Problem 1.

Solution. True derivative $f'(x_0 = \pi) = 2\pi \approx 6.283\,185$.

$$D_{h=10^{-11}}(x_0 = \pi) = \frac{(\pi + 10^{-11})^2 - \pi^2}{10^{-11}} \approx 6.283\,152,$$
$$D_{h=10^{-13}}(x_0 = \pi) = \frac{(\pi + 10^{-13})^2 - \pi^2}{10^{-13}} \approx 6.270\,537.$$

$$e_{h=10^{-11}} = |D_{10^{-11}}(\pi) - f'(\pi)| = 3.353\,054 \times 10^{-5},$$
$$e_{h=10^{-13}} = |D_{10^{-13}}(\pi) - f'(\pi)| = 1.264\,566 \times 10^{-2}.$$

We see that the error with smaller step size ($h = 10^{-13}$) results in a bigger error; the difference in terms of order of magnitude is 2.6. [2 or 3 are both acceptable]

This is because $f(x_0 + h)$ and $f(x_0)$ gets close as h gets small; in this problem h is so small such that we encounter catastrophic cancellation when computing their difference $f(x_0 + h) - f(x_0)$, resulting in loss of precision.

–OR–

$$D_h(x_0 = \pi) = \frac{(\pi + h)^2 - \pi^2}{h} = 2\pi + h \quad \rightsquigarrow \quad e_h = h.$$

Therefore,

$$e_{h=10^{-11}} = 10^{-11}, \quad e_{h=10^{-13}} = 10^{-13}.$$

The order of magnitude difference of error is exactly 2. This confirms that D_h is a first-order approximation.

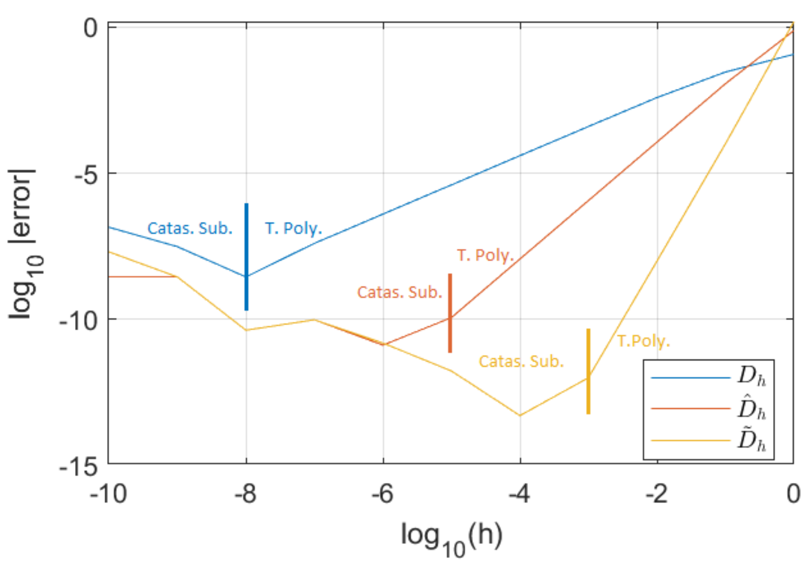
Problem 2.

Solution.

h	D_h	$\hat{D}_h$	$\tilde{D}_h$	e_h	$\hat{e}_h$	$\tilde{e}_h$
1	0.153 650 922	-0.490 842 181	-1.270004353	0.117 019 644	0.761 512 747	1.540674919
10^{-1}	0.299 552 175	0.259 142 595	0.270 569 003	0.028 881 608	0.011 527 972	0.000 101 563
10^{-2}	0.274 527 562	0.270 555 043	0.270 670 557	0.003 856 996	0.000 115 523	9.772×10^{-9}
10^{-3}	0.271 066 594	0.270 669 411	0.270 670 566	0.000 396 028	1.155×10^{-6}	1.006×10^{-12}
10^{-4}	0.270 710 273	0.270 670 555	0.270 670 566	3.971×10^{-5}	1.155×10^{-8}	5.079×10^{-14}
10^{-5}	0.270 674 538	0.270 670 566	0.270 670 566	3.972×10^{-6}	1.144×10^{-10}	1.707×10^{-12}
10^{-6}	0.270 670 964	0.270 670 566	0.270 670 566	3.972×10^{-7}	1.310×10^{-11}	1.541×10^{-11}
10^{-7}	0.270 670 606	0.270 670 566	0.270 670 566	3.973×10^{-8}	9.636×10^{-11}	9.636×10^{-11}
10^{-8}	0.270 670 569	0.270 670 567	0.270 670 567	2.818×10^{-9}	4.242×10^{-11}	4.242×10^{-11}
10^{-9}	0.270 670 597	0.270 670 569	0.270 670 569	3.057×10^{-8}	2.818×10^{-9}	2.818×10^{-9}
10^{-10}	0.270 670 708	0.270 670 569	0.270 670 546	1.416×10^{-7}	2.818×10^{-9}	2.031×10^{-8}

[Students need not to include the table.]

Error in derivative approximations



For $\hat{D}_h$, also accept cutoff at -6; $\tilde{D}_h$ at -4.

In the linear region, the slope indicates the order (i.e. the exponent in $\mathcal{O}(h^?)$).

Sample MATLAB code

Students should include extensive comments

```
% arguments
% x_0 = 1;
% f = @(x) exp(-2 * x) * cos(pi * x);

true_der_at_x_0 = -exp(-2 * x_0) * (pi * sin(pi * x_0) + 2 * cos(pi * x_0));
J = 10.^(-(0:10));
history = zeros(length(J), 1 + 3 + 3); % [h, D, e]
for idx = 1:length(J)
    h = J(idx)
    % [D, D_hat, D_tilde]
    D = [(f(x_0 + h) - f(x_0)) / h, ...
        (f(x_0 + h) - f(x_0 - h)) / (2 * h), ...
        (8 * (f(x_0 + h) - f(x_0 - h)) - (f(x_0 + 2 * h) - f(x_0 - 2 * h))) / (12 * h)]
    e = abs(D - true_der_at_x_0) % [e, e_hat, e_tilde]
    history(idx, :) = [h, D, e];
end

% plotting commands
% plot(log10(history(:, 1)), log10(history(:, 5:7)))
% ...
```

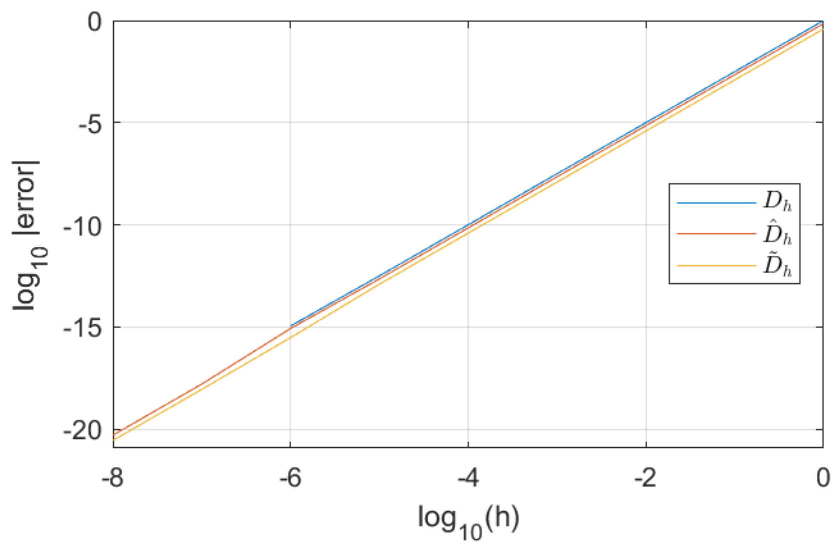
Problem 3.

Solution.

h	D_h	e_h	$\log_{10}(e_h)$
1	2	1	0
10^{-1}	1.003 162 277 660 17	$3.162\,277\,660\,168\,30 \times 10^{-3}$	-2.50
10^{-2}	1.000 010 000 000 00	$9.999\,999\,999\,843\,47 \times 10^{-6}$	-5.00
10^{-3}	1.000 000 031 622 78	$3.162\,277\,661\,949\,99 \times 10^{-8}$	-7.50
10^{-4}	1.000 000 000 100 00	$1.000\,000\,082\,740\,37 \times 10^{-10}$	-10.0
10^{-5}	1.000 000 000 000 32	$3.161\,915\,174\,132\,45 \times 10^{-13}$	-12.5
10^{-6}	1.000 000 000 000 00	$1.110\,223\,024\,625\,16 \times 10^{-15}$	-14.95
10^{-7}	1	0	//
10^{-8}	1	0	//

[Student may provide either table or plot; $\hat{D}_h$ and $\tilde{D}_h$ (errors thereof) are optional]

Error in derivative approximations



Students shall note at least one of the following:

[Lack of catastrophic subtraction] — the error is log-log linear with the step size, with no signs of catastrophic cancellation. This is because $f(x_0 = 0) = 0$.

[Slope is not 1, but 2.5; $\mathcal{O}(h^{5/2})$] — the log-log slope is such that decreasing h by a factor of 10 decreases the error by a factor of approximately $10^{2.5}$. This is because $f'(x_0 = 0) = 1$ and

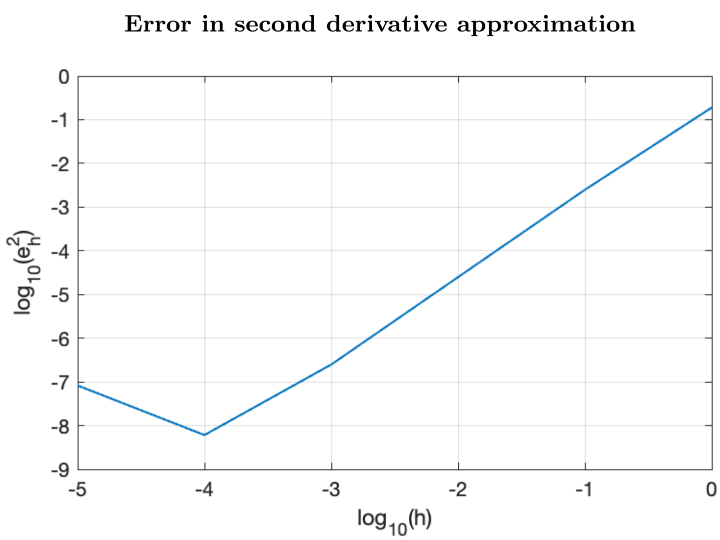
$$D_h(x_0 = 0) = \frac{f(h) - f(0)}{h} = \frac{h^{7/2} - 0}{h} + 1 \quad \rightsquigarrow \quad e_h = h^{5/2}.$$

Problem 4.

Solution.

h	D_h^2	e_h^2	$\log_{10}(e_h^2)$
1	0.810 930 216 216 329	0.189 069 783 783 671	-0.723
10^{-1}	0.997 508 302 207 794	$2.491\,697\,792\,205\,59 \times 10^{-3}$	-2.604
10^{-2}	0.999 975 000 834 930	$2.499\,916\,507\,048\,28 \times 10^{-5}$	-4.602
10^{-3}	0.999 999 750 117 553	$2.498\,824\,471\,786\,07 \times 10^{-7}$	-6.602
10^{-4}	0.999 999 993 922 529	$6.077\,470\,970\,922\,15 \times 10^{-9}$	-8.216
10^{-5}	1.000 000 082 740 37	$8.274\,037\,122\,113\,50 \times 10^{-8}$	-7.082

[Students need not to include the table.]



Judging from the slope in the linear region, the order is $\mathcal{O}(h^2)$. Catastrophic cancellation is encountered for small h.

Problem 5.

Solution.

Given $\dot{x} = \frac{x^2}{4}$, we have $f(t, x) = \frac{x^2}{4}$.

Step size $h = \frac{T}{n} = \frac{3/2}{3} = \frac{1}{2} = 0.5$.

We use the recurrence formula $x_k = x_{k-1} + hf(t_{k-1}, x_{k-1})$ and obtain

t	x	$f(t,x)$
0	2	1
0.5	2.5	1.5625
1	3.281 25	2.691 65
1.5	4.627 08	5.352 46

The true solution has the form $\bar{x}(t) = -\frac{1}{c + \frac{t}{4}}$, where c can is given by

$$\bar{x}(0) = -\frac{1}{c + \frac{0}{4}} = x(0) = 2 \quad \rightsquigarrow \quad c = -\frac{1}{2}.$$

Thus, $\bar{x}(T) = -\frac{1}{-\frac{1}{2} + \frac{3/2}{4}} = 8$ and

$$e_3^3 = |\bar{x}(T) - x_n| \approx 3.372\,92.$$