Kasey
Haman

HW 6

1. $F(s) = \frac{17}{(s-3)\cdot((s+1)^2+1)} = \frac{A}{(s-3)} + \frac{B(s+1)+C}{(s+1)^2+1}$ So,

$17 = A(s^2+2s+2) + B(s+1)(s-3) + C(s-3)$ so...

when $s=3$ $17=(9+6+2)A$, $A=\frac{17}{17}=1$, when

$s=-1$, $17=(1-2+2)+0-4C \to 16=-4C$, $C=-4$

lastly, when $s=0$, $17 = 2-3B+12 \to B=-1$

so $L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s-3}\right) - L^{-1}\left(\frac{s+1}{(s+1)^2+1}\right) - 4L^{-1}\left(\frac{1}{(s+1)^2+1}\right)$

$\boxed{= e^{3t} - e^{-t}\cos(t) - 4e^{-t}\sin(t)}$

Derivative to take laplace

2. a) $L^{-1}\left(\ln\left(\frac{s^2+9}{s^2+1}\right)\right) = L^{-1}(\ln(s^2+9) - \ln(s^2+1))$

Note: If $L(f(t)) = F(s)$ then $L(t f(t)) = -\frac{d}{ds}F(s)$

So, $f(t) = -\frac{1}{t}L^{-1}(\frac{d}{ds}F(s))$. So now, we can write our original problem as $f(t) = -\frac{1}{t}L^{-1}\left(\frac{d}{ds}(\ln(s^2+9) - \ln(s^2+1))\right)$.

So, $f(t) = \frac{1}{t}L^{-1}\left(\frac{2s}{s^2+9} - \frac{2s}{s^2+1}\right) = -\frac{1}{t}\cdot(2\cos(3t) - 2\cos(t))$

$\boxed{= \frac{-2\cos(3t) + 2\cos(t)}{t}}$

b) Because $L(t f(t)) = -\frac{d}{ds}F(s)$, $f(t) = -\frac{1}{t}L^{-1}(-F'(s))$, and we can write $f(t) = -\frac{1}{t}L^{-1}\left(\frac{1}{1+(\frac{1}{s})^2}\right)$, $L^{-1}\left(\frac{1}{1+\frac{1}{s^2}}\right) =$

$L^{-1}(1/(\frac{s^2+1}{s})) = L^{-1}\left(\frac{s}{s^2+1}\right) = \cos(t)$ so, $f(t) =$

$\boxed{-\frac{\cos(t)}{t}}$

3. i) $L(\sin(t) - t\cos(t)) = \frac{1}{s^2+1} - \frac{s^2-1}{(s^2+1)^2} = \frac{2}{(s^2+1)^2}$, $L(t\sin(t)) = -1\cdot\frac{d}{ds}\left(\frac{1}{s^2+1}\right) =$

$-1\cdot\left(\frac{-2s}{(s^2+1)^2}\right) = \frac{2s}{(s^2+1)^2}$,

ii) $ty'' - 2y' + ty = 0$, $y(0)=1$, $y'(0)=0$.

$L(ty'') = -\frac{d}{ds}(s^2y(s) - sy(0) - y'(0)) = -(2sy(s) + s^2y'(s)) + 1$

$L(-2y') = -2(sy(s)-1) = -2sy(s)+2$, $L(ty) = -\frac{d}{ds}y(s) = -y'(s)$

So, we have: $-s^2Y'(s) - Y'(s) - 2sY(s) - 2sY(s) + 3 = 0$

$(s^2+1)y' + 4sy = 3 \to \mu = e^{\int\frac{4s}{(s^2+1)}} = e^{2\ln(s^2+1)} = (s^2+1)^2$ So,

$(s^2+1)^2y + 4s(s^2+1) = 3(s^2+1)$ which simplifies to

$(s^2+1)^2y = \int 3(s^2+1)\,ds = s^3+3s+C \to y = \frac{s^3+3s+C}{(s^2+1)^2} =$

$\frac{A_1s+B_1}{s^2+1} + \frac{A_2s+B_2}{(s^2+1)^2}$ so $s^3+3s+C = A_1s(s^2+1) + B_1(s^2+1) + A_2s + B_2$

Problem 3 continued

$s^3 + 3s + c = A_1(s^3+s) + B_1(s^2+1) + A_2 s + B_2$,

$A_1 = 1$, $B_1 = 0$, $A_2 = 2$, $B_2 = c$ So, we have

$$L^{-1}\left(\frac{s}{s^2+1} + \frac{2s}{(s^2+1)^2} + \frac{c}{(s^2+1)^2}\right) = y = \cos(t) + t\sin(t) + c(\sin(t) - t\cos(t))$$

$$0 = \frac{16}{9} + 4B + \frac{8}{3}$$

$$\frac{40}{4} \qquad -\frac{40}{9} \cdot \frac{1}{4} = B = -\frac{10}{9}$$

Problem 4)

i) $y'' + 6y' + 9y = 1$, $y(0) = -1$, $y'(0) = 6$, using laplace:

$L(y'') + 6L(y') + 9y(s) = \frac{1}{s}$

$L(y'') = s^2 y(s) - s \cdot -1 - 6$, $L(y') = sy(s) + 1$ So,

$s^2 y(s) + s - 6 + 6sy(s) + 6 + 9y(s) = 1/s \longrightarrow$

$y(s)(s^2 + 6s + 9) = \frac{1-s^2}{s}$, $y(s) = \frac{1-s^2}{s(s+3)^2} = \frac{A}{s} + \frac{B}{(s+3)} + \frac{C}{(s+3)^2} \rightarrow$

$1 - s^2 = (s+3)^2 A + B(s+3)s + Cs$ When $s = 0$

$1 = 9A$, $A = 1/9$, when $s = -3$, $1 - 9 = -3C$, $C = \frac{8}{3}$,

and $\frac{1}{9}s^2 + Bs^2 = -s^2$ so $B = -\frac{10}{9}$

$$\boxed{y = \frac{1}{9} - \frac{10}{9}(e^{-3t}) + \frac{8}{3}(e^{-3t}t)}$$

ii) $y'' + 3ty' - 6y = 1$, $y(0) = 0$, $y'(0) = 0$

$L(y'') + 3L(ty') - 6L(y) = \frac{1}{s}$

$L(y'') = s^2 Y(s) - sy(0) - y'(0)$, $L(ty') = -\frac{d}{ds}(sY(s) - y(0))$, so

$s^2 Y(s) - 3(Y(s) + sY'(s)) - 6Y(s) = \frac{1}{s} \rightarrow -3sY'(s) + s^2 Y(s) - 9Y(s) = \frac{1}{s}$

$\rightarrow Y'(s) - \frac{Y(s)(s^2 - 9)}{3s} = -\frac{1}{3s^2}$, $u = \frac{(9-s^2)}{3s}$ $e^{\int \frac{(9-s^2)}{3s}} \rightarrow$

$e^{\int \frac{3}{s} - \int \frac{s}{3}}$ which is equal to $e^{3\ln(s) - \frac{s^2}{6}} = s^3 e^{-s^2/6} \rightarrow$

$s^3 e^{-s^2/6} Y(s) = \int -\frac{e^{-s^2/6} \cdot s}{3} = -\frac{1}{3}\int se^{-s^2/6} = -\frac{1}{3}\int -\frac{6}{2}e^{u} = e^{-s^2/6} + C$

now, dividing $s^3 e^{-s^2/6}$, we have $Y(s) = \frac{1}{s^3} + \frac{Ce^{s^2/6}}{s^3}$, when we take the lim, $\frac{Ce^{s^2/6}}{s^3}$ will go to infinity, so $C = 0$.

Therefore, $Y(s) = \frac{1}{s^3}$ and $\boxed{y(t) = \frac{t^2}{2}}$.