UGANDA MARTYRS UNIVERSITY

UNIVERSITY EXAMINATIONS

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS

END OF SEMESTER 2, 2022/23 FINAL ASSESSMENT

BSC GEN III, MATH 3

ABSTRACT ALGEBRA

DATE : 23^{th} May 2023

TIME : 2:00 PM - 5:00 PM

DURATION: 3 Hrs

Instructions

1. *Carefully read through ALL the questions before attempting.*
2. *ANSWER FOUR (4)Questions (All questions carry equal marks).*
3. *Ensure that your* **Reg. number Name** *and* **Course** *are indicated on all pages of your work.*
4. *Ensure that your work is* **clear and readable**. *Untidy work will be penalized.*
5. *Any type of examination Malpractice will lead to automatic disqualification.*

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1. (a) Let $G = (\mathbb{Z}_8, +) = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Then $H = \{0, 3, 6\}$ and $K = \{0, 2, 4, 5, 7\}$ are subgroups of G.

   (i) Draw the Cayley table for the group G. [5 Marks]

   (ii) Find $|G|$, $|H|$, and $|K|$. [3 Marks]

   (iii) Find all left cosets of K in G. [5 Marks]

   (iii) Show that H is a subgroup of G. [4 Marks]

   (b) Prove that the identity element of a group G is unique. [3 Marks]

   (c) (i) When is a group G said to be *cyclic*? [1 Marks]

   (ii) Prove that every cyclic group is abelian. [4 Marks]

2. (a) Consider the symmetric group $S_3 = \{(1), (2\ 3), (1\ 3), (1\ 2\ 3), (1\ 3\ 2), (1\ 3)\}$. Draw the Cayley table for S_3. [12 Marks]

   (b) (i) Giving an example in each case, differentiate between an *even* and an *odd* permutation. [4 Marks]

   (ii) Define an *alternating group* A_n. [1 Mark]

   (ii) Find all elements of A_n from the symmetric group S_3. [4 Marks]

   (c) Prove that A_n is a subgroup of S_n. [4 Marks]

3. (a) (i) Let S be a set and R be a relation on S. When is R said to be an equivalence relation? [3 Marks]

   (ii) Define an *equivalence class*, given a set S and an equivalence relation R on S. [2 Marks]

   (iii) Explain the meaning of the phrase "x is congruent to y modulo m. [2 Marks]

   (b) Let m be a fixed integer and $a, b, c, d \in \mathbb{Z}$ such that $a \equiv b(mod\ m)$ and $c \equiv d(mod\ m)$. Prove the following relations.

   (i) $a + c \equiv b + d(mod\ m)$ [4 Marks]

   (ii) $a - c \equiv b - d(mod\ m)$ [4 Marks]

   (ii) $ac \equiv bd(mod\ m)$ [4 Marks]

(c) Use mathematical induction to prove that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n}{6}(n+1)(2n+1), n \in \mathbb{N}.$$

[6 Marks]

4. (a) Define the following terms giving a well explained example in each case.

   (i) A mapping. [3 Marks]

   (ii) A well defined mapping. [4 Marks]

   (iii) A surjective mapping. [4 Marks]

   (iv) A bijective mapping. [5 Marks]

   (v) An onto mapping. [4 Marks]

   (vi) Domain of a mapping. [3 Marks]

   (vii) Cartesian product of two sets S_1, S_2. [2 Marks]

5. (a) Let G be a cyclic group of order 9 generated by $a \in G(a^9 = e)$.

   (i) Define all possible homomorphisms on G. [6 Mark]

   (ii) Which of them are epimorphisms; automorphisms? [3 Marks]

   (iii) Why is there no monomorphism $f : G \to G$? [3 Marks]

   (iv) Determine the kernel of each monomorphism that you have defined (if any). [3 Marks]

   (b) Let $f : A \to B$ and $g : B \to C$ be group homomorphisms. Prove that $g \circ f$ is a group homomorphism. [5 Marks]

   (c) Let G be an abelian group. Show that $f : G \to G$ given by $f(x) = x^{-1}$ is an isomorphism. [5 Marks]

End