UGANDA MARTYRS UNIVERSITY

NKOZI

UNIVERSITY EXAMINATIONS

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS AND STATISTICS

END OF SEMESTER FINAL ASESSMENT

SEMESTER II, 2022/2023

THIRD YEAR EXAMINATION FOR BACHELOR OF SCIENCE WITH EDUCATION

Number Theory

DATE: 24 MAY, 2023

TIME: 9:30PM – 12:30PM

Instructions:

1. *Carefully read through ALL the questions before attempting*
2. ***ANSWER only FIVE Questions from the Seven questions***
3. *Ensure that your **Reg number** is indicated on all pages of the examination answer booklet.*
4. *Ensure your work is **clear** and **readable**. Untidy work shall be penalized*
5. *Any type of examination Malpractice will lead to automatic disqualification*

1. (a) **[6 marks]** For $n \geq 1$, verify that $1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \binom{2n+1}{3}$.

   (b) **[6 marks]** In 1647, Mersenne noted that when s number can be written as a sum of two relatively prime squares in two distinct ways, it is composite and can be factored as follows: if $n = a^2 + b^2 = c^2 + d^2$, then $n = \frac{(ac+bd)(ac-bd)}{(a+d)(a-d)}$.

   Use this result to factor the number $38025 = 168^2 + 99^2 = 156^2 + 117^2$.

   (c) **[8 marks]** Find the highest power of 3 dividing 80! And the highest power of 5 dividing 1000!

2. **[10 marks]** (a) For $n \geq 1$, establish that the integer $n(7n^2 + 5)$ is of the form 6k.

   [6 marks] (b) Find all solutions in the integers of 15x + 12y + 30z = 24
   [Hint: Put y = 3s – 5t and z = -s + 2t]

   [4 marks] (c) If $p \geq q \geq 5$, and p and q are both primes, prove that $24 \mid p^2 - q^2$.

3. (a) (i) **[6 marks]** Use the Euclidean algorithm to compute the greatest common divisor of (2517, 2370)

   (ii) **[5 marks]** Find all integer solutions to the equation 2517x -2370y = 69 or explain why there are none

   (iii) **[5 marks]** Solve the linear congruence $2379x \equiv 69 (\text{mod} 2517)$ or explain why no solutions exist.

   (b)**[4 marks]** List the first four perfect numbers.

4. (a) (i) **[4 marks]** What can you say about the prime factorization of n if $\tau(n) = 8$?

   (ii) **[4 marks]** What is the smallest n with $\tau(n) = 8$?

   (iii) **[6 marks]** Find three n with $\phi(n) = 16$.

   (b) **[6 marks]** For $n \geq 1$, use mathematical induction to establish the following divisibility statement $5 \mid 3^{3n+1} + 2^{n+1}$.

5. (a) **[10 marks]** Find all solutions to the congruence $20x \equiv 30 \pmod{35}$.

   (b) **[5 marks]** An unanswered question is whether there exist an infinite number of sets of five consecutive odd integers of which four are primes. Find five sets if such integers.

   (c) **[5 marks]** From Fermat's Theorem deduce that, if any integer $n \geq 0$, $13 \mid 11^{12n+6} + 1$

6. (a) **[10 marks]** Use the Chinese Remainder Theorem to solve the simultaneous congruence

$$x \equiv -5 \pmod{7}$$
$$x \equiv 2 \pmod{5}$$
$$x \equiv 4 \pmod{6}$$

   (b) **[6 marks]** For $n \geq 1$, use congruence theory to establish the following divisibility statement $27 \mid 3^{5n+1} + 5^{n+2}$

   (c) **[4 marks]** List the first four Fermat numbers

7. (a)(i) **[3 marks]** Evaluate $\sigma(1500)$

   (ii) **[5 marks]** Use Euler's Theorem to find the remainder when 7^{1203} is divided by 1500

   (b) **[4 marks]** Prove that if 3 | a and 4 | b, then 12 | 8a – 9b.

   (c) **[8 marks]** Knowing that 2 is a primitive root of 19, find all the quadratic residues and quadratic non-residues of 19.