2. Using mathematical induction, prove that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Let $p(n) = 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

To prove $p(1)$ is true

Put $n=1$

$$P(1) = 1^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = 1$$

$$1 = 1$$

$P(1)$ is true

Assume $p(K)$ is true

$$1^2 + 2^2 + \dots + K^2 = \frac{K(K+1)(2K+1)}{6} \rightarrow \text{A}$$

To prove $p(K+1)$ is true

Add $(K+1)^2$ on equation A

$$1^2 + 2^2 + \dots + K^2 + (K+1)^2 = \frac{K(K+1)(2K+1)}{6} + (K+1)^2$$

$$= \frac{K(K+1)(2K+1) + 6(K+1)^2}{6}$$

$$= (K+1)\left[\frac{K(2K+1) + 6(K+1)}{6}\right]$$

$$= (K+1)\left[\frac{2K^2 + K + 6K + 6}{6}\right]$$

$$(K+1)\left[\frac{(2K^2+7K+6)}{6}\right] = \frac{(K+1)(K+2)(2K+3)}{6}$$

$$\boxed{\begin{aligned}&2K^2+7K+6\\&2K^2+4K+3K+6\\&2K(K+2)+3(K+2)\\&=(K+2)(2K+3)\end{aligned}}$$

$P(K+1)$ is true

$\therefore$ By principle of mathematical induction, $p(n)$ is true for all positive integers of n.

1. Solve the recurrence relation of Fibannoci series where $f_n = f_{n-1} + f_{n-2}, n > 2$ with initial condition $f_1 = 1$ and $f_2 = 1$

$F_1 = 1 ; F_2 = 1$

$F_n - F_{n-1} - F_{n-2} = 0$

$r^2 - r^{2-1} - r^{2-2} = 0$

$r^2 - r^1 - 1 = 0$

$a = 1, b = -1, c = -1$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= 1 \pm \frac{\sqrt{1 - 4(1)(-1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$= \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$$

$$f_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{(1-\sqrt{5})}{2}\right)^n \rightarrow (A)$$

Put $n = 1$ in (A)

$$f_1 = A\left(\frac{1+\sqrt{5}}{2}\right)^1 + B\left(\frac{1-\sqrt{5}}{2}\right)^1$$

$$1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$$

$$1 = \frac{A}{2} + \frac{\sqrt{5}A}{2\times} + \frac{B}{2} - \frac{B\sqrt{5}}{\times 2}$$

$$1 = \frac{1}{2}(A+B) + \frac{\sqrt{5}}{2}(A-B) \rightarrow (1)$$

$n = 2$ in (1)

$$F_2 = A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2$$

$$1 = \frac{A}{4}(1+\sqrt{5})^2 + \frac{B}{4}(1-\sqrt{5})^2$$

$$1 = \frac{A}{4}(1+(\sqrt{5})^2 + 2\sqrt{5}) + \frac{B}{4}(1+(\sqrt{5})^2 - 2\sqrt{5})$$

$$= \frac{A}{4}(6+2\sqrt{5}) + \frac{B}{4}(6-2\sqrt{5})$$

$$= \frac{2A}{4}(3+\sqrt{5}) + \frac{2B}{4}(3-\sqrt{5})$$

$$= \frac{A}{2}(3+\sqrt{5}) + \frac{B}{2}(3-\sqrt{5})$$

$$1 = \frac{3A}{2} + \frac{\sqrt{5}}{2}A + \frac{3}{2}B - \frac{\sqrt{5}}{2}B$$

$$1 = \frac{3}{2}(A+B) + \frac{\sqrt{5}}{2}(A-B) \rightarrow (2)$$

Subtracting (2) - (1)

$$1 = \frac{3}{2}(A+B) + \frac{\sqrt{5}}{2}(A-B)$$

(-)

$$1 = \frac{1}{2}(A+B) + \frac{\sqrt{5}}{2}(A-B)$$

$$0 = (A+B)\left(\frac{3}{2} - \frac{1}{2}\right)$$

$$0 = (A+B)(1)$$

$A + B = 0$

$B = -A$

$1 = \frac{1}{2}(+A - A) + \frac{\sqrt{5}}{2}(A - (-A))$

$1 = \frac{\sqrt{5}}{\cancel{2}}(\cancel{2}A)$

$\boxed{A = \frac{1}{\sqrt{5}}}$ $\boxed{B = -\frac{1}{\sqrt{5}}}$

$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

$$\boxed{f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)}$$

4. Using mathematical induction,

Prove that $1^2+3^2+5^2+\ldots+(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

Let $p(n) = 1^2+3^2+5^2+\ldots+(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

To prove $p(1)$ is true

$(2(1)-1)^2 = \frac{1(2(1)-1)(2(1)+1)}{2}$

$(2-1)^2 = \frac{1(1)(3)}{2}$

$1^2 = 1$

$1 = 1$

$p(1)$ is true

Assume $P(K)$ is true

$1^2+3^2+\ldots+(2K-1)^2 = \frac{K(2K-1)(2K+1)}{3} \rightarrow$ (A)

$P(K+1)$ is true

Add $(2K+1)^2$ on both sides (A)

$1^2+3^2+\ldots+2(K-1)^2+2(K+1)^2$

$= \frac{K(2K-1)(2K+1)}{3}+(2K+1)^2$

$= \frac{K(2K-1)(2K+1)+3(2K+1)^2}{3}$

$= (2K+1)\frac{[K(2K-1)+3(2K+1)]}{3}$

$= (2K+1)\left[\frac{2K^2-K+6K+3}{3}\right]$

$= (2K+1)\left[\frac{2K^2+5K+3}{3}\right]$

$= \frac{(2K+1)(K+1)(2K+3)}{3}$

$2K^2+5K+3$
$2K^2+2K+3K+3$
$2K(K+1)+3(K+1)$
$=(K+1)(2K+3)$

CONCLUSION:-

By principle of MI $p(n)$ is true for all positive integers of n.

1. Using method of generating function, solve the recurrence relation $a_{n+1} - 8a_n + 16a_{n-1} = 4^n$; $n \geq 1$, where $a_0 = 1$ and $a_1 = 8$

$a_{n+1} - 8a_n + 16a_{n-1} = 4^n \rightarrow$ (A)

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$

x by x^n on both sides (A)

$a_{n+1}x^n - 8a_n x^n + 16a_{n-1}x^n = (4x)^n$

Taking summation on both sides from 1 to ∞

$$\sum_{n=1}^{\infty} a_{n+1}x^n - \sum_{n=1}^{\infty} 8a_n x^n + \sum_{n=1}^{\infty} 16a_{n-1}x^n = \sum_{n=1}^{\infty} (4x)^n \rightarrow \text{(B)}$$

Multiply and divide $\sum_{n=1}^{\infty} a_{n+1}x^n$ by x

$$\frac{1}{x} \sum_{n=1}^{\infty} a_{n+1} x^{n+1}$$

$$\frac{1}{x}\left[a_2x^2 + a_3x^3 + \cdots\right] \text{ (Add \& sub } a_0, a_1x)$$

$$\frac{1}{x}\left[a_0 + a_1x + a_2x^2 + \cdots\right] - a_0 - a_1x$$

$$\frac{1}{x}[G(x)] - 1 - 8x \rightarrow (1)$$

Apply $G(x)$ on $8\sum_{n=1}^{\infty} a_n x^n$

$8[a_1x + a_2x^2 + \cdots]$

Add and subtract a_0

$8[a_0 + a_1x + a_2x^2 + \cdots] - a_0$

$8[G(x)] - 1 \rightarrow (2)$

Multiply and divide $16\sum_{n=1}^{\infty} a_{n-1}x^n$ by x

$$16x \sum_{n=1}^{\infty} a_{n-1}x^{n-1}$$

$16x\,[a_0 + a_1 x + a_2 x^2 + \cdots]$

$16x\,G(x) \rightarrow$ (3)

Substitute 1, 2, 3 in equation (B)

$$\sum_{n=1}^{\infty} a_{n+1} x^n - 8\sum_{n=1}^{\infty} a_n x^n + 16\sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} (4x)^n \rightarrow \text{(B)}$$

$$\frac{1}{x}[G(x) - 1 - 8x] - 8(G(x) - 1) + 16x\,G(x) = 4x + (4x)^2 + \cdots$$

$$\frac{1}{x}G(x) - \frac{1}{x} - \cancel{8} - 8G(x) + \cancel{8} + 16x\,G(x) = 1 + 4x + (4x)^2 + \cdots - 1$$

(Add 1, -1)

$$G(x)\left[\frac{1}{x} - 8 + 16x\right] - \frac{1}{x} = (1-4x)^{-1} - 1$$

$$G(x)\left[\frac{1 - 8x + 16x^2}{x}\right] - \frac{1}{x} = \frac{1}{1-4x} - 1$$

$$G(x)\left[\frac{(1-4x)^2}{x}\right] - \frac{1}{x} = \frac{1}{1-4x} - 1$$

$$= \frac{1 - (1-4x)}{1-4x}$$

$$G(x)\left[\frac{(1-4x)^2}{x}\right] - \frac{1}{x} = \frac{\cancel{1} - \cancel{1} + 4x}{(1-4x)}$$

$$G(x)\left[\frac{(1-4x)^2}{x}\right] = \frac{4x}{1-4x} + \frac{1}{x}$$

$$G(x)\ \frac{(1-4x)^2}{x} = \frac{4x^2+1-4x}{x(1-4x)}$$

$$G(x) = \frac{1-4x+4x^2}{(1-4x)^3}$$

$$= (1-4x+4x^2)(1-4x)^{-3}$$

$$= (1-4x+4x^2)\left[1-3(4x)+6(4x)^2+10(4x)^3+\dots\right]$$

$$= (1-4x+4x^2)\frac{1}{1.2}\left[\begin{array}{l}1.2+2.3(4x)+3.4(4x)^2\dots+\\ n(n-1)(4x)^{n-2}+n(n-1)\\ (4x)^{n-1}+n(n+1)(4x)^{n-2}+\\ (n+1)(n+2)(4x)^n\end{array}\right]$$

$$= \frac{1}{1.2}\left[(n+1)(n+2)4^n - 4n(n-1)4^{n-1} + 4n(n-1)4^{n-2}\right]$$

$$= \frac{1}{2}\left[4^n(n+1)(n+2) - 4^n n(n+1) + 4^{n-1}(n-1)n\right].$$

$$= \frac{4^{n-1}}{2}\left[4(n+1)(n+2) - 4n(n+1) + n(n-1)\right]$$

$$= \frac{4^{n-1}}{2}\left[4(n^2+3n+2) - 4n^2 - 4n + n^2 - n\right]$$

$$= \frac{4^{n-1}}{2}\left[\cancel{4n^2}+12n+8-\cancel{4n^2}-4n+n^2-n\right]$$

$$= \frac{4^{n-1}}{2}\left[n^2+7n+8\right]$$

7. Prove that $n^5 - n$ is devisible by 5.

Let $P(n)$ be $n^5 - n$ is devisible by 5

To prove $P(1)$ is true

$P(1) = 1^5 - 1 = 0$ is devisible by 5

$P(1)$ is true

Assume $P(K)$ is true

$P(K) = K^5 - K$ is devisible by 5

$\frac{K-K}{5} = r$, where r is the integer

$K-K = 5r$

$K^5 - K = 5r$

$K^5 = 5r + K$

To prove $P(K+1)$ is true

consider $(K+1)^5 - (K+1)$

$K^5 + 5C_1 K^4 (1) + 5C_2 K^3 (1)^2 + 5C_3 K^2 (1)^3 + 5C_4 K^1 (1)$
$+ 5C_5 (K)^0 (1)^5$

$= K^5 + 5K^4 + 10K^3 + 10K^2 + 10K + \cancel{1} - K - \cancel{1}$

$= K^5 + 5K^4 + 10K^3 + 10K^2 + 4K$

$= 5r + K + 5K^4 + 10K^3 + 10K^2 + 4K$

$= 5r + 5K^4 + 10K^3 + 10K^2 + 5K$

$= 5(r + K^4 + 2K^3 + 2K^2 + K)$

$\therefore 5(r + K^4 + 2K^3 + 2K^2 + K)$ is devisible by 5

$P(K+1)$ is true

By principle of mathematical induction

$P(n)$ is true

$\therefore \quad n \geq 1$

3. Find the number of integers between 1-250 both inclusive and that are divisible by 2, 3, 5.

Let A be the number divisible by 2

Let B be the number divisible by 3

Let C be the number divisible by 5

$|A| = \left|\frac{250}{2}\right| = 125$

$|B| = \left|\frac{250}{3}\right| = 83$

$|C| = \left|\frac{250}{5}\right| = 50$

$|A \cap B| = \left|\frac{250}{LCM(2,3)}\right| = \left|\frac{250}{6}\right| = 41$

$|B \cap C| = \left|\frac{250}{LCM(3,5)}\right| = \left|\frac{250}{15}\right| = 16$

$|A \cap C| = \left|\frac{250}{LCM(2,5)}\right| = \left|\frac{250}{10}\right| = 25$

$|A \cap B \cap C| = \left|\frac{250}{LCM(2,3,5)}\right| = \left|\frac{250}{30}\right| = 8$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$= 125 + 83 + 50 - 41 - 16 - 25 + 8$

$= 184$

184 numbers of integers between 1-250 which is inclusive and are divisible by 2, 3, 5.

1. Solve the recurrence relation $a_{n+2} - 2a_{n+1} + a_n = 2^n$. Given that $a_0 = 2$ and $a_1 = 1$

Given,

$$a_{n+2} - 2a_{n+1} + a_n = 2^n$$

$a_0 = 2$ and $a_1 = 1$

LHS:-

$$a_{n+2} - 2a_{n+1} + a_n = 0$$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$r = 1 \text{ (twice)}$$

General solution:-

$$a_n = (A+Bn)1^n$$

$$a_n = (A+Bn)$$

RHS:-

$$F(n) = 2^n$$

Particular solution:-

$$a_n = C2^n$$

$$a_{n+2} - 2a_{n+1} + a_n = 2^n$$

$$C2^{n+2} - 2C2^{n+1} + C2^n = 2^n$$

$$C2^n \cdot 2^2 - 2C2^n \cdot 2 + C2^n = 2^n$$

$$\cancel{2^n}[4C - 4C + C] = \cancel{2^n}$$

$$C = 1$$

Substitute value of C in particular solution

$a(n) = (1)2^n$

$a_n = 2^n$

General solution:-

$a_n^{(c)} + a_n^{(p)}$

$a_n = (A + Bn) + 2^n \rightarrow$ A

when $n = 0$ in A

$a_0 = (A + B(0)) + 2^0$

$2 = A + 1$

$2 - 1 = A$

$\boxed{A = 1}$

when $n = 1$ in A

$a_1 = (A + B(1)) + 2$

$1 = (A + B) + 2$

$A + B + 2 = 1$

$1 + B + 2 = 1$

$B + 3 = 1$

$\boxed{B = -2}$

$\boxed{\text{General solution:- } a_n = 1 - 2n + 2^n}$

on the recurrence

2. A survey of 100 students, where it was found that 30 studied mathematics and 54 students studied stastics and 25 students studied operation research where 1 student studied all 3 subjects, 20 studied mathematics & stastics and 3 studied mathematics and operation research, 15 studied stastics and operation research. Find how many students studied none of the subjects and how many studied only mathematics.

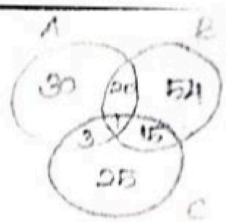
$U = 100$

Let A be the students who studied mathematics

Let B be the students who studied stastics

Let C be the students who studied operation research

$|A| = 30$, $|B| = 54$, $|C| = 25$



$|A \cap B \cap C| = 1$ $\quad$ $|A \cap B| = 20$

$|A \cap C| = 3$ $\quad$ $|B \cap C| = 15$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$= 30 + 54 + 25 - 20 - 15 - 3 + 1$

$= 110 - 38 = 72$

The no of students who studied none of the subjects $= 100 - 72$

$= 28$

The no. of students who studied only mathematics

$= 30 - 19 - 2 - 1$

$= 30 - 22$

$= 8$

5. Prove that $6^{n+2} + 7^{2n+1}$ is devisible by 43 for each positive integer n.

Let $P(n) = 6^{n+2} + 7^{2n+1}$ is devisible by 43

To prove $P(1)$ is true

$P(1) = 6^{n+2} + 7^{2+1} = 6^3 + 7^3$

$= 216 + 343 = 559$

$= 43 \times 13$

$\therefore$ $P(1)$ is true

$P(K) = 6^{K+2} + 7^{2K+1}$ is devisible by 43

$= \dfrac{6^{K+2} + 7^{2K+1}}{43} = C$

where C is an positive integer

$6^{K+2} + 7^{2K+1} = 43C$

To prove $P(K+1)$ is true

$P(K+1) = 6^{(K+1)+2} + 7^{2(K+1)+1}$

$= 6^{K+3} + 7^{2K+3} \cdot 2(K+1)$

$= 6^{K+2+1} + 7^{2K+1+2} * 2K+2+1$

$= 6^{K+2}(6) + 7^{2K+1} \cdot 7^2$

$= 6 \cdot 6^{K+2} + 49^{2K+1}$

$= 6 \cdot 6^{K+2} + (43+6)\, 7^{2K+1}$

$= 6 \cdot 6^{K+2} + 43(7)^{2K+1} + 6(7^{2K+1})$

$= 6 \cdot 6^{K+2} + 6(7^{2K+1}) + 43(7^{2K+1})$

$= 6(6^{K+2} + 7^{2K+1}) + 43(7^{2K+1})$

$= 6(43C) + 43(7^{2K+1})$

$= 43(6C + 7^{2K+1})$ is devisible by 43

$\therefore$ $P(K+1)$ is true

By principle of mathematical induction $P(n)$ is true for all positive integers n.

6. Suppose there are 6 boys & 5 girls
i) In how many ways, they can sit in a row?
ii) In how many ways, they can sit in a row if all boys sit together and all girls sit together.

iii) In how many ways they can sit in a row if just girls sit in a row?
iv) In how many ways they can sit in a row if just boys sit in a row?

6 - Boys and 5 - Girls

i) No of ways can sit in a row $= 11!$
$= 39916800$

ii)

B	B	B	B	B	B	G	G	G	G	G

$\underbrace{BBBBBB}_{1}\ \underbrace{GGGGG}_{1} = 2$

No of ways if boys all sit together and girls all sit together

$= 6! \times 5! \times 2!$

$= 6 \times 5 \times 4 \times 3 \times 2 \times 5 \times 4 \times 3 \times 2 \times 2$

$= 172800$

iii)

G	G	G	G	G	B	B	B	B	B	B

$\underbrace{GGGGG}_{5}\ \underbrace{BBBBBB}_{1}$

No of ways can be arranged if girls all sit together

$= 5! \times 7!$

$= 5 \times 4 \times 3 \times 2 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2$

$= 604800$

iv)

B_1	B_2	B_3	B_4	B_5	B_6		G_2	G_3	G_4	G_5

No of ways if all boys sit together

$= 6! \times 6!$

$= 6 \times 5 \times 4 \times 3 \times 2 \times 6 \times 5 \times 4 \times 3 \times 2$

$= 518400$

2. Find all the solution of the recurrence relation :- $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

$a_n - 5a_{n-1} + 6a_{n-2} = 7^n$

$a_n - 5a_{n-1} + 6a_{n-2} = 0$

$r^2 - 5r + 6 = 0$

$(r-2)(r-3) = 0$

$r = 2, 3$

Characteristic equation:-

$a_n = A2^n + B3^n$

$f_n = 7^n$

Particular solution:-

$a_n = C7^n \rightarrow (1)$

$a_n - 5a_{n-1} + 6a_{n-2} = 7^n$

$C7^n - 5C7^{n-1} + 6C7^{n-2} = 7^n$

$C7^n - 5C7^n . 7^{-1} + 6C7^n 7^{-2} = 7^n$

$$C7^n - \frac{5C7^n}{7} + \frac{6C7^n}{7^2} = 7^n$$

$$\cancel{7^n}\left(C - \frac{5C}{7} + \frac{6C}{49}\right) = \cancel{7^n}$$

$$\frac{7C - 5C}{7} + \frac{6C}{49} = 1$$

$$\frac{2C}{7} + \frac{6C}{49} = 1$$

$$\frac{14C + 6C}{49} = 1$$

$$\frac{20C}{49} = 1$$

$$\boxed{C = \frac{49}{20}}$$

$$a_n^{(P)} = \frac{49}{20} 7^n$$

General solution:-

$= a_n^{(C)} + a_n^{(P)}$

$$= A2^n + B3^n + \frac{49}{20} 7^n$$

$\boxed{\text{General solution:- } A2^n + B3^n}$

1. Using inclusion & exclusion principles, find the no. of students who play neither games if in a class of 50 students where 20 students play football; 16 students play hockey and 10 students who play both games.

Let A be no. of students who played football

Let B be no. of students who played hockey

$|A| = 20 \quad |B| = 16 \quad |A \cap B| = 10$

$$
\begin{aligned}
|A \cup B| &= |A| + |B| - (A \cap B) \\
&= 20 + 16 - 10 \\
&= 26
\end{aligned}
$$

No. of students who did not play any game $= 50 - 26 = 24$

4. Find the number of integers between 1 and 1000 that are both inclusive and are not divisible by any of integers 5, 7, 9

Let A be an integer divisible by 5

Let B be an integer divisible by 7

Let C be an integer divisible by 9

$|A| = \left|\frac{1000}{5}\right| = 200$

$|B| = \left|\frac{1000}{7}\right| = 142$

$|C| = \left|\frac{1000}{9}\right| = 111$

$|A \cap B| = \left|\frac{1000}{LCM(5,7)}\right| = \left|\frac{1000}{35}\right| = 28$

$|B \cap C| = \left|\frac{1000}{LCM(7,9)}\right| = \left|\frac{1000}{63}\right| = 15$

$|A \cap C| = \left|\frac{1000}{LCM(5,9)}\right| = 22$

$|A \cap B \cap C| = \left|\frac{1000}{LCM(5,7,9)}\right| = \left|\frac{100}{315}\right| = 3$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$= 200 + 142 + 111 + 3 - 28 - 15 - 22$

$= 391$

The number of integers that are not divisible by 5, 7, 9

$= 1000 - 391$

$= 609$

6. Prove that $3^n + 7^n - 2$ is divisible by 8 for $n \geq 1$ using mathematical induction.

Let $P(n) = 3^n + 7^n - 2$ is divisible by 8

To prove $P(1)$ is true

$P(1) = 3 + 7 - 2 = 8$

8 is divisible by 8

$\therefore P(1)$ is true

Assume $P(K)$ is true

$P(K) = 3^K + 7^K - 2$ is divisible by 8

$\frac{3^K + 7^K - 2}{8} = x$ where x is an integer

$3^K + 7^K - 2 = 8x$

$3^K = 8x - 7^K + 2$

To prove $P(K+1)$ is true

Consider

$P(K+1) = 3^{K+1} + 7^{K+1} - 2$

$= 3^K \cdot 3 + 7 \cdot 7^K - 2$

$= 3 \cdot 3^K + 7 \cdot 7^K - 2$

$= 3(8x - 7^K + 2) + 7 \cdot 7^K - 2$

$= 24x - 3(7^K) + 6 + 7(7^K) - 2$

$= 24x + 4 + 7^K(-3 + 7)$

$= 24x + 4 + 7^K(4)$

$= 24x + 4(7^K + 1)$

7^K is odd $7^K + 1$ is even

$7^K + 1 = 2y$ where y is a +ve integer

$24x + 4(2y)$

$= 24x + 8y$

$= 8(3x + y)$ which is divisible by 8

$P(K+1)$ is true

By principle of mathematical induction,

$P(n)$ is true for all $n \geq 1$

3. Suppose a department consists of 8 men and 9 women, in how many ways we can

select a committee with
i) 3 men and 4 women
ii) 4 persons that has atleast 1 women
iii) 4 persons that has atmost 1 men
iv) 4 persons that has persons of both gender

Men = 8 ; Women = 9

i) A committee of 3 men and 4 women

$= 8C_3 + 9C_4$

$= \frac{8\times7\times\cancel{6}}{\cancel{3}\times\cancel{2}\times1} + \frac{9\times\cancel{8}^{3}\times7\times6}{\cancel{4}\times3\times\cancel{2}}$

$= 56 + 126$

$= 182$

ii) A committee must have atleast 1 women

$3M1W_1$; $2M_2W_1$; $1M3W_1$; $0M4W$

$= 8C_3\times9C_1 + 8C_2\times9C_2 + 8C_1\times9C_3 + 8C_0\times9C_4$

$= \frac{8\times7\times\cancel{6}}{\cancel{3}\times\cancel{2}}\times9 + \frac{^{4}\cancel{8}\times7}{2\times1} \times \frac{9\times\cancel{8}^{4}}{\cancel{2}\times1} + 8\times\frac{\cancel{9}^{3}\times8\times7}{\cancel{3}\times2} + \frac{1\times9\times\cancel{8}^{2}\times7\times6}{\cancel{4}\times\cancel{3}\times\cancel{2}}$

$= 504 + 1008 + 672 + 126$

$= 2310$

iii) A committee must have atmost 1 men

$= 4W0M + 3W1M$

$= 9C_4\times8C_0 + 9C_3\,8C_1$

$= \frac{9\times8\times7\times6}{4\times3\times2}\times1 + \frac{^{3}9\times\cancel{8}^{4}\times7}{3\times\cancel{2}}\times8$

$= 126 + 672$

$= 798$

iv) A committe must have persons of both gender

$= 3M1W + 2M2W + 1M3W$

$= 8C_3\times9C_1 + 8C_2\times9C_2 + 8C_1\times9C_3$

$= \frac{8\times7\times\cancel{6}}{3\times\cancel{2}}\times9 + \frac{^{4}\cancel{8}\times7}{2\times1}\times\frac{9\times\cancel{8}^{4}}{2\times1} + 8\times\frac{\cancel{9}^{3}\times\cancel{8}^{4}\times7}{\cancel{3}\times\cancel{2}\times1}$

$= 504 + 1008 + 672$

$= 2184$

3. Solve $a_n = 3a_{n-1} + 1$, $n \geq 1$ with $a_0 = 1$

$$\boxed{\begin{matrix} n-(n-1) \\ n-n+1 \end{matrix}}$$

$a_n - 3a_{n-1} = 1$

$a_n - 3a_{n-1} = 0$

$r - 3 = 0$

$r = 3$

<u>Characteristic Equation</u>

$a_n = A3^n$

$F(n) = 1 = (1)^n$

<u>Particular solution</u>

$a_n = B(1)^n$

$a_n - 3a_{n-1} = 1$

$B(1)^n - 3B(1)^{n-1} = 1$

$1^n \left[B - \frac{3B}{1} \right] = 1$

$B - 3B = 1$

$-2B = 1$

$\boxed{B = -\frac{1}{2}}$

$a_n = -\frac{1}{2}(1)^n$

$a_n = -\frac{1}{2}$

<u>General solution:-</u>

$a_n^{(c)} + a_n^{(P)}$

$a_n = A3^n - \frac{1}{2} \rightarrow$ (A)

When $n = 0$ in (A)

$a_0 = A3^0 - \frac{1}{2}$

$1 = A(1) - \frac{1}{2}$

$1 = A - \frac{1}{2}$

$1 + \frac{1}{2} = A$

$A = \frac{3}{2}$

$$\boxed{\text{General Solution:- } a_n = \frac{3}{2} 3^n - \frac{1}{2} = \frac{1}{2}(3^{n+1} - 1)}$$