UGANDA MARTYRS UNIVERSITY
FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS AND STATISTICS

University Examinations, Semester I 2012/2013

Third Year Examination for the Degree of Bachelor of Science (FM)

MTF 3103 Functional Analysis

Thursday 13 December 2012 Time: 9:00 - 12:00 noon

Instructions

(i) Answer **Five** questions

(ii) Write on both sides of the paper but begin a new question on a fresh page.

1. Let X be a nonempty set.

(a) (i) Define a metric d on X. (3 Marks)

(ii) Define a metric on X by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Show that (X, d) is a metric space. (5 Marks)

(b) Let $f : X \longrightarrow Y$ be a mapping between metric spaces X and Y.

(i) What does it mean to say that f is continuous? (2 Marks)

(ii) Show that f is continuous if and only if for every open set U in Y, $f^{-1}(U)$ is open in X. (4 Marks)

(c) Let (x_n) be a sequence in a metric space X.

(i) Explain what it means to say that (x_n) converges in X. (2 Marks)

(ii) Prove that every convergent sequence in X is a Cauchy sequence. (4 Marks)

2. (a) Let X be a metric space.

(i) What does it mean to say that X is a complete metric space? (2 Marks)

(ii) Prove that if $X = \mathbb{R}^n$ with the usual metric, then X is a complete metric space. (5 Marks)

(b) Let X be a linear space.

(i) Define a norm $\| \cdot \|$ on X. (3 Marks)

(ii) When is X said to be a Banach space? (1 Mark)

(iii) Show that a metric d induced by a norm on X satisfies $d(x+a, y+a) = d(x, y)$ and $d(\alpha x, \alpha y) = |\alpha| d(x, y)$, for all x, y, a in X and any scalar α. (4 Marks)

(c) Prove that if $\dim X < \infty$, then X is complete normed space. (5 Marks)

3. (a) Let X be a normed linear space.

(i) What does it mean to say that the two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on X are equivalent? (2 Marks)

(ii) Show that if two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on X are equivalent, then $\| x_n - x \|_1 \longrightarrow 0$ implies $\| x_n - x \|_2 \longrightarrow 0$. (3 marks)

(iii) Prove that any two norms on a finite dimensional linear space X are equivalent. (5 Marks)

(b) Let $T : X \longrightarrow Y$ be an operator between normed spaces X and Y. Explain the following phrases:

(i) T is a bounded linear operator (2 Marks)

(ii) T is continuous (2 Marks)

(c) Let $T : X \longrightarrow X$ be a linear operator. Prove that if X is finite dimensional,then T is bounded. (6 Marks)

4. Let X be a linear space.

   (a) (i) Define an inner product on X. (4 Marks)

   (ii) What does it mean to say that X is a Hilbert space? (1 Mark)

   (b) (i) State and prove the parallelogram equality for a norm on an inner product space X. (5 Marks)

   (ii) With clear illustrations, show that the space l^p with $p \neq 2$ is not a Hilbert space. (5 Marks)

   (c) State and prove the Cauchy-Schwartz inequality. (5 Marks)

5. (a) (i) Explain what it means to say that a set M in a Hilbert space H is orthonormal. (2 Marks)

   (ii) Show that every orthonormal set is linearly independent. (5 Marks)

   (iii) Show that if $< x, y >=< x, z >$ for all x in a Hilbert space H, then $y = z$. (3 Marks)

   (b) Let $T : H_1 \longrightarrow H_2$ be a bounded linear operator between Hilbert spaces H_1 and H_2.

   (i) Define the Hilbert-adjoint operator T^* of T. (2 Marks)

   (ii) Show that $< T^*y, x >=< y, Tx >$. (2 Marks)

   (c) Let (x_n) be a sequence in a normed space X.

   (i) Define strong and weak convergence of (x_n). (2 Marks)

   (ii) Show that if (x_n) converges strongly to $x \in X$, then it converges weakly to $x \in X$. (4 Marks)

6. Let X be an inner product space.

   (a) (i) Explain what it means to say that $x, y \in X$ are orthogonal. (1 Mark)

   (ii) Show that if x is orthogonal to y in X, then $\| x + y \|^2 = \| x \|^2 + \| y \|^2$. (4 Marks)

   (b) (i) Show that $\| x \|^2 =< x, x >$ defines a norm on X. (5 Marks)

   (ii) Let (x_n) and (y_n) be sequences in X such that $x_n \longrightarrow x, y_n \longrightarrow y$ as $n \longrightarrow \infty$ with $x, y \in X$. Show that $< x_n, y_n > \longrightarrow < x, y >$ as $n \longrightarrow \infty$. (5 Marks)

   (c) Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal set in X with $y = \sum_{i=1}^{\infty} < x, e_i > e_i$. Show that for any $x \in X$ defined by $z = x - y$, then $z \perp y$. (5 Marks)