Surface Energy Balance Equation
Dimensional Analysis

Karun Datadien

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This document examines the dimensional consistency of Equation 35 from Liang et al. (1994), "A Simple Hydrologically Based Model of Land Surface Water and Energy Fluxes for GCMs."

1. Reconstructed Energy Balance Equation, Eq. 35

$$
\begin{aligned}
&\varepsilon[n]\sigma\left(T_s^+[n]\right)^4 + \left(\frac{\rho_a c_p}{r_h[n]} + \frac{\rho_a c_p z_a[n]}{2\Delta t} + \frac{\dfrac{\kappa[n]}{D_2} + \dfrac{C_s[n]D_2}{2\Delta t}}{1 + \dfrac{D_1}{D_2} + \dfrac{C_s[n]D_1D_2}{2\Delta t\kappa[n]}}\right)T_s^+[n] \\
&= (1-\alpha[n])R_s + \varepsilon[n]R_L + \frac{\rho_a c_p}{r_h[n]}T_a[n] - \rho_w L_e E[n] + \frac{\rho_a c_p z_a[n]T_s^-[n]}{2\Delta t} \\
&\quad + \frac{\dfrac{\kappa[n]T_2}{D_2} + \dfrac{C_s[n]D_2T_1^-[n]}{2\Delta t}}{1 + \dfrac{D_1}{D_2} + \dfrac{C_s[n]D_1D_2}{2\Delta t\kappa[n]}}
\end{aligned}
$$

2. Units and Dimensions

Base dimensions:

- Temperature $[T]$: K
- Length $[L]$: m
- Mass $[M]$: kg
- Time $[t]$: s
- Energy $[E]$: J = kg·m^2/s^2

Symbol Definitions and Units:

- σ (Stefan-Boltzmann constant): W/m^2/K^4 = [M][T]$^{-3}$[K]$^{-4}$
- ρ_a, ρ_w (air/water density): kg/m^3 = [M][L]$^{-3}$
- c_p (specific heat per mass): J/kg/K = [L]^2[T]$^{-2}$[K]$^{-1}$
- C_s (volumetric heat capacity): J/m^3/K = [M][L]$^{-1}$[T]$^{-2}$[K]$^{-1}$
- κ (thermal conductivity): W/m/K = [M][L][T]$^{-3}$[K]$^{-1}$
- L_e (latent heat of evaporation): J/kg = [L]^2[T]$^{-2}$

- E (evapotranspiration flux): m/s = [L][T]$^{-1}$
- R_s, R_L (radiative fluxes): W/m^2 = [M][T]$^{-3}$
- r_h (aerodynamic resistance): s/m = [T][L]$^{-1}$

3. Dimensional Consistency of All Terms

Left-Hand Side:

- $\varepsilon\sigma T_s^4$: $[1] \cdot [M][T]^{-3}[K]^{-4} \cdot [K]^4 = [M][T]^{-3}$
- $\left(\frac{\rho_a c_p}{r_h}\right) T_s$: $[M][L]^{-3} \cdot [L]^2[T]^{-2}[K]^{-1} \,/\, ([T][L]^{-1}) \cdot [K] = [M][T]^{-3}$
- $\left(\frac{\rho_a c_p z_a}{2\Delta t}\right) T_s$: $[M][L]^{-3} \cdot [L]^2[T]^{-2}[K]^{-1} \cdot [L] \,/\, [T] \cdot [K] = [M][T]^{-3}$
- $\left(\frac{\kappa}{D_2}\right) T_s$: $[M][L][T]^{-3}[K]^{-1} \,/\, [L] \cdot [K] = [M][T]^{-3}$
- $\left(\frac{C_s D_2}{2\Delta t}\right) T_s$: $[M][L]^{-1}[T]^{-2}[K]^{-1} \cdot [L] \,/\, [T] \cdot [K] = [M][T]^{-3}$

Right-Hand Side:

- $(1-\alpha)R_s + \varepsilon R_L$: W/m^2 = [M][T]$^{-3}$
- $\left(\frac{\rho_a c_p}{r_h}\right) T_a$: Same as LHS counterpart = [M][T]$^{-3}$
- $-\rho_w L_e E$: $[M][L]^{-3} \cdot [L]^2[T]^{-2} \cdot [L][T]^{-1} = [M][T]^{-3}$
- $\left(\frac{\rho_a c_p z_a}{2\Delta t}\right) T_s^-$: $[M][T]^{-3}$
- $\left(\frac{\kappa T_2}{D_2}\right)$: $[M][L][T]^{-3}[K]^{-1} \cdot [K] \,/\, [L] = [M][T]^{-3}$
- $\left(\frac{C_s D_2 T_1^-}{2\Delta t}\right)$: $[M][L]^{-1}[T]^{-2}[K]^{-1} \cdot [L] \,/\, [T] \cdot [K] = [M][T]^{-3}$

Conclusion: All terms on both sides have consistent dimensions:

$$\text{W/m}^2 = [\text{M}][\text{T}]^{-3}$$

4. Typical Values of Variables (Example Setup)

ε	0.95 (surface emissivity)
σ	5.67×10^{-8} W/m^2/K^4 (Stefan–Boltzmann constant)
α	0.3 (albedo)
R_s	100 W/m^2 (shortwave radiation)
R_L	300 W/m^2 (longwave radiation)
T_s^-	300 K (previous surface temperature)
T_1^-	295 K (first soil layer temp)
T_2	290 K (second soil layer temp)
T_a	298 K (air temperature)
z_a	0.3 m (depth of upper layer)
D_1	0.5 m (layer 1 depth)
D_2	1 m (layer 2 depth)
ρ_a	1.225 kg/m^3 (air density)
ρ_w	1000 kg/m^3 (water density)
c_p	1005 J/kg/K (specific heat of air)
C_s	2.13×10^6 J/m^3/K (volumetric soil heat capacity)
κ	0.6 W/m/K (thermal conductivity)
L_e	2.45×10^6 J/kg (latent heat)
E	1.16×10^{-5} m/s (evapotranspiration, 1 mm/day)
r_h	40.8 s/m (aerodynamic resistance)
Δt	86400 s (1 day timestep)