O. H. 1:00 pm

- Content on test, Dh^2?
- Q's on code for #5-7
- Q on gradescope incorrect answer.
- 3rd study problem

MAE 107
Assignment 4
Due 10:00pm, Saturday, 13 May

Note: You must show all your work (including your codes) in order to get credit!

Problems to hand in (Not all problems may be graded.)

1. Either using matlab, or with the aid of a calculator, apply Euler's method with $n = 5$ and $n = 10$ steps to solve the initial value problem,

$$\dot{x} = \exp(-x),$$
$$x(0) = 1,$$

up to terminal time $T = 2$. What are the actual errors you obtain for each n? Use the logs of these two errors to make a wild estimate of the order of the method. What was our estimate of the order of the errors in Euler's method? Compare these estimates. (A short sentence is sufficient!) You do NOT need to provide any code you use on this specific problem, but you do need to provide the results and the math you use. Note: The true solution (undoubtedly also to be found online!) may be obtained by integrating $\frac{dx}{\exp(-x)} = dt$, and then employing the initial condition.

$e^x dx = dt$

$e^x = T$

2. Obtain the linear interpolation for $f(x) = \exp(4x)$ with $x_0 = 0$ and $x_1 = 0.5$.

3. Suppose you will use a piecewise linear interpolation for $f(x) = \exp(4x)$ over the interval from $a = 0$ to $b = 0.5$. How many segments will be needed such that the maximum error over the entire interval is no greater than $\varepsilon = 0.02$? What is the equation of the last segment, and over what interval is it valid?

4. Suppose you will be writing code to integrate functions over $[0, 10]$. You know that the functions will be at least twice differentiable with $|f'(x)| \leq 1$ and $|f''(x)| \leq 30$ for all of the functions. Suppose the error in any of the integrals must be no greater than $\varepsilon = 10^{-1}$. How many steps would be required to guarantee that this error bound is met if

you were applying the left-endpoint rectangle rule? How many steps would be required for this guarantee in the cases of the trapezoid and midpoint rules? How would the numbers change if $\varepsilon = 10^{-6}$ instead?

5. Write matlab codes for computing

$$\int_0^3 \sin(\pi x) - \tfrac{1}{2}\cos(2\pi x)\, dx$$

   by the left-endpoint rule, the trapezoid rule and the corrected trapezoid rule. Obtain the approximations for $n = 10^k$ with $k = 1, 2, \ldots 4$. Plot the log of the error versus the log of the number of steps (i.e., $\log_{10}(n)$) for all three rules, all in the same plot window. Comment on the slopes. Also, by examination of the graph, very roughly estimate the minimal number of steps required by each method such that the error is no greater than 10^{-6}.

   For this problem, your code must be structured as separate functions. More specifically, one function will simply evaluate $f(x) = \tan(x)$ at either a given single value of x or for all x in an array of points, as you prefer. Three other functions will compute the left-endpoint, trapezoid and corrected-trapezoid integral estimates for a given input specific value of n. You are free to structure your codes so that these functions either take arrays of function values as inputs or call the above indicated function inside. You may write either a script or a function that will loop over the various values of n above, calling the left-endpoint and trapezoid method functions. This outer-loop code will also generate the requested log-log plot.

6. Apply your code to the case of

$$\int_0^3 \sin\left(\pi(1 - x^2)\right)/\sqrt{2 + x^2}\, dx$$

   and provide the analogous plot and comments. However, as you do not know the exact value of the integral, use the corrected trapezoid rule with $n = 10^5$ as truth.

7. Apply your code to the case of

$$\int_0^3 \left|x - \sqrt{2}\,\right| dx$$

and provide the analogous plot. Also, attempt explanations for the seemingly different behaviors observed in the resulting curves. A few sentences are likely to be sufficient.

Problems 1, 3, 4 and 5 are worth 10 points each. The other problems are worth 5 points each.

Study Problems (Will not be graded.)

- In Problem 2 above, what is the maximum actual error over the interval? What is our bound on the maximum error over the interval?
- Approximately compute $\int_0^\pi \cos(x)\,dx$ using the left-endpoint and trapezoid rules with $n = 3$ steps. Discuss the actual errors observed, specifically noting the relation to the bounds on these errors.
- Suppose you were using Euler's method to solve $\dot{x} = f(x)$ with a very computationally intensive function, f, from initial condition $x(0) = 1$ up to time $T = 500$. With $n = 25$, you obtained $x_{25} \simeq 7.13584$. With $n = 500$, you obtained $x_{500} \simeq 7.11584$. In the latter case, the code had an approximate run-time of 1.5 seconds. Roughly how long of a run-time do you estimate would be required so that the error with Euler's method would be no greater than 10^{-7}? (Of course, your answer will only be a rough approximation.)

$e_{25}^{25} \simeq |x_{25}^{25} - x_{500}^{500}| = 0.02 \simeq \frac{c}{n}$ $\quad c = 0.5$ with $n = 25$,

so, $\frac{0.5}{n} \leq 10^{-7}$, $n \geq 5 \times 10^6$

500 steps takes 1.5 sec, so, $\frac{500}{1.5} = \frac{5 \times 10^6}{x}$

$x = \underline{15{,}000 \text{ Seconds}}$

Problem 1)

$\dot{x} = e^{-x}$, $x(0) = 1$, $T = 2$,

for $n = 5$,

$h = \frac{T}{n} = \frac{2}{5}$ (step size)

$\frac{dx}{e^{-x}} = dt$

$\tilde{x} = \ln(t + c_1)$

$1 = \ln(c)$

$c = e$,

so $\tilde{x} = \ln(t + e)$

Also $\rightarrow t = e^x + c$

$\rightarrow 0 = e + c$

$c = -e$

$\rightarrow 2 = e^x - e$

$2 + e = e^x$

$\tilde{x} = \ln(2 + e)$

$t_0 = 0$	$x_0 = 1$
$t_1 = t_0 + 2/5 = 2/5$	$x_1 = x_0 + e^{-1} \cdot 2/5 \simeq 1.14715$
$t_2 = t_1 + 2/5 = 4/5$	$x_2 = x_1 + e^{-1.147} \cdot 2/5 \simeq 1.27416$
$t_3 = t_2 + 2/5 = \frac{6}{5}$	$x_3 = x_2 + e^{-1.274} \cdot 2/5 \simeq 1.38603$
$t_4 = t_3 + 2/5 = 8/5$	$x_4 = x_3 + e^{-1.386} \cdot 2/5 \simeq 1.48606$
$t_5 = t_4 + 2/5 = 2$	$x_5 = x_4 + e^{-1.486} \cdot 0.4 \simeq \boxed{1.5765}$

Actual error:

$|\ln(2+e) - 1.5765| \simeq \boxed{0.0251}$

for $n = 10$, $h = \frac{T}{n} = \frac{2}{10} = 1/5$

$t_0 = 0$	$x_0 = 1$
$t_1 = t_0 + 1/5 = 1/5$	$x_1 = x_0 + e^{-1} \cdot 1/5 \simeq 1.07357$
$t_2 = t_1 + 1/5 = 2/5$	$x_2 = x_1 + e^{-1.07357} \cdot 1/5 \simeq 1.14193$
$t_3 = t_2 + 1/5 = 3/5$	$x_3 = x_2 + e^{-1.14193} \cdot 1/5 \simeq 1.20577$
$\vdots$	$x_4 = x_3 + e^{-1.20577} \cdot 1/5 \simeq 1.26566$
	$x_5 = x_4 + e^{-1.26566} \cdot 1/5 \simeq 1.32207$
	$x_6 = x_5 + e^{-1.32207} \cdot 1/5 \simeq 1.37539$
	$x_7 = x_6 + e^{-1.37539} \cdot 1/5 \simeq 1.42594$
	$x_8 = x_7 + e^{-1.42594} \cdot 1/5 \simeq 1.47399$
	$x_9 = x_8 + e^{-1.47399} \cdot 1/5 \simeq 1.51979$
$t_{10} = t_9 + 1/5 = 2$	$x_{10} = x_9 + e^{-1.51979} \cdot 1/5 \simeq \boxed{1.5635}$

Actual error:

$|\ln(2+e) - 1.5635| \simeq \boxed{0.0121}$

Comparing our errors, $|0.0251 - 0.0121| \simeq 0.013$. We essentially cut our error in half by doubling the steps.

Aprox slope of log: $\frac{\log_{10}(0.0251) - \log_{10}(0.0121)}{\log_{10}(2/5) - \log_{10}(1/5)} \simeq 1.05$ Slope

Thus, we can estimate that the order of this method is 1.

Problem 2)

Linear interpolation of e^{4x}, $x_0 = 0$, $x_1 = 0.5$

$$P_1(x) = f(x_0)\left(\frac{x_1 - x}{x_1 - x_0}\right) + f(x_1)\left(\frac{x - x_0}{x_1 - x_0}\right)$$

$$P_1(x) = 1\left(\frac{0.5 - x}{0.5}\right) + e^2\left(\frac{x - 0}{0.5 - 0}\right)$$

$$= (1 - 2x) + e^2\left(\frac{x}{0.5}\right) = \boxed{1 + 12.778x}$$

Problem 3)

3. Suppose you will use a piecewise linear interpolation for $f(x) = \exp(4x)$ over the interval from $a = 0$ to $b = 0.5$. How many segments will be needed such that the maximum error over the entire interval is no greater than $\varepsilon = 0.02$? What is the equation of the last segment, and over what interval is it valid?

$y = e^{4x}$, 0 to 0.5, $\varepsilon < 0.02$, $y' = 4e^{4x}$, $y'' = 16e^{4x}$

$$h = \frac{x_1 - x_0}{n} = \frac{0.5}{n}$$

$$\text{error} \le \max_{\xi \in \left(\frac{0.5n - 0.5}{n}, 0.5\right)} \frac{|f''(\xi)|}{8} (\overbrace{x_1 - x_0}^{h})^2$$

$$\le 2e^2\left(\frac{0.5}{n}\right)^2 = 0.02$$

So, $n^2 = \frac{2e^2 \cdot 0.25}{0.02}$, $n = 13.59 \rightarrow$ needs to be integer $\rightarrow \boxed{n = 14}$

Problem 4)

4. Suppose you will be writing code to integrate functions over $[0, 10]$. You know that the functions will be at least twice differentiable with $|f'(x)| \le 1$ and $|f''(x)| \le 30$ for all of the functions. Suppose the error in any of the integrals must be no greater than $\varepsilon = 10^{-1}$. How many steps would be required to guarantee that this error bound is met if you were applying the left-endpoint rectangle rule? How many steps would be required for this guarantee in the cases of the trapezoid and midpoint rules? How would the numbers change if $\varepsilon = 10^{-6}$ instead?

$[0,10]$, $f'(x) \le 1$, $f''(x) \le 30$, $\varepsilon < 10^{-1} = 0.1$ and $\varepsilon < 10^{-6}$

L.E.

$$e_n^{LE} \le \frac{K_1}{2}(b-a)^2 \cdot n^{-1}, \quad K_1 = 1$$

So,

$$e_n^{LE} \le 0.1 = \frac{1}{2}(100) \cdot \frac{1}{n}, \quad n = 500 \text{ steps}$$

$$e_n^{LE} \le 10^{-6} = \frac{1}{2}(100) \cdot \frac{1}{n}, \quad n = 5 \times 10^7 \text{ steps}$$

Trap.

$$e_n^{trap} \le \frac{K_2(b-a)^3}{12} \cdot \frac{1}{n^2}$$

* rounded up

So,

$$e_n^{trap} \le 0.1 = \frac{30(10)^3}{12} \cdot \frac{1}{n^2}, \quad n = 159 \text{ steps}$$

$$e_n^{trap} \le 10^{-6} = \frac{30 \cdot (10)^3}{12} \cdot \frac{1}{n^2}, \quad n = 5 \times 10^4 \text{ steps}$$

Midpoint

$$e_n^{md} \le \frac{K_2(b-a)^3}{24} \cdot \frac{1}{n^2}$$

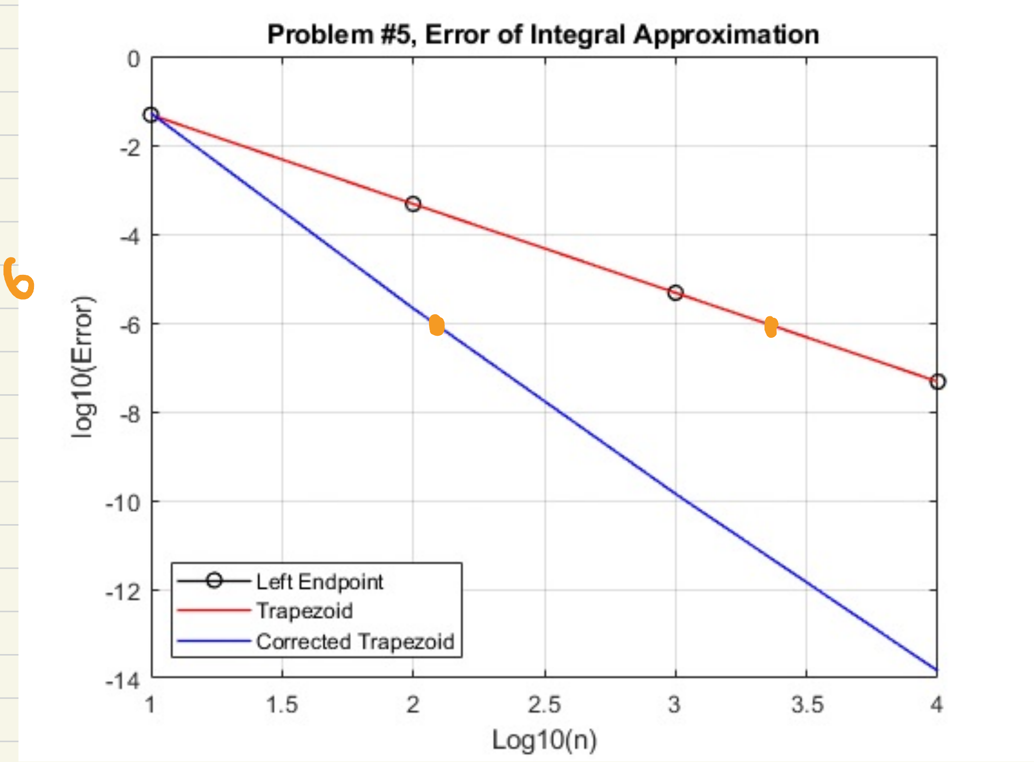
So,

$$e_n^{md} \le 0.1 = \frac{30 \cdot (10)^3}{24} \cdot \frac{1}{n^2}, \quad n = 112 \text{ steps}$$

$$e_n^{md} \le 10^{-6} = \frac{30 \cdot (10)^3}{24} \cdot \frac{1}{n^2}, \quad n = 35{,}356 \text{ steps}$$

Problem 5)

$\log_{10}(10^{-6}) = -6$



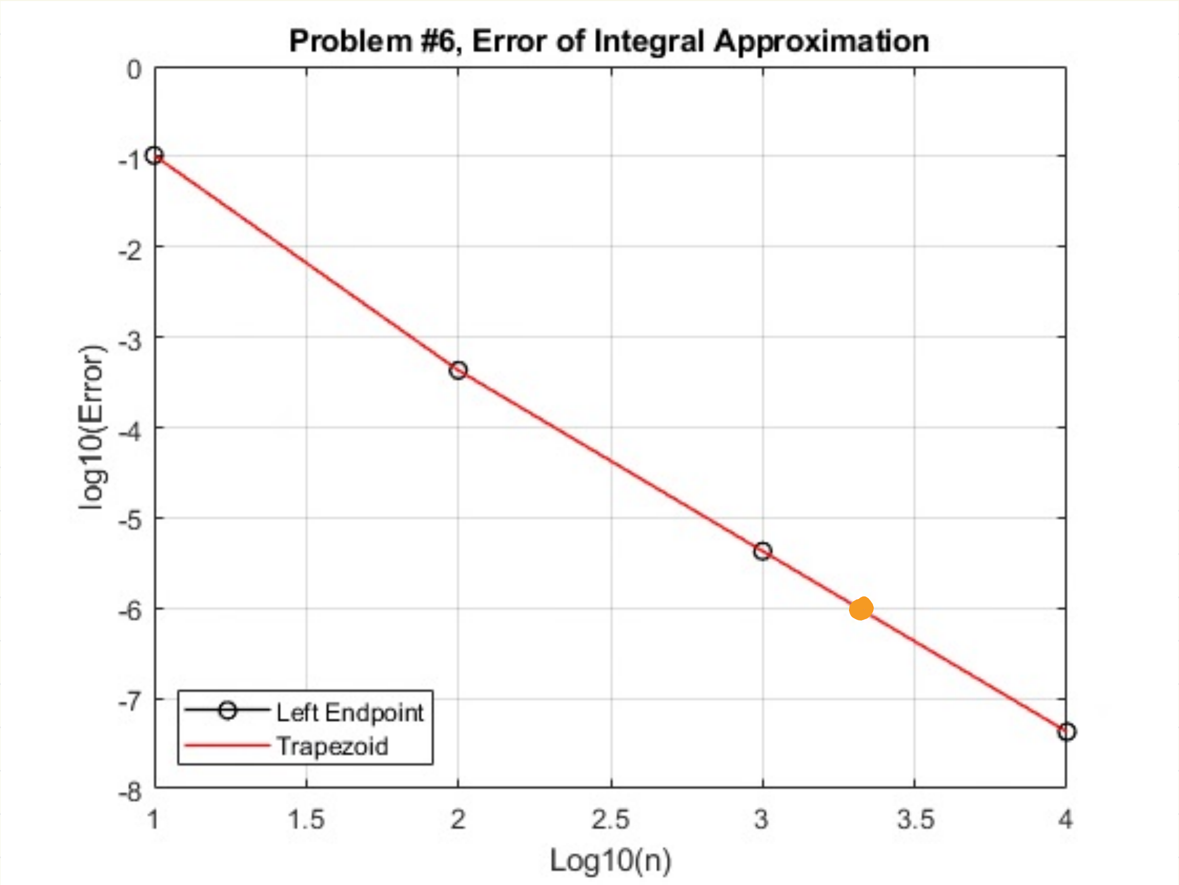
The corrected trapezoid is significantly steeper in slope than the LE & Trap methods. This means it's more accurate in fewer steps.

For error of 10^{-6}

L.E & trap: $10^{2.1} = n \approx 126$ steps

Ctrap: $10^{3.4} = n \simeq 2512$ steps

Problem 6)

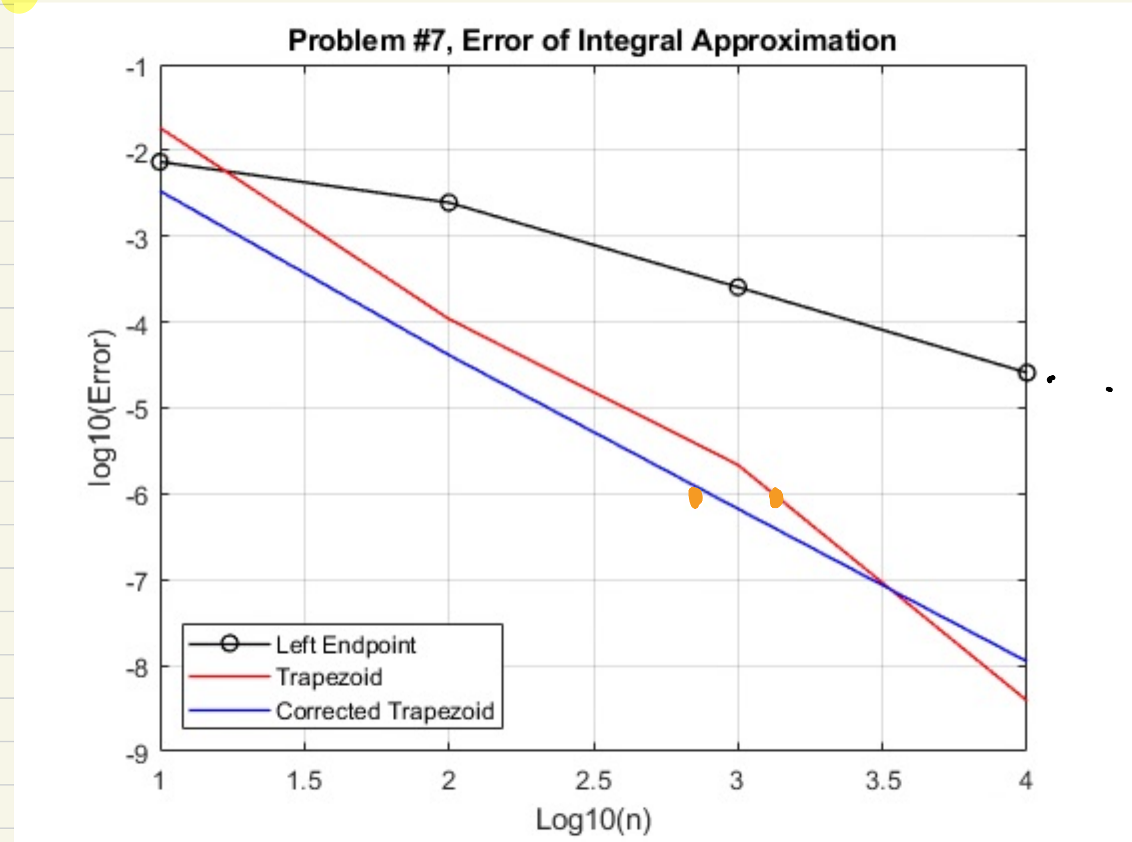


The slope for both graphs is the same; they have the same error. This is likely due to some symmetry in the equation.

For error of 10^{-6}

L.E. and trap: $10^{3.3} \approx n = 1996$ steps

Problem 7)



6 ?

The corrected trapezoid has a higher slope than the trapezoid method and the trap. method has a higher slope than L.E. method. Thus, CTrap has lowest error and the trap. method has the 2nd lowest.

For error of 10^{-6}:

LE: $10^{6} = n =$ 1Mil steps

Trap: $10^{3.2} = n \cong 1585$ steps

Ctrap: $10^{2.75} = n \cong 563$ steps.