UGANDA MARTYRS UNIVERSITY

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS AND STATISTICS

University Examinations, Semester I 2012/2013

Second Year Examination for the Degree of Bachelor of Science (FM, GEN)

MTC 2102 Linear Algebra

Wednesday, 19 December 2012 Time: 2:00 - 5:00 pm

Instructions

(i) Answer **Five** questions

(ii) Write on both sides of the paper but begin a new question on a fresh page.

1. Consider the following system of linear equations

$$\begin{aligned} x + y + 2z &= -1 \\ x - 2y + z &= -5 \\ 3x + y + z &= 3 \end{aligned}$$

(a) (i) Write down the augmented matrix A for the above system. **(1 mark)**

(ii) By elementary row operations reduce A to echelon form. **(6 marks)**

(iii) Use your answer in a(ii) above to solve the above ~~system~~ linear system. **(2 marks)**

(b) Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}$. Find the nonzero vector $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ such that $\mathbf{Au} = 3\mathbf{u}$. **(4 marks)**

(c) Find the basis and dimension for the general solution of the homogeneous system of linear equations

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 2x_4 - 4x_5 &= 0 \\ 2x_1 + 4x_2 - 5x_3 + x_4 - 6x_5 &= 0 \\ 5x_1 + 10x_2 - 13x_3 + 4x_4 - 16x_5 &= 0. \end{aligned}$$

(7 marks)

2. (a) Let $\mathbf{A}$ and $\mathbf{B}$ be n-square matrices. Prove that

(i) $\mathrm{tr}(k\mathbf{A}) = k\ \mathrm{tr}(\mathbf{A})$ for any scalar k. **(3 marks)**

(ii) $\mathrm{tr}(\mathbf{A}^T) = \mathrm{tr}(\mathbf{A})$ **(2 marks)**

(b) Prove that if $\mathbf{A}$ and $\mathbf{B}$ are n-square invertible matrices, then $\mathbf{AB}$ is also invertible. **(3 marks)**

(c) (i) Let $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 0 & k \end{bmatrix}$. Find the values of k for which $\mathbf{A}$ is a zero of the polynomial $f(x) = x^2 - 7x + 10$. **(4 marks)**

(ii) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. By block matrix multiplication compute $\mathbf{AB}$ **(5 marks)**

(iii) Let $\mathbf{M} = \mathrm{diag}(\mathbf{A}, \mathbf{B}, \mathbf{C})$ where $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\mathbf{B} = [5]$ and $\mathbf{C} = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$. Find $\mathbf{M}^2$. **(3 marks)**

3. (a) Let **A** square matrix. Prove that

   (i) $|A^{-1}| = |A|^{-1}$ **(2 marks)**

   (ii) If any two rows or columns of A are equal, then $|A| = 0$. **(3 marks)**

   (b) Let $\mathbf{A} = \begin{bmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & 4 & 6 \end{bmatrix}$. By cofactor expansion, find the determinant of matrix **A**. **(7 marks)**

   (d) (i) State Cramer's rule. **(2 marks)**

   (ii) Use Cramer's rule to solve the linear system

$$\begin{aligned} -2x + 3y - z &= 1 \\ x + 2y - z &= 4 \\ -2x - y + z &= -3 \end{aligned}$$

   (6 marks)

4. (a) (i) Define an eigenvalue and eigenvector of an $n \times n$ matrix **A**. **(2 marks)**

   (ii) Let $\mathbf{A} = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Verify that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of **A** and if so, find the corresponding eigenvalues. **(4 marks)**

   (b) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$

   (i) Find the characteristic polynomial of matrix **A**. **(3 marks)**

   (ii) Find all the eigenvalues and associated eigenvectors of matrix **A**. **(7 marks)**

   (iii) Check whether A is diagonalizable or not. **(4 marks)**

5. (a) Let V be a non-empty set and U be a subset of V. Explain what it means to say that

   (i) V is a vector space over a field K. **(4 marks)**

   (ii) U is a subspace of V. **(2 marks)**

   (b) Let V be a vector space over $\mathbb{R}$. Prove that for any $\mathbf{u}$ in V

   (i) $0\mathbf{u} = \mathbf{0}$ **(2 marks)**

   (ii) $(-1)\mathbf{u} = -\mathbf{u}$ **(2 marks)**

   (c) Let $S = \{u_1, u_2, ..., u_n\}$ be a set of vectors in a vector space V.

   (i) Explain what it means to say that
   - S is linearly independent. **(1 mark)**
   - S spans V. **(1 mark)**
   - S is a basis for V. **(1 mark)**

   (ii) Show that the set $S = \{t^2 + 1, t - 1, 2t + 2\}$ is a basis for the vector space P_2, the space of all polynomials of degree two. **(4 marks)**

   (d) Show that if $S = \{u_1, u_2, ..., u_n\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be written uniquely as a linear combination of the vectors in S. **(3 marks)**

6. (a) Let V and W be vector spaces and let $T : V \longrightarrow W$ be a mapping.

   (i) Explain what it means to say that T is a linear transformation. (**2 marks**)

   (ii) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a function defined by

$$T(x, y) = (2x, 3x + y, x - 2y).$$

   Show that T is a linear transformation. (**4 marks**)

(b) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a linear transformation defined by

$$T(x_1, x_2, x_3) = (x_1 + 2x_2, 3x_2 - 2x_3)$$

and let $\{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$, where $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. Find the standard matrix representing T. (**4 marks**)

(c) Let $T : V \longrightarrow W$ be a linear transformation between vectors spaces V and W.

   (i) Define the kernel and the range of T and show that the range of T is a subspace of W. (**5 marks**)

   (ii) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

   Find the basis and dimension of kernel of T. (**5 marks**)