MAE 107
Assignment 3
Due 10:00pm, Tuesday, 2 May

Note: You must show all your work (including your codes) in order to get credit!

Problems to hand in (Not all problems may be graded.)

In this assignment, we will use the notation:

$$D_h(x_0) = \frac{f(x_0+h) - f(x_0)}{h}, \qquad \hat{D}_h(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

and

$$\tilde{D}_h(x_0) = \frac{8[f(x_0+h) - f(x_0-h)] - [f(x_0+2h) - f(x_0-2h)]}{12h}.$$

1. Let $f(x) = x^2$. Let $x_0 = \pi$. Use $D_h(x_0)$ to compute approximations to the derivative, $f'(x_0)$, with $h = 10^{-11}$ and $h = 10^{-13}$. What are the resulting errors? In one or two sentences, provide a rough explanation of the behavior of the errors, noting the specific orders-of-magnitude difference between the two errors.

2. Consider
$$f(x) = e^{-2x}\cos(\pi x).$$
Suppose we want to compute the derivative at $x_0 = 1$ using difference approximations. Compute $D_h(1)$, $\hat{D}_h(1)$ and $\tilde{D}_h(1)$ for $h = 10^{-j}$, for $j \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let e_h, $\hat{e}_h$ and $\tilde{e}_h$ denote the error magnitudes in $D_h(1)$, $\hat{D}_h(1)$ and $\tilde{D}_h(1)$, respectively. Using matlab, plot $log_{10}(e_h)$, $log_{10}(\hat{e}_h)$ and $log_{10}(\tilde{e}_h)$ versus $log_{10}(h)$, all on the same graph, but in different colors. Write text on the graph (using matlab) to indicate which curve is for which approximation. Roughly indicate either with matlab or by hand, the regions where the Taylor-polynomial derived errors dominate, and where the catastrophic-subtraction errors dominate. In the linear sections of the curves, what do the slopes tell you? *Keep in mind that your codes must be commented in accordance with the examples appearing on the website.*

3. Do the same steps as in the first problem, but for the case of

$$f(x) = x^{(7/2)} + x$$

at $x_0 = 0$, and with $h = 10^{-j}$, for $j \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Can you suggest an explanation for the slope behavior that is different from that in the previous example. *Keep in mind that your codes must be commented in accordance with the examples appearing on the website.*

4. Consider the difference approximation to $f''(x_0)$ given by

$$D_h^2(x_0) = \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2}.$$

Consider $f(x) = \ln(x^2 + 2)$, and use $D_h^2(0)$ to approximate $f''(0)$. Compute $D_h^2(0)$ for $h = 10^{-j}$, for $j \in \{0, 1, 2, 3, 4, 5\}$. Let e_h^2 denote the error in $D_h^2(0)$. Using matlab, plot $log_{10}(e_h^2)$ versus $log_{10}(h)$, and comment on the result. In particular, what do you think the order might be? *Keep in mind that your codes must be commented in accordance with the examples appearing on the website.*

5. By hand, apply Euler's method with $n = 3$ steps to solve the initial value problem,

$$\dot{x} = x^2/4,$$
$$x(0) = 2,$$

up to terminal time $T = 3/2$. You may use a calculator if you like, but show each line as the algorithm progresses. What is the actual error in your solution at terminal time, $T = 3/2$? Note that the true solution has the form $\bar{x}(t) = -1/(c + t/4)$ for some value of $c < 0$ (that we'll let you find), and obviously would go off to infinity as $t \uparrow -4c$. Because of that, you might not expect the numerical method to perform well with so few steps.

Problems 1, 3 and 5 are worth 5 points each. Problems 2 and 4 are worth 10 points each.