Problem 1.

Solution.

In general, given $\dot{x} = f(t,x)$, Euler's method is given by $x_{n+1} = x_n + hf(T_0 + nh, x_n)$, $x_0 = x(T_0)$.

For $n = 5$, the step size is $h = \frac{2}{5} = 0.4$.

t	x	x'
0	1	0.367879441
0.4	1.147151776	0.317539907
0.8	1.274167739	0.279663624
1.2	1.386033189	0.25006530
1.6	1.486059310	0.226262532
2	$\boxed{1.576564322}$	0.206683979

For $n = 10$, the step size is $h = \frac{2}{10} = 0.2$.

t	x	$x' = f$
0	1	0.367879441
0.2	1.073575888	0.341784148
0.4	1.141932718	0.319201499
0.6	1.205773018	0.299460422
0.8	1.265665102	0.282051641
1	1.322075430	0.266581456
1.2	1.375391721	0.252740573
1.4	1.425939836	0.240282531
1.6	1.473996342	0.229008458
1.8	1.519798034	0.218756064
2	$\boxed{1.563549246}$	0.209391569

[Students need not to include the tables.]

The true solution can be found by the following procedure:

$$\dot{\bar{x}} = e^{-\bar{x}} \quad \rightsquigarrow \quad \int_0^t \frac{\dot{\bar{x}}(s)}{e^{-\bar{x}(s)}}\,\mathrm{d}s = \int_0^t 1\,\mathrm{d}s \quad \rightsquigarrow \quad \int_{\bar{x}(0)}^{\bar{x}(t)} e^x\,\mathrm{d}x = t \quad \rightsquigarrow \quad e^{\bar{x}(t)} - e = t.$$

Solving this (algebraic) equation for $\bar{x}(t)$ yields $\bar{x}(t) = \ln(e+t)$. Therefore, $\bar{x}(2) \approx 1.551\,444\,714$.

The errors are

$$e_5^5 = |1.576564322 - 1.551444714| \approx 0.025\,119\,608$$
$$e_{10}^{10} = |1.563549246 - 1.551444714| \approx 0.012\,104\,532.$$

In theory, Euler's method is $\mathcal{O}(h)$. To estimate the order from the calculations, we compare the ratio between errors vs. the ratio between the number of steps. In particular, we may expect the order is $\mathcal{O}(n^k)$ where k is given by

$$k = \frac{\log(e_{10}^{10}/e_5^5)}{\log(10/5)} = \frac{\log(e_{10}^{10}) - \log(e_5^5)}{\log(10) - \log(5)} \approx -1.053.$$

–OR–

If using h, this is $\mathcal{O}(h^{1.053})$.

Problem 2.

Solution. The linear interpolation for $f(x) = \exp(4x)$ for x between 0 and 0.5 is

$$p_1(x) = \frac{x_1 - x}{x_1 - x_0} f(x_0 = 0) + \frac{x - x_0}{x_1 - x_0} f(x_1 = 0.5) = \frac{0.5 - x}{0.5} + \frac{x}{0.5} e^2 = 1 + 2(e^2 - 1)x.$$

–OR–

$$p_1(x) = f(x_0) + \frac{(f(x_1) - f(x_0))(x - x_0)}{x_1 - x_0} = 1 + \frac{(e^2 - 1)x}{0.5} = 1 + 2(e^2 - 1)x.$$

Problem 3.

Solution. The error of piecewise linear interpolation is given by

$$\frac{1}{8} \max_{\zeta \in [0,0.5]} |f''(\zeta)| \left(\frac{x_1 - x_0}{n} \right)^2.$$

In the present case, $f'' = 16e^{4x}$, which is positive and increasing function, whose absolute value is maximized at the right endpoint.

Hence we shall find n such that

$$\frac{16e^2}{8} \left(\frac{0.5}{n} \right)^2 \leq 0.02 \quad \rightsquigarrow \quad n > 5e \approx 13.591\,41.$$

We see the minimal number of segments n (i.e. the smallest integer no less than 13.59) is 14.

The last interval is therefore $[\frac{13}{14} \times 0.5, 0.5] = [0.464\,286, 0.5]$, where the linear interpolation is given by

$$\frac{x_1 - x}{x_1 - x_0} f(x_0 = 0.464286) + \frac{x - x_0}{x_1 - x_0} f(x_1 = 0.5) = 28(0.5 - x)e^{13/7} + (28x - 13)e^2 = 27.542x - 6.382.$$

Problem 4.

Solution.

We work on the interval $[0, 10]$; that is, $a = 0$, $b = 10$.

We take $K_1 = \max_{x\in[0,10]} |f'(x)| = 1$ and $K_2 = \max_{x\in[0,10]} |f''(x)| = 30$.

Left-endpoint The error bound is given by $e_n^{\mathrm{LE}} \le \frac{K_1}{2}(b-a)h = \frac{K_1}{2}\frac{(b-a)^2}{n}$.

For $\epsilon = 10^{-1}$,

$$\frac{K_1}{2}\frac{(b-a)^2}{n} \le \epsilon \quad \rightsquigarrow \quad n \ge 500.$$

We may take $n = 500$.

For $\epsilon = 10^{-6}$,

$$\frac{K_1}{2}\frac{(b-a)^2}{n} \le \epsilon \quad \rightsquigarrow \quad n \ge 5 \times 10^7.$$

We may take $n = 5 \times 10^7$.

Trapezoid The error bound is given by $e_n^{\mathrm{TRAP}} \le \frac{K_2}{12}(b-a)h^2 = \frac{K_2}{12}\frac{(b-a)^3}{n^2}$.

For $\epsilon = 10^{-1}$,

$$\frac{K_2}{12}\frac{(b-a)^3}{n^2} \le \epsilon \quad \rightsquigarrow \quad n \ge 50\sqrt{10} \approx 158.1.$$

We may take $n = 159$.

For $\epsilon = 10^{-6}$,

$$\frac{K_2}{12}\frac{(b-a)^3}{n^2} \le \epsilon \quad \rightsquigarrow \quad n \ge 5 \times 10^4.$$

Round up.

We may take $n = 5 \times 10^4$.

Midpoint The error bound is given by $e_n^{\mathrm{TRAP}} \le \frac{K_2}{24}(b-a)h^2 = \frac{K_2}{24}\frac{(b-a)^3}{n^2}$.

For $\epsilon = 10^{-1}$,

$$\frac{K_2}{24}\frac{(b-a)^3}{n^2} \le \epsilon \quad \rightsquigarrow \quad n \ge 50\sqrt{5} \approx 111.8.$$

We may take $n = 112$.

For $\epsilon = 10^{-6}$,

$$\frac{K_2}{24}\frac{(b-a)^3}{n^2} \le \epsilon \quad \rightsquigarrow \quad n \ge \frac{5 \times 10^4}{\sqrt{2}} \approx 35\,355.3.$$

We may take $n = 35356$.

Problem 5.

Solution. The true integral is

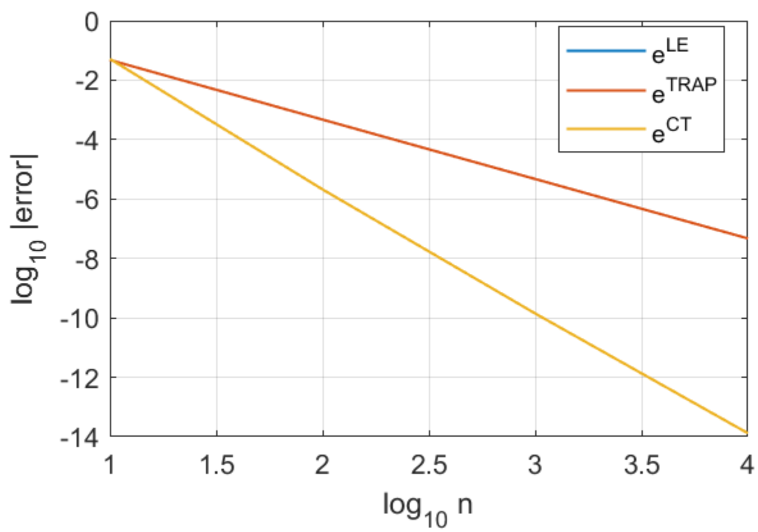
$$\int_0^3 \sin(\pi x) - \frac{1}{2}\cos(2\pi x)\,\mathrm{d}x = \left[-\frac{\cos(\pi x)}{\pi} - \frac{1}{4\pi}\sin(2\pi x)\right]\Bigg|_{x=0}^{3} = \frac{2}{\pi}.$$

The errors of numerical integration is given by

n	e^{LE}	e^{TRAP}	e^{CT}
10	4.78×10^{-2}	4.78×10^{-2}	5.21×10^{-2}
100	4.71×10^{-4}	4.71×10^{-4}	2.11×10^{-6}
1000	4.71×10^{-6}	4.71×10^{-6}	1.40×10^{-10}
10000	4.71×10^{-8}	4.71×10^{-8}	1.33×10^{-14}

[Students need not to include the table.]

The log-log plot of the error is



Blue and orange line overlap

[Students may plot error vs. h rather than n, in which case the plot should look like the mirror image of this, and slopes are negated.]

Both left-endpoint and trapezoid method has slope around -2. This is because the integrand are equal at both $x = 0$ and $x = 3$. Corrected trapezoid method has slope around -4 on the plot. These match the theoretical prediction that trapezoid method is $\mathcal{O}(n^{-2})$ and corrected trapezoid method is $\mathcal{O}(n^{-4})$.

Reading from the plot, for left-endpoint and trapezoid method, it would require around 2500 (accept anything from 1000 to 3000) steps so that error is less than 10^{-6}. For corrected trapezoid method, it would require around 130 steps to achieve the same (accept anything from 100 to 200).

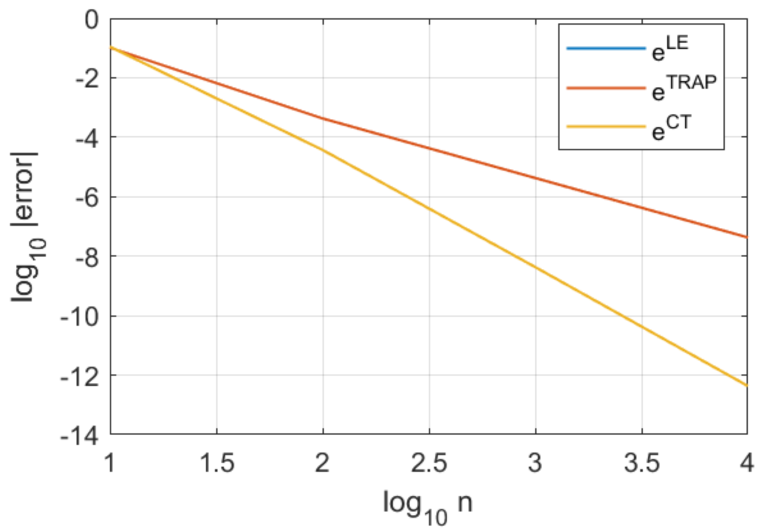
Problem 6.

Solution. The true integral is approximated by corrected trapezoid method with $n = 10^5$, which yields $I_{10^5}^{\mathrm{CT}} \approx 0.254\,914\,093$.

The errors of numerical integration is given by

n	e^{LE}	e^{TRAP}	e^{CT}
10	1.03×10^{-1}	1.03×10^{-1}	1.11×10^{-1}
100	4.29×10^{-4}	4.29×10^{-4}	3.75×10^{-5}
1000	4.26×10^{-6}	4.26×10^{-6}	4.30×10^{-9}
10000	4.26×10^{-8}	4.26×10^{-8}	4.31×10^{-13}

The log-log plot of the error is



Both left-endpoint and trapezoid method has slope around -2. This is again because the integrand are equal at $x = 0$ and $x = 3$. Corrected trapezoid method still has slope around -4 on the plot. These match the theoretical prediction that trapezoid method is $\mathcal{O}(n^{-2})$ and corrected trapezoid method is $\mathcal{O}(n^{-4})$.

Reading from the plot, for left-endpoint and trapezoid method, it would require around 2500 (accept anything from 1000 to 3000) steps so that error is less than 10^{-6}. For corrected trapezoid method, it would require around 230 steps to achieve the same (accept anything from 100 to 300).

Problem 7.

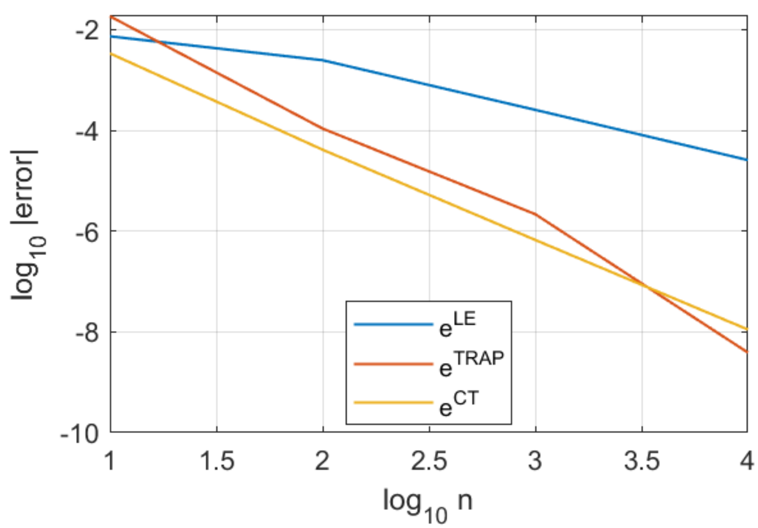
Solution. The true integral is given by

$$\int_0^3 |x - \sqrt{2}|\,\mathrm{d}x = \frac{13}{2} - 3\sqrt{2}.$$

The errors of numerical integration is given by

n	e^{LE}	e^{TRAP}	e^{CT}
10	0.007359313	-0.018376618	-0.003376618
100	0.00246494	-0.000108653	4.13E-05
1000	0.000255191	-2.17E-06	-6.68E-07
10000	2.57E-05	-3.88E-09	1.11E-08

The log-log plot of the error is



Left-endpoint method has slope around -1, whereas the slope for both trapezoid and corrected trapezoid method is around -2. Compare this to the theoretical prediction that left-endpoint method is $\mathcal{O}(n^{-1})$, trapezoid method is $\mathcal{O}(n^{-2})$, and corrected trapezoid method is $\mathcal{O}(n^{-4})$. However, these asymptotic orders are meant to hold when the integrand is sufficiently many times differentiable, which does not hold for the present case.

Reading from the plot, for left-endpoint method, it would require around 3200 (accept anything from 2000 to 5000) steps so that error is less than 10^{-6}. For trapezoid and corrected trapezoid method, it would require around 1000 steps to achieve the same (accept anything from 400 to 2500).