Introduction to Elasticity

1) Introduction

In this chapter we study the mechanical properties of materials (especially solids) exhibited by them when a deforming force is applied. When a force is applied on a solid, which is not allowed to move, it deforms the shape of the solid. This applied force is called the deforming force and when the deforming force is removed the solid may or may not regain its original shape. A material having the ability to regain its original shape and size upon the removal of the deforming force is called an elastic material and the property is known as elasticity. The mechanism of the elastic nature of a material is understood by considering the effect of deforming force on the arrangement of atoms of the material and the interatomic forces between them. The interatomic force which are predominately electrostatic in nature are affected by the deforming force and they become the action-reaction pair of the deforming force and oppose the deformation. When the deforming force is removed, the reaction forces which were set up inside the material by the deforming force tries to restore the original state of equilibrium, that is the original shape and size of the material. These reactionary forces set inside the body by the deforming force is called restoring force. Certain materials are more elastic in nature and some are not so. No material in the world is perfectly elastic and the best elastic material we get is quartz. A material lacking the ability to regain its original shape and size upon the removal of the deforming force is called a plastic material and the property is known as plasticity. No material in the world is perfectly plastic and the best plastic material we get is putty. The knowledge of elastic properties of materials are essential to choose suitable materials for a wide variety of engineering applications.

1.1) Stress and Strain

We know from our experience that the deformed objects regain their shape depending on the magnitude of the deforming force and the physical dimensions of the object. To quantify elasticity, we define two quantities namely, stress and strain, which are defined in terms of the physical dimensions and the force acting. Stress is defined as the restoring force per unit area which brings back the body to its original state from the deformed state. Unit of Stress is N/m^2. Depending on the direction of the applied force acting on the object, the stress can be classified into three types namely,

1. Normal Stress/longitudinal stress

When the force is applied perpendicular to the surface of the body, then the stress applied is normal Stress. Longitudinal stress is defined as the force per unit area,

$$longitudinal\ stress = \frac{Restoring\ Force\ acting\ along\ the\ length}{Area\ of\ cross\ section}\ \mathrm{Nm^{-2}}$$

$$= \frac{F}{A}$$

2. Tangential Stress/Shearing stress

When the force is applied along the surface of the body, then the stress applied is called as tangential stress. The tangential stress is also called as Shearing Stress.

$$Tangential\ stress\ \tau = \frac{Restoring\ force\ acting\ tangential\ to\ the\ surface}{Area\ of\ cross\ section}\ Nm^{-2}$$

3. **Volumetric stress/Compressive stress**

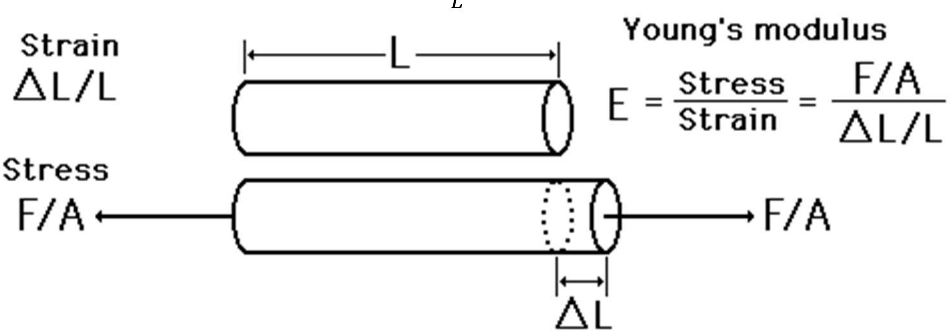
When the force is applied all around the body, then the stress applied is called as volumetric stress. The volumetric stress is also called as Compressive Stress.

Strain is defined as the change in dimension (fractional deformation) produced by the external force of the body. It can also be defined as the ratio of the change in dimension to the original dimension. Strain is a unitless physical quantity

Different types of stress cause different types of strains, namely,

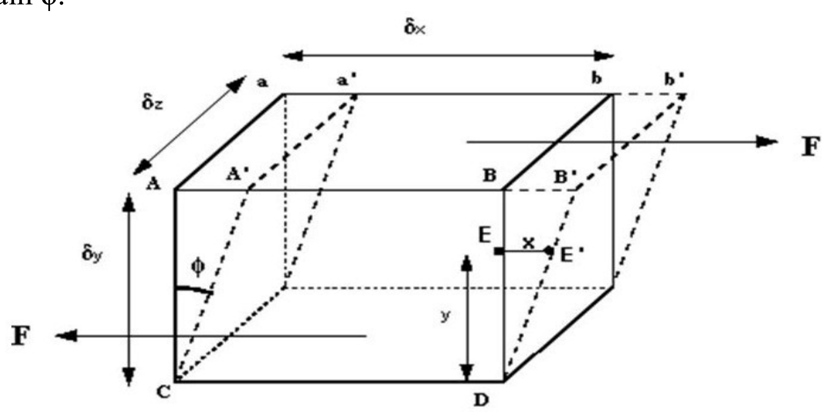
1. **Longitudinal or Tensile strain**

It is defined as the ratio between the changes in length to the original length without any change in length to the original length without any change in its shape, after the removal of the external forces. If the original length of the body is 'L' and the change in length due to applied force is 'ΔL', longitudinal strain $\varepsilon = \frac{\Delta L}{L}$.



2. **Shearing strain or Tangential strain**

It is defined as the angular deformation produced on the body due to the application of external tangential forces on it. Let ABCD be a body with its CD fixed as shown in the figure. A tangential force is applied on the upper surface AB of the body. Therefore, the body shears to angle and it goes to a new position. This angle measured in radians is called Shearing Strain ϕ.

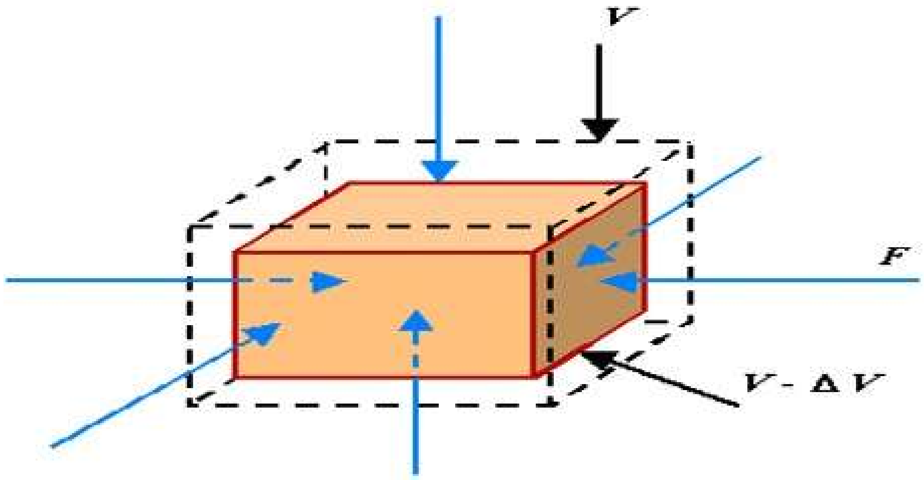


Shear stress $\tau = \frac{F}{A}$ Shear strain $\phi = \frac{x}{y}$

3. **Volumetric Strain**

It is defined as the ratio between the change in volume v to the original volume V.

Volumetric strain=$\frac{\Delta V}{V}$.



1.2) **Relationship between Stress and Strain**

Hooke's law

Robert Hooke proposed a relation between Stress and Strain called Hooke's law. According to this law, "Stress is directly proportional to the Strain produced, within the elastic limit"

$$Stress \propto Strain,$$
$$Stress = E \times Strain,$$

where E is the modulus of elasticity and has a unit Nm^{-2}

The modulus of elasticity is defined as the ratio of Stress and Strain,

$$E = \frac{Stress}{Strain}\ Nm^{-2}.$$

Depending on the kind of type of stress and strain, we have three moduli of elasticity, namely, Young's modulus of elasticity, Rigidity modulus of elasticity and Bulk modulus of elasticity.

1.2.1) Young's modulus of elasticity

The Young's modulus (Y) is defined as the ratio between the longitudinal stress to the longitudinal strain,

$$Young's\ modulus\ Y = \frac{longitudinal\ stress}{longitudinal\ strain}\ Nm^{-2}$$

Substituting the expressions for the longitudinal stress and longitudinal strain, we get

$$Young's\ modulus\ Y = \frac{F/A}{\Delta L/L}$$

$$Y = \frac{FL}{A\Delta\mathrm{L}} \; Nm^{-2}$$

1.2.2) Rigidity modulus of elasticity

The Rigidity modulus (η) is defined as the ratio between the longitudinal stress to the longitudinal strain,

$$Rigidity\ modulus\ \eta = \frac{tangential\ stress}{tangential\ strain} \; Nm^{-2}$$

Substituting the expressions for the tangential stress and tangential strain, we get

$$Rigidity\ modulus\ \eta = \frac{F/A}{\phi}$$

$$\eta = \frac{F}{A\phi} \; Nm^{-2}$$

1.2.3) Bulk modulus of elasticity

The Bulk modulus (K) is defined as the ratio between the longitudinal stress to the longitudinal strain,

$$Bulk\ modulus\ \eta = \frac{compressive\ stress}{compressive\ strain} \; Nm^{-2}$$

Substituting the expressions for the compressive stress and compressive strain, we get

$$Bulk\ modulus\ K = \frac{F/A}{\Delta V/V}$$

$$K = \frac{PV}{\Delta V} \; Nm^{-2}$$

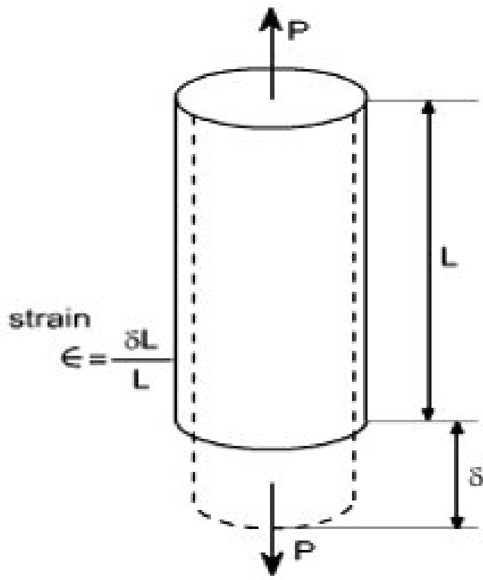
where P is the pressure.

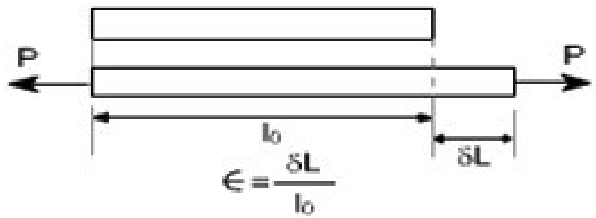
Stress-Strain curve

1.3) Stress-Strain Curve

The relationship between the stress and strain that a particular material displays is known as that particular material's stress–strain curve. It is unique for each material and is found by recording the amount of deformation (strain) at distinct intervals of a variety of loadings (stress). These curves reveal many of the properties of a material (including data to establish the Modulus of Elasticity, E).

Consider a wire of original cross-sectional area A being subjected to equal and opposite forces F pulling at the ends so the wire is under tension. The material is experiencing a stress defined to be the ratio of the force to the cross-sectional area of the wire, as well as an axial elongation:

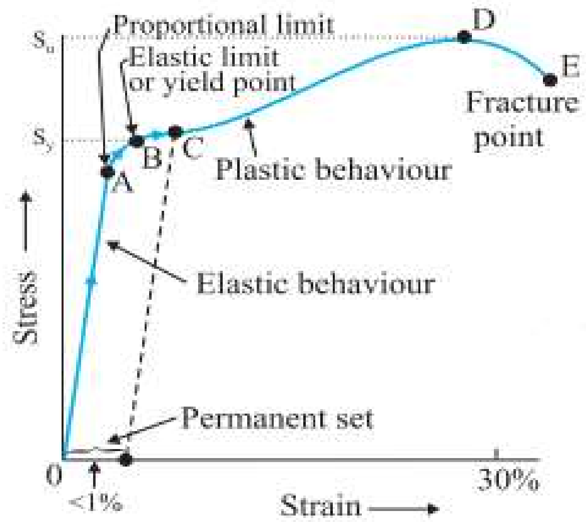




As the force is increased, the stress increases and thereby increasing the strain. The values of the stress and strain are plotted as a graph and the graph is a characteristic to the material of the wire and provides a great amount of information about the material such as its ductility, plasticity or brittleness. It is a convention to take the stress in the y-axis and the strain in the x-axis while plotting this stress-strain curve.

- **(i)** **Linear region:** The first stage OA is the linear elastic region. The stress is proportional to the strain, that is, obeys the general Hooke's law, and the slope is Young's modulus. In this region, the material undergoes only elastic deformation. The point A is called the proportional limit
- **(ii)** **Elastic limit:** The point B is the elastic limit and this is the maximum stress which the material can withstand without undergoing a permanent deformation or without losing its elastic property. In the region AB the stress and strain are proportional but they don't have a simple linear relationship.
- (iii) **Yield point:** The point C is called the yield point. Beyond the elastic limit plastic deformation occurs and strains are not totally recoverable. There will be thus permanent deformation or permanent set when load is removed. The stress at the yield point is called the yield strength.
- (iv) **Ultimate strength:** The point D is called the ultimate strength of the material and this is the maximum stress the wire can withstand before breaking. In the region CD, the material undergoes plastic deformation and even for a small increase in the stress, a huge strain is produced.
- (v) **Fracture point:** Beyond the point D, the necking (thinning of wire locally at different places in the wire) begins to appear and finally the wire breaks at the point E. The graph shown here is of a ductile material and we can find that it has a large plastic region BD. For a brittle material, the point D and C will be very close and the

material reaches the fracture point without undergoing the plastic deformation.



(vi) **Toughness:** The toughness of a material is its ability to absorb energy without getting broke or ruptured permanently. The area under the stress-strain curve represents the toughness of the material.

All engineering applications are designed to operate well below the yield strength of the material.

1.4) Factors affecting elasticity

It is found that bodies lose their elastic limit, due to elastic fatigue. Therefore, the manufacture should choose the material in such a way that it should regain its elastic property even when it is subjected to large number of cycles of stress.

For example, substances like quartz, phosphor, bronze etc. May be employed in manufacturing of galvanometers, electrometers etc., after knowing their elastic properties.

Apart from elastic fatigue some material will have change in their elastic property because of the following factors.

a) Effect of stress

b) Effect of annealing

c) Change in temperature

d) Presence of impurities

e) Due to the nature of crystals

a) **Effect of stress:** When a material is subjected to a constant stress (lesser than the elastic limit) for a prolonged time, the material loses its elasticity. For example, a stretched rubber band after a few days is permanently deformed even when the stretching is within the elastic limit.

b) **Effect of Annealing:** Annealing is a process by which the material is heated to a very high temperature and then it is slowly cooled. Usually this process is adopted for the material to increase the softness and ductility in the material. But if annealing is made to a material it results in the formation of large crystal grains, which ultimately reduces the elastic property of the material.

c) **Effect of temperature:** The elastic property of the materials changes with the temperature. Normally the elasticity increases with the decrease in temperature and vice-versa.

Examples

1. The elastic property of lead increases when the temperature is decreased.

2. The carbon filament becomes plastic at higher temperatures.

d) **Effect of impurities:** The addition of impurities produces variation in the elastic property of the materials. The increase and decrease of elasticity depend upon the type of impurity added to it.

Examples:

1. When potassium is added to gold, the elastic property of gold increases.

2. When carbon is added to molten iron, the elastic property of iron decreases provided the carbon content should be more than 1% in iron.

e) **Effect of nature of crystals:** The elasticity also depends upon the types of the crystals, whether it is a single crystal or poly crystals. For a single crystal the elasticity is more and for a poly crystal the elasticity is less.

Beams and Bending of Beams

1.7) Beams

A beam is defined as a rod or bar. Circular or rectangular of uniform cross section whose length is very much greater than its other dimensions, such as breadth and thickness. It is commonly used in the construction of bridges to support roofs of the buildings etc. Since the length of the beam is much greater than its other dimensions the shearing stresses are very small.

Assumptions:

While studying about the bending of beams, the following assumptions have to be made.

1. The length of the beam should be large compared to other dimensions.

2. The load(forces) applied should be large compared to the weight of the beam

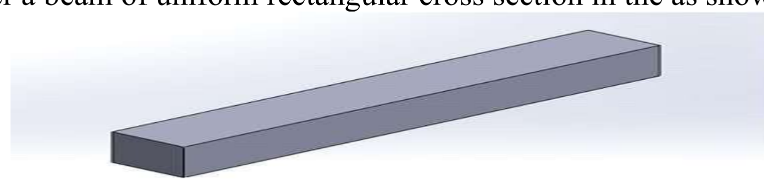
3. The cross section of the beam remains constant and hence the geometrical moment of inertia also remains constant

4. The shearing stresses are negligible

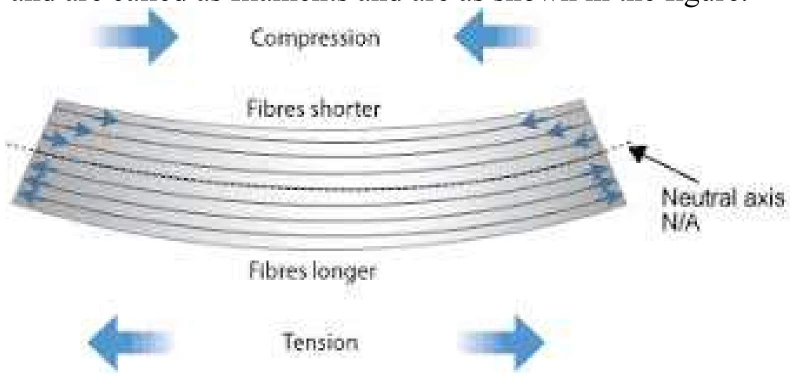
5. The curvature of the beam is very small

1.7.1) Bending of beams

Let us consider a beam of uniform rectangular cross section in the as shown figure.



A beam may be assumed to consist of a number of parallel longitudinal fibers placed one over the other and are called as filaments and are as shown in the figure.



Consider the beam to be bent into an arc of a circle by the application of an external couple. We know the beam consist of many filaments. Let us consider the filament represented by the dotted lines in the middle of the beam. It is found that the filaments(layers) lying above this filament gets compressed, while the filaments lying below it gets elongated. Therefore, the filaments i.e. **the layer which remains unaltered is taken as the reference axis called neutral axis and the plane is called neutral plane.** Further, the deformation of any filaments can be measured with reference to the neutral axis. About the neutral axis, one can find pairs of filaments which experience equal amount of compression and tension. Such a pair forms a couple and has a moment of couple. One can find many such pairs and each of them has a moment. The resultant of all these moments of all the internal couples are called the internal bending moment. In equilibrium condition, internal bending moment is equal to the external bending moment.

1.8) Internal Bending Moment of Beams

Let us consider a beam under the action of deforming forces. The beam bends into a circular arc as shown in the figure. Let AB be the neutral axis of the beam. Here the filaments above AB are elongated and the filaments below AB are compressed. The filament AB remains unchanged.

P and Q are two points on the neutral axis AB. R is the radius of curvature of the neutral axis and θ is the angle subtended by bent beam at its centre of curvature C.

$$\text{i.e. } \angle PCQ = \theta$$

Consider two corresponding points P_1 and Q_1 on a parallel layer at a distance 'x' from the neutral axis. From the figure,

$$PQ = R \times \theta \tag{1}$$

Corresponding length on the parallel layer upon bending

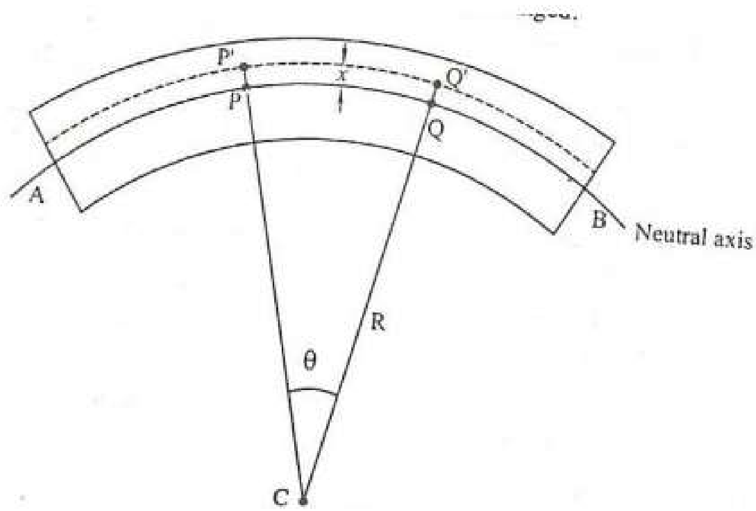
$$P_1Q_1 = (R + x)\theta$$

Increase in length of P_1Q_1 due to bending = P_1Q_1-PQ

$$= (R + x)\theta - R\theta$$

$$= x\theta \tag{2}$$

Before bending P_1Q_1=PQ



$$Longitudinal\ strain\ produced = \frac{Increase\ in\ length}{Original\ length}$$

$$Longitudinal\ strain = \frac{x\theta}{R\theta} = \frac{x}{R}$$

If Y is the Young's modulus of the material,

$$Y = \frac{Longitudinal\ stress}{Longitudinal\ strain}$$

$$Longitudinal\ stress = Y \times Longitudinal\ strain$$

$$Longitudinal\ stress = Y \times \frac{x}{R}$$

If δA is the area of cross-section of the filament, then

the force acting on the area δA, $F = stress \times area$

$$F = Y \times \frac{x}{R}\delta A$$

Moment of this force about the neutral axis AB $= Y \times \frac{x}{R}\delta A \times x = \frac{Y}{R}\delta A x^2$

The sum of the moments of forces acting on all the filaments $= \sum \frac{Y}{R}\delta A x^2$

$$= \frac{Y}{R}\sum \delta A x^2$$

Let $I = \sum \delta A x^2$, and I is called the geometrical moment of inertia of the cross-section of the beam.

The sum of moments of forces acting on all the filaments is the internal bending moment and therefore,

The internal bending moment $= \frac{YI}{R}$

The geometrical moment of inertia I depends upon the shape of the beam and for a beam having a rectangular cross section

$$I = \frac{bd^3}{12}$$

where b is the breadth of the beam and d is the thickness of the beam.

For a beam with a circular cross section

$$I = \frac{\pi r^4}{4}$$

where r is the radius of the beam.

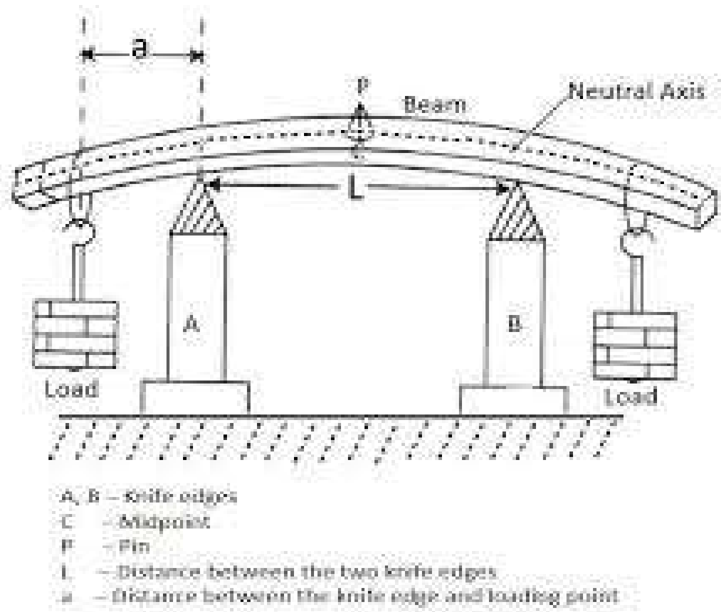
1.9) Young's modulus by uniform bending

Let us consider a beam of negligible mass, supported symmetrically on the two knife edges A and B as shown in figure.

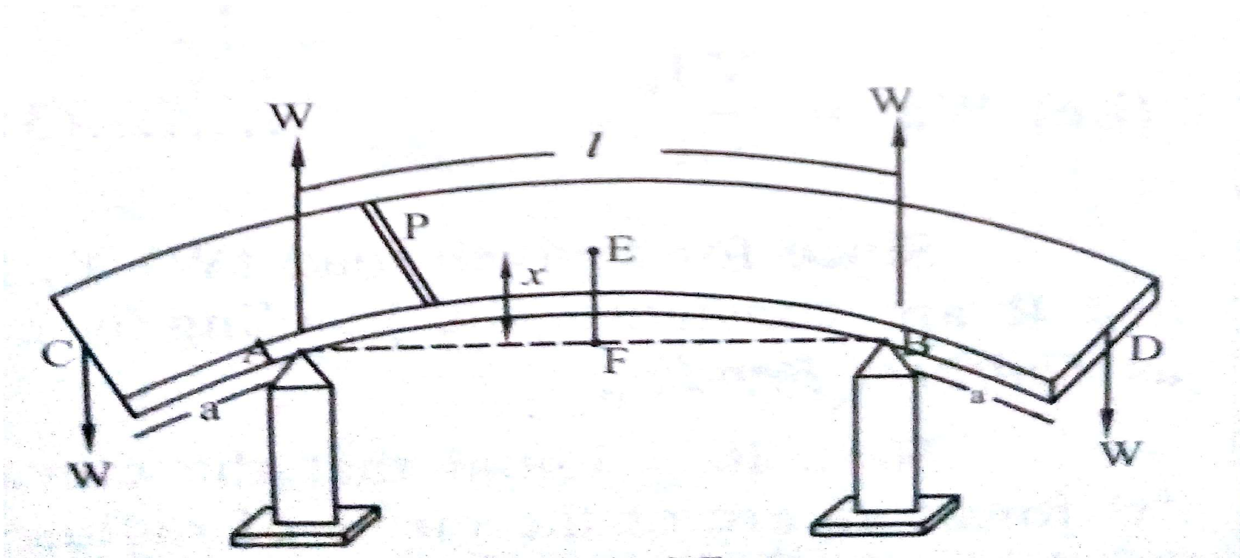
Let the length between A and B be l. Let equal weights W, be added to either end of the beam C and D.

Let the distance CA=BD=a.

Due to the load applied, the beam bends from position F to E into an arc of a circle and produces as elevation x from position F to E. Let W be the reaction produced at the point A and B which acts vertically upwards as shown in figure.



Consider a point P on the cross-section of the beam. Then the forces acting on the part PC of the beam are (i) force W at C and (ii) reaction W at A as shown in figure.



Let the distance PC=a_1 and PA=a_2, then the external bending moment about P is

$$M_p = W \times a_1 - W \times a_2$$

Here the clockwise moment is taken as negative and anticlockwise moment is taken as positive.

External bending moment about P can be written as

$$M_p = W(a_1 - a_2)$$

$$M_p = Wa \qquad (1)$$

We know the

$$\text{internal bending moment} = \frac{YI}{R} \qquad (2)$$

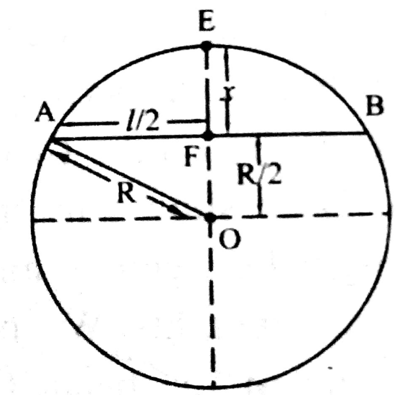
Under equilibrium condition

External bending moment= Internal bending moment

∴ We can write equation (1) = equation (2)

$$i.e. Wa = \frac{YI}{R} \qquad (3)$$

The beam bends with a uniform (constant throughout the beam) radius of curvature and forms an arc of a circle of radius R, and produces an elevation x at the midpoint of the beam.



Using the property of cords of a circle, we have

$$AF \times FB = EF \times FG$$

$$\frac{l}{2} \times \frac{l}{2} = x \times (2R - x)$$

$$\frac{l^2}{4} = 2Rx - x^2 \quad (4)$$

If the elevation x is very small, then the term x^2 can be neglected.

∴ We can write $\frac{l^2}{4} \approx 2xR$

$$x = \frac{l^2}{8R}$$

∴ Radius of curvature

$$R = \frac{l^2}{8x}$$

Substituting the value of R value in equation (3) we have

$$W\ a = \frac{YI}{l^2/8x}$$

Rearranging

$$Y = \frac{Wal^2}{8xI} \quad (5)$$

1.10.1) Experimental determination of Young's modulus by uniform bending

Description

It consists of a beam, symmetrically supported on the two knife edges A and B. Two weight hangers are suspended on either side of the beam at the position C and D. The distance between AC and BD are adjusted to be equal. A pin is fixed vertically at the centre of the beam as shown in figure. A travelling microscope is placed in front of the whole set up for finding the position of the pin.

Procedure

Taking the weight hanger as the dead load W, the microscope is adjusted and the tip of the pin is made to coincide with the vertical cross wire. The reading is noted from the vertical scale of the microscope.

Now the load on each hanger is increased in equal steps of $m, 2m, 3m, 4m,$ etc. kilogram and the corresponding readings are noted from the vertical scale of the microscope. The same procedure is repeated during unloading. The readings are noted form the vertical scale of the microscope. The readings are tabulated in the tabular columns as shown.

S.No.	Load (M)	Microscope readings			Elevation (x)	M/x
		Increasing load	Decreasing load	Mean		
	kg	$\times 10^2$ m	$\times 10^2$ m	$\times 10^2$ m	$\times 10^2$ m	kg m^{-1}
1.	W			x_0		
2.	W+m			x_1		
3.	W+2m			x_2	$x_2 - x_0$	
4.	W+3m			x_3	$x_3 - x_1$	
5.	W+4m			x_4	$x_4 - x_2$	
6.	W+5m			x_5	$x_5 - x_3$	

The mean elevation x of the centre for M kg is found. The distance between the two knife edges is measured as l and the distance from the point of suspension of the load to the knife edge is measured as a.

Then, we know the Young's modulus

$$Y = \frac{Wal^2}{8Ix}$$

The weight W=Mg and the geometric moment of inertia for a beam of rectangular cross-section is $I = \frac{bd^3}{12}$. Substituting all these in the above equation we get

$$Y = \frac{3}{2}\frac{Mgal^2}{bd^3x}$$

Substituting the mean value of $\frac{M}{x}$ from the tabular column the Young's modulus Y of the material of the given beam can be calculated.

Young's modulus of a beam used as cantilever

Cantilever is a beam supported at one end and carrying a load at the other end or distributed along the unsupported portion. The upper half of the thickness of such a beam is subjected to tensile stress, tending to elongate the fibres, the lower half to compressive stress, tending to crush them.

$$y = \frac{W}{YI}\frac{l^3}{3}$$

Special cases

(i) Rectangular cross-section

If b is the breadth and d is the thickness of the beam then we know that

$$I = \frac{bd^3}{12}$$

Substituting the value of I in equation (7), we can write the depression produced at free end for a rectangular cross-section

$$y = \frac{Wl^3}{3Y\left({bd^3}/{12}\right)}$$

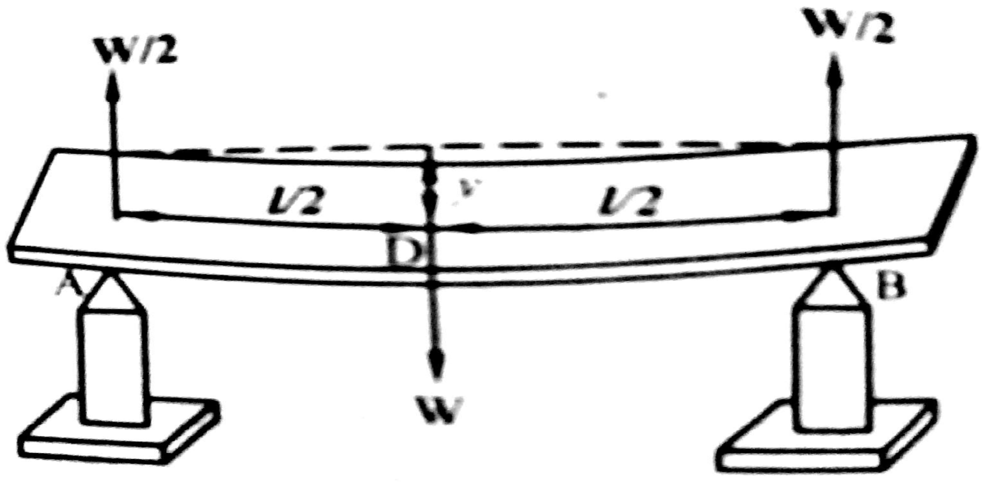
$$\therefore y = \frac{4\,Wl^3}{Ybd^3}$$

Rearranging equation (11) we get, Young's modulus

$$Y = \frac{4Mgl^3}{bd^3y}$$

1.10) Determination of Young's modulus of a beam by nonuniform bending

Let us consider a beam of length l (distance between two knife edges) supported on the two knife edges A and B as shown in figure. The load of weight W is suspended at the centre C. It is found that the beam bends and the maximum displacement is at the center of the beam.



Due to the load W applied, at the middle of the beam the reaction $\frac{W}{2}$ is acted vertically upwards at each knife edges. The bending is called as non-uniform bending, as the radius of curvature of the beam is not uniform throughout the bend.

The beam may be considered as two cantilevers, whose free end carries a load $\frac{W}{2}$ each of length $\frac{l}{2}$ and fixed at the point C.

Hence, we can say the elevation of A above C as the depression of C below A. We know the depression of a cantilever

$$y = \frac{Wl^3}{3YI}$$

Therefore, substituting the value of l as $^{l}/_{2}$ and W as $^{W}/_{2}$ in the expression for the depression of a cantilever, we have

$$Depression\ of\ C\ below\ A\ is\ y = \frac{(W/2)(l/2)^3}{3YI}$$

$$y = \frac{Wl^3}{48YI}$$

1.11.1) Experimental determination of Young's modulus- nonuniform bending

Description

It consists of a beam, symmetrically supported on the two knife edges A and B. A weight hanger is suspended at the centre C of the beam by means of a loop or thread. A pin is fixed vertically at C by some wax. In order to measure the depression, the tip of the pin is focused using a travelling microscope M.

Procedure

Taking the weight hanger as the dead load W, the microscope is adjusted and the tip of the pin is made to coincide with the horizontal cross wire. The reading is noted from the vertical scale of the microscope.

The weights are added in steps of $m, 2m, 3m$ kgs and the corresponding readings are noted from the vertical scale of the microscope. The same procedure is repeated while unloading and the readings are tabulated in the tabular column as shown. The mean depression y is found for a load of M kg.

S.No.	Load (M)	Microscope readings			Depression (y)	M/y
		Increasing load	Decreasing load	Mean		
	kg	$\times 10^2$ m	$\times 10^2$ m	$\times 10^2$ m	$\times 10^2$ m	kg m^{-1}
1.	W			y_0		
2.	W+m			y_1		
3.	W+2m			y_2	$y_2 - y_0$	
4.	W+3m			y_3	$y_3 - y_1$	
5.	W+4m			y_4	$y_4 - y_2$	
6.	W+5m			y_5	$y_5 - y_3$	

Theoretically, we know the depression produced is

$$y = \frac{Wl^3}{48\, Y\, I}$$

where l be the length of the beam, that is, the distance between the knife edges.
If b is the breadth of the beam and d is the thickness of the beam, then the geometrical moment of inertia is

$$I = \frac{bd^3}{12}$$

and $W = Mg$.

Substituting these in the expression for depression and rearranging we get

$$Y = \frac{Mgl^3}{4bd^3y}$$

Substituting the mean value of M/y, from the tabular column, the Young's modulus Y can be determined.