Lesson Four: Matrix Determinants & Inverses

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DETERMINANTS OF 2x2 AND 3x3 MATRICES

- In this lesson we will cover determinants of and matrices.
- In each case, several worked out examples will be given.
- We will also deal with a general formula for finding the determinant of *n*x*n* matrices.
- We will introduce the inverse of a 2x2 matrix
- Assignment – Inverse of a 3x3 matrix

Lesson learning outcomes

By the end of the lesson, you will be able to;

- ✓Find the determinants of a matrix
- ✓Use the general formula to finding the determinant of *n*x*n* matrices e.g. to find the determinants of a 4 X 4 and 5 X 5 matrix
- ✓Find matrix Inverses

3.2.1 Determinants of 2×2 and 3×3 matrices

Let $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Det $A=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}\,a_{22}-a_{12}\,a_{21}$. The determinant is a scalar.

Examples:

1. Find the value of λ such that $\begin{vmatrix} \lambda & \lambda \\ 3 & \lambda-2 \end{vmatrix}=0$.

2. Let $\begin{vmatrix} 1+x & 1 \\ 2+2x & 2 \end{vmatrix}=$ find x.

3. Let $\begin{vmatrix} x & 3 \\ 2 & 2x+1 \end{vmatrix}=4$. Find x.

3×3 Matrices

$$\text{Let } A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det A=\begin{vmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$=a_{11}\left(a_{22}\,a_{33}-a_{23}a_{32}\right)-a_{12}\left(a_{21}a_{33}-a_{23}\,a_{31}\right)+a_{13}\left(a_{21}a_{32}\right)$$

$$=a_{11}\,a_{22}\,a_{33}-a_{11}\,a_{23}\,a_{32}-a_{12}\,a_{21}\,a_{33}+a_{12}\,a_{23}\,a_{31}+a_{13}\,a_{21}\,a_{32}-a_{13}\,a_{22}\,a_{31}$$

$$=a_{11}\,a_{22}\,a_{33}+a_{12}\,a_{23}\,a_{31}+a_{13}\,a_{21}\,a_{32}-a_{11}\,a_{23}\,a_{32}-a_{12}\,a_{21}\,a_{33}-a_{13}\,a_{22}\,a_{31}$$

Note: This is a sum of 6 products, *3 positive* and *3 negative.* Each product has exactly one factor from each row and column.

3.2.2 Determinant of an *n*x*n* matrix

- For a large square matrix the determinant is a sum of products, half of which have minus signs added.
- Each product will have exactly one factor from each row and one factor from each column.
- There are $n!$ summations.

Finding Matrix Determinant - Examples

1. Evaluate $\begin{vmatrix} 3 & 1 & 2 \\ -1 & -1 & 4 \\ -2 & -1 & 1 \end{vmatrix}$

Solution: $\begin{vmatrix} 3 & 1 & 2 \\ -1 & -1 & 4 \\ -2 & -1 & 1 \end{vmatrix} = 3[(-1)1+1(4)]-1[-1(1)+8]+2[1-2] = 0$

2. Evaluate $\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 2 \\ 1 & -1 & 2 \end{vmatrix}$

Solution: $\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 2 \\ 1 & -1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = 1(4) - 3(2) + 1(-3) = -5$

3\. Evaluate $\begin{vmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 2 & 0 \\ -1 & -2 & -1 & 1 \\ 1 & -1 & 3 & 3 \end{vmatrix}$

Solution: $\begin{vmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 2 & 0 \\ -1 & -2 & -1 & 1 \\ 1 & -1 & 3 & 3 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 & 0 \\ -2 & -1 & 1 \\ -1 & 3 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 & 0 \\ -1 & -1 & 1 \\ 1 & 3 & 3 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 & 0 \\ -1 & -2 & 1 \\ 1 & -1 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 & 2 \\ -1 & -2 & -1 \\ 1 & -1 & 3 \end{vmatrix}$

$= 1(4) - 2(-4) + 3(-6) - 1(-6) = 0$

(vi) $\begin{bmatrix} +1 & 0 & 3 \\ -2 & 1 & 3 \\ 5 & 2 & 4 \end{bmatrix} = +1 \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} - 0 \begin{bmatrix} -2 & 3 \\ 5 & 4 \end{bmatrix} + 3 \begin{bmatrix} -2 & 1 \\ 5 & 2 \end{bmatrix}$

$= +1 \begin{bmatrix} -2 \end{bmatrix} - 0 \begin{bmatrix} -23 \end{bmatrix} + 3 \begin{bmatrix} -9 \end{bmatrix}$

$-2 - 0 + -27 = \cancel{28}$

$-2 + 0 - 27 = \cancel{-28}\ -29$

Easier way for 3x3 matrix

$\begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & 3 \\ 5 & 2 & 4 \end{bmatrix} \begin{matrix} 1 & 0 & 3 \\ -2 & 1 & 3 \\ 5 & 2 & 4 \end{matrix}$

$(4 + 0 + -12) - (15 + 6 + 0)$

$-8 - 21 = -29$

4x4 Matrix Determinant

$$\begin{bmatrix} 1 & 0 & 4 & -6 \\ 2 & 5 & 0 & 3 \\ -1 & 2 & 3 & 5 \\ 2 & 1 & -2 & 3 \end{bmatrix}$$

R_1 & C_3 have same No. of zero
we pick R_1 for simplicity

Eliminate R_1 C_1 and pick coefficients.

$$+1\begin{bmatrix} 5 & 0 & 3 \\ 2 & 3 & 5 \\ 1 & -2 & 3 \end{bmatrix} - \overset{\text{zero}}{0}\begin{bmatrix} 2 & 0 & 3 \\ -1 & 3 & 5 \\ 2 & -2 & 3 \end{bmatrix} + 4\begin{bmatrix} 2 & 5 & 3 \\ -1 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix} - 6\begin{bmatrix} 2 & 5 & 0 \\ -1 & 2 & 3 \\ 2 & 1 & - \end{bmatrix}$$

$$+1\left[+5\begin{bmatrix} 3 & 5 \\ -2 & 3 \end{bmatrix} - \overset{\text{zero}}{0}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} + 3\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}\right] = 1\left[5(19) - 0 + 3(-7\right.$$

$$= 1(95 - 21) = \underline{74}$$

$$+4\left[2\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} - 5\begin{bmatrix} -1 & 5 \\ 2 & 3 \end{bmatrix} + 3\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}\right] = 4\left[2(1) - 5(-13) + 3(-5\right.$$

$$= 4[(2 + 65 - 15)] = (52)$$

$$4(52) = \underline{208}$$

$$+6\left[2\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} - 5\begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix} + \overset{\text{zero}}{0}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}\right] = 6[2(-7) - 5(-4) + 0]$$

$$= 6(-14 + 20)$$

$$= 6(6) = \underline{36}$$

$$74 + 208 + 36 = \underline{318}$$

Matrix Inverse

If A is a non-singular square matrix, there is an existence of n x n matrix A^{-1}, which is called the **inverse matrix** of A such that it satisfies the property:

$AA^{-1} = A^{-1}A = I$, where I is the Identity matrix

The identity matrix for the 2 x 2 matrix is given by

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverse of a Matrix Formula

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the 2 x 2 matrix. The inverse matrix of A is given by the formula,

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$$

$$= \frac{1}{8-7} = \frac{1}{1} \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 8-7 & -2+2 \\ 28-28 & -7+8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$3 \times 2 = 6$

$4 \times 1 = 4$

$$B^{-1} = \frac{1}{4-6} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{-2+1.5}\ -2 & \cancel{1+}\ 1 \\ 1.5 & -0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix} = \begin{bmatrix} -2+3 & 1-1 \\ -6+6 & 3-2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$$

Solution: Let A = IA

Or $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Applying $R_1 \longrightarrow (1/2)R_1$, we have

$$\begin{bmatrix} 1 & 1/2 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $R_2 \longrightarrow R_2 - 7R_1$

$$\begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ -7/2 & 1 \end{bmatrix} A$$

Applying $R_2 \longrightarrow 2R_2$, we have

$$\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ -7 & 2 \end{bmatrix} A$$

Applying $R_1 \longrightarrow R_1 - (1/2)R_2$, we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix} A$$

Thus, the inverse of matrix A is given by:

I= A^{-1} A

Therefore,

$$A^{-1} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$$

Assignment

Find the inverse of a matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 2 & 9 \end{pmatrix}$

Find the inverse of a matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 1 & 1 \end{bmatrix}$

Inverse Matrix 3 x 3 Example

Problem:

Find the inverse of a matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 2 & 9 \end{pmatrix}$

Solution:

Determinant of the given matrix is

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 2 & 9 \end{pmatrix} = 1\cdot 33 - 2(-6) + 3(-27)$$

$$= -36$$

Let us find the minors of the given matrix as given below:

$$M_{1,1} = \det\begin{pmatrix} 5 & 6 \\ 2 & 9 \end{pmatrix} = 33$$

$$M_{1,2} = \det\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} = -6$$

$$M_{1,3} = \det\begin{pmatrix} 4 & 5 \\ 7 & 2 \end{pmatrix} = -27$$

$$M_{2,1} = \det\begin{pmatrix} 2 & 3 \\ 2 & 9 \end{pmatrix} = 12$$

$$M_{2,2} = \det\begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} = -12$$

$$M_{2,3} = \det\begin{pmatrix} 1 & 2 \\ 7 & 2 \end{pmatrix} = -12$$

$$M_{3,1} = \det\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} = -3$$

$$M_{3,2} = \det\begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix} = -6$$

$$M_{3,3} = \det\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}: -3$$

$$\text{cofactors:} \begin{pmatrix} 33 & 6 & -27 \\ -12 & -12 & 12 \\ -3 & 6 & -3 \end{pmatrix}$$

Now, find the adjoint of a matrix by taking the transpose of cofactors of the given matrix.

$$\begin{pmatrix} 33 & 6 & -27 \\ -12 & -12 & 12 \\ -3 & 6 & -3 \end{pmatrix}^T = \begin{pmatrix} 33 & -12 & -3 \\ 6 & -12 & 6 \\ -27 & 12 & -3 \end{pmatrix}$$

Now,

$A^{-1} = (1/|A|)$ Adj A

Hence, the inverse of the given matrix is:

$$= \begin{pmatrix} -\frac{11}{12} & \frac{1}{3} & \frac{1}{12} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{3}{4} & -\frac{1}{3} & \frac{1}{12} \end{pmatrix}$$

Questions? Comments?