallel to $\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$ since $\begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix} = -2\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 10 \\ 15 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix}$; (10,15,-3) and (2,3,1) are on this line.

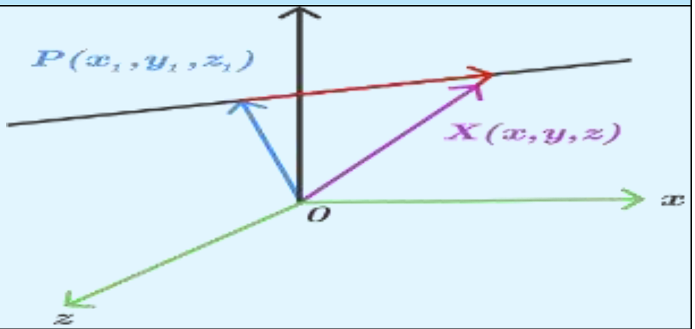
The equation of a line through a point $A(x_1, y_0, z_0)$ and parallel to vector $\begin{pmatrix} p \\ q \\ s \end{pmatrix}$ is given by

$$\vec{r} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\begin{pmatrix} p \\ q \\ s \end{pmatrix}. \text{ i.e. } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\begin{pmatrix} p \\ q \\ s \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 + tp \\ y_0 + tq \\ z_0 + ts \end{pmatrix}$$

This can be written as

$$t=\frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{s}$$

E-tivity 13.2.1 - The vector equation of a line

Numbering, pacing and sequencing	13.2.1
Title	The equation of a line in 3D
Purpose	Find the vector, parametric and symmetric equation of a line in 3D.
Brief summary of overall task	Watch the video on the equation of a line in 3D
Spark	
Individual task	After reading notes in section 13.2.1 and watching the video above, write the equation $\frac{x-1}{2}+\frac{y-3}{3}+\frac{z-4}{5}$ in the form $\vec{r}=\vec{a}+t\vec{u}$ and show that it passes through $\begin{pmatrix}7\\12\\19\end{pmatrix}$.
Interaction begins	Follow the solutions posted by at least three of your colleagues in Discussion forum 13.2.1. Exchange ideas in case your methods are differing. The answer should be the same though regardless of the method used.
E-moderator interventions	Note that most of the times you will need to start by looking for the vector equation of a line, which you then convert to symmetric and/or parametric form
Schedule and time	Lecture 13. This activity will take one hour
Next	**Equation of a plane**

13.2.2 Equation of a plane

It is always possible to find a plane through 3 points. A fourth point may not lie on the plane.
A plane is uniquely determined by 3 points.
The general equation of a plane through points A,B and C is given by
$\overrightarrow{AP}=m\overrightarrow{AB}+n\,\overrightarrow{AC}$ i.e. $\overrightarrow{OP}=\alpha\,\overrightarrow{OA}+\beta\,\overrightarrow{OB}+\gamma\,\overrightarrow{OC}$

where $\alpha+\beta+\gamma=1$

Since $\overrightarrow{OP}-\overrightarrow{OA}=m\left(\overrightarrow{OB}-\overrightarrow{OA}\right)=n\left(\overrightarrow{OC}-\overrightarrow{OA}\right)$

$\overrightarrow{OP}=(1-m-n)\,\overrightarrow{OA}+m\,\overrightarrow{OB}+n\,\overrightarrow{OC}$

$(1-m-n)+m+n=1$

Examples:

Find the equation of the plane through $A(1,1,1),\ B(5,0,0)\ and\ C\ (3,2,1)$

Solution:

$\overrightarrow{AP}=m\overrightarrow{AB}+n\,\overrightarrow{AC}$ where *P (x,y,z)*

$$\begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix}=m\begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}+n\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}+m\begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}+n\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$x = 1 + 4m + 2n$

$y = 1- m +n$

$z = 1-m$

Eliminate *n* from equation 1 and 2 to get

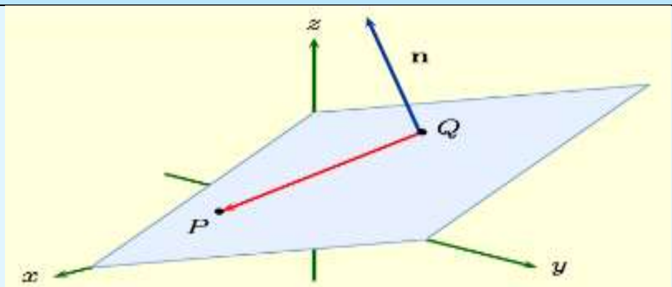
$x - 2y = -1 + 6m$

$z = 1- m$

Eliminate *m* to get

$x - 2y + 6z = 5$ which is the equation of the plane

E-tivity 13.2.2 Equation of a plane

Numbering, pacing and sequencing	13.2.2
Title	Equation of a plane
Purpose	To find the equation of a plane in 3D.
Brief summary of overall task	Watch the video on the equation of a plane
Spark	
Individual task	After watching the video and reading noted in section 13.2, find the equation of the plane passing through the points $P(-4,-1,-1), Q(-2,0,1)$ and

	$R(-1,-2,-3)$.
Interaction begins	Go to the discussion forum 13.2.2 and follow the steps used by two of your classmates. If your answers are different, exchange ideas on your methods and learn from each other.
E-moderator interventions	Note that when the equation of the plane is written in the form *ax+by+cz=d*, where a, b, c and d are constants, then the normal vector to the plane is *(a,b,c)*.
Schedule and time	Week 13. This activity will take 0ne hour
Next	**Intersection of two planes**

13.2.3 Intersection of two planes

Two planes always meet in a straight line.

Examples:

1. Find the equation of the line of intersection for the planes $3x-5y+z=8$ and $2x-3y+z=3$

Solution: At the line of intersection, the values of *x,y,z* satisfy both equations.

3x-5y+z = 8

<u>2x-3y+z = 3</u> -

x-2y = 5

i.e.eliminate *z* from equation 1and 2 to get

$x-2y=5$, or $x=5+2y$. Thus the equation of line is x – 2y – 5 = 0

Since the line can also be obtained by eliminating x or y, we get it in the form

$\vec{r}=\vec{a}+t\vec{u}$, we Let $y=t$, $\Rightarrow x=5+2t$

Substitute in plane $3x-5y+z=8$ to get $3(5+2t)-5t+z=8 \Rightarrow z=-7-t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5+2t \\ t \\ -7-t \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -7 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

Note that the point $(5+2t, t, -7-t)$ lies in both planes. i.e.

3x -5y+z=3(5+2t) – 5t+(-7-t) = 8 and

2x – 3y+ z = 2(5+2t) – 3t + (-7-t) = 3

2. Show that the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ lies on the plane $2x+3y-5z=-7$.

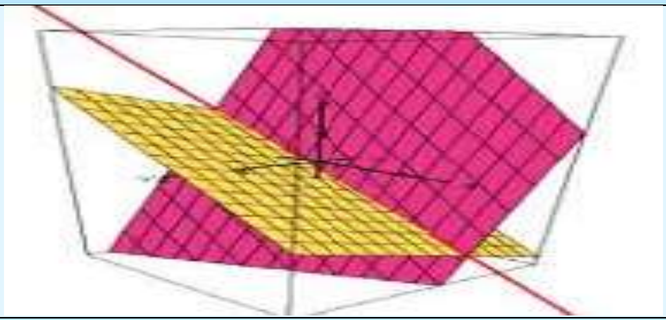
Solution: Substitute the point $(1+t, 2+t, 3+t)$ on line in plane equation $2x+3y-5z=-7$ to get

2x+3y – 5z = 2(1+t) +3(2+t)-5(3+t)=2+2t+6+3t-15-5t=-7

3. Show that the lines $\vec{r} = \begin{pmatrix}3\\5\\7\end{pmatrix} + m\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\vec{s} = \begin{pmatrix}1\\2\\3\end{pmatrix} + n\begin{pmatrix}2\\3\\5\end{pmatrix}$ do not meet i.e. they are skew lines.

Solution: Suppose they meet. Then, $\left.\begin{aligned}3+m&=1+2n\\5+2m&=2+3n\\7+m\ &=3+5n\end{aligned}\right\}$. Solving equation 1 and 2 gives $n=1$, and $m=0$. Substituting these values in equation 3 gives $7+0 \neq 3+5 \Rightarrow$ There is no solution. Hence the lines do not meet

E-tivity 13.2.3 - Intersection of two planes

Numbering, pacing and sequencing	13.2.3
Title	Intersection of two planes
Purpose	To find the equation of the line of intersection of two planes
Brief summary of overall task	Watch the video on the line of intersection of two planes
Spark	
Individual task	After reading the notes in section 13.2.3 and watching the video, find the equation of the line of intersection for the planes 4x+3y + z=10 and x + y + z = 6
Interaction begins	Go to discussion forum 13.2.3 and see what you can learn from the solutions of two of your classmates. Offer constructive criticism and also be ready to receive the same in case your answer is not correct
E-moderator interventions	Observe that you will first get the equation of the straight line in parametric form, then you can convert it into symmetric or vector form as the case may be.
Schedule and time	Week 13. This activity will take one hour
Next	**This is the last lesson**

13.3 Assessment Questions

1. Find an equation of the plane passing through the point $(3,-1,7)$ and perpendicular to the vector $\mathbf{n}=(4,2,-5)$.

2. Find the equation of the plane passing through the points $P_1(1,2,-1)$, $P_2(2,3,1)$ and $P_3(3,-1,2)$.

3. Find The line through the point $(1,2,-3)$ and parallel to the vector $\mathbf{v}=(4,5,-7)$ has parametric equations

4. (a)Find parametric equations for the line l passing through the points $P_1(2,4,-1)$ and $P_2(5,0,7)$.

   (b)Where does the line intersect the xy-plane?

5. Find parametric equations for the line of intersection of the planes $3x+2y-4z-6=0$ and $x-3y-2z-4=0$

13.4 References

1. https://youtu.be/2sZKZHyaQJ8
2. https://youtu.be/SoSTdgqknvY

3. Linear Algebra by Michael O'nan, Herbert Enderton
4. A First Course in Linear Algebra by Daniel Zelisky
5. Elementary Linear Algebra by Bennard Kolman
6. Elementary Linear Algebra by Howard Anton

ANSWERS TO LESSON ASSESSMENT QUESTIONS

1.3 Answers

1 (a) $AB=\begin{pmatrix}11 & -6 & 14\\ 1 & 2 & -14\end{pmatrix}$ 1(b) BA is not defined.

2. (a) $AB=(6,1,-3)$ (b) BA is not defined.

3. (a) $AB=\begin{pmatrix}-1 & -8 & -10\\ 1 & -2 & -5\\ 9 & 22 & 15\end{pmatrix}$ (b) $BA=\begin{pmatrix}15 & -21\\ 10 & -3\end{pmatrix}$

2.3 Answers

1. $A^t=\begin{pmatrix}1 & 2 & 4\\ 0 & 3 & 4\\ 1 & 4 & 4\\ 0 & 5 & 4\end{pmatrix}$ 2. (a) $A^t=\begin{pmatrix}1 & 3\\ 2 & -1\\ 0 & 4\end{pmatrix}$, Then $AA^t==\begin{pmatrix}5 & 1\\ 1 & 26\end{pmatrix}$;

(b) $A^tA=\begin{pmatrix}10 & -1 & 12\\ -1 & 5 & -4\\ 12 & -4 & 16\end{pmatrix}$

3.3 Answers

1. (a) $\begin{vmatrix}3 & -2\\ 4 & 5\end{vmatrix}=23$ (b) $\begin{vmatrix}a-b & a\\ b & a+b\end{vmatrix}=-b^2$

2. $k=0$;and $k=2$.That is, if $k=0$ or $k=2$

3. (a) $\begin{vmatrix}1 & 2 & 3\\ 4 & -2 & 3\\ 2 & 5 & 1\end{vmatrix}=79$ (b) $\begin{vmatrix}2 & 0 & 1\\ 4 & 2 & -3\\ 5 & 3 & 1\end{vmatrix}=24$ (c) $\begin{vmatrix}2 & 0 & 1\\ 3 & 2 & -3\\ -1 & -3 & 5\end{vmatrix}=-5$ (d) $\begin{vmatrix}1 & 0 & 0\\ 3 & 2 & -4\\ 4 & 1 & 3\end{vmatrix}=10$

4.3 Answers

1. $\begin{vmatrix}1 & 2 & 3\\ 2 & -1 & 3\\ 1 & 0 & 1\end{vmatrix}=\begin{vmatrix}5 & 0 & 9\\ 2 & -1 & 3\\ 1 & 0 & 1\end{vmatrix}$,obtained by adding twice the second row to the first row Det=4.

2. -120

5.5 Answers

$$1. A^{-1} = \frac{1}{|A|}(adjA) = \begin{pmatrix} -18/-46 & -11/-46 & -10/-46 \\ 2/-46 & 14/-46 & -4/-46 \\ 4/-46 & 5/-46 & -8/-46 \end{pmatrix} = \begin{pmatrix} 9/23 & 11/46 & 5/23 \\ -1/23 & -7/23 & 2/23 \\ -2/23 & -5/46 & 4/23 \end{pmatrix}$$

6

6.3 Answers

1. $\begin{pmatrix} 1 & -2 & 1 \\ 0 & 5 & -3 \\ 0 & 0 & 0 \end{pmatrix}$

2. *b,e,f* are in echelon form; *a* and *f* are in canonical form. *c* and *d* are not in echelon form.

7.3 Answers

1. $x_1 = \frac{13}{2}, x_2 = 4, x_3 = 9/2, x_4 = -3$

2. $x_1 = 4_1 \quad x_2 = 1, x_3 = -3, \; x_4 = 2$

3. $x_1 = 2, x_2 = 0, x_3 = 1, x_4 = -1$

4. $x_3 = k, \;\; x_2 = 2 + k, \;\; x, = 4 - k$

8.3 Answers

9.3 Answers

1. $\|\mathbf{v}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

2. $0 \le \theta \le \prod, \theta = 60^0$

3. $\mathbf{u}.\mathbf{v} = (2)(4) + (-4)(2) = 0$

10.3 Answers

1. $L = \{(x, y) / y = 2x\} = \{(x, 2x) / x \in R\}$
2. Let $w_1, w_2 \in W$. $w_1 = (x_1, x_1 + 1), \;\; w_2 = (x_2, x_2 + 1)$

$w_1 + w_2 = (x_1 + x_2, x_1 + x_2 + 2) \notin W$ .OR: $\alpha\,(x_1, x_1 + 1) = (\alpha x_1, \alpha x_1 + \alpha) \notin W$, since $\alpha \neq 1$.
Therefore the line $y = x + 1$ is not a subspace of $\Re$

3. For any invertible n × n matrix *A*, -*A* is also invertible.
But $A + (-A) = 0$ not invertible and therefore the set of all **invertible** n × n matrices is not a subspace of the set $M_n\,(R)$

11.3 Answers

1. Set v as a linear combination of the e_1 using the unknowns x, y and z : $v = xe_1 + ye_2 + ze_3$.
v cannot be written as a linear combination of the vectors e_1, e_2 and e_3.
2. Set $u = xv + yw$; $k = -8$.
3We need to show that an arbitrary vector $(a,b,c) \in \Re^3$ is a linear combination of u, v and w.
$x = a, y = b - 2a, z = c - 2b + a$ is a solution. Thus u, v and w generate $\Re^3$.
4. Set (a,b,c) as a linear combination of u, v and w using unknowns x, y and z :
$(a,b,c) = xu + yv + zw$; $2a - 4b - 3c = 0$.

12.3 Answers

1. Two vectors u and v are dependent if and only if one is a multiple of the other.
(a)No (b) Yes; for $v = 3u$ (c)No (d)Yes; for $u = -2v$ (e)Yes; for $v = 2u$ (f)No (g)No (h)Yes;
2. (a) Since the echelon matrix has a zero row the vectors are dependent. (The three given vectors generate a space of dimension 2.)
(b)Yes, since any four (or more) vectors in $\Re^3$ are dependent.
(c)Since the echelon matrix has no zero rows, the vectors are independent. (The three given vectors generate a space of dimension 3.)
(d)Since $0 = (0,0,0)$ is one of the vectors, the vectors are dependent
3. (a)and (b). No; for a basis of $\Re^3$ must contain exactly 3 elements, since $\Re^3$ is of dimension 3.
(c)The vectors form a basis if and only if they are independent. The echelon matrix has no zero rows; hence the three vectors are independent and so form a basis for $\Re^3$.
(d)The echelon matrix has a zero row, i.e. only two non zero rows; hence the three vectors are dependent and so do not form a basis for $\Re^3$.

13.3Answers

1. $4x + 2y - 5z + 25 = 0$
2. $9x + y - 5z - 16 = 0$

3. $$\begin{aligned} x &= 1+4t \\ y &= 2+5t \\ z &= -3-7t \end{aligned}$$

4.(a) $$\begin{aligned} x &= 2+3t \\ y &= 4-4t \\ z &= -1+8t \end{aligned}$$ *(b)* $(x, y, z) = \left(\frac{19}{8}, \frac{7}{2}, 0\right)$

5. $$\begin{aligned} x &= \frac{26}{11}+\frac{16}{11}t \\ y &= -\frac{6}{11}-\frac{2}{11}t \\ z &= t \end{aligned}$$ where $-\infty < t < +\infty$