UGANDA MARTYRS UNIVERSITY

UNIVERSITY EXAMINATIONS

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS

END OF SEMESTER 1, 2021/22 FINAL ASSESSMENT

BSc. Gen III & BSc. EDUC III

Functional Analysis

MTC 3102

DATE : 19^{th} January 2022

TIME : 2:00 AM - 5:00 PM

DURATION: 3 Hrs

Instructions

1. *Carefully read through ALL the questions before attempting.*
2. *ANSWER FOUR (4) Questions (All questions carry equal marks).*
3. *Ensure that your* ***Reg. number Name*** *and* ***Course*** *are indicated on all pages of your work.*
4. *Ensure that your work is* ***clear*** *and* ***readable****. Untidy work will be penalized.*
5. *Any type of examination Malpractice will lead to automatic disqualification.*

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1. (a) Suppose X is a non-empty set with $x, y \in X$.

   (i) Let $d : X \times X \to \mathbb{R}$. State the conditions necessary for (X, d) to be a *metric space.* [5 Marks]

   (ii) Let $X = \mathbb{R}^n$ and define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}},$$

   for $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$. Prove that (X, d) is a metric space. [10 Marks]

   (iii) Give any two other examples of metric spaces. [4 Marks]

   (iv) Find the distance $d(x, y) - d(p, q)$ between vectors $x = (6, 8, 9), y = (4, 7, 3), p = (1, 1, 1)$ and $q = (1, 7, 8)$. [6 Marks]

2. (a) Give the difference between each of the following properties of metric spaces.

   (i) *Open ball* and *closed ball.* [3 Marks]

   (ii) *Accumulation point* and *isolated point.* [4 Marks]

   (iii) *Open set* and *closed set.* [3 Marks]

   (b) Prove that every open ball $\mathcal{B}(a, r)$ is an open set. [5 Marks]

   (c) Let $u, v \in V$ be vectors in an inner product space V. Prove the following inequalities.

   (i) $|\langle u|u \rangle| \leq ||u|| \; ||v||$. [6 Marks]

   (ii) $||u + v|| \leq ||u|| \; ||v||$. [4 Marks]

3. (a) (i) Let $\mathcal{N} = (X, || \cdot ||)$, where X is a vector space over a field $\mathbb{F}$ and $|| \cdot || : X \to \mathbb{R}$ is the norm. State the conditions that must be satisfied for $\mathcal{N}$ to be a *normed space.* [5 Marks]

   (ii) Let $X = l^p = \{x = (x_n) : \sum_{i=1}^{\infty} |x_n|^p < \infty\}$. Define $|| \cdot || : X \to \mathbb{R}$ by $||x||_p = ||(x_n)||_p = \left(\sum_{i=1}^{n} |x_n|^p \right)^{\frac{1}{p}}$. Prove that $(X; || \cdot ||_p)$ is a normed space. [12 Marks]

(b) (i) When do we say that two norms $||\cdot||_1$ and $||\cdot||_2$ are equivalent? [2 Marks]

(ii) Let $X = l^2$ and for each $x = (x_1, x_2, \ldots, x_n) \in X$, define $T : X \to X$ by $T(x_1, x_2, \ldots,) = \left(0, \frac{x_1}{1}, \frac{x_2}{2}, \ldots, \frac{x_n}{n}\right)$. Prove that T is a linear operator on X. [6 Marks]

4. (a) (i) Define a **Hilbert space**. [1 Marks]

(ii) Give any two examples of Hilbert spaces [4 Marks]

(b) (i) Inner product spaces satisfy the following identity (Parallelogram identity); $2||x||^2 + 2||y||^2 = ||x+y||^2 + ||x-y||^2$. Prove that this identity is true. [5 Marks]

(ii) Using the parallelogram identity, explain why a continous function $f(t) = t^2$ on $[0,1]$ whose norm is given by $||x|| = max_{t\in[a,b]}|x(t)|$ is not a Hilbert space [4 Marks]

(c) Show that the vectors $\{e_1, e_2, e_3\} = \{(1,0,0), (0,1,0), (0,0,0)\}$ are *orthogonal* to each other. [5 Marks]

(d) Consider the orthogonal vectors $S = \{u_1, u_2, u_3\}$ where $u_1 = (1,2,1), u_2 = (2,1,-4), u_3 = (3,-2,1)$. Write the vector $v = (7,1,9)$ as a linear combination of the set of vectors of S. [5 Marks]

5. (a) (i) What is an orthonormal set?. [2 Marks]

(ii) Prove that an orthonormal sequence $\{e_1, e_2, \cdots, e_n\}$ is linearly independent. [7 Marks]

(b) The space of all bounded linear functionals on a vector space X is called the *dual space* of X. Consider the vector space $\mathbb{C}[a,b]$ and define f by $f(x) = \int_a^b x(t)dt;\ x(t) \in \mathbb{C}[a,b]$. Show that f is a dual space. [6 Marks]

(c) Consider the sequence $x_1 = t^2, x_2 = t, x_3 = 1$. A set of vectors from the space of continuous integrable functions on the interval $[-1, 1]$ with respect to the inner product $\langle x, y \rangle = \int_{-1}^{1} x(t)y(t)dt$. Compute the orthonormal vectors corresponding to the vectors above. **[6 Marks]**

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