1.

EXPONENTIAL DISTRIBUTION:

pdf:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

MGF, Mean, Variance

$$MGF = M_x(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} e^{tx-\lambda x} dx$$

$$= \lambda \int_0^{\infty} e^{-x(\lambda - t)} dx$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - t)x} dx$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right]_0^{\infty}$$

$$= \frac{\lambda}{-(\lambda-t)} \left[e^{-(\lambda-t)x} \right]_0^{\infty}$$

$$= \frac{\lambda}{-(\lambda-t)} \left[e^{-\infty} - e^0 \right]$$

$$= \frac{\lambda}{-(\lambda-t)} (0-1)$$

$$MGF = \lambda / \lambda - t$$

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_0^{\infty} x \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} \underset{u}{x}\ \underset{v}{e^{-\lambda x}}$$

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \dots$$

$u = x$ $\quad$ $v = e^{-\lambda x}$

$u' = 1$ $\quad$ $v_1 = \dfrac{e^{-\lambda x}}{-\lambda}$

$v_2 = \dfrac{e^{-\lambda x}}{\lambda^2}$

$$= \lambda \left[x \left(\frac{e^{-\lambda x}}{-\lambda} \right) - (1) \left(\frac{e^{-\lambda x}}{\lambda^2} \right) \right]_0^{\infty}$$

$$= \lambda \left[\cancel{\infty} \; 0 - 0 + 0 + \frac{1}{\lambda^2} \right]$$

$$\text{Mean} = \frac{1}{\lambda}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} x^2 e^{-\lambda x} dx$$

$$\int u\,dv = uv_1 - u'v_2 + v''v_3 - \dots$$

$u = x^2$	$v = e^{-\lambda x}$ | $v_3 = e^{-\lambda x}/\lambda$
$u' = 2x$	$v_1 = e^{-\lambda x}/-\lambda$
$u'' = 2$	$v_2 = e^{-\lambda x}/\lambda^2$
$u''' = 0$	$v_3 = e^{-\lambda x}/-\lambda^3$

$$= \lambda \left[x^2 \left(\frac{e^{-\lambda x}}{-\lambda} \right) - (2x) \left(\frac{e^{-\lambda x}}{\lambda^2} \right) + (2) \left(\frac{e^{-\lambda x}}{-\lambda^3} \right) \right]_0^\infty$$

$$= \lambda \left[0 - 0 - 0 - 0 + 0 - 2 \frac{(1)}{-\lambda^3} \right]$$

$$= \lambda \left[\frac{2}{\lambda^3} \right]$$

$$= \frac{2}{\lambda^2}$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda^2}$$

MEMORYLESS PROPERTY OF EXPONENTIAL DISTRIBUTION

PROPERTY:

RV x is memoryless

if $P(x > s+t \,/\, x > t) = P(x > s), \ \forall\ s, t \neq 0$

PDF of Exponential distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , \text{otherwise.} \end{cases}$$

$$P(x > m) = \int_m^\infty f(x)\, dx$$

$$= \int_m^\infty \lambda e^{-\lambda x} dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_m^\infty$$

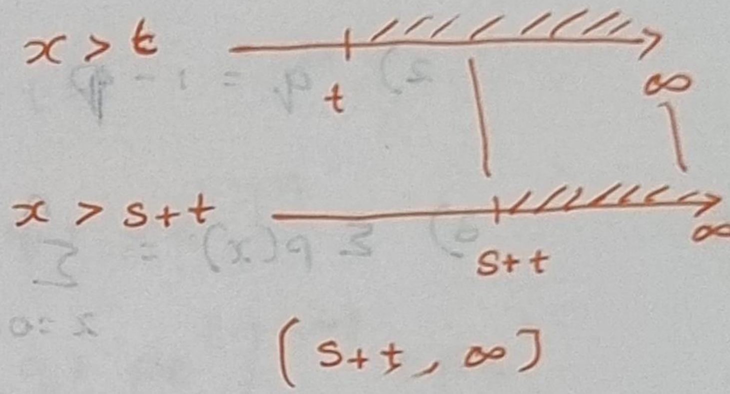
$$= - \left[0 - e^{-\lambda m} \right]$$

$$\therefore P(x > m) = e^{-\lambda m}$$

$$\Rightarrow P(x > s) = e^{-\lambda s} \quad \text{— (1)}$$

$$LHS = P(x > s+t \,/\, x > t) \qquad \left[\because P(A/B) = \frac{P(A \cap B)}{P(B)} \right.$$

$$= \frac{P(x > s+t \cap x > t]}{P(x > t)}$$



$$= \frac{P[x > s+t]}{P[x > t]}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}}$$

$$= \frac{e^{-\lambda s - \lambda t}}{e^{-\lambda t}}$$

$$= \frac{e^{-\lambda s} \cdot e^{-\lambda t}}{e^{-\lambda t}}$$

$$= e^{-\lambda s} = P(x > s) = RHS$$

DISCRETE DISTRIBUTION :

2. BINOMIAL DISTRIBUTION:

pmf

$$p(x) = nC_x \, p^x q^{n-x} \quad , \quad x = 0, 1, 2, \dots n$$

Note:

1) $p + q = 1$

2) $q = 1 - p$ / $p = 1 - q$

3) $\Sigma p(x) = \sum_{x=0}^{n} nC_x \, p^x q^{n-x}$

$$\Sigma p(x) = (nC_0 \, p^0 q^{n-0}) + (nC_1 \, p^1 q^{n-1}) + (nC_2 \, p^2 q^{n-2}) \dots + (nC_n \, p^n q^{n-n})$$

$$= 1 q^n + n p q^{n-1} + n(n-1) p^2 q^{n-2} + \dots + p^n$$

$$= (q + p)^n$$

$$= 1$$

$$\Sigma p(x) = 1$$

MGF, Mean, Variance

MGF:

$$M_x(t) = E[e^{tx}]$$

$$= \Sigma e^{tx} p(x)$$

$$= \Sigma e^{tx} (nC_x \, p^x q^{n-x})$$

$$= \Sigma nC_x (e^t p)^x q^{n-x}$$

$= nC_0 (e^t p)^0 q^n + nC_1 (e^t p)^1 q^{n-1} + \dots + nC_n (e^t p)^n q^{n-n}$

$= q^n + n(e^t p) q^{n-1} + \dots + (e^t p)^n$

$MGF = [q + pe^t]^n$

<u>Mean:</u> $= E(x) = [M_x'(t)]_{t=0} = \mu'$

$\cancel{= \sum x p(x)}$

$M_x(t) = [q + pe^t]^n$

$M_x'(t) = \underline{n} [q + pe^t]^{n-1} \times \underline{pe^t}$

$M_x''(t) = np\left[(q+pe^t)^{n-1} e^t + e^t (n-1)(q+pe^t)^{n-2} pe^t \right]$

$M_x'(0) = n [q + pe^0]^{n-1} \times pe^0$

$= n \underbrace{[q + p]}_{1}^{n-1} \times p$

$= n [1]^{n-1} \times p$

$\boxed{M_x'(0) = \mu_1' = np}$

$M_x''(0) = np\left[(q+pe^0)^{n-1} e^0 + e^0 (n-1)(q+pe^0)^{n-2} pe^0 \right]$

$= np[\underbrace{(q+p)}_{1}^{n-1} + (n-1) \underbrace{(q+p)}_{1}^{n-1} p]$

$= np[1 + (n-1)p]$

$= np[1 + np - p]$

$\boxed{M_x''(0) = \mu_2' = np + n^2p^2 - np^2}$

$$Var = \mu_2' - (\mu_1')^2$$

$$= np + (np)^2 - np^2 - (np)^2$$

$$= np - np^2$$

$$= np(1-p)$$

$$\boxed{Var = npq}$$

3. POISSON DISTRIBUTION

pmf: $p(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$; $x = 0, 1, \ldots \infty$

MGF, Mean, Variance

MGF:

$$M_x(t) = E(e^{tx})$$

$$= \Sigma e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= e^{-\lambda} \left[\sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!} \right]$$

$$= e^{-\lambda} \left[\frac{(e^t\lambda)^0}{0!} + \frac{(e^t\lambda)^1}{1!} + \frac{(e^t\lambda)^2}{2!} + \ldots + \infty \right]$$

$$= e^{-\lambda} \left[1 + \frac{(e^t\lambda)^1}{1!} + \frac{(e^t\lambda)^2}{2!} + \ldots \right] \qquad 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \ldots = e$$

$$= e^{-\lambda} \left[e^{e^t\lambda} \right]$$

$$\text{MGF} = e^{-\lambda + e^t\lambda} = e^{\lambda(e^t - 1)}$$

$$\boxed{M_{(x)}(t)}$$

$$M_x'(t) = e^{x(e^t-1)} \times \lambda e^t$$

$$M_x''(t) = \lambda \left[e^{x(e^t-1)} e^t + e^t e^{x(e^t-1)} \lambda e^t \right]$$

$$M_x'(0) = e^0 \lambda e^0 = \lambda$$

$$M_x''(0) = \lambda \left[e^0 e^0 + e^0 e^0 \lambda e^0 \right]$$

$$= \lambda \left[1 + \lambda \right]$$

$$= \lambda + \lambda^2$$

$$Var = \mu_2' - (\mu_1')^2$$

$$= \lambda + \lambda^2 - \lambda^2$$

$$= \lambda$$

4.

x	0	1	2	3	4	5	6	7	8
$p(x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

i) find a (ii) $p(x \geq 3)$ (iii) $p(x<3)$ (iv) $p(x \leq 7)$ (v) $p(0<x<3)$

Sol:

Let x be the discrete random variable.

$\therefore$ $p(x)$ is the pmf

(i) $p(x) \geq 0 \quad \forall x \quad , \quad k \geq 0$

(ii) $\Sigma p(x) = 1$

$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$

$81a = 1$

(i) $\boxed{a = \frac{1}{81}}$

x	0	1	2	3	4	5	6	7	8
$p(x)$	$\frac{1}{81}$	$\frac{3}{81}$	$\frac{5}{81}$	$\frac{7}{81}$	$\frac{9}{81}$	$\frac{11}{81}$	$\frac{13}{81}$	$\frac{15}{81}$	$\frac{17}{81}$

(ii) $p(x \geq 3) = p(x=3) + p(x=4) + p(x=5) + p(x=6) + p(x=7) + p(x=8)$

$= \frac{7}{81} + \frac{9}{81} + \frac{11}{81} + \frac{13}{81} + \frac{15}{81} + \frac{17}{81} = \frac{72}{81} = \frac{8}{9}$

(iii) $p(x<3) = p(x=0) + p(x=1) + p(x=2)$

$= \frac{1}{81} + \frac{3}{81} + \frac{5}{81} = \frac{9}{81} = \frac{1}{9}$

(iv) $p(x \leq 7) = 1 - p(x>7)$

$= 1 - \frac{17}{81} = \frac{81-17}{81} = \frac{64}{81}$

(v) $p(0<x<3) = p(x=1) + p(x=2)$

$= \frac{3}{81} + \frac{5}{81}$

$= \frac{8}{81}$

CUMULATIVE DISTRIBUTIVE FUNCTION (cdf):

If X is a discrete random variable then $F(x) = P(X \le x)$ is called cdf of X.

X	0	1	2	3	4	5	6	7	8
$p(x)$	$\frac{1}{81}$	$\frac{3}{81}$	$\frac{5}{81}$	$\frac{7}{81}$	$\frac{9}{81}$	$\frac{11}{81}$	$\frac{13}{81}$	$\frac{15}{81}$	$\frac{17}{81}$

(i) Find CDF of X

cdf of X is $F(x) = P(X \le x)$

$$F(0) = P(X \le 0)$$
$$= P(X = 0)$$
$$= \frac{1}{81}$$

$$F(1) = P(X \le 1)$$
$$= P(X=0) + P(X=1) \quad \text{or} \quad P(X \le 0) + P(X = 1)$$
$$= \frac{1}{81} + \frac{3}{81} \qquad\qquad = \frac{1}{81} + \frac{3}{81}$$
$$= \frac{4}{81} \qquad\qquad = \frac{4}{81}$$

$$F(2) = P(X \le 2)$$
$$= P(X \le 1) + P(X = 2)$$
$$= \frac{4}{81} + \frac{5}{81}$$
$$= \frac{9}{81}$$

$$F(3) = P(X \le 3)$$
$$= P(X \le 2) + P(X = 3)$$
$$= \frac{9}{81} + \frac{7}{81}$$
$$= \frac{16}{81}$$

$F(4) = p(x \le 4)$
$= p(x \le 3) + p(x = 4)$
$= \frac{16}{81} + \frac{9}{81}$
$= \frac{25}{81}$

$F(5) = p(x \le 5)$
$= p(x \le 4) + p(x = 5)$
$= \frac{25}{81} + \frac{11}{81}$
$= \frac{36}{81}$

$F(6) = p(x \le 6)$
$= p(x \le 5) + p(x = 6)$
$= \frac{36}{81} + \frac{13}{81}$
$= \frac{49}{81}$

$F(7) = p(x \le 7)$
$= p(x \le 6) + p(x = 7)$
$= \frac{49}{81} + \frac{15}{81}$
$= \frac{64}{81}$

$F(8) = p(x \le 8)$
$= p(x \le 7) + p(x = 8)$
$= \frac{64}{81} + \frac{17}{81}$
$= \frac{81}{81} = 1$

cdf of x,

$$F(x) = \begin{cases} 0, & x < 0 \\ 1/81, & 0 \le x < 1 \\ 4/81, & 1 \le x < 2 \\ 9/81, & 2 \le x < 3 \\ 16/81, & 3 \le x < 4 \\ 25/81, & 4 \le x < 5 \\ 36/81, & 5 \le x < 6 \\ 49/81, & 6 \le x < 7 \\ 64/81, & 7 \le x < 8 \\ 1, & x \ge 8 \end{cases}$$

5. If $P(X = x) = \frac{x}{15}$, $x = 1, 2, 3, 4, 5$

(i) Find $P(x = 1$ or $x = 2)$

(ii) $P\left(\frac{1}{2} < x < \frac{5}{2} \,/\, x > 1\right)$

(iii) Distribution function of x

Solution :

x	1	2	3	4	5
$P(x)$	$1/15$	$2/15$	$3/15$	$4/15$	$5/15$

(i) $P(x = 1$ or $x = 2)$

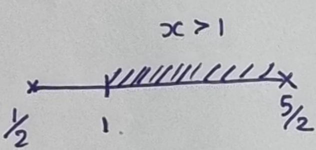
$= \frac{1}{15} + \frac{2}{15}$

$= \frac{3}{15}$

$\because$ or $\rightarrow$ $+$

and $\rightarrow$ $\times$

(ii) $P\left(\frac{1}{2} < x < \frac{5}{2} \,/\, x > 1\right)$



$$= \frac{P\left(\frac{1}{2} < x < \frac{5}{2} \cap x > 1\right)}{P(x > 1)}$$

$$= \frac{P(1 < x < 5/2)}{P(x > 1)}$$

$$= \frac{P(x = 2)}{1 - P(x \leq 1)}$$

$$= \frac{\frac{2}{15}}{1 - \frac{1}{15}}$$

$$= \frac{2/15}{14/15}$$

$$= \frac{2}{15} \times \frac{15}{14}$$

$\boxed{\text{Ans} = \frac{1}{7}}$

(iii) $F(1) = P(x = 1)$

$= 1/15$

$F(2) = P(x \leq 2)$

$= P(x=1) + P(x=2)$

$= \frac{1}{15} + \frac{2}{15}$

$= \frac{3}{15}$

$F(3) = P(x \leq 3)$

$= P(x \leq 2) + P(x = 3)$

$= \frac{3}{15} + \frac{3}{15}$

$= \frac{6}{15}$

$F(4) = P(x \leq 4)$

$= P(x \leq 3) + P(x = 4)$

$= \frac{6}{15} + \frac{4}{15}$

$= \frac{10}{15}$

$F(5) = P(x \leq 5)$

$= P(x \leq 4) + P(x = 5)$

$= \frac{10}{15} + \frac{5}{15}$

$= \frac{15}{15}$

Distribution function

$$f(x) = \begin{cases} 0 & , \quad x > 1 \\ 1/15 & , \quad 1 \leq x < 2 \\ 3/15 & , \quad 2 \leq x < 3 \\ 6/15 & , \quad 3 \leq x < 4 \\ 10/15 & , \quad 4 \leq x < 5 \\ 15/15 & , \quad x \geq 5 \end{cases}$$

6. The distribution function of RV 'x' is given by $F(x) = 1-(1+x)e^{-x}$, $x \geq 0$ Find the density function, mean, variance of x

<u>Solution:</u>

$$F(x) = 1-(1+x)e^{-x}$$

$$= 1 - e^{-x} - xe^{-x}$$

$$f(x) = F'(x)$$

$$= e^{-x} - (-xe^{-x} + (1)e^{-x})$$

$$= e^{-x} + xe^{-x} - e^{-x}$$

$$f(x) = xe^{-x}$$

<u>Mean:</u>

$$E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{0}^{\infty} x \cdot xe^{-x}\, dx$$

$$= \int_{0}^{\infty} x^2 e^{-x}\, dx$$

$\int uv\, dx = uv_1 - u'v_2 + u''v_3 + \dots$

$u = x^2$ | $v = e^{-x}$

$u' = 2x$ | $v_1 = e^{-x}/-1$

$u'' = 2$ | $v_2 = e^{-x}$

$v_3 = e^{-x}/-1$

$$= \left[x^2\left(\frac{e^{-x}}{-1}\right) - (2x)(e^{-x}) + (2)\left(\frac{e^{-x}}{-1}\right) \right]_0^{\infty}$$

$$= 0 - 2\left(\frac{1}{-1}\right)$$

$$= 2$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{0}^{\infty} x^2 \cdot x e^{-x}\, dx$$

$$= \int_{0}^{\infty} \underline{x^3} \cdot \underline{e^{-x}}\, dx$$

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \dots$$

$$u = x^3 \qquad v_1 = e^{-x}/_{-1}$$

$$u' = 3x^2 \qquad v_2 = e^{-x}$$

$$u'' = 6x \qquad v_3 = e^{-x}/_{-1}$$

$$u''' = 6 \qquad v_4 = e^{-x}$$

$$\cancel{v_5}.$$

$$= \left[x^3\left(\frac{e^{-x}}{-1}\right) - (3x^2)(e^{-x}) + (6x)\left(\frac{e^{-x}}{-1}\right) - (6)(e^{-x}) \right]_0^{\infty}$$

$$= 0 + 6(1)$$

$$= 6$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= 6 - 4$$

$$= 2$$

7. If X is a poisson random variable such that $P(X=2) = 9\,P(X=4) + 90\,P(X=6)$, then find

(i) Variance. (ii) Mean, $E(X^2)$, $P(X \geq 2)$

Solution:

Poisson Distribution is $P(X=x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$, $x = 0, 1, 2, 3, \ldots$

Given:

$$P(X=2) = 9\,P(X=4) + 90\,P(X=6)$$

$$\frac{e^{-\lambda}\lambda^2}{2!} = 9 \cdot \frac{e^{-\lambda}\lambda^4}{4!} + 90\,\frac{e^{-\lambda}\lambda^6}{6!}$$

$$\cancel{e^{-\lambda}\lambda^2}\left(\frac{1}{2}\right) = \cancel{e^{-\lambda}\lambda^2}\left(\frac{9\,\lambda^2}{4\times3\times2\times1} + \frac{90\,\lambda^4}{6\times5\times4\times3\times2\times1}\right)$$

$$\frac{1}{\cancel{2}} = \frac{3\lambda^2}{\cancel{8}\,4} + \frac{\lambda^4}{\cancel{8}\,4}$$

$$1 = \frac{1}{4}(3\lambda^2 + \lambda^4)$$

$$3\lambda^2 + \lambda^4 - 4 = 0$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

$$(\lambda^2 + 4)(\lambda^2 - 1) = 0$$

$$\cancel{\lambda = \sqrt{-4}} \;\&\; \lambda = 1 \qquad \because \lambda > 0$$

WKT,

Mean $= \lambda = 1$, $Var(X) = \lambda = 1$

$$E(X^2) = \lambda^2 + \lambda = 1 + 1 = 2$$

$$P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - [P(X = 0) + P(X = 1)]$$

$$= 1 - \left[e^{-1} + \frac{e^{-1} \cdot 1}{1!} \right]$$

$$P(X \geq 2) = 1 - 2e^{-1}$$