Problem 1.

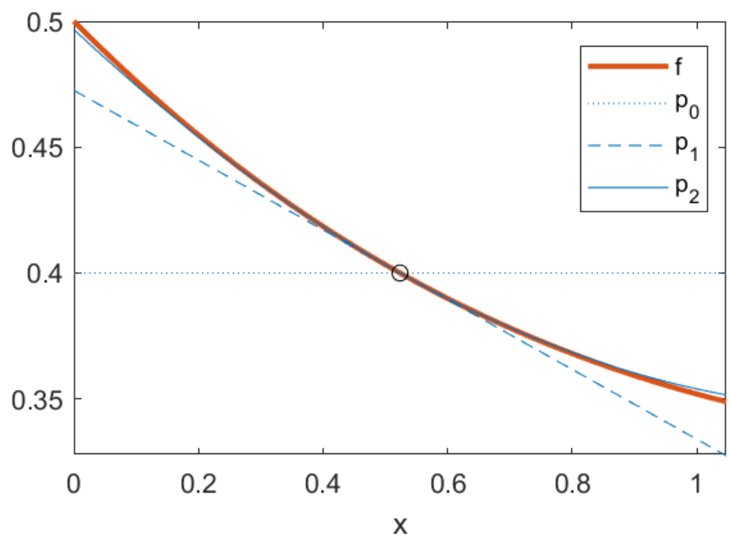
Solution 1.

n	$f^{(n)}(x)$	$f^{(n)}(x_0 = \pi/6)$
0	$\frac{1}{\sin(x)+2}$	$\frac{2}{5}$
1	$-\frac{\cos(x)}{(\sin(x)+2)^2}$	$-\frac{2\sqrt{3}}{25}$
2	$\frac{\cos(x)^2+2\sin(x)+1}{(\sin(x)+2)^3}$	$\frac{22}{125}$

$$P_0(x) = f^{(0)}(x_0)(x-x_0)^0 = \frac{2}{5}$$

$$P_1(x) = f^{(0)}(x_0)(x-x_0)^0 + f^{(1)}(x_0)(x-x_0)^1 = \frac{2}{5} - \frac{2\sqrt{3}}{25}\left(x - \frac{\pi}{6}\right)$$

$$P_2(x) = f^{(0)}(x_0)(x-x_0)^0 + f^{(1)}(x_0)(x-x_0)^1 + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 = \frac{2}{5} - \frac{2\sqrt{3}}{25}\left(x - \frac{\pi}{6}\right) + \frac{11}{125}\left(x - \frac{\pi}{6}\right)^2.$$



Problem 2.

Solution 2.

n	$f^{(n)}(x)$	$f^{(n)}(x_0=0)$
0	$\exp(1+x/2)$	e
1	$2^{-1}\exp(1+x/2)$	2^{-1}e
2	$2^{-2}\exp(1+x/2)$	2^{-2}e
3	$2^{-3}\exp(1+x/2)$	2^{-3}e
4	$2^{-4}\exp(1+x/2)$	//

$$P_3(x)=\sum_{k=0}^{3}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k=\mathrm{e}+\frac{\mathrm{e}}{2}x+\frac{\mathrm{e}}{2^2\times 2!}x^2+\frac{\mathrm{e}}{2^3\times 3!}x^3=\mathrm{e}+\frac{\mathrm{e}}{2}x+\frac{\mathrm{e}}{8}x^2+\frac{\mathrm{e}}{48}x^3.$$

Actual error at $x=1$: $|f(1)-P_3(1)|=|\mathrm{e}^{3/2}-\left(\mathrm{e}+\frac{\mathrm{e}}{2}+\frac{\mathrm{e}}{8}+\frac{\mathrm{e}}{48}\right)|\approx 0.00785$.

Error bounds:

$$|R_3(x)|\le\frac{\max_{\zeta\in[0,1]}\overbrace{2^{-4}\exp(1+\zeta/2)}^{f^{(4)}(\zeta)}}{4!}|1-x_0|^4\overset{\zeta=1}{=}\frac{\mathrm{e}^{3/2}}{384}\times 1^4=\frac{\mathrm{e}^{3/2}}{384}\approx 0.011\,671 \text{ for all } x\in\overline{x_0,1}=[0,1];$$

$$|R_3(x)|\le\frac{\max_{\zeta\in[-2,0]}\overbrace{2^{-4}\exp(1+\zeta/2)}^{f^{(4)}(\zeta)}}{4!}|-2-x_0|^4\overset{\zeta=0}{=}\frac{\mathrm{e}}{384}\times 2^4=\frac{\mathrm{e}}{24}\approx 0.113\,262 \text{ for all } x\in\overline{x_0,-2}=[-2,0].$$

Therefore, $|R_3(x)|\le 0.113262$ for all $x\in[-2,1]$.

OR [not commended]

$$|R_3(x)|\le\frac{\max_{\zeta\in[-2,1]}2^{-4}\exp(1+\zeta/2)}{4!}\max\{|1-x_0|,|-2-x_0|\}^4\overset{\zeta=1}{=}\frac{\mathrm{e}^{3/2}}{384}\times 2^4=\frac{\mathrm{e}^{3/2}}{24}\approx 0.186\,737 \text{ for all } x\in[-2,1].$$

Problem 3.

Solution.

n	$f^{(n)}(x)$	$f^{(n)}(x_0=1)$	$\max_{\zeta\in[0,1]}\lvert f^{(n)}(\zeta)\rvert$	$\max_{\zeta\in[1,2]}\lvert f^{(n)}(\zeta)\rvert$
0	$\ln(4-x)$	$\ln(3)$	$\ln(4)$	$\ln(3)$
1	$-(4-x)^{-1}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$
2	$-(4-x)^{-2}$	$-\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{4}$
3	$-2(4-x)^{-3}$	$-\frac{2}{27}$	$\frac{2}{27}$	$\frac{1}{4}$
4	$-6(4-x)^{-4}$	$-\frac{2}{27}$	$\frac{2}{27}$	$\frac{3}{8}$
5	$-24(4-x)^{-5}$	$-\frac{8}{81}$	$\frac{8}{81}$	$\frac{3}{4}$
6	$-120(4-x)^{-6}$	//	$\frac{40}{243}$	$\frac{15}{8}$

For $x \in [0,1]$, observe that

$$|R_3(x)| \leq \frac{\max_{\zeta\in[0,1]}|f^{(4)}(\zeta)|}{4!}|0-1|^4 = \frac{2}{27\times 4!} \approx 0.0031;$$

whereas

$$|R_2(x)| \leq \frac{\max_{\zeta\in[0,1]}|f^{(3)}(\zeta)|}{3!}|0-1|^3 = \frac{2}{27\times 3!} \approx 0.0123.$$

Hence the minimum order for $x \in [0,1]$ (according to the bounds above) is $n=3$.

For $x \in [1,2]$, observe that

$$|R_5(x)| \leq \frac{\max_{\zeta\in[1,2]}|f^{(6)}(\zeta)|}{6!}|2-1|^6 = \frac{15}{8\times 6!} \approx 0.0026;$$

whereas

$$|R_4(x)| \leq \frac{\max_{\zeta\in[1,2]}|f^{(5)}(\zeta)|}{5!}|2-1|^5 = \frac{3}{4\times 5!} \approx 0.0063.$$

Hence the minimum order for $x \in [1,2]$ (according to the bounds above) is $n=5$.

The minimum required order for $x \in [0,2]$ is therefore $n=5$, and the Taylor polynomial is

$$P_5(x) = \ln(3) - \frac{1}{3}(x-1) - \frac{1}{9\times 2!}(x-1)^2 - \frac{2}{27\times 3!}(x-1)^3 - \frac{2}{27\times 4!}(x-1)^4 - \frac{8}{81\times 5!}(x-5)^5.$$

The actual error is $|f(0)-P_5(0)| \approx 0.000\,178$ at $x=0$, $|f(1)-P_5(1)|=0$ at $x=1$, and $|f(2)-P_5(2)| \approx 0.000\,321$ at $x=2$.

Problem 4.

Solution. Precompute the coefficients. Let

$$a_0 = \ln(3) = 1.0986123, \quad a_1 = -\frac{1}{3} = -0.3333333, \quad a_2 = -\frac{1}{9\times 2!} = -0.0555555,$$

$$a_3 = -\frac{2}{27\times 3!} = -0.0123457, a_4 = -\frac{2}{27\times 4!} = -0.0030864, a_5 = -\frac{8}{81\times 5!} = -0.0008230.$$

Let $\hat{x} = x - x_0 (= x-1)$. Then using Horner's method, we have

$$P_5(x) = (((((a_5\hat{x} + a_4)\hat{x}) + a_3)\hat{x} + a_2)\hat{x} + a_1)\hat{x} + a_0,$$

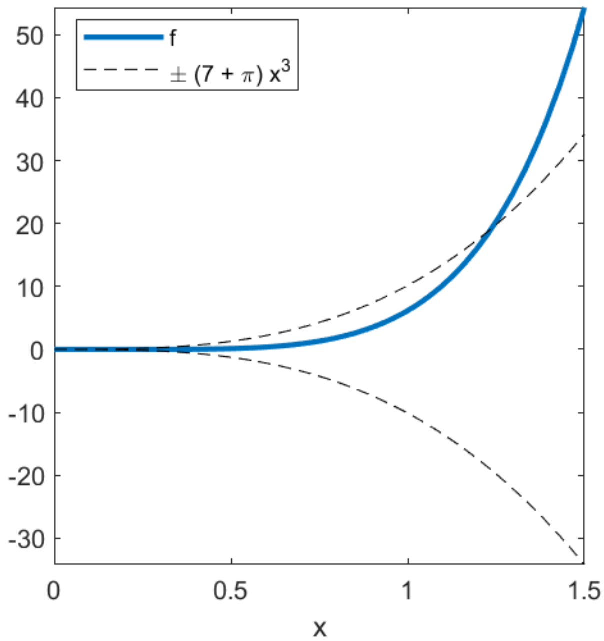
which requires 5 multiplications and $1+5=6$ additions/subtractions.

Problem 5.

Solution. $f(x) = \mathcal{O}(x^3)$.

Let $\delta = 1$. For $0 < x < \delta = 1$, we have $x^4 < x^3$ and $x^6 < x^3$. Therefore,

$$|f(x)| = |5x^4 + \pi x^6 - 2x^3| \leq |5x^4| + |\pi x^6| + |2x^3| \leq \underbrace{(5 + \pi + 2)}_{C} x^3 \qquad \text{for all } x \in (0, \delta).$$



OR

Observe that $|f(x)| = |5x^4+\pi x^6-2x^3| = |5x+\pi x^3-2|x^3$ and $5x+\pi x^3-2$ is monotonically decreasing as $x \downarrow 0$. Therefore, $|5x + \pi x^3 - 2| < \max\{|5 \cdot 1 + \pi \cdot 1^3 - 2|, |5 \cdot 0 + \pi \cdot 0^3 - 2|\} = |5 + \pi - 2| = 3 + \pi$ for all $x \in (0, 1)$. Then letting $\delta = 1$, we have

$$|f(x)| = |5x + \pi x^3 - 2|x^3 \leq \underbrace{(3 + \pi)}_{C} x^3 \qquad \text{for all } x \in (0, \delta).$$

[$C = 3 + \pi$ is the best (smallest) C for $\delta = 1$.]