1. Prove by mathematical induction method $2^n > n$ for all $n \geq 0$

To prove the inequality $2^n > n$ for all $n \geq 0$ using mathematical induction, follow these steps:

Step 1: Base Case (n = 0)

For $n = 0$:

$$2^0 = 1, \quad \text{and} \quad 0 = 0$$

Clearly, $1 > 0$, so the base case holds.

Step 2: Induction Hypothesis

Assume that the inequality holds for some $n = k$, i.e.,

$$2^k > k$$

for some $k \geq 0$.

Step 3: Inductive Step (Prove for $n = k + 1$)

We need to prove that:

$$2^{k+1} > k + 1$$

Using the induction hypothesis $2^k > k$, we multiply both sides by 2:

$$2^{k+1} = 2 \cdot 2^k > 2 \cdot k$$

Now, we need to show that:

$$2 \cdot k \geq k + 1$$

This simplifies to:

$$2k \geq k + 1$$

Rearranging:

$$2k - k \geq 1$$

$$k \geq 1$$

Since the inequality holds for $k \geq 1$, we check separately for $k = 0$:

$$2^1 = 2 > 1$$

which is true.

Conclusion

By the principle of mathematical induction, we have proven that:

$$2^n > n, \quad \text{for all } n \geq 0.$$

2. Prove by Induction that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 3 and n>0 .

We will prove by mathematical induction that

$$n^3 + (n+1)^3 + (n+2)^3$$

is divisible by 3 for all $n > 0$.

Step 1: Base Case (n = 1)

For $n = 1$:

$$1^3 + (1+1)^3 + (1+2)^3 = 1^3 + 2^3 + 3^3$$

$$= 1 + 8 + 27 = 36$$

Since 36 is divisible by 3, the base case holds.

Step 2: Induction Hypothesis

Assume that for some $n = k$, the statement holds:

$$k^3 + (k+1)^3 + (k+2)^3 \text{ is divisible by } 3.$$

That is,

$$k^3 + (k+1)^3 + (k+2)^3 = 3m$$

for some integer m.

Step 3: Inductive Step

We need to prove that the statement holds for $n = k+1$, i.e.,

$$(k+1)^3 + (k+2)^3 + (k+3)^3$$

Expanding:

$$(k+1)^3 + (k+2)^3 + (k+3)^3 = [k^3 + 3k^2 + 3k + 1] + [(k+1)^3 + 3(k+1)^2 + 3(k+1) + 1] + [(k+2)^3 + 3(k+2)^2 + 3(k+2) + 1]$$

Rearrange terms:

$$(k^3 + (k+1)^3 + (k+2)^3) + 3(k^2 + (k+1)^2 + (k+2)^2) + 3(k + (k+1) + (k+2)) + 3$$

By the induction hypothesis, we know that

$$k^3 + (k+1)^3 + (k+2)^3 = 3m.$$

Factor out 3:

$$3[m + (k^2 + (k+1)^2 + (k+2)^2) + (k + (k+1) + (k+2)) + 1]$$

Since the term inside the brackets is an integer, the expression is divisible by 3.

3. Prove the following by the deductive Proof, let x be a real number then $\lfloor x \rfloor = \lceil x \rceil$ if and only if x is an integer.

To prove the statement using **deductive proof**, we need to show that:

$$\lfloor x \rfloor = \lceil x \rceil \text{ if and only if } x \text{ is an integer.}$$

Understanding the Floor and Ceiling Functions

- The **floor function** $\lfloor x \rfloor$ gives the greatest integer less than or equal to x.

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \le x\}$$

- The **ceiling function** $\lceil x \rceil$ gives the smallest integer greater than or equal to x.

$$\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \ge x\}$$

Now, we proceed with the proof in two directions:

($\Longrightarrow$) If $\lfloor x \rfloor = \lceil x \rceil$, then x is an integer

- By definition, $\lfloor x \rfloor$ is the largest integer $\leq x$, and $\lceil x \rceil$ is the smallest integer $\geq x$.
- If these two values are equal, say $\lfloor x \rfloor = \lceil x \rceil = n$ for some integer n, then we must have:

$$n \leq x \leq n$$

  which implies:

$$x = n.$$

- Since n is an integer, it follows that x is also an integer.

Example 1 (Integer Case)

Let $x = 5$:

- $\lfloor 5 \rfloor = 5$
- $\lceil 5 \rceil = 5$
- Since $\lfloor x \rfloor = \lceil x \rceil$, and $x = 5$ is an integer, the proof holds.

Example 2 (Non-Integer Case)

Let $x = 3.7$:

- $\lfloor 3.7 \rfloor = 3$
- $\lceil 3.7 \rceil = 4$
- Here, $\lfloor x \rfloor \neq \lceil x \rceil$, so x is **not** an integer.

($\Longleftarrow$) If x is an integer, then $\lfloor x \rfloor = \lceil x \rceil$

- Suppose $x = n$ for some integer n.
- Then, by definition of the floor and ceiling functions:

$$\lfloor n \rfloor = n \quad \text{and} \quad \lceil n \rceil = n.$$

- Since both values are equal, we conclude that:

$$\lfloor x \rfloor = \lceil x \rceil.$$

Example 3 (Integer Case)

Let $x = -2$:

- $\lfloor -2 \rfloor = -2$
- $\lceil -2 \rceil = -2$
- Since $\lfloor x \rfloor = \lceil x \rceil$, x is an integer.

Example 4 (Non-Integer Case)

Let $x = -4.5$:

- $\lfloor -4.5 \rfloor = -5$
- $\lceil -4.5 \rceil = -4$
- Since $\lfloor x \rfloor \neq \lceil x \rceil$, x is **not** an integer.

We have proved both directions:

1. **If** $\lfloor x \rfloor = \lceil x \rceil$, **then** x must be an integer.
2. **If** x is an integer, **then** $\lfloor x \rfloor = \lceil x \rceil$.

Thus, we conclude:

$$\lfloor x \rfloor = \lceil x \rceil \iff x \text{ is an integer.}$$

4. Write the converse and contrapositive of the following statements. "If a function is differentiable then it is continuous".

"If a function is differentiable, then it is continuous."

This is in the form of a conditional statement:

"If P, then Q."

where:

- P = "A function is differentiable."
- Q = "A function is continuous."

1. Converse

The **converse** of a conditional statement is obtained by swapping P and Q, i.e.,

"If Q, then P."

Converse:

"If a function is continuous, then it is differentiable."

👉 This statement is **false** in general because there exist continuous functions that are not differentiable. Example: $f(x) = |x|$ is continuous but not differentiable at $x = 0$.

🔹 **Example:**

Consider the function $f(x) = |x|$, which is defined as:

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

✅ **Continuous everywhere**, but ❌ **Not differentiable at $x = 0$** because of the sharp corner.

➡️ This shows that a function can be continuous but not differentiable, proving that the **converse is false**.

2. Contrapositive

The **contrapositive** of a conditional statement is obtained by negating both P and Q and reversing their order, i.e.,
"If not Q, then not P."

Contrapositive:
"If a function is not continuous, then it is not differentiable."

👉 This statement is **true** because differentiability implies continuity. If a function is not continuous, it cannot be differentiable.

🔹 **Example:**

Consider the step function:

$$f(x) = \begin{cases} 1, & x < 0 \\ 2, & x \geq 0 \end{cases}$$

✅ **Not continuous at $x = 0$** (because of the jump).

❌ **Not differentiable at $x = 0$** (since it's not even continuous).

➡️ Since a function must be continuous to be differentiable, this confirms that the **contrapositive is always true.**

1. **Converse:** "If a function is continuous, then it is differentiable." (**False**)
   - Example: $f(x) = |x|$ (Continuous but not differentiable at $x = 0$).
2. **Contrapositive:** "If a function is not continuous, then it is not differentiable." (**True**)
   - Example: A step function (Discontinuous at $x = 0$, so not differentiable).

5. Prove by mathematical induction $n \geq 0 \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Step 1: Base Case ($n = 1$)

For $n = 1$, the left-hand side of the equation is:

$$\sum_{i=1}^{1} i^2 = 1^2 = 1$$

The right-hand side of the equation is:

$$\frac{1(1+1)(2(1)+1)}{6} = \frac{1 \times 2 \times 3}{6} = \frac{6}{6} = 1$$

✅ **Since both sides are equal, the base case holds.**

Step 2: Inductive Hypothesis

Assume the formula is true for some $n = k$, i.e.,

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

We need to prove that the formula holds for $n = k + 1$:

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

Step 3: Inductive Step

Starting from the assumption:

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

Using the inductive hypothesis:

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Adding $(k+1)^2$ to both sides:

$$\sum_{i=1}^{k+1} i^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

Factor $(k+1)$:

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

Expanding inside the bracket:

$$= \frac{(k+1)[2k^2+k+6k+6]}{6}$$

$$= \frac{(k+1)[2k^2+7k+6]}{6}$$

Factor $2k^2+7k+6$:

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

which is exactly the required formula for $n=k+1$.

By the principle of **mathematical induction**, we have proved that:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

for all $n \geq 0$. ✅

6.) Prove that √2 is not rational.

Theorem:

$\sqrt{2}$ is not a rational number.

Proof:

1. **Assumption (Contradiction Method):**
   Assume that $\sqrt{2}$ is rational. This means that it can be expressed as a fraction of two integers:

   $$\sqrt{2} = \frac{a}{b}$$

   where a and b are integers with **no common factors** other than 1 (i.e., the fraction is in its simplest form).

2. **Squaring Both Sides:**

$$2 = \frac{a^2}{b^2}$$

   Multiplying both sides by b^2:

$$2b^2 = a^2$$

   This implies that a^2 is **even** (since it is equal to $2b^2$, which is clearly even).

3. **Implication for a:**
   Since a^2 is even, it follows that a itself must be even (because the square of an odd number is always odd).
   Let $a = 2k$ for some integer k.

4. **Substituting $a = 2k$ into the equation:**

$$2b^2 = (2k)^2$$

$$2b^2 = 4k^2$$

   Dividing both sides by 2:

$$b^2 = 2k^2$$

   This shows that b^2 is also even, which means that b is even.

5. **Contradiction:**
   Since both a and b are even, they share a common factor of 2. This contradicts our assumption that $\frac{a}{b}$ was in its simplest form.

6. **Conclusion:**
   Our initial assumption that $\sqrt{2}$ is rational must be false.
   Therefore, $\sqrt{2}$ is **irrational**.