UGANDA MARTYRS UNIVERSITY

FACULTY OF SCIENCE

DEPARTMENT OF NATURAL SCIENCES

BSc.III FINAL ASSESSMENT

REAL ANALYSIS I

Date: 20/01/2022
Time: 9:30 AM – 3:30 PM

INSTRUCTIONS

1. Carefully read through **ALL** the questions before attempting
2. Question 1 is compulsory and Attempt questions 2 and 3 following the given instructions.
3. Ensure that your **Reg number** is indicated on all pages of the examination answer booklet
4. Ensure your work is **clear** and **readable**. Untidy work shall be penalized
5. Any type of examination malpractice will lead to automatic disqualification
6. Calculators and mathematical tables may be used

1. **[28 marks]** Indicate which f the following statements are **TRUE (T)** or **FALSE (F)**
   (a) A function is said to be continuous on a set A if it continuous at most one point in A.
   (b) A monotone function f with $D(f) \supset (a,b)$ can have at most countably many discontinuities in (a, b).
   (c) If A is a closed bounded set of real numbers, then A is compact.
   (d) A sequence converges if and only if it is a Cauchy sequence.
   (e) Let A be bounded set of real numbers and suppose c is a real number. If c < 0, then g.l.b.(cA) = c(g.l.b.A).
   (f) A sequence of real numbers can converge to at most one number.
   (g) A sequence $\{a_n\}$ converges if and only if it is bounded and has at least one subsequential limit point.
   (h) Suppose f is continuous on [a, b] and differentiable on (a, b). If $f'(x) = 0$ on (a, b), then f is constant on [a, b].
   (i) A sequence $\{a_n\}$ converges to L if some subsequences of $\{a_n\}$ converge to L.
   (j) A bounded function f defined on [a, b] is Riemann integrable on (a, b) if and only if, given $\varepsilon > 0$, there is a partition P of [a, b] such that $U(f,P) < L(f,P) + \varepsilon$.

(k) If f is Riemann integrable on [a, b], then |f| is Riemann integrable on [a, b] and $\left|\int_a^b f\right| \geq \int_a^b |f|$.

(l) Every bounded infinite set of real numbers has at least one limit point.

(m) If B and C are sets, then $(B \cup C)^c = B^c \cup C^c$

(n) Continuity of f at x = c implies differentiability of f at x = c.

2. Attempt only **SIX** of the following:

(a) **[6 marks]** Given that $\{a_n\} = \left\{\frac{n^2+1}{n^2}\right\}$.

(i) Find the number to which the sequence converges.

(ii) If L is the number to which the sequence converges, find how large n must be so that $|a_n - L| < 0.01$

(iii) For $\varepsilon > 0$, find $N(\varepsilon)$ such that $|a_n - L| < \varepsilon$ if $n > N(\varepsilon)$.

(b) **[6 marks]** Find the value of c where $\int_2^6 f(x).dx = f(c)(6-2)$ if $f(x) = x^2 + x$

(c) **[6 marks]** Let A and B be sets of real numbers. Define a set by $A + B = \{a + b : a \in A, b \in B\}$. Let {-4, -1, 2, 4, 6} and B = {-2, 5, 8, 11, 14}. Find A + B. Show that if A and B are bounded sets, then l.u.b (A + B) = l.u.b.A + l.u.b.B and g.l.b. (A + B) = g.l.b.A + g.l.b.B

(d) **[6 marks]** Let f and g be defined by $f(x) = \sqrt{2x-1}$ and $g(x) = x^3 + 4$. Find (i) $(f \circ g)(2)$, (ii) $(g \circ f)(5)$, (iii) $f^{-1}(5)$, (v) $g^{-1}(12)$.

(e) **[6 marks]**Use l'Hopital's rule and any formulas from elementary calculus to evaluate:

(i) $\lim_{x \to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \tan x$ (ii) $\lim_{x \to 0} \left(e^{2x} - x\right)^{1/x}$

(f) **[6 marks]** Let $f(x) = x^2 - x$ and let $P = \{1, 3/2, 2, 5/2, 3, 7/2, 4\}$

(i) Find the upper and lower Riemann sums of f with respect to the partition P.

(ii) From part (i), calculate the Riemann sum of f.

(g) **[6 marks]** Suppose $f(x) = 4x^3 - 7x^2$ and $g(x) = x^4 - 5$ are functions that are continuous on [0, 2] and differentiable on (0, 2). Find $c \in (0, 2)$ where $[f(2) - f(0)]g'(c) = [g(2) - g(0)]f'(c)$.

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(h) **[6 marks]** Find a number c in the interval (-1, 3) that satisfies the Mean-Value Theorem for the function $f(x) = x^3 - x^2 + 3$.

3. Attempt only **SIX** of the following:

   (a) **[6 marks]** Show that Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$

   (b) **[6 marks]** Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers such that $\{a_n\} \to a$ and $\{b_n\} \to b$. Prove that $\{a_n + b_n\} \to a + b$

   (c) **[6 marks]** Let f and g be functions defined on (a, b) that are differentiable at $c \in (a,b)$. Show that $(fg)'(c) = f(c)g'(c) + g(c)f'(c)$.

   (d) **[6 marks]** Use the definition of limit to prove that $\lim\limits_{x \to 2} 4x^2 + 3x - 1 = 21$

   (e) **[6 marks]** Let $f(x) = \dfrac{3x^2 - 27}{x + 3}$.

   (i) Find L such that $\lim\limits_{x \to -3} f(x) = L$

   (ii) Use the $(\varepsilon - \delta)$ definition of limit to prove that $\lim\limits_{x \to -3} f(x) = L$.

   (f) **[6 marks]** Let $f : X \to Y$ with $A, B \subset Y$. Show that $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B)$.

   (g) **[6 marks]** Show that for any positive integer n, $\sum\limits_{j=1}^{n} j^2 = \dfrac{n(n+1)(2n+1)}{6}$

   (h) **[6 marks]** Suppose $f : A \to B$ and $g : B \to C$ are functions, show that if both f and g are bijection, then $g \circ f$ is bijection. [Note: bijection means both 1-1 and onto]