INVERSE OF 3X3 Matrix

- In this lesson we find the inverses of 3 x 3 matrices using the adjoint of a matrix
- We later find the matrix inverses using Row Reduction Method

Definition:

- **Nonsingular Matrix** A square matrix that is not singular, i.e., one that has a matrix inverse.
- Nonsingular matrices are sometimes also called regular matrices.
- A square matrix is nonsingular iff its determinant is nonzero

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{32} & a_{33} \end{bmatrix}; \text{Adj (A)} = \begin{bmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31}' & a_{32}' & a_{33}' \end{bmatrix}^T = \begin{bmatrix} |A_{11}| & -|A_{12}| & |A_{13}| \\ -|A_{21}| & |A_{22}| & -|A_{23}| \\ |A_{31}| & -|A_{32}| & |A_{33}| \end{bmatrix}^T$$

$$|A_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\ a_{33} - a_{23}\ a_{32}; \quad |A_{12}| = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}\ a_{33} - a_{23}\ a_{31}$$

$$|A_{13}| = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\ a_{32} - a_{22}\ a_{31}; \quad |A_{21}| = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}\ a_{33} - a_{13}a_{32}$$

$$|A_{22}| = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11}\ a_{33} - a_{31}\ a_{13}; \quad |A_{23}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}\ a_{32} - a_{12}\ a_{31}$$

$$|A_{31}| = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = a_{12}\ a_{23} - a_{13}\ a_{22}; \quad |A_{32}| = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = a_{11}\ a_{23} - a_{13}\ a_{21}$$

$$|A_{33}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\, a_{22} - a_{12}\, a_{21}\,; \qquad \text{Det } A = a_{11}\,|A_{11}| - a_{12}\,|A_{12}| + a_{13}\,|A_{13}|\,; \qquad A^{-1} = \frac{1}{\det A}$$

$adj(A)$

Examples: Find the inverse of $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}$

Solution:

Det A = $4\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} = 4(2) - 0 + 1\,(2-6) = 8 - 4 = 4$; Therefore, A^{-1} exists.

$$adj(A) = \begin{bmatrix} |A_{11}| & -|A_{12}| & |A_{13}| \\ -|A_{21}| & |A_{22}| & -|A_{23}| \\ |A_{31}| & -|A_{32}| & |A_{33}| \end{bmatrix}^T ; \quad |A_{11}| = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2 \; ; \quad |A_{12}| = \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = 2 \; ; \quad |A_{13}| = \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} = -4 \; ;$$

$$|A_{21}| = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1 \; ; \quad |A_{22}| = \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} = 1 \; ; \quad |A_{23}| = \begin{vmatrix} 4 & 0 \\ 3 & 1 \end{vmatrix} = 4 \; ; \quad |A_{31}| = \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} = -2$$

$$|A_{32}| = \begin{vmatrix} 4 & 1 \\ 2 & 0 \end{vmatrix} = -2 \qquad |A_{33}| = \begin{vmatrix} 4 & 0 \\ 2 & 2 \end{vmatrix} = 8 = 8 \; adj(A) = \begin{bmatrix} 2 & -2 & -4 \\ 1 & 1 & -4 \\ -2 & 2 & 8 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 & -2 \\ -2 & 1 & 2 \\ -4 & -4 & 8 \end{bmatrix};$$

$$A^{-1} = \begin{bmatrix} 1/2 & \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ -1 & -1 & 2 \end{bmatrix}$$

ROW-ECHELON FORM

- Here, we discover the inverse of a matrix using row reduction method, the general concept of reducing a matrix to echelon form and finally introduce the concept of the canonical form of a matrix.
- Each concept is illustrated by several examples.

Learning outcomes

By the end of this sub-section, you will be able to;

- Find the inverse of a matrix using row reduction method
- Reduce a given matrix to echelon form.
- Reduce given matrix to canonical form (reduced row echelon).

Inverse of a matrix (row reduction method)

- The inverse of a matrix a can be found using row reduction to echelon form of the augmented matrix (A|I) to get (I | A^{-1}) .

Example:

Find the inverse of $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 4 & 0 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ 2R_2 - R_1 \\ 4R_3 - 3R_1 \end{array} \quad \rightarrow \quad \left[\begin{array}{ccc|ccc} 4 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & -1 & -1 & 2 & 0 \\ 0 & 4 & 1 & -3 & 0 & 4 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 4 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & -1 & -1 & 2 & 0 \\ 0 & 0 & 2 & -2 & -2 & 4 \end{array}\right] \begin{array}{l} 2R_1 - R_3 \\ 2R_2 + R_3 \\ \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 8 & 0 & 0 & 4 & 2 & -4 \\ 0 & 8 & 0 & -4 & 2 & 4 \\ 0 & 0 & 2 & -2 & -2 & 4 \end{array}\right] \begin{array}{l} R1/8 \\ R2/8 \\ R3/2 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right]; \quad \text{Inverse} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ -1 & -1 & 2 \end{bmatrix}$$

Note:

- For a matrix in echelon form, for subsequent rows, the non-zero entries occur in later and later columns.
- For a **matrix in echelon form, all entries below the main diagonal are zero.**
- Given any matrix B (not in echelon form) we perform the following **elementary row operations** to change it to echelon form:

1. Change the order of the rows (interchange some rows)
2. Multiply one row by a nonzero constant.
3. Add a multiple of one row to a nonzero multiple of another row.

6.2.3 Reduced row-echelon form

- **Definition:** A matrix is said to be **in reduced** row echelon form (canonical form) if:

1. Each nonzero row begins with a **pivot entry 1**. (Leading 1 of the row)
2. Each leading entry of a row is in a column to the rights of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros i.e. The rest of the columns containing the pivot entry 1 consists of 0s.
4. In subsequent rows, the **pivot entries** occur in later and later columns.
5. All nonzero rows are above any rows of all zeros
6. The all-zero rows are at the bottom (they are the unused rows).

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

are in echelon form. In fact, the second matrix is in reduced echelon form.

Examples

1. Reduce $\begin{bmatrix} 3 & 4 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ 4 & -3 & 11 & 2 \end{bmatrix}$ to reduced row-echelon (canonical) form.

Solution: $\begin{bmatrix} 3 & 4 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ 4 & -3 & 11 & 2 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & -1 & 3 & 1 \\ 3 & 4 & -1 & 1 \\ 4 & -3 & 11 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 4R_1 \end{matrix}$

$$\to \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 7 & -10 & -2 \\ 0 & -1 & -1 & -2 \end{bmatrix} R_2/7 \to \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & -\frac{10}{7} & -\frac{2}{7} \\ 0 & -1 & -1 & -2 \end{bmatrix} \begin{matrix} R_1 + R_2 \\ \\ R_3 + R_2 \end{matrix}$$

$$\to \begin{bmatrix} 1 & 0 & \frac{11}{7} & \frac{5}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{2}{7} \\ 0 & 0 & -\frac{17}{7} & -\frac{16}{7} \end{bmatrix} \begin{matrix} \\ \\ -\frac{7}{17}R_3 \end{matrix} \to \begin{bmatrix} 1 & 0 & \frac{11}{7} & \frac{5}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{2}{7} \\ 0 & 0 & 1 & \frac{16}{17} \end{bmatrix} \begin{matrix} R_1 - \frac{11}{7}R_3 \\ R_2 + \frac{10}{7}R_3 \\ \\ \end{matrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{13}{17} \\ 0 & 1 & 0 & \frac{18}{17} \\ 0 & 0 & 1 & \frac{16}{17} \end{bmatrix}$$

2. Reduce to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 2 & 0 \\ -1 & -2 & -1 & 1 \\ 1 & -1 & 3 & 4 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R1 \\ R_3 + R_1 \\ R_4 - R_1 \end{array} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -4 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & -3 & 0 & 2 \end{bmatrix} \begin{array}{l} \\ \\ \\ R_4 - R_2 \end{array} \rightarrow$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -4 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 4 & 4 \end{bmatrix} \begin{array}{l} \\ \\ \\ R_4 - 2R_3 \end{array} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -4 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Rows Echelon form

Recall - Gaussian elimination outputs an upper triangular matrix

Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$ use pivots to eliminate below and upper pivots

Take Pivot 1 - To eliminate Nos. below it

$R_2 = -4R_1 + R_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix}$

To elimate 6, $R_3 = -6R_1 + R_3$

Take pivot -3. Notice R_2 has a common factor of 3 R_3 a 5. Remember we can switch rows, multiply rows by constants or divide by constants

Divide R_2 by -3 and R_3 by -5

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ R_3 and R_2 are similar. Eliminate one

Take pivot 1 - $-2R_2 + R_1 = R_1$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = rref(X)$$

$\underbrace{}_{\text{Pivot columns}}$ $\underbrace{}_{\text{Non pivot columns}}$

Pivots become one

6.3 Assessment Questions

1. Reduce the matrix $\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 7 & -4 & 1 \end{pmatrix}$ to echelon form

2. Determine if the following matrices are in echelon canonical form or not.

(a) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & 5 & -2 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 8 & 7 & -3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Questions?
Comments?