INTRODUCTION

The main goal of this thesis is to define an interacting particle system: coalescing Brownian motions on the real line R. This system is the continuous space analogue of a discrete space system: coalescing random walks on the lattice of integers Z, which is discussed in Refs. [13], [14], [12], [9], [4], and [10]. We construct the system with arbitrary initial sets of particles. The system has, up to rescaling, an invariant equilibrium measure, a point process on R which we call π^c. Using coalescing Brownian motions, one can analyze the limiting behavior of some nearest neighbor interacting particle systems on the one dimensional lattice Z, such as coalescing random walks, annihilating random walks (see Refs. [7], [1], [19], [20], [9], [4], and [10]), and the basic voter model (see Refs. [13], [6], [17], [9], and [4]).

Intuitively, coalescing Brownian motions is a system of particles undergoing independent, identical diffusions, with coalescing interference. When two particles arrive at the same position, they coalesce--they are replaced by a single particle, identical with the other particles. Alternatively, one can imagine that whenever two particles collide, one vanishes and the remaining particle continues its Brownian motion. Another possibility is to imagine that particles which collide become glued together, but continue to diffuse at the same speed as a single particle.

In our model on the line, particles are represented as points--they do not influence each other until they occupy exactly the same position. In more than one dimension, there is probability zero that two points executing independent Brownian motions will ever collide, so a more complicated collision mechanism must be specified.

In 1916 Smoluchowski [21] gave a heuristic analysis for a model of coagulation in colloids based on coalescing Brownian motions in R^3. This model is discussed by Chandrasekhar [5], who includes references to the comparison of Smoluchowski's predictions and experimental data. A rigorous version of Smoluchowski's result was given in 1979 by Lang and Xuan Xanh [18]. In these models, particles follow independent Brownian motion paths until they come within some fixed collision radius of each other. When particles of mass m and n collide, a single particle of mass m+n replaces them. A fair coin is used to decide which of the two predecessors' locations will be the location of the new particle. Regardless of mass, all particles have the same diffusion rate and collision radius.

In all of the models discussed so far, all particles behave identically. In a real physical system, such as the droplets of an aerosol spray, particles of widely differing sizes are formed by coalescing. The diffusion coefficient varies with the size of the particle; for spheres of radius a in a fluid with coefficient of viscosity η, the diffusion speed is $kT/6\pi a\eta$ ([5], formula 173).

It would be interesting to analyze a model which takes this into account.

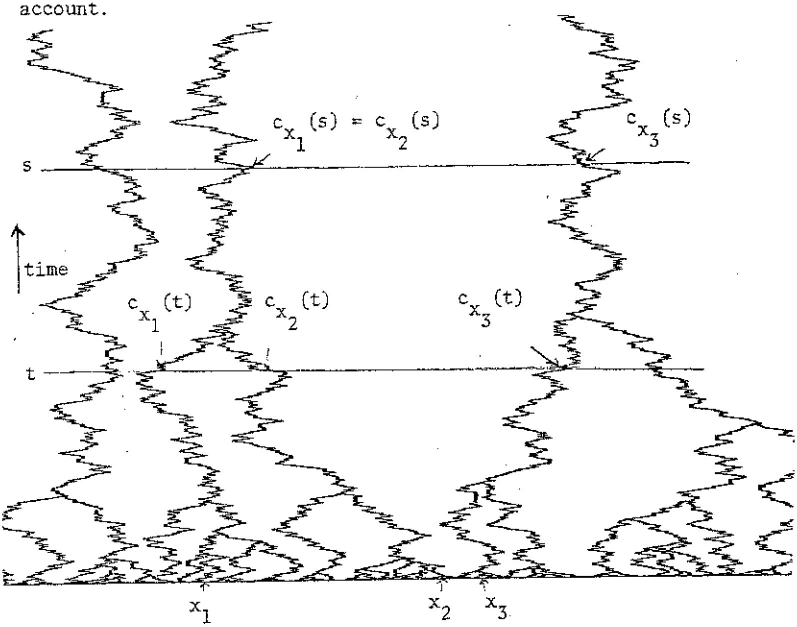


Figure - The coalescing system $c = (c_x(t),\ x \in R,\ t \geq 0)$

The system of coalescing Brownian motions on the line can be formulated in two ways. Consider a system with distinct, indesctructible particles, one for each point $x \in R$, which become glued together upon collision. Write $c_x(t) = c_x(t,\omega)$ for the position at time t of the particle which started at x. Thus, the system $c = c(\omega)$ is a real valued stochastic process indexed by $(x,t) \in R \times [0, \infty)$. Every sample path $c(\omega)$ has the following properties: for $x, y \in R;\ t \geq 0$,

$$c_x(0) = x; \tag{1}$$

$c_x(\cdot) \in C_R[0, \infty)$, i.e., each particle (2)

follows a continuous path in the line;

$c_x(t) = c_y(t)$ implies that (3)

$$c_x(s) = c_y(s) \text{ for all } s > t.$$

This last condition is the coalescing property: after two particles collide, they stay together. Finally, there is a requirement on the finite dimensional distributions of c:

> for every finite $A \subset R$, the family of paths $(c_x(\cdot),\ x \in A)$ must be a family of coalescing Brownian motions. (4)

The distribution of a finite system of coalescing Brownian motions is specified in Chapter 1. It is the distribution of a system which is derived from a system of independent Brownian motions, using some collision rule.

The second formulation of a system of Brownian motions involves identical, mortal particles. Whenever two particles collide, one vanishes and the other survives. As time goes on, there are fewer and fewer particles. At any time t, the state X_t of the system is a subset of R; X_t is the set of positions occupied by the surviving particles. This system is a projection of the system c: for $t \geq 0$,

$$X_t = \{c_x(t) : x \in R\}. \tag{5}$$

This set-valued evolution $(X_t, t \geq 0)$ is a Markov process; it is analogous to the system ξ_t^Z of coalescing random walks.

Here are the main results of this thesis: A system c, satisfying (1) through (4), exists (Theorem (2.1), i.e., the first numbered item in Chapter 1). For every $t > 0$, X_t is a discrete set, almost surely (Theorem (3.12)). As an evolution in $t > 0$, X_t is a Markov process (Theorem (3.51)). Up to rescaling, the distribution of X_t is invariant, i.e., $X_t \stackrel{d}{=} \sqrt{t}\, X_1$ (Theorem (4.34)). Here, $\stackrel{d}{=}$ denotes equality in distribution; for $A \subset R$, $r\,A$ means $\{rx : x \in A\}$.

The Percolation Substructure

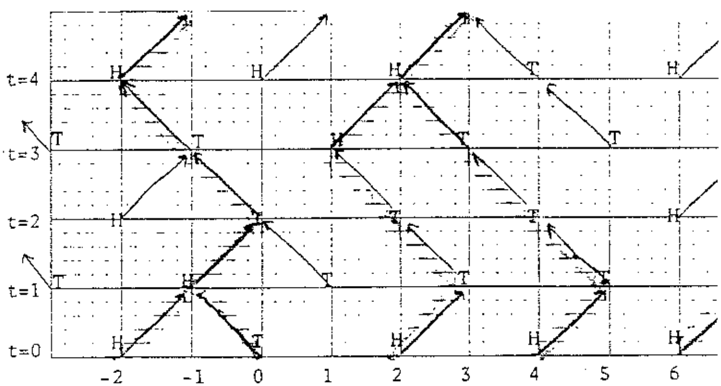
Continuous time, discrete space interacting particle systems can be studied using a graphical representation due to Harris ([12], [10]). The representation features a random percolation substructure, and it produces a coupling: sample paths with each possible initial configuration are constructed simultaneously on a single probability space.

For the system of coalescing simple random walks on Z, this substructure is as follows; to simplify the exposition we present a discrete-time version.

The substructure for coalescing random walks on Z is a random diagram in the upper half plane. Let

$$L_o = \{(x,t) \in Z^2 : t \geq 0, \quad x \equiv t \equiv 0 \pmod 2\};$$

$$L_1 = \{(x,t) \in Z^2 : t \geq 0, \quad x \equiv t \equiv 1 \pmod 2\}.$$



The percolation substructure for coalescing random walks on Z.

Figure 6

The percolation substructure is formed with a collection of fair coins $(A(x,t))$ indexed by $(x,t) \varepsilon L_o \cup L_1$. If $A(x,t)$ is heads, draw an arrow from (x,t) to $(x+1, t+1)$; if it is tails, draw the arrow from (x,t) to $(x-1, t+1)$. Particles are started at $t = 0$ on (some subset of) the even integers, and follow the arrows upward. This produces the same distribution of paths as would independent random walks and a collision rule.

The system c is an analogue of the percolation substructure. For example, coalescing Brownian motions starting from an arbitrary subset A of the line can be defined by:

$$X_t^A = \{c_x(t) \ : \ x \ \varepsilon \ A\} \ . \tag{7}$$

This immediately yields a coupling such that for arbitrary $A, B \quad R$, and for all ω,

$$X_t^{A \cup B} = X_t^A \cup X_t^B \ . \tag{8}$$

A closer analogue of the percolation substructure would be a coalescing system $c = (c_x(s,t) \ : \ x \ \varepsilon \ R, \ t \geq s \geq 0)$ where $c_x(s,t)$ gives the position at time t of a particle, which is "born" at time s at location x. The coalescing property is:

$$\begin{gathered} c_x(u,t) = c_y(r,t) \text{ implies that} \\ c_x(u,s) = c_y(r,s), \text{ for all } s > t \ . \end{gathered} \tag{9}$$

To specify the distribution of c, require first that the paths of particles born at a fixed time u should form a system c^u like c. Formally, for each $u \geq 0$ define a system c^u by

$$(c^u)_x(t) = c_x(u, u+t),$$

and require that c^u satisfy (1) through (4). The second requirement is that for any t, the part of c before t,

$$(c_x(r,s) : 0 \leq r \leq s \leq t, x \in R),$$

must be independent of the part of c after t,

$$(c_x(u,v) : t \leq u \leq r < \infty, x \in R).$$

It should be possible to construct c by piecing together a countable family of independent versions of c.

The entire percolation substructure for coalescing random walks can be rescaled to approximate a substructure for coalescing Brownian motions. Imagine Fig. 6 compressed by the factor s along the time axis, and by the factor $\sqrt{s}$ along the space axis. Thus, the individual random walks take steps of $\pm \frac{1}{\sqrt{s}}$, with time $\frac{1}{s}$ between jumps. For s large, the path of a single particle is "almost" a Brownian motion. Using non-standard analysis, one can take s to be infinite, and work in the hyperreal numbers with a lattice of infinitesimal spacing. This is carried out rigorously, to define a single Brownian motion from a random walk on the hyperreal line, by Anderson [2]; see also [6]. It should be possible to give rigorous proofs of the results in this thesis using an infinitesimal substructure. Although all proofs in this thesis are standard, some of the results were discovered by considering the infinitesimal substructure.

A Stationary "Gaussian" Process

Define π^c to be the distribution of the random set X_1. Theorem (4.34) states that for any $t > 0$, the distribution of the rescaled set $\frac{1}{\sqrt{t}} X_t$ is π^c. Write $(X_t^{\pi^c}, t \geq 0)$ for the mixture of the processes X_t^A, where A is distributed according to π^c independently of the "substructure" c. Then, for $t > 0$,

$$\frac{1}{\sqrt{t}} X_t^{\pi^c} \stackrel{d}{=} X_0^{\pi^c} . \tag{10}$$

In words: particles start on a discrete subset of R, whose distribution is π^c. Independently of this starting configuration, the particles undergo independent Brownian motions with coalescing interference, for time t. Dividing the locations of each particle by $\sqrt{1+t}$, the resulting set again has distribution π^c.

As an evolution in $t \geq 0$, the Markov process $\frac{1}{\sqrt{1+t}} X_t^{\pi^c}$ is in equilibrium, but its transition mechanism is not homogeneous in time. This leads us to consider a time change; define

$$Y_t = e^{-t/2} X_{(e^t)} , \tag{11}$$

for $-\infty < t < \infty$. This is a Markov process with a time homogenous transition mechanism (Theorem (4.55)). For every $t \in R$, the distribution of Y_t is π^c. It is a system of coalescing diffusions. Every particle has diffusion coefficient 1, and when a particle is at x, its drift is $-\frac{x}{2}$. Thus, particles come in toward the origin from plus and minus infinity, but when two particles collide, one is destroyed. It may seem surprising that this evolution, which focuses on the origin, has an equilibrium, π^c, which is translation invariant in space. The transition mechanism is not as spatially inhomogeneous as it appears to be: view the system from a frame of reference moving down the line with velocity $\frac{a}{2}$. Then the particles appear to move toward the point $-a$; the drift coefficient of a particle is proportional to its distance from $-a$.

Duality

Methods involving time reversal are referred to as "duality methods". Duality relations for interacting particle systems on the lattice, such as coalescing and annihilating random walks, and the voter model, appear in [22], [13], [11], [14], [12], and [10]. The duality for coalescing Brownian motions is a joint realization of two processes on the same time interval [0, T], with one running forward in time and the other running backward--see Fig. 13. The paths of the two systems never cross. Theorem (5.48) states this precisely: There is a joint realization of c and c', two systems of coalescing Brownian motions as specified by (1) through (4), such that, for all $\omega \in \Omega$; x, y ε R; and $0 \leq s, t \leq T$,

$$(c_x(s) - c'_y(T-s))(c_x(t) - c'_y(T-t)) \geq 0. \qquad (12)$$

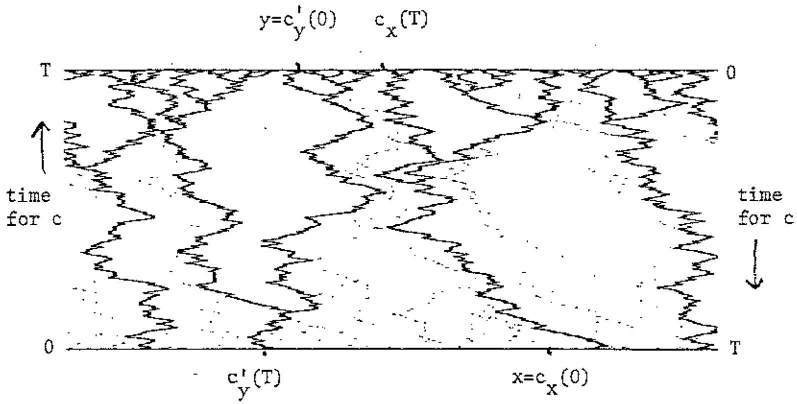


Figure 13

This duality relation is derived by taking coalescing Brownian motions as the limit of coalescing random walks. It is easy to construct two systems of coalescing random walks in duality, using the percolation substructure; just "reverse the arrows". In terms of the construction given with Fig. 6, draw an arrow from $(x, t+1)$ to $(x-1, t)$ if $A(x,t)$ is heads; and from $(x, t+1)$ to $(x+1, t)$ otherwise. Thus, each coin picks one of two possible building blocks:

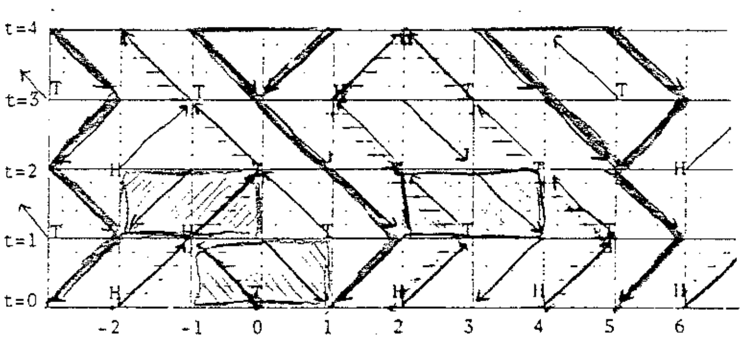
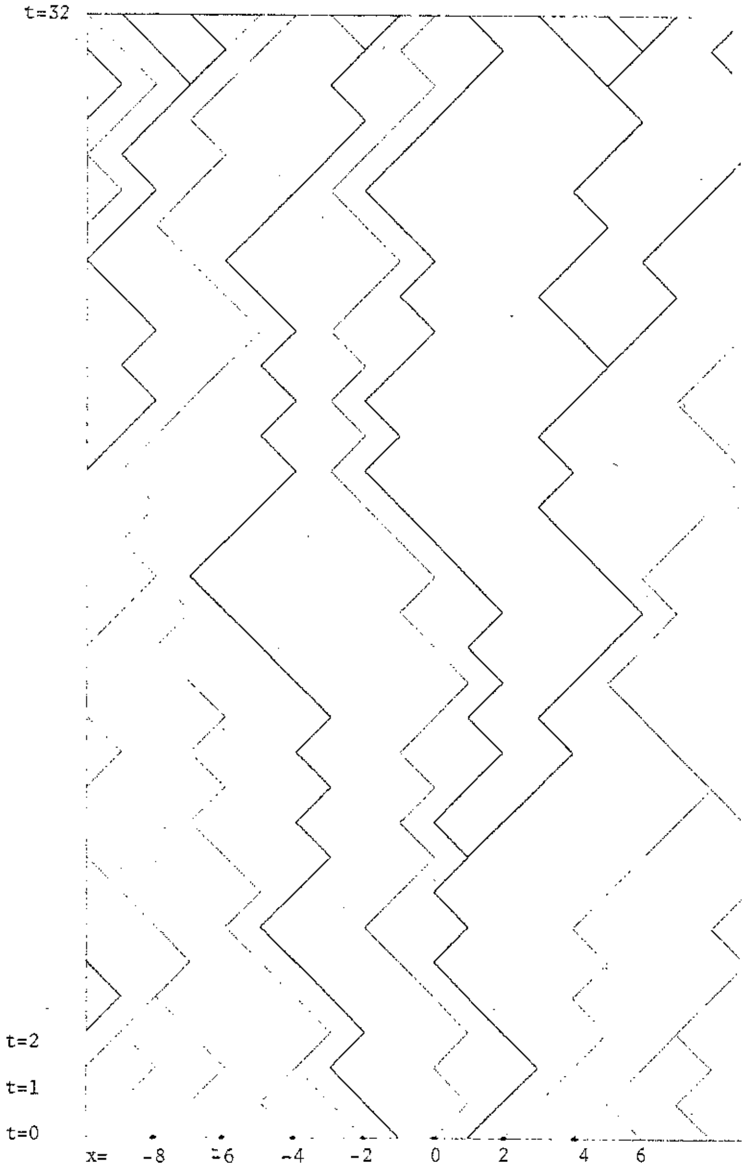
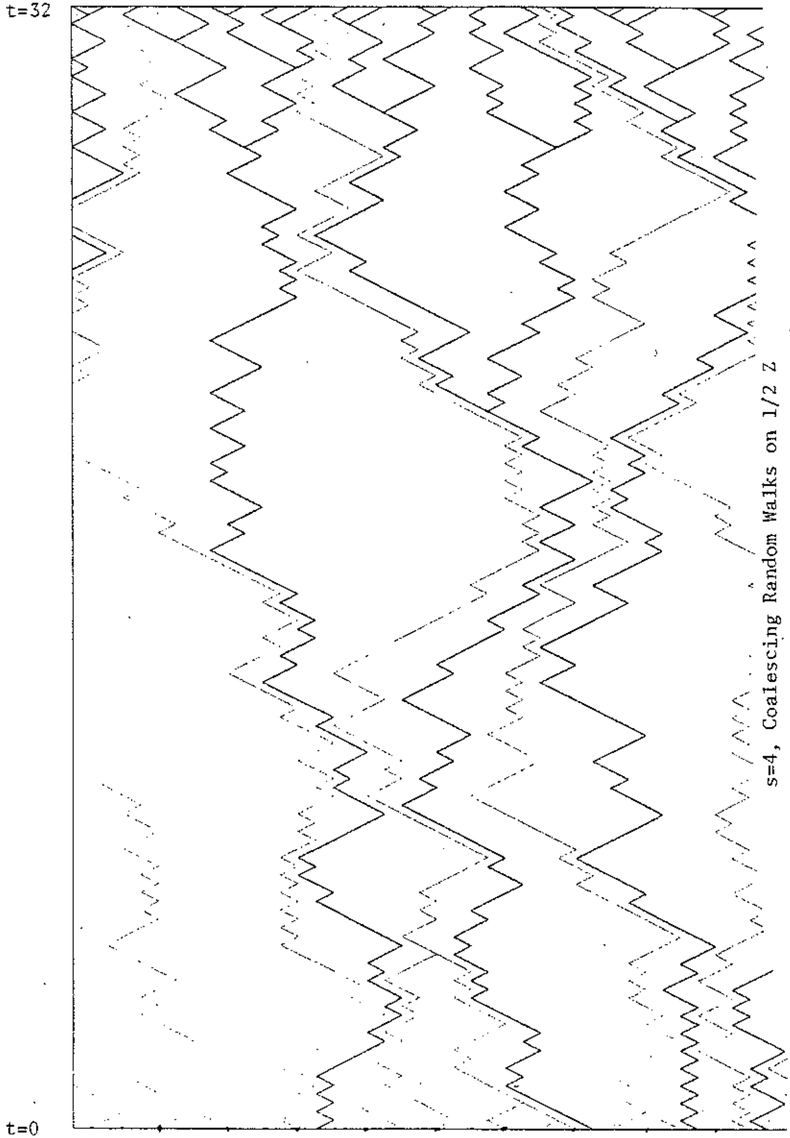


Figure 14. Getting duality from the substructure.

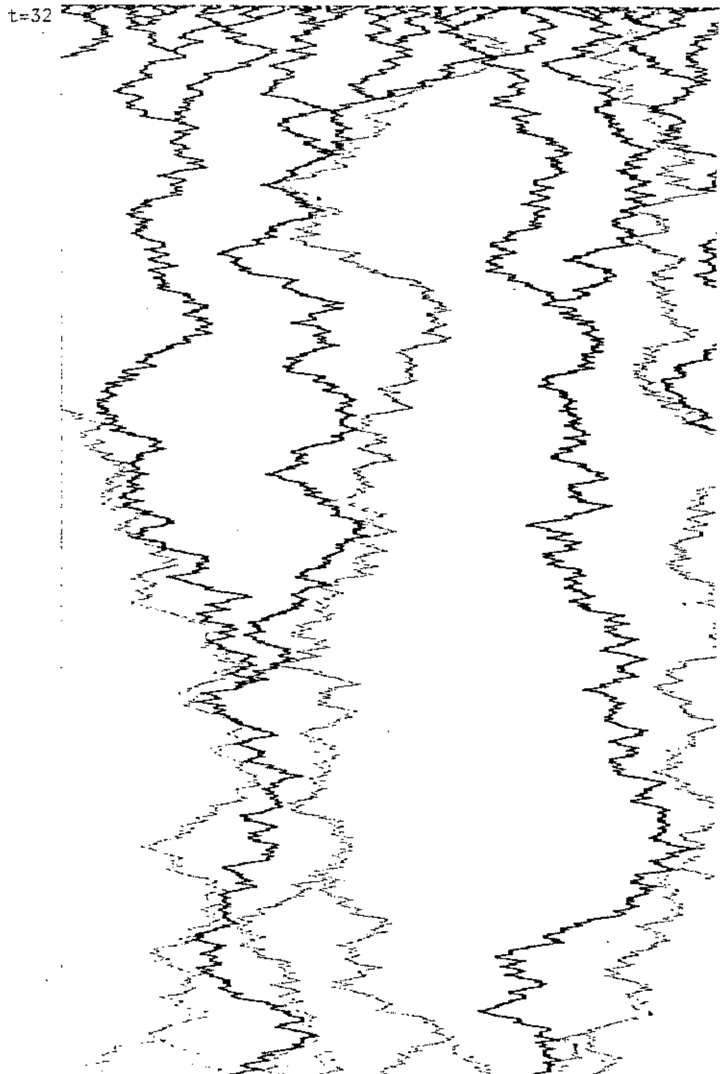
Take an even integer $N \approx s\,T$, and start particles moving upward from even integers at time 0, and downward from odd integers, starting at time N. Rescale the picture, by s along the time axis and $\sqrt{s}$ along the space (x) axis, to get an approximation to two systems of coalescing Brownian motions in duality on [0, T]. We next present two applications of this duality.



s=1, Coalescing Random Walks on Z

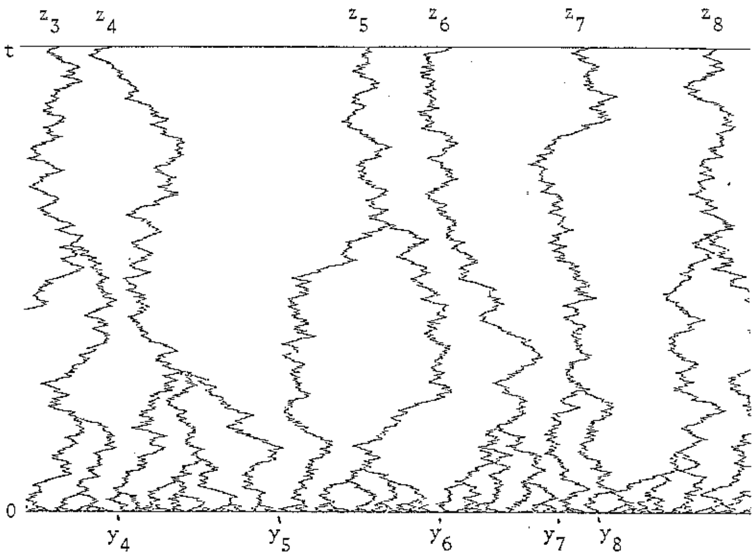


s=4, Coalescing Random Walks on 1/2 Z



s=64, Coalescing Random Walks on 1/8 Z

$X_t = \{\ldots,\ z_3, z_4, z_5, \ldots\}$



$D_t = \{\ldots,\ y_3, y_4, y_5, \ldots\}$

Figure 15

For any $t > 0$, enumerate the set of positions occupied by coalescing Brownian motions: $X_t = \{\ldots, z_{-2}, z_{-1}, z_0, z_1, z_2, \ldots\}$. For $z_i \in X_t$, the starting locations of particles that are at z_i at time t form an interval. The system c will be defined so that this interval is closed on the left and open on the right; thus there are partition points $\ldots, y_{-2}, y_{-1}, y_0, y_1, \ldots$ such that

$$\{x : c_x(t) = z_i\} = [y_i, y_{i+1}). \tag{16}$$

The set $D_t = \{y_{-2}, y_{-1}, y_o, y_1, \ldots\}$ is the set of "borders" between particles that are not coalesced together by time t. Thus

$$D_t = \{y \in R : \lim_{x \to y^-} c_x(t) \neq \lim_{x \to y^+} c_x(t)\} . \tag{17}$$

The points of D_t do not move:

$$\text{for} \quad s \geq t > 0, \qquad D_s \subset D_t . \tag{18}$$

What is the distribution of D_t? Construct c and c' in duality on the time interval [0,t]. Write $X_t' = \{c_x'(t) : x \in R\}$. Now

$$D_t = X_t' \stackrel{d}{=} X_t . \tag{19}$$

As a second application of duality, there is a formula for π^c, the distribution of X_1. An orderly point process, i.e., a point process with no multiple points, such as X_t, is determined by its zero function, $\phi_t(B) = P(X_t \cap B = \phi)$ for $B \subset R$. Let $I_o = \{B \subset R : B = [a_1,b_1] \cup \ldots \cup [a_n,b_n]$, for some $n, a_i, b_i\}$; the distribution of an orderly point process is determined by the values $\phi(B)$ of its zero function for $B \in I_o$. Now take systems c and c' in duality on the time interval [0,t]. We have,

$$\begin{aligned}\phi_t(B) &= P(X_t \cap B = \phi) \qquad (20)\\ &= P(c'_{a_i}(t) = c'_{b_i}(t) \quad \text{for } i=1,2,\ldots,k)\end{aligned}$$

For a finite $F \subset R$, let A^F be a system of annihilating Brownian motions starting with particles on F: when two particles collide, they both vanish. The duality equation (20) can be expressed, for $B \in I_o$, as

$$\phi_t(B) = P(X_t \cap B = \phi) = P(A_t^{\partial B} = \phi).$$

Here, ∂B is the boundary of B. Contrast this with the zero function of another point process: the Poisson process with intensity λ:

$$\phi(B) = e^{-\lambda|B|} ,$$

where $|B|$ is the Lebesque measure of B.

For a reference on point processes, see Kallenberg [15]. For material related to the Skorohod space $D_E(R)$ used in Chapters 4 and 5, see [8] and [3].

CHAPTER 1

SYSTEMS WITH A FINITE NUMBER OF PARTICLES

This chapter serves as a foundation to the study of infinite particle systems of coalescing motion on the line. By specifying a transition function, we define rigorously the notion of a finite system of Brownian motions which are independent except for coalescing interference upon collisions. We show that this system can be constructed by "collision precedence" rules from a system of independent Brownian motions, and that the distribution of the resulting system is the same regardless of which collision precedence rule is used.

Figure (1) shows two different constructions, based on two different collision precedence rules, of a coalescing system with three particles.

There are two distinct interacting particle systems which we call "coalescing Brownian motions". Corresponding to the notion of distinct, indestructible particles which stick together, as in the infinite system c, we have an n-particle system C. The vector C(t) specifies the n positions, not all distinct after a collision has occurred, of the n particles at time t. Corresponding to the notion of identical particles, where two particles are replaced by

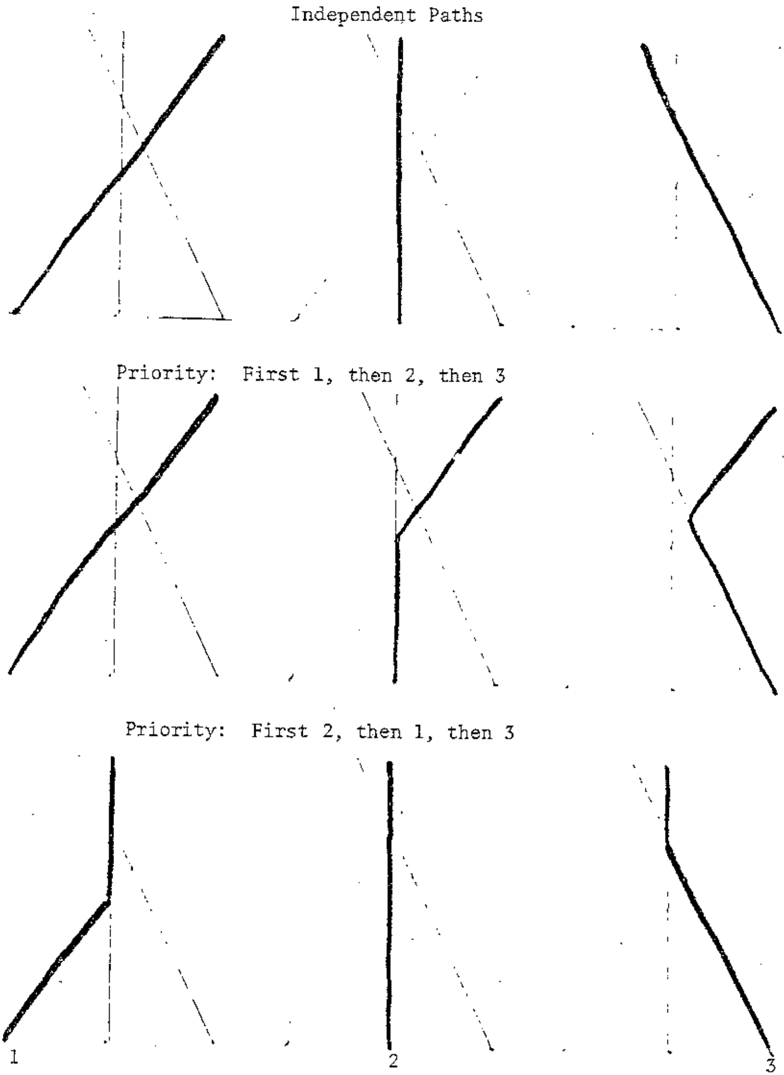

Independent Paths
Priority: First 1, then 2, then 3
Priority: First 2, then 1, then 3
1
2
3

Figure 1

a single particle upon collision, as in the infinite set valued evolution X, we have the finite system U. $U(t)$ specifies the positions of the surviving particles; after each collision there is one less position to specify.

<u>Notation: S_n, H_n^i, U_n, $\underline{U}$, ι, $\hat{\pi}$, #</u>

It is convenient to list the positions of the particles in increasing order. Thus we define, for $n \geq 1$, the region

$$S_n = \{x \in R^n\colon\ x_1 \leq x_2 \leq \ldots \leq x_n\}\ , \tag{2}$$

and its interior

$$U_n = \{x \in R^n\colon\ x_1 < x_2 < \ldots < x_n\}\ , \tag{3}$$

and n-1 boundary hyperplanes:

$$\text{for } 1 \leq i < n,\quad H_n{}^i = \{x \in R^n\colon\ x_i = x_{i+1}\}\ . \tag{4}$$

Notice that the boundary of U_n or S_n is the union of n-1 faces:

$$\partial U_n = \bigcup_{i=1}^{n-1} (H_n{}^i \cap S_n)\ . \tag{5}$$

The system C with n distinct indestructible particles will have S_n as its state space. For the system U involving a decreasing set of surviving particles, the state space will be

$$\underline{U} = \bigcup_{n \geq 1} U_n \quad . \tag{6}$$

Give U the disjoint union topology: each U_n is both open and closed as a subset of $\underline{U}$. The natural bijection between finite subsets of R and elements of $\underline{U}$ will be denoted ι (iota):

$$\iota : \{A \subset R: |A| < \infty\} \to \underline{U} \tag{7}$$

$\iota(A) = (u_1,\ldots,u_n)$ if $A = \{u_1,u_2,\ldots,u_n\}$ and $u_1 < n_2 < \ldots < u_n$.

The system U with state space $\underline{U}$, starting with n particles, is naturally given as a projection of the system C with state space S_n. Formally, we define:

$$\pi : \bigcup_{n \geq 1} R^n \to \underline{U} \; ; \tag{8}$$

$$\pi((x_1,\ldots,x_n)) = \iota(\{x_1,x_2,\ldots,x_n\}) \; .$$

Thus,

$$\pi((x_1,\ldots,x_n)) = (u_1,\ldots,u_m), \quad \text{where}$$

$u_1 < u_2 < \ldots < u_m$ and $\{u_1,\ldots,u_m\} = \{x_1,\ldots,x_n\}$.

It is useful to have a notation for the number of distinct components of an n-tuple x. Formally:

$$\# : \bigcup R^n \to \{1,2,\ldots\} \; ,$$

$$\#x = m \text{ if } \pi(x) \in U_m .$$

The System C: Distinct, Indestructible Particles

A system C with n distinct particles can be viewed as a single point moving in R^n, where the components of the state are interpreted as the positions of n particles on the line. We always follow the convention of labeling the particles in order of their initial position from left to right on the line, i.e., $C(0) \in S_n$. The coalescing interference guarantees that particles cannot cross over each other, so:

$$\text{for } t \geq 0, \quad C(t) \in S_n \,, \tag{9}$$

i.e.,

$$\text{for } t \geq 0, \quad C_1(t) \leq C_2(t) \leq \ldots \leq C_n(t) \,.$$

The coalescing property is, formally:

$$\begin{gathered} \text{for } s > t \geq 0; \quad 1 \leq i,\ j \leq n, \\ C_i(t) = C_j(t) \text{ implies } C_i(s) = C_j(s). \end{gathered} \tag{10}$$

Our goal is to specify, for each $x = (x_1, x_2, \ldots, x_n) \in S_n$, the measure P_x on $C_{S_n}[0, \infty)$ which describes coalescing Brownian motions for n particles starting at $x_1, x_2, \ldots, x_n$. $C_{S_n}[0, \infty)$ denotes the space of continuous functions from $[0, \infty)$ into S_n. The family of measures $(P_x, x \in S_n)$ is a Markov process, with transition function given by formula (85) in this chapter.

Intuitive Description of C

Suppose that the initial point x lies in the interior of S_n, i.e. $C(0) = x \varepsilon U_n$, representing particles at $x_1 < x_2 <\ldots< x_n$. The system C initially performs n dimensional diffusion in the interior of S_n until it hits the boundary of S_n, say at time τ_{n-1}. Let I_{n-1} be the index of the hyperplane $H_n^{I_{n-1}}$ which contains $C(\tau_{n-1})$. (The probability of n dimensional Brownian motion ever hitting the intersection of two n-1 dimensional hyperplanes is zero.)

After τ_{n-1}, the process is trapped in $H_n^{I_{n-1}}$. It performs an n-1 dimensional Brownian motion. The system continues with n-1 degrees of freedom until τ_{n-2}, when another hyperplane $H_n^{I_{n-2}}$ is hit. Now the process is trapped in $H_n^{I_{n-1}} \cap H_n^{I_{n-2}}$ and performs an n-2 dimensional Brownian motion.

The process continues, with each hyperplane H_n^i, $1 \leq i < n$ a trap, until all n-1 hyperplanes have been hit. We list the indices $i = I_{n-1}, I_{n-2},\ldots,I_2, I_1$ of the hyperplanes H_n^i in the order they are hit, at times $0 < \tau_{n-1} < \tau_{n-2} < \ldots < \tau_2 < \tau_1$.

After τ_1, the system is trapped in the line

$$\bigcap_{k=1}^{n-1} H_n^{I_k} = \bigcap_{k=1}^{n-1} H_n^k = \{x \varepsilon R^n : x_1 = x_2 = \ldots x_n\} .$$

The system, representing n particles glued together, performs a one dimensional Brownian motion for all time after τ_1.

The System U: Identical Particles, Distinct Locations

The system U involving identical particles (one fewer particle after each collision) keeps track only of the finite set of positions occupied by the particles. We list the positions of n particles as an n-tuple, in increasing order, so our state space is

$$\underline{U} = \bigcup_{n \geq 1} U_n ,$$

where

$$U_n = \{x \in R^n : x_1 < x_2 < \ldots < x_n\} .$$

Our overall strategy is to define the process C, from independent Brownian motions, using a collision-precedence rule, to construct the process U using the projection π (defined in (8)):

$$U(t) \equiv \pi(C(t)) , \tag{11}$$

and then to show that U is a Markov process.

Intuitive Description of U

A system starting with particles in n distinct locations has $U(0) \in U_n$. It starts performing n dimensional Brownian motion until time τ_{n-1}, when the boundary of U_n is hit. At τ_{n-1}, the state jumps from U_n to U_{n-1}; we have

$$U(\tau_{n-1}) = \pi(U(\tau_{n-1} -)),$$

which is almost surely in U_{n-1} and not in a lower dimensional space. This jump at τ_{n-1} corresponds to the first collision among the n particles; almost surely, there is no multiple collision.

After τ_{n-1} the system performs n-1 dimensional Brownian motion in U_{n-1} until the boundary is hit, say at time τ_{n-2}. The system jumps at τ_{n-2} to

$$U(\tau_{n-2}) = \pi(U(\tau_{n-2} -)) \varepsilon U_{n-2} \qquad \text{(a.s.)}$$

The process continues. When there are two particles left, we have $U(t) \varepsilon U_2$. The system performs two dimensional Brownian motion in this half plane until τ, when the boundary is hit. The system jumps to $U(\tau_1) = \pi(U(\tau_1 -)) \varepsilon U_1$, and performs Brownian motion on the line $(U_1 = R)$ for all future time. τ_1 is the time of the final collision, when the last two particles are replaced by a single particle.

Construction of C Using Collision Precedence Rules

From n arbitrary deterministic paths on the line, we can construct a system of n coalescing paths. Intuitively, each path is assigned one of n ranked particles. Each particle would like to follow the path that it starts on, but after colliding with a particle of higher rank, it must follow that particle along its preferred path. At any time, we have "leader" particles still using their initially assigned path, and "follower" particles tagging along. When two "leader" particles collide, the particles of lower rank, and all of the followers of the particle of lower rank, become followers of the particle of higher rank.

Each of the n! possible rankings gives rise to a different collision precedence rule. We will show that if the initial paths have the distribution of n independent Brownian motions, then the choice of ranking has no effect on the law of the resulting coalescing system.

The Deterministic Collision Precedence Rule

Number the particles $1,2,\ldots,n$, according to their starting position, from left to right on the line. Write $j = m_i(t)$ to indicate that, at time t, particle i is a follower of particle j. If particle j is a leader at time t, then $m_j(t) = j$. If the n arbitrary paths are $B_1,B_2,\ldots,B_n$, and the coalescing paths are $C_1,C_2,\ldots,C_n$, then, for $t \geq 0$; $i = 1,2,\ldots,n$,

$$C_i(t) = B_{m_i(t)}(t) \quad . \tag{14}$$

For $t \geq 0$, write

$$B(t) = (B_1(t),\ B_2(t),\ldots,B_n(t))$$

and

$$C(t) = (C_1(t),\ C_2(t),\ldots,C_n(t)),$$

so that our arbitrary input is

$$B \in C_{R^n}[0, \infty),$$

and the result of applying a collision precedence rule is

$$C \in C_{R^n}[0, \infty).$$

To specify a precedence order, we use

$$\sum_n = \{\sigma : \sigma \text{ is a permutation on } \{1,2,\ldots,n\}\ \}.$$

Given $\sigma \in \sum_n$, we interpret $\sigma(1)$ as the index of highest priority,..., $\sigma(n)$ as the index of lowest priority. If the first collision involves the paths of index i and j, we compare $\sigma^{-1}(i)$ and $\sigma^{-1}(j)$; if $\sigma^{-1}(i) < \sigma^{-1}(j)$ then i had higher priority and both particles continue on the i path.

We now can give a rigorous definition of collision precedence rules which captures the preceding heuristic ideas. The complexity of the definition is partly to handle systems with multiple simultaneous collisions, or with particles starting from coinciding positions.

(15) Definition. For $\sigma \in \sum_n$, $B = (B_1, \ldots, B_n) \in C_{R^n}[0, \infty)$, and $C = (C_1, \ldots, C_n) \in C_{R^n}[0, \infty]$, we say that "C is derived from B using a collision rule with precedence σ," and we write

$$C = \Phi_\sigma(B) \ , \tag{16}$$

if there exist

$$m_1, m_2, \ldots, m_n \in D_{\{1,2,\ldots,n\}}[0, \infty) \ , \tag{17}$$

(D denotes right continuous functions having left limits) such that, for all $t \geq 0$ and $i,j \in \{1,2,\ldots,n\}$ the following hold:

$$C_i(t) = B_{m_i(t)}(t) \ ; \tag{18}$$

$$C_i(0) = B_i(0); \tag{19}$$

$$\sigma^{-1}(m_i(0)) \leq \sigma^{-1}(i), \text{ and } \sigma^{-1}(m_i(\cdot)) \text{ is non-increasing;} \tag{20}$$

$$\text{If } j = m_i(t), \text{ then } j = m_j(t) \ ; \tag{21}$$

$$\text{If } C_i(t) = C_j(t), \text{ then } m_i(s) = m_j(s) \text{ for all } s \geq t. \tag{22}$$

If $C = \Phi_\sigma(B)$, then the coalescing property (10) for C is an immediate consequence of (18) and (22). The continuity of C implies that if $C_i(0) \leq C_j(0)$, then for all $t \geq 0$, $C_i(t) \leq C_j(t)$. Using (19), we can conclude:

$$\text{If } B(0) \in S_n, \text{ then for all } t \geq 0,\ C(t) \in S_n. \tag{23}$$

We want to show that, for any $\sigma \in \sum_n$ and $B \in C_{R^n}[0, \infty)$, there is exactly one $C \in C_{R^n}[0, \infty)$ such that $C = \Phi_\sigma(B)$. The "follower data" $m = (m_1,\ldots,m_n)$ in (17) through (22) is also unique.

$C(t)$ is a function of just $m(t)$ and $B(t)$, by formula (18). We can also determine $m(t)$ from just $C(t)$.

Definition: For $\sigma \in \sum_n$, $i = 1,2,\ldots,n$, let

$$M_{\sigma,i} : R^n \to \{1,2,\ldots,n\} ;$$

$$M_{\sigma,i}(x) = \sigma(\min\{\sigma^{-1}(i) : x_j = x_i,\ j = 1,2,\ldots,n\}). \tag{24}$$

(25) Lemma. If B, C, and m satisfy (17) through (22), then for $t \geq 0$, $i = 1,2,\ldots,n$,

$$m_i(t) = M_{\sigma,i}(C(t)).$$

Proof: Fix $t \geq 0$ and $r \in R$. Let $I_r = \{i : C_i(t) = r\}$. Suppose $I_r \neq \phi$. By (22), $\ell = m_i(t)$ is the same for all $i \in I_r$. By (21), $\ell = m_\ell(t)$. Taking some $i \in I_r$, we have $r = C_i(t) = B_{m_i(t)}(t) = B_\ell(t) = B_{m_\ell(t)}(t) = C_\ell(t)$, so that $\ell \in I_r$ also. For every $i \in I_r$, $\sigma^{-1}(\ell) = \sigma^{-1}(m_i(t)) \leq \sigma^{-1}(m_i(0)) \leq \sigma^{-1}(i)$. Since $\ell \in I_r$, this forces:

$$\ell = \text{the element } i \text{ of } I_r \text{ for which } \sigma^{-1} \text{ is minimized}$$

$$= M_{\sigma,i}(C(t)). \qquad \text{q.e.d.}$$

(26) Lemma. For any $B \in C_{R^n}[0, \infty)$ and $\sigma \in \sum_n$, there is at most one $C \in C_{R^n}[0, \infty)$ which satisfies (17) through (22).

Proof. Suppose $C, C' \in C_{R^n}[0, \infty)$; C satisfies (17) through (22) with $m = (m_1, m_2, \ldots, m_n)$, and C satisfies (17) through (22) with $m' = (m_1', m_2', \ldots, m_n')$. Since $C(0) = B(0) = C'(0)$, (25) implies that $m(0) = m'(0)$. Assume that $m' \neq m$. Let $\tau = \inf\{t : m(t) \neq m'(t)\}$. $\sigma^{-1}(m_i(\cdot))$ is constant apart from negative integer jumps, so there is an $\varepsilon > 0$ such that m and m' are constant in $(\tau, \tau + \varepsilon)$. By the right continuity of m and m', $m(\tau) = \lim_{t \to \tau^+} m(t) = m(\tau + \frac{\varepsilon}{2}) \neq$

$m'(\tau + \frac{\varepsilon}{2}) = m'(\tau)$. Thus, $\tau > 0$. Since $m(s) = m'(s)$ implies $C(s) = C'(s)$, left continuity of C and C' at τ implies $C(\tau) = C'(\tau)$. But (25) then implies $m(\tau) = m'(\tau)$, a contradiction. Thus $m = m'$, and $C = C'$ follows from (18). q.e.d.

(27) Lemma. For any $B \in C_{R^n}[0, \infty)$ and $\sigma \in \sum_n$, there exist $C \in C_{R^n}[0, \infty)$ and $m = (m_1, m_2, \ldots, m_n)$ satisfying (17) through (22).

Proof. We proceed recursively, defining the highest priority components of m and C first. Let

$$m_{\sigma(1)}(\cdot) \equiv \sigma(1),$$

so that $C_{\sigma(1)}(\cdot) = B_{\sigma(1)}(\cdot)$.

Let δ_2 be the first time that the path of second priority hits the path of higher priority; δ_2 is infinite if they never hit. Thus, let

$$\delta_2 = \inf\{t : B_{\sigma(2)}(t) = C_{\sigma(1)}(t)\}.$$

Let

$$\begin{aligned} m_{\sigma(2)}(t) &= \sigma(2) \qquad \text{if } t < \delta_2 ; \\ &= m_{\sigma(1)}(t) \quad \text{if } t \geq \delta_2. \end{aligned}$$

Suppose we already have defined the components of m and C with indices $\sigma(1)$, $\sigma(2), \ldots, \sigma(i)$, and that the properties (17) through

(22) hold for these components. We then define

$$\delta_{i+1} = \inf\{t : B_{\sigma(i+1)}(t) \in \{C_{\sigma(j)}(t),\ j \le i\}\} \ .$$

By continuity, $B_{\sigma(i+1)}(\delta_{i+1}) = C_{\sigma(j)}(\delta_{i+1})$ for some $j \le i$, and by (22), the value, call it n_{i+1}, of $m_j(\delta_{i+1})$ does not depend on the choice of j. Let

$$\begin{aligned} m_{\sigma(i+1)}(t) &= \sigma(i+1) \quad \text{if } t < \delta_{i+1} \ ; \\ &= m_{n_{i+1}}(t) \quad \text{if } t \ge \delta_{i+1} \ . \end{aligned}$$

and

$$C_{\sigma(i+1)}(t) = B_{m_{\sigma(i+1)}(t)}(t) \ .$$

Finally, one checks that these i+1 components of m and C satisfy (17) through (22). We omit the routine verification. q.e.d.

Thus, the relation (16), $C = \Phi_\sigma(B)$, actually defines a function

$$\Phi_\sigma : C_{R^n}[0, \infty) \to C_{R^n}[0, \infty) \ ; \qquad (28)$$

for each B there is a unique C (and unique $m_1, m_2, \ldots, m_n$) such that conditions (17) through (22) are satisfied.

From the construction used to prove the existence of $\Phi_\sigma(B)$ we can observe the following.

(29) Lemma. If $C = \Phi_\sigma(B)$, then for any $t \geq 0$, $C(t)$ is a measurable function of $(B(s), s \leq t)$.

To construct an infinite system in the next chapter, we need a sequence of systems involving more and more particles, such that each particle follows the same path in all of the systems that contain it. The following lemma is the key to this construction; the lemma is an easy corollary of the proof of Lemma (27).

(30) Lemma. Suppose $n' > n$, $\sigma' \in \sum_{n'}$, $\sigma \in \sum_n$, $B' \in C_{R^{n'}}[0, \infty)$ and $B \in C_{R^n}[0, \infty)$. Let $C' = \Phi_\sigma(B')$ and $C = \Phi_\sigma(B)$. If, for $i = 1,2,\ldots,n$,

$$B_{\sigma(i)}(\cdot) = B'_{\sigma'(i)}(\cdot) , \tag{31}$$

then, for $i = 1,2,\ldots,n$,

$$C_{\sigma(i)}(\cdot) = C'_{\sigma'(i)}(\cdot) . \tag{32}$$

Remark. Eq. (31) means that the first n stages in the construction of C and C' agree. The remaining components of C' involve particles of priority lower than n, and do not interfere with the n components of C', indexed by $\sigma'(1),\ldots,\sigma'(n)$, which are of highest priority.

The next lemma, which still considers a single deterministic input B, can be applied in the case where B is random and has independent, identically distributed increments, to show that $C = \Phi_\sigma(B)$ is then a Markov process.

(33) Lemma. Let $t \geq 0$, $\sigma \in \sum_n$, and $B \in C_{R^n}[0, \infty)$. Let $C = \Phi_\sigma(B)$. Define $B' \in C_{R^n}[0, \infty)$ by:

$$\text{for } s \geq 0, \; B'(s) = B(t+s) - B(t) + C(t). \tag{34}$$

Then

$$(\Phi_\sigma(B'))(s) = C(t+s) \; . \tag{35}$$

Proof. Take m such that B, C, and m satisfy conditions (17) through (22). Define C' and m' by:

$$\text{for } s \geq 0, \; C'(s) = C(t+s) \; ; \tag{36}$$

$$m'(s) = m(t+s). \tag{37}$$

In terms of C', this lemma states that

$$C' = \Phi_\sigma(B') \; .$$

By the uniqueness lemma, Lemma (26), we only have to check that B', C', and m' satisfy conditions (17) through (22).

The trickiest property to verify is (18), that $C'_i(t) = B'_{m'_i(t)}(t)$. We will need the following observation:

$$\text{for } i = 1,2,\ldots,n;\ s \geq 0, \quad C_{m_i(t+s)}(t) = B_{m_i(t+s)}(t). \tag{38}$$

<u>Proof of (38)</u>: Intuitively, the particles indexed by $m_i(t+s)$ are leaders at time $t + s$, and thus were leaders at time t. Therefore, $C(t)$ agrees with $B(t)$ for those particles. Formally, write $\ell = m_i(t+s)$. Using (21), $\ell = m_\ell(t+s)$. Since

$$\begin{aligned}
\sigma^{-1}(\ell) &= \sigma^{-1}(m_\ell(t+s)) \\
&\leq \sigma^{-1}(m_\ell(t)) && \text{(by (20))} \\
&\leq \sigma^{-1}(m_\ell(0)) \leq \sigma^{-1}(\ell) && \text{(by (20))},
\end{aligned}$$

we see that $m_\ell(t) = \ell$. Thus

$$\begin{aligned}
C_{m_i(t+s)}(t) &= C_\ell(t) \\
&= B_{m_\ell(t)}(t) && \text{(by (18))} \\
&= B_\ell(t) \\
&= B_{m_i(t+s)}(t) .
\end{aligned}$$

Thus, (38) is established.

Verification of (18) for B', C', m' :

$$
\begin{aligned}
C_i'(s) &= C_i(t+s) && \text{(by (36))}\\
&= B_{m_i(t+s)}(t+s) && \text{(by (18) for } B,C,m)\\
&= B'_{m_i(t+s)}(s) + B_{m_i(t+s)}(t) - C_{m_i(t+s)}(t) && \text{(by (34))}\\
&= B'_{m_i(t+s)}(s) && \text{(by (38))}\\
&= B'_{m_i'(s)}(s) && \text{(by (37))}
\end{aligned}
$$

The remaining conditions, (17), and (19) through (22), are easily verified. q.e.d.

Applying Collision-Precedence Rules to Brownian Motion

So far, we have defined, for each permutation σ on $\{1,2,\ldots,n\}$, a deterministic function Φ_σ which can be applied to any $B \in C_{R^n}[0, \infty)$. B represents n arbitrary paths on the line R. Φ_σ produces $C = \Phi_\sigma(B) \in C_{R^n}[0, \infty)$, a system of n coalescing paths on the line. Now we let B be Brownian motion.

From now on, B will always be random. We assume:

$$\text{for } i = 1,2,\ldots,\ B_i(\cdot) = B_i(\cdot,\ \omega) \in C_R[0,\ \infty); \tag{39}$$

$$B_1, B_2, \ldots \text{ are independent Brownian motions.} \tag{40}$$

We define F_t to be the σ-algebra generated by observing these Brownian motions up to time t:

$$\text{for } t \geq 0, \quad F_t = \sigma(B_i(s),\ 0 \leq s \leq t;\ \ i = 1,2,\ldots) \tag{41}$$

Define I_t to be the σ-algebra generated by observing the increments of these Brownian notions, from time t on:

$$\text{for } t \geq 0, \ \ \ell_t = \sigma((B_i(t+s) - B_i(t))\ ,\ \ s \geq 0;\ i = 1,2,\ldots.). \tag{42}$$

For any $t \geq 0$, F_t and I_t are independent σ-algebras.

For any $x \in S_n$, i.e., with $x_1 \leq x_2 \leq \ldots \leq x_n$, let $B^x = B^x(\omega) = (B_1,\ldots,B_n)$, with $B^x(0) = x$. Fix any precedence order $\sigma \in \sum_n$. Define coalescing Brownian notion in S_n, starting from x, to be any process with the same distribution as

$$C^x = \Phi_\sigma(B^x). \tag{43}$$

We will show, in Theorem (86) that this distribution is not affected by the choice of $\sigma \in \sum_n$. Define a transition function

p_n by:

for $x \in S_n$, G a Borel subset of S_n, and $t \geq 0$,

$$p_n(x,t,G) = P(C^x(t) \in G). \tag{44}$$

Remark. To claim that p_n is a transition function, we need to show that $p_n(\cdot,t,G)$ is a Borel measurable function of x. This will follow from Theorem (46), which says that the distribution of $C^x(t)$ is a continuous function of $x \in S_n$.

(45) Theorem. $(C^x, x \in S_n)$ is a Markov process, with paths in $C_{S_n}[0, \infty)$ and stationary transition function p_n.

Proof. Let $t, s \geq 0$ and G be a Borel subset of S_n.

$$\begin{aligned} &P(C^x(t+s) \in G \mid F_t) \\ &\quad = P((\Phi_\sigma[B^x(t+\ ) - B(t) + C^x(t)])\ (s) \in G \mid F_t) \\ &\quad = p_n(C^x(t), s, G) . \end{aligned}$$

To justify the last equality, notice that, for any $y \in S_n$, $[B(t+\cdot) - B(t) + y]$ is I_t measurable and has the same distribution as B^y, $C^x(t)$ is F_t measurable, and F_t and I_t are independent.

q.e.d.

The Feller Property For C

(46) Theorem. For each $x \in S_n$, $t \geq 0$, as y converges to x in S_n, $C^y(t)$ converges in distribution to $C^x(t)$.

In order to prove this theorem, and also to show, in Chapter 5, that coalescing random walks, appropriately normalized, approach coalescing Brownian motions, we need the following lemma. This lemma is essentially the same as Lemma (4) of Bramson and Griffeath [3].

(47) Lemma. Give $C_{R^n}[0, \infty)$ the topology of uniform convergence on compact subsets of $[0, \infty)$. Then, for any $\sigma \in \sum_n$, and for any $x \in U_n$

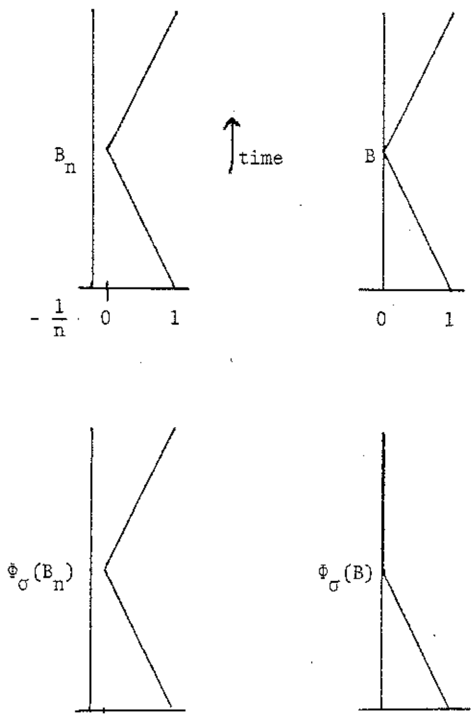
$$\Phi_\sigma : C_{R^n}[0, \infty) \to C_{R^n}[0, \infty)$$

is continuous a.s. with respect to the distribution of Brownian motion in R^n starting from x. An example, Fig.(48), shows that this lemma has content; Φ is not continuous at every B.

Proof. Let $B' \in C_{R^n}[0, \infty)$ be close to B. Write $C = \Phi_\sigma(B)$, $C' = \Phi_\sigma(B')$, and $m_i = M_{\sigma,i}(C)$, $m_i' = M_{\sigma,i}(C')$, for $i = 1,2,\ldots,n$.

Figure 48. An example in $C_{R^2}[0, \infty)$ to show that Φ_σ is not always continuous.

$B_n \to B$ as $n \to \infty$, but $\Phi_\sigma(B_n) \not\to \Phi_\sigma(B)$.



(49) <u>Claim</u>: It is enough to show that m_i' is close to m_i, for $i = 1,2,\ldots,n$, in the topology of $D_{\{1,2,\ldots,n\}}[0, \infty)$.

<u>Sketch of proof of (49)</u>: Fix i. Since m_i and m_i' are integer valued, $m_i \approx m_i'$ means that there is a function λ, uniformly close to the identity function on compact subsets of $[0, \infty)$, such that $m_i'(t) = m_i(\lambda(t))$. Thus

$$\begin{aligned}
C_i'(t) &= B'_{m_i'(t)}(t) \\
&= B'_{m_i(\lambda(t))}(t) \\
&\approx B_{m_i(\lambda(t))}(\lambda(t)) && \text{(since } B \approx B' \text{ and } B \text{ is continuous)} \\
&= C_i(\lambda(t)) \\
&\approx C_i(t). && \text{(since } C_i \text{ is continuous)}
\end{aligned}$$

This establishes the claim, (49).

To show that $m_i' \approx m_i$, [3] argues as follows: Consider the $n(n-1)/2$ first collision times:

$$\tau_{ij} = \tau_{ij}(B) = \inf\{t : B_i(t) = B_j(t),\ 1 \le i < j \le n\}\ . \tag{50}$$

The jump times of m are a subset of $\{\tau_{ij},\ 1 \le i < j \le n\}$, and the sequence of values of m can be determined from the relative order of the $n(n-1)/2$ collision times. The Brownian motion B^x has two properties:

(51) A.s., the $\frac{n(n-1)}{2}$ times τ_{ij} are distinct. (This uses $x \in U_n$.)

(52) $\tau_{ij}(\cdot) : C_{R^n}[0, \infty) \to [0, \infty]$ is a.s. continuous with respect to the distribution of Brownian motion.

Putting these together, we get, for $i = 1,2,\ldots,n$, as $y \to x$ in U_n, $m_i' \to m_i$. This completes the proof of the lemma.

Proof that C^x is Feller: For $x \in U_n$, it follows from Lemma (47) that, for any $t \geq 0$, $C^y(t)$ converges in distribution to $C^x(t)$ as $y \to x$. To handle the general case $x \in S_n$, we use some notation that is introduced in (69) through (71).

Let τ_x be the hitting time for C^y to $\bar{S}_x$, i.e., τ_x is the first time that C^y has all of the coalescence of x:

$$\begin{aligned}\tau_x &= \inf\{t : C^y(t) \in \bar{S}_x\} \qquad (53)\\ &= \inf\{t : \text{for } 1 \leq i < j \leq n,\ x_i = x_j \text{ implies } C_i^y(t) = C_j^y(t)\}.\end{aligned}$$

As y converges to x, τ_x converges in probability to 0, $C^y(\tau_x)$ converges in probability to x, and $\pi(C^y(\tau_x))$ converges in probability to $\pi(x)$. In particular, the probability that $C^y(\tau_x) \in S_x$, i.e., that $\#C^y(\tau_x) = \#x$, approaches 1. Say $\#x = k$, and write $\bar{C}$ for a k particle system of coalescing Brownian motion, a system with state space S_k. On the event $\{\tau_x \leq t,\ C^y(\tau_x) \in S_x\}$, we have

$$C^y(t) = \pi_x^{-1} \left(\bar{C}_{t-\tau_x}^{\pi(C^y(\tau_x))}\right). \tag{54}$$

(This is done in detail in (71) through (73).) The processes $\bar{C}^z$, $z \in S_k$, are all constructed by a collision rule from Brownian motions, so there is uniform stochastic continuity: for any $\varepsilon > 0$

$$\lim_{s\to t} \sup_{z\in S_k} P(|\bar{C}_s^z - \bar{C}_t^z| > \varepsilon) = 0. \tag{55}$$

Putting this together with the Feller property for $\bar{C}^{\pi(x)}$ starting from $\pi(x) \in U_k$, we get convergence in distribution:

$$\text{as } y \to x,\ C_{t-\tau_x}^{\pi(C^y(\tau_x))} \to \bar{C}_t^{\pi(x)} . \tag{56}$$

Since π_x is an isometry,

$$C^y(t) \to \pi_x^{-1} (\bar{C}_t^{\pi(x)}) = C^x(t) . \tag{57}$$

q.e.d.

The Distribution of $\Phi_\sigma(B)$ Does Not Depend on σ

To show that the transition probability p_n is not affected by the choice of a priority rule σ, and to calculate p_n, we will condition on the times and places of collisions.

Fix some priority rule $\sigma \in \sum_n$, and simply write $C = \Phi(B)$ to denote $C = \Phi_\sigma(B)$. Let τ be the time of the first collision for the process C^x starting at $x \in S_n$:

$$\tau = \inf \{t : \#C(t) < \#C(0)\} . \tag{58}$$

Write μ_x for the joint distribution on $[0, \infty) \times S_n$ of $(\tau, C^x(\tau))$. Thus, for I a Borel subset of $[0, \infty)$ and G a Borel subset of S_n,

$$\mu_x(I \times G) = P_x(\tau \in I, C(\tau) \in G). \tag{59}$$

Partition S_n into n subsets S_n^i according to the number of distinct locations occupied by the n particles:

$$S_n^i = \{x \in S_n : \#x = i\} \tag{60}$$

(61) <u>Lemma</u>. Suppose G is a Borel subset of S_n^j, $t \geq 0$, $x \in S_n$, and $\#x > j$. Then

$$p_n(x,t,G) = \int_{(r,y)\in[0,t]\times S_n} p_n(y,t-r,G)\, \mu_x(dr,dy) \tag{62}$$

Proof. In order to apply the strong Markov property to both the time and place of collision, attach a component T, with zero variance and drift one, to serve as a clock. Write T^t for T started as t; thus $T^t(s) = t + s$. Given $t \geq 0$, $x \in S_n$, write $P_{(t,x)}$ for the distribution on $C_{R^{n+1}}[0, \infty)$ of (T^t, C^x). Since (T,C) is a Feller process it has the strong Markov property; it is an n+1 dimensional diffusion. Define a Borel subset A of $C[0, \infty)$ which indicates that C is in G when the clock hits t:

$$A = \{(T,C) : \text{for some } s \geq 0,\ T(s) = t,\ C(s) \in G\}. \tag{63}$$

Since $\#x > j$, there must be a collision before C reaches G:

$$\{(T^o, C^x) \in A\} = \{(T^o, C^x)(\tau + \cdot) \in A,\ \tau \leq t\}. \tag{64}$$

Denote the probability of A, starting from $r \geq 0$ and $y \in S_n$, by

$$F(r,y) = P_{(r,y)}\ ((T,C) \in A). \tag{65}$$

Thus

$$\begin{aligned} F(r,y) &= P(T^r(s) = t,\ C^y(s) \in G, \text{ for some } s) \\ &= P(r+s=t,\ C^y(s) \in G, \text{ for some } s) \\ &= P(C^y(t-r) \in G) \\ &= p_n(y,\ t-r,\ G). \end{aligned}$$

The strong Markov property yields:

$$P_{(0,x)}((T,C)(\tau+\cdot) \in A \mid F_\tau) = F(\tau, C^x(\tau)) \, . \tag{66}$$

Integrating over the event $\{\tau \leq t\} \in F_T$, we get

$$\begin{aligned} &\int_{\{\tau\leq t\}} 1_{(T,C)(\tau+\cdot) \in A} \, dP_{0,x} \\ &= \int_{\{\tau\leq t\}} F(T(\tau), C(\tau)) \, dP_{0,x} \\ &= \int_{\{\tau\leq t\}} p_n(C(\tau), t-T(\tau), G) \, dP_{0,x} \\ &= \int_{0\leq r\leq t, y\in S_n} p_n(y, t-r, G) \, \mu_x(dr, dy). \end{aligned} \tag{67}$$

Since expression (67) is equal to

$$\begin{aligned} &P_{(0,x)}((T,C)(\tau+\cdot) \in A, \tau \leq t) \\ &= P_{(0,x)}((T,C) \in A) \\ &= p_n(x,t,G), \end{aligned} \tag{68}$$

we are done.

q.e.d.

Expressing C in Terms of Brownian Motion in U_k

For any $x \in S_n$, consider the set of all points in S_n which have the same coalescence--i.e., those which lie in exactly the same hyperplanes H_n^i, $i = 1,2,\ldots,n-1$, that x does:

Definition: for $x \in S_n$,

$$S_x = \{y \in S_n : \text{for } 1 \leq i < j \leq n, \quad y_i = y_j \text{ iff } x_i = x_j\}. \tag{69}$$

Since $S_n = \{y \in R^n : y_1 \leq y_2 \leq \cdots \leq y_n\}$, S_x is the collection of $y \in S_n$ such that the strict inequalities $y_i < y_{i+1}$ hold for exactly the same indices in y as in x. Thus the closure of S_x is

$$\bar{S}_x = \{y \in S_n : \text{for } 1 \leq i < j \leq n, \quad x_i = x_j \text{ implies } y_i = y_j\}. \tag{70}$$

$\bar{S}_x$ is precisely the set of states in S_n accessible to C^x.

Suppose $\#x = k$, i.e., $\pi(x) \in U_k$. The restriction $\pi|_{S_x}$ is an isometry between $S_x \subset R^n$ and $U_k \subset R^k$. Define:

$$\pi_x = \text{ the closure of } \pi|_{S_x} , \tag{71}$$

so that π_x is an isometry between $\bar{S}_x \subset R^n$ and $\bar{U}_k \subset R^k$.

Before the time τ of the first collision, C^x is basically k dimensional Brownian motion in U_k, starting at $u = \pi(x)$. To be precise, the data m appearing in formula (18),

$$C_i(t) = B_{m_i(t)}(t) ,$$

is constant before τ. We have

$$m_i(t) = m_i(0) = M_{\sigma,i}(x), \quad \text{for } t < \tau,$$

for the process C^x starting at $x \varepsilon S_n$. The n-tuple $m(0) = (m_1(0), m_2(0), \ldots, m_n(0))$ contains k distinct values. We list them in increasing order by writing $(n_1,\ldots,n_k) = \pi(m(0))$. We can conclude that

$$\pi_x(C(t)) = (B_{n_1}(t), B_{n_2}(t),\ldots,B_{n_k}(t)), \tag{72}$$

for $t < \tau$. This equality extends to $t = \tau$ by continuity.

Thus, the construction of C^x can be described as follows: From $x \varepsilon S_n$ with $\#x = k$, determine the indices $(n_1,n_2,\ldots n_k)$. Write $B' = (B_{n_1}, B_{n_2} \ldots, B_{n_k}) \varepsilon C_{R^k}[0, \infty)$. Note that $B'(0) = \pi(x) \varepsilon U_k$. τ is the time that B' first exits U_k. We have

$$C^x(t) = (\pi_x)^{-1}(B'(t)), \text{ for } t \leq \tau . \tag{73}$$

Given $u \varepsilon U_k$, let B^u be a k-dimensional Brownian motion starting from u. Let τ_u be the hitting time to ∂U_k. Write ν_u for the joint distribution, on $[0, \infty) \times \partial U_k$, of $(\tau_u, B^u(\tau))$: for I a Borel subset of $[0, \infty)$ and H a Borel subset of $\partial U_k = S_k - U_k$,

$$\nu_u(I \times H) = P(\tau_u \in I,\ B^u(\tau) \in H). \tag{74}$$

(75) <u>Lemma</u>. Let $x \in S_n^k$, $u = \pi(x) \in U_k$. Let I be a Borel subset of $[0, \infty)$ and let G be a Borel subset of S_n. Then

$$\mu_x(I \times G) = \nu_u(I \times \pi_x(G \cap \bar{S}_x)), \tag{76}$$

i.e.,

$$\mu_x(ds, dy) = \nu_u(ds,\ d(\pi_x(y))), \tag{77}$$

for $y \in \bar{S}_x$. Furthermore, μ_x is concentrated on $[0, \infty) \times (S_n^{k-1} \cap \bar{S}_x)$.

<u>Proof</u>: Use the special case $t = \tau$ of formula (73):

$$C^x(\tau) = (\pi_x)^{-1}(B'(\tau)), \tag{78}$$

where B' is a Brownian motion in U_k, starting from u. We have:

$$\begin{aligned} \mu_x(I \times G) &= P(\tau \in I,\ C^x(\tau) \in G) \\ &= P(\tau \in I,\ B'(\tau) \in \pi_x(G \cap \bar{S}_x)) \\ &= \nu_u(I \times \pi_x(G \quad \bar{S}_x)). \end{aligned}$$

The probability that a k dimensional Brownian motion ever hits a k-2 dimensional subspace is zero. Thus,

$$\nu_u([0,\infty) \times \{v \in \bar{U}_k : \#v \neq k-1\}) = 0 . \tag{79}$$

Since $x \in S_n^k$, we have:

$$\pi_x^{-1}(\{v \in \bar{U}_k : \#v = k-1\}) = S_n^{k-1} \cap \bar{\bar{S}}_x . \tag{80}$$

Thus, $\mu_x(ds,dy)$ is concentrated on $S_n^{k-1} \cap \bar{\bar{S}}_x$.

q.e.d.

An Explicit Expression for $p_n(x,t,G)$

Lemma (61) enables the transition function $p_n(x,t,dy)$, for y with $\#y < \#x$ to be expressed in terms of the time and place of the first collision of C^x. For y with $\#y = \#x$, there is a direct expression for $p_n(x,t,dy)$.

Let B^u denote k dimensional Brownian motion starting from $u \in U_k$. Given $t \geq 0$, let $\nu_{u,t}$ be the distribution of $B^u(t)$ restricted to U_k. Thus, for H a Borel subset of U,

$$\nu_{u,t}(H) = P(B^u(t) \in (H \cap U_k)) . \tag{81}$$

For $t > 0$, $\nu_{u,t}$ is a defective distribution:

$$\nu_{u,t}(U) < 1.$$

For $x \in S_n^k$, $u = \pi(x) \in U_k$, and G a Borel subset of S_n^k we have $\pi_x(G \cap S_x) \subset U_k$, and

$$p_n(x,t,G) = P(C^x(t) \in G,\ \tau > t) \tag{82}$$

$$= P(C^x(t) \in (G \cap S_x))$$

$$= P(B'(t) \in \pi_x(G \cap S_x) \qquad \text{(by (73))}$$

$$= \nu_{u,t}(\pi_x(G \cap S_x)).$$

The last equality follows since B' is a Brownian motion in U_k starting at u. The relation

$$p_n(x,t,G) = \nu_{u,t}(\pi_x(G \cap S_x)) \tag{83}$$

can be expressed as:

$$\text{for } y \in S_x \quad p_n(x,t,dy) = \nu_{\pi(x),t}(d\pi_x(y)).$$

For with $\#y = \#x$, but $y \notin S_x$, C^x can never reach y.

Using Lemma (75), we can iterate (62). As before, take $x \varepsilon S_n^k$ and G a Borel subset of S_n^j, with $j < k$. A restatement of (62), using the fact that $\mu_x(dt,dy) = 0$ for $y \notin S_n^{k-1}$, is:

$$p_n(x,t,G) = \int_{\substack{t_{k-1}\varepsilon\, [0,t], \\ x_{k-1}\varepsilon\, S_n^{k-1}}} p_n(x_{k-1},t-t_{k-1},G)\mu_x(dt_{k-1},dx_{k-1}) \tag{84}$$

Iterating this formula k-j times yields a formula which expresses the fact that, to get from x to G in time t, C^x must leave S_n^k and enter S_n^{k-1}, at x_{k-1} at time $t_{k-1} > 0$; enter S_n^{k-2} at x_{k-2}, at time $t_{k-2} > t_{k-1}$; continuing one collision at a time until leaving S_n^{j+1} and entering S_n^j at x_j, at time $t_j > t_{j-1}$; and finally staying in S_n^j for $t-t_j$ units of time, to be in G at time t.

$$p_n(x,t,G) = \tag{85}$$

$$\int_{\substack{t_{k-1}\varepsilon[0,t], \\ x_{k-1}\varepsilon S_n^{k-1}}} \int_{\substack{t_{k-2}\varepsilon[t_{k-1},t], \\ x_{k-2}\varepsilon S_n^{k-2}}} \cdots \int_{\substack{t_j\varepsilon[t_{j+1},t] \\ x_j\varepsilon S_n^j}} p_n(x_j,t-t_j,G) \times$$

$$d\mu_{x_{j+1}}(t_{j+1}-t_j,x_j) \ \ldots \ d\mu_{x_{k-1}}(t_{k-2}-t_{k-1},x_{k-2}) \ \ d\mu_x(t_{k-1},x_{k-1}).$$

Lemma (75) and formula (82) express the integrand of the preceding expression in terms of measures ν_u and $\nu_{u,s}$, in a way that does not depend on the choice of a priority rule σ. Thus $p_n(x,t,G) = P(\Phi_\sigma(B^x)(t) \in G)$ really is well-defined. In summary,

(86) Theorem. The process $C^x = \Phi_\sigma(B^x)$, where B^x is Brownian motion starting from $x \in S_n$, is the same distribution, regardless of the priority rule $\sigma \in \sum_n$. Any process with this distribution can be called "coalescing Brownian motions".

Markov Property of U

The process U involving fewer and fewer identical particles is defined as a projection of the process C involving a fixed number of distinguishable particles:

Definition: For $u \in U_n$, U^u is the process in $D_{\underline{U}}[0, \infty)$ given by

$$U^u(t) = \pi(C^u(t)) \ . \tag{87}$$

Suppose H is a Borel subset of U_j, and

$$G = \{y \in S_n : \pi(y) \in H\}. \tag{88}$$

Then G is a Borel subset of S_n^j, and for any $x \in S_n$, $t \geq 0$, $p_n(x,t,G)$ is given by formula (85). Suppose $x \in S_n$, and $u = \pi(x)$, possibly $u = x$. We have

$$P(\pi(C^x(t+s)) \in H \mid F_t) = P(C^x(t+s) \in G \mid F_t) \qquad (89)$$

$$= p_n(C^x(t), s, G).$$

We would like to conclude that this last expression is a function of the form $p_U(\pi(C^x(t)), s, H)$. Setting $t = 0$, for the case $x = u$, would then show that $P(U^u(s) \in H) = p_U(u, s, H)$, so that p_U is the transition function for the Markov process U.

For $t \geq 0$, $u \in U_k$, H a Borel subset of U_j, and $j < k$, we define:

$$p_U(u, t, H) = \qquad (90)$$

$$\int\limits_{\substack{t_{k-1}\in[0,t],\\ u_{k-1}\in\bar{U}_k\cap S_k^{k-1}}} \int\limits_{\substack{t_{k-2}\in[t_{k-1},t],\\ u_{k-2}\in\bar{U}_{k-1}\cap S_{k-1}^{k-2}}} \cdots \int\limits_{\substack{t_j\in[t_{j+1},t],\\ u_j\in \bar{U}_{j+1}\cap S_{j+1}^{j}}} \nu_{\pi(u_j),t-t_j}(H) \,\times$$

$$d\nu_{\pi(u_{j+1})}(t_j-t_{j+1},u_j)\ldots d\nu_{\pi(u_{k-1})}(t_{k-2}-t_{k-1},u_{k-2})\; d\nu_u(t_{k-1},u_{k-1}).$$

For $j = k$, we simply let

$$p_U(u, t, H) = \nu_{u,t}(H). \qquad (91)$$

(92) Lemma. For $x \in S_n^k$, $u = \pi(x)$, $n \geq k \geq j$, H a Borel subset of U_j, and $G = \{y \in S_n : \pi(y) \in H\}$, we have

$$p_n(x,t,G) = p_U(u,t,H). \tag{93}$$

Proof. Proceed by induction on $k-j$. If $k-j = 0$, this is a trivial combination of (82) and (91), since $G \cap S_x = \pi_x^{-1}(H)$. In fact,

$$p_n(x,t,G) = \nu_{u,t}(H) = p_U(u,t,H).$$

The inductive step, to derive the lemma for $k-j > 0$ as a consequence of the lemma for $k-j-1$, depends on

$$p_n(x,t,G) = \int\limits_{\substack{0 \leq s \leq t, \\ y \in S_n^{k-1} \cap \bar{S}_x}} p_n(y,t-s,G)\mu_x(ds,dy) \; . \tag{94}$$

This formula is Lemma (61) combined with Lemma (75). By the induction hypothesis, $p_n(y,t-s,G) = p_U(\pi(y), t-s, H)$. By Lemma (75), $\mu_x(ds,dy) = \nu_u(ds,d(\pi_x(y)))$. Thus

$$p_n(x,t,G) = \int\limits_{\substack{0 \leq s \leq t \\ y \in S_n^{k-1} \cap \bar{S}_x}} p_U(\pi(y),t-s,H) \; \nu_u(ds,d\pi_x(y)).$$

As y runs over $S_n^{k-1} \cap \bar{S}_x$, $w = \pi_x(y)$ ranges over $\bar{U}_k \cap S_k^{k-1}$. Notice that $\pi(w) = \pi(\pi_x(y)) = \pi(y)$. Thus

$$p_n(x,t,G) = \int_{\substack{0 \le s \le t,\\ w \in \bar{U}_k \cap S_k^{k-1}}} p_U(\pi(w),t-s,H)\ \nu_u(ds,dw) . \tag{94}$$

By formula (90), the definition of p_U as an integral, this right hand side is $p_U(u,t,H)$. This completes the induction. q.e.d.

Now, for H a Borel subset of U, define

$$p_U(u,t,H) = \sum_{j=1}^{k} p_U(u,t,H \cap U_j)$$

The previous lemma immediately extends to

(95) Lemma. For $x \in S_n$, $n = \pi(x)$, H a Borel subset of U, and $G = \{y \in S_n : \pi(y) \in H\}$,

$$p_n(x,t,G) = p_U(u,t,H) . \tag{96}$$

(97) Theorem. For any $x \in S_n$, $\pi(C^x)$ is a Markov process on U, adapted to the filtration $(F_t, t \geq 0)$, with transition function $p_U(u, t, H)$.

Proof. For H a Borel subset of U, t, s, ≥ 0, and $x \in S_n$, let $G = \{y \in S_n : \pi(y) \in H\}$. Then

$$\begin{aligned} P(\pi(C^x(t+s)) \in H \mid F_t) &= P(C^x(t+s) \in G \mid F_t) \\ &= p_n(C^x(t), s, G) \\ &= p_U(\pi(C^x(t)), s, H). \end{aligned}$$

q.e.d.

CHAPTER 2

THE INFINITE SYSTEM OF COALESCING BROWNIAN MOTIONS, c

The objective in this chapter is to construct the infinite particle system c, coalescing Brownian motions on R starting with a particle at each point of the line.

(1) Theorem. There is a random process

$$c(\omega) = c = (c_x(t);\ x \in R,\ t \in [0,\infty)) \qquad (2)$$

such that, for any finite $A \subset R$, the system of paths $(c_x(\cdot),\ x \in A)$ is a finite system of coalescing Brownian motions starting from A. In particular, each path $c_x(\cdot)$ is Brownian motion starting at x.

Notation

There is already a problem in notation. The finite system of paths $(c_x(\cdot),\ x \in A)$ lives in $(C[0, \infty))^A$. If $A = n$, "coalescing Brownian motions starting from A" specifies the distribution of a random element C in $C_{R^n}[0, \infty)$, where $C(0) = \iota(A)$. Recall that the map ι lists the finite set A as an n-tuple in increasing order:

$$\imath(A) = (x_1, x_2, \ldots, x_n) \quad \text{if}$$

$$A = (x_1, x_2, \ldots, x_n) \quad \text{and} \quad x_1 < x_2 < \ldots < x_n .$$

To handle this small difference in notation, we agree on the following convention.

A family of paths starting from the points of $A \subset R$ will be denoted $(f_x(\cdot),\ x \in A)$, where for each $x \in A$, $f_x \in C_R[0, \infty)$ and $f_x(0) = x$.

If A is finite and $|A| = n$, then F^A will be the element of $C_{R^n}[0, \infty)$ such that, with

$$\imath(A) = (x_1, x_2, \ldots, x_n) ,$$

we have, for $i = 1,2,\ldots,n$,

$$F_i^A(t) = f_{x_i}(t)$$

Thus

$$F^A(0) = \imath(A).$$

Given a finite family of paths $(f_x,\ x \in A)$, we can form the corresponding $F^A \in C_{R^{|A|}}[0, \infty)$. Conversely, given $F \in C_{R^n}[0, \infty)$, with $F(0) = (x_1, x_2, \ldots, x_n)$ and $x_1 < x_2 < \ldots < x_n$, $A = \{x_1, x_2, \ldots, x_n\}$, we can write $F = F^A$ and we have $(f_x,\ x \in A)$, the family of paths such that, for $i = 1,2,\ldots,n$,

$$f_{x_i}(t) = F_i(t) .$$

With this notation, Theorem (1) can be restated precisely:

<u>Theorem</u>: There exists $c(\omega) = c = (c_x(\cdot),\ x \in R)$, such that, for any $A \subset R$ with $|A| = n$, the system C^A, corresponding to $(c_x(\cdot),\ x \in A)$, has the distribution of coalescing Brownian motion in R^n, starting from $C^A(0) = \imath(A)$.

<u>The Construction of $(c_q,\ q \in Q)$</u>

Rather than work with the rational points of the line, we use the dyadic rationals.

(3) <u>Definition</u>: $Q = \{ \frac{i}{2^n},\ i,\ n \in Z\}$.

The basis for our construction is a countable family

$$b(\omega) = b = (b_q(\cdot),\ q \in Q) \tag{4}$$

of independent Brownian motions, where $b_q(0) = q$ for all $q \in Q$, $\omega \in \Omega$.

For any $t \geq 0$, define the σ-algebra

$$F_t = \text{the completion of } \sigma(b_q(s),\ s \leq t,\ q \in Q). \tag{5}$$

Fix any enumeration $q_1, q_2, q_3, \ldots$ of the dyadic rationals Q.

The idea is to construct, for each $n > 0$, a coalescing system based on the Brownian motions $B_{q_1}, B_{q_2}, \ldots, B_{q_n}$, with the priority rule "q_i before q_j if $i < j$". These priority rules insure that each smaller system is a subsystem of any larger system.

For $n > 0$, define

$$I_n = \{q_1, q_2, \ldots, q_n\} \quad . \tag{6}$$

Thus $(b_q,\ q \in I_n)$ is a family of n independent Brownian motions, with $b_q(0) = q$ for each $q \in I_n$. There is the corresponding element B^{I_n} of $C_{R^n}[0, \infty)$. Formally, take ρ to be the permutation on $\{1,2,\ldots,n\}$ which induces the natural ordering of the line:

$$q_{\rho(1)} < q_{\rho(2)} < \ldots < q_{\rho(n)}. \tag{7}$$

Then $B^{I_n} = (B_1^{I_n}, \ldots, B_n^{I_n})$ where $B_i^{I_n} = b_{q_{\rho(i)}}$, so that $B^{I_n}(0) \in U_n$. Let $\sigma = \rho^{-1}$ be our priority rule. Then the component of B^{I_n} of ith priority is

$$B_{\sigma(i)}^{I_n} = b_{q_{\rho(\sigma(i))}} = b_{q_i} \ , \tag{8}$$

i.e., the particle starting at q_i has ith highest priority. Let $C^{I_n} = \Phi_\sigma(B^{I_n})$, so that $C_i^{I_n}(0) = q_{\sigma(i)}$. Define $c^n = (c_{q_i}(\cdot),\ i = 1,2,\ldots,n)$ to be the family of paths corresponding to C^{I_n}. This means setting $c_{q_i}^n = C_{(i)}^{I_n}$, so that

$$c_{q_i}^n(0) = C_{\sigma(i)}^{I_n}(0) = q_{\rho(\sigma(i))} = q_i. \tag{9}$$

As a consequence of Lemma (1.30) the systems $c^n = (c^n_{q_i}(\cdot), i \leq n)$ agree on their overlap. The following consistency condition holds:

(10) Lemma. For $n > m \geq i \geq 1$,

$$c^n_{q_i} = c^m_{q_i} , \tag{11}$$

so that

$$c^n \text{ is an extension of } c^m. \tag{12}$$

Thus, it makes sense to set, for $i = 1,2,3,\ldots,$

$$c_{q_i} = c^n_{q_i}, \quad \text{where } n \geq i \tag{13}$$

(14) Lemma: $c_{q_i}(t)$ is measurable with respect to $(b_{q_j}(s), s \leq t, j \leq m) \subset F_t$.

Proof. Apply Lemma (1.29) to the system $C^{I_m} = (c_{q_j}, j \leq m)$, which is constructed via a priority rule from $(b_{q_j}, j \leq m)$.

(15) Lemma: For any finite $A \subset Q$, $|A| = m$, the family of paths $(c_r(\cdot), r \in A)$ has the distribution of coalescing Brownian motions. For every ω, and for $r \in A$, $c_r(0) = r$.

Proof. Pick n so large that $A \subset I_n = \{q_1, q_2, \ldots, q_n\}$. The family of paths $C = (c_x, x \in I_n)$ is given by $C = \Phi_\sigma(B^{I_n})$, for a certain priority rule σ. (This is done in detail starting with formula (7).) Let σ' be another priority rule, with the property that the m particles starting in A are the m particles of highest priority. More precisely, suppose that $\iota(A) = (r_1, r_2, \ldots, r_m)$. Then choose σ' to have the property:

$$\text{for } i = 1,2,\ldots,m, \quad B^{I_n}_{\sigma'(i)}(0) = r_i.$$

By Theorem (1.86) the system $(c'_x, x \in I_n) = C' = \Phi_{\sigma'}(B^{I_n})$ has the same distribution as C. In particular, the sub-family of paths $(c'_x, x \in A)$ from C' has the same distribution as the sub-family $(c_x, x \in A)$ of paths from C.

Let $\bar{\sigma}$ be the identity permutation on $\{1,2,\ldots,m\}$. By definition, the system $(\bar{c}_x, x \in A) = \bar{C}^A = \Phi_{\bar{\sigma}}(B^A)$ has the distribution of coalescing Brownian motions.

Lemma (1.30) says that, for each ω, $(\bar{c}_x, x \in A) = (c'_x, x \in A)$. Thus $(c_x, x \in A)$ is equal in distribution to $(c'_x, x \in A)$, which is equal to $(\bar{c}_x, x \in A)$, a system of coalescing Brownian motions.

q.e.d.

(16) Lemma. Suppose $A \subset Q$, A is finite, and $u = \iota(A) \in U_m$. The U-valued evolution defined by

$$U^u(t) = \iota(\{c_q(t),\ q \in A\})$$

is a Markov process adapted to the filtration $(F_t,\ t \geq 0)$, with transition function p_U given by formula (1.90).

Comment. We could conclude that

$$P(U^u(t+s) \in H \mid U^u(r),\ r \leq t) \qquad (17)$$

$$= p_U(U^u(t),s,H)$$

just from Theorem (1.97) and Lemma (15). But we want to condition on F_t, and not just the past as observed by $\sigma(U^u(r),\ r \leq t)$.

Proof. Pick n large enough that $A \subset I_n = \{q_1,q_2,\ldots,q_n\}$. Suppose $q_{\rho(1)} < q_{\rho(2)} < \ldots < q_{\rho(n)}$. Let $J = \{i : q_{\rho(i)} \in A\}$. Let C correspond to $(c_q,\ q \in I_n)$, so that for $t \geq 0$, $\{C_j(t) : j \in J\} = \{c_q(t) : q \in A\}$. Let H be a Borel subset of U, and define

$$G_n = \{x \in S_n : \iota(\{x_j,\ j \in J\}) \in H\} \qquad (18)$$

and

$$G_m = \{y \in S_m : \pi(y) \in H\}. \qquad (19)$$

Lemma (1.95) says, for $y \in S_m$,

$$p_m(y,t,G_m) = p_U(\pi(y),t,H). \tag{20}$$

We want to show that, for $x \in S_n$,

$$p_n(x,t,G_n) = p_U(\iota(\{x_j,\ j \in J\}),\ t,H). \tag{21}$$

Fix $x \in S_n$ and let

$$y = \iota(\{x_j,\ j \in J\}). \tag{22}$$

Let B^x be Brownian motion in R^n starting at x. Let $\sigma' \in \sum_n$ be a priority rule such that

$$J = \{\sigma'(1),\ \sigma'(2),\ldots,\ \sigma'(m)\}.$$

(Thus, if $x = \iota(I_n)$ we have $y = \iota(A)$, and for σ', the particles starting in A have higher priority than those starting in I_n - A.) Let C^y be the system extracted from C^x by taking only the components C_i^x for $i \in J$. By Lemma (1.30) this C^y is the same as a system we could construct with some priority rule in $\sum_m$.

Now we have

$$\begin{aligned} p_n(x,t,G_n) &= P(C^x(t) \in G_n) \\ &= P(\iota(\{C_j^x(t),\ j \in J\}) \ \in H) \\ &= P(\pi(C^y(t)) \in H) \end{aligned}$$

$$= P(C^y(t) \in G_m)$$

$$= p_m(y,t,G_m)$$

$$= p_U(\pi(y),t,H).$$

The last equality is an application of Lemma (1.95). This establishes formula (21).

The proof of the lemma is to compute

$$P(U^u_{t+s} \in H \mid F_t)$$

$$= P(C(t+s) \in G_n \mid F_t)$$

$$= p_n(C(t),\ s,\ G_n)$$

$$= p_U(U^u_t,\ s,\ H). \qquad \text{(using (21))}$$

q.e.d.

The Extension to Paths Starting Everywhere on R

For $x \in R$, define:

$$c_x(t) = \inf \{c_q(t) : q \in Q,\ q \geq x\} ; \qquad (23)$$

$$c_{x-}(t) = \sup \{c_q(t) : q \in Q,\ q < x\} . \qquad (24)$$

This family $(c_x(\cdot),\ x \in R)$ is the object whose existence is claimed in Theorem (1). For $x \varepsilon Q$, this new definition of c_x agrees with the old one because of monotonicity: if $x = q < q'$, then $c_q(t) \leq c_{q'}(t)$ for all t.

As a consequence of the monotonicity of $c_q(\cdot)$ in q, formula (23) is equivalent, for $x \notin Q$, to:

$$c_x(t) = \lim_{q \in Q, q \to x^+} c_q(t). \tag{25}$$

Since the pointwise limit of measurable function is measurable, Lemma (14) yields:

(26) Lemma. For all $x \varepsilon R$ and $t \geq 0$, $c_x(t)$ is F_t measurable.

Monotonicity by itself, however, does not guarantee the uniformity of the convergence for $t \varepsilon [0, \infty)$, or even for t ranging over compact subsets of $[0, \infty)$. In fact, $c_x(\cdot)$ need not even be continuous, as an example shows.

(27) Example. We start with a family $(c_x(\cdot), x \varepsilon (0,1))$ of coalescing trajectories, and define

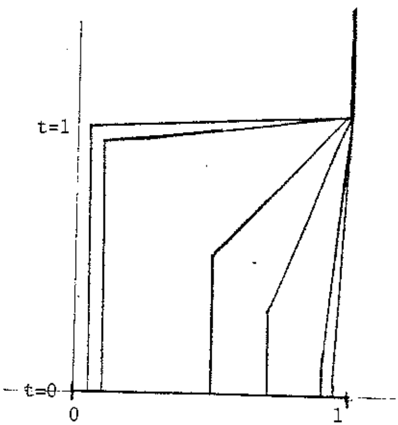
$$c_o(t) = \lim_{x \to 0^+} c_x(t) \quad \text{for each } t \geq 0.$$

For $x \in (0,1)$, $c_x(t) = x$, if $t \leq 1 - x$

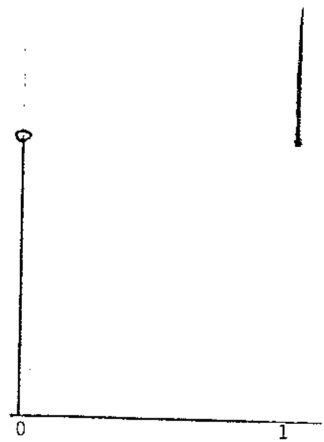
1, if $t \geq 1$

linear, for t between $1-x$ and 1.

Then $c_o(t) = 0$ if $t < 1$, 1 if $t \geq 1$.



$c_x(t)$ for $0<x<1$



$c_o(t) \equiv \lim_{x \to 0^+} c_x(t)$

(28) Lemma. For each $x \in R$, with probability 1, $c_x(\cdot)$ is continuous and

$$\lim_{q \in Q, q \to x^+} \sup_{t \in [0,\infty)} (c_q(t) - c_x(t)) = 0. \tag{29}$$

Remark: The exceptional sets of probability zero, indexed by $x \in R$, have as their union a set of probability zero. We prove this in Chapter 4, Lemma (5). But the easier, weaker lemma here is all that is needed for this chapter and it has a very quick proof.

Proof. Fix $x \in R$ and let $\varepsilon > 0$ be arbitrary. We need to show that

$$P\Big(\lim_{\substack{q \to x^+ \\ q \in Q}} \sup_{t \geq 0} (c_q(t) - c_x(t)) > \varepsilon \Big) \leq \varepsilon. \tag{30}$$

Choose q and $r \in Q$, $r < x < q$, and $q-r < \varepsilon^2$. By monotonicity,

$$\text{for } t \geq 0 \quad c_r(t) \leq c_x(t) \leq c_q(t) .$$

Apply the gambler's ruin formula to $X(\cdot) = c_q(\cdot) - c_r(\cdot)$, which is a Brownian motion (with speed 2) starting from $q-r < \varepsilon^2$, with absorption at 0. The probability that X hits ε before it hits 0 is $\frac{q-r}{\varepsilon} < \frac{\varepsilon^2}{\varepsilon} = \varepsilon$. We have

$$\sup_t (c_q(t) - c_x(t)) \leq \sup_t (c_q(t) - c_r(t)),$$

which is less than ε with probability at least $1-\varepsilon$. q.e.d.

Next, define for each $n \geq 0$, a step function approximation to $f(x) = x$:

$$^{n}\cdot : R \to Q_n$$

$$^{n}x = \min \{\frac{i}{2^n}, \ i \in Z, \ \frac{i}{2^n} \geq x\}. \tag{31}$$

Lemma (28) implies that

$$\lim_{n\to\infty} c_{(^{n}x)}(\cdot) = c_x(\cdot) \quad , \tag{32}$$

in the sup norm, and thus also in the topology of uniform convergence on compact sets. We can now prove Theorem (1).

Proof. Let A be a finite subset of R, say $|A| = m$. Let $(b'_x, \ x \in A)$ be a family of independent Brownian motions with $b'_x(0) = x$. (No relation to the original family $(b_q, \ q \in Q)$ is assumed.) Write B' for the corresponding element of $C_{R^m}[0, \infty)$. Fix some priority rule $\sigma \in \sum_m$. Let $C' = \Phi_\sigma(B')$; by definition C' is coalescing Brownian motion. Let C^A be the element of $C_{R^m}[0, \infty)$ corresponding to the family of paths $(c_x(\cdot), \ x \in A)$. We need to show that C^A has the same distribution as C', i.e., that

$$C^A \stackrel{d}{=} C' \ . \tag{33}$$

Let ${}^nA = \{{}^nx : x \in A\}$. For n large enough, ${}^nA$ has m distinct elements. From the family $(b_q, q \in {}^nA)$ we form the element $B^{{}^nA}$ of $C_{R^m}[0, \infty)$. Take $C^{{}^nA,\sigma} = \Phi_\sigma(B^{{}^nA})$ to get a family of coalescing trajectories starting from ${}^nA$. By Lemma (15), the family of paths $C^{{}^nA} = \{c_q, q \in {}^nA)$ has the same distribution as $C^{{}^nA,\sigma}$. As $n \to \infty$, $B^{{}^nA}$ converges in distribution to B' :

$$B^{{}^nA} \xrightarrow{d} B' \quad . \tag{34}$$

Lemma (1.47) states that Φ_σ is continuous, a.s., with respect to the distribution of B'. Thus

$$\Phi_\sigma(B^{{}^nA}) \xrightarrow{d} \Phi_\sigma(B') \quad . \tag{35}$$

We have

$$C^{{}^nA} = C^{{}^nA,\sigma} = \Phi_\sigma(B^{{}^nA}) \xrightarrow{d} \Phi_\sigma(B') = C' \quad . \tag{36}$$

Lemma (28) implies that,with probability one,

$$\lim C^{{}^nA} = C^A \; ,$$

as elements of $C_{R^m}[0, \infty)$ with the topology of uniform converges on compact sets. Thus

$$C^{{}^nA} \xrightarrow{d} C^A.$$

With (36) this yields

$$C^A \overset{d}{=} C' \quad .$$

This completes the proof of Theorem (1).

CHAPTER 3

THE SYSTEM X_t OF IDENTICAL PARTICLES

For any $A \subset R$, define the set of sites occupied at time $t \geq 0$ by coalescing Brownian motions starting on A to be

$$X_t^A = \{c_x(t),\ x \in A\} . \tag{1}$$

For the special case A = R, with particles starting everywhere, write

$$X_t = X_t^R = \{c_x(t),\ x \in R\} . \tag{2}$$

Thus, for any A, $B \subset R$, and for all $t \geq 0$,

$$X_t^{A \cup B} = X_t^A \cup X_t^B \subset X_t . \tag{3}$$

The first goal of this chapter is to show, for any $A \subset R$ and $t > 0$, that X_t^A is a discrete subset of R -- i.e., that it is a point process. The second goal is to show that, as an evolution in $t > 0$, X_t^A is a Markov process.

Technicality: The Space N for Orderly Point Processes on R

Let the collection of discrete subsets of R be N:

$$N = \{A \subset R : \text{ for } \ell > 0, \ |[-\ell,\ell] \cap A| \text{ is finite}\}. \tag{4}$$

An element A of N will be identified with the measure on R having an atom of mass one at each point of A.

(5) <u>Notation</u>. Write $C_c^+(R)$ for the collection of non-negative continuous functions on R with compact support. For $f \in C_c^+(R)$ and $A \in N$ write

$$(f,A) = \sum_{x \in A} f(x) \ , \tag{6}$$

so that (f,A) is always finite. We will use the vague topology on N, that is, the weakest topology such that, for every $f \in C_c^+(R)$, the map

$$A \to (f,A) \tag{7}$$

is continuous. Then the Borel σ-algebra on N is the smallest σ-algebra such that for every bounded interval $I \subset R$, the map

$$A \to |A \cap I| \tag{8}$$

is measurable.

A Borel measurable map from a probability space into N is a <u>point process</u>. N contains only measures with no multiple points, i.e., the measures in N are composed entirely of atoms of mass one. Thus, an N-valued random variable is an <u>orderly</u>

point process. Our first goal is to prove:

(12) Theorem. With probability one, for every $t > 0$, X_t is a discrete subset of R. For any $A \subset R$ and $t > 0$, X_t^A is an orderly point process on R; X_t^A is F_t-measurable.

Define:

$$\begin{aligned} N_o &= \{A \in N : |A| < \infty\} \\ &= \{A \subset R : |A| < \infty\} \end{aligned} \tag{9}$$

The map $\iota : N_o \to \underline{U}$ introduced in (1.7) is a homeomorphism. We take the liberty of identifying $A \in N_o$ with $\iota(A) \in \underline{U}$.

Suppose $A \in N_o$. With this identification, the $\underline{U}$-valued Markov process $U^{\iota(A)}$ given by

$$U^{\iota(A)}(t) = \iota(\{c_x(t),\ x \in A\})$$

is the same as the N-valued process X^A defined by

$$X_t^A = \{c_x(t),\ x \in A\}.$$

The translation of the Markov property for $U^{\iota(A)}$, established in Lemma (2.16) for $A \subset Q$, A finite, is:

(10) Lemma. For H a Borel subset of N_o, $A \in N_o$, $A \subset Q$, and $t, s \geq 0$,

$$P(X^A_{t+s} \in H \mid F_t) = p_U(X^A_t(\omega),\ s,\ H). \tag{11}$$

Heuristics of the Proof that X_t is Discrete

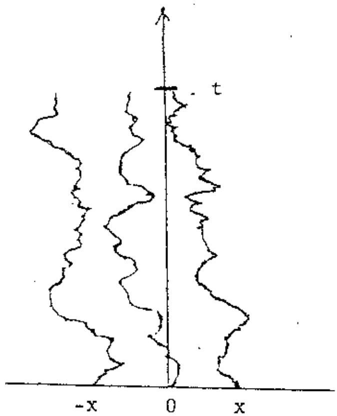
In order to prove that X_t is discrete for $t > 0$, we construct X_t in stages, looking first at X^Z_t, the set of points reached at time t by particles starting on the integers. At the nth stage we consider $X^{Q_n}_t$, the set of points reaches by particles starting on the nth level dyadic rationals—numbers of the form $\frac{i}{2^n}$. For a given compact set $[-\ell,\ell]$ (a window through which to observe the infinite set X_t), we show that the probability of finding any new points in going from the nth stage to the $(n+1)^{st}$ stage is $O(2^{-n})$. Thus, in that fixed compact set, the probability of finding any new points after the nth stage is also $O(2^{-n})$.

An Estimate on the Probability of Coalescing

The following lemma is the key to proving Theorem (3.8). The analogous lemma, for coalescing simple random walks, appears in Bramson and Griffeath [4]. The Brownian motion version is easier to prove.

(12) Lemma. For $t > 0$, $x > 0$,

$$P(|X_t^{\{-x,0,x\}}| = 3) < (\frac{2}{\pi t})\ x^2 .$$



Proof. Start with independent Brownian motions b_{-x}, b_o, and b_x starting from $-x$, 0, and x. Construct a coalescing family $(c'_{-x},\ c'_o,\ c'_x)$ from these using a collision rule. There is an equality of events:

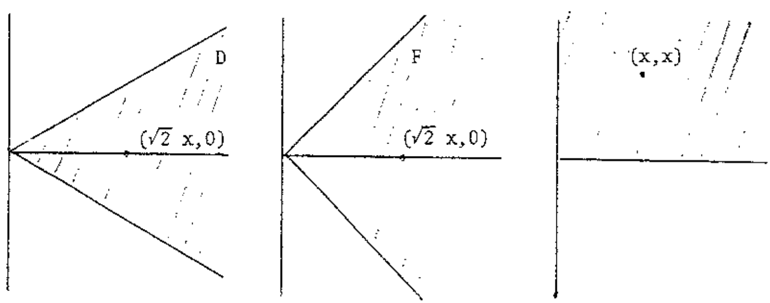
$$\begin{aligned} &\{|\{c'_{-x}(t),\ c'_o(t),\ c'_x(t)\}| = 3 \\ &= \{|\{b_{-x}(s), b_o(s), b_x(s)\}| = 3 \text{ for all } s \le t\}. \end{aligned} \tag{13}$$

Since $X_t^{\{-x,0,x\}} = \{c_{-x}(t),\ c_o(t),\ c_x(t)\}$ has the same distribution as $\{c_{-x}(t),\ c'_o(t),\ c'_x(t)\}$, our goal is to show that the right side of (13) has probability $< (\frac{2}{\pi t})x^2$. Let

$$a_1 = (b_x - b_{-x})/\sqrt{2}, \quad a_2 = (b_{-x} - 2b_o + b_x)/\sqrt{6} \quad ; \tag{14}$$

The correlation matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that (a_1, a_2) is standard Brownian motion in R^2 starting at $(\sqrt{2}x, 0)$. Let D be the wedge

$$D = \{(x_1, x_2) : \sqrt{3}\ |x_2| < x_1\} \tag{15}$$



and let F be the larger wedge

$$F = \{(x_1,x_2)\ ;\ |x_1| < x_2\}. \tag{16}$$

The last event in (13) is equal to

$$\{(a_1(s),\ a_2(s)) \in D \quad \text{for all } s \leq t\} \tag{17}$$

and this is contained in the event

$$\{(a_1(s),\ a_2(s)) \in F \quad \text{for all } s \leq t\}. \tag{18}$$

Rotate the picture of F by 45°, to see that the event (18) says that each of two independent Brownian motions on the line, starting from x, runs for time t without hitting zero. The probability of each of these independent events, by the reflection principle, is

$$\int_{-x}^{x} \frac{1}{\sqrt{2\pi t}}\ e^{-\frac{y^2}{2t}}\ dy < \frac{2x}{\sqrt{2\pi t}}\quad . \tag{19}$$

Thus,

$$\begin{aligned} P(|X_t^{\{-x,0,x\}}| = 3) &= P(\ (a_1(s),a_2(s)) \in D \text{ for all } s \leq t\ ) \\ &\leq P(\ (a_1(s),a_2(s)) \in F \text{ for all } s \leq t) \\ &< \left(\frac{2x}{\sqrt{2\pi t}}\right)^2 = \frac{2}{\pi t}\, x^2 \quad . \end{aligned}$$

q.e.d.

Proof that X_t is Discrete

For the remainder of this chapter, fix $t > 0$. Now fix a compact interval $[-m,m]$ and consider, for $n = 0,1,2,\ldots$, the sets

$$X_t^{[-m,m] \cap Q_n} = \{c_q(t) : q \in [-m,m] \text{ and } 2^n q \in Z\} . \tag{20}$$

We calculate

$$P(X_t^{[-m,m] \cap Q_n} \neq X_t^{[-m,m] \cap Q_{n+1}})$$

$$= P\Big(\bigcup_{-m\cdot 2^n \leq i < m\cdot 2^n} \{ \Big| X^{\{\frac{i}{2^n}, \frac{i+1/2}{2^n}, \frac{i+1}{2^n}\}} \Big| = 3\} $$

$$\leq \quad 2m\cdot 2^n \cdot \frac{2}{\pi t} (2^{-(n+1)})^2$$

$$= \frac{m}{\pi t} \; 2^{-n} .$$

Thus

$$P(X^{[-m,m] \cap Q} \neq X^{[-m,m] \cap Q_k})$$

$$= P(\bigcup_{n \geq k} \{X^{[-m,m] \cap Q_n} \neq X^{[-m,m] \cap Q_{n+1}}\})$$

$$\leq \sum_{n \geq k} \frac{m}{\pi t} 2^{-n}$$

$$= \frac{2m}{\pi t} 2^{-k} .$$

Thus there is a finite random variable $K = K(m,t,\omega)$ such that

$$X_t^{[-m,m]} \cap Q = X_t^{[-m,m]} \cap Q_K \qquad \text{a.s.} \qquad (21)$$

To conclude that X_t^Q is a discrete set almost surely, we need an easy estimate expressing the local nature of coalescing Brownian motions.

For $m, \ell > 0$, define an event $E_{m,\ell}$, on the complement of which there is no influence from outside $[-m,m]$ that is visible at time t inside the window of observation $[-\ell,\ell]$:

$$E_{m,\ell} = \{c_m(t) \leq \ell \quad \text{or} \quad c_{-m}(t) \geq -\ell\}$$

Because of the monotonicity of $c_x(\cdot)$ in x, these events form a decreasing sequence as m increases:

$$E_{m',\ell} \subset E_{m,\ell} \quad \text{if } m' > m.$$

Using Theorem (2.1), which asserts that each path c_x is a Brownian motion, we can compute:

$$P(E_{m,\ell}) \leq 2\, P\{c_m(t) \leq \ell\}$$

$$= 2 \int_{-\infty}^{\ell-m} \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}}\, dx \qquad \to 0, \text{ as } m \to \infty\ .$$

Since

$$P(\bigcap_{m>0} E_{m,\ell}) = \lim_{m\to\infty} P(E_{m,\ell}) = 0,$$

there is a finite random variable $M = M(\ell,t,\omega)$, such that $E_{M,\ell} = \phi$ a.s. In other words,

$$c_{-M}(t) < -\ell \quad \text{and} \quad c_M(t) > \ell \qquad \text{a.s.} \qquad (22)$$

The monotonicity of c_x in x implies

$$[-\ell,\ell] \cap X_t^Q = [-\ell,\ell] \cap X_t^{Q\,[-M,M]} \qquad \text{a.s.}$$

Combining this with (3.24), for $K = K(M(\ell,t,\omega),\ t,\ \omega)$,

$$[-\ell,\ell] \cap X_t^Q = [-\ell,\ell] \cap X_t^{Q_K\ [-M,M]} \qquad \text{a.s.} \qquad (23)$$

The set of starting locations $Q_K \cap [-M,M]$ has $(2M \cdot 2^K)+1$ elements, so that

$$|[-\ell,\ell] \cap X_t^Q| \leq (2M \cdot 2^K) + 1 \qquad \text{a.s.}$$

This proves that with probability one X_t^Q is a discrete set. Recall that for $x \notin Q$, $c_x(t) = \lim_{\substack{q\in Q \\ q\to x^+}} c_q(t)$. Thus from "$X_t^Q$ is discrete, a.s.", we conclude that, actually,

$$(\text{for all } x \in R,\ c_x(t) = \min\{c_q(t) : q \in Q,\ q \geq x\}) \qquad \text{a.s.} \qquad (24)$$

and

$$X_t = X_t^Q \qquad \text{a.s.} \qquad (25)$$

Technicality: Measurability

Recall the enumeration $Q = \{q_1, q_2, \ldots\}$ and the sequence of finite sets I_n approximating $Q : I_n = \{q_1, q_2, \ldots, q_n\}$. For n large enough, depending on ℓ, t, and ω,

$$I_n \supset Q_K \cap [-M,M] \ . \tag{26}$$

Thus (23) and (25) imply

$$[-\ell,\ell] \cap X_t = \lim_{n\to\infty} ([-\ell,\ell] \cap X_t^{I_n}) \qquad \text{a.s.} \tag{27}$$

in the sense that equality is eventually obtained. Since this is true for all compact intervals $[-\ell,\ell]$, we get

$$X_t = \lim_{n\to\infty} X_t^{I_n} \qquad \text{a.s.,} \tag{28}$$

in the vague topology on N. Because the limit of measurable maps is also measurable,

$$X_t = \lim X_t^{I_n} \quad \text{is a point process.} \tag{29}$$

To show that X_t^A is a measurable map into N, for any (not necessarily measurable) $A \subset R$ is more work. Define, for $y \in R$, $t \geq 0$,

$$\ell_t(y) = \inf \{r \in Q : c_r(t) = c_y(t)\} \tag{30}$$

and $$r_t(y) = \sup \{r \in Q : c_r(t) = c_y(t)\} \ . \tag{31}$$

Notice that for any $t \geq 0$ and $y \in R$, $\ell_t(y)$ and $r_t(y)$ are F_t - measurable. From

$$X_t = X_t^Q \quad \text{is discrete} \qquad \text{a.s.} \qquad (32)$$

and the definition

$$c_x(t) = \inf \{c_q(t) : q \in Q,\ q \geq x\}$$

it follows, first with the aid of Lemma (2.28), that,

$$\text{for } q \in Q,\ \ell_t(q) < r_t(q) \qquad \text{a.s.,} \qquad (33)$$

and then, that

$$X_t^A = \{c_q(t) : q \in Q,\ \ell_q(t) \leq x < r_q(t) \quad \text{for some } x \in A\} \qquad \text{a.s.} \qquad (34)$$

We want to show, for $q \in Q$, that

$$\{\omega : \ell_q(t) \leq x < r_q(t) \quad \text{for some } x \in A\} \in F_t. \qquad (35)$$

Given $A \subset R$ define $f_A : R \to R$ by

$$f_A(x) = \inf \{y : y \in A,\ y \geq x\}\ . \qquad (36)$$

Since f_A is monotone, it is Borel measurable. Define

$$S_A = \{(x_1, x_2) \in R^2 : \text{for some } q \in A,\ x_1 \leq a < x_2\}. \qquad (37)$$

S_A is Borel measurable because

$$S_A = \{(x_1,x_2) \in R^2 : x_2 > f_A(x_1)\} .$$

We have, for any $q \in Q$,

$$\begin{aligned} \{\omega : c_q(t) &= c_x(t) \text{ for some } x \in A\} \\ &= \{\ell_q(t) \leq x < r_q(t) \text{ for some } x \in A\} \quad \text{a.s.} \\ &= \{\ell_q(t),\ r_q(t)) \in S_A\} \in F_t . \end{aligned}$$

Since F_t is complete, we conclude that

$$\{\omega : c_q(t) = c_x(t) \quad \text{for some } x \in A\} \quad \in F_t . \tag{38}$$

Recall that $I_n = \{q_1, q_2, \ldots, q_n\}$. Define a random subset of I_n:

$$A(n,t) = \{q \in I_n : c_q(t) = c_x(t) \text{ for some } x \in A\} \tag{39}$$

The measurability statement (38) shows that, for any of the 2^n possible $I \subset I_n$,

$$\{A(n,t) = I\} \in F_t . \tag{40}$$

Thus

$$X_t^{A(n,t)} = \{c_q(t) : q \in A(n,t,\omega)\}$$

is an F_t-measurable point process. For n large enough that

$$[-\ell,\ell] \cap X_t = [-\ell,\ell] \cap X_t^{I_n} ,$$

we have

$$[-\ell,\ell] \cap X_t^A = [-\ell,\ell] \cap X_t^{A(n,t)} .$$

Thus, in the vague topology on N,

$$X_t^A = \lim_{n\to\infty} X_t^{A(n,t)} \qquad \text{a.s.} \qquad (41)$$

This shows that X_t^A is a point process and is F_t - measurable.

Formula (28), ($X_t = \lim_{n\to\infty} X_t^{\{q_1,q_2,\ldots,q_n\}}$ a.s.), expresses X_t in terms of one particular countable dense set of starting locations. A generalization of this will be useful in Chapter 4.

(42) Lemma. Suppose $\{a_1,a_2,\ldots\}$ is any countable dense subset of R. For any $t > 0$,

$$X_t = \lim_{m\to\infty} X_t^{\{a_1,a_2,\ldots,a_m\}} \qquad \text{a.s.}$$

Proof. Fix a compact interval $[-\ell,\ell]$. With probability one,

$$[-\ell,\ell] \cap X_t = [-\ell,\ell] \cap X_t^{I_N} ,$$

where N is random and finite. With probability one, each of the half open intervals

$$[\ell_q(t),\ r_q(t)) \quad \text{for } q \in I_N$$

is actually non-empty. Since $\{a_1, a_2, \ldots\}$ is dense in R, there is a finite m, depending on the 2N random variables ($\ell_{q_i}(t)$, $r_{q_i}(t)$, for $i = 1,2,\ldots,N$), such that

$$\text{for } i = 1,2,\ldots,N,$$

$$[\ell_{q_i}(t),\ r_{q_i}(t)) \cap \{a_1, a_2, \ldots, a_m\} \neq \phi$$

For this value of m we have

$$[-\ell,\ell] \cap X_t = [-\ell,\ell] \cap X_t^{\{a_1,\ldots,a_m\}} \quad \text{a.s.}$$

q.e.d.

The Markov Property for X_t

Let H be a Borel subset of N which only depends on the compact interval $[-\ell,\ell] \subset R$, i.e., suppose that there is a Borel subset G of N_o such that

$$H = \{A \in N : A \cap [-\ell,\ell] \in G\} . \tag{43}$$

Let

$$H_o = H \cap N_o . \tag{44}$$

The distribution of a point process such as X_t is determined by the quantities

$$P(X_t \in H) = P(X_t \cap [-\ell,\ell] \in G)$$

for such "tame Borel sets" H.

Let $t > 0$. We compute the transition function for X.

$$P(X_{t+s} \in H \mid F_t)$$

$$= \lim_{n\to\infty} P(X_{t+s}^{I_n} \in H_o \mid F_t) \qquad \text{(by (27))}$$

$$= \lim_{n} p_U(X_t^{I_n}, s, H_o) \qquad \text{(by (10))}$$

$$= \lim_{m\to\infty} \lim_{n} p_U(X_t^{I_n} \cap (-m,m), s, H_o) \qquad \text{(see below)}$$

$$= \lim_{m} p_U(X_t \cap (-m,m), s, H_o). \tag{45}$$

To justify the next-to-last equality, we reason as follows: Let $\varepsilon > 0$ be arbitrary. Pick m so large that

$$P(c_m(s) < \ell \text{ or } c_{-m}(s) > -\ell) < \varepsilon .$$

Then

$$\sup_{B\in N} P(X_s^B \cap [-\ell,\ell] \neq X_s^{B\cap(-m,m)} \cap [-\ell,\ell]) < \varepsilon$$

Thus

$$\sup_{B\in N} \left| p_U(B \cap (-m,m),s,H_o) - p_U(B,s,H_o)\right| < \varepsilon .$$

This shows that

$$\lim_n p_U(X_t^{I_n},s,H_o)$$

and

$$\lim_n p_U(X_t^{I_n} \cap (-m,m),s,H_o)$$

are at most ε apart, for large enough m. Since $\varepsilon > 0$ was arbitrary, the justification is complete.

The Markov Property for X_t^A

The notation, comments, and justifications, from the previous section on the Markov property of X_t, apply again here.

We first need to establish that, for $t > 0$, $s \geq 0$,

$$X_{t+s}^A = \lim_{n\to\infty} X_{t+s}^{A(n,t)} . \tag{46}$$

Note that the time involved in choosing the set $A(n,t)$ of starting locations is t, not t+s. To prove this claim, fix an arbitrary compact window $[-\ell,\ell]$. By (22) there is a finite random $M = M(\ell,t+s,\omega)$ such that

$$c_{-M}(t+s) < -\ell \quad \text{and} \quad c_M(t+s) > \ell \qquad \text{a.s.}$$

There is a finite random integer N by (26) and (39) such that

$$X_t^A \cap [-M,M] = X_t^{A(N,t)} \cap [-M,M] \qquad \text{a.s.}$$

These last two statements combine to yield

$$X_{t+s}^A \cap [-\ell,\ell] = X_{t+s}^{A(N,t)} \cap [-\ell,\ell] \qquad \text{a.s.} \quad (47)$$

Since ℓ was arbitrary, this establishes the claim, formula (46).

We now show that X_t^A is Markov. For $t > 0$, $s \geq 0$,

$$P(X_{t+s}^A \in H \mid F_t) \tag{48}$$

$$= \lim_{n\to\infty} P(X_{t+s}^{A(n,t)} \in H_o \mid F_t)$$

$$= \lim_{n} \sum_{I \subset I_n} P(X_{t+s}^{A(n,t)} \in H_o,\ A(n,t) = I \mid F_t)$$

$$= \lim_{n} \sum_{I \subset I_n} P(X_{t+s}^{I} \in H_o,\ A(n,t) = I \mid F_t)$$

$$= \lim_{n} \sum_{I \subset I_n} 1_{\{A(n,t)=I\}} \cdot P(X_{t+s}^{I} \in H_o \mid F_t)$$

$$= \lim_{n} \sum_{I \subset I_n} 1_{\{A(n,t)=I\}}\ p_U(X_t^I, s, H_o)$$

$$= \lim_{n} \; p_U(X_t^{A(n,t)}, \; s, \; H_o)$$

$$= \lim_{m\to\infty} \lim_{n} \; p_U(X_t^{A(n,t)} \cap (-m,m), \; s, \; H_o)$$

$$= \lim_{m\to\infty} \; p_U(X_t^A \cap (-m,m), \; s, \; H_o). \tag{49}$$

This establishes the Markov property of X_t^A for any $A \subset R$. The same function of the present state occurs in (49) regardless of the choice of A. Thus, we can define p, the transition function of the processes $((X_t^A, \; t > 0), \; A \subset R)$.

For H a tame Borel subset of N (see (43)), for $s \geq 0$, and for $B \in N$, let

$$p(B,s,H) = \lim_{m\to\infty} \; p_U(B \cap (-m,m), s, H \cap N_o) \tag{50}$$

We can summarize the result of this chapter as follows:

(51) Theorem. For any $A \subset R$, $(X_t^A, \; t > 0)$ defined by $X_t^A = \{c_x(t) : x \in A\}$ is an N-valued Markov process, adapted to the filtration $(F_t, \; t > 0)$, with transition function p.

CHAPTER 4

THE SELF-SIMILARITY OF c

In this chapter, we consider $c = (c_x(\cdot),\ x \in R)$ as an evolution in the starting location x. For each $x \in R$, the path $c_x(\cdot)$ followed by the particles starting at x is an element of

$$E = C_R[0,\ \infty), \tag{1}$$

with the topology of uniform convergence on compact subsets of $[0,\ \infty)$. Thus, for each ω, c can be viewed as a map

$$\begin{aligned} c : R &\to E , \\ x &\to c_x(\cdot) . \end{aligned} \tag{2}$$

We will show that, as a function of x, c is right continuous and has left limits. Thus,

$$c \in D_E(R) . \tag{3}$$

It is a general result that a sequence of processes in $D_E(R)$ converges in distribution to a process in $D_E(R)$ if and only if the sequence is "tight" and the finite dimensional distributions converge. The finite dimensional distributions of $c \in D_E(R)$ refer to the families of trajectories

$$(c_x(\cdot),\ x \in A) \text{ for } A \subset R,\quad |A| \text{ finite},$$

which we have already studied. We can exploit this in the next chapter to show that c is the limit of appropriately normalized systems of coalescing simple random walks on the lattice of integers. In this chapter, we use the self similarity of Brownian motion, together with the general fact that the distribution of a process in $D_E(R)$ is determined by its finite dimensional distributions, to establish the self-similarity of c. Recall the definitions of c_x and c_{x-}

$$\begin{aligned} c_x(t) &= \inf\{c_q(t),\ q \in Q,\ q \geq x\} \\ c_{x-}(t) &= \sup\{c_q(t),\ q \in Q,\ q < x\}. \end{aligned} \tag{4}$$

(5) Lemma. $c = (c_x,\ x \in R) \in D_E(R)$ a.s. In fact, with probability one, c is right continuous and has left limits, in the sup norm on $E = C_R[0, \infty)$, at every $x \in R$:

$$\lim_{y\to x^+} \sup_{t\geq 0} (c_y(t) - c_x(t)) = 0 \quad \text{and} \quad \lim_{y\to x^-} \sup_{t\geq 0} (c_{x-}(t) - c_y(t)) = 0. \tag{6}$$

Proof. For each $q \in Q$, we apply Lemma (2.28) to conclude that in the sup norm on E, $c_q = \lim_{y\to q^+} c_y$, a.s. Let

$$\Omega_1 = \bigcap_{q\in Q} \{\omega : c_q = \lim_{y\to q^+} c_y\} \cap \bigcap_{\substack{t>0, \\ t \text{ rational}}} \{\omega : X_t \text{ is discrete}\}. \tag{7}$$

As the countable intersection of almost sure events, Ω_1 has probability one. Fix $\omega \in \Omega_1$ and $x \in R$. We will show that in the sup norm on E, $c_x = \lim_{y \to x^+} c_y$. For $x \in Q$, this follows from the assumption that $\omega \in \Omega_1$. For $x \notin Q$, argue in the following way.

Given $\varepsilon > 0$, pick n such that $2^{-n} < \varepsilon$. Write $a = {}^n x$ and $b = a + 2^{-n}$. Thus

$$a, b \in Q, \quad a \leq x < b, \text{ and } \quad b - a < \varepsilon. \tag{8}$$

Since c_a and c_b are continuous, there is a rational $t_o > 0$ such that

$$\text{for } s \in [0, t_o], \quad c_a(s) > a - \varepsilon \quad \text{and} \quad c_b(s) < b + \varepsilon. \tag{9}$$

Since X_{t_o} is discrete, the definition of c_x implies that we can choose q with

$$q \in Q, \ x \leq q \leq b, \quad c_x(t_o) = c_q(t_o). \tag{10}$$

From the assumption that $x \notin Q$, we conclude that $x < q$. For y in the interval (x, q),

$$\sup_{t \geq 0} |c_y(t) - c_x(t)| \tag{11}$$

$$\leq \sup_{t \geq 0} \ c_q(t) - c_x(t)$$

$$= \sup_{0 \leq t \leq t_o} c_q(t) - c_x(t)$$

$$\leq \sup_{0 \leq t \leq t_o} c_b(t) - c_a(t)$$

$$\leq (b+\varepsilon) - (a-\varepsilon)$$

$$< 3\varepsilon \quad .$$

Since $\varepsilon > 0$ is arbitrary, this establishes the right continuity of c at x.

A similar argument shows that,

$$\text{for all } x \in R, \; c_{x-} = \lim_{y \to x^-} c_y \; . \tag{12}$$

There is no need here for a special argument if $x \in Q$, since the definition $c_{x-}(t) = \sup \{c_q(t) : q \in Q, \, q < x\}$ involves the strict inequality $q < x$.
q.e.d.

Technicality: $c\colon \Omega \to D_E(R)$ is Measurable

For any $f \in D_E(R)$ we can define step functions f_n which approximate f:

$$^n f(x) = f(\frac{i}{2^n}) \quad \text{for} \quad \frac{i}{2^n} \leq x < \frac{i+1}{2^n} \; , \quad i \in Z, \tag{13}$$

or equivalently

$$^n f(x) = f(^n x) \; .$$

It is a general result that as elements of $D_E(R)$,

$$^n f \to f. \tag{14}$$

Applying this to c, we obtain approximations $^n c \in D_E(R)$, defined by

$$^n c_x(t) = c_{(^n x)}(t). \tag{15}$$

Notice that, in contrast to our convention about starting locations, $^n c_x(0) \neq x$ unless $x \in Q_n$:

$$^n c_x(0) = {}^n x. \tag{16}$$

The range of $^n c$ is $\{c_q(\cdot) : q \in Q_n\}$. Since each c_q is a measurable function

$$c_q : \Omega \to E,$$

one also sees that

$$^n c : \Omega \to D_E(R) \text{ is measurable.} \tag{17}$$

Since $^n c \in D_E(R)$ a.s., we immediately have:

(18) <u>Lemma.</u> In $D_E(R)$,

$$^n c \to c \qquad \text{a.s.} \tag{19}$$

Thus,

$$c : \Omega \to D_E(R) \text{ is measurable.} \tag{20}$$

Self Similarity

The self similarity of Brownian motion is expressed by:

If b is standard Brownian motion, then for any $s > 0$, the process $\bar{b}$ defined by

$$\bar{b}(t) = \frac{1}{\sqrt{s}}\, b(ts) \tag{20}$$

is also standard Brownian motion.

It is easy to show that this property is inherited by finite coalescing systems. We introduce a multiplicative notation for rescaling subsets of R:

$$\text{For } A \subset R,\ r > 0,\ r\,A = \{rx : x \in A\}\ . \tag{21}$$

(22) Lemma. Let $s > 0$, $A \subset R$, $|A| = n$, and suppose that $C = (c_x(\cdot),\ x \in \sqrt{s}\, A)$ is a system of coalescing Brownian motions starting from $\sqrt{s}\, A$. Define another family of paths $\bar{C} = (\bar{c}_y(\cdot),\ y \in A)$ by

$$\bar{c}_y(t) = \frac{1}{\sqrt{s}}\ c_{y\sqrt{s}}\,(ts) \quad \text{for } y \in A,\ t \geq 0. \tag{23}$$

Then $\bar{C}$ is a system of coalescing Brownian motions starting from A.

<u>Proof</u>. By definition, C is equal in distribution to a system C' formed by a collision rule from a family B of independent Brownian motions starting from $\sqrt{s}\, A$:

$$C \overset{d}{=} C' = \Phi(B) ,$$

where

$$B = (b_x(\cdot),\ x \in \sqrt{s}\, A)$$

Let $\bar{C}' = (c_y,\ y \in A)$ be given by normalizing C':

$$\bar{c}'_y(t) = \frac{1}{\sqrt{s}}\, c'_{y\sqrt{s}}(ts) \quad \text{for } y \in A,\ t \geq 0;$$

so that

$$\bar{C} \overset{d}{=} \bar{C}' .$$

Define

$$\bar{B} = (\bar{b}_y,\ y \in A) \text{ by}$$

$$\bar{b}_y(t) = \frac{1}{\sqrt{s}}\, b_{y\sqrt{s}}(ts), \quad \text{for } y \in A,\ t \geq 0,$$

so that $\bar{B}$ is a family of independent Brownian motions starting from A. It can be seen that

$$\bar{C}' = \Phi(\bar{B}).$$

Since $\bar{C} \overset{d}{=} \bar{C}'$, this shows that $\bar{C}$ is a system of coalescing Brownian motions starting from A.

q.e.d.

For any $s > 0$ we define c^s, a space-time normalization of c, by

$$c_x^s(t) = \frac{1}{\sqrt{s}} \; c_{(x\sqrt{s})}(ts), \quad \text{for } x \in R,\ t \geq 0 \ . \tag{24}$$

Notice that, in agreement with our convention for notation,

$$c_x^s(0) = x \quad \text{for } x \in R \ . \tag{25}$$

(26) Theorem. For any $s > 0$, the processes c and c^s in $D_E(R)$ have the same distribution. In symbols,

$$c^s \overset{d}{=} c \quad \text{for } s > 0. \tag{27}$$

Proof. We only have to check that c^s and c have the same finite dimensional distributions. Suppose $A \subset R$, $|A| = n$. The family of paths $C = (c_x,\ x \in \sqrt{s}\,A)$ is a system of coalescing Brownian motions, starting from $\sqrt{s}\,A$. By the previous lemma, the family $\bar{C} = (c_y^s,\ y \in A)$ is a system of coalescing Brownian motions starting from A. Thus

$$(c_y^s,\ y \in A) \overset{d}{=} (c_y,\ y \in A) \ . \tag{28}$$

q.e.d.

Technical Aside: The "Distribution of c"

Implicit in the phrase "the distribution of c" is an underlying measurable space. The distribution of c is the family of values ($P\{\omega : c(\omega) \in B\}$, B a measurable set).

If c is viewed as a random element of the complete separable metric space $D_E(R)$, then the measurable sets are the Borel sets of $D_E(R)$. It is a general result about the Skorohod space $D_E(R)$ that the Borel σ-algebra is equal to the σ-algebra generated by the finite dimensional cylinder sets, i.e., sets of the form

$$\{c \in D_E(R) : \quad c_{x_i} \in H_i, \quad i = 1,2,\ldots,k\}, \tag{29}$$

where each $x_i \in R$ and each H_i is a Borel subset of E.

For $E = C_R[0, \infty)$, the Borel σ-algebra is equal to the σ-algebra generated by the finite dimensional cylinder sets, i.e., sets of the form

$$\{f \in C_R[0, \infty) : f(t_i) \in B_i, \quad i = 1,2,\ldots,k\}, \tag{30}$$

where each $t_i \geq 0$ and each B_i is a Borel subset of R.

By combining these statements, it can be seen that the Borel σ-algebra on $D_E(R)$ is equal to the σ-algebra generated by sets of the form

$$G_1 = \{c \in D_E(R) : c_{x_i}(t_i) \in B_i, \quad i = 1,2,\ldots,k\}, \tag{31}$$

where each pair $(x_i, t_i) \in R \times [0, \infty)$, and each B_i is a Borel subset of R.

c can also be viewed as a family of real valued random variables indexed by $(x,t) \in R \times [0, \infty)$:

$$c = (c_x(t),\ x \in R,\ t \geq 0) \in R^{R\times[0,\infty)} .$$

The usual σ-algebra for a stochastic process (here, with parameter set $R \times [0, \infty)$) is that generated by the finite dimensional cylinder sets:

$$G_2 = \{c \in R^{R\times[0,\infty)} : c_{x_i}(t_i) \in B_i, \quad i = 1,2,\ldots,k\}, \tag{32}$$

where each $(x_i, t_i) \in R \times [0, \infty)$ and each B_i is a Borel subset of R.

Since $c \in D_E(R)$, a.s., we have

$$\begin{aligned} &P\ \{c(\omega) \in D_E(R),\ c \in G_1\} = \\ &\qquad P\ \{c(\omega) \in G_2\} \quad , \end{aligned} \tag{33}$$

in which we continue the notational abuse of identifying $c(\omega) \in D_E(R)$ a.s., with $c(\omega) \in R^{R\times[0,\infty)}$. Thus, for either of the two spaces that c lives in, the distribution of c has the same meaning.

Similarity of X_1 and X_s for Any $s > 0$

From the self similarity of c, it is easy to show that the distribution of X_s, up to a multiplicative rescaling of the line, is constant:

(34) Theorem. For $s > 0$, $X_1 \overset{d}{=} \frac{1}{\sqrt{s}} X_s$

(Reminder: for $A \subset R$, $r > 0$, $rA = \{rx : x \in A\}$.)

Proof. Use (3.28) in the special case $t = 1$:

$$X_1 = \lim_{n\to\infty} X_1^{I^n} = \lim_{n\to\infty} \{c_{q_1}(1), c_{q_2}(1), \ldots, c_{q_n}(1)\} \quad \text{a.s.} \tag{35}$$

Fix $s > 0$.

Since $c^s = c$, we get the same distribution on N by replacing c with c^s:

$$X_1 \overset{d}{=} \lim_{n\to\infty} \{c^s_{q_1}(1), \ldots, c^s_{q_n}(1)\} . \tag{36}$$

Expand this using the definition of c^s:

$$X_1 \overset{d}{=} \lim_{n\to\infty} \{\frac{1}{\sqrt{s}} c_{q_1\sqrt{s}}(s), \ldots, \frac{1}{\sqrt{s}} c_{q_n\sqrt{s}}(s) \tag{37}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{s}} X_s^{\{q_1\sqrt{s},\ldots,q_n\sqrt{s}\}} . \tag{38}$$

Since $\{q_1\sqrt{s}, q_2\sqrt{s},\ldots\}$ is a countable dense subset of R, Lemma (3.42) identifies this last limit (38) as $\frac{1}{\sqrt{s}} X_s$, with probability one. Thus

$$X_1 \stackrel{d}{=} \frac{1}{\sqrt{s}} X_s . \tag{39}$$

q.e.d.

Translation Invariance

Given any $y \in R$, define a translation $T_y c$ of the system c:

$$\text{for } x \in R,\ t \geq 0,\ (T_y c)_x(t) = c_{x+y}(t)-y. \tag{40}$$

Use an additive notation for the translation of a set:

$$\text{for } y \in R,\ A < R, \tag{41}$$

$$y+A = \{y + x : x \in A\}.$$

(42) Theorem. c is translation invariant. For any $t > 0$, X_t is translation invariant. More precisely, for any $y \in R$,

$$T_y c \stackrel{d}{=} c \tag{43}$$

and

$$(y+X_t) \stackrel{d}{=} X_t \; . \tag{44}$$

Proof. (43) is shown by checking the equality of the finite dimensional distributions of $T_y c$ and c. This is similar to the proof that $c^s \stackrel{d}{=} c$, and we omit the details.

To get $y + X_t \stackrel{d}{=} X_t$ from $T_y c \stackrel{d}{=} c$, write

$$\begin{aligned} X_t &= \lim_{n\to\infty} \{c_{q_1}(t),\ldots,c_{q_n}(t)\} \qquad (45)\\ &\stackrel{d}{=} \lim_{n\to\infty} \{(T_y c)_{q_1}(t),\ldots,(T_y c)_{q_n}(t)\} \\ &= \left|\lim_{n\to\infty} \{c_{y+q_1}(t),\ldots,c_{y+q_n}(t)\}\right| - y \\ &= X_t - y. \end{aligned}$$

The last equality follows from Lemma (3.42) since $\{q_1+ y, q_2+ y,\ldots\}$ is a countable dense subset of R.

Theorem (34), that $\frac{1}{\sqrt{t}} X_t = X_1$, can be restated as an "invariance" property for the family of Markov processes $(X^A, A \subset R)$.

Introduce a symbol for the distribution of X_1:

$$\pi^c = L(X_1) = \text{the law of } X_1, \tag{46}$$

so that π^c is a translation invariant probability measure on N.

The Markov process X^{π^c}, by definition, is the mixture of the processes X^A where $A \varepsilon N$ has distribution π^c and A is independent of F_t for every $t \geq 0$. Thus, for a Borel subset G of N, and for $t \geq 0$,

$$P(X_t^{\pi^c} \varepsilon G) = \int_{A \varepsilon N} P(X_t^A \varepsilon G) \, \pi^c(dA) \tag{47}$$

$$= \int p(A,t,G) \, \pi^c(dA).$$

Use the Markov property of X:

$$P(X_{t+1} \varepsilon G \mid F_1) = P(X_{t+1} \varepsilon G \mid X_1) \tag{48}$$

$$= p(X_1,t,G) \ .$$

Integrating over $X_1 \varepsilon N$,

$$P(X_{t+1} \varepsilon G) = \int_{A \varepsilon N} p(A,t,G) \, P(X_1 \varepsilon dA) \tag{49}$$

$$= \int_{A \varepsilon N} p(A,t,G) \, \pi^c(dA)$$

$$= P(X_t^{\pi^c} \varepsilon G),$$

Thus,

$$X_t^{\pi^c} \stackrel{d}{=} X_{t+1} \ . \tag{50}$$

Combining this with Theorem (34), we obtain

$$\frac{1}{\sqrt{t+1}} X_t^{\pi^c} = \frac{1}{\sqrt{t+1}} X_{t+1} = X_1 , \tag{51}$$

or

$$L(\frac{1}{\sqrt{t+1}} X_t^{\pi^c}) = L(X_1) = \pi^c . \tag{52}$$

Theorem (34), that $\frac{1}{\sqrt{t}} X_t = X_1$, and Theorem (3.51), that X_t is a Markov process adapted to $(F_t, t > 0)$, can also be combined to yield an N-valued Markov process $(\frac{1}{\sqrt{t}} X_t, t > 0)$ with an invariant measure π^c, In the process $\frac{1}{\sqrt{t}} X_t$, particles undergo coalescing Brownian motions, but a particle at x at time t has diffusion coefficient $\frac{1}{t}$ and drift $-\frac{x}{2t}$. By changing the time scale, we can get a process with a stationary transition mechanism. Define an N-valued process Y by

$$\text{for } t \in R,\ Y_t = e^{-\frac{t}{2}} X_{(e^t)} . \tag{53}$$

Define σ-fields G_t by

$$\text{for } t \in R,\ G_t = F_{(e^t)} . \tag{54}$$

(55) Theorem. $(Y_t, -\infty < t < \infty)$, is a stationary Markov process adapted to the filtration $(\mathcal{G}_t, t \in R)$. The invariant distribution for Y is π^c, the distribution of X_1. The transition function q for Y can be expressed in terms of the transition function p for X by:

$$q(A,s,G) \doteq p(A,e^s-1,\ e^{s/2}\ G). \tag{56}$$

Proof. Fix $u \in R$, and let $\bar{c} = c^{(e^u)}$, i.e., define $\bar{c} = (\bar{c}_x(t),$ $x \in R,\ t \geq 0)$ by

$$\bar{c}_x(t) = e^{-u/2}\ c_{x\,e^{u/2}}\ (t\ e^u)\ . \tag{57}$$

Theorem (26) says that $\bar{c} \overset{d}{=} c$. Define an N-valued evolution $\bar{X}$ by:

$$\text{for } t > 0,\ \bar{X}_t = \{\bar{c}_x(t),\ x \in R\}\ . \tag{58}$$

Thus,

$$\bar{X}_t = \{e^{-u/2}\ c_y(t\ e^u),\ y \in R\}. \tag{59}$$

$$= \ e^{-u/2}\ X_{te^u}\ .$$

Notice that $\bar{X}_1 = Y_u$.

Technicality. For any $t > 0$, the formation of X_t from c involves a measurable function

$$\pi_t : D_E(R) \to N \tag{60}$$

$$\pi_t(c) = \lim_{n\to\infty} \{c_t(q_1),\ldots,c_t(q_n)\}, \text{ if the limit exists.}$$

$$= \phi, \text{ if the limit is not in } N.$$

We have $X_t = \pi_t(c)$ a.s. and $\bar{X}_t = \pi_t(\bar{c})$ a.s. Thus, for any $t, s > 0$, the joint distributions on $N \times N$ of (X_t, X_{t+s}) and $(\bar{X}_t, \bar{X}_{t+s})$ are equal. Thus $\bar{X}$ and X have the same transition mechanism p.

For G a Borel subset of N, $u \in R$, and $s > 0$, we compute

$$P(Y_{u+s} \in G \mid G_u) \tag{61}$$

$$= P(e^{-(u+s)/2} \cdot X_{e^{u+s}} \in G \mid F_{e^u})$$

$$= P(e^{-u/2} \cdot X_{e^s e^u} \in e^{s/2}\, G \mid X_{e^u})$$

$$= P(\bar{X}_{e^s} \in e^{s/2}\, G \mid e^{u/2}\, \bar{X}_1)$$

$$= p(\bar{X}_1,\ e^s - 1,\ e^{s/2}\, G)$$

$$= p(Y_u,\ e^s - 1,\ e^{s/2}\, G)$$

$$= q(Y_u,\ s,\ G).$$

q.e.d.

Each particle in Y moves according to a diffusion with speed 1 and drift $-\frac{x}{2}$. Here is a partial justification:

Suppose, for some $t > 0$, that $x \in Y_t$. Then $e^{t/2} x \in X_{e^t}$, so for some $q \in Q$, $b_q(e^t) = e^{t/2} x$. Define b' by $b'(t) = e^{-t/2} b_q(e^t)$. Between t and $t + \Delta t$, the expected change in position in Y of the particle that was at x at time t is

$$E(b'(t+\Delta t) - b'(t) | b'(t) = x) =$$

$$E(e^{-(t+\Delta t/2)} b_q(e^{t+\Delta t}) - x | b_q(e^t) = e^{t/2} x)$$

$$= e^{-(t+\Delta t)/2} e^{t/2} x - x \quad = \quad x(e^{-\Delta t/2} - 1)$$

$$= \quad (-\frac{x}{2}) \Delta t + o(\Delta t).$$

Thus, a particle in Y at x has drift $-\frac{x}{2}$. A similar calculation shows that b', the path of the particle at x, has diffusion coefficient 1.

Y_t may be described as a system of particles undergoing independent speed 1 diffusions and drift $-\frac{x}{2}$, with coalescing interference. The system is not spatially homogeneous--the origin is a center of attraction, but its equilibrium, π^c, is translation invariant.

CHAPTER 5

COALESCING RANDOM WALKS ON THE LATTICE Z

References to coalescing random walks are given on page 1. We will show that, under an appropriate normalization, the system of coalescing random walks converges in distribution to the system of coalescing Brownian motions. Using a duality relation for coalescing random walks, a duality relation for coalescing Brownian motions can be derived. Specifically, we handle one-dimensional simple random walks: a particle at $x \in Z$ waits for an exponentially distributed, mean 1 holding time, and then jumps to $x-1$ or $x+1$, with probability 1/2 each.

Following the notation of [9] and [10], we write ξ_t^z, for $z \in Z$ and $t \geq 0$, to denote the position at time t of the particle starting from z. The system of coalescing paths, $\xi = (\xi_t^z,\ z \in Z,\ t \geq 0)$, can be constructed from a family $d = (d_z(t), z \in Z,\ t \geq 0)$ of independent random walks using a collision precedence rule, based on an arbitrary enumeration $z_1, z_2, z_3, \ldots$ of Z, which supplies the precedence order. The construction parallels exactly the construction of the coalescing family $(c_q(t),\ q \in Q,\ t \geq 0)$, which was then extended to a family $c = (c_x(t),\ x \in R,\ t \geq 0)$ with particles starting everywhere, using the formula

$$c_x(t) = \inf \{c_q(t) : q \geq x,\ q \in Q\}. \qquad (1)$$

It required an argument to show that, a.s., $c \in D_E(R)$, where $E = c_R[0, \infty)$. Now, the paths ξ^x of a single particle are in the space

$$E' = D_R[0, \infty) \quad . \tag{2}$$

Define a system e, which is coalescing random walks with particles starting everywhere, by

$$e_x(t) = \xi_t^{\lfloor x \rfloor}, \tag{3}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Define

$$S = D_{E'}(R) \quad . \tag{4}$$

That $e \in S$ is immediate since, for $x, y \in [z, z+1)$ and $z \in Z$, $e_x(\cdot) = e_y(\cdot)$.

Detail: Extending the Collision Rule

The collision rule Φ_σ in Chapter 1 applies to $C_{R^n}[0, \infty) = (E)^n$, where $E = C_R[0, \infty)$. To extend this to $(E')^n$, where $E' = D_R[0, \infty)$, use the recursive construction given in the proof of (Lemma 1.27); the construction requires only that the individual paths be right-continuous.

Lemma (1.47), that Φ_σ is continuous almost everywhere with respect to the distribution of Brownian motion, is no longer true. The problem is that for two paths in E', the first collision time is <u>never</u> a continuous function of the paths. Say that $f, g \in E'$ have the "<u>nearest neighbor property</u>" iff

$$\tau(f,g) \equiv \inf \{t \geq 0 : f(t) = g(t)\} \tag{5}$$

and

$$\sigma(f,g) \equiv \inf \{t \geq 0 : (f(t)-g(t))(f(0)-g(0)) \leq 0\} \tag{6}$$

are equal. If f and g do not have the nearest neighbor property, then they cross over each other without colliding. Any two simple random walks on the same lattice $\frac{1}{\sqrt{s}} Z$ have this nearest neighbor property.

(7) <u>Lemma</u>. For $\sigma \in \sum_n$, the map $\Phi_\sigma : (E')^n \to (E')^n$, restricted to families of paths such that the nearest neighbor property holds for every pair of paths in the family, is continuous a.s. with respect to the distribution of independent Brownian motions starting from distinct locations.

<u>Proof</u>. The proof of Lemma (1.47) is valid, if we replace the first collision times, $\tau_{ij} : C_{R^n} \to [0, \infty]$ appearing in (1.50) and (1.52), by the first crossing times $\sigma_{ij} = \sigma(B_i, B_j)$.

Rescaling the Coalescing Random Walks

Recall the notation of Chapter 4 for a space-time normalization of c: for $s > 0$, c^s is given by;

$$\text{for } x \in R,\ t \geq 0, \quad c_x^s(t) = \frac{1}{\sqrt{s}}\, c_{x\sqrt{s}}(ts). \tag{8}$$

Theorem (4.26) states that

$$c^s \stackrel{d}{=} c. \tag{9}$$

Defining e^s in the same way, we obtain a system "living" on the lattice $\frac{1}{\sqrt{s}}Z$; the random walks jump $\pm\frac{1}{\sqrt{s}}$ after mean $\frac{1}{s}$ exponential holding times. We will show

(10) <u>Theorem.</u> As $s \to \infty$, $e^s \xrightarrow{d} c$, and for each $t > 0$, $\{e_x^s(t) : x \in R\} \xrightarrow{d} X_t$. Here, $e^s \xrightarrow{d} c$ involves the Skorohod topology on $S = D_{E'}[R)$, and $\xrightarrow{d} X_t$ involves the vague topology on N.

<u>Proof.</u> Let c be constructed from a family $b = (b_q,\ q \in Q)$ of independent Brownian motions indexed by the dyadic rationals. This is done in detail in Chapter 2; an arbitrary enumeration $q_1, q_2, \ldots$ supplies the precedence order. For each $s \uparrow 1$, take $d^s = (d_q^s,\ q \in Q)$ to be a family of independent simple random walks on the lattice $\frac{1}{\sqrt{s}}Z$, with $d_q^s(0) = \frac{1}{\sqrt{s}}\lfloor q\sqrt{s}\rfloor$. In the separable metric space $(E')^Q$, $d^s \xrightarrow{d} b$ as $s \to \infty$. By the Skorohod representation, we can take d^s, $s \geq 1$, and b on a single probability space,

such that $d^s \xrightarrow{a.s.} b$. For each s, construct a system $(e^s_x,\ x \in \frac{1}{\sqrt{s}} Z)$ of coalescing random walks on the lattice $\frac{1}{\sqrt{s}} Z$, using d^s and the precedence order derived from the enumeration $q_1, q_2, \ldots$. In detail, the path $d^s_{q_1}(\cdot)$, starting at $\frac{1}{\sqrt{s}} \lfloor q_1 \sqrt{s} \rfloor$ has highest priority; the path $d^s_{q_2}$ is next highest, and so on. If $\lfloor q_2 \sqrt{s} \rfloor = \lfloor q_1 \sqrt{s} \rfloor$ then $d^s_{q_1}$ and $d^s_{q_2}$ collide at time zero, and the information in $d^s_{q_2}$ does not appear anywhere in the coalescing system e^s. For every $z \in Z$, these will be an infinite family of indices i for which $\lfloor q_i \sqrt{s} \rfloor = z$; let z(s) be the smallest of these. Then the particle starting at $\frac{z(s)}{\sqrt{s}}$ follows the path $d^s_{q_{z(s)}}$ for some positive time before it is involved in a collision. Now extend $(e^s_x,\ x \in \frac{1}{\sqrt{s}} Z)$ to a family with particles starting everywhere, $e^s \in S = D_{E'}(R)$, by

$$e^s_x(t) = e^s_{\frac{1}{\sqrt{s}} \lfloor x\sqrt{s} \rfloor}(t) \tag{11}$$

for $x \in R$, $t \geq 0$.

There is a metric $d_{E'}$ for the Skorohod topology on $E' = D_R[0, \infty)$ such that, for f, g $\in$ E', and for any T,

$$d_{E'}(f,g) \leq e^{-T} + \sup_{0 \leq t \leq T} |f(t) - g(t)|. \tag{12}$$

For a metric on $(E')^Q$, we will use

$$d_{(E')^Q}(b, b') = \sum_{i \geq 1} 2^{-i} d_{E'}(b_{q_i}, b'_{q_i}). \tag{13}$$

Recall that $U_k = \{(u_1,u_2,\ldots,u_k) \in R^k : u_1 < u_2 < \ldots < n_k\}$. Define a metric on $\underline{U} = \bigcup_{k \geq 1} U_k$ by

$$d_{\underline{U}}(u,r) = 1 \qquad \text{if } \#u \neq \#r$$
$$= \min\left(1, \max_{1 \leq i \leq k} |u_i - r_i|\right) \qquad \text{if } \#u=\#r=k. \tag{14}$$

In (1.7), we defined a map ι from finite subsets of R onto $\underline{U}$. There is a metric d_N on N such that for every L,

$$d_N(A,B) \leq e^{-L} + d_{\underline{U}}(\iota(A \cap [-L,L]), \iota(B \cap [-L,L]) \ . \tag{15}$$

To specify a metric d_S for the Skorohod topology on $S = D_{E'}(R)$, define

$$\Lambda = \{\lambda: \ \lambda \text{ is a strictly increasing map from R onto R,}$$
$$\text{with } \lambda(0) = 0\}. \tag{16}$$

There is a metric d_S such that, for c, c' ε S, and for every $\lambda \in \Lambda$, $L > 0$,

$$d_S(c,c') \leq e^{-L} + \sup_{-L\leq x\leq L} \max(|\lambda(x)-x|, \ d_{E'}(c_x, c'_{\lambda(x)})). \tag{17}$$

To simplify notation, write

$$b' = d^S; \ c' = e^S; \ X_t^{'A} = \{c'(t) : x \in A\} \ . \tag{18}$$

We will show, that for any $\varepsilon > 0$ and $t_o > 0$, there is an s_o, such that for all $s > s_o$ there is an event E_o (depending on s) with

$$P(E_o) < 9\varepsilon, \tag{19}$$

$$\text{for } \omega \in E_o^C, \qquad d_N(X_{t_o}, X'_{t_o}) < 2\varepsilon \quad \text{and} \quad d_S(c,c') < 7\varepsilon. \tag{20}$$

Pick $L > t_o$ such that $e^{-L} < \varepsilon$.

Pick $M > L + 1$ such that the event

$$\begin{aligned} E_1 = \{&c'_M(t) < L \text{ or } c_M(t) < L \text{ or} \\ &c'_{-M}(t) > -L \text{ or } c_{-M}(t) > -L\} \end{aligned} \tag{21}$$

has probability less than ε. An estimate like (3.22) shows that this is possible for the coalescing random walks.

Pick n_1 so that $2^{-n_1} < \varepsilon$.

Pick t_1, $0 < t_1 < t_o$, such that the event

$$E_2 = \{\sup_{t \le t_1} \max_{q \in Q_{n_1} \cap [-M-1,M+1]} |c_q(t) - q| > \frac{2^{-n_1}}{4}\} \tag{22}$$

has probability less than ε.

Pick n_2 such that the events

$$E_3 = \{X_{t_1}^{[-M,M]} \neq X_{t_1}^{[-M,M] \cap Q_{n2}}\}\ , \tag{23}$$

$$E_4 = \{X'^{[-M,M]}_{t_1} \neq X'^{[-M,M]}_{t_1} \cap Q_{n_2}\} \tag{24}$$

each have probability less than ε. That this can be done for E_3 follows from (3.21). The analogous estimate, for coalescing random walks, appears in [4].

Pick n so large that

$$Q_{n_2} \cap [-M,M] \subset \{q_1,q_2,\ldots,q_n\} \equiv I_n. \tag{25}$$

Write $b^n = (b_q,\ q \in I_n)$, $c^n = (c_q,\ q \in I_n)$, $b'^n = (b'_q,\ q \in I_n)$, and $c'^n = (c'_q,\ q \in I_n)$.

Let d_1 be the function on $(E') \times (E')^n$ given by

$$d_1(b^n,b'^n) = \sup_{0\le t\le L}\ \max_{i\le n} \left|b_i(t) - b'_i(t)\right| . \tag{26}$$

Let d_2 be the metric on $(E')^n$ given by

$$d_2(b^n,b'^n) = \sum_{i\le n} 2^{-i} \quad d_{E'}(b_i,b'_i), \tag{27}$$

so that

$$d_2(b^n,b'^n) \le d_{(E')Q}\ (b,b') . \tag{28}$$

Pick a, $0 < a < 2^{-n_1}/4$, so small that the events

$$E_5 = \{\{b_q(t_1),\ q \in I_n\} \text{ has two points within } 3a \text{ of each other}\} , \tag{29}$$

$$E_6 = \{\{b_q(t_o),\ q \in I_n\} \text{ has two points within } 3a \text{ of each other}\} \tag{30}$$

have probability less than ε.

By Lemma (7) and the fact that $d_1(b^n, \cdot)$ is continuous on $(E')^n$, there exists $\delta(\omega,a)$ which is > 0 almost surely, such that $d_2(b^n,b'^n) < \delta(\omega,a)$ implies that

$$d_1(c^n,c'^n) < a, \tag{31}$$

and

$$d_1(b^n,b'^n) < a. \tag{32}$$

Pick $\delta > 0$ so small that the event

$$E_7 = \{\delta(\omega,a) \leq \delta\} \tag{33}$$

has the probability less than ε.

Finally, pick s_o so large that for $s > s_o$, the events

$$E_8 = \{d_{(E')Q}(b,b') \geq \delta\}$$

and

$$E_9 = \{\lim_{x\to 0^-} c_x(t_1) \neq c_o(t_1) \text{ or } \lim_{x\to 0^-} c'_x(t_1) \neq c'_o(t_1)\}$$

have probability less than ε.

Let $E_o = E_1 \cup \ldots \cup E_9$, so that $P(E_o) < 9\varepsilon$. E_o is the exceptional set in formula (20), which we now verify. Fix $\omega \in \bar{E}_o^C$.

Since $\omega \notin E_8 \cup E_7$, $d_2(b^n, b'^n) \leq d_{(E')^Q}(b, b') < \delta < \delta(\omega, a)$. Thus $d_1(b^n, b'^n) < a$ and $d_1(c^n, c'^n) < a$, i.e.,

$$\sup_{0 \leq t \leq L} \max_{q \in I_n} |b_q(t) - b'_q(t)| < a , \tag{34}$$

$$\sup_{0 \leq t \leq L} \max_{q \in I_n} |c_q(t) - c'_q(t)| < a . \tag{35}$$

Now we show that $c^n(t_1)$ and $c'^n(t_1)$ have the same coalescence, i.e., that for $q, r \in I_n$,

$$c_q(t_1) = c_r(t_1) \text{ iff } c'_q(t_1) = c'_r(t_1). \tag{36}$$

We always have, for $q \in I_n$, $c_q(t_1) \in \{b_r(t_1), r \in I_n\}$ and $c'_q(t_1) \in \{b'_r(t_1), r \in I_n\}$. Since $\omega \notin E_5$, the set $\{b_r(t), r \in I_n\}$ has spacing at least 3a between points. Combine this with (34) and (35) to conclude that (36) holds. The same argument, using $\omega \notin E_6$, shows that $c^n(t_o)$ and $c'^n(t_o)$ have the same coalescence, i.e., that for $q, r \in I_n$

$$c_q(t_o) = c_r(t_o) \quad \text{iff} \quad c'_q(t_o) = c'_r(t_o). \tag{37}$$

As a consequence of (37), the sets $X_{t_o}^{I_n \cap [-M,M]}$ and $X'^{I_n \cap [-M,M]}_{t_o}$ have the same cardinality and using (35),

$$d_U(X_{t_o}^{I_n \cap [-M,M]}, X'^{I_n \cap [-M,M]}_{t_o}) < a < \varepsilon .$$

Since $\omega \notin E_3 \cup E_4 \cup E_1$, $X_{t_o}^{I_n \cap [-M,M]} = X_{t_o}^{[-M,M]} = X_{t_o} \cap [-L,L]$, and $X'^{I_n \cap [-M,M]}_{t_o} = X'^{[-M,M]}_{t_o} = X'_{t_o} \cap [-L,L]$. Thus

$$d_N(X_{t_o}, X'_{t_o}) \leq e^{-L} + d_{\underline{U}}(X_{t_o} \cap [-L,L],\ X'_{t_o} \cap [-L,L]) < 2\varepsilon. \tag{38}$$

This establishes the first part of (20).

The same argument for t_1 instead of t_o shows that $X_{t_1}^{[-M,M]}$ and $X'^{[-M,M]}_{t_1}$ are in one to one correspondence, with distance < a between corresponding points z_i and z'_i. List these sets in increasing order; say $X_{t_1}^{[-M,M]} = \{z_1, z_2, \ldots, z_k\}$ and $X'^{[-M,M]}_{t_1} = \{z'_1, z'_2, \ldots, z'_k\}$. Let y_i and y'_i be the corresponding "borders", i.e., for $i = 1,2,\ldots,k-1$, let

$$y_i = \inf \{x : c_x(t_1) = z_i\}, \qquad y'_i = \inf \{x : c'_x(t_1) = z'_i\}. \tag{39}$$

Each interval $[y_i, y_{i+1})$ for $i < k$, contains a point $r_i \in Q_{n_2}$ (using $\omega \notin E_3$). Thus

$$c_{t_1}(r_i) = z_i. \tag{40}$$

The argument used to show (36) implies that $c'_{t_1}(r_i) = z'_i$. Write r'_i and r''_i for the elements of Q_{n_1} such that

$$r_i' + 2^{-n_1} \leq r_i < r_i'' - 2^{-n_1}, \qquad r_i'' - r_i = 3 \cdot 2^{-n_1} \tag{41}$$

Using $\omega \notin E_2$, $c'_{r_i}(t_1) < c_{r_i}(t_1) < c_{r_i''}(t_1)$ and $c'_{r_i'}(t_1) < c'_{r_i}(t_1) < c_{r_i}(t_1)$. Thus $r_i' < \min(y_i, y_i')$ and $r_i'' > \max(y_{i+1}, y'_{i+1})$. Pick any $\lambda \in \Lambda$ for which $\lambda(y_i) = y_i'$ for $i = 1,2,\ldots,k$ and $\lambda(0) = 0$. This uses $\omega \notin E_9$. Now we have, for $i < k$, $x \in [r_i, r_{i+1})$,

$$c'_{\lambda(x)}(t_1) = c'_{r_i}(t_1) \text{ and } c_x(t_1) = c_{r_i}(t) . \tag{42}$$

The coalescing property implies that this also holds for all $t \geq t_1$. Using (35) we get

$$\text{for } t_1 \leq t \leq L, \ |c'_{\lambda(x)}(t) - c_x(t)| < a < \varepsilon . \tag{43}$$

Before time t_1 we can use monotonicity: for any $t < t_1$, $x \in [y_i, y_{i+1})$, we have $r_i' < x$, $\lambda(x) < r_i''$, and using $\omega \notin E_2$,

$$c_x(t) \geq c_{r_i'}(t) > r_i' - \varepsilon; \tag{44}$$

$$c'_{(x)}(t) \leq c'_{r_i''}(t) < c_{r_i''}(t) + \varepsilon < r_{i''} + 2\varepsilon. \tag{45}$$

Combining these, using a similar estimate for the opposite direction, and noting that the intervals $[y_i, y_{i+1})$ cover $[-L,L]$,

$$\sup_{-L\leq x\leq L}\ \sup_{t\leq t_1}\left|c_x(t) - c'_{\lambda(x)}(t)\right| < 6\varepsilon. \tag{46}$$

Combining this with (43), we obtain

$$\sup_{-L\leq x\leq L} d_{E'}(c_x, c'_{\lambda(x)}) < 6\varepsilon .$$

Thus, using $e^{-L} < \varepsilon$,

$$d_S(c,c') < 7\varepsilon .$$

This completes the proof of (20).

Duality

For any $T > 0$, it is possible to construct two systems of coalescing walks, e and $\bar{e}$, which are in "duality" on the interval $[0, T]$. "Duality" here means that the paths of particles in e do not cross the paths of particles in $\bar{e}$, if we run $\bar{e}$ backwards in time, from T down. More precisely, for any $x, y \in Z$ and $r, t \in [0, T]$,

$$(e_x(r) - \bar{e}_y(T-r))(e_x(t) - \bar{e}_y(T-t)) \geq 0 . \tag{47}$$

This construction is described in the Introduction, starting on page 10. We can approximate coalescing Brownian motions with a sequence of rescaled coalescing random walks to prove

(48) Theorem. For any $T \geq 0$, there exist two systems c and $\bar{c}$ of coalescing Brownian motions, in duality on the time interval $[0,T]$. Thus, for every ω,

$$\text{for all } x,\ y \in R;\ r,t \in [0,T], \quad (c_x(r)-\bar{c}_y(T-r))(c_x(t)-\bar{c}_y(T-t)) \geq 0 \tag{49}$$

Proof. For each positive integer s, let e^s and $\bar{e}^s$ be systems of rescaled coalescing random walks, with steps of $\pm \frac{1}{\sqrt{s}}$ and mean time $\frac{1}{s}$ between jumps, in duality on the interval $[0, T]$. This is done by constructing coalescing random walks on the lattice Z, in duality on $[0, Ts]$, and then rescaling. Consider the sequence $(e^s, \bar{e}^s) \in S^2$. Since $\bar{e}^s \overset{d}{=} e^s \overset{d}{\to} c$ as $s \to \infty$, each of the sequences (e^s) and $(\bar{e}^s)$, is tight in $D_{E'}(R)$. Thus the sequence $(e^s, \bar{e}^s)$ is tight, and we may extract a convergent subsequence:

$$(e^{s_i},\ \bar{e}^{s_i}) \overset{d}{\to} (c,\ \bar{c})\ . \tag{50}$$

It would be interesting to determine whether or not the entire sequence $(e^s,\ \bar{e}^s)$ must converge in distribution. All we claim, using Theorem (10), is that c and $\bar{c}$, the marginal limits, are each equal in distribution to the system of coalescing Brownian motions constructed in Chapter 2. Use the Skorohod representation to get versions of $(e^{s_i},\ \bar{e}^{s_i})$ and $(c,\ \bar{c})$ on a single probability space, such that,

$$(e^{s_i}, \bar{e}^{s_i}) \xrightarrow{a.s.} (c, \bar{c}) \ . \tag{51}$$

To show that the duality relation, (49), holds almost surely for $(c, \bar{c})$, assume that it does not. Then there is an $L > 0$ and $\varepsilon > 0$ such that

$$\begin{aligned} \varepsilon < P(&\text{ for some } x, y \in [-L,L];\ r, t \in [0, T], \\ &c_x(r) - \bar{c}_y(T-r) > \varepsilon \ \text{ and } \ c_x(t) = \bar{c}_y(T-t) < -\varepsilon) \end{aligned} \tag{52}$$

(A similar proof works if this holds with the signs reversed.) Pick δ so small that $d_S(c, e^s) < \delta$ implies that there is $\lambda \in \Lambda$ such that

$$P(\sup_{-L\leq x\leq L} \ \sup_{t\leq T} |c_x(t) - e^s_{\lambda(x)}(t)| < \frac{\varepsilon}{2}) > 1 - \frac{\varepsilon}{2} \ .$$

This uses the continuity of $c_x(\cdot)$. Now pick i so large that

$$P(d_S(c, e^{s_i}) \geq \delta \ \text{ or } \ d_S(\bar{c}, \bar{e}^{s_i}) \geq \delta) < \frac{\varepsilon}{2} \ .$$

Then for some ω, the duality of e^{s_i}, $\bar{e}^{s_i}$ contradicts

$$e^{s_i}_{\lambda(x)}(r) - \bar{e}^{s_i}_{\lambda(y)}(T-r) > 0 \ \text{ and } \ e^{s_i}_{\lambda(x)}(t) - \bar{e}^{s_i}_{y}(T-t) < 0 \ .$$

q.e.d.

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