Vectors and Vector Spaces

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Introduction

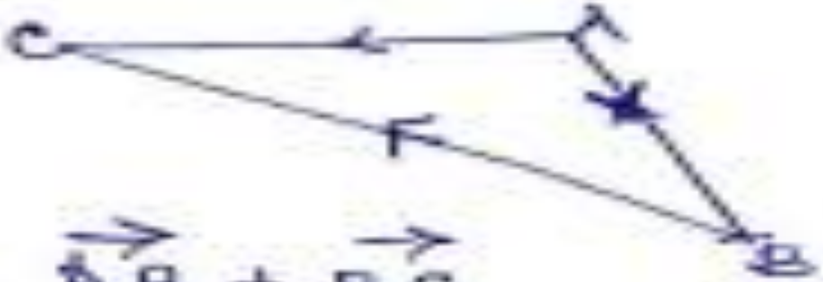
A vector is an ordered list of numbers.

A vector has direction and length (magnitude)

$\overrightarrow{AB}$

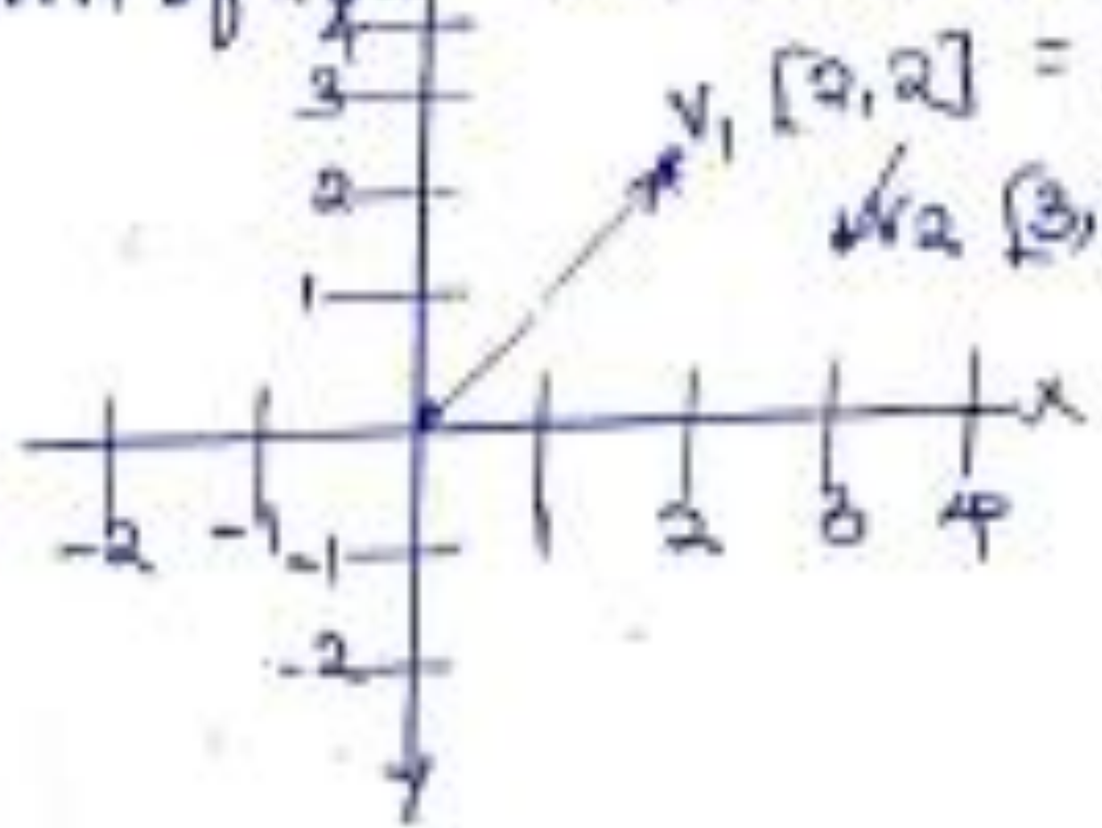
$\overline{BC}$

Joining the two



$\bar{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

Vector components can take the form of row vectors or column vectors



$v_1\ [2,2] = \begin{bmatrix}2\\2\end{bmatrix}$

$v_2\ [3,1] = \begin{bmatrix}3\\1\end{bmatrix}$

A scalar quantity has magnitude only.

Speed is a scalar - 40m/s

Velocity is a vector - 40m/s North

Force is a vector 300newtons exerted East

Question

Suppose vector v has initial point A (-4, 1) and Terminal point B(8, 6)

vector u has initial pnt C(7, -15) and terminal points D (3, 9)

a) Determine if the two vectors are equivalent.

$v = \overrightarrow{AB}$ $u = \overrightarrow{CD}$

$v = (8 - -4), (6 - 1) = (12, 5)$
$u = (9 - -15)(3 - -7), (9 - -15) = (10, 24)$

Magnitudes

$\|v\| = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$

$\|u\| = \sqrt{10^2 + 24^2} = \sqrt{676} = 26$

Not equal

(b) Do they have the same direction?

- The slope should tell us

$$m_v = \frac{y_2 - y_1}{x_2 - x_1} \begin{array}{l} \rightarrow \text{change in } y \text{ values} \\ \rightarrow \text{change in } x \text{ values} \end{array}$$

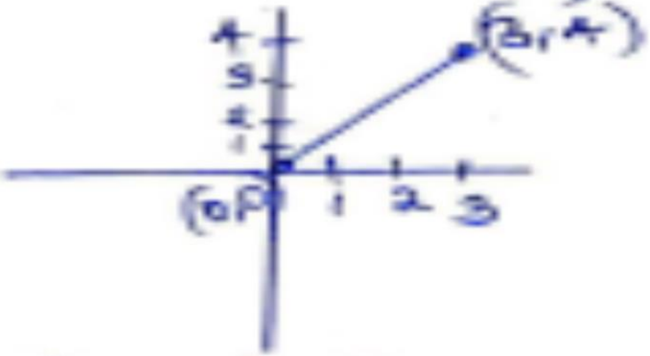
$$m_v = \frac{6-1}{8 - -4} = 5/12 = \frac{v_y}{v_x}$$

$$m_u = \frac{9 - -15}{3 - -7} = \frac{24}{10} = \frac{12}{5}.$$

$$\therefore 5/12 \neq \frac{12}{5}$$

Position Vector

Any vector with initial points at point (0,0)



Unit Vector

$\|u\| = 1$

Any vector with a magnitude of 1.

$$\frac{v}{\|v\|} = v = u \cdot \|v\|$$

Direction Magnitude

Position and Unit Vectors

Question

Find the unit vector in the same direction of $v = (4, -3)$

$$u = \frac{v}{\|v\|} = \frac{(4,-3)}{\sqrt{4^2+(-3)^2}} = \frac{4,-3}{5} = \left(\frac{4}{5}, \frac{-3}{5}\right)$$

Unit vector of vector $v = \left(\frac{4}{5}, \frac{-3}{5}\right)$

Proof: Is the magnitude = 1?

$$\sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{-3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = \sqrt{\frac{25}{25}} = \sqrt{1} = 1$$

Eigen Vectors and Eigen Values

Eigen Vectors and Eigen Values defined:

Let A be an $n \times n$ matrix.

A scalar λ is called an Eigen Value of A if there's a non-zero Vector $\bar{x}$ such that $A\bar{x} = \lambda\bar{x}$.

The vector $\bar{x}$ is called an Eigen vector of A corresponding to Eigen Value λ.

Example:

Show that $\bar{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an Eigen vector of $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$, corresponding to $\lambda = 4$

Solution

Recall: $A\bar{x} = \lambda\bar{x}$

$$\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{pmatrix} 3\cdot 2 + 2\cdot 1 \\ 3\cdot 2 + (-2)\cdot 1 \end{pmatrix} \stackrel{?}{=} \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$\begin{pmatrix} 6+2 \\ 6+-2 \end{pmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$A\bar{x} = \lambda\bar{x} \quad \triangle$$

$$\overrightarrow{v_{\lambda=4}} \rightarrow \{2, 1\}$$

$$\overrightarrow{v_{\lambda=-3}} \rightarrow \{-1, 3\}$$

Scalars or Multiplies apply (e.g. vector 4,2, vector 6,3, Vector 200,100)

Note If λ is an Eigen value of A and $\bar{x}$ is an Eigen vector belonging to λ, any non-zero multiple of $\bar{x}$ will be an Eigen vector.

Eg from the vector above $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is we multiply $\bar{x}$ with a scalar 3 or 100 we get $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 200 \\ 100 \end{bmatrix}$ respectively. These are also applicable eigen vectors of $\bar{x}$.

Finding Eigen Values and Eigen Vectors

The steps of computing Eigen Values and Eigen vectors

To solve the eigenvalues λ_i and the corresponding Eigen vectors $\bar{x}_i$ of an $n \times n$ matrix A, we follow the following steps: e.g. if $A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$

i) Multiply the $n \times n$ Identity matrix by the scalar λ

$$\lambda I = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

ii) Subtract the Identity matrix multiple from A

$$A - \lambda I = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{bmatrix}$$

iii) Find the determinant of the matrix from step (ii)

$$\det \begin{bmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{bmatrix} = [(7-\lambda)(-\lambda-1)] - [(3)(3)]$$

$$= -7\lambda - 7 + \lambda^2 + \lambda - 9$$

$$= \lambda^2 - 6\lambda - 16$$

Factors λ and 8, 2

iv) Factorization

$$(\lambda - 8)(\lambda + 2) = 0$$

$$\left.\lambda = 8 \quad \lambda = -2\right\}$$ Eigen values.

Solve the values of λ that satisfy the equation $\det(A - \lambda I) = 0$

vi)

equation $\det(A - \lambda I) = 0$

vi) Solve for the corresponding vectors for each λ

for $\lambda_1 = 8$

$$\begin{bmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{bmatrix}$$

Replace λ with 8

$$\begin{bmatrix} 7-8 & 3 \\ 3 & -1-8 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}$$ Matrix B

To solve $B\vec{x} = 0$

$$\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Row Reduction

$3R_1 + R_2 = \text{New } R_2$

$$\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -x_1 + 3x_2 = 0$$

$$3x_2 = x_1$$

Suppose $x_2 = 1$

$$3(1) = x_1 = 3$$

Vector $= \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

for $\lambda_2 = -2$

$$\begin{bmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{bmatrix}$$ Replace λ with -2

$$\begin{bmatrix} 7-{-2} & 3 \\ 3 & -1-{-2} \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-3R_2 + R_1 = \text{New } R_1$

$$\begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 3x_1 + x_2 = 0$$

$$x_2 = -3x_1$$

suppose $x_1 = 1$

$$x_2 = -3(1) = -3$$

vector $= \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Check: $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

Example Two

Finding Eigen Values and Eigen vectors.

a) $A = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$

$A\bar{x} = \lambda \bar{x}$

$\lambda I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\lambda = \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}$

determine $A - \lambda I$

$\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} = \begin{bmatrix}1-\lambda & 0\\ 0 & 2-\lambda\end{bmatrix}$

Determinant of $A - \lambda I$

$\det\begin{pmatrix}1-\lambda & 0\\ 0 & 2-\lambda\end{pmatrix} = (1-\lambda)(2-\lambda) - 0$
$= 2 - \lambda - 2\lambda + \lambda^2$
$= 2 - 3\lambda + \lambda^2$

Already factored!

$\lambda = 1$ or $\lambda = 2$

If $\lambda_1 = 1$

$\begin{bmatrix}1-1 & 0\\ 0 & 2-1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = 0$

$\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0\\ x_2\end{bmatrix} = 0$

If $\lambda_2 = 2$

$\begin{bmatrix}1-2 & 0\\ 0 & 2-2\end{bmatrix}$

$\begin{bmatrix}-1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ 0\end{bmatrix} = 0$

$\overrightarrow{v_{\lambda=2}} \to \{0,1\}$

$\overrightarrow{v_{\lambda=1}} \to \{1,0\}$

Example Three

b) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ②

$A\bar{x} = \lambda \bar{x}$

$\lambda I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$

Determinant of $(A - \lambda I)$

$= \lambda^2 + 1$

$\lambda^2 = -1$

Eigen values ± 1

If $\lambda_1 = -1$

$\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $\quad v_1 + v_2 = 0$

$v_1 + v_2 = 0$

$v_2 = -v_1$ $\quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

If $\lambda_2 = 1$

$\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-v_1 + v_2 = 0$

$v_1 - v_2 = 0$

$v_1 = v_2$ $\quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Check: $\lambda = 1$

$A\bar{x} = \lambda \bar{x}$

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 0 & -1 \\ 1 & -0 \end{bmatrix} \begin{bmatrix} +1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0-1 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{v}_{\lambda=i} \to \{i, 1\}$

$\vec{v}_{\lambda=-i} \to \{-i, 1\}$

0	-1
-1	0

$$\overrightarrow{v_{\lambda=-1}} \rightarrow \{1, 1\}$$
$$\overrightarrow{v_{\lambda=1}} \rightarrow \{-1, 1\}$$

Real life applications of Eigen values and Eigen vectors

- https://www.youtube.com/watch?v=R13Cwgmpuxc

Questions & Comments