PROB

- **Funz caratteristiche**

- **Prop.** (funz caratt. somma) sia X_j $j=1,\dots,n$ v.a reali indipendenti, sia $S_n = \sum_{j=1}^n X_j$ $\rightsquigarrow$ $\underbrace{\varphi_{S_n}}_{\text{funz caratt}} = \prod_{j=1}^n \varphi_X(t)$

dimostrazione

- $\varphi_{S_n}(t) = \mathbb{E}\left[e^{it\sum_{j=1}^n X_j}\right] = \mathbb{E}\left[\prod_{j=1}^n e^{itX_j}\right] \overset{\text{indipendenza}}{=} \prod_{j=1}^n \mathbb{E}\left[e^{itX_j}\right] = \prod_{j=1}^n \varphi_{X_j}(t)$

Prop $X = (X_1,\dots,X_d)^T$ vett. aleatorio allora

(1) $\varphi_X(\underline{0}) = 1$

(2) $t \mapsto \varphi_X(t)$ è uniformemente continua su $\mathbb{R}^d$

(3) $\underset{\in\mathbb{R}^{n\times n}}{A}, b \in \mathbb{R}^d$ fissi $\rightsquigarrow$ $\varphi_{AX+b}(t) = e^{it^Tb}\varphi_X(A^Tt)$

dim

(1) $\varphi_X(\underline{0}) = \mathbb{E}\left[e^{i0^TX}\right] = \mathbb{E}[1] = 1$

(2) $\left|\varphi_X(t+h) - \varphi_X(t)\right| \underset{\text{linearità valore atteso}}{=} \left|\mathbb{E}\left[e^{it^TX}\left(e^{ih^TX}-1\right)\right]\right| \overset{\text{disuguaglianza}}{\le} \mathbb{E}\left[\left|e^{it^TX}\right| \cdot \left|e^{ih^TX}-1\right|\right]$

$h \in \mathbb{R}^d$

$\Rightarrow \left|\varphi_X(t+h) - \varphi_X(t)\right| \le \mathbb{E}\left[\left|e^{ih^TX}-1\right|\right]$ (≤ 2) $\rightsquigarrow$ per conv. dominata $\rightsquigarrow \lim_{h\to0} \mathbb{E}\left[\left|e^{ih^TX}-1\right|\right] = \mathbb{E}\left[\underbrace{\lim_{h\to0}\left|e^{ih^TX}-1\right|}_{=0}\right] = 0$

quindi $\dots \le \mathbb{E}\left[\left|e^{ih^TX}-1\right|\right] = R(h)$ con $R(h) \xrightarrow{h\to0} 0$

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(3) $\varphi_{AX+b}(t) = \mathbb{E}\left[e^{it^T(AX+b)}\right] = \mathbb{E}\left[e^{it^TAX}e^{it^Tb}\right] \overset{\text{linearità valore atteso}}{=} e^{it^Tb}\,\mathbb{E}\left[e^{it^TAX}\right] = e^{it^Tb}\,\mathbb{E}\left[e^{i(A^Tt)^TX}\right] = e^{it^Tb}\varphi_X(A^Tt)$

(3 prop) $\underline{X} = (X_1,\dots,X_d)$ vett. aleatori

$(X_1,\dots,X_d)$ ha componenti indipendenti sse $\varphi_{\underline{X}}(t_1,\dots,t_d) = \prod_{j=1}^d \varphi_{X_j}(t_j)$ # non confondere con quello della somma! $\forall\, t = (t_1,\dots,t_d) \in \mathbb{R}^d$

$\Updownarrow$

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oss senza assumere indipendenza: $\varphi_{\sum_{j=1}^d X_j}(t) = \varphi_{\underline{X}}\underbrace{(t,t,\dots,t)}_{\text{sulla "diagonale"}}$

dimostrazione

$[\Rightarrow]$ $\varphi_X(t_1,\dots,t_d) = \mathbb{E}\left[e^{i\sum_{j=1}^d t_jX_j}\right] = \mathbb{E}\left[\prod_{j=1}^d e^{it_jX_j}\right] \overset{\text{indipendenza}}{=} \prod_{j=1}^d \mathbb{E}\left[e^{it_jX_j}\right] = \prod_{j=1}^d \varphi_{X_j}(t_j)$

prod. scala t^Tx

$[\Leftarrow]$ $\varphi_X(t) = \prod_{j=1}^d \varphi_{X_j}(t_j) = \prod_{j=1}^d \int_{\mathbb{R}} e^{it_jx_j} P_X(dx_j) \overset{\text{Fubini}}{=} \int_{\mathbb{R}^d} \prod_{j=1}^d e^{it_jx_j} P_{X_1}(dx_1) \otimes \dots \otimes P_{X_n}(dx_n)$

$\underset{d=2}{=} \int e^{it_1x_1}e^{it_2x_2} P_{X_1}(dx_1) \otimes P_{X_2}(dx_2)$

$= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{it_1x_1+it_2x_2} P_{X_2}(dx_2)\right)P_{X_1}(dx_1) = \int_{\mathbb{R}} e^{it_1x_1}\left(\int_{\mathbb{R}} e^{it_2x_2}P_{X_2}(dx_2)\right)P_{X_1}(dx_1)$

$= \int_{\mathbb{R}} e^{it_1x_1}P_{X_1}(dx_1)\int_{\mathbb{R}} e^{it_2x_2}P_{X_2}(dx_2)$ $\rightsquigarrow$ $\varphi_X(t) = \int_{\mathbb{R}^d} e^{i\sum_{j=1}^d t_jx_j} P_{X_1} \otimes \dots \otimes P_{X_d}(dx_1\dots dx_d)$

$\rightsquigarrow$ il th di unicità dice che $P_{\underline{X}} = P_{X_1} \otimes \dots \otimes P_{X_d}$ $\longrightarrow$ $\underline{X}$ ha componenti indipendenti

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Funzione caratteristica e momenti

- **Theorema** sia X v.a reale, supponiamo che $\mathbb{E}\left[|X|^{n+\delta}\right]$ con n intero non negativo (incluso 0), $\delta \in [0,1]$ $\Rightarrow$ $\underset{\text{sviluppo con taylor in un punto}}{\varphi_X^{(k)}(t)} = i^k\,\mathbb{E}\left[X^ke^{itX}\right]$ $1 \le k \le n$ con

$\varphi^{(k)}(t=0) = i^k\,\mathbb{E}\left[X^k\right]$

(2) $\varphi_X(t) = \sum_{j=0}^n \frac{(it)^j\,\mathbb{E}\left[X^j\right]}{j!} + R_n(t)$ con $R_n(t) = o\left(|t|^n\right)$

- se $\delta > 0$ $\rightsquigarrow$ $\left|R_n(t)\right| \le C_\delta\,|t|^{n+\delta}\,\mathbb{E}\left[|X|^{n+\delta}\right]$

$\frac{\partial}{\partial t}\varphi_X(t) = \frac{\partial}{\partial t}\mathbb{E}\left[e^{itX}\right] \overset{!}{=} \mathbb{E}\left[\frac{\partial}{\partial t}e^{itX}\right] = i\,\mathbb{E}\left[Xe^{itX}\right]$ $\rightsquigarrow$ $\varphi^{(1)}(0) = i\,\mathbb{E}[X]$

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esempi sia $X_j \sim Ber(p)$ con $\varphi_{X_j}(t) = 1-p+pe^{it}$, allora se $X \sim Bin(n,p)$ con $X = \sum_{j=1}^n X_j$ $\rightsquigarrow$ $\varphi_X(t) = \left(1-p+pe^{it}\right)^n$

sia $\underset{\text{frequenza di successi}}{S_n} = \frac{1}{n}\sum_{i=1}^n X_j$ $\varphi_{S_n}(t) \overset{?}{=}$

- S_n è una binomiale? no $\quad\varphi_{aX+b}(t) = e^{itb}\varphi_X(at)$
- S_n fun caratt? $\varphi_{\sum_{j=1}^n X_j}\left(\frac{1}{n}t\right) = \varphi_{X_j}\left(\frac{1}{n}t\right)^n = \left(1-p+pe^{it/n}\right)^n$

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esempio 2 $X_j \overset{IND}{\sim} Pos(\lambda_j)$ $\quad\varphi_{X_j}(t) = e^{-\lambda(1-e^{it})}$ $\quad\varphi_{\sum_{j=1}^n X_j}(t) = \prod_{j=1}^n e^{-\lambda_j(1-e^{it})} = e^{-\left(\sum_{j=1}^n \lambda_j\right)(1-e^{it})} = e^{-\bar{\lambda}(1-e^{it})}$

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esempio 3 / proposizione $X \sim N(\mu,\sigma^2)$ $\Rightarrow$ $\varphi_X(t) = e^{i\mu t}e^{-\sigma^2\frac{t^2}{2}}$

- dim $\rightsquigarrow$ sia $X = \mu + \sigma X_0$ con $X_0 \sim N(0,1)$, se dimostro che $\varphi_{X_0}(t) = e^{-t^2/2}$ $\rightsquigarrow$ $\varphi_{\mu+\sigma X_0}(t) = e^{i\mu t}e^{-\frac{\sigma^2t^2}{2}}$

"dim" $\varphi_{X_0}^{(1)}(t) = \frac{\partial}{\partial t}\int_{\mathbb{R}} e^{itx}\underbrace{\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}}_{f(x_0)}dx = \int_{\mathbb{R}}\frac{\partial}{\partial t}e^{itx}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx = \int_{\mathbb{R}} ixe^{itx}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx = i\int\frac{e^{itx}}{\sqrt{2\pi}}\frac{\partial}{\partial x}\left(-e^{-\frac{x^2}{2}}\right)dx \underset{\text{per parti}}{=} i\left[\int_{\mathbb{R}} -it\,e^{itx}\frac{\left(-e^{-x^2/2}\right)}{\sqrt{2\pi}}dx + \left[e^{itx}\left(-e^{-\frac{x^2}{2}}\right)\right]_{-\infty}^{+\infty}\right]$

$= -t\,\varphi_{X_0}(t) + 0$ $\Rightarrow$ $\begin{cases}\dot{\varphi}_{X_0}(t) = -t\,\varphi_{X_0}(t)\\ \varphi_{X_0}(0) = 1\end{cases}$ $\Rightarrow$ $\varphi_{X_0}(t) = e^{-\frac{t^2}{2}}$

Prop $X_j \overset{ind}{\sim} N(\mu_j,\sigma_j^2)$ $\rightsquigarrow$ $\sum_{j=1}^n X_j \sim N\left(\sum_{j=1}^n \mu_j, \sum_{j=1}^n \sigma_j^2\right)$ $\underset{\text{dim}}{\overset{\text{somma gaussiane rimane gaussiana}}{}}$ $\varphi_{\sum_{j=1}^n X_j}(t) = \prod_{j=1}^n \varphi_{X_j}(t) = \prod_{j=1}^n e^{it\mu_j - \frac{\sigma_j^2t^2}{2}} = e^{it\sum_{j=1}^n \mu_j - \frac{1}{2}\left(\sum_{j=1}^n \sigma_j^2\right)t^2}$ "la seconda cazzia diretta è pessima"

Prop $X \sim Gamma(\alpha,\beta)$ $\quad\varphi_X(t) = \frac{1}{\left(1-\frac{it}{\beta}\right)}$ $\quad f_X(x) \propto x^{\alpha-1}e^{-\beta x}$ (densità) $\quad f_X(x) = c_{\alpha\beta}\,x^{\alpha-1}e^{-\beta x}\,\mathbb{1}_{(0,+\infty)}(x)$

$X \sim exp(\lambda)$ $\quad\varphi_X(t) = \frac{1}{\left(1-\frac{it}{\lambda}\right)}$

Esercizio $X_i \overset{IND}{\sim} Gamma(\alpha_i, \underset{\text{uguale per tutti}}{\beta})$ determinare la legge della somma $\sum_{j=1}^n X_j \sim ?$