UGANDA MARTYRS UNIVERSITY

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS

MTF 3201 STOCHASTIC CALCULUS

Year 2015/2016: Third Year BSc FM

Date: 06th May 2016

Time: 9:30 AM - 12:30 PM

INSTRUCTIONS

(i) Attempt ALL questions

(ii) Read through the paper carefully and follow the instructions on each respective question.

(iii) Calculators and mathematical tables may be used.

1. **[24 marks]** State which of the following statements are **TRUE** or **FALSE**.

(a) If X and Y are independent random variables, then E(XY) = E(X) + E(Y)

(b) If $\int_0^T E(X^2(t))dt < \infty$, then $E\left(\int_0^T X(t)dB(t)\right) \neq 0$

(c) A Brownian path (motion), B(t), is a continuous function of t.

(d) The covariance function of the process X(t) is defined by
$Cov(X(t), X(s)) = E(X(t)X(s)) - EX(t)EX(s)$

(e) If X(t) is a differentiable function, then the stochastic integral

$$\int_0^T X(t)dB(t) = X(t)B(t) - X(0)B(0)$$

(f) If X and Y are uncorrelated random variables, then Var(X + Y) = Var(X) + Var(Y)

(g) A Brownian motion is differentiable at some points

(h) If $\int_0^T E(X^2(t))dt < \infty$, then $E\left(\int_0^T X(t)dB(t)\right)^2 = \int_0^T E(X^2(t))dt$

2. ***Choose (a) OR (b)***

(a) (i) **[6 marks]** Given

$$f(t) = \begin{cases} kt(1-t), \ldots\ldots 0 < t < 1 \\ 0, \ldots\ldots\ldots\ldots\ldots\ldots otherwise \end{cases}$$

find the value of k so that this function is a probability density.

(ii) **[6 marks]** For what values of β is the *Itô* integral $\int_0^1 (1-t)^{2\beta - 1/3} dB(t)$ defined?

(iii) **[6 marks]** The *Itô* formula for stochastic differential is given by

$d(f(B(t))) = f'(B(t)dB(t) + \frac{1}{2} f''(B(t))dt$. Hence find $d(e^{2tB(t)})$ for $f(x) = e^{2tx}$

(b) (i) **[6 marks]** Given

$$f(t) = \begin{cases} kt^2(1-t), \ldots\ldots 0 < t < 1 \\ 0, \ldots\ldots\ldots\ldots\ldots otherwise \end{cases}$$

find the value of k so that this function is a probability density.

(ii) **[6 marks]** For what values of β is the *Itô* integral $\int_0^1 (1-t)^{2/3 - 1/5\beta} dB(t)$ defined?

(iii) **[6 marks]** If B(t) is a Brownian motion on [0, T], then for any $t \leq T$,

$f(B(t)) = f(0) + \int_0^t f'(B(s))dB(s) + \frac{1}{2}\int_0^t f''(B(s))ds$. Hence find

ln B(t) for f(x) = ln x

3. ***Choose (a) OR (b)***

(a) **[7 marks]** Take $f(t) = (2-t)^{\alpha/3 - 2}$. For what value of α does the condition that $\int_0^1 Ef^2(t)dt < \infty$ fail?

(b) **[7 marks]** Take $f(t) = (t+3)^{3-\alpha/5}$. For what value of α does the condition that $\int_0^1 Ef^2(t)dt < \infty$ fail?

4. ***Choose (a) OR (b)***

(a) **[18 marks]** If the probability density of a random variable X is given by

$$f(x) = \begin{cases} x+1, \ldots\ldots 1 \leq x < 2 \\ 3-x, \ldots\ldots 2 \leq x < 3 \\ 0, \ldots\ldots\ldots\ldots otherwise \end{cases}$$

Find the probabilities that a random variable having this probability density will take on a value (i) between 1.2 and 1.8, (ii) between 1.5 and 2.5. Also find (iii) E(X), (iv) Var(X)

(b) **[18 marks]** A certain random variable X has probability density function $f(x)=k\left(\frac{x+1}{\sqrt{x}}\right)$ on $[1,4]$. Find the value of k and calculate the mean, the variance and the standard deviation. Also find $P(2<X<3)$.

5. ***Choose (a) OR (b)***

(a) **[15 marks]** Let $f(t)=\begin{cases}2t+5, & -2\le t<-1\\ 1-t, & -1\le t<1\\ t^2-3, & 1\le t<3\\ 0, & otherwise\end{cases}$

Find the value under the condition that $\int_0^1 E(X^2(t))dt<\infty$ is satisfied.

(b) **[15 marks]** Let $f(t)=\begin{cases}\frac{1}{2}(t-2), & -1\le t<1\\ 1-2t, & 1\le t<3\\ \frac{1}{t}, & 3\le t<4\\ 0, & otherwise\end{cases}$

Find the value under the condition that $\int_0^1 E(X^2(t))dt<\infty$ is satisfied.

6. ***Choose (a) OR (b)***

(a) **[18 marks]** Given P(A) = 0.30, P(B) = 0.78 and $P(A\cap B)=0.16$, find

(i) $P(A\cup B)$, (ii) $P(A\cap\overline{B})$, (iii) $P(\overline{A}\cap B)$, (iv) $P(\overline{A}\cap\overline{B})$, (v) $P(\overline{A}\cup\overline{B})$

(vi) Are A and B independent?

(b) **[18 marks]** Suppose that a company has 50 employees who are classified according to their marital status (married M and not married $\overline{M}$) and according to whether they care college graduates (G) or not $(\overline{G})$. Given that 30 employees are married, 15 employees are college graduates and 10 are married employees who are college graduates. And suppose that each employee has a probability of 1/50 of being elected. Find the probability that

(i) a married employee is elected

(ii) an employee who is a college graduate is elected

(iii) an employee who is elected is married, a college graduate or both.

(iv) an employee who is a married college graduate is elected.

(v) a college graduate is elected, given that he/she is married.

(vi) a married employee is elected, give that he/she is a college graduate