第一次作业 12211834 沈[illegible]

2024年3月18日 星期一 15:39

Ch1 1(e)

1(e) $a=m+ni,\ z=x+yi,\ b=p+qi$

$(m+ni)(x+yi)+(p+qi)=(mx-ny+p)+(xn+my+q)i$.

$Re(az+b)=mx-ny+p>0 \Rightarrow ny<mx+p$.

若 $n<0$ 则 $y>\frac{mx+p}{n}$ 即在直线 $y=\frac{mx+p}{n}$ 上半部分.

$n=0$ $mx+p>0$. 直线（据m正负）

$n>0$ $y<\frac{mx+p}{n}$ 直线下部分.

7. (a) $\left|\frac{w-z}{1-\bar{w}z}\right|\le 1$.

根据提示即证 $|z|^2+|w|^2\le 1+|w|^2|z|^2$.

而 $(1-|w|^2)(1-|z|^2)\ge 0$ 显然成立.

当取等号 iff $|z|=1$ or $|w|=1$

(b) 有 $|w|<1$ $|F(z)|<1$ 考虑其结论，$\forall z\in D$.

(i) $\lim_{h\to 0}\frac{F(z+h)-F(z)}{h}=\lim_{h\to 0}\frac{\frac{w-(z+h)}{1-\bar{w}(z+h)}-\frac{w-z}{1-\bar{w}z}}{h}=\frac{w\bar{w}-1}{(1-\bar{w}z)^2}$ 存在.

由(a)显然 $F:D\to D$. 故是全纯的.

(ii) $F(0)=w,\ F(w)=0$. 显然成立

(iii) 由(a)有 若 $|w|<1$ 则 $|F(z)|=1$ if $|z|=1$

(iv) $F\cdot F(z)=\frac{w-\frac{w-z}{1-\bar{w}z}}{1-\bar{w}\left(\frac{w-z}{1-\bar{w}z}\right)}=\frac{w(1-\bar{w}z)-w+z}{1-\bar{w}z-\bar{w}(w-z)}=\frac{z-zw\bar{w}}{1-w\bar{w}}=z$.

故其为双射

8.

假设 U,V 为复平面上开集，— — — $h=g\cdot f$. $h: U\to \mathbb{C}$.

证明：考虑 h 以 z_1 为变量 $z_1=f(z)$ 再考虑 h 以 z 为变量 $h(z_1)=g(f(z_1))$.

有 z 与 $z_1=f(z)$ 和 $\bar{z}_1=\bar{f}(z)$ 有关.

$\Rightarrow \frac{\partial h}{\partial z}=\frac{\partial h}{\partial z_1}\cdot\frac{\partial z_1}{\partial z}+\frac{\partial h}{\partial \bar{z}_1}\cdot\frac{\partial \bar{z}_1}{\partial z}=\frac{\partial g}{\partial z}\cdot\frac{\partial f}{\partial z}+\frac{\partial g}{\partial \bar{z}}\cdot\frac{\partial \bar{f}}{\partial z}$.

对 $\frac{\partial h}{\partial \bar{z}}$ 同理可证.

$f: U\to V,\ g: V\to \mathbb{C}$ $h=g\cdot f$

$\frac{\partial h}{\partial z}=\frac{\partial g\cdot f}{\partial z}=$

9.

$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}=\cos\theta\cdot\frac{\partial u}{\partial x}+\sin\theta\cdot\frac{\partial u}{\partial y}$.

$\frac{\partial v}{\partial \theta}=\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta}=-r\sin\theta\frac{\partial v}{\partial x}+r\cos\theta\cdot\frac{\partial v}{\partial y}$

$=r\cos\theta\frac{\partial u}{\partial x}+r\sin\theta\cdot\frac{\partial u}{\partial y}$.

$\Rightarrow \frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$ 同理 $\frac{1}{r}\frac{\partial u}{\partial \theta}=-\frac{\partial v}{\partial r}$.

~~$f=u(x,y)$~~

$f=u(x,y)+iv(x,y)$

$=r\cos\theta\cdot u(x,y)+ir\sin\theta\, v(x,y)$.

10

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial z}\cdot\frac{\partial z}{\partial x}+\frac{\partial}{\partial \bar z}\cdot\frac{\partial \bar z}{\partial x}=\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar z}.$$

$$\frac{\partial}{\partial y}=\frac{\partial}{\partial z}\cdot\frac{\partial z}{\partial y}+\frac{\partial}{\partial \bar z}\cdot\frac{\partial}{\partial \bar z}\cdot\frac{\partial \bar z}{\partial y}=i\frac{\partial}{\partial z}-i\frac{\partial}{\partial \bar z}.$$

$$\frac{\partial}{\partial x^2}=\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial z\partial \bar z}+\frac{\partial^2}{\partial \bar z^2}+\frac{\partial^2}{\partial \bar z\partial z}. \quad ③$$

$$\frac{\partial}{\partial y^2}=-\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial z\partial \bar z}+\frac{\partial^2}{\partial \bar z\partial z}-\frac{\partial^2}{\partial \bar z^2} \quad ④$$

③+④ $\Delta=2\frac{\partial^2}{\partial z\partial \bar z}+2\frac{\partial^2}{\partial \bar z\partial z}.=4\frac{\partial}{\partial z}\cdot\frac{\partial}{\partial \bar z}=4\frac{\partial}{\partial \bar z}\cdot\frac{\partial}{\partial z}.$

11,

f 正则, Ω 开集. $f=u+iv$ 为正则的若具连续.

$\Rightarrow \frac{\partial^2 v}{\partial x\partial y}=\frac{\partial^2 v}{\partial y\partial x}$ $\quad \frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$

$\Delta u=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u=\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)=\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 v}{\partial y\partial x}=0$

同理 $\Delta v=0$ 即证.

13(a),

f 在开集 Ω 上正则. 求证: 若 $Re(f)$ 为常数, f 为常数.

有: $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=0.$ 又正则则 $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=0.$

$\Rightarrow$ 在 Ω 中 $f'(z)=\frac{\partial f}{\partial x}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=0.$ 即 $f(z)$ 为常数.

$$\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{bmatrix}.$$

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x}=-1\frac{\partial u}{\partial y}.$$

14,

$\{a_n\}_{n=1}^{N}\{b_n\}_{n=1}^{N}$ 有限复数对. $B_k=\sum_{n=1}^{k}b_n$ $\quad B_0=0.$

求证: $\sum_{n=M}^{N}a_nb_n=a_NB_N-a_MB_{M-1}-\sum_{n=M}^{N-1}(a_{n+1}-a_n)B_n$

证明: $RHS=a_N\sum_{n=1}^{N}b_n-a_M\sum_{n=M-1}^{N}b_n-\sum_{n=M}^{N-1}a_{n+1}B_n+\sum_{n=M}^{N-1}a_nB_n$

$=a_N\sum_{n=1}^{N}b_n-a_M\sum_{n=M-1}^{N}b_n-\sum_{n=M}^{N-1}a_{n+1}B_n+\sum_{n=M}^{N-1}a_n(B_{n-1}+b_n)\Leftarrow$ 有 $B_n-B_{n-1}=b_n.$

$=a_N\sum_{n=1}^{N}b_n-a_M\sum_{n=M-1}^{N}b_n-\sum_{n=M}^{N-1}a_{n+1}B_n+\sum_{n=M}^{N-1}a_nB_{n-1}+\sum_{n=M}^{N-1}a_nb_n.$

$=a_N\sum_{n=1}^{N}b_n-a_M\sum_{n=M-1}^{N}b_n-\sum_{n=M}^{N-1}a_{n+1}B_n+\sum_{n=M-1}^{N-2}a_{n+1}B_n+\sum_{n=M}^{N-1}a_nb_n$ $\quad n\to N-1.$ $\quad n-1\to N-2.$

$=a_NB_N-a_MB_{M-1}+a_MB_{M-1}-a_NB_{N-1}+\sum_{n=M}^{N-1}a_nb_n$

$=a_N(B_N-B_{N-1})+\sum_{n=M}^{N-1}a_nb_n=a_Nb_N+\sum_{n=M}^{N-1}a_nb_n=\sum_{n=M}^{N}a_nb_n=RHS.$

即证

16(a)(c)(d),

(a) $\frac{1}{R}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left(\frac{\log(n+1)}{\log n}\right)^2=\left(\lim_{n\to\infty}\frac{\log(n+1)}{\log n}\right)^2=1.$

$\Rightarrow R=1$

(c) $a_n=\frac{n^2}{4^n+3n}$ $\quad \frac{1}{R}=\lim_{n\to\infty}\sqrt[n]{\frac{n^2}{4^n+3n}}=\frac{1}{4}\Rightarrow R=4$

(c) $a_n=\frac{n^2}{4^n+3n}$ $\quad \frac{1}{R}=\lim_{n\to\infty}\sqrt[n]{\frac{n^2}{4^n+3n}}=\frac{1}{4} \Rightarrow R=4$

(d) $a_n=\frac{(n!)^3}{(3n!)}$. $\frac{1}{R}=\lim_{n\to\infty}\sqrt[n]{\frac{(cn^{n+\frac{1}{2}}e^{-n})^3}{C\cdot(3n)^{3n+\frac{1}{2}}\cdot e^{-3n}}}$

$=\lim_{n\to\infty}\sqrt[n]{\frac{c^3 n^{3n+\frac{3}{2}}\cdot \cancel{e^{-3n}}}{\cancel{c}\cdot(3n)^{3n+\frac{1}{2}}\cdot \cancel{e^{-3n}}}}=\lim_{n\to\infty}\sqrt[n]{\frac{c^2 n}{3^{3n+\frac{1}{2}}}}=\frac{1}{27}$,

$\therefore R=27$.

17.

若$\{a_n\}$ $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=L$. 则 $\lim_{n\to\infty}|a_n|^{\frac{1}{n}}=L$.

对 $\forall\varepsilon>0$ $\exists N$ s.t. $n>N$ $L-\varepsilon<\left|\frac{a_{n+1}}{a_n}\right|<L+\varepsilon$ 累乘有:

故 $(L-\varepsilon)^{n+1}a_0<|a_{n+1}|<c(L+\varepsilon)^{n+1}|a_0|$.

$\Rightarrow L-\varepsilon<\sqrt[n+1]{a_{n+1}}<L+\varepsilon$. 当 $n\to\infty$ $\sqrt[n]{a_n}\to L$.

19.

(a) $\sum nz^n$. $\quad z=\cos\theta+i\sin\theta$ $\quad z^n=\cos n\theta+i\sin n\theta$.

$\Rightarrow \sum nz^n=\sum n\cos n\theta+i\sum n\sin n\theta$. 发散.

(b) $\sum\left|\frac{z^n}{n^2}\right|=\sum\left|\frac{\cos n\theta+i\sin n\theta}{n^2}\right|<\sum\frac{1}{n^2}$ 后者明显收敛

故收敛.

(c) $z=1$时 $\sum\frac{z^n}{n}=\sum\frac{1}{n}$ 明显发散.

$z\neq1$时, $z=\cos\theta+i\sin\theta$. $a_n=\frac{z^n}{n}=\frac{\cos n\theta}{n}+\frac{i\sin n\theta}{n}=a_n+ib_n$.

原式 $=\sum\frac{\cos n\theta}{n}+i\sum\frac{\sin n\theta}{n}$

$\sum\cos n\theta=\frac{\sin(n+\frac{1}{2}\theta)}{2\sin\frac{\theta}{2}}-\frac{1}{2}$. 在 $\theta\in(0,2\pi)$ 时其有界.

$\{\frac{1}{n}\}$单减且趋近于0, 故 $\sum\frac{\cos n\theta}{n}$ 收敛. 同理 $\sum\frac{\sin n\theta}{n}$ 收敛.

故 $\sum\frac{z^n}{n}$ 收敛除了$z=1$时.

25.

(a) $z=r\cdot e^{i\theta},\theta\in[0,2\pi)$

$\int_\gamma z^n dz=\int_\gamma r^n e^{i\theta n}d\,re^{i\theta}=r^{n+1}\int_0^{2\pi}ie^{i\theta(n+1)}d\theta$.

$=\frac{r^{n+1}}{n+1}\cdot\int_0^{2\pi}de^{i(n+1)\theta}=\frac{r^{n+1}}{n+1}[e^{i(n+1)2\pi}-1]$.

$=\begin{cases}2\pi i & n=-1\\ 0 & \text{else}.\end{cases}$

(b) $z=pe^{i\theta}+qe^{i\varphi}$ $\quad \theta,\varphi\in[0,2\pi)$

$\int_\gamma z^n dz=\left.\frac{z^{n+1}}{n+1}\right|_\gamma=\left.\frac{(pe^{i\theta}+qe^{i\varphi})^{n+1}}{n+1}\right|_0^{2\pi}=\frac{(pe^{i2\pi}+qe^{i\varphi})^{n+1}}{n+1}-\frac{(p+qe^{i\varphi})^{n+1}}{n+1}$

$=\frac{(p+qe^{i\varphi})^{n+1}}{n+1}-\frac{(p+qe^{i\varphi})^{n+1}}{n+1}=0$.

(c) $|a|<r<|b|$ $\quad \int_r\frac{1}{(z-a)(z-b)}dz=\frac{2\pi i}{a-b}$

$\int \frac{1}{\quad} d\,) = \frac{1}{\quad}\cdot\int\frac{a-b}{\quad}dz\ \frac{1}{\quad}\cdot\int_r\frac{(z-b)-(z-a)}{\quad}dz$

②. $|a|<r<|b|$ $\int_\gamma \frac{1}{(z-a)(z-b)}dz = \frac{2\pi i}{a-b}$

$$\int_\gamma \frac{1}{(z-a)(z-b)}dz = \frac{1}{a-b}\cdot\int_\gamma \frac{a-b}{(z-a)(z-b)}dz = \frac{1}{a-b}\cdot\int_\gamma \frac{(z-b)-(z-a)}{(z-a)(z-b)}dz$$

$$= \frac{1}{a-b}\cdot\int_\gamma\left(\frac{1}{z-a}-\frac{1}{z-b}\right)dz$$

∴命题即证：$\int_\gamma \frac{1}{z-a}dz-\int_\gamma\frac{1}{z-b}dz=2\pi i$

$$\int_\gamma \frac{1}{z-a}dz = \int_\gamma \frac{1}{z(1-\frac{a}{z})}dz = \int_{|z|=r} z^{-1}\left(1+\frac{a}{z}+\left(\frac{a}{z}\right)^2+\cdots\right)dz.$$

$$= \sum_{n=0}^{\infty}\int_{|z|=r} z^{-1}\left(\frac{a}{z}\right)^n dz = \int_{|z|=r} z^{-1}dz = 2\pi i.$$

同样的我们有：$\int_\gamma \frac{1}{z-b}dz = \sum_{n=0}^{\infty}\int_{|z|=r} z^{-1}\left(\frac{b}{z}\right)^n dz = 0.$

$\Rightarrow$ $\int_\gamma \frac{1}{z-a}dz-\int_\gamma\frac{1}{z-b}dz=2\pi i$ 即证.

复变第二次作业　　　　12211834

2024年4月2日　19:09　　　　沈博涵

1、证明 $\int_0^\infty \sin x^2 dx = \int_0^\infty \cos x^2 dx = \frac{\sqrt{2\pi}}{4}$

令 $f(z)=e^{iz^2}$ 对 $f(z)$ 积分．由 $\theta: 0\to\frac{\pi}{4}$，半径 R．

沿 x 轴积分：$\int_0^R e^{ix^2}dx$ ①

沿曲线部分 $z=Re^{i\theta}$ 积分：

$$\int_0^{\frac{\pi}{4}} e^{iR^2e^{2i\theta}} iRe^{i\theta}d\theta = iR\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2\theta}\cdot e^{i(\theta+R^2\cos 2\theta)}d\theta \quad ②$$

$\frac{\pi}{4}$ 时 $z=re^{\frac{\pi i}{4}}$：积分为 $\int_R^0 e^{ir^2e^{\frac{\pi i}{2}}} e^{\frac{\pi i}{4}} dr$ ③

$R\to\infty$ 时．第三部分积分为 $-e^{\frac{i\pi}{4}}\int_0^\infty e^{-x^2}dx = -e^{\frac{i\pi}{4}}\cdot\frac{\sqrt{\pi}}{2} = -\frac{\sqrt{2\pi}}{4}-\frac{\sqrt{2\pi}}{4}i$

且 ①+②+③ $=0$．

下对曲线部分进行计算（即②）

$$\left|iR\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2\theta}\cdot e^{i(\theta+R^2\cos 2\theta)}d\theta\right| \le R\int_0^{\frac{\pi}{4}}\left|e^{-R^2\sin 2\theta}\cdot e^{i(\theta+R^2\cos 2\theta)}\right|d\theta.$$

$$=R\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2\theta}d\theta \le R\int_0^{\frac{\pi}{4}} e^{-\frac{4R^2\theta}{\pi}}d\theta = -\frac{\pi}{4R}e^{-\frac{4R^2\theta}{\pi}}\Big|_0^{\frac{\pi}{4}} = \frac{\pi(1-e^{-R^2})}{4R}$$

由于 $\lim_{R\to\infty}\frac{\pi(1-e^{-R^2})}{4R}=0$．即 ② $=0$

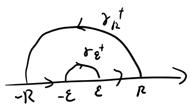
故 ①+③ $=0 \Rightarrow \int_0^\infty e^{ix^2}dx - \frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}i = 0.$

$\Rightarrow \int_0^\infty e^{ix^2}dx = \frac{\sqrt{2\pi}}{4}+\frac{\sqrt{2\pi}}{4}i$　考虑其实部与虚部；

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}$$

2、求证 $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$．

证明：考虑函数 $f(z)=\frac{e^{iz}}{z}$



实轴上：$\int_{-R}^{-\varepsilon}\frac{\cos x+i\sin x}{x}dx+\int_\varepsilon^R\frac{\cos x+i\sin x}{x}dx = 2i\int_\varepsilon^R\frac{\sin x}{x}dx$

$z=Re^{i\theta}$ 弧上，$\int_0^\pi \frac{e^{iRe^{i\theta}}}{Re^{i\theta}}iRe^{i\theta}d\theta = i\int_0^\pi e^{-R\sin\theta}e^{iR\cos\theta}d\theta.$

考虑 $\left|i\int_0^\pi e^{-R\sin\theta}e^{iR\cos\theta}d\theta\right| \le \int_0^\pi\left|e^{-R\sin\theta}e^{iR\cos\theta}\right|d\theta$

$$=\int_0^\pi e^{-R\sin\theta}d\theta.$$

$$=2\int_0^{\frac{\pi}{2}}e^{-R\sin\theta}d\theta.$$

$\sin\theta\ge\frac{2\theta}{\pi}$ 当 $0\le\theta\le\frac{\pi}{2}$ 成立

$$\le 2\int_0^{\frac{\pi}{2}}e^{-\frac{2R\theta}{\pi}}d\theta = -\frac{\pi e^{-\frac{2R\theta}{\pi}}}{R}\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{\pi(1-e^{-R})}{R}$$

$R\to\infty$ 时 上式 $\to 0$．

内半圆积分为：$\int_\pi^0\frac{1}{\varepsilon e^{i\theta}}i\varepsilon e^{i\theta}d\theta = -\int_0^\pi i\,d\theta = -\pi i$．

由总体积分有 $2i\int_0^\infty\frac{\sin x}{x}dx - \pi i = 0 \Rightarrow \int_0^\infty\frac{\sin x}{x}dx = \frac{\pi}{2}$

3．计算 $\int_0^\infty e^{-ax}\cos bx\,dx$ 与 $\int_0^\infty e^{-ax}\sin bx\,dx$．$a>0$

对 $f(z)=e^{-Az}$ 积分 $A=\sqrt{a^2+b^2}$，$\theta: 0\to w$

有 $\int_0^R e^{-Ax}dx + \int_\gamma e^{-Az}dz + \int_{l^-} e^{-Az}dz = 0.$

$\int_0^R e^{-Ax}dx = -\frac{1}{A}e^{-Ax}\Big|_0^R$　$R\to\infty$ 时上式 $\to\frac{1}{A}$．

$|e^{-Az}| = \frac{1}{e^{AR}|e^{i\theta}|} = \frac{1}{e^{AR}}$

$\therefore \left|\int_\gamma f(z)dz\right| \le \sup_{z\in\gamma}|f(z)|\cdot \text{length}\,\gamma = \frac{1}{e^{AR}}\cdot Rw$

$R\to\infty$　上式 $\to 0$

$-\int_{l^-}e^{-Az}dz = \int_0^R e^{-Are^{iw}}e^{iw}dr = \int_0^R e^{-Ar(\cos w+i\sin w)}e^{iw}dr$

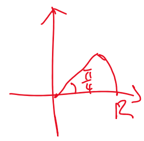
$$=\int_0^R e^{-Ar\cos w}e^{-iAr\sin w}e^{iw}dr.$$

$\left\{\begin{array}{l}\text{对 } A=\sqrt{a^2+b^2},\\ \cos w=\frac{a}{\sqrt{a^2+b^2}}\end{array}\right.$ $=\int_0^R e^{iw}e^{-ar}\cdot e^{-ibr}dr$

$=e^{iw}\int_0^R e^{-ar}(\cos br - i\sin br)dr.$

$\int_0^\infty \sin x^2dx = \int_0^\infty \cos x^2 dx = \frac{\sqrt{2\pi}}{4}$

$f(z)=e^{iz^2} = \cos z^2 + i\sin z^2$



$\int_0^R e^{ix^2}dx$

曲线对 θ 积分，$z=Re^{iR^2}$

$\int_0^{\frac{\pi}{4}} e^{iz^2}$　　$z'(r)=e^{\frac{\pi i}{4}}\cdot r$

$\frac{\pi}{4}$ 时 $z=a\,Re^{\frac{\pi i}{4}}$　$\int_R^0 z(r)\cdot z'(r)dr$

$=\int_R^0$

$|z|=|R\cdot e^{i\theta}| = |R\cos\theta + iR\sin\theta|$

$=\sqrt{R^2}$

$\frac{dz}{d\theta}=iRe^{i\theta}$

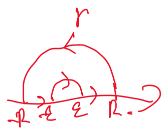
$z=Re^{i\theta}$　$z^2=R^2e^{2\theta i}$

$e^{iz^2}=e^{iR^2e^{2\theta i}}$

$\int_0^{\frac{\pi}{4}} z(\theta)\cdot z'(\theta)d\theta \le (\sup |z(\theta)|)\cdot\int_0^{\frac{\pi}{4}} z'(\theta)d\theta.$

$= R\cdot z(\theta)\Big|_0^{\frac{\pi}{4}}$

$=R.$



$f(z)=\frac{e^{iz}}{z}$

$\int_{-R}^{-\varepsilon}\frac{\cos x+i\sin x}{x}dx+\int_\varepsilon^R\frac{\cos x+i\sin x}{x}dx = 2i\int_\varepsilon^R\frac{\sin x}{x}dx.$

$\int_0^\pi\frac{e^{iRe^{i\theta}}}{Re^{i\theta}}$

$$= \int_0^{R} e^{\cdots} e^{\cdots}$$

$$\begin{cases} \text{记} A=\sqrt{a^2+b^2}, \\ \cos w=\frac{a}{\sqrt{a^2+b^2}}, \\ \sin w = \frac{b}{\sqrt{a^2+b^2}}. \end{cases} = \int_0^R e^{iw} e^{-ar} \cdot e^{-ibr} dr$$

$$= e^{iw}\int_0^R e^{-ar}[\cos br - i\sin br) dr.$$

$$= (\cos w + i\sin w)\left[\int_0^R e^{-ar}\cos br\,dr - i\int_0^R e^{-ar}\sin br\,dr\right).$$

$R\to\infty$ 有 上式 $=\frac{1}{A}(a+bi)[I_1 - iI_2]$

$I_1 = \int_0^R e^{-ar}\cos br\,dr,\ I_2 = \int_0^R e^{-ar}\sin br\,dr.$

代入原等式中：$\frac{1}{A}(a+bi)[I_1 - iI_2] = \frac{1}{A}$

$$\Rightarrow \begin{cases} aI_1 + bI_2 = 1 \\ aI_2 = bI_1 \end{cases}$$

即：$\begin{cases} \int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2} \\ \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}. \end{cases}$

6. 令 γ_ε 为半径为 ε 的圆，圆心在 w. ε 足够小使 γ_ε 在 T 中.

f 在 R 上全纯 $\int_{\partial R} f(z)dz = \int_T f(z)dz - \int_{\gamma_\varepsilon} f(z)dz = 0$

$\Rightarrow \int_T f(z)dz = \int_{\gamma_\varepsilon} f(z)dz.$

f 在 w 附近有界，且当 $\varepsilon\to 0$, length $\gamma_\varepsilon \to 0$

$\Rightarrow \int_{\gamma_\varepsilon} f(z)dz \to 0$ 故 $\int_T f(z)dz = 0.$

9. Ω 为 $\mathbb{C}$ 上有界开集，$\varphi: \Omega\to\Omega$ 为全纯函数. 求证若 $\exists z_0\in\Omega$ 使

$\varphi(z_0) = z_0,\ \varphi'(z_0) = 1 \Rightarrow \varphi$ 为线性的

证明：令 $f(z) = \varphi(z+z_0) - z_0$. $z\in\Omega - z_0 \Rightarrow f(z)\in\Omega - z_0$.

则若 φ 线性，f 线性. 假设 $z_0 = 0$. $\varphi(z) = z + a_2z^2 + \cdots$

假设 a_n 为第一个非零系数. $\varphi(z) = z + a_nz^n + O(z^{n+1})$.

下用归纳法证明：$\varphi_k(z) = z + ka_nz^n + O(z^{n+1})$.

若对 k 成立，考虑 $\varphi_{k+1}(z) = (z + ka_nz^n + O(z^{n+1})) + a_n(z + ka_nz^n + O(z^{n+1}))^n$

$+ O((z + ka_nz^n + O(z^{n+1}))^{n+1}) = z + (k+1)a_nz^n + O(z^{n+1}).$

令 $r>0$, $z\in\Omega$ 且 $|z|\le r$. 由柯西不等式：

$|\varphi_k^{(n)}(0)| \le \frac{n!\,\|\varphi_k\|_r}{r^n}$ (其中 $\|\varphi_k\|_r = \sup_{|z|=r}|\varphi_k(z)|$).

而 $\varphi_k(z)\in\Omega$ 有界. 故存在独立于 n,k 常数 M,

若对 $\varphi_k^{(n)}$ 计算 有 $\varphi_k^{(n)} = kn!a_n \Rightarrow kn!a_n \le \frac{n!M}{r^n} \Rightarrow a_n \le \frac{M}{kr^n}$ 对所有 k 成立

$\Rightarrow k\to\infty$ 时有 $a_n = 0$. 故不可能存在非零系数

即 仅存在 z 情况下 即 φ 线性

10. 考虑 $f(z) = \bar{z}$, 连续且非全纯.

若 $P_n(z)\to\bar{z}$ 在 $n\to\infty$ 时成立.

而 $P_n(z)$ 在紧集 D 上全纯，故 $\bar{z}$ 亦全纯，矛盾

得证.

11. $\frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\,\mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)d\varphi.$

$= \frac{1}{4\pi}\int_0^{2\pi} f(Re^{i\varphi})\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z} + \frac{Re^{-i\varphi}+\bar{z}}{Re^{-i\varphi}-\bar{z}}\right)d\varphi.$

$= \frac{1}{4\pi}\int_0^{2\pi} f(Re^{i\varphi})\left(\frac{2Re^{i\varphi}}{Re^{i\varphi}-z} - \frac{2\bar{z}}{\bar{z}-Re^{-i\varphi}}\right)d\varphi$

$= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\frac{Re^{i\varphi}d\varphi}{Re^{i\varphi}-z} - \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\frac{\bar{z}\,d\varphi}{\bar{z}-Re^{-i\varphi}}$

$= \frac{1}{2\pi i}\int_0^{2\pi} f(Re^{i\varphi})\frac{d(Re^{i\varphi})}{Re^{i\varphi}-z} - \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\frac{Re^{i\varphi}d\varphi}{Re^{i\varphi}-\frac{R^2}{\bar{z}}}$

$= \frac{1}{2\pi}\int_0^{2\pi}\frac{f(Re^{i\varphi})}{Re^{i\varphi}-z}Re^{i\varphi}dz - \frac{1}{2\pi}\int_0^{2\pi}\frac{f(Re^{i\varphi})}{Re^{i\varphi}-\frac{R^2}{\bar{z}}}Re^{i\varphi}dz$

$= f(z) - 0 = f(z).$

(c) $$\mathrm{Re}\left(\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}\right) = \mathrm{Re}\left(\frac{R\cos\gamma + r + iR\sin\gamma}{R\cos\gamma - r + iR\sin\gamma}\right)$$

$$= \mathrm{Re}\left(\frac{(R\cos\gamma + r + iR\sin\gamma)(R\cos\gamma - r - iR\sin\gamma)}{[R\cos\gamma - r]^2 + R^2\sin^2\gamma}\right)$$

$$= \mathrm{Re}\left(\frac{R^2\cos^2\gamma - r^2 + R^2\sin^2\gamma}{R^2 - 2Rr\cos\gamma + r^2}\right)$$

$$= \frac{R^2 - r^2}{R^2 - 2Rr\cos\gamma + r^2} \quad 得证.$$

15, Problem 1(a).

12. u 实值函数 定义于单位圆 D 上 假设 u 二次连续可微 $\Delta u(x,y) = 0$ 对 $(x,y) \in D$ 都成立.

(a) 试证存在全纯函数 f 在单位圆上使 $\mathrm{Re}(f) = u$.

假设 $P(x,y) = -\frac{\partial u}{\partial y}$, $Q(x,y) = \frac{\partial u}{\partial x}$

故 $\frac{\partial P}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial x^2}$ $\quad \Delta u(x,y) = 0 \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

根 $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

故在单位圆 D 中令 $v(x,y)$ 满足 $dv = Pdx + Qdy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy.$

故 $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ 满足柯西-黎曼等式

$\Rightarrow f(z) = u(x,y) + iv(x,y)$ $\quad f(z)$ 为全纯的 且 $\mathrm{Re}(f) = u(x,y)$

v 由 u 决定 若 c 为常数, 有 $d(v+c) = dv$.

(b) 由 11 题知有: $u(z) + iv(z) = \frac{1}{2\pi}\int_0^{2\pi} [u(e^{i\varphi}) + iv(e^{i\varphi})]\,\mathrm{Re}\left(\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\right)d\varphi.$

故 $$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi})\,\mathrm{Re}\left(\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\right)d\varphi.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi})\,\mathrm{Re}\left(\frac{e^{i(\varphi-\theta)}+r}{e^{i(\varphi-\theta)}-r}\right)d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi})\,\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\varphi)\,u(e^{i\varphi})\,d\varphi.$$

15. 假设 f 为不消失 连续函数在 $\bar{D}$ 上 且在 D 上全纯.

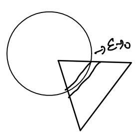
试证 若 $|z| = 1$ 时 $|f(z)| = 1$ 则 f 为常数.

证明: 定义 $$F(z) = \begin{cases} f(z) & |z| < 1 \\ \dfrac{1}{\overline{f(\frac{1}{\bar z})}} & |z| > 1. \\ f(z) = \dfrac{1}{\overline{f(\frac{1}{\bar z})}}, & |z| = 1 \end{cases}$$

当 $|z| = 1$, $z = \cos\theta + i\sin\theta$

$f(\frac{1}{\bar z}) = f\left(\frac{1}{\cos\theta - i\sin\theta}\right) = f(\cos\theta + i\sin\theta) = f(z)$

故 $F(z)$ 在 C 上连续.



考虑三角形与圆相交情况 $|z| = 1$

令 $\varepsilon \to 0$ 使其分为内部与外部.

由 Morera's theorem 可证 $F(z)$ 全纯

1(a) f 定义在单位圆 D 上. $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ 对 $|z| < 1$. 收敛半径为 1.

$\theta = \frac{2\pi p}{2^k}$ $\quad p, k$ 正整数 $\quad z = re^{i\theta}$ $\quad |f(z)| \to \infty$ 当 $r \to 1$

$$\left|\sum_{n=0}^{\infty} r^{2^n} e^{i2\pi p 2^{n-k}}\right| = \left|\sum_{n=0}^{k} e^{\frac{i2\pi p}{2^{k-n}}} + \sum_{n=k+1}^{\infty} e^{i2\pi p 2^{n-k}}\right|$$

对 $r \in [0,1)$ 有 $$\left|\sum_{n=0}^{\infty} r^{2^n} e^{i2\pi p 2^{n-k}}\right| \geq \underbrace{\left|\sum_{n=k+1}^{\infty} r^{2^n} e^{i2\pi p 2^{n-k}}\right|}_{r^{2^n}} - \left|\sum_{n=0}^{k} r^{2^n} e^{\frac{i2\pi p}{2^{k-n}}}\right|.$$

$$\Rightarrow \sum_{n=k+1}^{\infty} r^{2^n} - \sum_{n=0}^{k} r^{2^n} \geq \sum_{n=k+1}^{\infty} r^{2^n} - (k+1)$$

若右边极限为 ∞ 则可证. 令 $N \in \mathbb{N}$, $N > k+1$. 则:

$$\sum_{n=k+1}^{\infty} r^{2^n} > \sum_{n=k+1}^{N} r^{2^n} \Rightarrow \lim_{r \to 1} \sum_{n=k+1}^{\infty} r^{2^n} \geq N - k.$$

由于 N 任意大. 上面可以取到 ∞, 得证

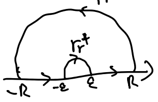
做题

2024年4月17日 星期三 11:09

$\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$.

取 $f(z) = \frac{e^{iz}}{z}$ 则 $\underline{\int_0^\infty \frac{\sin x}{x}dx = \frac{1}{2i}\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx}$.

γ_R^+ γ_r^+



$\int_{\gamma_R^+} f(z)\,dz + \int_{\gamma_r^+}\frac{e^{ix}}{x}dx + \int_{-R}^{-\varepsilon} f(z)\,dz$

$+\int_{\varepsilon}^{R}\frac{e^{ix}}{x}dx = 0$

$iRe^{i\theta} = iR(\cos\theta + i\sin\theta)$.

$= iR\cos\theta - R\sin\theta$.

$f(z) = \frac{e^{iz}}{z}$ $z = Re^{i\theta}$.

$\int_0^\pi \frac{e^{iRe^{i\theta}}}{Re^{i\theta}}\cdot iRe^{i\theta}d\theta = i\int_0^\pi e^{iRe^{i\theta}}d\theta$

$= i\int_0^\pi e^{iR\cos\theta}\cdot e^{-R\sin\theta}d\theta$

$\le \int_0^\pi |e^{-R\sin\theta}\cdot e^{iR\cos\theta}|d\theta = \int_0^\pi e^{-R\sin\theta}d\theta$

$= 2\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}d\theta$.

$\int_\pi^0 \frac{e^{i\varepsilon e^{i\theta}}}{\cancel{\varepsilon e^{i\theta}}}\cdot i\cancel{\varepsilon e^{i\theta}}d\theta =$

$\sin\theta \ge \frac{\theta}{\frac{\pi}{2}}$

$R\to\infty$时

$\le 2\int_0^{\frac{\pi}{2}} e^{-\frac{2R\theta}{\pi}}d\theta$.

$w=0$ $\int_{-\infty}^{\infty} e^{-\pi x^2}dx = 1$

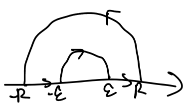
$= -\frac{\pi e^{-\frac{2R\theta}{\pi}}}{R}\Big|_0^{\frac{\pi}{2}} = \frac{\pi(1-e^{-R})}{R} \to 0$

$\int_{-\infty}^{\infty} e^{-\pi x^2}e^{2\pi i x w}dx = e^{-\pi w^2}$.

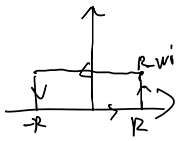
$e^{-\pi(x-wi)^2}$

$= e^{-\pi(x^2-2xwi-w^2)}$

$= e^{-\pi x^2}\cdot e^{2\pi x w i}\cdot e^{\pi w^2}$



$\Leftrightarrow \int_{-\infty}^{\infty} e^{-\pi(x-wi)^2}dx = e^{\pi w^2}\cdot e^{-\pi w^2} = e^0 = 1$.



即证 $\int_{-\infty}^{\infty} e^{-\pi(x-wi)^2}d(x-wi) = 1$.

$z = x - wi$ $f(z) = e^{-\pi z^2}$.

$\Leftrightarrow \int_{-\infty}^{\infty} e^{-\pi z^2}dz = 1$

$\int_{-R}^{R} e^{-\pi z^2}dz + \underbrace{\int_0^{w} e^{-\pi(R+yi)^2}i\,dy}_{\downarrow} + \int_R^{-R} e^{-\pi(x-wi)^2}dx + \underbrace{\int_{-w}^{0} e^{-\pi(yi-R)^2}i\,dy}_{\downarrow} = 0$

$\Rightarrow \int_{-R}^{R} e^{-\pi z^2}dz$

$= \int_{-R}^{R} e^{-\pi(x-wi)^2}dx$.

$\int_0^{-w} e^{-\pi R^2 - 2\pi R y i + \pi y^2}i\,dy$

$R\to\infty$

$\le \int_0^{-w}\left|e^{\pi(y^2-R^2)}\right|dy$

$z = x+yi$

$u(x,y)+v(x,y)i$

$\frac{\partial u}{\partial x}$ $\frac{\partial u}{\partial y}$ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{\partial v}{\partial x}$ $\frac{\partial v}{\partial y}$ $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

$\Rightarrow \int_{-\infty}^{\infty} e^{-\pi(x-wi)^2}dx$

$= \int_{-\infty}^{\infty} e^{-\pi z^2}dz$.

$\le \int_0^{-w}\frac{e^{\pi y^2}}{e^{R^2}}dy$

$= 0 \quad (R\to\infty)$.

$= 1$

$a^2 = v$ $2a\,da = dv$

$da = \frac{dv}{2a}$

$z = a+bi$

$\int_{-\infty}^{\infty} e^{-\pi(a+bi)^2}d(a+bi)$.

即证对复数 a. $\int_{-\infty}^{\infty} e^{-\pi a^2}da = 1$.

$= \int_{-\infty}^{\infty}$

$\int_{-\infty}^{\infty} e^{-a^2}da = 1$

$= \int_{-\infty}$

$\int_{-\infty}^{+\infty} e^{-a^2} da = 1$

$\frac{d}{da} \int_{-\infty}^{+\infty} \frac{e^{-a^2}}{2a} da^2 = 1$

n: ≥ 2 的整数.

$-\pi x^2 = u \quad du = -2\pi x dx \quad d^2u = -2\pi dx^2$

$\int_{-\infty}^{\infty} e^u d$

$n \geq 2$ 整数, α 为实数 $n > 1+\alpha > 0$.

$\int_0^{\infty} \frac{x^\alpha}{1+x^n} dx. \quad f(z) = \frac{z^\alpha}{1+z^n}.$

$z^8 - 7z^3 + 2z + 1 = 0 \quad w, \cdots$

$$\frac{\frac{(z+h)^\alpha}{1+(z+h)^n} - \frac{z^\alpha}{1+z^n}}{h}$$

$i\varepsilon(\cos\theta + i\sin\theta)$

$\varepsilon_\pi^{i\theta}$

$z=$

$\int_0^{\infty} \frac{1-\cos x}{x^2} dx = \frac{\pi}{2} \quad f(z) = \frac{1-e^{iz}}{z^2}$

$2\int_\varepsilon^R \frac{1-e^{iz}}{z^2} dz + \int_{\gamma_\varepsilon} f(z)dz + \int_{\gamma_R} f(z)dz = 0.$

$\int_\pi^0 \frac{1-e^{i\varepsilon\pi^{\theta i}}}{\varepsilon^2 \pi^{2\theta i}} \cdot i \, d\theta$

$= i\int_\pi^0 \frac{1-e^{i\varepsilon\pi^{\theta i}}}{\varepsilon\pi^{\theta i}} d\theta.$

$2^7 \times 8 + 2 -$

$\le i\int \frac{2}{\varepsilon\pi^{\theta i}}d\theta.$

$f(z)=z^8-7z^3+2z+1$

$z^8-7z^3+2z+1=0$

$f'(z)=8z^7-21z^2+2$

$f''(z)=56z^6-42z\ge 0.$

$f'(z)$

$z=r$

z实：~~z~~ $f(1)=-$

$f(z)=z^8-7x8+24+1$

∴一线根

$=$

$z=re^{i\theta}$ $r\in(1,2)$ $|z|=r\in(1,2)$

$z^4-7\bar{z}+2(\bar{z})^3+(\bar{z})^8=0.$

a^2b a^3-2ab

$1\ a$

$1-a\ b)\ 1\ 0\ 0\ 0\ 0\ 7\ 0\ 2\ 1$

$1-ab$

$a-b$

$a-a^2\ ab$

$a^2b\ -ab$

$a^2b\ ab\ a^3\ a^2b-b^2$

$a^3-2ab\ \ b^2-a^2b\ \ 7.$

a^3-2ab

第四次作业　12211834 沈博涵

2024年5月14日 星期二　20:47

Ch3　9, 10, 15, 17, 19, 22.

9. 证明：$\int_0^1 \log(\sin\pi x)\,dx = -\log 2$

证明：$\int_0^1 \log(\sin\pi x)\,dx = \int_{-\frac12}^{\frac12} \log(\cos\pi x)\,dx$

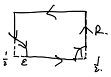
$= \int_{-\frac12}^{\frac12} \log 2\cos\pi x\,dx - \log 2 = \int_{-\frac12}^{\frac12} \log(e^{\pi i z} + e^{-\pi i z})\,dz - \log 2.$

∴ 即证：$\int_{-\frac12}^{\frac12} \log(e^{\pi i z} + e^{-\pi i z})\,dz = 0$

$\log(e^{\pi i z} + e^{-\pi i z}) = \log\left(\frac{1+e^{2\pi i z}}{e^{\pi i z}}\right) = \log(1+e^{2\pi i z}) - \pi i z.$

$\because \int_{-\frac12}^{\frac12}(-\pi i z)\,dz = 0$ ∴ 证 $\int_{-\frac12}^{\frac12} \log(1+e^{2\pi i z})\,dz = 0$ 即可.

有极点于 $z = \pm\frac12$. 又 $1+e^{2\pi i z}$ 周期为1. 故左右部分相反.



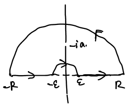
上：$\int_{\frac12}^{-\frac12} \log(1+e^{2\pi i(r+iR)})\,dr$

$= \int_{\frac12}^{-\frac12} \log(1+e^{-2\pi R} e^{2\pi i r})\,dr \to 0$ 当 $R\to\infty$ 时.

四分之一圆：$\left|\int_0^{\frac{\pi}{2}} \log(1+e^{2\pi i(\frac12+\varepsilon e^{i\theta})})\, i\varepsilon e^{i\theta} d\theta\right| \le \varepsilon\left|\int_0^{\frac{\pi}{2}} \log(1+e^{-2\pi\varepsilon\sin\theta})\,d\theta\right| \to 0$

当 $\varepsilon\to 0$ 时

故 $\int_{-\frac12}^{\frac12} \log(1+e^{2\pi i z})\,dz = 0.$



10. 求证：若 $a>0$ 则 $\int_0^{+\infty} \frac{\log x}{x^2+a^2}\,dx = \frac{\pi}{2a}\log a.$

$x = a\tan\theta$　$x^2+a^2 = a^2(\tan^2\theta+1) = a^2\cdot\frac{1}{\cos^2\theta}$

$\log x = \log a + \log\tan\theta$.　$\because x\in(0,+\infty),\ a>0 \Rightarrow \theta\in(0,\frac{\pi}{2})$

$LHS = \int_0^{\frac{\pi}{2}} \frac{\log a\tan\theta}{\frac{a^2}{\cos^2\theta}}\, d\,a\tan\theta$　a为常数.

$= \int_0^{\frac{\pi}{2}} \frac{\log a\tan\theta}{a^2}\cdot \cancel{\cos^2\theta}\cdot\frac{1}{\cancel{\cos^2\theta}}\,da\theta = \int_0^{\frac{\pi}{2}} \frac{\log a + \log\tan\theta}{a}\,d\theta.$

$= \left.\frac{\log a\cdot\theta}{a}\right|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\log\tan\theta}{a}\,d\theta = \frac{\pi}{2}\cdot\frac{\log a}{a} + \int_0^{\frac{\pi}{2}}\frac{\log\tan\theta}{a}\,d\theta.$

$= \frac{\pi}{2a}\cdot\log a + \frac{1}{a}\cdot\int_0^{\frac{\pi}{2}}\log\tan\theta\,d\theta.$

故原命题 $\Leftrightarrow \int_0^{\frac{\pi}{2}} \log\tan\theta\,d\theta = 0 \Leftrightarrow \int_0^{\frac{\pi}{2}}\log\sin\theta\,d\theta = \int_0^{\frac{\pi}{2}}\log\cos\theta\,d\theta$

$\Leftrightarrow \int_0^{\frac{\pi}{2}}\log\sin\theta\,d\theta = \int_0^{\frac{\pi}{2}}\log\sin(\theta+\frac{\pi}{2})\,d\theta = \int_{\frac{\pi}{2}}^{\pi}\log\sin\theta\,d\theta.$

由于 $\sin\theta$ 在 $\theta\in[0,\pi]$ 上关于 $\frac{\pi}{2}$ 对称. 故左式=右式

即 $\int_0^{\frac{\pi}{2}}\log\tan\theta\,d\theta = 0$

15. 求证：(a) 若 f 为整函数，满足 $\sup_{|z|=R}|f(z)| \le AR^k + B$. 对 all $R>0$ 且对某些整数 $k\ge 0$. 常数 $A, B>0$. f 为 deg ≤ k 的多项式

由 Cauchy inequalities, $|f^{(n)}(0)| \le \frac{n!(AR^k+B)}{R^n}$　若 $n>k$, 令 $R\to\infty$, $f^{(n)}(z) = 0$.

故 f 为 degree ≤ k 的多项式

(b) 若函数 f 为定义在单位圆盘内的全纯函数，且函数有界，存在的区域 $\theta < \arg z < \varphi$ 内当 $|z|\to 1$ 时一致收敛于0，则 $f = 0$.

证明：假设 $\theta = 0$　f 一致收敛至0. $\Leftrightarrow \forall\varepsilon>0, \exists\delta>0$ 使 $\forall z$, $w\in\partial D$,

$|z-w|<\delta$, 有 $|f(z)|<\varepsilon$.

对 $\frac{\varphi}{2}$, 有 $N\in\mathbb{N}$ 使 $N\frac{\varphi}{2} > 2\pi$　覆盖整个 ∂D.

考虑 $F(z) = f(z) f(e^{-\frac{i\varphi}{2}}z)\cdots f(ze^{-\frac{iN\varphi}{2}})$ 在 D 上有界.

$\forall w\in\partial D$, $\exists k$ 使 $k\frac{\varphi}{2} < \arg(w) < \frac{(k+1)\varphi}{2}$　故任意 $\varepsilon>0, \exists\delta>0$ 使 $|z - e^{-\frac{ik\varphi}{2}}w| < \delta$

有即 $|f(z)|>\varepsilon$. 且 $|f(e^{-\frac{ij\varphi}{2}}w)| < M$ 有界 由于 f 有界 对 $j\ne k$ 成立

故 $|F(z)| = \prod_{j=0}^{N}|f(e^{-\frac{ij\varphi}{2}}z)| < M^N\varepsilon.$

故可以定义 $F(z)$ 为0在 ∂D 上. 由最大模原理，有 $F\equiv 0$.

若 $f\ne 0$，则 f 有可数零点，故 $F(z)$ 有可数零点，与 $F\equiv 0$ 矛盾.

故 $f\equiv 0$.

(c) 定义 $F(z) = \prod_{i=1}^{n}(z-w_i)$, $|F(0)|=1$, $|F(w_j)| = 0$.　$\forall j\in\{1,2,\cdots,n\}$.

由 最大模定理 $\exists z\in\partial D$ 使 $|F(z)|\ge 1$

由于 $|F(e^{i\theta})|$ 为连续的，$\exists z\in\partial D$ 使 $|F(z)| = 1$.

(d) $|e^f| = e^{\mathrm{Re} f}$ 有界　设 $\mathrm{Re}(f)\le M$　令 $g(z) = e^{f(z)} \Rightarrow g$ 为整函数.

$|g| = e^{\mathrm{Re}(f)} < e^M$　由 Liouville's theorem, $g(z) = C$ 对常数 C 成立.

$\Rightarrow f(z) = \log C$,　e^f 为整函数且连续 $\Rightarrow f$ 为常数.

17. (a) 由 Rouché's theorem, $f(z)$ 与 $f(z)-w_0$ 在单位圆中有相同个数的零点 $(|w_0|<1)$.

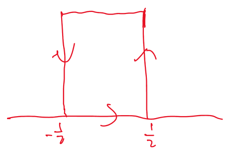
$\Rightarrow$ 若 f 有零点则其像包含单位圆. 若 f 非零，则 $\frac{1}{f}$ 全纯. 则 $\left|\frac{1}{f(z)}\right|\le 1$ 对 $z\in\bar{D}$ 成立（由最大模定理知）. 但当 $|f(z)|\ge 1$ 对 $z\in\bar D$，取 $\forall z, |z|=1$.

$\Rightarrow f(\bar D)$ 包含 $f(z)$ 邻域，即有点 w, $|w|<1$，与开映射定理矛盾.

故 $f(z) = w_0$ 对 $\forall w\in D$ 有一个根.

(b) 令 $w_0 = f(z_0)$, $|z_0|<1$, $|w_0|<1$. 由 Rouché's theorem, $f(z)$ 与 $f(z)-w$ 有相同个数的零点. 对 $\forall|w|<1$ 成立. 由于存在 w_0 使 $f(z)-w_0=0$ 有一个根.

9. $\int_0^1 \ln(\sin\pi x)\,dx = -\log 2.$



$\log re^{i\theta} = \log r + i\theta.$

$\sin\pi x = \cos\pi(x-\frac12)$.

$e^{ix} = \cos x + i\sin x$

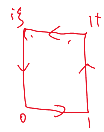
$\cos x = \frac{e^{ix}+e^{ix}}{2}$

$\cos\pi x = \frac{e^{i\pi x}+e^{-i\pi x}}{2}$

$\int_{-\frac12}^{\frac12} \ln(\cos\pi x) = -\ln 2.$

$\int_0^1 \ln\left(\frac{e^{i\pi x}-e^{-i\pi x}}{2}\right)dx = \int_0^1\left(\ln(e^{i\pi x}-e^{-i\pi x}) - \ln 2\right)dx$

$= \int_0^1 \ln\frac{e^{2i\pi x}-1}{e^{i\pi x}}\,dx - \ln 2 = -\ln 2.$



$\int_0^1 \ln(e^{i\pi x}-e^{-i\pi x})\,dx = 0.$

$\Leftrightarrow \int_0^1 \ln(2e^{i\pi x}-1)\,dx = 0$

下：$\int_0^1 \ln(2e^{i\pi x}-1)\,dx$

$\int_1^0 \ln(2e^{i\pi(x+i\xi)}-1)\,dx$

$= \int_1^0 \ln(2e^{i\pi x-\pi\xi}-1)\,dx.$

$= \int_1^0 \ln\left(\frac{2e^{i\pi x}}{e^{\pi\xi}}-1\right)dx$　$\xi\to+\infty$ 时

故，$f(z)=w_0$ 对 $\forall w\in D$ 有一个解。

由令 $w_0=f(z_0)$，$|z_0|<1$，$|w_0|<1$。由 Rouché's theorem，$f(z)$ 与 $f(z)-w$ 有相同个数的零点。对 $\forall |w|<1$ 成立。由于存在 w_0 使 $f(z)-w_0=0$ 有一个根。

w 任取时都有一个根，即 f 的像包含 D。

19. (a) 假设 u 在 z_0 处取得 max，$z_0\in\Omega$。令 f 为一全纯函数于 z_0 邻域上。

$u=\mathrm{Re}(f)$　$f(z_0)=u(z_0)+iv(z_0)$　u 不全为常数，故 f 不为常数。

由开映射定理，f 从 $U_\delta(z_0)\subset\Omega$ 映射到一个开集中。设取 δ' 使 $U_\delta(f(z_0))\subset f(U_{\delta'}(z_0))$。

对 $\forall z\in U_\delta(z_0)$，$u(z)\le u(z_0)$，故 $\forall\delta'>0$，$u(z_0)+\frac{\delta'}{2}+iv(z_0)\in U_{\delta'}(f(z_0))$ 且

$u(z_0)+\frac{\delta'}{2}+iv(z_0)\notin f(U_{\delta'}(z_0))$。即有 u 在 Ω 上无法取得 max。

同理 u 无法取得 min

(b) u 在 $\bar\Omega$ 上连续，$\bar\Omega$ 为紧集。则可以在 $\bar\Omega$ 上取得 max。由 (a) 知则无法在 Ω 上取得 max。

否则 u 为常数。即 $\sup_{z\in\Omega}|u(z)|\le\sup_{z\in\bar\Omega-\Omega}|u(z)|$。

22. 试证：不存在全纯函数 f 定义于单位圆盘 D 上，且在 D 边界 ∂D 上连续使对 $z\in\partial D$ 满足 $f(z)=\frac{1}{z}$

证明：由于 f 在紧集 ∂D 上连续，故一致连续。

$\forall\varepsilon>0$，$\exists\delta>0$，使 $|z_1-z_2|<\delta$，$|f(z_1)-f(z_2)|<\varepsilon$。

故 $\left|\oint_{\partial D}f(z)-f((1-\delta)z)dz\right|\le\left|\oint_{\partial D}\varepsilon dz\right|=2\pi\varepsilon$。

则 $\left|\oint_{\partial D}f(z)dz-\oint_{C_{1-\delta}}f(z)dz\right|\le 2\pi\varepsilon$。

$\because$ $C_{1-\delta}$ 包含于 D 中。$\oint_{\partial D}f(z)dz=0$　而 $\oint_{|z|=1}\frac{1}{z}dz=2\pi i$ 故矛盾

故 $f(z)=\frac{1}{z}$ 不存在。

第8章作业 2,4,5,8,10,11,12,14.

2024年6月12日 星期三 17:18

2. $f(z)$在$z=z_0$附近全纯且 $F(z_0)=F'(z_0)=0$, $F''(z_0)\neq 0$ 在 z_0 下不为单射.
若$F(z)$为单射且全纯 则 $F'(z_0)\neq 0$. 又 $F'(z_0)\neq 0$

4. 对 $f(z)=-\frac{1}{2}(z+\frac{1}{z})$ 单位圆盘 $D=\{z=u+iv, |z|\le 1\}$. ($u^2+v^2\le 1$) $\frac{1}{u+iv}=\frac{u-iv}{u^2+v^2}$

$g(z)=f'(z)=\frac{1}{2}(z^2+\frac{1}{z^2})=\frac{1}{2}(z+\frac{1}{z})^2$ $u=r\cos\theta, v=r\sin\theta$, ($r\le 1$) $\theta\in[0,2\pi]$.

$u+iv+\frac{u-iv}{u^2+v^2}$
$=\frac{u^3+uv^2+uv^2i+v^3i+u-iv}{u^2+v^2}$
$=\frac{u^3+uv^2+u+i(u^2v+v^3-v)}{u^2+v^2}$
$\frac{1}{\cos\theta+i\sin\theta}=\cos\theta-i\sin\theta$.

$r\cos\theta+ir\sin\theta+\frac{1}{r\cos\theta+ir\sin\theta}$
$=r(\cos\theta+i\sin\theta)+\frac{1}{r}(\cos\theta-i\sin\theta)$
$=(r+\frac{1}{r})\cos\theta+(r-\frac{1}{r})i\sin\theta=\sqrt{(r+\frac{1}{r})^2+(r-\frac{1}{r})^2}$
$=\frac{1}{2}\sqrt{2r^2+\frac{2}{r^2}}$

5. 证明: $f(z)=-\frac{1}{2}(z+\frac{1}{z})$ 将 $\{z=x+iy; |z|<1, y>0\}\to\{z=x+iy, x\in\mathbb{R}, y>0\}$.

$f(z)=-\frac{1}{2}(x+iy+\frac{x-iy}{x^2+y^2})=(-\frac{1}{2}x-\frac{x}{2(x^2+y^2)})-\frac{1}{2}i(y-\frac{y}{x^2+y^2})$.

$\mathrm{Re}f(z)=-\frac{1}{2}(x+\frac{x}{x^2+y^2})$. $x=y=0^+$ 取 $-\infty$ $x=y=0^-$ 取 $+\infty$.

$\mathrm{Im}f(z)=\frac{y}{x^2+y^2}-y=(\frac{1}{x^2+y^2}-1)y$ $y\to 0, x\to 1$ 时取 0.

$-\frac{1}{2}(z+\frac{1}{z})=w\Rightarrow z^2+2wz+1=0$ 的解 $\mathrm{Re}(w)=\mathrm{Re}(-\frac{1}{2}z)+\mathrm{Re}(-\frac{1}{2z})$

证共形映射: 双射且全纯.
证满射: $\mathrm{Im}(w)=(\frac{1}{|z|^2}-1)\mathrm{Im}(z)$ $|z|<1\Rightarrow\frac{1}{|z|^2}-1>0\Rightarrow \mathrm{Im}(w)>0$.
单射: 若 z_1,z_2 有 $f(z_1)=f(z_2)=w\Rightarrow z_1,z_2$ 为 $z^2+2wz+1=0$ 的解
$\Rightarrow z_1z_2=1$ $|z_1||z_2|=1\Rightarrow$ 必有一个 ≥ 1.

8. $F_1=\frac{z-1}{z+1}$ 将第一象限 → 上半圆盘. $\frac{z-1}{z+1}$ 将上半圆盘 → 第一象限.
$F_2=\log z$
$F_3=z-\frac{\pi}{2}$ (先转 $\frac{\pi}{2}$ 再平移 $\frac{\pi}{2}$)
$F_4=\frac{1+\sin z}{2}$ $F_5=z-1$

10. $F: H\to C$ 为全纯函数. 满足 $|F(z)|\le 1$, $F(i)=0$. 证明 $|F(z)|\le\left|\frac{z-i}{z+i}\right|$.
定义 $G: D\to D$ $G(w)=F(i\frac{1-w}{1+w})$ G为全纯, $G(0)=F(i)=0$.
由Schwarz lemma $|G(w)|\le|w|$ 对 $w\in D$
$|F(z)|=\left|G(\frac{i-z}{i+z})\right|\le\left|\frac{z-i}{z+i}\right|$

$G(w)\le|w|$.
$w=\frac{z-i}{z+i}\Rightarrow wz+wi=z-i$
$(w-1)z=-(w+1)i$
$z=i\frac{w+1}{w-1}$

11. 证明: 若 $f: D(0,R)\to C$ 为全纯的 且存在某个 $M>0$ 使 $|f(z)|\le M$
则 $\left|\frac{f(z)-f(0)}{M^2-\overline{f(0)}f(z)}\right|\le\frac{|z|}{MR}$.

对$z\in D$, 令 $g(z)=\frac{f(Rz)}{M}$ $Rz\in D(0,R)$, $|f(Rz)|\le M$ 故 $g(z)\in D$.
易 $g: D\to D$ 且为全纯的. 令 $\alpha=g(0)=\frac{f(0)}{M}$ $\psi_\alpha\circ g: D\to D$ 满足 $\psi_\alpha(g(0))=0$.
$\psi_\alpha(w)=\frac{\alpha-w}{1-\bar\alpha w}$ 将圆盘映到自身的映射 ($w=0$时)
对 $\psi_\alpha\circ g$ 用 Schwarz lemma. $|\psi_\alpha\circ g(t)|\le|t|\Rightarrow\left|\frac{\alpha-g(t)}{1-\bar\alpha g(t)}\right|\le|t|\Rightarrow\left|\frac{\frac{f(0)}{M}-\frac{f(Rt)}{M}}{1-\frac{\overline{f(0)}}{M}\frac{f(Rt)}{M}}\right|\le|t|$.
$\Rightarrow M\left|\frac{f(0)-f(Rt)}{M^2-\overline{f(0)}f(Rt)}\right|\le|t|\Rightarrow\left|\frac{f(0)-f(z)}{M^2-\overline{f(0)}f(z)}\right|\le\frac{|z|}{MR}$ $(z=Rt)\in D(0,R)$.

$g(z)=\frac{f(Rz)}{M}$ $z\in D(0,1)$
$|g(z)|\le 1$
$\Rightarrow g(z)$ 为 $D\to D$ 上的自同构.
有 $|g(z)|\le|z|$ $\psi_\alpha=\frac{\alpha-z}{1-\bar\alpha z}$ α为D中一点.
令 $\alpha=\frac{f(0)}{M}$
$\bar\alpha=\frac{\overline{f(0)}}{M}$
$\frac{\frac{f(0)}{M}-g(z)}{1-\bar\alpha g(z)}=\frac{\frac{f(0)}{M}-\frac{f(Rz)}{M}}{1-\frac{f(Rz)\cdot\overline{f(0)}}{M^2}}$

12. 映射 $f: D\to D$ 及点 $w\in D$ 满足 $f(w)=w$ 称w为一个固定点.
(a) 证明: 若 $f: D\to D$ 解析 且有两个不同固定点, 则f为单位元 吗
对$\forall z\in D$ 有 $f(z)=z$.
Proof: 可以先通过伸缩及平移使其中一个固定点到 $z=0$处 $f(z)=0$.
又由Schwarz $\exists z_0\neq 0$ $f(z_0)=z_0$ 则f为旋转, 可设为 $f(z)=e^{i\theta}z=z\Rightarrow\theta=0$
即 $f(z)=z$.
(b) $z\mapsto z+1$ 无固定点, 但是全纯.

$w=re^{i\theta}$ $r\le 1$, $\theta\in[0,2\pi)$.
$z=i\frac{1-w}{1+w}$
$=i\frac{}{1+w}$

14. 证明: 所有从上半平面$H\to$单位圆盘D上共形映射 形如 $e^{i\theta}\frac{z-\beta}{z-\bar\beta}$.
令 $g: H\to D$ 为共形映射. $\phi: D\to H$ 为 $z=\phi(w)=i\frac{1-w}{1+w}$.
ϕ为共形映射, 逆为 $w=\psi(z)=\frac{i-z}{i+z}$. $g\circ\phi: D\to D$ 为圆盘上的自同构.
故$\exists\mu\in\mathbb{R}$, $\alpha\in D$ 使 $g(z)=g(i\frac{1-w}{1+w})=e^{i\mu}\frac{\alpha-w}{1-\bar\alpha w}=e^{i\mu}\frac{\alpha-\frac{i-z}{i+z}}{1-\bar\alpha\frac{i-z}{i+z}}$
$=e^{i\mu}\frac{z(1+\alpha)-i(1-\alpha)}{z(1+\bar\alpha)-i(1-\bar\alpha)}=e^{i\mu}\frac{z-i\frac{1-\alpha}{1+\alpha}}{z\frac{1+\bar\alpha}{1+\alpha}+i\frac{1-\bar\alpha}{1+\alpha}}=e^{i\mu}\frac{z-\beta}{e^{ir}(z-\bar\beta)}$

$D\to H$ $z=i$
$H\to D$ $w=$
圆盘上自同

$\frac{1+\bar\alpha}{1+\alpha}=\frac{1+e^{i\theta}}{1+e^{-i\theta}}$

$|z| < R t \Rightarrow z = \frac{t}{R}$.

$\cancel{z=Rt}$ $\cancel{|z| = Rt}$.

$= \frac{M(f(0)-f(Rz))}{M^2-\overline{f(0)}\cdot f(Rz)} \le |z|$.

$\Rightarrow \frac{M(f(0)-f(t))}{M^2-\overline{f(0)}\cdot f(t)} \le \left|\frac{t}{RM}\right|$

$\frac{1-w}{1+w}$

$\frac{i-z}{i+z}$

和 $e^{i\theta}\cdot\boxed{\frac{\alpha-z}{1-\bar{\alpha}z}}$. 映射.

旋转.

故对 $\mu \in R$, ...

$$= e^{i\mu} \frac{z(1+\alpha) - i(1-\alpha)}{z(1+\alpha) - i(1-\overline{\alpha})} = e^{i\mu} \frac{z - i\frac{1-\alpha}{1+\alpha}}{z\frac{1+\overline{\alpha}}{1+\alpha} + i\frac{1-\overline{\alpha}}{1+\alpha}} = e^{i\mu} \frac{z-\beta}{e^{i\gamma}(z-\overline{\beta})}$$

$$= e^{i\theta} \frac{z-\beta}{z-\overline{\beta}} \quad \theta = \mu - \gamma \in R \quad e^{i\gamma} = \frac{1+\overline{\alpha}}{1+\alpha} \quad \beta = i\frac{1-\alpha}{1+\alpha} \in H \quad \therefore \beta = \phi(\alpha),\ \alpha \in D.$$

$D \to H$: $z = \varphi(w) = i\frac{1-w}{1+w}$ 逆为 $w = g(z) = \frac{i-z}{i+z}$.

对 $\mu \in R$, $\alpha \in D$ $\quad g(z) = g\left(i\frac{1-w}{1+w}\right) = e^{i\mu}\frac{\alpha - w}{1-\overline{\alpha}w} = e^{i\mu}\frac{\alpha - \frac{i-z}{i+z}}{1-\overline{\alpha}\cdot\frac{i-z}{i+z}}$

$$= e^{i\mu} \cdot \frac{(i+z)\alpha - (i-z)}{(i+z) - (i-z)\overline{\alpha}} = e^{i\mu} \cdot \frac{(\alpha+1)z + (\alpha-1)i}{(\overline{\alpha}+1)z + (\overline{\alpha}-1)i}$$

$$= e^{i\mu} \cdot \frac{z + \frac{\alpha-1}{\alpha+1}i}{\frac{\overline{\alpha}+1}{\alpha+1}z + \frac{\overline{\alpha}-1}{\alpha+1}i}$$

$$\frac{1+\overline{\alpha}}{1+\alpha} = \frac{1+e^{-i\theta}}{1+e^{i\theta}}$$

$=$

上半平面

对 $f(0) = 0$

$|f(z)| \le |$

$|f'(0)| \le$

旋转

$\frac{a\cancel{z}+b}{cz+d}$ $ad-bc=1$

, $D \to D$.

$|z|$ 若 $z_0 \neq 0$ $f(z_0)=z_0$ 旋转

$=1$. 旋转

定理总结。

2024年6月13日 星期四 02:43

Ch3 亚纯函数

3.3 可去奇点 z_0 f在 $\Omega-\{z_0\}$ 上有界

极点: f有孤立奇点于 z_0, iff $|f(z)|\to\infty$ 当 $z\to z_0$ 则 z_0 为 pole.

除以上两个之外的为本质奇点 essential singularity

Casorati-Weierstrass

第五次作业 12211834 沈鸿阳

2024年5月29日 星期三 08:38

Chs

4. $t>0$, $F(z)=\prod_{n=1}^{\infty}(1-e^{-2\pi nt}e^{2\pi iz})$.

证: $|F(z)|\le Ae^{a|z|^2}$. F order 2

(a) 对 z, 令 $c>\frac{1}{t}$, 令 $N=[c|z|]$

令 $F_1(z)=\prod_{n=1}^{N}(1-e^{-2\pi nt}e^{2\pi iz})$, $F_2(z)=\prod_{n=N+1}^{\infty}(1-e^{-2\pi nt}e^{2\pi iz})$

有 $e^{-2\pi(N+1)t}e^{2\pi|z|}\le\frac{1}{2}$. (由于 $|z|$ 足够大).

∵ $|\log(1+y)|\le 2|y|$ 对 $|y|\le\frac{1}{2}$ 成立. 有:

$|\log F_2(z)|=\left|\sum_{n=N+1}^{\infty}\log(1-e^{-2\pi nt}e^{2\pi iz})\right|$

$\le\sum_{n=N+1}^{\infty}|\log(1-e^{-2\pi nt}e^{2\pi iz})|\le\sum_{n=N+1}^{\infty}2|e^{-2\pi nt}e^{2\pi iz}|$

$\le 2\sum_{n=N+1}^{\infty}e^{-2\pi nt}e^{2\pi|z|}=2e^{-2\pi(N+1)t+2\pi|z|}\sum_{k=0}^{\infty}e^{-2\pi kt}$

$=C$ 故 $|F_2(z)|$ 有界且上界为常数.

对于 n. $|1-e^{-2\pi nt}e^{2\pi iz}|\le 1+e^{2\pi|z|}\le 2e^{2\pi|z|}$

∵ F_1 为 N 个次的积, 故能找到 c', 使:

$|F_1(z)|\le(2e^{2\pi|z|})^N=2^Ne^{2\pi N|z|}\le 2^{c|z|}e^{2\pi c|z|^2}\le e^{c'|z|^2}$

故 $F=F_1F_2$, F_2 有界. F order at most 2 (∵ F_1 order 最大为2).

(b) 对某些整数 $n\ge 1$ $e^{-2\pi(nt+iz)}=1\Rightarrow z=-int+m$.

m 为整数. 故若 z_n 表示所有零点, 有 $\sum\frac{1}{|z_n|^2}<\infty$

由 Thm 2.1 $\sum\frac{1}{|z_n|^2}=\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^2+t^2n^2}=\frac{\pi^2}{6t^2}+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^2+t^2n^2}$

$\ge 2\sum_{m,n\ge1}\frac{1}{m^2+t^2n^2}\ge 2\int_1^{\infty}\int_1^{\infty}\frac{1}{x^2+t^2y^2}dxdy=\frac{2}{t}\int_1^{\infty}\int_1^{\infty}\frac{1}{x^2+u^2}dxdu$

$=\frac{2}{t}\int_0^{\frac{\pi}{2}}\int_1^{\infty}\frac{1}{r^2}rdrd\theta=\frac{\pi}{t}\int_1^{\infty}\frac{dr}{r}=\infty$ (...)

由于所有零点都刚能, 故明使 ... 成立.

9. 若 $|z|<1$. 证 $\prod_{k=0}^{\infty}(1+z^{2^k})=\frac{1}{1-z}$.

证明: 积为收敛的. 且在紧子闭盘 $|z|\le R<1$ 上一致收敛.

∵ $\sum|z|^{2^k}\le\sum R^{2^k}\le\sum R^k=\frac{1}{1-R}<\infty$.

下面由归纳证明 $\prod_{k=0}^{N}(1+z^{2^k})=\sum_{j=0}^{2^{N+1}-1}z^j$

$N=0$ 时显然成立

$\prod_{k=0}^{N+1}(1+z^{2^k})=(1+z^{2^{N+1}})\prod_{k=0}^{N}(1+z^{2^k})=(1+z^{2^{N+1}})\sum_{j=0}^{2^{N+1}-1}z^j$

$=\sum_{j=0}^{2^{N+1}-1}z^j+z^{2^{N+1}}\sum_{j=0}^{2^{N+1}-1}z^j=\sum_{j=0}^{2^{N+1}-1}z^j+\sum_{j=2^{N+1}}^{2^{N+2}-1}z^j$

$=\sum_{j=0}^{2^{N+2}-1}z^j$ 其收敛于: $\lim_{N\to\infty}\sum_{j=0}^{2^{N+1}-1}z^j=\lim_{n\to\infty}\sum_{j=0}^{n}z^j=\frac{1}{1-z}$

故对原式左右两边取极限:

$\prod_{k=0}^{\infty}(1+z^{2^k})=\lim_{N\to\infty}\prod_{k=0}^{N}(1+z^{2^k})=\lim_{N\to\infty}\sum_{j=0}^{2^{N+1}-1}z^j=\frac{1}{1-z}$.

2.

(a) 注意到 $|z^n|=O(e^{|z|^\varepsilon})$ 对所有的 $\varepsilon>0$. 故 P 的增长阶为0.

(b) e^{bz^n} 增长阶为 n.

(c) e^{e^z} ∵ $|z|^k=O(e^z)$, 可知 设有 k 与常数 A,B

使 $e^{e^{|z|}}\le Ae^{B|z|^k}$ 故其增长阶为无穷.

6. $\sin(\frac{\pi}{2})=\frac{\pi}{2}\prod_{n=1}^{\infty}(1-\frac{1}{4n^2})=\frac{\pi}{2}\cdot\prod_{n=1}^{\infty}\left(\frac{(2n+1)(2n-1)}{2n\cdot 2n}\right)=1$

故 $\frac{\pi}{2}=\prod_{n=1}^{\infty}\frac{2n\cdot 2n}{(2n-1)(2n+1)}=\frac{2\cdot2}{1\cdot3}\cdot\frac{4\cdot4}{3\cdot5}\cdots\frac{2m\cdot2m}{(2m-1)(2m+1)}\cdots$

10. 求下列函数 Hadamard 乘积:

(a) e^z-1.

e^z-1 为1阶且有零点 $z=2\pi in$, $n\in\mathbb{Z}$. 故有:

$e^z-1=e^{Az+B}z\prod_{n\neq0}(1-\frac{z}{2\pi in})e^{\frac{z}{2\pi in}}=e^{Az+B}z\prod_{n\ge1}(1+\frac{z^2}{4\pi^2n^2})$.

左右乘 $e^{-\frac{z}{2}}$ 可知两边都能为奇函数. $\Rightarrow A=\frac{1}{2}$ 考虑 $\lim_{z\to0}\frac{e^z-1}{z}=1$.

可知 $B=0$. 即为 $e^{\frac{z}{2}}\prod_{n\ge1}^{\infty}(1+\frac{z^2}{4n^2\pi^2})$.

(b) $\cos\pi z$ $\frac{e^{i\pi z}+e^{-i\pi z}}{2}$

$\cos\pi z$ 也为1阶. 且有零点于 $\frac{2n+1}{2}$ 对所有的 $n\in\mathbb{Z}$. 故有:

$\cos(\pi z)=e^{Az+B}\prod_{n\in\mathbb{Z}}(1-\frac{2z}{(2n+1)})e^{\frac{2z}{2n+1}}=e^{Az+B}\prod_{n\ge1}(1-\frac{4z^2}{(2n+1)^2})$

cos 偶, 故 $A=0$. 令 $z=0$ 可知 $B=0$ 即为 $\prod_{n\ge0}^{\infty}(1-\frac{4z^2}{(2n+1)^2})$.

11. 据 Hadamard 定理. 故若 f 滴去 a,b 则有 $f(z)-a=e^{p(z)}$, $f(z)-b=e^{q(z)}$ 对多项式 p,q 成立.

设有两数 a,b 不同.

$\Rightarrow e^{p(z)}-e^{q(z)}=C$ 对某常数 C 成立. 令 $z\to\infty\Rightarrow p,q$ 的首项

必须一致. 设作 a_nz^n. 再考虑 z 左边的无穷的极限.

$e^{p(z)-a_nz^n}-e^{q(z)-a_nz^n}=Ce^{-a_nz^n}$

故有下一项也相同. 由归纳. 可知每项相同, 故 $p=q$. 此时 $a=b$

3. $\sum_{n=-\infty}^{+\infty}e^{\pi in^2\tau}e^{2\pi inz}$ 为2阶 $\le Ae^{b|z|^\rho}$ 阶为 $\inf\rho$.

$e^{i\pi(n^2\tau+2nz)}=\cos\pi(n^2\tau+2nz)+i\sin(n^2\tau+2nz)$

增长阶: $\le Ae^{B|z|^\rho}$ 阶为 $\inf\rho$.

$\le\rho$ 阶

性质: (1) 对 $C>0$, 足够大的 r, $n(r)\le C\cdot r^\rho$.

(2) $z_1,z_2\cdots$ 表示零点, $s>\rho$ 时 $\sum\frac{1}{|z_n|^s}<+\infty$.

无穷乘积 复数列 $\{a_n\}_{n=1}^{+\infty}$, $\prod(a_n+1)$ 收敛

~~若 ... 存在 C_n 使 $|a_n-1|\le C_n$ 对~~

对 F_n $(|F_n(z)-1|\le C_n$ 时$)$; F_n 收敛到 F.

$\frac{F'}{F}=\sum_{n=1}^{+\infty}\frac{F_n'(z)}{F_n(z)}$

W 乘积.

2 4 6 8 10 11 13

2 (a) 0

(b) n

(c) $+\infty$

4. $F(z)=\prod_{n=1}^{+\infty}(1-e^{-2\pi nt}e^{2\pi iz})$

$=\underbrace{\prod_{n=1}^{N}(1-e^{-2\pi nt}e^{2\pi iz})}_{\le e^{a|z|^2}}\cdot\underbrace{\prod_{N+1}^{+\infty}(1-e^{-2\pi nt}e^{2\pi iz})}_{\le1}$

5. $F_a(z)=\int_{-\infty}^{+\infty}e^{-|t|^a}e^{2\pi izt}dt$

$=\lim_{c\to\infty}\int_{-c}^{c}e^{-|t|^a}e^{2\pi izt}dt$ $\sin\pi z=\frac{e^{i\pi z}-e^{-i\pi z}}{2}$

6. $1=\sin\frac{\pi}{2}=\frac{e^{\frac{i\pi}{2}}-e^{-\frac{i\pi}{2}}}{2}$ 故阶为 $\frac{1}{2}$.

$\sin\pi z=\frac{e^{i\pi z}-e^{-i\pi z}}{2}$

$\sin z+\sin(-z)=0$. $z+\frac{z^2}{2}+\cdots+\frac{z^k}{k}=$

$z=n\pi$ 为零点

$z=0$ 为1阶

$\sin z=e^{a}\cdot z\prod_{n=1}^{+\infty}E_k(\frac{z}{a_n})$.

0阶

对数序列

$E_0(z)=1-z$

$E_k(z)=(1-z)e^{z+\frac{z^2}{2}+\cdots+\frac{z^k}{k}}$

$E_k(z)=(1-\frac{z}{a_n})e^{z+\frac{z^2}{2}+\cdots+\frac{z^k}{k}}$

$e^{p(z)}\cdot z^n\cdot\prod_{n=1}^{+\infty}E_k(\frac{z}{a_n})$.

$(1-n\pi)$

$k=0$

$1-z$

$\sin\pi z$ 奇.

零点 $z=2\pi in$. 当 $z=0$ 时 1阶零点.

$e^{Az+b}\cdot z\cdot\prod_{n=1}^{+\infty}E_k(\frac{z}{a_n})=$

$\cos\pi z=\frac{e^{i\pi z}+e^{-i\pi z}}{2}$

令 $f(z)=\sin\pi z$. 零点为 $z=n$, $z=0$ 为1阶零点.

$f(z)=\frac{e^{i\pi z}-e^{-i\pi z}}{2i}$ $p(z)$ 阶为1 取 $\rho_0=1$, $k=1$.

$m=1$ $\deg p(z)=1$

$\sin\pi z=e^{\cancel{Az+B}}\cdot z^m\prod_{n=1}^{+\infty}E_k(\frac{z}{a_n})$.

$=e^{az+b}\cdot z\prod_{n=1}^{+\infty}E_1(\frac{z}{a_n})$ $a=0$

$=e^{b}\cdot z\prod_{n=1}^{+\infty}(1-\frac{z}{n})e^{\frac{z}{n}}$

$e^{\frac{z}{n\pi}}$

$z=\frac{\pi}{2}$. $a_n=n\pi$ $p=1=k$. $n=1$

$\sin z=e^{p(z)}\cdot z^m\cdot\prod_{n=1}^{+\infty}E_k(\frac{z}{a_n})$.

$=e^{Az+B}\cdot z^1\cdot\prod_{n=1}^{+\infty}(1-\frac{z}{a_n})e^{\frac{z}{a_n}}$

$\frac{\frac{\pi}{2}}{n\pi}$ $\frac{\frac{\pi}{2}}{n\pi}=\frac{1}{2n}$

$1=\sin\frac{\pi}{2}=\frac{\pi}{2}\prod_{n=1}^{+\infty}(1-\frac{1}{2n})e^{\frac{1}{2n}}$.

$e^{\frac{1}{2n}}=1+\frac{1}{2n}$.

$\sin z$ 为奇函数 $\Rightarrow A=0$

$e^{B}=\lim_{z\to0}\frac{\sin z}{z}=1\Rightarrow B=0$.

$-i$, $\cos\frac{-i}{2n}+i\sin\frac{-i}{2n}$

必须要一致，设作 $a_n z^n$，再考虑

$$e^{p(z)-a_n z^n} - e^{q(z)-a_n z^n} = Ce^{-a_n z^n}$$

故有下一项也相同，由归纳，可见每项相同，故 $p=q$，此时 $a=b$

这与假设矛盾，故原命题不成立

13. $e^z - z$ 为1阶整函数，故 $e^z - z = e^{Az+B}\prod_n (1-\frac{z}{a_n})e^{\frac{z}{a_n}}$.

若只有有限个零点，则 $e^z - z = e^{Az+B}Q(z)$ 对某个多次式 Q 成立。

而 $Q(z) = \frac{e^z - z}{e^{Az+B}} = O(e^{(1-A)z})$ 这仅当 Q 为常数时可能成立。

此时 $1-A=0$，这个情况下有 $z = Ce^z$ 对常数 $C = 1-e^B$ 而这显然不成立。

故 $e^z - z = 0$ 在复数集 C 上有无穷多个解。

$\Rightarrow A=0$

$e^B = \lim_{z\to 0}\frac{\sin z}{z} = 1 \Rightarrow B=0$.

$\sin z = z\cdot\prod_{n\geq 1}^{+\infty}(1-\frac{z}{2n})e^{\cdot(\frac{i}{2n})}$

$\cos\frac{-i}{2n} + i\sin\frac{-i}{2n}$

$= 1+\frac{1}{2n}$

$\cos\pi z$.

e $e^{\frac{z}{a_n}} = 1+$ $e^{\frac{z}{a_n}}$

对函数 $f(z) \leq Ae^{B|z|^\rho}$ 增长阶为 $\rho_0 = \inf$

对充分大的 r $n(r) \leq Cr^\rho$

$z_1, z_2, \cdots z_n$ 表示零点 $\sum\frac{1}{|z_n|^s} < +\infty$ 对 s

$\{c_n\}$ $\prod(a_n+1)$ 收敛 $|a_n - 1| \leq c_n$ $\sum c_n <$

$\prod F_n \to F$

$$\frac{F'}{F} = \sum_{n=1}^{+\infty}\frac{F_n'}{F_n}.$$

$E_0(z) = 1$

$E_k(z) =$

W $f_1(z) = e^{g(z)}\cdot f_2(z)$.

Hadamard. 对 $f(z)$ 阶为 ρ_0 零点 $a_1, \cdots$

$f(0)=0$ 的零点阶为 m.

$$f(z) = e^{P(z)}\cdot z^m\cdot\prod_{n=1}^{\infty}E_k(\frac{z}{a_n})$$

$k \leq \rho_0 < k+1$ $\deg P(z) = \rho_0$.

$$\sin z = e^{Az+B}\cdot z^1\cdot\prod_{n\in\mathbb{Z}}(1-\frac{z}{n})$$

$B = \lim_{z\to 0}\frac{\sin z}{z} = 1$

$f\,l$ ·

l 截 z

$f\lambda$

$-z$

$(1-z)\cdot e^{z+\frac{z^2}{2}+\cdots+\frac{z^n}{n}}$ ·

$n\,\cancel{\theta}$

c

$m)$

$e^{\frac{z}{n}}$ ·

第三次作业 12211834 沈博涵

2024年4月16日 星期二 19:13

1, 2, 4, 5, 6, 8, 12, 13

1. $\sin\pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}$ $\sin\pi z$ 在C上全纯 $\because e^{i\theta} = \cos\theta + i\sin\theta$

$\sin\pi z = 0 = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}$ 设 $z = x + iy$.
$\begin{cases} e^{i\pi x} = \cos\pi x + i\sin\pi x \\ e^{\pi y} = \cos\pi y - i\sin\pi y \\ e^{-i\pi x} = \cos\pi x - i\sin\pi x \\ e^{-\pi y} = \cos\pi y - i\sin\pi y \end{cases}$

$\Rightarrow e^{i\pi z} = e^{-i\pi z} \Rightarrow e^{i\pi x}e^{-\pi y} = e^{-i\pi x}e^{\pi y}$

$\Rightarrow \cos\pi x\,[e^{2\pi y} - 1] = 0$; $\sin\pi x \cdot \underbrace{[e^{2\pi y} + 1]}_{>0} = 0$.

$\Rightarrow \sin\pi x = 0 \Rightarrow e^{2\pi y} = 1 \Rightarrow y = 0,\ x \in \mathbb{Z} \Rightarrow z = x$ is an integer.

To find the order of z. $\because 2i \frac{d}{dz}\Big|_{z=n} \sin\pi z = i\pi(e^{in\pi} + e^{-in\pi}) = 2i\pi e^{in\pi} \neq 0$.

$\Rightarrow$ order is 1.

when $z = n$, $res_{z_0} f = \lim_{z\to n}(z-n)\frac{1}{\sin\pi z} = \lim_{z\to n}\frac{z-n}{\sin\pi n}$

由 Hopital : $= \lim_{z\to n}\frac{1}{\pi\cos\pi n} = \frac{1}{\pi\cos\pi n} = \frac{(-1)^n}{\pi}$.

2. evaluate $\int_{-\infty}^{+\infty}\frac{dx}{1+x^4}$. and its poles.

for $x^4 + 1 = 0 \Rightarrow$ it's poles are w, w^3, w^5, w^7, $w = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i = e^{\frac{\pi}{4}i}$

suppose $f(z) = \frac{1}{1+z^4}$ we have for poles z_1, z_2, z_3, z_4

$\int_\gamma f(z)dz = 2\pi i\sum_{k=1}^{4} res_{z_k} f = 2\pi i(res_{z_1}f + res_{z_2}f + res_{z_3}f + res_{z_4}f)$,

$res_{z_1} f = \lim_{z\to z_1}(z - z_1)\cdot\frac{1}{1+z^4} \underset{\text{Hopital}}{=} \frac{1}{4z_1^3} = \frac{1}{4e^{\frac{3}{4}\pi i}} = -\frac{e^{\frac{\pi}{4}i}}{4}$

$res_{z_2} f = -\frac{e^{\frac{3}{4}\pi i}}{4}$ $res_{z_3} f = -\frac{e^{\frac{5}{4}\pi i}}{4}$ $res_{z_4} f = -\frac{e^{\frac{7}{4}\pi i}}{4} \Rightarrow res_{z_1}f + res_{z_2}f = -\frac{\sqrt{2}i}{4}$.

$\Rightarrow \int_{-\infty}^{+\infty}\frac{dx}{x^4+1} = 2\pi i(res_{z_1}f + res_{z_2}f) = -2\pi i\cdot\frac{\sqrt{2}}{4}i = \frac{\pi}{\sqrt{2}}$.

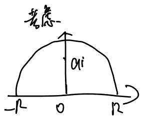
4. 求证 $\int_{-\infty}^{+\infty}\frac{x\sin x}{x^2+a^2}dx = \pi e^{-a}$ for all $a > 0$.

令 $f(z) = \frac{-ize^{iz}}{z^2+a^2} = \frac{-iz\cos z + z\sin z}{z^2+a^2}$ 其极数为 $z = \pm ai$

对上半圆: $res_{z=ai} f = \lim_{z\to ai}(z - ai)\frac{-ize^{iz}}{z^2+a^2}$

$= \lim_{z\to ai}\frac{-ize^{iz}}{z+ai} = \frac{-ie^{-a}}{2}$

考虑



$\Rightarrow$ 回路积分 $\int_\gamma f(z)dz = 2\pi i\cdot\frac{-ie^{-a}}{2} = \frac{\pi}{e^a} = \pi\cdot e^{-a}$

$\frac{e^{-a}}{\gamma}\cdot 2\pi i = \pi e^{-a}$

实数部分: $\left|\int_{-R}^{R}\frac{-ize^{iz}}{z^2+a^2}dz\right| = \left|\int_0^\pi\frac{-iRe^{i\theta}e^{iRe^{i\theta}}}{R^2e^{2i\theta}+a^2}iRe^{i\theta}d\theta\right| \le R^2C\int_0^\pi\frac{e^{-R\sin\theta}}{R^2+a^2}d\theta \to 0$ as $R\to\infty$

而原式与 $f(z)$ 实数积分相等. | 故 $\int_{-\infty}^{+\infty}\frac{x\sin x}{x^2+a^2}dx = \int_{-\infty}^{+\infty}f(z)dz = \int_\gamma f(z)dz = \pi\cdot e^{-a}$.

$e^{ix} \to 1$ 此时模长, 模长为1.

$\frac{1}{R^2} \to 0$.

5. 求证: $\int_{-\infty}^{+\infty}\frac{e^{-2\pi ix\xi}}{(1+x^2)^2} = \frac{\pi}{2}(1 + 2\pi|\xi|)e^{-2\pi|\xi|}$ 对 $\xi \in \mathbb{R}$.

极点 $(x^2+1)^2 = 0 \Rightarrow x = \pm i$ 对 $\xi \ge 0$. 取 $u = -x$ 则可得 $\xi < 0$ 时情况.

$\therefore$ 仅对 ξ 加进行讨论: 令 $-2\pi\xi = \theta$

则需证: $\int_{-\infty}^{+\infty}\frac{e^{\theta xi}}{(1+x^2)^2} = \frac{\pi}{2}(1+\theta)e^{-\theta}$.

令 $f(z) = \frac{e^{\theta zi}}{(1+z^2)^2}$ $z = i$ 为其极点.

$res_{z=i} f = \lim_{z\to i}\frac{d}{dz}(z-i)^2\cdot\frac{e^{z\theta i}}{(1+z^2)^2} = \lim_{z\to i}\frac{d}{dz}\frac{e^{z\theta i}}{(z+i)^2}$

$= \lim_{z\to i}\frac{\theta i\cdot e^{z\theta i}(z+i)^2 - e^{z\theta i}\cdot 2(z+i)}{(z+i)^4} = \lim_{z\to i}\frac{\theta i\cdot e^{z\theta i}(z+i) - 2e^{z\theta i}}{(z+i)^3}$

$= \frac{\theta i\cdot e^{-\theta}\cdot 2i - 2e^{-\theta}}{-8i} = \frac{(1+\theta)e^{-\theta}}{4i} \Rightarrow \int_\gamma f(z)dz = 2\pi i\, res_{z=i} f$

$= 2\pi i\cdot\frac{(1+\theta)e^{-\theta}}{4i}$

$= \frac{\pi}{2}(1+\theta)e^{-\theta}$.

and $\int_{\gamma_R} f(z)dz \le \int_{-R}^{R}|f(z)|dz \le \int_0^\pi\left|\frac{R}{(1+R^2e^{2i\theta})^2}\right|d\theta \le \frac{R}{(R^2-1)^2}\cdot\pi$.

$R\to\infty$ 时, 其 $\to 0$.

$\Rightarrow \int_{-\infty}^{\infty}\frac{e^{-2\pi ix\xi}}{(1+x^2)^2}dx = \int_\gamma f(z)dz = \frac{\pi}{2}(1 + 2\pi|\xi|)e^{-2\pi|\xi|}$.

6. 求证: $\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^{n+1}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot\pi = \frac{(2n)!}{2^{2n}\cdot(n!)^2}\cdot\pi$.

Let $f(z) = \frac{1}{(1+z^2)^{n+1}} \Rightarrow z = i$ 为其极点. 其阶为 $n+1$.

$res_{z=i} f = \frac{1}{n!}\lim_{z\to i}\left(\frac{d}{dz}\right)^n(z-i)^{n+1}\cdot\frac{1}{(z^2+1)^{n+1}}$

$\frac{1}{\ }\ \lim\ \frac{d}{\ }\ \frac{1}{\ }$

$res_{z_0} f = \frac{1}{n!}\lim\left(\frac{d}{dz}\right)^n(z - z_0)^{n+1}\cdot\frac{1}{f(z)}$.

n 阶: $res_{z_0} f = \frac{1}{(n-1)!}\lim\left(\frac{d}{dz}\right)^{n-1}(z - z_0)^n\frac{1}{f(z)}$

$$\mathrm{res}_{z=i} f = \frac{1}{n!} \lim_{z\to i} \left(\frac{d}{dz}\right)^n (z-i)^{n+1} \cdot \frac{1}{(z^2+1)^{n+1}}$$
$$= \frac{1}{n!} \lim_{z\to i} \frac{d}{dz^n} \frac{1}{(z+i)^{n+1}}$$
$$= \frac{1}{n!} \cdot \lim_{z\to i} (-n-1)(-n)\cdots(-2n) \cdot \frac{1}{(z+i)^{2n+1}}$$
$$= \frac{(-1)^n \cdot (2n)!}{(n!)^2} \cdot \lim_{z\to i} \frac{1}{(z+i)^{2n+1}}$$
$$= \frac{\cancel{(-1)^n} \cdot (2n)!}{(n!)^2} \cdot \frac{1}{2^{2n+1} \cdot \cancel{(-1)^n}\, i}$$
$$= \frac{(2n)!}{2^{2n+1} \cdot (n!)^2 \cdot i}$$

$\Rightarrow \int_\Gamma f(z)\,dz = 2\pi i \cdot \mathrm{res}_{z=i} f = 2\pi i \cdot \frac{(2n)!}{2^{2n+1} (n!)^2 \cdot i} = \frac{(2n)!}{2^{2n} (n!)^2} \cdot \pi$.

再考虑其 $\int_{-R}^{R} \frac{dz}{(z^2+1)^{n+1}} \leq \frac{\pi R}{(R^2-1)^{n+1}}$ 当 $R\to\infty$ 时其 $\to 0$ 故为 0

$\Rightarrow \int_{-\infty}^{\infty} f(z)\,dz = \int_\Gamma f(z)\,dz = \frac{(2n)!}{2^{2n} \cdot (n!)^2} \pi$ 即证.

8. 求证 $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$, 若 $a>|b|$, $a,b\in R$.

对整圆 C (单位圆).

$z = e^{i\theta} = \cos\theta + i\sin\theta$ $\quad \frac{1}{z} = \cos\theta - i\sin\theta$.

$\Rightarrow \cos\theta = \frac{z+\frac{1}{z}}{2}$ $\quad i\theta = \ln z \Rightarrow i\,d\theta = \frac{dz}{z} \Rightarrow d\theta = \frac{dz}{iz}$.

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \oint_C \frac{\frac{dz}{iz}}{a+b\cdot\frac{z+\frac{1}{z}}{2}} = \oint_C \frac{2\,dz}{i(bz^2+2az+b)}$$

$\frac{-a+\sqrt{a^2-b^2}}{b}$

对 $f(z) = \frac{1}{bz^2+2az+b}$ $\quad bz^2+2az+b=0$ 得 $\frac{-2a+\sqrt{4a^2-4b^2}}{2b} = \frac{-a+\sqrt{a^2-b^2}}{b}$ 为其一根 z_0 记 $=z_1$.

$$\Rightarrow \mathrm{res}_{z=z_0} f = \lim_{z\to z_0} \frac{d}{dz}(z-z_0)^2 \frac{1}{bz^2+2az+b} = \lim_{z\to z_0} \frac{1}{2bz+2a} = \frac{1}{2\sqrt{a^2-b^2}}$$

$$\Rightarrow \oint_C f(z)\,dz = 2\pi i \cdot \frac{2}{i} \cdot \mathrm{res}_{z=z_0} f$$
$$= 4\pi \cdot \frac{1}{2\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}.$$

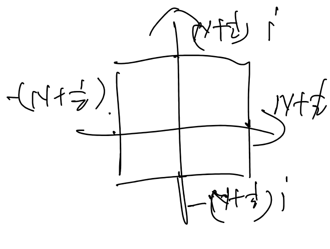
12. 假设 u 非整数, 求证 $\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{(\sin\pi u)^2}$.

由 对 $f(z) = \frac{\pi\cot\pi z}{(u+z)^2}$ 在圆 $|z| = R_N = N+\frac{1}{2}$ (N 整, $N \geq |u|$).

$f(z) = \frac{\pi\cot\pi z}{(u+z)^2}$. poles 为 $-u$, 阶为 2; $n\in N$, 阶为 1.

① $\lim_{z\to -u} \frac{d}{dz}(z+u)^2 f(z) = \lim_{z\to -u} \frac{d}{dz} \pi\cot\pi z = -\frac{\pi^2}{(\sin\pi u)^2}$

② $\lim_{z\to n} (z-n) \frac{\pi\cos\pi z}{(z+u)^2 \sin\pi z} = \frac{1}{(u+n)^2}$



考虑 $|y|>1$ $\quad |\cot\pi z| \leq \frac{1+e^{-2\pi y}}{|1-e^{-2\pi|y|}|} \leq \frac{1+e^{-2\pi}}{1-e^{-2\pi}} = C_1$

若 $|y| \leq 1$, 则 $x = \pm(N+\frac{1}{2})$

$|\cot\pi z| = |\cot\pi(N+\frac{1}{2}+iy)| = |\tanh(y)| \leq \tanh(\frac{\pi}{2}) = C_2$.

$\Rightarrow |\cot\pi z|$ 在 $\{(x+iy) \mid |x|, |y| \leq N+\frac{1}{2}\}$ 上有界 即 $N\to\infty$ 时趋于 0.

$\Rightarrow |\cot \pi z|$ 在 $\{(x+iy) \mid |x|, |y| \le N+\frac{1}{2}\}$ 上有界 при $N\to\infty$ 时趋于0.

由①.②式可知: $\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{(\sin \pi u)^2}$.

13. 假设 $f(z)$ 在 $D_r(z_0)-\{z_0\}$ 上全纯. $\underbrace{|f(z)| \le A|z-z_0|^{\varepsilon-1}}_{\text{有界}}$ 对 $\varepsilon>0$.

和所有 z 在 z_0 附近成立. 求证其可去掉.

即3.1 黎曼定理.

令 $g(z)=(z-z_0)f(z) \Rightarrow g$ 为 analytic $\because g(z)\to 0$ as $z\to z_0$.

且 g 有界. z_0 对 $g(z)$ 来说为可去奇点. 故 g 在 z_0 处可解析:

由于 g 全纯 且 $g(z_0)=0$ 则 $g(z)=(z-z_0)h(z)$ 在 z_0 邻域上, 有 h 在 z_0 处全纯.

$\Rightarrow f(z)=h(z)$ 在 z_0 的一个去掉邻域上. 通过定义 $f(z_0)=h(z_0)$

我们可以扩展 f 使其在 z_0 处全纯. 即该奇点为可去的