UNIT : 2

LINEAR TRANSFORMATION AND DIAGONALIZATION

Defn : Linear Transformation

Let V and W be vector spaces over F. A fn $T : V \to W$ is called a linear transformation of V into W if (i) $T(u+v) = T(u) + T(v) \quad \forall u, v \in V$

(ii) $T(\alpha u) = \alpha T(u) \quad \forall u \in V \ \& \ \alpha \in F$

(or) both (i) & (ii) together can be written as

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v), \quad \forall u, v \in V, \ \alpha, \beta \in F$$

Note : This defn means T preserves addition and multiplication, ie, algebraic structures, it is also known as vector space homomorphism.

Problems :

Check whether T is a linear transformation

(i) $T : R \to R$ defined by $T(x) = x+3 \quad \forall x \in R$

$\longrightarrow$ To check: $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$

LHS: $T(\alpha x + \beta y) = \alpha x + \beta y + 3$ by defn

RHS: $\alpha T(x) + \beta T(y) = \alpha(x+3) + \beta(y+3)$

$= \alpha x + 3\alpha + \beta y + 3\beta$

$\Rightarrow$ LHS $\neq$ RHS $\Rightarrow$ T is not linear.

(ii) $T : R^3 \to R^3$ defined as $T(x,y,z) = (x,0,0) \quad \forall (x,y,z) \in R^3$

$\longrightarrow$ To check: $T(\alpha u + \beta v) = \alpha T(u) + \beta(T(v))$

Here $u = (x_1, y_1, z_1)$ & $v = (x_2, y_2, z_2) \in R^3$

$\alpha u + \beta v = (\alpha x_1 + \beta x_2, \ \alpha y_1 + \beta y_2, \ \alpha z_1 + \beta z_2)$

LHS : $T(\alpha u+\beta v) = T(\alpha x_1+\beta x_2,\ \alpha y_1+\beta y_2,\ \alpha z_1+\beta z_2)$

$= (\alpha x_1+\beta x_2,\ 0,\ 0)$ By defn

RHS : $\alpha T(u)+\beta T(v) = \alpha T(x_1,y_1,z_1) + \beta T(x_2,y_2,z_2)$

$= (\alpha x_1, 0, 0) + (\beta x_2, 0, 0)$ By defn

$= (\alpha x_1+\beta x_2,\ 0,\ 0)$

$\Rightarrow$ LHS = RHS $\Rightarrow$ T is Linear.

(iii) $T : R^3 \to R^3 \ni T(1,0,3) = (1,1)$ and $T(-2,0,-6) = (2,1)$

$\rightarrow$ To check : T is linear.

$T(-2,0,-6) = -2\,T(1,0,3)$

$= -2(1,1)$

$= (-2,-2) \neq (2,1)$

$\Rightarrow$ T is not linear.

(iv) $T : R^2 \to R^3$ by $T(x,y) = (x+1,\ 2y,\ x+y)$

$\rightarrow$ To check : T is not linear or linear.

Let $u = (x_1, y_1)$ & $v = (x_2, y_2)$, $u, v \in R^2$

LHS : $T(\alpha u+\beta v) = T(\alpha(x_1,y_1) + \beta(x_2,y_2))$

$= T(\alpha x_1+\beta x_2,\ \alpha y_1+\beta y_2)$

$= (\alpha x_1+\beta x_2+1,\ 2\alpha y_1+2\beta y_2,\ \alpha x_1+\beta x_2+\alpha y_1+\beta y_2)$

by defn

RHS $\alpha T(u)+\beta T(v) = \alpha T(x_1,y_1) + \beta T(x_2,y_2)$

$= \alpha[x_1+1,\ 2y_1,\ x_1+y_1] + \beta[x_2+1,\ 2y_2,\ x_2+y_2]$

$= (\alpha x_1+\beta x_2+\alpha+\beta,\ 2\alpha y_1+2\beta y_2,\ \alpha x_1+\beta x_2+\alpha y_1+\beta y_2)$

$\Rightarrow$ LHS $\neq$ RHS $\Rightarrow$ T is not linear

Practice Problems:

① check T is linear (a) $T: R^2 \to R^2$, $T(a_1, a_2) = (2a_1 + a_2, a_2)$

(b) $T: R^3 \to R^2$, $T(a_1, a_2, a_3) = (a_1 - a_2, a_1 - a_3)$

(c) $T: R^2 \to R^2$, $T(a, b) = (2a - 3b, a + 4b)$

Properties: If $T: V \to W$ be a L.T

(i) $T(0) = 0$

proof: T is L.T

$\Rightarrow T(cu) = cT(u)$

put $c = 0$

$T(0.u) = 0\,T(u)$

$T(0) = 0$

Hence proved

(ii) $T(-V) = -T(V)$

Proof: put $c = -1$ in $T(cV) = cT(V)$

$T(-1V) = -1\,T(V)$

$T(-V) = -T(V)$

Hence proved

(iii) $T(x - y) = T(x) - T(y)$

Proof: Given T is L.T

$\Rightarrow T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$

put $\alpha = 1$, $\beta = -1$

$T(1 \cdot x + (-1) \cdot y) = 1\,T(x) - 1\,T(y)$

$T(x - y) = T(x) - T(y)$

Hence proved.

(iv) $T(cx + y) = cT(x) + T(y)$

To prove: $T(cx + y) = cT(x) + T(y)$

put $\alpha = c$ & $\beta = 1$

in $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$

$T(cx + y) = cT(x) + T(y)$

(v) P.T if T is linear iff $T(cx + y) = cT(x) + T(y)$

Proof: Given T is linear $\Rightarrow T(x + y) = T(x) + T(y)$

$T(cx) = cT(x)$

$T(cx + y) = T(cx) + T(y)$

$= c\,T(x) + T(y)$

Conversely, given $T(cx+y) = cT(x) + T(y)$

put $c=1$ $\quad T(x+y) = T(x) + T(y)$ ——— *

put $y=0$ $\quad T(cx) = c\,T(x)$ ——— *

from (*), T is linear.

Matrix Representation of a Linear Transformation

Let $T: V \to W$ be a L·T, Let $B = \{V_1, V_2, \ldots V_m\}$ and $B' = \{W_1, W_2, \ldots W_n\}$ be the ordered basis of V & W.

Let $T(V_1) = a_{11} W_1 + a_{12} W_2 + \cdots + a_{1n} W_n$

$T(V_2) = a_{21} W_2 + a_{22} W_2 + \cdots + a_{2n} W_n$

$\vdots$

$T(V_m) = a_{m1} W_1 + a_{m2} W_2 + \cdots + a_{mn} W_n$

Then the matrix $T = [a_{ij}]^T_{m\times n}$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}^T$$

Note: Special linear transformations

(i) Zero Transformation : $T: V \to W$ defined by $T(v) = 0 \;\forall v \in V$

(ii) Identity Transformation : $T: V \to V$ defined by $T(v) = v \;\forall v \in V$

(iii) Linear functional : $T: V \to F$

(iv) Isomorphism : If $T: V \to W$ is 1—1 and onto

Some Standard Basis

(i) For R^2, $B = \{e_1, e_2\}$ where $e_1 = (1,0)$, $e_2 = (0,1)$

(ii) For R^3, $B = \{e_1, e_2, e_3\}$ where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$
$e_3 = (0,0,1)$

(iii) $P_2(R)$, $B = \{1, x, x^2\}$

(iv) $P_3(R)$, $B = \{1, x, x^2, x^3\}$

Problems:

① Find the matrix of the transformation $T : P_3(R) \to P_2(R)$ by $T(f(x)) = f'(x)$ w.r. to the std basis.

Std Basis B of $P_3(R) = \{1, x, x^2, x^3\}$

Std Basis of $P_2(R)$ is $B' = \{1, x, x^2\}$

$$T(1) = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2$$
$$T(x) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$$
$$T(x^2) = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2$$
$$T(x^3) = 3x^2 = 0\cdot 1 + 0\cdot x + 3\cdot x^2$$

$$\text{Matrix } T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

② Find the matrix of the transformation $T : R^3 \to R^3$ given by $T(a,b,c) = (3a+c, -2a+b, a+2b+4c)$ w.r.to the std basis.

std Basis of R^3 : $\{e_1, e_2, e_3\}$, $e_1 = (1,0,0)$ $e_2 = (0,1,0)$
$e_3 = (0,0,1)$

$$T(e_1) = T(1,0,0) = (3(1)+0, -2(1)+0, 1+2(0)+4(0)) = (3,-2,1)$$
$$T(e_2) = T(0,1,0) = (0, 1, 2)$$
$$T(e_3) = T(0,0,1) = (1, 0, 4)$$

$$\text{Matrix } T = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 4 \end{bmatrix}^T = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$

3. Let $T: R^2 \to R^3$ be defined by $T(x,y) = (x-y, x, x+2y)$ $\forall (x,y) \in R^2$, $B = \{e_1, e_2\}$ where $e_1 = (1,2)$ $e_2 = (2,3)$ and $B' = \{V_1, V_2, V_3\}$ where $V_1 = (1,1,0)$, $V_2 = (0,1,1)$, $V_3 = (2,2,3)$ then find the matrix of T.

Z

(ii) →

$B = \{e_1, e_2\}$ ∋ $e_1 = (1,2)$ $e_2 = (2,3)$

$B' = \{V_1, V_2, V_3\}$, $V_1 = (1,1,0)$ $V_2 (0,1,1)$ $V_3 = (2,2,3)$

$T(e_1) = T(1,2) = (1-2, 1, 1+4) = (-1, 1, 5)$ — ①

$T(e_2) = T(2,3) = (2-3, 2, 2+6) = (-1, 2, 8)$ — ②

Now representing ① & ② in terms of B'

$(-1,1,5) = \alpha(1,1,0) + \beta(0,1,1) + \gamma(2,2,3)$

$(-1,1,5) = (\alpha+2\gamma, \alpha+\beta+2\gamma, \beta+3\gamma)$

on solving $\alpha + 0\beta + 2\gamma = -1$; $\alpha+\beta+2\gamma = 1$;

$0\alpha + \beta + 3\gamma = 5$

We get $\alpha = -3$, $\beta = 2$, $\gamma = 1$ — ③

Now $(-1,2,8) = \alpha(1,1,0) + \beta(0,1,1) + \gamma(2,2,3)$

$(-1,2,8) = (\alpha + 0\beta + 2\gamma, \alpha+\beta+2\gamma, 0\alpha+\beta+3\gamma)$

on solving $\alpha + 0\beta + 2\gamma = -1$; $\alpha+\beta+2\gamma = 2$; $0\alpha+\beta+3\gamma = 8$

We get $\alpha = -4.33$, $\beta = 3$, $\gamma = 1.67$ — ④

From ③ & ④

$$\text{Matrix } T = \begin{bmatrix} -3 & 2 & 1 \\ -4.33 & 3 & 1.67 \end{bmatrix}^T = \begin{bmatrix} -3 & -4.33 \\ +2 & 3 \\ 1 & 1.67 \end{bmatrix}$$

4. Find the matrix representation $T : V_3 \rightarrow V_3$ given by $T(a,b,c) = \{3a+c,\ -2a+b,\ a+2b+4c\}$ w.r.to the basis $(1,0,1)$, $(-1,2,1)$, $(2,1,1)$

$T(1,0,1) = (3(1)+1,\ -2(1)+0,\ 1+2(0)+4(1)) = (4,-2,5)$ — ①

$T(-1,2,1) = (3(-1)+1,\ -2(-1)+2,\ -1+2(2)+4(1)) = (-2,4,7)$ — ②

$T(2,1,1) = (3(2)+1,\ -2(2)+1,\ 2+2(1)+4(1)) = (7,-3,8)$ — ③

Now representing ①, ②, ③ in terms of Basis

$(4,-2,5) = \alpha(1,0,1) + \beta(-1,2,1) + \gamma(2,1,1)$

$4 = \alpha - \beta + 2\gamma$; $-2 = 0\alpha + 2\beta + \gamma$; $5 = \alpha + \beta + \gamma$

on solving weget $\alpha = \frac{27}{4}$ $\beta = -1/4$ $\gamma = -3/2$

$\alpha = 6.75$ $\beta = -0.25$ $\gamma = -1.5$ — ④

$(-2,4,7) = \alpha(1,0,1) + \beta(-1,2,1) + \gamma(2,1,1)$

$-2 = \alpha - \beta + 2\gamma$; $4 = 0\alpha + 2\beta + \gamma$; $7 = \alpha + \beta + \gamma$

on solving, we get $\alpha = 6.25$ $\beta = 3.25$ $\gamma = -2.5$ — ⑤

$(7,-3,8) = \alpha(1,0,1) + \beta(-1,2,1) + \gamma(2,1,1)$

$7 = \alpha - \beta + 2\gamma$; $-3 = 0\alpha + 2\beta + \gamma$; $8 = \alpha + \beta + \gamma$

on solving, We get $\alpha = 10.5$ $\beta = -0.5$ $\gamma = -2$ — ⑥

From ④, ⑤, ⑥ Matrix $T = \begin{bmatrix} 6.75 & -0.25 & -1.5 \\ 6.25 & 3.25 & -2.5 \\ 10.5 & -0.5 & -2 \end{bmatrix}^T$

$$= \begin{bmatrix} 6.75 & 6.25 & 10.5 \\ -0.25 & 3.25 & -0.5 \\ -1.5 & -2.5 & -2 \end{bmatrix}$$

Practice problems :

1. If $T : R^2 \to R^2$ is defined as $T(a, b) = (a+2, b+3)$ then find the matrix of T in the std basis
2. If $T : R^2 \to R^2$ given by $T(a,b) = (-b, a)$ w.r.to basis $(1, 2)$, $(+1, -1)$, find the $[T]$
3. Find the matrix $[T]_e$ whose linear operator is $T(x, y) = (5x + y, 3x - 2y)$

Transforming Matrix to Linear Transformation

1. Find the L.T $T : R^3 \to R^3$ determined by $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ w.r.to the std bases. What is $T(-2, 2, 3)$?

$T = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ coefficient matrix $= \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$

Std basis : $e_1 = (1, 0, 0)$ $e_2 = (0, 1, 0)$ $e_3 = (0, 0, 1)$

$T(e_1) = (1, 0, -1)$ $T(e_2) = (2, 1, 3)$ $T(e_3) = (1, 1, 4)$

Let $(x, y, z) \in R^3$ Then

$$(x, y, z) = xe_1 + ye_2 + ze_3$$

$$T(x, y, z) = xT(e_1) + yT(e_2) + zT(e_3)$$

$$= x(1, 0, -1) + y(2, 1, 3) + z(1, 1, 4)$$

$$= (x + 2y + z, \ y + z, \ -x + 3y + 4z)$$

$$T(-2, 2, 3) = (-2 + 4 + 3, \ 2 + 3, \ -(-2) + 3(2) + 4(3))$$

$$= (5, 5, 20)$$

2. Find the linear transformation $T : R^2 \to R^3$ determined by $\begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}$ w.r.to the Std bases. What is $T(2, 3)$?

(10) Given

$$T = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix} \quad \text{coefficient matrix} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & -4 \end{pmatrix}$$

$e_1 = (1,0) \quad e_2 = (0,1)$

$T(e_1) = (1,0,2) \quad T(e_2) = (3,0,-4)$

$(x,y) = x(1,0) + y(0,1)$

$(x,y) = xe_1 + ye_2$

$T(x,y) = xT(e_1) + yT(e_2)$

$= x(1,0,2) + y(3,0,-4)$

$= (x+3y,\ 0,\ 2x-4y)$

$T(2,3) = (2+3(3),\ 0,\ 2(2)-4(3)) = (11,0,-8)$

Practice Problems

(1) Find the L.T of $A = \begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix}$ defined from $T: R^2 \to R^2$ w.r.to the std. basis

(2) Find the L.T $T: R^2 \to R^3$ given by $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$ w.r.to the std basis.

Eigen Values and Eigen vectors

Let T be a linear operator on a vector space V, a non-zero vector $v \in V$ is called an Eigen Vector of T if $\exists$ a scalar $\lambda \ni T(v) = \lambda v$.

The scalar λ is called the Eigen Value of the corresponding Eigen vector v.

Properties of Eigen values

(i) A square matrix A and its transpose A^T have the same eigen values

(ii) Sum of Eigen values of a square matrix A is equal to the sum of the elts on its main diagonal

(iii) sum of E.V of A = trace of A

(iv) Product of E.V of A = $|A|$

(v) If $\lambda_1, \lambda_2 \dots \lambda_n$ are E.V of A then the E.V of A^{-1} is $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots \frac{1}{\lambda_n}$

(vi) If $\lambda_1, \lambda_2, \dots \lambda_n$ are E.V of A then the E.V of CA, $C \neq 0$ is $C\lambda_1, C\lambda_2, \dots C\lambda_n$

(vii) If $\lambda_1, \lambda_2, \dots \lambda_n$ are E.V of A then the E.V of A^m is $\lambda_1^m, \lambda_2^m, \dots \lambda_n^m$

(viii) If $\lambda_1, \lambda_2, \dots \lambda_n$ are E.V of A then the E.V of $A \pm KI$ is $\lambda_1 \pm K, \lambda_2 \pm K, \dots \lambda_n \pm K$.

Properties of Eigen vector

(i) Eigen vector corresponding to an Eigen value λ is not unique.

(ii) Eigen vector corresponding to different Eigen values are Linearly independent.

Note: The vectors are Linearly independent if $|B| \neq 0$, ie B = matrix formed from the E. vectors written column-wise and hence form a Basis.

6

① Let $T: P_2(R) \to P_2(R)$ be a linear operator given $T(f(x)) = f(x) + (x+1)f'(x)$. Let $B = \{1, x, x^2\}$ be an ordered basis of $P_2(R)$ with $A = [T]_B$

Find (i) the matrix A

(ii) the Eigen values and Eigen vectors of T

(i) $T(1) = 1 + (x+1)(0) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$

$T(x) = x + (x+1)(1) = 1+2x = 1\cdot 1 + 2\cdot x + 0\cdot x^2$

$T(x^2) = x^2 + (x+1)2x = 3x^2 + 2x = 0\cdot 1 + 2\cdot x + 3\cdot x^2$

Coefficient Matrix $= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 2 & 3 \end{pmatrix}$

$[T]_B = A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

(ii) To find the Eigen values and Eigen vectors

Since A is a upper triangular matrix, its Eigen values are its principal diagonal.

$\therefore$ Eigen values $\lambda = 1, 2, 3$

NKT $(A - \lambda I)X = 0$

$$\begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

When $\lambda = 1$ $\quad 0x_1 + x_2 + 0x_3 = 0 \Rightarrow x_2 = 0$

$x_2 + 2x_3 = 0$

$2x_3 = 0 \Rightarrow x_3 = 0$

$\therefore x_1 = K, \; K \neq 0, \; K \in R$

Let $x_1 = 1$ $\quad \therefore X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

When $\lambda = 2$ $\quad -x_1 + x_2 = 0$ $\quad \therefore X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$2x_3 = 0$

$x_3 = 0$

$\cancel{+} x_1 = \cancel{+} x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{1}$

When $\lambda = 3$ $\quad -2x_1 + x_2 = 0$

$-x_2 + 2x_3 = 0$

$$\begin{matrix} 1 & 0 & -2 & 1 \\ -1 & 2 & 0 & -1 \end{matrix} \qquad \frac{x_1}{2} = \frac{x_2}{4} = \frac{x_3}{2}$$

$\therefore X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$\therefore$ Eigen values are $1, 2, 3$

Corresponding Eigen vectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Since E. vectors are elts of $P_2(R)$ they are polynomials 1, $1+x$, $1+2x+x^2$.

Practice problem

(4) Let $V = R^3$, T is a linear operator on V given by $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a+2b-2c \\ -4a-3b+2c \\ -c \end{pmatrix}$ and an ordered basis of R^3 is $B = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$

Find (i) $A = [T]_B$

(ii) eigen values and Eigen vectors of T

(iii) Whether the Eigen vectors form a basis of R^3

Diagonalization

Defn: A linear operator T on a finite dimensional vector Space V is called diagonalizable if $\exists$ an ordered basis B for V $\ni$ the matrix $[T]_B$ is a diagonal matrix.

Theorem: A L.T T on a finite dimensional Vector space V is diagonalizable iff $\exists$ an ordered basis $B = \{v_1, v_2, \ldots v_n\}$ consisting of Eigen vectors of T.

More over if T is diagonalizable, B is an ordered basis of Eigen vectors of T and $D = [T]_B$ then D is a diagonal matrix and D_{ii} are the E. value corresponding to v_i

Note: ① To diagonolise we have to find a basis consisting of independent eigen Vectors.

② We can diagonolise by both orthogonal and similarity transformation.

③ $D = P^{-1}AP$, where P is modal matrix. under Similar transformation

④ $D = N^T A N$, where N is normalized modal matrix. under orthogonal transformation

Problems:

① For the linear operator $T : P_2(R) \rightarrow P_2(R)$ defined as $T(f(x)) = x f'(x) + f(2)x + f(3)$. Find the matrix of T in an ordered basis B Such that matrix of $[T]_B$ is diagonalizable.

$$T(f(x)) = x f'(x) + f(2)x + f(3)$$

$$B = \{1, x, x^2\}$$

$f(x) = 1 \Rightarrow f(2) = 1, \quad f(3) = 1$

$f(x) = x \Rightarrow f(2) = 2, \quad f(3) = 3$

$f(x) = x^2 \Rightarrow f(2) = 4, \quad f(3) = 9$

$$T(1) = 0 + 1 \cdot x + 1 = x + 1 = 1 \cdot 1 + 1 \cdot x + 0 \cdot x^2$$

$$T(x) = x \cdot 1 + 2 \cdot x + 3 = 3x + 3 = 3 \cdot 1 + 3 \cdot x + 0 \cdot x^2$$

$$T(x^2) = x \cdot 2x + 4 \cdot x + 9 = 2x^2 + 4x + 9 = 9 \cdot 1 + 4 \cdot x + 2 \cdot x^2$$

$$[T]_B = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 3 & 0 \\ 9 & 4 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix} = A$$

The characteristic eqn is $|A - \lambda I| = 0$

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$$

$$S_1 = 1 + 3 + 2 = 6$$

$$S_2 = \begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 9 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 6 + 2 + 0 = 8$$

$$S_3 = 2 \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0.$$

$$\lambda^3 - 6\lambda^2 + 8\lambda = 0$$

$$\lambda(\lambda^2 - 6\lambda + 8) = 0 \qquad \lambda = 0, \quad (\lambda - 4)(\lambda - 2) = 0$$

$$\lambda = 2, 4$$

$\therefore$ The Eigen values are $0, 2, 4$

To find the Eigen vectors $(A - \lambda I)X = 0$

$$\begin{pmatrix} 1-\lambda & 3 & 9 \\ 1 & 3-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

When $\lambda = 0$

$$x_1 + 3x_2 + 9x_3 = 0$$
$$x_1 + 3x_2 + 4x_3 = 0$$
$$2x_3 = 0$$

$$\begin{array}{cccc} 3 & 9 & 1 & 3 \\ 3 & 4 & 1 & 3 \end{array}$$

$$X_1 = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \sim \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \qquad \frac{x_1}{12-27} = \frac{x_2}{9-4} = \frac{x_3}{3-3}$$

When $\lambda = 2$

$$-x_1 + 3x_2 + 9x_3 = 0$$
$$x_1 + x_2 + 4x_3 = 0$$

$$\begin{array}{cccc} 3 & 9 & -1 & 3 \\ 1 & 4 & 1 & 1 \end{array}$$

$$X_2 = \begin{pmatrix} 3 \\ 13 \\ -4 \end{pmatrix} \qquad \frac{x_1}{3} = \frac{x_2}{13} = \frac{x_3}{-4}$$

When $\lambda = 4$

$$-3x_1 + 3x_2 + 9x_3 = 0$$
$$x_1 - x_2 + 4x_3 = 0$$
$$-2x_3 = 0$$

$$\begin{array}{cccc} 3 & 9 & -3 & 3 \\ -1 & 4 & 1 & -1 \end{array}$$

$$X_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \frac{x_1}{12+9} = \frac{x_2}{9+12} = \frac{x_3}{3-3}$$

Since the Eigen values are different, the Eigen vectors are linearly independent and forms the basis.

$$M = \begin{bmatrix} -3 & 3 & 1 \\ 1 & 13 & 1 \\ 0 & -4 & 0 \end{bmatrix} \text{ and } M^{-1} \text{ exists}$$

To prove $D = M^{-1} A M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, it is

enough to prove $AM = MD$

$$AM = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -3 & 3 & 1 \\ 1 & 13 & 1 \\ 0 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 4 \\ 0 & 26 & 4 \\ 0 & -8 & 0 \end{bmatrix}$$

$$MD = \begin{bmatrix} -3 & 3 & 1 \\ 1 & 13 & 1 \\ 0 & -4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 4 \\ 0 & 26 & 4 \\ 0 & -8 & 0 \end{bmatrix}$$

$$AM = MD \Rightarrow M^{-1}AM = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Practice Problem :

① For the linear operator $T : P_2(R) \rightarrow P_2(R)$ defined as $T(f(x)) = f(x) + xf'(x) + f''(x)$. find the eigen values of T in an ordered basis B for $P_2(R)$ such that the matrix of the given transformation w.r.to the new resultant basis B is a diagonal matrix.

② Find the eigen values of T and an ordered basis β for $T \ni [T]_\beta$ is the diagonal matrix and the linear transformation is given by $T[f(x)] = xf'(x) + f''(x) - f(2)$ $V = P_3(R)$.

Problems :

① Diagonalise the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$. by orthogonal transformation

(ii)

A is a symmetric matrix.

The characteristic Eqn of A is $|A - \lambda I| = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = 9$

$$S_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 8 + 8 + 8 = 24$$

$$S_3 = |A| = 3(8) - 1(4) + 1(-4)$$
$$= 16$$

$$\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$$

$\lambda = 1, 4, 4$ are the Eigen values of A

To find Eigen vectors $(A - \lambda I)X = 0$

$$\begin{pmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

When $\lambda = 1$

$$2x_1 + x_2 + x_3 = 0$$
$$x_1 + 2x_2 - x_3 = 0$$
$$x_1 - x_2 + 2x_3 = 0$$

$$\begin{matrix} 1 & 1 & 2 & 1 \\ 2 & -1 & 1 & 2 \end{matrix}$$

$$\frac{x_1}{-3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

When $\lambda = 4$

$$-x_1 + x_2 + x_3 = 0$$
$$x_1 - x_2 - x_3 = 0$$
$$x_1 - x_2 - x_3 = 0$$

$x_1 - x_2 - x_3 = 0$

Let $x_3 = 0$

$x_1 = x_2$

$$X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $X_3^T \cdot X_1 = 0 \Rightarrow -a+b+c = 0$

$X_3^T \cdot X_2 = 0 \Rightarrow a+b = 0$

$$\begin{matrix} 1 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{matrix}$$

$$\frac{a}{-1} = \frac{b}{1} = \frac{c}{-2}$$

$$X_3 = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

The vectors are pair-wise orthogonal.

Normalised Modal matrix $N = \begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}$

$$D = N^T A N$$

$$AN = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & +1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} = \begin{pmatrix} -1/\sqrt{3} & 4/\sqrt{2} & -4/\sqrt{6} \\ 1/\sqrt{3} & 4/\sqrt{2} & 4/\sqrt{6} \\ 1/\sqrt{3} & 0 & -8/\sqrt{6} \end{pmatrix}$$

$$D = N^T AN = \begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} -1/\sqrt{3} & 4/\sqrt{2} & -4/\sqrt{6} \\ 1/\sqrt{3} & 4/\sqrt{2} & +4/\sqrt{6} \\ 1/\sqrt{3} & 0 & -8/\sqrt{6} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Practice problem

① Diagonalize by orthogonal transformation. $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$

② Diagonalise $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ and find A^3

Hint:

$$D = N^T AN \Rightarrow A = NDN^T \Rightarrow A^3 = ND^3N^T$$

① Let $V = P_1(\mathbb{R})$, $T(a+bx) = (6a-6b) + (12a-11b)x$ and $\beta = \{3+4x, 2+3x\}$, T is a linear operator on V and β is ordered basis. Compute $[T]_\beta$ and determine whether the basis consisting of Eigen vectors of T

$$T(3+4x) = [6(3)-6(4)] + [12(3)-11(4)]x = -6-8x.$$
$$T(2+3x) = [6(2)-6(3)] + [12(2)-11(3)]x = -6-9x$$

$$-6-8x = -2[3+4x] + 0\cdot[2+3x]$$
$$-6-9x = -3[2+3x] + 0[3+4x]$$

$$[T]_\beta = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}^T = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$

Since the matrix is a diagonal matrix,

$\beta = \{3+4x, 2+3x\}$ is a basis containing the Eigen vectors of T

Practice problems

(1)

Let $V = R_2$, $T(a,b) = (10a-6b, 17a-10b)$

$\beta = \{(1,2), (2,3)\}$ is ordered basis. T is a linear operator on V and determine whether β is a basis consisting of Eigen vectors of $[T]_\beta$.

(2) Find the Eigen values of T and an ordered basis β for T $\ni$ $[T]_\beta$ is diagonal, basis consist of Eigen vectors of T given by $T[f(x)] = xf'(x) + f''(x) - f(2)$ under std basis of $P_3(R)$

Range and Null space

Defn Let V and W be vector spaces over F and $T: V \to W$ be a linear transformation.

The Range of T is $R(T) = \{T(v) / \forall v \in V\}$
= set of all images.

The null space or Kernel of T is $N(T) = \{v \in V / T(v) = 0\}$

Example Let $V: R^3 \to R^2$ be a L.T defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$. Find $N(T)$ & $R(T)$

2
(9)

$$N(T) = \{v \in V / T(v) = 0\}$$
$$= \{(a_1, a_2, a_3) \in R^3 / T(a_1, a_2, a_3) = 0\}$$
$$= \{(a_1, a_2, a_3) \in R^3 / (a_1 - a_2, 2a_3) = 0\}$$
$$= \{(a_1, a_2, a_3) \in R^3 / a_1 = a_2, a_3 = 0\}$$

$N(T) = \{(a_1, a_1, 0),\ a_1 \in R\}$

$$R(T) = \{T(v) \mid v \in V\}$$
$$= \{T(a_1, a_2, a_3) \mid (a_1, a_2, a_3) \in R^3\}$$
$$= \{(a_1 - a_2, 2a_3) \mid (a_1, a_2, a_3) \in R^3\}$$

$R(T) = R^2$

Theorem : Let V and W be the vector spaces and $T: V \to W$ be a linear transformation then

(i) $R(T)$ is a subspace of W

(ii) $N(T)$ is a subspace of V

Proof :

(i) $R(T) = \{T(v) : v \in V\}$

Since $T(0) = 0 \in R(T)$, $R(T)$ is non-empty

Let $x, y \in R(T)$ then $\exists$ vectors $u, v \in V$

$\ni T(u) = x$ & $T(v) = y$

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$$
$$= \alpha x + \beta y, \quad \alpha, \beta \in F$$
$$\in R(T)$$

$\Rightarrow R(T)$ is a subspace of W

(ii) $N(T) = \{v \mid T(v) = 0\}$

Since $T(0) = 0 \in N(T)$, $N(T)$ is non empty

Let $x, y \in N(T)$ then $T(x) = 0$, $T(y) = 0$, $\alpha, \beta \in F$

Then $T(\alpha x + \beta y) = \alpha T(x) + \beta (T(y))$
$$= 0 + 0$$
$$= 0 \in N(T)$$

$\Rightarrow N(T)$ is a subspace of V

Defn: Rank and Nullity

Let V, W are vector spaces over F and $T: V \to W$ be a linear transformation. If $R(T)$ and $N(T)$ are finite dimensional, then we define $\text{rank}(T) = \dim[R(T)]$ and $\text{nullity}(T) = \dim[N(T)]$

Thm: Let V and W be vector spaces over F and $T: V \to W$ be a linear transformation. If $S = \{v_1, v_2, \dots v_n\}$ is a basis for V. Then $R(T) = L(\{T(v_1), T(v_2) \dots T(v_n)\})$

$= \text{Span}\{T(v_1), T(v_2) \dots T(v_n)\}$

Proof: Given $T: V \to W$ be a L.T

$S = \{v_1, v_2, \dots v_n\}$ is a basis of V

Then $T(v_1), T(v_2) \dots T(v_n) \in R(T)$

We know $R(T)$ is a subspace of W

$$L(\{T(v_1), T(v_2) \dots T(v_n)\}) \subseteq R(T) \quad \text{--- } (1)$$

Now let $w \in R(T)$ Then $\exists$ $v \in V \ni T(v) = w$

and since $v_1, v_2, \dots v_n$ is a basis of V, $\alpha_i \in F$

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

$$w = T(v) = \alpha_1 T(v_1) + \alpha_2 T(v_2) + \dots + \alpha_n T(v_n)$$

$$w \in L(\{T(v_1), T(v_2) \dots T(v_n)\})$$

$$\Rightarrow R(T) \subseteq L(\{T(v_1), T(v_2) \dots T(v_n)\}) \text{ --- } (2)$$

from (1) & (2) $\quad R(T) = \text{span}\{T(v_1), \dots T(v_n)\}$

Dimension Theorem

Let V and W be vector spaces over F and $T: V \to W$ be a linear transformation. If V is finite dimensional then $\text{rank}(T) + \text{nullity}(T) = \dim V$

3

Proof: Given V is a finite dimensional over F and let $\dim V = n$

Since $N(T)$ is a subspace of V, $N(T)$ has a basis (finite)

$S = \{v_1, v_2, \dots v_k\}$.

$\dim[N(T)] = k \Rightarrow \text{Nullity}(T) = k$

$\therefore \{v_1, v_2, \dots v_k\}$ is a linearly independent set in V

This can be expanded to a basis $\beta = \{v_1, v_2 \dots v_n\}$ for V

Now we claim $S_1 = \{T(v_{k+1}), T(v_{k+2}) \dots T(v_n)\}$ is a basis of $R(T)$.

$\because \beta = \{v_1, v_2, \dots v_n\}$ is a basis of V

$$R(T) = \text{Span}\{(T(\beta)\}$$
$$= \text{Span}\{T(v_1), T(v_2) \dots T(v_n)\}$$
$$= \text{Span}\{T(v_{k+1}), T(v_{k+2}) \dots T(v_n)\}$$

$R(T) = \text{Span}\, S_1$ ——— ①

Now, we prove that S_1 is L.I

To prove this, Let $b_{k+1}T(v_{k+1}) + b_{k+2}T(v_{k+2}) + \dots + b_n(T(v_n)) = 0$

$$T(b_{k+1}v_{k+1} + b_{k+2}v_{k+2} + \dots b_n v_n) = 0$$

$$\Rightarrow b_{k+1}v_{k+1} + b_{k+2}v_{k+2} + \dots b_n v_n \in N(T)$$

Since $v_1, v_2, \dots v_k$ is a basis for $N(T)$

$b_{k+1}v_{k+1} + b_{k+2}v_{k+2} + \dots + b_n v_n = b_1 v_1 + b_2 v_2 + \dots + b_k v_k$

$(-b_1)v_1 + (-b_2)v_2 + \dots + (-b_k)v_k + b_{k+1}v_{k+1} + \dots + b_n v_n = 0$

$\Rightarrow b_1 = b_2 = \dots = b_n = 0$ [$\because \{v_1, v_2, \dots v_n\}$ are linear indt]

$\Rightarrow S_1$ is L.I ——— ②

from ① & ② S_1 is a basis for $R(T)$

Now $\dim[R(T)] = n-k$

$$\dim(R(T)) = \dim V - \dim(N(T))$$

$$\dim R(T) + \dim N(T) = \dim V$$

$$\dim V = \text{Rank } T + \text{nullity } T$$

Problems:

① If $T : R^2 \to R^3$ be defined by $T(x,y) = (2x-y, 3x+4y, x)$ Compute the matrix T in the std. basis of R^2 and R^3. Find $N(T)$ and $R(T)$. Is T one-one and onto ?

Given $T(x,y) = (2x-y, 3x+4y, x)$

std basis of R^2 is $\{\underset{e_1}{(1,0)}, \underset{e_2}{(0,1)}\}$

$T(e_1) = T(1,0) = (2,3,1)$

$T(e_2) = T(0,1) = (-1,4,0)$

$$\text{Matrix } T = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 4 & 0 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}$$

$N(T) = \{ x \in R^2 / T(x) = 0 \}$

$= \{(x,y) \in R^2 / T(x,y) = 0\}$

$= \{(x,y) \in R^2 / (2x-y, 3x+4y, x) = 0\}$

$2x-y=0 \quad 3x+4y=0 \quad x=0$

Put $x=0 \Rightarrow -y=0$

$y=0 \qquad \Rightarrow x=0, y=0$

$N(T) = \{(0,0)\} = \{0\}$

Hence T is one to one

$R(T) = \{ T(v) : v \in R^2 \}$

$= \text{Span } T(\beta)$

$= \text{span}\{T(1,0), T(0,1)\}$

$= \text{span}\{(2,3,1), (-1,4,0)\}$

We check $(2,3,1)$ and $(-1,4,0)$ are L.I

$$\alpha_1(2,3,1)+\alpha_2(-1,4,0)=(0,0,0)$$

$$2\alpha_1-\alpha_2=0 \qquad 3\alpha_1+4\alpha_2=0 \qquad \alpha_1=0$$

put $\alpha_1=0$, $-\alpha_2=0 \Rightarrow \alpha_2=0$

$\therefore \alpha_1=0,\ \alpha_2=0$

$\Rightarrow (2,3,1)$ and $(-1,4,0)$ are L.I

$\therefore$ $\dim R(T)=2$

$\dim R^3=3$

$\because$ $\dim(R(T)) \neq \dim R^3$, T is not onto.

4

② Let $T: P_2(R) \to P_3(R)$ be defined by $T(f(x)) = 2f'(x) + \int_0^x 3f(t)\,dt$. Find bases for N(T) and R(T) and hence verify the dimension theorem. Is T 1−1 and onto. Justify.

$$T: P_2(R) \to P_3(R)$$

std β of $P_2(R) = \{1, x, x^2\}$

$R(T) = \text{span}\{T(1), T(x), T(x^2)\}$

$$T(1) = 2(0) + 3\int_0^x 1\,dt = 3\,t\Big]_0^x = 3x$$

$$T(x) = 2(1) + 3\int_0^x \cancel{3}x\,d\cancel{x} = 2+\frac{3x^2}{2}\Big]_0^x = 2+\frac{3x^2}{2}$$

$$T(x^2) = 2(2x) + 3\int_0^x x^2\,dx = 4x + 3\frac{x^3}{3} = 4x+x^3$$

$$R(T) = L\left(\left\{3x,\ 2+\frac{3x^2}{2},\ 4x+x^3\right\}\right)$$

We check for L.I

$$\alpha_1(3x)+\alpha_2\left(2+\frac{3x^2}{2}\right)+\alpha_3(4x+x^3)=0$$

$$2\alpha_2 + (3\alpha_1 + 4\alpha_3)x + \frac{3}{2}\alpha_2 x^2 + \alpha_3 x^3 = 0$$

Equating like coefficients, we get

$$\alpha_2 = 0, \quad \alpha_3 = 0, \quad 3\alpha_1 + 4\alpha_3 = 0 \Rightarrow \alpha_1 = 0$$

Hence $3x, \ 2 + \frac{3}{2}x^2, \ 4x + x^3$ are L.I

$\dim R(T) = 3 \quad \Rightarrow \text{rank } T = 3$

$\dim P_3(\mathbb{R}) = 4$

$\therefore$ T is not onto.

We know $N(T) = \{ f(x) \in P_2(\mathbb{R}) : T(f(x)) = 0 \}$

$$T(f(x)) = 0 \Rightarrow 2f'(x) + 3\int_0^x f(t)\,dt = 0 \quad \text{--- (1)}$$

Diff w.r.to x, $\quad 2f''(x) + 3f(x) = 0$

$$(2D^2 + 3) f(x) = 0$$

$$2m^2 + 3 = 0$$

$$m^2 = -3/2 \qquad m = \pm i\sqrt{3/2}$$

$$f(x) = A\cos\sqrt{3/2}\,x + B\sin\sqrt{3/2} \notin P_2(\mathbb{R})$$

So this is not a solution

Clearly, only $f(x) = 0$ satisfies (1)

$\therefore \quad N(T) = \{0\} \quad \text{nullity}(T) = 0 \Rightarrow T$ is 1-1

$$\text{Nullity } T + \text{rank}(T) = 0 + 3 = 3 = \dim P_2(\mathbb{R})$$

Hence dimension theorem is verified.

(3) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined as $T(x,y,z) = (2x - y, 3z)$

Verify whether T is linear or not. Find N(T) and R(T) and hence verify dimension theorem.

5

Let $u = (a_1, b_1, c_1)$ $v = (a_2, b_2, c_2)$ $\in R^3$

$$\alpha u + \beta v = \alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2$$

$$T(u) = (2a_1 - b_1, 3c_1) \quad T(v) = (2a_2 - b_2, 3c_2)$$

To show: T is linear

$$T(\alpha u + \beta v) = T(\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2)$$

$$= (2\alpha a_1 + 2\beta a_2 - \alpha b_1 - \beta b_2, 3\alpha c_1 + 3\beta c_2)$$

$$= ((2\alpha a_1 - \alpha b_1, 3\alpha c_1) + (2\beta a_2 - \beta b_2, 3\beta c_2))$$

$$= \alpha(2a_1 - b_1, 3c_1) + \beta(2a_2 - b_2, 3c_2)$$

$$= \alpha T(u) + \beta T(v)$$

$\therefore$ T is linear.

$$R(T) = \text{span } T(\beta)$$

$$= \text{Span } \{T(1,0,0), T(0,1,0), T(0,0,1)\}$$

$$= \text{Span } \{(2,0), (-1,0), (0,3)\}$$

$$= \text{Span } \{(-1,0), (0,3)\}$$

To check $(-1,0), (0,3)$ are L.I

$$\alpha_1(-1,0) + \alpha_2(0,3) = 0$$

$$-\alpha_1 = 0 \qquad 3\alpha_2 = 0$$

$$\alpha_1 = 0 \qquad \alpha_2 = 0$$

$\Rightarrow$ $(-1,0), (0,3)$ are L.I

$\therefore$ $\dim R(T) = 2 = \text{Rank } T$

$$N(T) = \{T(x,y,z) = 0 : (x,y,z) \in R^3\}$$

$$= \{2x - y, 3z = 0 : (x,y,z) \in R^3\}$$

$2x = y$, $3z = 0 \Rightarrow z = 0$ $\quad \therefore \{(\frac{y}{2}, y, 0), y \in R\}$

$\therefore \dim N(T) = 1 = \text{Nullity } T$

Now, $\text{Rank } T + \text{Nullity } T = 2+1 = 3 = \dim R^3$

Hence dimension theorem is verified

(4) A linear transformation $T: R^3 \to R^2$ over R defined on the basis $T(\bar{i}) = (0,0)$ $T(\bar{j}) = (1,1)$ $T(\bar{k}) = (1,-1)$ Compute $T(4\bar{i} - \bar{j} + \bar{k})$. determine nullity T & Rank T

Let $\vec{a} = x\bar{i} + y\bar{j} + z\bar{k}$

$$T(\vec{a}) = T(x\bar{i} + y\bar{j} + z\bar{k}) = x\,T(\bar{i}) + y\,T(\bar{j}) + z\,T(\bar{k})$$

$$= x(0,0) + y(1,1) + z(1,-1)$$

$$T(x,y,z) = (y+z,\ y,-z)$$

$$T(4\bar{i} - \bar{j} + \bar{k}) = (0,-2)$$

$$N(T) = \{T(x,y,z) = 0 : (x,y,z) \in R^3\}$$

$$= \{(y+z,\ y-z) = 0 : (x,y,z) \in R^3\}$$

$$y = -z \qquad y = z \quad \Rightarrow \quad y = 0 \ \Rightarrow \ z = 0$$

$$N(T) = \{(x,0,0) : x \in R\}$$

$\text{Nullity } T = 1$, $\quad \text{Dim } R^3 = 3$

By Dimension thm, $\quad \text{Rank } T + \text{Nullity } T = \text{Dim } R^3$

$$\text{Rank } T = 3 - 1$$
$$= 2$$

<u>Practice</u> <u>Problem</u> :

(1) Suppose $T: R^2 \to R^2$ is linear, $T(1,0) = (1,4)$, $T(1,1) = (2,5)$ What is $T(2,3)$ and $T(-2,3)$? Is T 1-1 ?

(2) $T: R^2 \longrightarrow P_2(R)$ be a linear transformation such that $T(1,1) = 2 - 3x + x^2$. $T(2,3) = 1 - x^2$. Determine T and find $T(-1,2)$, $T(-5,2)$. Also find the rank and nullity of T.

(3) Let $T: R^2 \rightarrow R^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$ verify dimension Theorem.