UGANDA MARTYRS UNIVERSITY

UNIVERSITY EXAMINATIONS

FACULTY OF SCIENCE

DEPARTMENT OF NATURAL SCIENCES

END OF SEMESTER FINAL ASSESSMENT

SEMESTER II 2022/23

THIRD YEAR EXAMINATIONS FOR BACHELOR OF SCIENCE GENERAL

(BSc GEN. 3 & BSc EDUC. 3)

Biomathematics and Modelling

MTH 3205

DATE : Monday 15^{th} May 2023

TIME : 9:30 AM - 12:30 PM

DURATION: 3 Hrs

Instructions

1. *Carefully read through ALL the questions before attempting.*
2. *ANSWER FOUR (4)Questions (All questions carry equal marks).*
3. *No **names** should be written anywhere on the examination booklet.*
4. *Ensure that your **Reg. number** and **Course** are indicated on all pages of your work.*
5. *Ensure that your work is **clear** and **readable**. Untidy work will be penalized.*
6. *Any type of examination Malpractice will lead to automatic disqualification.*

Question 1

In a pure birth process, let λ be the birth rate. Define $\lambda\delta t$ as the probability that an individual gives birth to an offspring in a time interval $(t, t+\delta t)$. Assume that we have the population $N(t)$ at time t and let $P_n(t) = P(N(t) = n)$; for $P_n(0) = 0$ when $n \neq N_0$ and $P_n(0) = 1$ when $n = N_0$

(a) Show that the probability function satisfies the equation

$$P_n'(t) = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t).$$

[10 Marks]

(b) Using $G(x,t) = \sum_{n=0}^{\infty} x^n P_n(t)$ show that $G(x,t)$ satisfies the PDE

$$\frac{\partial G}{\partial t} = \lambda x(x-1)\frac{\partial G}{\partial x}$$

[5 Marks]

(c) Subject to $G(x,0) = x^{N_0}$ obtain the expression for $G(x,t)$.
Hence prove that $E[N(t)] = N_0 e^{\lambda t}$. [10 Marks]

Question 2

The interaction of two types of animals competing for the same ecological resources in a game park is described as

$$\frac{dX}{dt} = X(3 - X - 2Y)$$

$$\frac{dY}{dt} = Y(2 - X - Y)$$

where X and Y are densities of the animals measured at appropriate time.

(a) What kind of interaction is represented by the system? [4 Marks]

(b) Find and classify all the critical points of the system
(using the Jacobian Method). [11 Marks]

(c) Draw a phase diagram to illustrate the behaviour of the system and indicate all the critical points. [10 Marks]

Question 3

(a) In a single species population size $N(t)$ at any time t. If r is the intrinsic rate of growth rate of the population, ϕ is the intraspecific competition rate and γ is the immigration rate of the ecosystem that contains this population. Formulate but do not solve a suitable ecological model that can be used to explain the dynamics of the population of the species in this ecosystem. [6 **Marks**]

(b) A simple model for logistically changing population of fish in a pond undergoing a constant restocking rate is given by

$$\frac{dP}{dt} = rP(1 - \frac{P}{K}) + Q, \qquad P(0) = P_0$$

where $P(t)$ is the fish stock at any time t, $r > 0$, $K > 0$, $P_0 > 0$ and $Q > 0$ are constants.

(i) Describe all parameters and variables in the model. [5 **Marks**]

(ii) Taking $r = 1$ and $K = 1$, find the equilibrium levels of the system and examine how this equilibrium changes with Q. [7 **Marks**]

(iii) Plot the function $f(P, Q)$ for $Q = 0$ and explain the solution behaviour over time as the population approaches the carrying capacity of the system. [7 **Marks**]

Question 4

The interaction between two types of cells is given by the following system of equations

$$\frac{dN}{dt} = N(a + bN + cM)$$

$$\frac{dM}{dt} = M(d + eN + dM)$$

where a,b,c,d,e and f are positive parameters.

(a) State the signs on the parameters if the interaction is

(i) typical symbiosis with logistic growth, [5 **Marks**]

(ii) obligatory competition without intraspecific competition, [5 **Marks**]

(ii) facultative association. [5 Marks]

(b) In case of (iii) above obtain the co-existence equilibrium point and the conditions necessary for it to hold. [10 Marks]

Question 5

(a) State the stages involved in mathematical modelling. [5 Marks]

(b) Define what is meant by the following ecological modelling terms

(i) Carrying capacity K, [2 Marks]

(ii) Intrinsic growth rate r, [2 Marks]

(iii) Mutualism, [2 Marks]

(iv) Parasitism. [2 Marks]

(c) In an SIR model,the sum of the susceptibles $S(t)$, the infectives $I(t)$ and removals $R(t)$ is assumed to be constant such that $S+I+R=1$. The differential equations of the system are given by

$$\begin{aligned}\frac{dS}{dt} &= -\alpha SI\\ \frac{dI}{dt} &= \alpha SI-\beta I\\ \frac{dR}{dt} &= \beta I\end{aligned}$$

where α and β are positive constants.

(i) Explain clearly all constants in the model. [4 Marks]

(ii) Find the threshold density of the susceptibles. [3 Marks]

(ii) Considering the above system where the recruitment into the susceptible group is ΛS, write down the set of differential equations describing the new system. Hence find the steady state $(S^{\star}, I^{\star})$. [5 Marks]

End