positive integer are special rational functions. They are analytic every where except at z_0.

2.13. CAUCHY—RIEMANN (C–R) EQUATIONS [JNTU 2003, 2008]

Having defined analytical function, we will prove a very important result to test the analyticity of a complex function.

Theorem : The necessary and sufficient condition for the derivative of the function $f(z) = w = u(x, y) + iv(x, y)$ to exist for all values of z in domain R are

(i) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ are continuous functions of x and y in R.

(ii) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ **[JNTU 2009S, JNTU (A) Nov. 2009 (Set No. 1)]**

The above relations are known as **Cauchy–Riemann equations.**

(Or) Derive the necessary and sufficient condition for $f(z)$ to be analytic in cartesian co-ordinates.

Proof : Necessity.

[JNTU 1998, 1996, 2002S, Nov. 2006, 2007S, Nov. 2008S, Nov. 2008 (Set No. 4)]

If $f(z) = u + iv$ is analytic in domain R, then u and v satisfy the equations

$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ provided the partial derivatives exist.

In order for $f(z)$ to be analytic, the limit

$$\underset{\Delta z \to 0}{\mathrm{Lt}} \frac{f(z+\Delta z) - f(z)}{\Delta z} = f'(z)$$

$$= \underset{\substack{\Delta x \to 0 \\ \Delta y \to 0}}{\mathrm{Lt}} \frac{\{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)\} - \{u(x, y) + iv(x, y)\}}{\Delta x + i\Delta y}$$

must be existing ...(1)

Hence the theorem.

PROPERTIES OF ANALYTIC FUNCTIONS

1. If $f(z)$ and $g(z)$ are analytic functions, then $f \pm g$, $f.g$ and $\frac{f}{g}$ are also analytic functions, provided $g(z) \neq 0$.
2. Analytic function of an analytic function is analytic.
3. An entire function of an entire function is entire.
4. Derivative of an analytic function is itself analytic.

2.14. HARMONIC FUNCTIONS — LAPLACE EQUATION

Theorem : If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then u and v satisfy Laplace equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$, respectively in D, and have continuous second order partial derivatives in D.

2.15. HARMONIC FUNCTIONS

Solutions of Laplace equations having continuous second order partial derivatives are called **Harmonic functions.** Their theory is called **Potential theory.** Hence, the real and imaginary parts of an analytic function are harmonic functions.

Thus the functions satisfying the Laplace equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

are known as **Harmonic functions.**

2.16. CONJUGATE HARMONIC FUNCTION

If two harmonic functions u and v satisfy the Cauchy–Riemann equations in a domain D and they are the real and imaginary parts of an analytic function f in D then v is said to be a conjugate Harmonic function of u in D. Two harmonic functions, u and v which are such that $u + iv$ is an analytic function are called conjugate harmonic functions. In other words, if $f(z) = u + iv$ is analytic and if u and v satisfy Laplace's equation, then u and v are called conjugate harmonic functions.

2.17. POLAR FORM OF CAUCHY–RIEMANN EQUATIONS

[JNTU 1998 Sept., 2001S, 2003, 2005, JNTU (H) Nov. 2009 (Set No. 1)]

If $f(z)=f\left(re^{i\theta}\right)=u(r,\theta)+iv(r,\theta)$ and f (z) is derivable at $z_0=r_0e^{i\theta_0}$ then.

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Example 14 : Prove that the function $f(z)$ defined by

$$f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2},\ (z\neq 0)$$
$$=0,\ (z=0)$$

is continuous and the Cauchy–Riemann equations are satisfied at the origin , yet $f'(0)$ does *not exist.*

[JNTU 99, Nov. 2006, 2007S, Nov. 2008S, Nov. 2008, JNTU (K) Nov. 2009 (Set No. 4)]

Solution : $\underset{z\to 0}{\text{Lt}}\ f(z)=\underset{\substack{x\to 0\\ y\to 0}}{\text{Lt}}\ \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}=\underset{y\to 0}{\text{Lt}}\ -\frac{y^3(1-i)}{y^2}=\underset{y\to 0}{\text{Lt}}\ [-y(1-i)]=0$

and $\underset{z\to 0}{\text{Lt}}\ f(z)=\underset{\substack{y\to 0\\ x\to 0}}{\text{Lt}}\ \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}=\underset{x\to 0}{\text{Lt}}\ \frac{x^3(1+i)}{x^2}=\underset{x\to 0}{\text{Lt}}\ x(1+i)=0$

Also $f(0)=0$ by given data.

Thus, we get $\underset{z\to 0}{\text{Lt}}\ f(z)=f(0)$

When $x\to 0,\ y\to 0$ and $y\to 0,\ x\to 0$

Now take both x and y tend to 0 simultaneously along the path $y = mx$, then

$$\underset{z\to 0}{\text{Lt}}\ f(z)=\underset{x\to 0}{\text{Lt}}\ \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}=\underset{x\to 0}{\text{Lt}}\ \frac{x^3(1+i)-m^3x^3(1-i)}{(1+m^2)x^2}$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{x\left[1+i-m^3(1-i)\right]}{1+m^2}=0$$

$\therefore$ Whatever is the manner in which $z\to 0$, we have $\underset{z\to 0}{\text{Lt}}\ f(z)=0=f(0)$

$\therefore$ The function is continuous at $z = 0$.

$$\left(\frac{\partial u}{\partial x}\right)_{(0,0)}=\underset{x\to 0}{\text{Lt}}\ \frac{u(x,0)-u(0,0)}{x}=\underset{x\to 0}{\text{Lt}}\ \frac{x}{x}=1$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \underset{y\to 0}{\text{Lt}} \frac{u(0,y)-u(0,0)}{y} = \underset{y\to 0}{\text{Lt}} -\frac{y}{y} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \underset{x\to 0}{\text{Lt}} \frac{v(x,0)-v(0,0)}{x} = \underset{x\to 0}{\text{Lt}} \frac{x}{x} = 1$$

$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \underset{y\to 0}{\text{Lt}} \frac{v(0,y)-v(0,0)}{y} = \underset{y\to 0}{\text{Lt}} \frac{y}{y} = 1$$

$\therefore$ We have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$\therefore$ Cauchy–Riemann equations are satisfied at the origin

$$f'(0) = \underset{z\to 0}{\text{Lt}} \frac{f(z)-f(0)}{z} = \underset{z\to 0}{\text{Lt}} \frac{f(z)}{z} = \underset{z\to 0}{\text{Lt}} \frac{\left(x^3-y^3\right)+i\left(x^3+y^3\right)}{\left(x^2+y^2\right)(x+iy)}$$

Let $z \to 0$ along the path $y = mx$, then $f'(0) = \dfrac{1-m^3+i\left(1+m^3\right)}{\left(1+m^2\right)(1+im)}$

which depends on m and hence is not unique. Thus, $f'(z)$ does not exist at (0, 0).

Example 17 : *Show that* $f(z)=\begin{cases}\dfrac{x\,y^2\,(x+i\,y)}{x^2+y^4}, & z\neq 0\\ 0, \quad \text{if} & z=0\end{cases}$

is not analytic at $z = 0$ *although* $C - R$ *equations are satisfied at the origin.*

[*JNTU 2003, 2007S, Nov. 2008 (Set No. 3)*]

Solution : $\dfrac{f(z)-f(0)}{z-0}=\dfrac{f(z)-0}{z}=\dfrac{f(z)}{z}$

$$=\frac{x\,y^2\,(x+i\,y)}{(x^2+y^4)\cdot z}=\frac{x\,y^2(z)}{(x^2+y^4)\cdot z}=\frac{x\,y^2}{x^2+y^4}$$

Clearly $\displaystyle\lim_{\substack{x\to 0\\ y\to 0}}\frac{x\,y^2}{x^2+y^4}=\lim_{\substack{y\to 0\\ x\to 0}}\frac{x\,y^2}{x^2+y^4}=0$

Along the path $y = mx$,

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{x\to 0}\frac{x\,(m^2\cdot x^2)}{x^2+m^4\cdot x^4}=\lim_{x\to 0}\frac{m^2\,x}{1+m^4}=0$$

Also along the path $x = m\,y^2$,

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{y\to 0}\frac{(m\,y^2)\,y^2}{m^2\,y^4+y^4}=\lim_{y\to 0}\frac{m}{m^2+1}\neq 0$$

Limit value depends on m *i.e.* on the path of approach and is different for the different paths followed and therefore limit does not exist. Hence $f(z)$ is not differentiable at $z = 0$. Thus $f(z)$ is not analytic at $z = 0$.

To prove that $C - R$ conditions are satisfied at the origin.

Let $f(z)=u+i\,v=\dfrac{x\,y^2(x+i\,y)}{x^2+y^4}$

Then $u\,(x, y)=\dfrac{x^2\,y^2}{x^2+y^4}$ and $v\,(x, y)=\dfrac{x\,y^3}{x^2+y^4}$, for $z\neq 0$

Also $u\,(0, 0) = 0$ and $v\,(0, 0) = 0$ $\quad [\because\ f(z)=0\ \ at\ \ z=0]$

Now $\dfrac{\partial u}{\partial x}=\displaystyle\lim_{x\to 0}\frac{u\,(x,0)-u\,(0,0)}{x}=\lim_{x\to 0}\frac{0-0}{x}=0$

$$\frac{\partial u}{\partial y}=\lim_{y\to 0}\frac{u(0,y)-u\,(0,0)}{y}=\lim_{y\to 0}\frac{0-0}{y}=0$$

$$\frac{\partial v}{\partial x}=\lim_{x\to 0}\frac{v(x,0)-v\,(0,0)}{x}=\lim_{x\to 0}\frac{0-0}{x}=0$$

and $\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0, y) - v(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0$

Thus Cauchy – Riemann equations are satisfied at the origin.

Hence $f(z)$ is not analytic at $z = 0$ eventhough C-R equations are satisfied at the origin.

Example 35 : Prove that, if $u = x^2 - y^2$, $v = \frac{-y}{x^2 + y^2}$ both u and v satisfy Laplace's equation, but $u + iv$ is not a regular (analytic) function of z. **[JNTU Aug. 2007S (Set No. 4)]**

Solution : Given $u = x^2 - y^2$ and $v = \frac{-y}{x^2 + y^2}$

$$\therefore \quad \frac{\partial u}{\partial x} = 2x, \; \frac{\partial u}{\partial y} = -2y, \; \frac{\partial^2 u}{\partial x^2} = 2, \; \frac{\partial^2 u}{\partial y^2} = -2$$

and
$$\frac{\partial v}{\partial x} = -y\left[\frac{-2x}{(x^2+y^2)^2}\right] = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = 2y\left[\frac{(x^2+y^2)^2 - 2x(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}\right] = \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{-(x^2+y^2) - 2y(-y)}{(x^2+y^2)^2} = \frac{-(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2(2y) - (y^2-x^2)2(x^2+y^2)(2y)}{(x^2+y^2)^4} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

Clearly $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Hence both u and v satisfies the Laplace's equation.

We observe that $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

Since u and v do not satisfy the Cauchy – Riemann equations, therefore $u + i\,v$ is not an analytic (regular) function of z.

Example 36 : Show that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic.

Solution : We have $u = e^{-x}(x \sin y - y \cos y)$

Differentiating partially, w.r.t x and y, we get

$$\frac{\partial u}{\partial x} = e^{-x}(\sin y) - e^{-x}(x \sin y - y \cos y)$$

$$\frac{\partial^2 u}{\partial x^2} = -2e^{-x} \sin y + x\,e^{-x} \sin y - y\,e^{-x} \cos y \qquad ...(1)$$

and $$\frac{\partial u}{\partial y} = e^{-x}(x \cos y + y \sin y - \cos y)$$

$$\frac{\partial^2 u}{\partial y^2} = e^{-x}(-x \sin y + 2 \sin y + y \cos y) \qquad ...(2)$$

Adding (1) & (2) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\therefore$ u is a harmonic function.

Example 37 : Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find a conjugate harmonic function v of u.

Solution : Given $u = x^2 - y^2 - y$

$$\therefore \quad \frac{\partial u}{\partial x} = 2x \text{ and } \frac{\partial^2 u}{\partial x^2} = 2 \text{ and } \frac{\partial u}{\partial y} = -2y - 1 \text{ and } \frac{\partial^2 u}{\partial y^2} = -2$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence u is a harmonic function.

To find the conjugate harmonic function v, we must have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x \quad ...(1) \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y + 1 \qquad ...(2)$$

We will integrate the first equation w.r.t. y and then differentiate w.r.t. x, we get

$$v = 2xy + h(x) \text{ and } \frac{\partial v}{\partial x} = 2y + \frac{dh}{dx}$$

Comparing with (2), we get

$$\frac{dh}{dx} = 1 \Rightarrow h(x) = x + c$$

where c is any real constant. Hence, $v = 2xy + x + c$ is the most general conjugate harmonic function of given u.

Example 38 : Show that the function $u(x, y) = e^x \cos y$ is harmonic. Determine its harmonic conjugate $v(x, y)$ and the analytic function $f(z) = u + iv$. [JNTU 1996]

Solution : Given $u(x, y) = e^x \cos y$

Differentiating with respect to x and y, we get

$$\frac{\partial u}{\partial x} = e^x \cos y \quad \text{and} \quad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

Hence, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Thus, u is a harmonic function. Let v be the harmonic conjugate of u. Then, by Cauchy-Riemann equations

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^x \sin y$$

Integrating, $v = e^x \sin y + f(y)$ ...(1)

$$\therefore \quad \frac{\partial v}{\partial y} = e^x \cos y + f'(y) \quad ...(2)$$

Again $$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^x \cos y \quad ...(3)$$

From (2) and (3), we get

$$e^x \cos y = e^x \cos y + f'(y)$$

or $$f'(y) = 0 \Rightarrow f(y) = c$$

Hence, from (1), we get

$$v = e^x \sin y + c$$

$$\therefore \quad f(z) = u + iv = e^x \cos y + ie^x \sin y + ic$$
$$= e^x (\cos y + i \sin y) + ic = e^x e^{iy} + ic = e^z + ic$$

Example 39 : If $f(z)$ is a regular function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4|f'(z)|^2$$

[JNTU 2004 (Set No. 1), 2005, Aug. 2007, Nov. 2008S (Set No. 1)]

or $$\nabla^2 \left[|f(z)|^2\right] = 4|f'(z)|^2$$

Solution : Let $f(z) = u(x, y) + iv(x, y)$ is a regular function. Then

$$|f(z)|^2 = u^2 + v^2 = \phi(x, y) \quad \text{(say)}$$

$$\therefore \quad \frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

and $$\frac{\partial^2 \phi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

Similarly, $$\frac{\partial^2 \phi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

Adding, we get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right]$$

$$+ 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \qquad ...(1)$$

Since u and v have to satisfy Cauchy-Riemann equations and the Laplace equation, we have

$$\left(\frac{\partial u}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2, \left(\frac{\partial u}{\partial y}\right)^2 = \left(-\frac{\partial v}{\partial x}\right)^2$$

and $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad ...(2)$$

Thus, $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$ [using (2) in (1)]

Hence, $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$