Homework 8

1. $y'' + 5y' + 6y = g(t)$. $y(0) = 0$, $y'(0) = 2$

$L(g(t)) = s^2 y(s) - s y(0) - y'(0) + 5(s y(s) - y(0)) + 6 y(s)$

$L(g(t)) = L(t((u-1) - (u-5)) + (u-5))$

$= L(t(u-1) - t(u-5) + (u-5))$

$= e^{-s} L(t+1) - e^{-5s} L(t+5) + e^{-5s}/s$

$= e^{-s}\left(-\frac{1}{s^2} + \frac{1}{s}\right) - e^{-5s}\left(\frac{1}{s^2} + \frac{5}{s}\right) + \frac{e^{-5s}}{s}$

$= (e^{-s} + e^{-s}\cdot s - e^{-5s} - e^{-5s}\cdot 5s + e^{-5s}\cdot s) / s^2$

$= (-e^{-5s} - 4s e^{-5s} + e^{-s} + e^{-s}\cdot s)/s^2$

So, $s^2 y(s) - 2 + 5s y(s) + 6 y(s) = L(g(t))$ which equals

$y(s)(s^2 + 5s + 6) - 2 = L(g(t))$, so,

$y(s) = (-e^{-5s} - 4s e^{-5s} + e^{-s} + e^{-s}\cdot s + 2s) / (s+3)(s+2)s^2$

if $\theta = \frac{1}{(s+3)(s+2)s^2}$, $y(s) = -e^{-5s}\cdot\theta - e^{-5s}\cdot 4s\cdot\theta + e^{-s}\cdot\theta + e^{-s}\cdot s\cdot\theta + 2s\theta$

Partial fraction expansion:

$\frac{1}{(s+3)(s+2)(s^2)} = \frac{A}{(s+3)} + \frac{B}{(s+2)} + \frac{C}{s} + \frac{D}{s^2}$, $A(s+2)s^2 + B(s+3)s^2 + C((s+3)(s+2))s + D(s+3)(s+2)$

$D6 = 1$, $D = (1/6)$, $A + B + C = 0$, $2A + 3B + 5C + D = 0$, $6C + 5D = 0$

$6C = -5/6$, $C = (-5/36)$, $A + B = 5/36$, $2A + 3B = 19/36$, $A = (-1/9)$, $B = (\frac{1}{4})$

And for $\frac{s}{(s+3)(s+2)s^2}$ we have $A = (\frac{1}{3})$, $B = (-1/2)$, $C = (1/6)$, $D = 0$

So $L^{-1}(-e^{-5s}\theta) = -u(t-5)\cdot\left(-\frac{1}{9}e^{-3(t-5)} + \frac{1}{4}e^{-2(t-5)} + \frac{(t-5)}{6} - \frac{5}{36}\right)$

$L^{-1}(-e^{-5s}\cdot 4s) = -4\cdot u(t-5)\cdot\left(\frac{1}{3}e^{-3(t-5)} - \frac{1}{2}e^{-2(t-5)} + \frac{1}{6}\right)$

$L^{-1}(e^{-s}\theta) = u(t-1)\left(\frac{t-1}{6} - 11/36 + \frac{e^{-2(t-1)}}{4} - \frac{e^{-3t}}{9}\right)$

$L^{-1}(e^{-s}\theta s) = u(t-1)\left(\frac{e^{2-2t}}{2} + \frac{e^{3-3t}}{3} + 1/6\right)$

$L^{-1}(2s\theta) = 1/3 - e^{-2t} + \frac{2e^{-3t}}{3}$

Final soln is all laplace inverses added together.

2. i) $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$, $\det \begin{bmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{bmatrix}$ b/c $(A - I\lambda)v = 0$

$-4 + \lambda^2 + 3 = 0$, So $\lambda^2 - 1 = 0$. $\lambda = 1$, $\lambda = -1$

$\begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $x_1 = -3x_2$, $x_2\begin{bmatrix} -3 \\ 1 \end{bmatrix} = E_1$

$\begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_1 = -x_2$, $x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = E_{-1}$

ii) $(1-\lambda)\begin{vmatrix} -\lambda & 3 \\ 3 & -\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 2 & -\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & -\lambda \\ 2 & 3 \end{vmatrix}$

$(1-\lambda)(\lambda^2 - 9) - 2(-2\lambda - 6) + 2(6 + 2\lambda)$

$\lambda^2 - 9 - \lambda^3 + 9\lambda + 4\lambda + 12 + 12 + 4\lambda = -\lambda^3 + \lambda^2 + 17\lambda + 15$

$= -(\lambda+1)\cdot(\lambda^2 - 2\lambda - 15) = -(\lambda+1)(\lambda+3)(\lambda-5) = 0$ $\lambda = -1, -3, 5$

$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$, $A - I\lambda_1 = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix}$, $A - I\lambda_3 = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{bmatrix}$ $A - I\lambda_5 = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -5 & 3 \\ 2 & 3 & -5 \end{bmatrix}$

$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \lambda_3\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = E_{-1}$, $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = x_3\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = E_{-3}$

$\begin{bmatrix} -4 & 2 & 2 \\ 2 & -5 & 3 \\ 2 & 3 & -5 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = x_3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = E_5$

3. $\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} -1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 3 \\ 0 & 3 & -1-\lambda \end{bmatrix}$

$(-1-\lambda)\big((2-\lambda)(-1-\lambda) - 9\big) - 1(-1-\lambda)$

$(-1-\lambda)\big((2-\lambda)(-1-\lambda) - 10\big)$

$-2 - \lambda + \lambda^2 - 10 \to \lambda^2 - \lambda - 12 \to (\lambda - 4)(\lambda + 3)$

So, $\lambda = -1$, 4 and -3.

$E_{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 3 & 3 \\ 0 & 3 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{matrix} x_1 = -3x_3 \\ x_2 = 0 \end{matrix}$ So $E_{-1} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, $E_4 = \begin{bmatrix} -5 & 1 & 0 \\ 1 & -2 & 3 \\ 0 & 3 & -5 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to$

$x_1 = 1/3\, x_3$, $x_2 = 5/3\, x_3$ so, $E_4 = \begin{bmatrix} 1/3 \\ 5/3 \\ 1 \end{bmatrix}$, $E_{-3} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 5 & 3 \\ 0 & 3 & 2 \end{bmatrix}$ $x_1 = 1/3\, x_3$, $x_2 = -2/3\, x_3$ So $E_{-3} = \begin{bmatrix} 1/3 \\ -2/3 \\ 1 \end{bmatrix}$

$$\boxed{x(t) = C_1 e^{-t}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + C_2 e^{4t}\begin{bmatrix} 1/3 \\ 5/3 \\ 1 \end{bmatrix} + C_3 e^{-3t}\begin{bmatrix} 1/3 \\ -2/3 \\ 1 \end{bmatrix}}$$