Problem 1.

Solution. Given $|g'| \le 0.9$, we have $|g(x_2) - g(x_1)| \le 0.9|x_2 - x_1|$; we may take $K = 0.9$. The error bound is given by

$$e_n = |x_n - \hat{x}| \le \frac{K^n}{1-K}|x_1 - x_0|.$$

Therefore, we need to solve $\dfrac{K^n}{1-K}|x_1 - x_0| \le 1 \times 10^{-8}$ for n.

$$\frac{0.9^n}{1-0.9}|100 - 0| \le 1 \times 10^{-8} \quad \rightsquigarrow \quad 0.9^n \le 1 \times 10^{-11} \quad \rightsquigarrow \quad n \ge 240.4.$$

Hence it takes 241 iterations to guarantee the desired precision.

Problem 2.

Solution. $f(t,y) = 2 + y$. $h = \dfrac{T}{n} = 0.5$.

True solution $\bar{y} = 2e^t - 2$, which gives $\bar{y}(T = 1) = 3.436\,563\,66$.

$\beta_1 = \beta_2 = \frac{1}{2}, \alpha = \nu = 1$. $y_{k+1} = y_k + \dfrac{h}{2}(f(t_k, y_k) + f(t_k + h, y_k + hf(t_k, y_k)))$

t_k	y_k	$f(t_k, y_k)$	$y^{\text{guess}} = y_k + hf(t_k, y_k)$	$f(t_k + h, y^{\text{guess}})$
0	0	2	1	3
0.5	1.25	3.25	2.875	4.875
1	$\boxed{3.281\,25}$	//	//	//

$\beta_1 = \frac{1}{4}, \beta_2 = \frac{3}{4}, \alpha = \nu = \frac{2}{3}$. $y_{k+1} = y_k + \dfrac{h}{4}(f(t_k, y_k) + 3f(t_k + \dfrac{2}{3}h, y_k + \dfrac{2}{3}hf(t_k, y_k)))$

t_k	y_k	$f(t_k, y_k)$	$y^{\text{guess}} = y_k + \frac{2}{3}hf(t_k, y_k)$	$f(t_k + \frac{2}{3}h, y^{\text{guess}})$
0	0	2	$\frac{2}{3}$	$\frac{8}{3}$
0.5	1.25	3.25	$\frac{7}{3}$	$\frac{13}{3}$
1	$\boxed{3.281\,25}$	//	//	//

Actual error (in both cases) is $|3.281\,25 - 3.436\,563\,66| = 0.155\,31$.