PROB

- **Funz caratteristiche**

- **Prop.** (funz caratt. somma) sia X_j $j=1,\dots,n$ v.a reali indipendenti, sia $S_n=\sum_{j=1}^n X_j$ $\rightsquigarrow$ $\varphi_{S_n}=\prod_{j=1}^n \varphi_X(t)$ funz caratt

dimostrazione

- $\varphi_{S_n}(t)=\mathbb{E}\left[e^{it\sum_{j=1}^n X_j}\right]=\mathbb{E}\left[\prod_{j=1}^n e^{itX_j}\right]\overset{\text{indipendenza}}{=}\prod_{j=1}^n \mathbb{E}\left[e^{itX_j}\right]=\prod_{j=1}^n \varphi_{X_j}(t)$

Prop $X=(X_1,\dots,X_d)^T$ vett. aleatorio allora

(1) $\varphi_X(0)=1$

(2) $t\mapsto \varphi_X(t)$ è uniformemente continua su $\mathbb{R}^d$

(3) $A\in\mathbb{R}^{d\times d}$, $b\in\mathbb{R}^d$ fissi $\rightsquigarrow$ $\varphi_{AX+b}(t)=e^{it^Tb}\varphi_X(A^Tt)$

dim

1) $\varphi_X(0)=\mathbb{E}[e^{i0^TX}]=\mathbb{E}[1]=1$

(2) $|\varphi_X(t+h)-\varphi_X(t)|=\left|\mathbb{E}\left[e^{it^TX}\left(e^{ih^TX}-1\right)\right]\right|\le \mathbb{E}\left[|e^{it^TX}|\cdot|e^{ih^TX}-1|\right]$ disuguaglianza; $h\in\mathbb{R}^d$; linearità valore atteso

$\Rightarrow |\varphi_X(t+h)-\varphi_X(t)|\le\mathbb{E}\left[|e^{ih^TX}-1|\right]$ ≤ 2 $\rightsquigarrow$ per conv. dominata $\rightsquigarrow \lim_{h\to0}\mathbb{E}[|e^{ih^TX}-1|]=\mathbb{E}[\lim_{h\to0}|e^{ih^TX}-1|]=0$

quindi $\dots\le\mathbb{E}\left[|e^{ih^TX}-1|\right]=R(h)$ con $R(h)\xrightarrow{h\to0}0$

(3) $\varphi_{AX+b}(t)=\mathbb{E}\left[e^{it^T(AX+b)}\right]=\mathbb{E}\left[e^{it^TAX}e^{it^Tb}\right]\overset{\text{linearità del valore atteso}}{=}e^{it^Tb}\,\mathbb{E}\left[e^{it^TAX}\right]=e^{it^Tb}\,\mathbb{E}\left[e^{i(A^Tt)^TX}\right]=e^{it^Tb}\varphi_X(A^Tt)$

(3 prop) $X=(X_1,\dots,X_d)$ vett. aleatori

$(X_1,\dots,X_d)$ ha componenti indipendenti sse $\varphi_X(t_1,\dots,t_d)=\prod_{j=1}^d \varphi_{X_j}(t_j)$ # non confondere con quello della somma! $\forall t=(t_1,\dots,t_d)\in\mathbb{R}^d$

$\Updownarrow$

oss senza assumere indipendenza: $\varphi_{\sum_{j=1}^n X_j}(t)=\varphi_X(t,t,\dots,t)$ stesso "t" uguale

dimostrazione

$[\Rightarrow]$ $\varphi_X(t_1,\dots,t_d)=\mathbb{E}\left[e^{i\sum_{j=1}^d t_jX_j}\right]=\mathbb{E}\left[\prod_{j=1}^d e^{it_jX_j}\right]\overset{\text{indipendenza}}{=}\prod_{j=1}^d\mathbb{E}\left[e^{it_jX_j}\right]=\prod_{j=1}^d\varphi_{X_j}(t_j)$ prod. scalare t^TX

$[\Leftarrow]$ $\varphi_X(t)=\prod_{j=1}^d\varphi_{X_j}(t_j)=\prod_{j=1}^d\int_{\mathbb{R}}e^{it_jx_j}P_X(dx_j)=$ Fubini $\int_{\mathbb{R}^d}\prod_{j=1}^d e^{it_jx_j}P_{X_1}(dx_1)\otimes\dots\otimes P_{X_d}(dx_d)$ $\vdots$ $\int e^{it_1x_1}e^{it_2x_2}P_{X_1}(dx_1)\otimes P_{X_2}(dx_2)$ $d=2$

$=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}e^{it_1x_1+it_2x_2}P_{X_2}(dx_2)\right)P_{X_1}(dx_1)=\int_{\mathbb{R}}e^{it_1x_1}\left(\int_{\mathbb{R}}e^{it_2x_2}P_{X_2}(dx_2)\right)P_{X_1}(dx_1)$

$=\int_{\mathbb{R}}e^{it_1x_1}P_{X_1}(dx_1)\int_{\mathbb{R}}e^{it_2x_2}P_{X_2}(dx_2)$ $\rightsquigarrow$ $\varphi_X(t)=\int_{\mathbb{R}^d}e^{i\sum_{j=1}^d t_jx_j}P_{X_1}\otimes\dots\otimes P_{X_d}(dx_1\dots dx_d)$

$\rightsquigarrow$ il th di unicità dice che $P_X=P_{X_1}\otimes\dots\otimes P_{X_d}$ $\rightarrow$ X ha componenti indipendenti

Funzione caratteristica e momenti

- **Teorema** sia X v.a reale, supponiamo che $\mathbb{E}[|X|^{n+\delta}]$ con n intero non negativo (incluso 0), $\delta\in[0,1]$ $\Rightarrow$ $\varphi_X^{(k)}(t)=i^k\,\mathbb{E}[X^ke^{itX}]$ $1\le k\le n$ e $\varphi^{(k)}(0)=i^k\,\mathbb{E}[X^k]$ sviluppo con taylor in un punto

(2) $\varphi_X(t)=\sum_{j=0}^n\frac{(it)^j\,\mathbb{E}[X^j]}{j!}+R_n(t)$ con $R_n(t)=o(|t|^n)$

- se $\delta>0$ $\rightsquigarrow$ $|R_n(t)|\le C_\delta|t|^{n+\delta}\,\mathbb{E}[|X|^{n+\delta}]$

$\frac{d}{dt}\varphi_X(t)=\frac{d}{dt}\mathbb{E}\left[e^{itX}\right]\overset{!}{=}\mathbb{E}\left[\frac{d}{dt}e^{itX}\right]=i\,\mathbb{E}\left[Xe^{itX}\right]$ $\rightsquigarrow$ $\varphi^{(1)}(0)=i\,\mathbb{E}[X]$

esempi sia $X_j\sim Ber(p)$ con $\varphi_{X_j}(t)=1-p+pe^{it}$, allora se $X\sim Bin(n,p)$ con $X=\sum_{j=1}^n X_j$ $\rightsquigarrow$ $\varphi_X(t)=(1-p+pe^{it})^n$

sia $S_n=\frac{1}{n}\sum_{j=1}^n X_j$ frequenza dei successi $\varphi_{S_n}(t)=?$

- S_n è una binomiale? no $\varphi_{aX+b}(t)=e^{itb}\varphi_X(at)$

- S_n fun caratt? $\varphi_{\sum_{j=1}^n X_j}\left(\frac{1}{n}t\right)=\varphi\left(\frac{1}{n}t\right)^n=\left(1-p+pe^{it/n}\right)^n$

esempio 2 $X_j\overset{IID}{\sim}Pos(\lambda_j)$ $\varphi_{X_j}(t)=e^{-\lambda_j(1-e^{it})}$ $\varphi_{\sum_{j=1}^n X_j}(t)=\prod_{j=1}^n e^{-\lambda_j(1-e^{it})}=e^{-\left(\sum_{j=1}^n\lambda_j\right)(1-e^{it})}=e^{-\bar\lambda(1-e^{it})}$

esempio 3 / proposizione $X\sim N(\mu,\sigma^2)$ $\Rightarrow$ $\varphi_X(t)=e^{i\mu t}e^{-\sigma^2\frac{t^2}{2}}$

- dim $\rightarrow$ sia $X=\mu+\sigma X_0$ con $X_0\sim N(0,1)$, se dimostro che $\varphi_{X_0}(t)=e^{-t^2/2}$ $\rightsquigarrow$ $\varphi_{\mu+\sigma X_0}(t)=e^{i\mu t}e^{-\frac{\sigma^2t^2}{2}}$

"dim" $\varphi_{X_0}^{(1)}(t)=\frac{d}{dt}\int_{\mathbb{R}}e^{itx}\underbrace{\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}}_{f(x_0)}dx=\int_{\mathbb{R}}\frac{d}{dt}e^{itx}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx=\int_{\mathbb{R}}ix\,e^{itx}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx=i\int\frac{e^{itx}}{\sqrt{2\pi}}\frac{d}{dx}\left(-e^{-\frac{x^2}{2}}\right)dx\overset{\text{per parti}}{=}i\left[\int_{\mathbb{R}}-it\,e^{itx}\frac{\left(-e^{-x^2/2}\right)}{\sqrt{2\pi}}dx+\left[e^{itx}\left(-e^{-\frac{x^2}{2}}\right)\right]_{-\infty}^{+\infty}\right]$

$=-t\varphi_{X_0}(t)+0$ $\Rightarrow$ $\begin{cases}\varphi'_{X_0}(t)=-t\varphi_{X_0}(t)\\ \varphi_{X_0}(0)=1\end{cases}$ $\Rightarrow$ $\varphi_{X_0}(t)=e^{-\frac{t^2}{2}}$

Prop $X_j\overset{ind}{\sim}N(\mu_j,\sigma_j^2)$ $\rightsquigarrow$ $\sum_{j=1}^n X_j\sim N\left(\sum_{j=1}^n\mu_j,\sum_{j=1}^n\sigma_j^2\right)$ somma gaussiane resta gaussiana dim $\varphi_{\sum_{j=1}^n X_j}(t)=\prod_{j=1}^n\varphi_{X_j}(t)=\prod_{j=1}^n e^{it\mu_j-\frac{\sigma_j^2t^2}{2}}=e^{it\sum_{j=1}^n\mu_j-\frac{1}{2}\left(\sum_{j=1}^n\sigma_j^2\right)t^2}$ "la seconda dimostrazione è pessima"

Prop $X\sim Gamma(\alpha,\beta)$ $\varphi_X(t)=\frac{1}{\left(1-\frac{it}{\beta}\right)^\alpha}$ $f_X(x)\propto x^{\alpha-1}e^{-\beta x}$ (genrica) $f_X(x)=c_{\alpha\beta}x^{\alpha-1}e^{-\beta x}\mathbb{1}_{(0,\infty)}(x)$

$X\sim exp(\lambda)$ $\varphi_X(t)=\frac{1}{\left(1-\frac{it}{\lambda}\right)}$

Esercizio $X_i\overset{IID}{\sim}Gamma(\alpha_i,\beta)$ uguale per tutti; determinare la legge della somma $\sum_{j=1}^n X_j\sim ?$