# UGANDA MARTYRS UNIVERSITY

UNIVERSITY EXAMINATIONS

FACULTY OF SCIENCE

DEPARTMENT OF NATURAL SCIENCES

END OF SEMESTER FINAL ASSESSMENT

SEMESTER II 2021/22

THIRD YEAR EXAMINATIONS FOR BACHELOR OF SCIENCE WITH EDUCATION

(BSc EDUC. 3)

**General Topology**

**MTH 3206**

DATE :WED 13$^{th}$ July 2021

TIME : 2:00 PM - 5:00 PM

TIME : 3 Hours

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Instructions

1. *Carefully read through ALL the questions before attempting.*
2. *ANSWER FOUR (4)Questions (All questions carry equal marks).*
3. *No* ***names*** *should be written anywhere on the examination booklet.*
4. *Ensure that your* ***Reg. number*** *and* ***Course*** *are indicated on all pages of your work.*
5. *Ensure that your work is* ***clear*** *and* ***readable****. Untidy work will be penalized.*
6. *Any type of examination Malpractice will lead to automatic disqualification.*

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1. (a) Topology consists of the study of the collection of objects that posses a mathematical structure. An example of such is the set of natural numbers $\mathbb{N} = \{1, 2, 3, ....\}$ which has a successor function (S).

   (i) State the conditions that the successor function (S) must satisfy (Peanos Axiom). [3 **Marks**]

   (ii) Which of the conditions yield the principle of mathematical induction? [1 **Marks**]

(b) (i) Define the equality of two sets as used in topology. [2 **Marks**]

   (ii) Determine whether each of the following statements as used in topology is **True** or **False**.

   (iia) For each set $B$, then $B \in 2^B$.

   (iib) For each set $B$, then $B \subset 2^B$.

   (iic) There are no members of the set $\{\phi\}$. [3 **Marks**]

(c) Let A, B and C be sets, Prove that if $A \subset B$ and $B \subset C$ then $A \subset C$. [4 **Marks**]

(d) Let A and B be two finite sets.

   (i) Define the cartesian product $A \times B$. [2 **Marks**]

   (ii) Do you think $A \times B = B \times A$? If No, justify your answer. [2 **Marks**]

(e) A relation $h : P \to Q$ is a function

$$P \times Q = \{(x, y), (x, z) \in h \Rightarrow y = z\}$$

i.e each object has at most one image. From this definition which of the following is a function?

   (i) $h(x) = x^2 + 3$

   (ii) $f(x) = \pm\sqrt{x} + 3$. [2 **Marks**]

(e) The possible characteristics of a mapping include; injective, surjective and bijective. Define with examples what each characteristic means. [6 Marks]

2. (a) With an aid of an example define what is meant by a metric space.[3 Marks]

(b) (i) Let $(X, d)$ be a metric space and let $X$ be a set of all continuous functions $f : (a, b) \to \mathbb{R}$ for $, g \in X$. Define a metric

$$d(f, g) = \int_a^b |f(t) - g(t)|dt.$$

Prove that $(X, d(f, g))$ is a metric space. [8 Marks]

(ii) If $f(t) = t^2 + 1$and $g(t) = 1 - t^2$ on (0,1), use the above metric to compute the distance between $f(t)$ and $g(t)$. [3 Marks]

(c) Let $f : X \to Y$ be a function between two metric spaces $X$ and $Y$.

(i) What is meant by $f$ being continuous at $a \in X$? [2 Marks]

(ii) Let $(X, d)$ and $(X, d')$ be metric spaces, and assume $f : X \to Y$ is an identity function. Show that $f$ is a continuous function. [3 Marks]

(d) (i) Differentiate between an open ball and a neighborhood of a point $a \in X$ where $X$ is a metric space. [4 Marks]

(ii) What is meant by a set $M \subset \mathbb{R}^2$ being closed and compact in $\mathbb{R}^2$ [2 Marks]

3. (a) Let $(X, d)$ be a metric space and $A \subseteq X$;

(i) What is meant by $A$ being bounded? [2 Marks]

(ii) Define the diameter of $A$. [2 Marks]

(iii) When is $A$ an open set? [2 Marks]

(iv) What makes $A$ a closed set? [2 Marks]

(v) when is $y$ a limit point of $A$. [2 Marks]

(b) Using clear examples show that;

(i) It is false to generalise that the intersection of an infinite number of open sets is open. [1 Mark]

(ii) A set can be simultaneously open and closed. [1 Mark]

(c) Let $f : (A_1, d_1) \to (A_2, d_2)$. Explain what is meant by $f$ being continuous at point $c \in A_1$ in terms of;

(i) open sets [2 Marks]

(ii) sequences [2 Marks]

(iii) closed sets [2 Marks]

(d) (i) Define a homeomorphism $g$ from metric space $(X, d_1)$ to metric space $(Y, d_2)$. [2 Marks]

(ii) Two metrics $d_1$ and $d_2$ are (Lipschitz) equivalent if there are constants $K\ k > 0$ such that for every $x, y \in A$

$$kd_2(x,y) \leq d_1(x,y) \leq Kd_2(x,y)$$

. Deduce that for any $x, y \in A$

$$\frac{1}{K}d_1(x,y) \leq d_2(x,y) \leq \frac{1}{k}d_1(x,y).$$

[3 Marks]

(iii) Let $A_1 = [0, 2\pi]$ and $A_2 = \{x \in \mathbb{R}^2,\ x_1^2 + x_2^2 = 1\}$ and take as metrics the restrictions of the usual metric on $\mathbb{R}$. Define $f : A_1 \to A_2$ by $f(t) = (\cos t, \sin t)$, show that $f$ is a continuous bijection but $f^{-1}$ is not continuous at the point $(1, 0)$. [4 Marks]

4. (a) Define and give at least two examples of a topological space. [6 Marks]

(b) (i) What is meant by a topological space being Hausdorff? [2 Marks]

(ii) What other name is given to a Hausdorff space? [2 Marks]

(c) With an example, differentiate between the interior and the closure of a subset of a topological space. [6 Marks]

(d) Prove that the subset $A$ of a topological space $X$ is closed if $A = \overline{A}$.[4 Marks]

(e) Prove that given a subset $A$ of a topological space and an open set $O$ contained in $A$ then $O \subset$ interior of $O$. [5 Marks]

5. (a) When is a topological space said to be connected? [2 Marks]

(b) (i) Define an interval $I$ on a real line. [2 Marks]

(ii) Prove that a subset $A$ of a real line that contains at least two distinct points is connected if and only if it is continuous in the interval. [10 Marks]

(c) (i) State and prove the intermediate value theorem a an application of connectedness of topological spaces. [3 Marks]

(ii) Using an example demonstrate the intermediate value theorem as stated in $4c(i)$ above. [2 Marks]

(d) Let $(X, \tau)$ be a topological space, $x, y \in X$.

(i) Define a path from $x$ to $y$. [2 Marks]

(ii) When is the path said to be connected? [2 Marks]

(e) Determine whether the following statements are true.

(i) Any path connected topological space is connected. [1 Mark]

(ii) A connected open set in $\mathbb{R}^n$ is path connected. [1 Mark]

*End*