# UGANDA MARTYR'S UNIVERSITY

FACULTY OF SCIENCE

DEPARTMENT OF NATURAL SCIENCES

Second year Supplementary Examination

Semester I 2020/2021

**First Year, MAT 1102:Linear Algebra**

## Instructions

1. Attempt only five questions questions
2. Give clear steps to earn more marks

### Question 1

(i) What is meant by vectors $\mathbf{v_i}, \mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}$ being linearly independent [2 marks]

(ii) Let $V = \mathbb{R}^n$ and consider the following elements in

$V$:$\{e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots e_n = (0, 0, 0, \ldots, 1)\}$

Show that $e_1, e_2, \ldots, e_n$ are linearly independent. [3 marks]

(iii) Determine whether the following matrices are linearly dependent or linearly independent in $M_2(\mathbb{R})$

$$A_1 = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \quad [4marks]$$

b(i) If $v_1 = (1, 2, -1), v_2 = (2, -1, 1), v_3 = (8, 1, 1)$, show that $\{v_1, v_2, v_3\}$ is linearly dependent in $\mathbb{R}^3$, and determine the linear dependency relationship [5 marks]

(ii) Let $v_1 = (1, 2, 3), v_2 = (-1, 1, 4), v_3 = (3, 3, 2)$ and $v_4 = (-2, -4, -6)$. Determine a linearly independent set of vectors that spans the same subspace of $\mathbb{R}^3$ as span $\{v_1, v_2, v_3, v_4\}$ [6 marks]

## Question 2

(a) State the Wroskian method for linear dependency. hence find the Wroskian of $f_1(x) = 2x$, $f_2(x) = x^2$ and $f_3(x) = x^3$ [6 marks]

(b) Determine whether the following functions are linearly dependent or linearly independent on $I = (-\infty, \infty)$ by

(i) $f_1(x) = e^x$, $f_2(x) = x^2e^x$ [4 marks]

(ii)

$$f_1(x) = x^2, f_2(x) = \begin{cases} 2x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$$

[5 marks]

(c) Show that the functions $f_1(x) = e^{r_1 x}, f_2(x) = e^{r_2 x}, f_3 = e^{r_3 x}$ have a Wroskian

$$W[f_1, f_2, f_3](x) = e^{(r_1+r_2+r_3)(x)} \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{bmatrix}$$
$$= e^{(r_1+r_2+r_3)(x)}(r_3 - r_1)(r_3 - r_2)(r_2 - r_1)$$

[5 marks]

## Question 3

(a) Determine whether the following functions are linear transformation. If they are, prove it; if not provide a counter example to one of the properties

(i) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ y \end{bmatrix} [4marks]$$

(ii) $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix} [4marks]$$

(b) For the following linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, find a matrix $A$ such that $T(\mathbf{x}) = \mathbf{Ax}$ for all $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$

(i) $T : \mathbb{R}^2 \to \mathbb{R}^3$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ 3y \\ 4x + 5y \end{bmatrix} [4marks]$$

(ii) $T : \mathbb{R}^2 \to \mathbb{R}^2$, satisfying $T : \mathbb{R}^n \to \mathbb{R}^3$

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, T\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} [4marks]$$

(c) Construct a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}$, $T(x) = Bx$, where $V = \ker(B)$. Then use $B$ to construct a basis for $\ker(B)$. You will need to verify that what you have is a basis[4 marks]

## Question 4

(a) (i) Define a bilinear form on a vector space$V$ over afield $K$. (3 marks)

(ii) Let $\phi$ and $\sigma$ be any functionals on a vector space $V$. Let $f : VxV \to K$ be defined by $f(u,v) = \phi(u)\sigma(v)$. Show that $f$ is bilinear form. (4marks)

(b) Given a function $f(u,v)$ where $u = (x_1, x_2)$ and $v = (y_1, y_2)$ determine whether $f(u,v) = 3x_2y_2$ is a bilinear form on $\Re^2$, hence rewrite a function in matrix form.(3 marks)

(c) Describe the conic $C$ whose equation is $5x^2 - 4xy + 8y^2 - 36 = 0$ (6 marks)

(d) Find the quadratic form $q(x_1, x_2, x_3)$ corresponding to the matrices below

(i)

$$\begin{bmatrix} 1 & 2 & -4 \\ 2 & 3 & 5 \\ -4 & 5 & -7 \end{bmatrix}$$

(2 marks)

(ii)

$$\begin{bmatrix} 2 & -5 & -1 \\ -5 & -6 & -7 \\ 1 & -7 & -9 \end{bmatrix}$$

(2 marks)

## Question 5

(a)

(i) Define a dual space of a vector space $V$ (3 marks).

(ii) Consider the following basis of$\Re^2$: $V_1 = (2,1), V_2 = (3,1)$ .Find the dual basis $(\phi_1, \phi_2\ )$.(4 marks)

(b) Consider basis of $\Re^2$: $S_1 = \quad V_1 = (1,1), V_2 = (1,0)$ and $S_2 = \quad w_1 = (4,3), w_2 = (3,2)$ . Find the change of basis matrix $P$ from $s_1$ to $s_2$ (5 marks)

(c) Determine the invariant subspaces of $A =$

$$\begin{bmatrix} 2 & -4 \\ 5 & -2 \end{bmatrix}$$

viewed as a linear operator on $\ell^2$

(d) Given a subspace $W$ which is invariant under $s : V \to V$ and $T : V \to V$ show that $W$ is invariant under $S + T$

(3marks)

## Question 6

(a) Define the matrix representation of a linear operator.(4 marks)

(b) Find the matrix representation of the linear map $T : \Re^2 \rightarrow \Re^2$ defined by $T(x,y) = 3x - 4y, x + 5y$ relative to the basis $B = \ V_1 = (1,3), V_2 = (2,5)$ ( 5marks)

(c) Let $V$ be the vector space of 2 $x$2 matrices over $\Re$ and

$$M = \begin{bmatrix} -1 & 2 \\ 2 & 5 \end{bmatrix}$$

Let $D$ denote the differential operator that is,$D(f(t)) = \frac{df}{dt}$.

Each of the following sets is a basis of a vector space $V$ of functions. Find the matrix representing $D$

(i) $(e^t, e^{2t}, te^{2t})$ [3 marks]

(ii) $(1, t, \sin 3t, \cos 3t)$ [3 marks]