UGANDA MARTYRS UNIVERSITY

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS AND STATISTICS

University Examinations 2012/2013

Examination for the Degree of Bachelor of Science Financial Mathematics and for Bachelor of Science General

Tuesday, December 18, 2012

MTC 3103 COMPLEX VARIABLES

Time allowed: 3 hours

Instructions

(i) Answer **FIVE** questions.

(ii) Write both sides of the paper but begin a new question on a fresh page.

(iii) Only approved basic scientific calculators may be used in this Examination.

1 (a) State the *Residue theorem.*

(2 marks)

(b) Find the residues of $f(z) = \frac{2z^2+5}{(z+2)(z^2+4)(z^2)}$. Hence evaluate

$$\oint \frac{2z^2+5}{(z+2)(z^2+4)(z^2)} dz$$

using the residue theorem.

(8 marks)

(c) (i) State De Moivre's theorem.

(1 mark)

(ii) Solve the equation $z^6 + 729 = 0$ using De Moivre's theorem.

(5 marks)

(d) Find the fourth roots of $z = 2\sqrt{(3)} - 2i$.

(4 marks)

2 (a) (i) Define the limit of a function $f(z)$ at infinity.

(1 mark)

(ii) Prove that $\lim_{z \to 1+i} \frac{z^2-z+1-i}{z^2-2z+2} = 1 - \frac{1}{2}i$.

(5 marks)

(b) If $f(z) = \frac{2z-1}{3z+2}$, prove that, at $z = z_0$, $\lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h} = \frac{7}{(3z_0+2)^2}$ where $z_0 \neq \frac{-2}{3}$.

(4 marks)

(c) (i) When is a complex function $f(z)$ said to be continuous at a point $z = z_0$.

(2 marks)

(ii) Find the points at which the function $f(z)$ below is discontinuous.

$$f(z) = \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$$

Redefine the function at the points where the function is discontinuous to remove the discontinuity and show that the limit of the function at that point is $4 + 4i$.

(8 marks)

3 (a) What does it mean to say that $f(z) = u(x, y) + iv(x, y)$ is analytic?

(2 marks)

(b) Prove that a necessary condition for $f(z)$ to be analytic is that it must satisfy the Cauchy-Reimann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(6 marks)

(c) Using the definition, find the derivative of $f(z) = z^3 - 2z$ at z_0. Hence determine the derivative at $z = -1$.

(4 marks)

(d) (i) State Green's theorem.

(2 marks)

(ii) Verify Green's theorem in the plane for $\oint_c (3xy + x^2)dx + (y^2 + 2x)dy$ where c is the closed curve containing $y = x^2$ and $y^2 = x$.

(6 marks)

4 (a) Define the following terms:

(i) an isolated singularity,

(ii) a pole of order n.

(4 marks)

(b) Locate and name all the singularities of the following function.

$$f(z) = \frac{z^2 - 3z}{(z^2 + 2z + 2)(z + 5)}$$

(8 marks)

(c) Use L'Hopitals rule to evaluate:

$$\lim_{z \to i} \frac{z^{10}+1}{z^6+1}$$

(4 marks)

(c) Let Z_1 and Z_1 be complex numbers. Prove that $|Z_1 Z_2| = |Z_1||Z_2|$.

(4 marks)

5 (a) (i) Define the complex line integral of a function $f(z) = u(x,y) + iv(x,y)$ along a curve C.

(2 marks)

(ii) Find $\int_{(0,3)}^{(2,4)} (2y+x^2)dx + (3x-y)dy$ along the parabola $x = 2t,\ y = t^2+3$. Also evaluate the integral along a straight line from $(0,3)$ to $(2,4)$.

(7 marks)

(b) Evaluate $\int_c (\overline{z} + 4z)dz$ from $z = 4+4i$ to $z = 9+6i$ along the curve given by $z = t^2 + 2it$.

(6 marks)

(c) Expand $f(z) = \cos\, z$ in a Taylor series about $z = \frac{\pi}{2}$.

(5 marks)

6 (a) What do you understand by a *multiply connected region.*

(2 marks)

(b) Prove Cauchy's theorem that states that if a function $f(z)$ is analytic in the region D and on its boundary C then the closed curve integral $\oint_c f(z)dz = 0$.

(5 marks)

(c) State *Cauchy's integral formulae.*

(2 marks)

(d) Evaluate the following using Cauchy's integral formulae.

(i) $\oint_c \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)}$ dz.

(6 marks)

(ii) $\oint_c \frac{3z-2}{z^2-z}$ dz.

(5 marks)

END