# UGANDA MARTYRS UNIVERSITY EXAMINATIONS

## FACULTY OF SCIENCE

## FINAL ASSESSMENT

STAT 2201 : ADVANCED PROBABILITY

Monday $28^{th}$ April, 2008 2:00 PM - 5:00 PM

INSTRUCTIONS:

*(i) Answer any* **FOUR** *questions.*

*(ii) Read all the instructions on the Answer book.*

### Question 1

(a) (i) Define what is meant by two events $A$ and $B$ being independent. (1 Marks)

(ii) If events $A$ and $B$ are independent, show that events $A^c$ and $B^c$ are also independent. (3 Marks)

(iii) Three men Smith, Brown and Jones independently participate in a shooting competition. Their respective chances of hitting the target are $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{2}$ respectively. Find the probability that the target will be hit by exactly two of the candidates. (3 Marks)

(b) (i) State Baye's theorem. (2 Marks)

(ii) In a bolt factory, 30%, 50% and 20% of production is manufactured by machines $A$, $B$ and $C$ respectively. If 4%, 5% and 3% of the output of these machines are defective, what is the probability that a randomly selected bolt that is found to be defective is manufactured by machine $C$. (3 Marks)

(c) For a Binomial random variable $X$ with a probability mass function given by

$$f(x) = \binom{n}{x} p^x q^{n-x}, \ x = 0, 1, 2, \ldots, n$$

(i) Derive the expressions for $E(X)$ and $E[X(X-1)]$ (2+2 Marks)

(ii) Using your results in $a(i)$ above, show that $Var(X) = npq$ (3 Marks)

(d) A random variable $X$ is said to be a Poisson distributed if its probability mass function is given by $f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \ldots$ Show that $E(X) = \lambda$ (3 Marks)

(e) A random variable $X$ is said to be uniformly distributed over an interval $(a, b)$ if its probability density function is given by $f(x) = \frac{1}{b-a}, \quad a < x < b$ Find the expected value of $X$, $E(X)$. (3 Marks)

**Question** 2

(a) (i) Define a probability generating function of a random variable $X$. 2Marks

(ii) State any two properties of a probability generating function. 2Marks

(b) Given the moment generating function of a random variable $X$ is $M_X(t) = \frac{5}{(5-t)}$.

State the density function of $X$. Deduce $E[X]$. 4Marks

(c) Let $Y$ have a *pdf*

$$f(y) = \begin{cases} 2y, & 0 \leq y \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find $M_Y(t)$ the moment-generating function of $Y$. 5Marks

(d) (i) Let $X$ be a gamma random variable with parameters $\alpha$ and $\beta$, show that

$$E(X^k) = \frac{(\alpha + k - 1)(\alpha + k - 2) \ldots \alpha}{\beta^k}$$

5 Marks

(ii) Find an expression for $E(X^2)$ if the moment generating function for $X$ is given by

$$M_X(t) = (1 - p_1 - p_2) + p_1 e^t + p_2 e^{2t}$$

3Marks

(e) The probability density function of a random variable $X$ is

$$f(x) = kx^{14}(1-x)^{13}, \qquad 0 < x < 1$$

Find the constant $k$. 4 Marks

## **Question** 3

(a) Assume the probabilistic behavior of a pair of discrete random variables $X$ and $Y$ is described by the joint probability density function

$$f(x,y) = \frac{xy^2}{39}$$

defined over the four points $(1,2),(1,3),(2,2),(2,3)$

(i) Determine the probability mass function of $X+Y$ and use it to compute $P(X+Y \leq 4)$. 4Marks

(ii) Find the expectation of $XY$. 3Marks

(iii) Find the conditional probability that $X = 1$ given that $Y = 2$. 2Marks

(iv) Calculate the correlation coefficient of $X$ and $Y$. 5Marks

(b) If $Y$ is the number of heads obtained in two tosses of a balanced coin, find the probability distribution of $X = Y^2 + 4$. 3Marks

(c) A supermarket has two express lines. Let $X$ and $Y$ denote the number of customers in the first and second lines respectively, at any given time. During non-rush hours, the joint *pdf* of $X$ and $Y$ is summarized by the following table:

| | | X | | | |
|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 |
| Y | 0 | 0.1 | 0.2 | 0 | 0 |
| | 1 | 0.2 | 0.25 | 0.05 | 0 |
| | 2 | 0 | 0.05 | 0.05 | 0.025 |
| | 3 | 0 | 0 | 0.025 | 0.05 |

(i) Find $P(|X-Y| = 1)$, the probability that $X$ and $Y$ differ by exactly one. 3Marks

(ii) Compute $E(X/Y = 2)$. 5Marks

## **Question** 4

(a) Let $X$ and $Y$ be continuous random variables with joint pdf

$$f(x,y) = \begin{cases} \frac{1}{8}(6-x-y), & 0<x<2,\ 2<y<4 \\ 0, & \text{elsewhere} \end{cases}$$

Compute

(i) $f_X(x)$ the marginal probability of $X$. 3Marks

(ii) $f_{Y/x}(y)$. 3Marks

(iii) $P\,(2<Y<3/X=1)$. 3Marks

(b) Show whether the two random variables $X$ and $Y$ with the following joint probability density functions are independent or not. 5Marks

$$f(x,y) = \begin{cases} 8xy; & 0 \le x \le y \le 1 \\ 0; & \text{otherwise} \end{cases}$$

(c) Suppose that

$$f_{Y/x}(y) = \frac{2y+4x}{1+4x} \quad \text{and} \quad f_X(x) = \frac{1}{3}(1+4x)$$

For $0<x<1$ and $0<y<1$. Find the marginal pdf of $Y$. 5Marks

(d) (i) Let $X$ be a random variable with the set of possible values $\{-1,0,1\}$ and probability mass function $P(-1) = P(0) = P(1) = \frac{1}{3}$. Letting $Y = X^2$, find $cov(X,Y)$ 3Marks

(ii) A student in mathematics argues that the concept of $cov(X,Y) = 0$ and $X, Y$ being independent events are really the same, and that if $cov(X,Y) = 0$, they must be independent. Do you agree with this statement? 3Marks

## **Question** 5

(a) Let $X \sim N(0,1)$ and $Y \sim N(0,1)$ be independent random variables. Find the joint probability density function of $R = \sqrt{X^2+Y^2}$ and $\theta = \arctan\left(\frac{Y}{X}\right)$. Show that $R$ and $\theta$ are independent. 10Marks

(b) Let $X \sim \text{Gamma}(\alpha_1, \beta)$ independent of $Y \sim \text{Gamma}(\alpha_2, \beta)$. Define

$$U = \frac{X}{X+Y} \quad \text{and} \quad V = X+Y$$

Find, by the Jacobian method, the joint probability density function of $\Phi(u,v)$. 5Marks

(c) Suppose that the p.d.f of a random variable $X$ is as follows:

$$f(x) = \begin{cases} \frac{1}{2}x, & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Determine the distribution and probability density functions of $Z = 4 - X^3$ 5Marks

(d) If the probability density function of $X$ is given by

$$f(x) = \frac{kx^3}{(1+2x)^6} \quad , \qquad x > 0$$

Show that the probability density of a random variable $Y = \frac{2X}{1+2X}$ is

$$f(y) = \frac{k}{16}y^3(1-y) \quad ; \quad 0 < y < 1$$

5Marks

## **Question** 6

(a) Suppose that $\{X_1, X_2 \ldots, X_8\}$ are independent and identically distributed random variables, from an exponential density function with mean $\frac{1}{4}$.

Let $Y_8 = \max\{X_1, \ldots, X_8\}$ and $Y_1 = \min\{X_1, \ldots, X_8\}$

- (i) Find the probability density function of $Y_8$. 5Marks
- (ii) Find the joint probability density function of $Y_1$ and $Y_8$. 5Marks

(b) (i) Let $\{X_1, \ldots, X_n\}$ be a random sample from a $N(\mu, \sigma^2)$. Show that the sample mean $\bar{X}$ is also normally distributed. State its mean and variance. 5Marks

(ii) Show that the sum of identically independent Geometric random variables is a negative binomial. 5Marks

(c) Let $X$ and $Y$ be independent random variables with Poisson distributions of parameters $\lambda_1$ and $\lambda_2$ respectively. Use the moment generating function technique to find the distribution of $Z = X + Y$. 5Marks

**END**