UGANDA MARTYRS UNIVERSITY

FACULTY OF SCIENCE

University Examinations 2014/2015

Second Year Supplementary Examination for the Degree of Bachelor of Science (FM and GENERAL)

MTC 2103 Real Analysis

Monday, 3 August 2015 2:00 - 5:00 Pm

Instructions

(i) Answer **five** questions

(ii) Write on both sides of the paper but begin a new question on a fresh page.

1. Let $X = \mathbb{R}$ and let d be a real valued function defined on $X \times X$.

   (a) (i) What does it mean to say that d is a metric on X? (4 Marks)

   (ii) Show that the function $d : X \times X \longrightarrow \mathbb{R}$ defined by $d(x, y) =| x - y |$ defines a metric on X. (5 Marks)

   (b) Let x be in $E \subset X$. Define the following terms:

   (i) a set U is a closed subset of E, (1 Mark)

   (ii) x is an accumulation point of E, (1 Mark)

   (iii) the set E is open, (1 Mark)

   (iv) x is an isolated point of E. (1 Mark)

   (c) (i) Define a neighborhood of a point $x \in E \subset X$. (2 Marks)

   (ii) Let $x \in X$. Prove that every neighborhood of x contains infinitely many points of X. (5 Marks)

2. (a) (i) Define the following terms:
   - greatest lower bound of a set (g.l.b) (2 Marks)
   - least upper bound of a set (l.u.b) (2 Marks)
   - a bounded set (1 Mark)

   (ii) Let $S = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$. Find the g.l.b and l.u.b of S. (2 Marks)

   (b) State the Bolzano-Weierstrass theorem. (2 Marks)

   (c) (i) With an example, explain what you understand by a sequence. (2 Marks)

   (ii) When is a sequence (a_n) of real numbers said to converge? (1 Mark)

   (iii) Use the definition of a limit to prove that $\lim\limits_{n \longrightarrow \infty} \frac{3n+1}{n+2} = 3$ (4 Marks)

   (d) Determine the limits in each of the following cases;

   (i) $\lim\limits_{n \longrightarrow \infty} \frac{n^3 - n^2 \cos n + 2}{4n^3 + n^2 - 4 \sin n}$ (2 Marks)

   (ii) $\lim\limits_{n \longrightarrow \infty} \frac{n^5 + 7n^3 + 5n^2 + 8}{5n^5 + 3n^4 + 27}$ (2 Marks)

3. Let (x_n) be a sequence of real numbers.

   (a) (i) Explain what it means to say that (x_n) absolutely converges to $x \in \mathbb{R}$. (2 Marks)

   (ii) Show that if (x_n) converges, then its limit is unique. (4 marks)

   (b) Show that if $\lim\limits_{n \longrightarrow \infty} a_n = a$ and $\lim\limits_{n \longrightarrow \infty} b_n = b$, then $\lim\limits_{n \longrightarrow \infty} (a_n + b_n) = a + b$.

   (4 Marks)

   (c) (i) Explain what it means to say that (x_n) is a Cauchy sequence. Suppose that a sequence (x_n) converges to a point x, prove that (x_n) is Cauchy. (5 Marks)

(d) If $b > 0$ and $B > 0$ and $\frac{a}{b} < \frac{A}{B}$, prove that $aB < bA$. Hence deduce that $\frac{a}{b} < \frac{a+A}{b+B} < \frac{A}{B}$. (5 Marks)

4. (a) Differentiate from first principles $f(x) = 2x^2 + 1$. (4 Marks)

(b) Let $a_n = \frac{1}{n(n+1)(n+2)}$; $n \in \mathbb{Z}^+$ and let $S_n = a_1 + a_2 + ... + a_n$. Express a_n in partial fractions and hence or otherwise show that $S_n = \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$.

Deduce that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ converges and find its sum. (6 Marks)

(c) Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (7 Marks)

(d) Let $f(x) = x^2 - x$ and $P = \{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\}$. Find the upper and lower Riemann sums of f with respect to the partition P. Hence or otherwise, calculate the Riemann sum of f. (3 Marks)

5. (a) (i) Let E be a subset of $\mathbb{R}$ and let (f_n) be a sequence of functions on E. What does it mean to say that
   - (f_n) converges uniformly,
   - (f_n) is pointwise convergent. (4 Marks)

(ii) State and prove Cauchy's criterion for uniform convergence. (7 Marks)

(b) State the Wierstrass-M test for uniform convergence. (2 Marks)

(c) (i) Use l'Hopital's rule to evaluate $\lim_{x \to 0} \frac{e^{x^2}-1}{\sin x^2}$ (5 Marks)

(ii) Explain why l'Hopital's rule fails in evaluation of $\lim_{x \to 0} \frac{\sin x}{\cosh x}$. (2 Marks)

6. Let $\sum_{n=1}^{\infty} a_n$ be an infinite series of real numbers

(a) (i) Define the following terms as applied to infinite series:
   - partial sum,
   - convergent sequence. (2 Marks)

(ii) Let $a_n = \frac{1}{n(n+1)}$; $n \in \mathbb{Z}^+$. Show that the series $\sum_{n=1}^{\infty} a_n$ converges and find its sum. (4 Marks)

(b) (i) Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$. (3 Marks)

(ii) Show that $\sum_{n=1}^{\infty} \frac{3n^2+2n+1}{7n^2-5n+3}$ diverges. (3 Marks)

(d) Discuss the convergence of the following series.

(i) $\sum_{n=1}^{\infty}(\frac{3n+1}{5n+1})^4$ (2 Marks)

(ii) $\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$ (3 Marks)

(iii) $\sum_{n=1}^{\infty}(-1)^{n+1}(\frac{1}{n})$ (3 Marks)

END