Homework 7

Problem 1:

i) $g(t) = 0(u(t) - u(t-1)) + 2(u(t-1) - u(t-5)) +$
$1(u(t-5) - u(t-11)) + 3(u(t-11))$

ii) $g(t) = e^{-t}(u(t) - u(t-3)) + t(u(t-1) - u(t-7))$
$+ 1(u(t-7))$

Problem 2:

i) $f(t) = t^2 u(t-2)$, $L\{f(t)\} = e^{-2s} \cdot L\{(t+2)^2\} =$
$e^{-2s} \cdot (L(t^2) + L(4t) + L(4)) = \boxed{e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)}$

ii) $f(t) = e^{-t}(u(t) - u(t-3)) + 1(u(t-3)) =$
$e^{-t}u(t) - e^{-t}u(t-3) + u(t-3)$
$L\{f(t)\} = \frac{1}{s+1} - e^{-3s} L(e^{-(t+3)}) + \frac{e^{-3s}}{s}$
$L(e^{-t-3}) = \frac{1}{e^3} L(e^{-t}) = \frac{1}{e^3 \cdot (s+1)}$. So, $L(f(t)) =$
$\boxed{\frac{1}{s+1} - \frac{e^{-3s}}{e^3(s+1)} + \frac{e^{-3s}}{s}}$

Problem 3:

i) inverse of $\frac{e^{-3s}(s-5)}{(s+1)(s+2)} = g(s)$. Using $L^{-1}(e^{-as}F(s)) = f(t-a)u(t-a)$,
$L^{-1}(g(t)) = u(t-3) \cdot L^{-1}\left(\frac{s-5}{(s+1)(s+2)}\right)$. Using partial fraction expansion:
$\frac{s-5}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \rightarrow s-5 = A(s+2) + B(s+1)$, so when
$s = -2$, $-7 = -B$, so $B = 7$. When $s = -1$, $A = -6$.
So our final answer is:
$\boxed{L^{-1}(g(t)) = u(t-3) \cdot (-6e^{-(t-3)} + 7e^{-2(t-3)})}$

Problem 3 continued

ii)

inverse laplase of $g(t) = \frac{e^{-s}(3s^2-s+2)}{(s-1)(s^2+1)}$

$L^{-1}(g(t)) = u(t-1) \cdot L^{-1}((3s^2-s+2)/((s-1)(s^2+1)))$

Using partial fraction expansion:

$\frac{3s^2-s+2}{(s-1)(s^2+1)} = \frac{A}{(s-1)} + \frac{Bs+C}{(s^2+1)}$, so $3s^2-s+2 = A(s^2+1) + \overset{B(s^2-s)}{Bs\cdot(s-1)} + C(s-1)$

$A+B=3$, $C-B=-1$, $A-C=2$ so $C=A-2$, $A-2-B=-1$,

$A-2-(3-A)=-1$, $2A=4$, $A=2$, $B=1$, $C=0$, So, our answer is:

$\boxed{L^{-1}(g(t)) = u(t-1)\cdot(2e^{t-1}+\cos(t-1))}$

Problem 4)

$y''+4y = \sin(t)\cdot(u(t)-u(t-\pi))$ ~~$+0(u(t-2\pi))$~~ $= g(t)$

Note $\sin 2\pi = \sin$

$L(g(t)) = s^2y(s) - sy(0) - y'(0) + 4(y(s)) = L(\sin(t)\cdot u(t) - \sin(t)\cdot u(t-2\pi)) = \frac{1}{s^2+1} - e^{-2\pi s}\cdot\frac{1}{s^2+1} = \frac{1}{s^2+1}(1-e^{-2\pi s})$

Simplified: $s^2y(s) - s - 3 + 4y(s) = \frac{1}{s^2+1}(1-e^{-2\pi s})$

and $y(s)(s^2+4) - s - 3 = \frac{1}{s^2+1}(1-e^{-2\pi s})$ so

$y(s) = \frac{1-e^{-2\pi s}}{(s^2+1)(s^2+4)} + \frac{s}{s^2+4} + \frac{3}{s^2+4} = \frac{1}{(s^2+1)(s^2+4)} - e^{-2\pi s}\cdot\frac{1}{(s^2+1)(s^2+4)} + \frac{s}{s^2+4} + \frac{3}{s^2+4}$

Using partial fraction expansion:

$\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{(s^2+4)}$ so $1 = A(s^3+4s) + B(s^2+4) + C(s^3+s) + D(s^2+1)$

$A+C=0$, $B+D=0$, $4A+C=0$, $4B+D=1$ $B=\frac{1}{3}$, $D=-\frac{1}{3}$ so,

our final answer is:

$$\boxed{y = \frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t) - \left(u(t-2\pi)\cdot\left(\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin(2t-4\pi)\right)\right) + \cos(2t) + \frac{2}{3}\sin(2t)}$$

Problem 4 continued

ii) $y'' + 2y' + 10y = g(t)$, $y(0) = -1$, $y'(0) = 0$

Laplace: $s^2 y(s) - s y(0) - y'(0) + 2(s y(s) - y(0)) + 10 y(s) =$

$10(u(t) - u(t-10)) + 20(u(t-10) - u(t-20)) + 0 =$

$\frac{10}{s} - e^{-10s} \cdot \frac{10}{s} + e^{-10s} \cdot \frac{20}{s} - e^{-20s} \cdot \frac{20}{s} = \frac{10}{s}(1 - e^{-10s}) + \frac{20}{s}(e^{-10s} - e^{-20s})$,

$(s^2 + 2s + 10) y(s) + s + 2 = \frac{10(1 - e^{-10s}) + 20(e^{-10s} - e^{-20s})}{s}$

So, $y(s) = \frac{10 - 10e^{-10s} + 20e^{-10s} - 20e^{-20s} - s^2 - 2s}{s \cdot ((s+1)^2 + 9)}$

$= \frac{10}{s((s+1)^2+9)} + e^{-10s} \frac{10}{s((s+1)^2+9)} - e^{-20s} \frac{20}{s((s+1)^2+9)} - \frac{s^2}{s((s+1)^2+9)} - \frac{2s}{s((s+1)^2+9)}$

Partial fraction expansion:

$\frac{1}{s((s+1)^2+9)} = \frac{A}{s} + \frac{B(s+1)+C}{(s+1)^2+9}$,

So, $1 = A(s^2+2s+10) + B(s^2+s) + C \cdot s$

$A = 1/10$, $A + B = 0$, $B = -1/10$, $2A + B + C = 0$, $C = -1/10$

So, $\frac{1}{s((s+1)^2+9)} = \frac{-(s+1)}{10((s+1)^2+9)} + \frac{1}{10s} - \frac{1}{10((s+1)^2+9)}$

inverse laplace

So $\frac{10}{s(s^2+2s+10)} = e^{-t}\cos(3t) - \frac{e^{-t}}{3}\sin(3t) + 1$

and $e^{-10s}\frac{10}{s(s^2+2s+10)} = u(t-10)\left(-e^{-(t-10)}\cos(3(t-10)) - \frac{1}{3}e^{-(t-10)}\sin(3(t-10)) + 1\right)$

and $\frac{e^{-20s}\,20}{s(s^2+2s+10)} = u(t-20)\left(-2e^{-(t-20)}\cos(3(t-20)) - \frac{2}{3}e^{-(t-20)}\sin(3(t-20)) + 1\right)$

Simplify $\frac{s^2-2s}{s(s^2+2s+10)}$ to $\frac{s-2}{s^2+2s+10}$ and get $\frac{e^{-t}\sin(3t)}{3} + e^{-t}\cos(3t)$

Our final solution is:

$$\boxed{\begin{aligned} y(t) = {} & \left(-2e^{-t}\cos(3t) - \tfrac{2}{3}e^{-t}\sin(3t) + 1\right) + u(t-10) \cdot \\ & \left(-e^{-(t-10)}\cos(3(t-10)) - \tfrac{1}{3}e^{-(t-10)}\sin(3(t-10)) + 1\right) \\ & - u(t-20)\left(-2e^{-(t-20)}\cos(3(t-20)) - \tfrac{2}{3}e^{-(t-10)}\sin(3(t-20)) + 1\right) \end{aligned}}$$