Multi-particle dynamics: Center of mass (CM) – CM of continuous bodies – motion of the CM – kinetic energy of system of particles. Rotation of rigid bodies: Rotational kinematics – rotational kinetic energy and moment of inertia - theorems of M.I – moment of inertia of continuous bodies – M.I of a diatomic molecule - torque – rotational dynamics of rigid bodies – conservation of angular momentum – rotational energy state of a rigid diatomic molecule - gyroscope - torsional pendulum – double pendulum – Introduction to nonlinear oscillations.

Mechanics

- static [object at rest]
- Dynamic [object at motion]

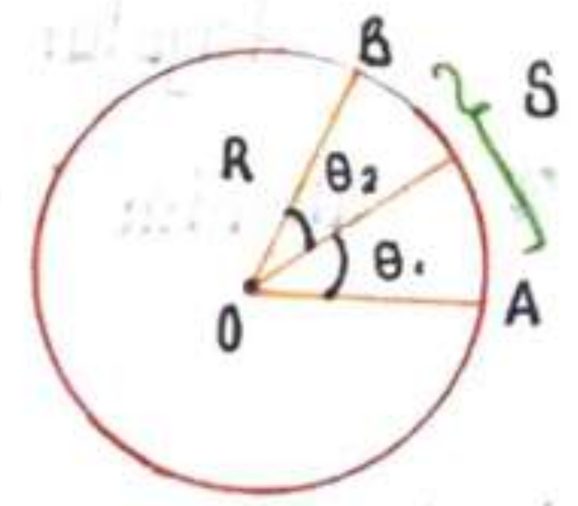
Rigid body : A rigid body is a solid body in which deformation is zero or so small it can be neglected.

Rigid body

- Translation Motion
  - Displacement
  - Velocity
  - Acceleration
  - Momentum
  - Force
  - Mass
- Rotational Motion
  - Angular _ Displacement
  - Angular - Velocity
  - Angular - Acceleration
  - Angular Momentum

Angular Displacement [θ]

Difference of Angular positions at two instance of rotational state of rigid body



$$\text{Angle} = \frac{\text{Arc}}{\text{Radius}}$$

S = Displacement

R = Radius

$$\theta = \frac{S}{R}$$

The unit of Angular Displacement is Radians

Angular Velocity [ω]

It is the rate of change of Angular displacement

$$\omega = \frac{d\theta}{dt} \text{ radians / sec}$$

Angular Acceleration [α]

It is the rate of change of Angular velocity

$$\alpha = \frac{d\omega}{dt} \quad \text{where} \quad \omega = \frac{d\theta}{dt}$$

$$\alpha = \frac{d^2\theta}{dt}$$

Relation Between Linear velocity (V) and Angular velocity (ω)

when a rotating object has angular displacement θ a point on object at a radius R travels a distance is given as

$$\theta = \frac{s}{r}$$

$$s = r\theta$$

Differentiate with respect to time

$$\frac{ds}{dt} = r \cdot \frac{d\theta}{dt} + \theta \cdot \frac{dr}{dt}$$

since r is constant $\frac{dr}{dt}$ is zero

$$\frac{ds}{dt} = r \cdot \frac{d\theta}{dt}$$

$$V = r\omega$$

Angular Momentum: [L]

Angular momentum is defined as moment of inertia of angular velocity of the particle

$$L = I\omega \quad kgm^2/s$$

Inertia:

The tendency of an object to maintain it's state of rest are of uniform motion

Rigid body:-

An object which has definite shape and size and doesnot change due to external force

Centre of Mass

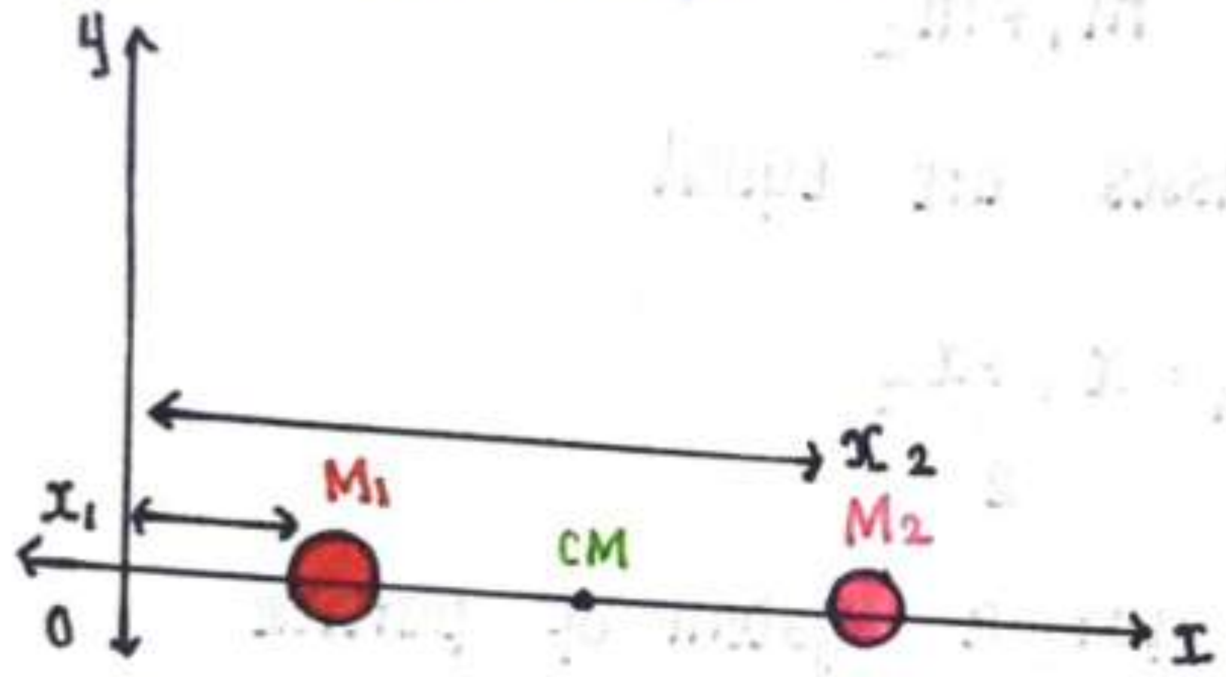
The point in the body at which the whole mass of the body is concentrated

Example symmetrical object like ball has it centre of mass at its geometrical centre Irregular shape of body like baseball bat . centre of mass is more towards thicker end

For solid can the centre of mass appears one fourth of the way up from the base

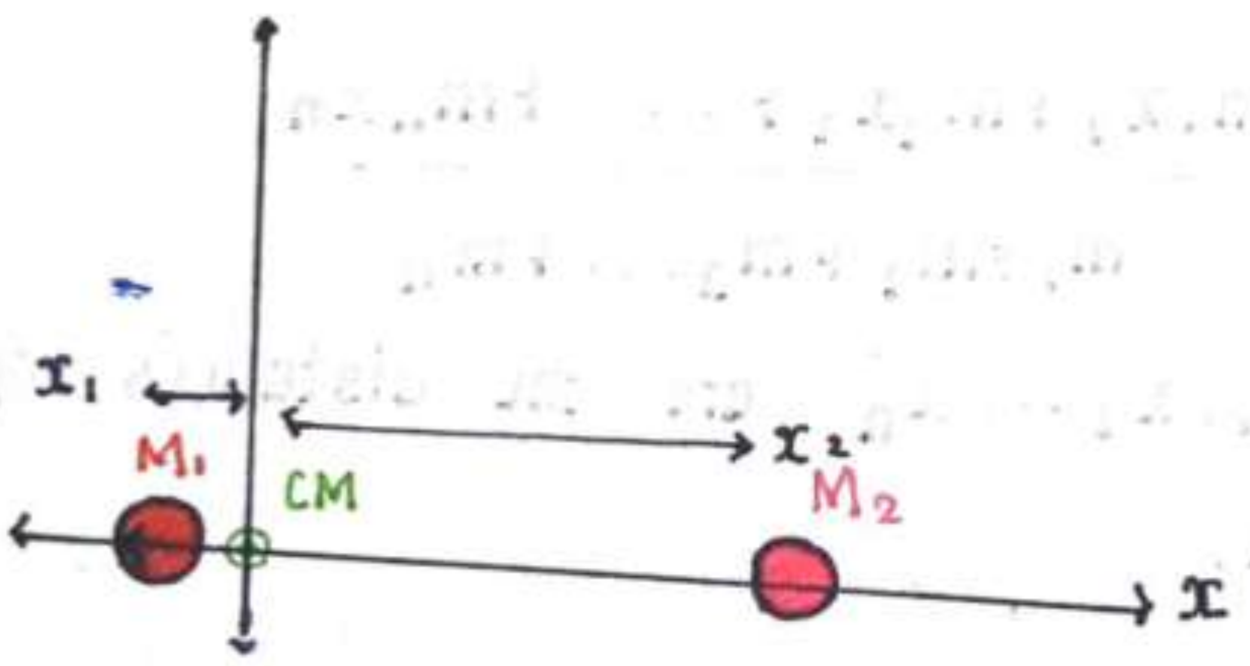
Centre of mass of a two particle system:

Let us consider a system made up of particle of masses m_1 and m_2 lying on x axis at a distance x_1 and x_2 respectively from the origin 'o'



$$X_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

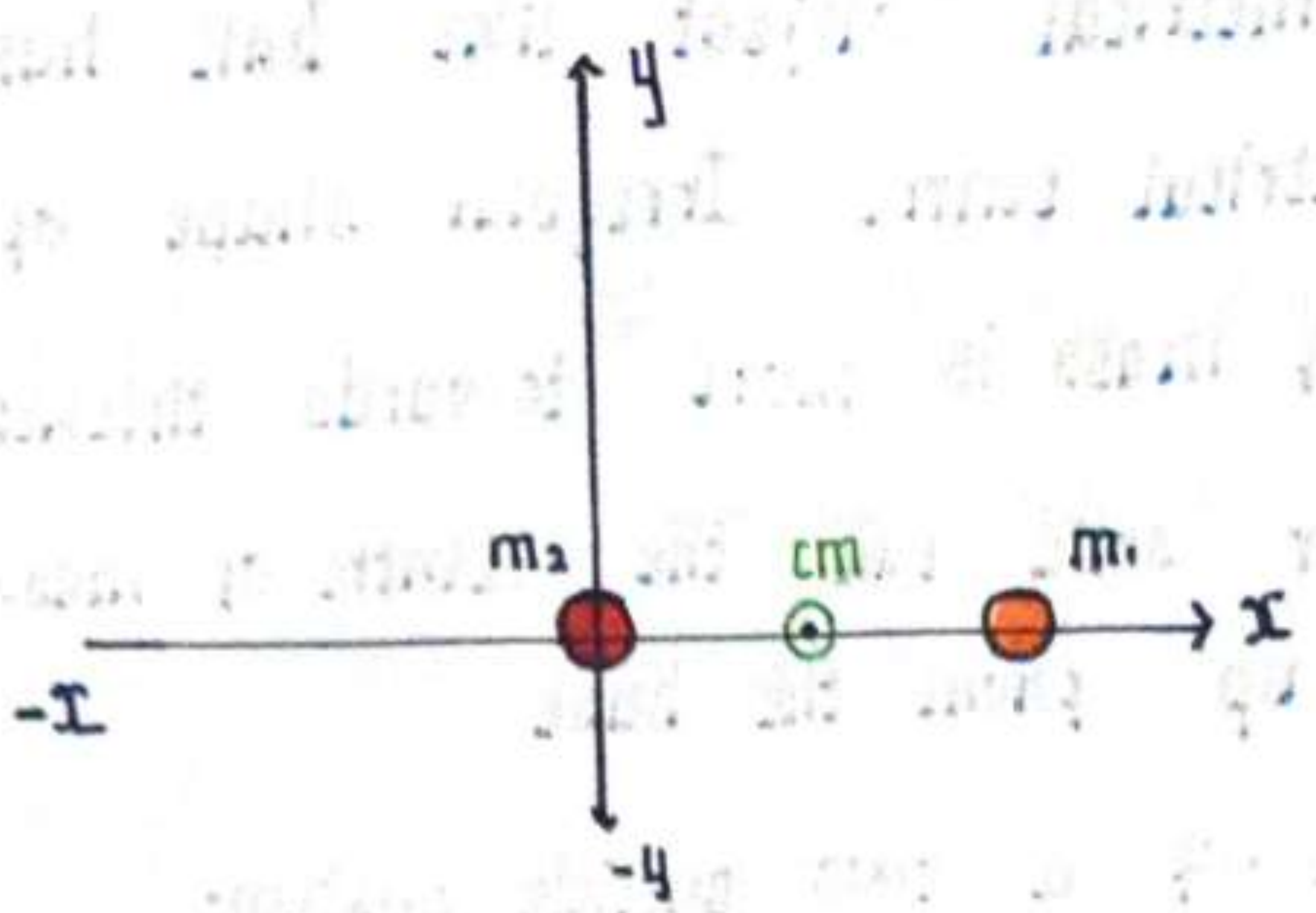
When the origin conincides with centre of mass



$$X_{cm} = \frac{m_1(+x_1) + m_2 x_2}{m_1 + m_2}$$

$$0 = \frac{m_1(-x_1) + m_2 x_2}{m_1 + m_2}$$

$$m_1 + m_2 = m_2 x_2$$



$$X_{cm} = \frac{m_1 x_1 + m_2(0)}{m_1 + m_2}$$

$$X_{cm} = \frac{m_1 x_1}{m_1 + m_2}$$

when both masses are equal

$$X_{cm} = \frac{x_1 + x_2}{2}$$

Centre of mass for a system of particle

consider a system consisting of 'n' no. of particle this system can be considered as a continuous body made up of tiny particles if $m_1, m_2, m_3 \ldots m_n$ are the mass of n particles along a straight line centre of mass can be written as

$$X = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{m_1 + m_2 + m_3 \ldots + m_n}$$

Where $x_1, x_2 \ldots x_n$ are the distances of the particle from the origin.

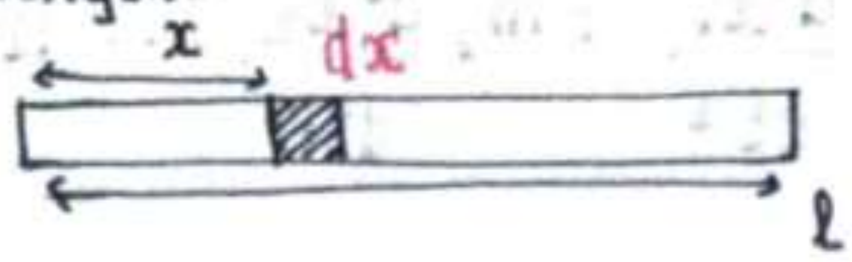
Centre of mass of continous bodies:-

centre of mass of a body or system of particles is defined as a point at which the whole mass of the body appears to be concentrated.

$$X_{cm} = \sum_{i=0}^{n} \frac{m_i x_i}{M}$$

Where $\sum m_i = M$

For enitre Length:-



$$X_{cm} = \int \frac{x\,dm}{X} \quad --(1)$$

Assume centre of mass lies on x-axis and consider breadth as negligible

$$dm = \frac{M}{l}\,dx \quad \text{[Mass per unit length]}$$

$$X_{cm} = \frac{\int x \frac{M}{l}\,dx}{M}$$

$$= \int_0^l \frac{xM}{Ml}\,dx \qquad = \int_0^l x\,dx$$

$$= \frac{1}{2l}[x^2]_0^l \qquad = \frac{1}{l}\frac{x^2}{2}$$

$$= \frac{1}{2l}\,l^2$$

$$= \frac{l}{2}$$

Centre of mass can be found for any geometrical shape

Motion of centre of mass

The motion of centre of mass is the force required to accelerate the system of particle with respect to centre of mass

Let us consider a external force F acting on the system of particle along x axis

$$X_{cm} = \sum_{i=0}^{n} \frac{m_i x_i}{M}$$

$$\Sigma m_i = M$$

$$M X_{cm} = m_1 x_1 + m_2 x_2 \cdot m_n x_n - (1)$$

Differentiate equation 1 with respect to time for (2 times)

$$M \frac{d}{dt} X_{cm} = m_1 \frac{dx}{dt} + m_2 \frac{dx}{dt} \ldots$$

$$M\frac{d^2}{dt^2} x_{cm} = m_1 \frac{d^2x_1}{dt^2} + m_2 \frac{d^2x_2}{dt^2} \ldots (2)$$

since acceleration or $a = \frac{d^2x}{dt^2}$

equation (2) becomes

$$M a_{cm} = m_1 a_1 + m_2 a_2 + \ldots (3)$$

According to Newton's II Law

$$F = ma \;--\; (4)$$

Sub eq (4) in (3)

$$F_{CM} = F_1 + F_2 + \ldots (5)$$

$$F_{CM} = \Sigma F_i \;--\; (6)$$

Force acting on centre of mass is equal to the sum of the forces acting on the system of particles this force is required to move the particles with respect to centre of mass This is called as motion of centre of mass.

Kinetic energy of the system of particles:

There are n number of particles in a system of particles and these have same motion. The motion of i^{th} particle depends on external force $\vec{F_P}$ acting on it

Let the velocity of i^{th} particle be $\vec{v_i}$ then KE,

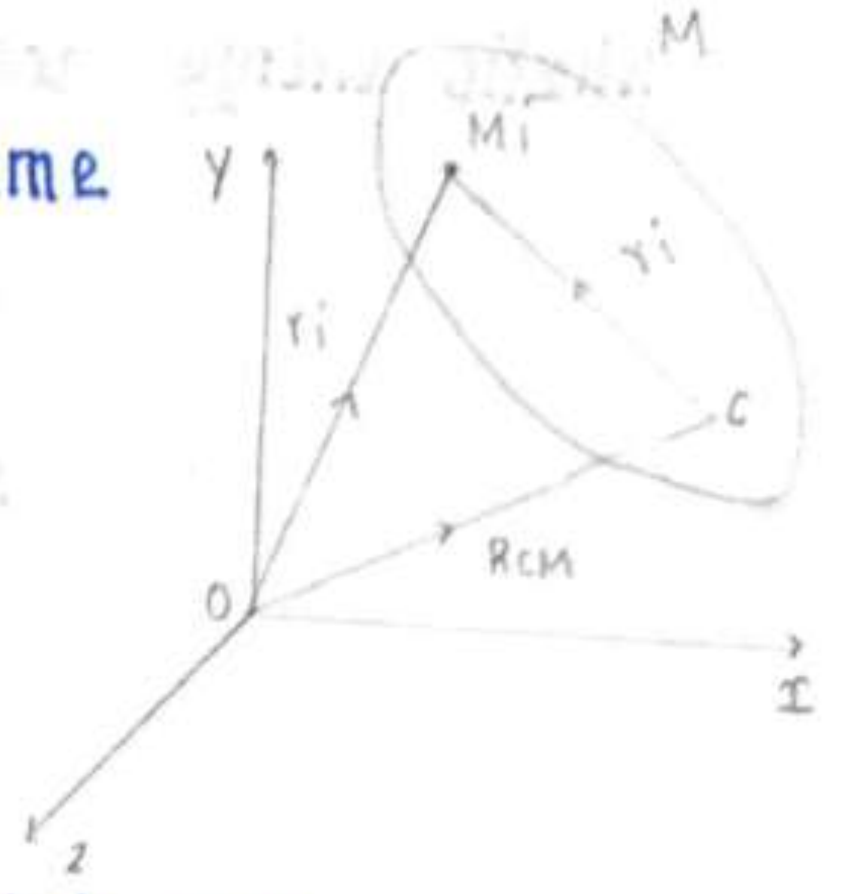
$$E_{KP} = \frac{1}{2} m v_i^2 \;--\; (1)$$

Let $\vec{r_i}$ be the position vector of i^{th} particle, with respect to 'o' and $\vec{r_i}'$ be the position vector of CM then

$$\vec{r_i} = \vec{r_i'} + R_{CM} \quad --(2)$$

Where R_{cm} is the position vector of CM of the system with respect to 'o'

Differentiating (2) eq with respect to time



$$\frac{d\vec{r_i}}{dt} = \frac{d\vec{r_i'}}{dt} + \frac{dR_{CM}}{dt}$$

$$V_i = V_i' = V_{CM} \quad --(3)$$

Where $V_i \rightarrow$ is the velocity of i^{th} particle and

$V_{CM} \rightarrow$ is the velocity of centre of mass of system of particle

sub (3) eqn in (1) eqn

$$E_{ki} = \frac{1}{2} m_i (V_i' + V_{CM})^2$$

$$E_{ki} = \frac{1}{2} m_i (V_i'^2 + 2V_i' V_{CM} + V_{CM}^2)$$

$$E_{ki} = \frac{1}{2} m_i v_i'^2 + m_i v_i' V_{CM} + \frac{1}{2} m_i V_{CM}^2 \quad --(4)$$

The sum of kinetic energy of all the particle can be given as

$$E_K = \sum_{i=1}^{n} E_{ki}$$

$$E_K = \frac{1}{2} V_{CM}^2 \sum_{i=1}^{n} m_i + \frac{1}{2} \sum_{i=1}^{n} m_i v_i'^2 + V_{CM} \sum_{i=1}^{n} m_i v_i'$$

$$E_K = \frac{1}{2} V_{CM}^2 M + \sum_{i=1}^{n} \frac{1}{2} m_i v_i'^2 + V_{CM} \frac{d}{dt} \sum_{i=1}^{n} m_i \vec{r_i}'$$

$$E_K = \frac{1}{2} M \vec{V}_{CM}^2 + \frac{1}{2} \sum_{i=1}^{n} m_i \vec{v_i}'^2$$

$$E_K = E_{KCM} + E'_K$$

$E'_K \longrightarrow$ K.E of the system of particles with respect to CM

KE of the system of particles consists of two parts like E_K' and E_{KCM}

Conclusion:

If there are no external force acting on the particle system then the velocity of CM of the system will remain constant and kinetic energy would also remain constant

$$V = u + at$$

$$S = ut + \frac{1}{2}at^2$$

$$V^2 = u^2 + 2as.$$

Rotational motion:

$$\omega = \omega_0 + \alpha t$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

First equation on rotational motion

$$\alpha = \frac{d\omega}{dt}$$

$$\alpha dt = d\omega$$

$$\int \alpha dt = \int d\omega$$

$$\alpha \int_0^t dt = \int_{\omega_0}^{\omega} d\omega$$

$$\alpha [t]_0^t = [\omega]_{\omega_0}^{\omega}$$

$$\Rightarrow \alpha t = \omega - \omega_0$$

$$\omega = \omega_0 + \alpha t \quad --(1)$$

Second equation of rotational motion:

$$\omega = \frac{d\theta}{dt}$$

$$d\theta = \omega dt \quad --(2)$$

Sub (1) in (2)

$$d\theta = (\omega_0 + \alpha t) dt$$

$$d\theta = \omega_0 dt + \alpha t dt$$

$$\int_0^\theta d\theta = \omega_0 \int_0^t dt + \alpha \int_0^t t\,dt$$

$$\theta = \omega_0 t + \alpha \frac{t^2}{2} \quad \text{--- (3)}$$

Third equation of rotational motion:-

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta} \cdot \frac{d\theta}{dt}$$

$$\alpha = \omega \cdot \frac{d\omega}{d\theta}$$

$$\alpha\, d\theta = \omega \cdot d\omega$$

$$\alpha \int_0^\theta d\theta = \int_{\omega_0}^{\omega} \omega \cdot d\omega$$

$$\alpha\theta = [\omega^2/2]_{\omega_0}^{\omega}$$

$$2\alpha\theta = [\omega^2]_{\omega_0}^{\omega}$$

$$2\alpha\theta = \omega^2 - \omega_0^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta \quad \text{--- (4)}$$

Moment of Interia:

Moment of interia of a rigid body about a fixed axis is defined as the sum of product of mass of all particles in the body and square of the respective distance from the axis of rotation $I = mr^2$ unit is kgm^2

Moment of interia doesnot depend on

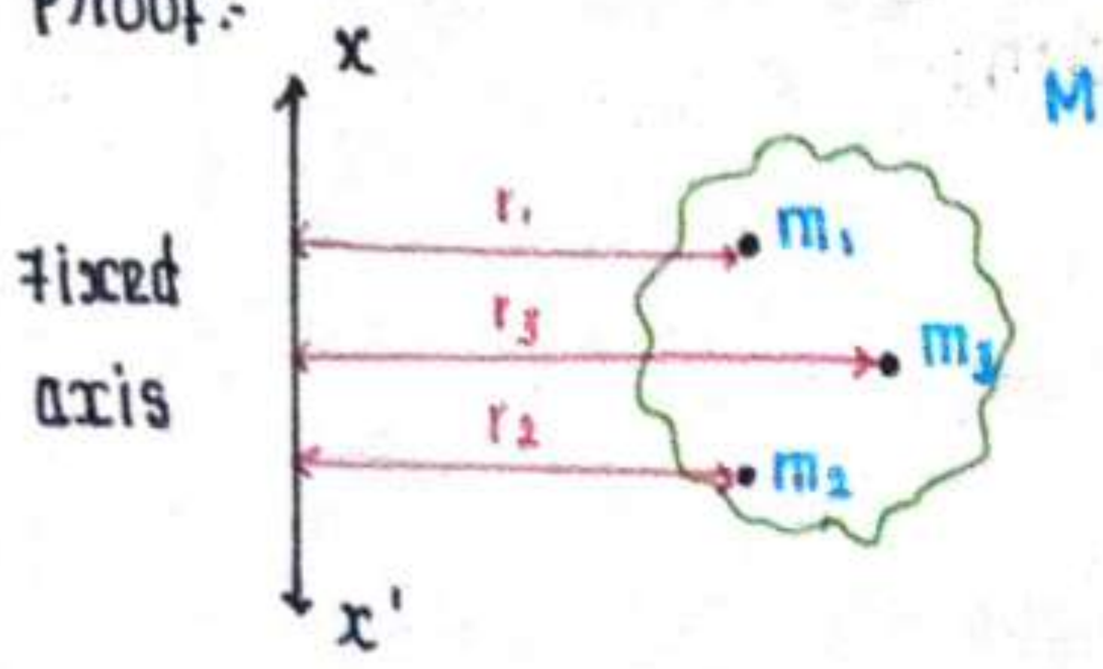
* Angular velocity
* Angular acceleration
* Angular momentum
* Torque
* Rotational kinetic energy

Moment of interia depends on

* Mass of the body

* Distribution of mass about axis of rotation

Proof:-



Let us consider a rigid body which consists of n number of particles located at different distances from the axis of rotation (x , x')

Moment of interia for first particle = $m_1 r_1^2$

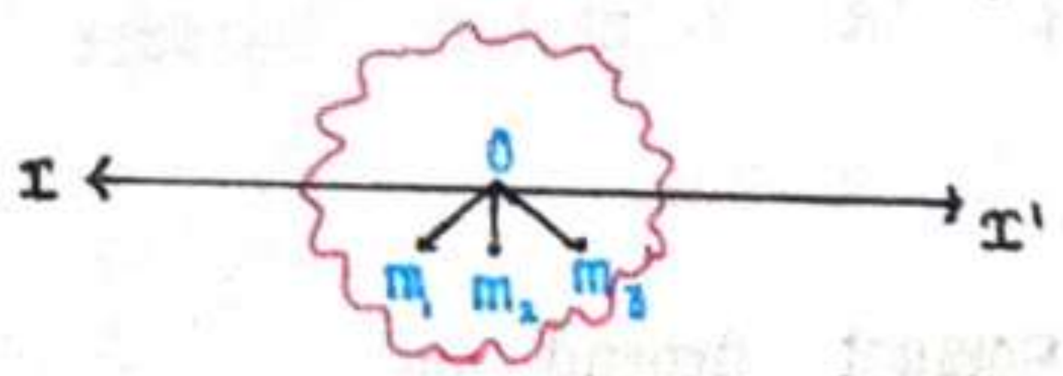
Moment of interia for second particle = $m_2 r_2^2$

Moment of interia for the entire body can be obtained by suming up of all particles

Hence $I = \sum_i m_i r_i^2$

Moment of interia of a rigid body rotating about an axis

consider a rigid body with large number of particles rotating about the final axis X O X'



Let m_1, m_2, m_2 be the masses of the particles suitated at distance r_1 ω, r_2 and r_3 from the firxed axis

A particle of mass m_1 located at distance r_1 from the axis of rotation has kinetic energy

$$K.E = \frac{1}{2} m_1 v_1^2$$

where v_1 is linear velocity of the particle similarly K.E of second particle.

$$K.E = \frac{1}{2} m_2 V_2^2$$

similarly K.E of third particle

$$K.E = \frac{1}{2} m_3 V_3^2$$

The total K.E is given the sum of K.E of indivuals particles

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \dots + \frac{1}{2} m_n v_n^2 \quad \text{--- (1)}$$

when a body rotating about the fixed axis has same linear velocity. each particle of the body moves in a circle to the linear velocity $v = r\omega$

$$\left.\begin{aligned} V_1 &= r_1 \omega \\ V_2 &= r_2 \omega_2 \\ V_n &= r_n \omega_n \end{aligned}\right\} \text{--- (2)}$$

sub 2 in (1)

$$K = \frac{1}{2} m_1 r_1^2 \omega_1^2 + \frac{1}{2} r_2^2 m_1 \omega_2^2 + \dots \frac{1}{2} m_n r_n^2 \omega_n^2$$

$$K = \sum_{i=1}^{n} \frac{1}{2} m_i r_i^2 \omega^2 \quad \text{-- (3)}$$

In an rotating rigid body each particle rotates with the same angular velocity

$$K = \frac{1}{2} \omega^2 \sum_{i=1}^{n} m_i r_i^2 \quad \text{-- (4)}$$

$$k = \frac{1}{2} I \omega^2 \quad \text{-- (5)}$$

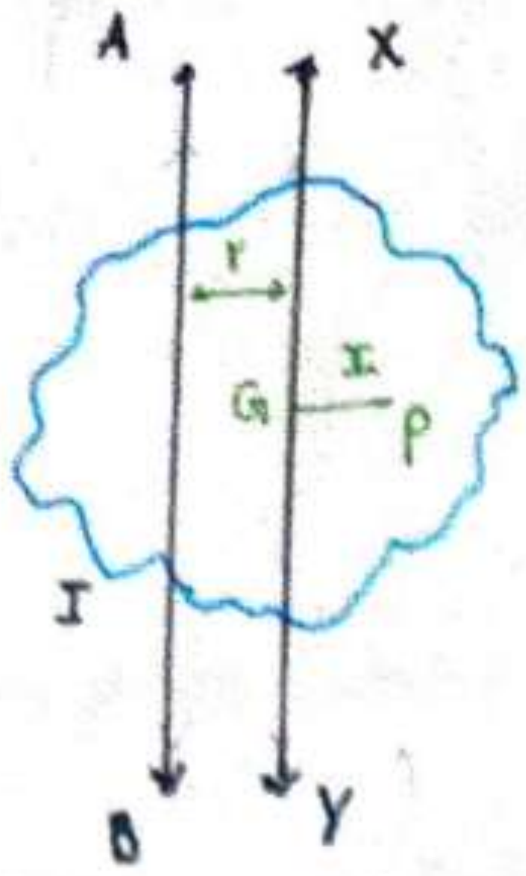
where I is the moment of interia of a rotating body.

16.12.2021

ii) Parallel axis theorem

statement

Momentum of interia about any axis is equal to the sum of its momentum of interia about a parallel axis through its centre of gravity and the product of the mass of the body and the square of the distance between two parallel axis.



$I = I_G + mr^2$ Parallel axis theorem

Proof:

consider a rigid body of mass m and centre of gravity is G. Let the body rotate about AB and I be the moment of interia of it. Let XY be the parallel axis passing through centre of gravity G. Let r be the seperation between two axes. consider a particles of mass 'm' at P distance x from XY axis

Distance of the particle from AB = $(r+x)$

moment of interia of the particle about AB

$$= m(r+x)^2$$

moment of interia of the whole body about AB

$$I = \Sigma m(r+x)^2$$

$$I = \Sigma mr^2 + \Sigma mx^2 + 2r\Sigma mx \quad \text{-- (1)}$$

$\Sigma m = M$ is the total mass of the body

$\Sigma mx^2 = I_G$ is a moment of inertia of the body about the axis through centre of gravity

Equation (1) becomes

$$I = Mr^2 + I_G + 2r\Sigma mx \quad \text{-- (2)}$$

sinu $2\Sigma mx = 0$

$$\boxed{I = mr^2 + I_G}$$

parallel axis theorem is proved

Not for exam

force acting on the particle = mg

moment of force about x G Y = (F × ⊥r distance)

$= mgx$

sum of moments of all particles $(g \neq 0)$

$(g = 9.8 \times 10^{"})$

$$\Sigma mgx$$

$$\Sigma mx = 0$$

Because sum of movements of all the particle = 0

so, $2r\Sigma mr = 0$

equation (2) becomes $I = I_G + mr^2$

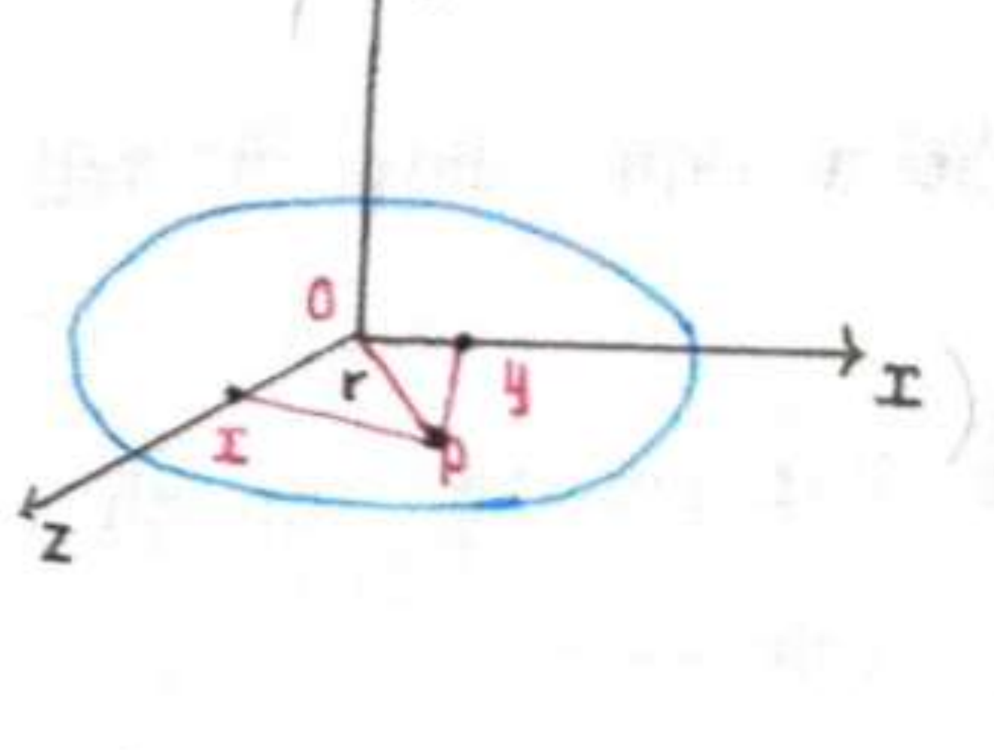
The algrebic sum of moments of all the force about an axes passing through centre of gravity of a body is zero so

$$\Sigma mx = 0$$

Perpendicular axes theorem

statement

(moment of interia of a plane Laminan body about an axis perpendicular to its plane and passing through the point of intersection of two mutually perpendicular axes in equal to the sum of m.I about two mutually perpendicular axes lying in the same plane)



consider a plane lamina having the axis ox and oy in the plane. The axes oz passes through o and perpendicular to the plane. Let the Lamina be divided into Large each of mass m. A particle p is at a distance r from o.

$$r^2 = x^2 + y^2 \quad \text{--- (1)}$$

moment of the particle p about oz $= mr^2$

m.I of whole lamina about oz is

$$I_z = \Sigma mr^2 \quad \text{--- (2)}$$

m.I of whole lamina about ox $= \Sigma mx^2$ ...

m.I of whole lamina about oy $= \Sigma my^2$

using eqn (1) & (2)

$$I_z = \Sigma m(x^2+y^2)$$

$$= \Sigma mx^2 + \Sigma my^2$$

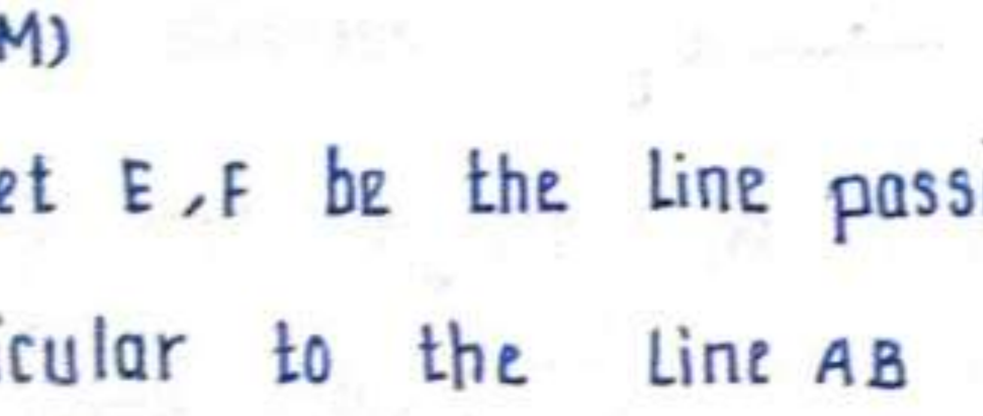
$$I_z = I_x + I_y$$

perpendicular axis theorem is proved

Moment of interia for continuous bodies :-

i) Thin rod
ii) circular disc
iii) solid sphere
iv) solid cylinder

1) Thin rod :



Let A and B be a thin uniform rod of mass per unit length (M)

Let E, F be the line passing through centre of mass O, and perpendicular to the line AB

Let us consider an elemental area of length dr at a distance r from centre of mass

mass of the element dm = density x l

$$dm = m.dr \quad \text{--- (1)}$$

Density $= \frac{M}{l}$

$M =$ Density $\times l$

$M = \frac{m}{l} \times l$

moment of interia about axis EF through 'O' for element = dmr²

$= mr^2\,dr$ --(2)

m.I of interia of whole rode about

$$EF = \int_{-L/2}^{L/2} mr^2\,dr$$

$$I = 2\int_0^{L/2} mr^2\,dr$$

$$= 2m\int_0^{L/2} r^2\,dr$$

$$= 2m\left[\frac{r^3}{3}\right]_0^{L/2}$$

$$= 2m\left[\frac{[L/2]^3}{3}\right]$$

$$= \frac{2mL^3}{3\times 8}$$

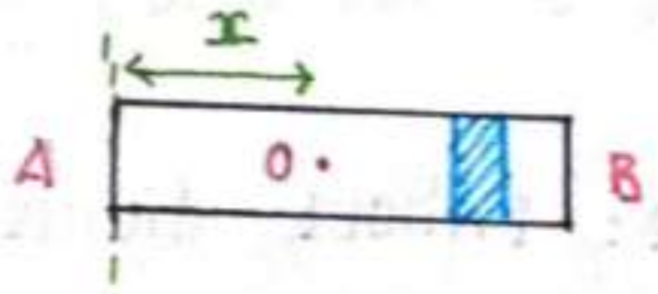
$$= \frac{mL^3}{12}$$

$$I = \frac{(mL)L^2}{12}$$

$$I = \frac{ML^2}{12}$$

M is the total mass of the rod

About axis passing through one of its ends and perpendicular to the length



$$I_G = I_G + mr^2$$

$$= \frac{mL^2}{12} + mx^2$$

$$= \frac{mL^2}{12} + m\left(\frac{L}{2}\right)^2$$

$$= \frac{ML^2}{12} + \frac{mL^2}{4}$$

$$= \frac{4mL^2}{12}$$

$$I = \frac{ML^2}{3}$$

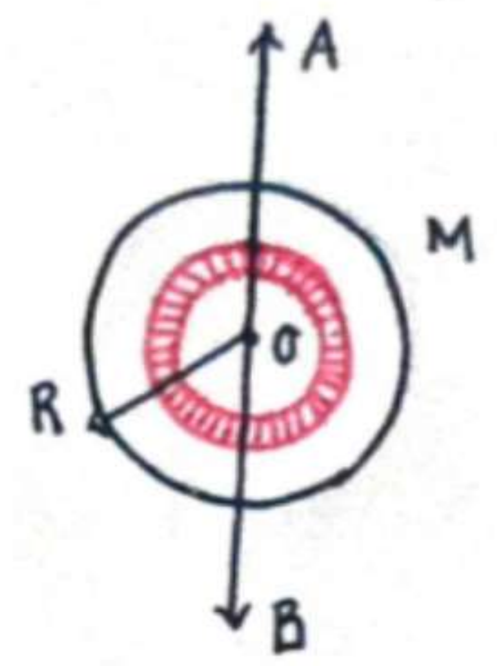
3.01.2022

2] circular disc:

case 1:

Let us consider thin circular disc of mass (m) and radius (r) The disc is free to rotate about a axes ab passing through centre o and perpendicular to the plane

consider a elementary disc with centre o and radius x. Let dx be the radial width



M.I of narrow strip $= Mx^2$

mass of the strip $= \frac{M}{\pi R^2} \cdot 2\pi x \cdot dx$

M.I of narrow strip $= \frac{M}{\pi R^2} \cdot 2\pi x^3 \cdot dx$

M.I of circular disc $= \int_0^R \frac{M}{\pi R^2} 2\pi x^3 dx$

$$= \frac{2M}{R^2}\int_0^R x^3 \cdot dx$$

$$= \frac{2M}{R^2}\left[\frac{x^4}{4}\right]_0^R$$

$$= \frac{\cancel{2}M}{R^2}\left[\frac{R^{\cancel{4}\,2}}{\cancel{4}_{2}}\right]$$

Density $= \frac{\text{mass}}{\text{Area}}$

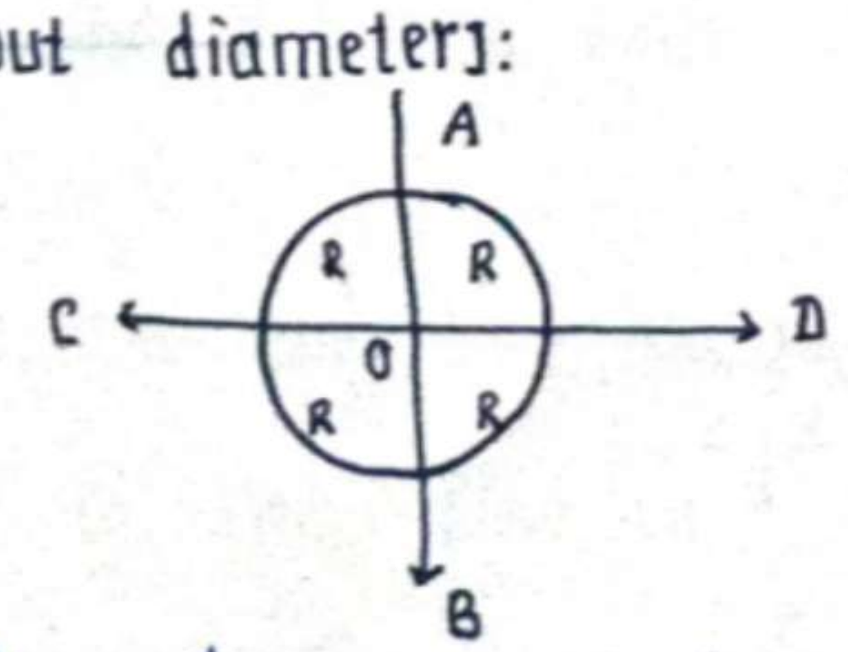
mass = Density x area

$= \left(\frac{M}{\text{Area}}\right) \times$ area of strip

$= \frac{M}{\pi R^2} \times 2\pi x\, dx$

M.I of circular disc $= \frac{MR^2}{2}$

case 2: [About diameter]:



Both diameter AB and CD are equal by using perpendicular axes theorem

$$I_z = I_x + I_y$$

since $I_x = I_y = I$

$$I_z = I + I$$

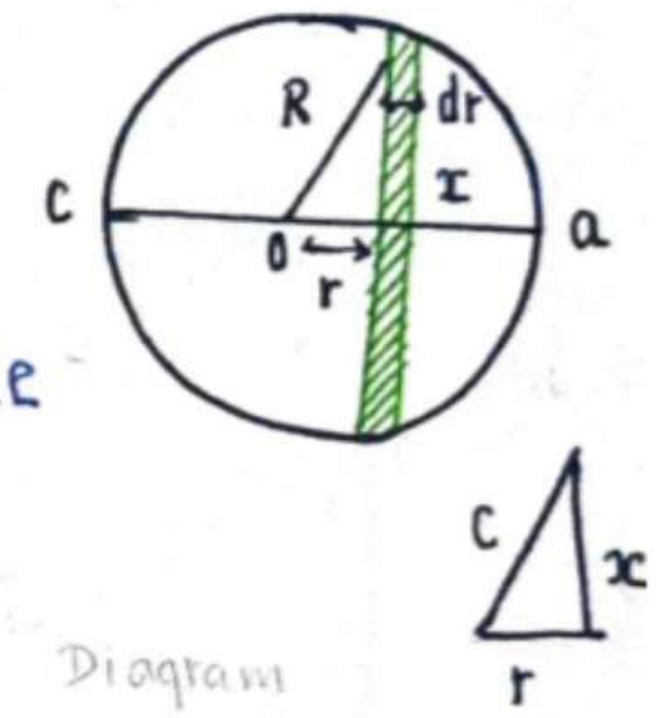
$$I_z = 2I$$

$$\frac{MR^2}{2} = 2I$$

$$I = \frac{MR^2}{4}$$

3) sphere

Let us consider a sphere of radius R and mass m with centre O let ρ be density of the sphere. Let us divide it into number of thin disc with diameter CD and consider one such elementary disc of thickness dr and radius x. Let small r be the distance from the centre O.



M.I of disc about CD = $\frac{MR^2}{2}$

mass of the elementary disc = density x volume

$$= \rho \pi x^3 dr$$

Radius of the disc $x \Rightarrow R^2 = x^2 + r^2$

$$x^2 = R^2 - r^2$$

$$x = \sqrt{R^2 - r^2}$$

Vol = l×b×h
Area
= $\pi x^2 \cdot dr$.

mass of the disc = $\rho \pi (R^2 - r^2) dr$

M.I of disc about CD = $\frac{\rho \pi (R^2 - r^2) dr . x^2}{2}$

M.I of a sphere = $\int_{-R}^{R} \frac{\rho \pi (R^2 - r^2) dr . x^2}{2}$

$$= \int_0^R \frac{2\pi \rho (R^2 - r^2) dr . x^2}{2}$$

$$= \int_0^R \frac{2\pi \rho (R^2 - r^2) dr \cdot (R^2 - r^2) dr}{2}$$

$$= \pi \varphi \int_0^R (R^2 - r^2) \cdot (R^2 - r^2)\, dr$$

$$= \pi \varphi \int_0^R (R^2 - r^2)^2\, dr$$

$$= \pi \varphi \int_0^R (R^2 - r^2)^2\, dr$$

$$= \pi \varphi \int_0^R (R^4 + r^4 - 2R^2 r^2)\, dr$$

$$= \pi \varphi \left\{ [R^4 r]_0^R + \left[\frac{r^5}{5}\right]_0^R - 2R^2 \left[\frac{r^3}{3}\right]_0^R \right\}$$

$$= \pi \varphi \left[R^5 + \frac{R^5}{5} - 2\frac{R^5}{3} \right]$$

$$= \pi \varphi \left[\frac{15R^5 + 3R^5 - 10R^5}{15} \right]$$

$$= \pi \varphi \frac{8R^5}{15}$$

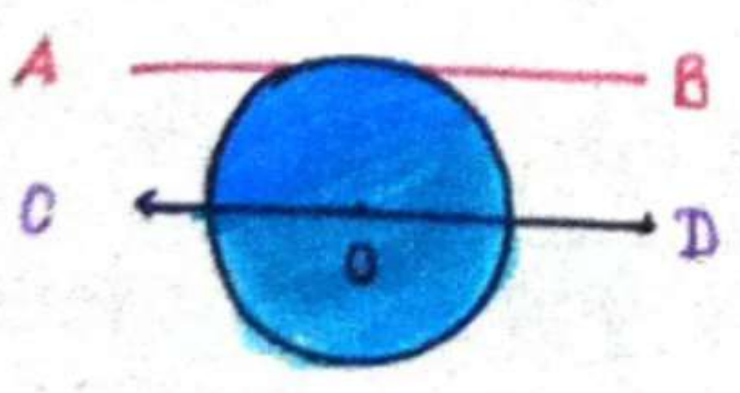
$$I = \pi \frac{M}{V} \frac{8R^5}{15}$$

$$= \pi \frac{M}{\frac{4}{3}\pi R^3} \frac{8R^5}{15}$$

$$= \frac{3\pi M\, 8R^5}{4\pi R^3\, 15}$$

$$= \frac{2MR^2}{5}$$

case 2:



By parallel axis theorem

$$I = I_G + MR^2$$

$$I = \frac{2}{5} MR^2 + MR^2$$

$$I = \frac{7}{5} MR^2$$

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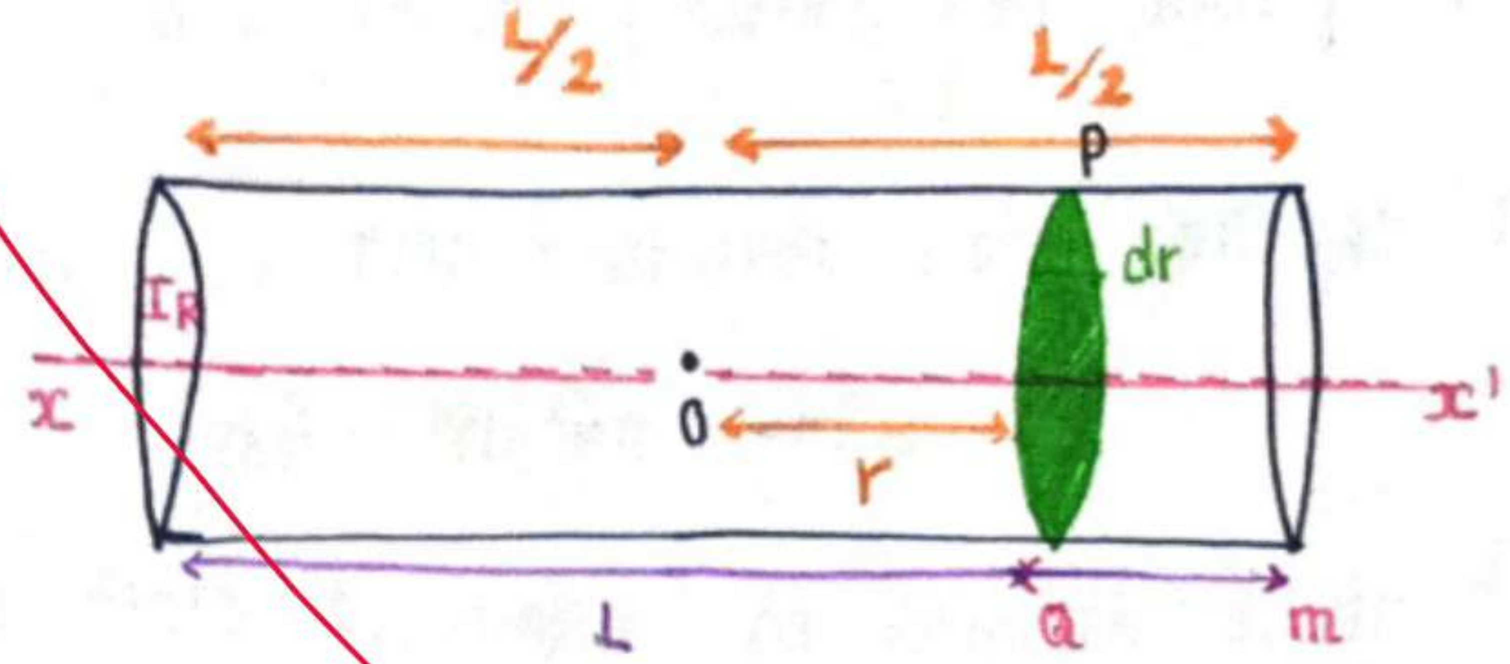
4] cylinder.



consider a solid cylinder of mass (M) radius (R) and length (L) The geometrical axis is x and x'. we consider one is of mass (m) which is at a distance r from centre o

centre 1: Along geometric axis

Density of like cylinder

$$\rho = \frac{M}{V} = \frac{M}{\pi R^2 L} \quad \text{--- (1)}$$

M.I of about geometric axis x, x'

$$= \frac{MR^2}{2}$$

M.I of the whole cylinder

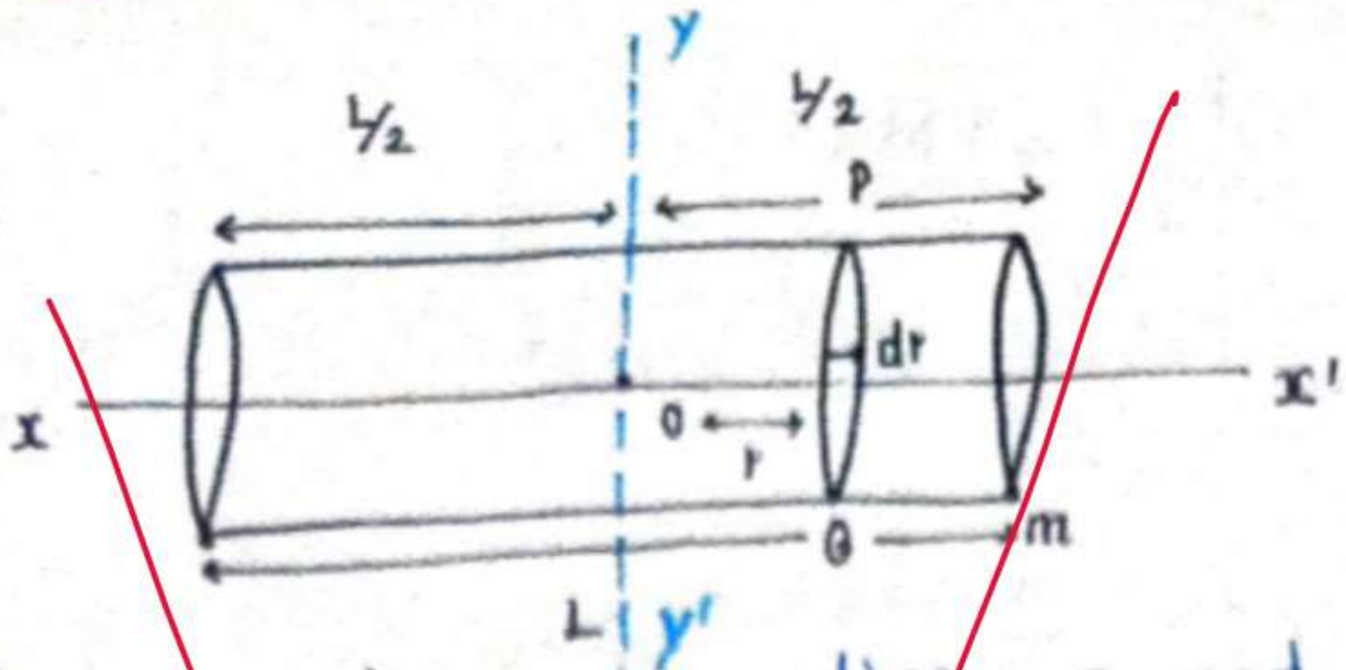
$$I_x = \Sigma \frac{1}{2} MR^2$$

$$= \frac{1}{2} R^2 \Sigma m$$

$$I_x = \frac{1}{2} MR^2 \quad \text{--- (2)}$$

case 2: M.I about an axis passing through centre and perpendicular to xx'

Let us consider an axis yy' perpendicular to geometrical axis to the xx' passing through the centre o of the cylinder.



consider one disc of radius R and thickness dr at a distance r from the axis.

M.I of the disc = density × area

$\Rightarrow m = \pi R^2 dr$ .. (4)

M.I about diameter $PQ = \frac{mR^2}{4}$ .. (3)

sub eq (4) in (3)

$PQ = \frac{\rho \pi R^4 dr}{4}$ .. (5)

using parallel axis theorem, about yy'

$$I = \frac{\pi R^4 \rho dr}{4} + mr^2$$

$$= \frac{\pi R^4 \rho dr}{4} + (\pi R^2 dr \rho) r^2$$

$$= \pi R^2 \rho dr \left[\frac{R^2}{4} + r^2\right] \text{ .. (6)}$$

M.I of whole cylinder,

$$I = \int_{-L/2}^{L/2} \pi R^2 \rho . \left[\frac{R^2}{4} + r^2\right] dr$$

$$I = \int_{0}^{L/2} 2\pi R^2 \rho \left[\frac{R^2}{4} + r^2\right] dr$$

$$= 2\pi R^2 \rho \int_{0}^{L/2} \left(\frac{R^2}{4} + r^2\right) dr$$

$$= 2\pi R^2 \rho \left[\frac{R^2 r}{4} + \frac{r^3}{3}\right]_0^{L/2}$$

$$= 2\pi R^2 \varphi \left[\frac{R^2 L}{4 \times 2} + \frac{L^3}{8 \times 3} \right]$$

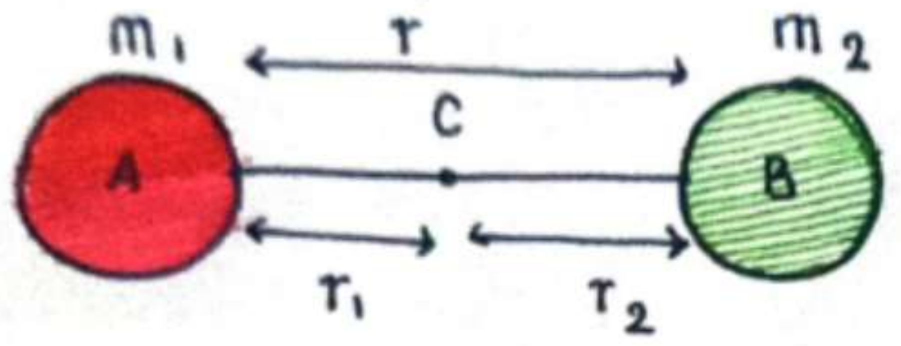
$$= \frac{2\pi R^2 \varphi L}{2} \left[\frac{R^2}{4} + \frac{L^2}{12} \right]$$

$$8 = \pi R^2 \varphi L \left[\frac{R^2}{4} + \frac{L^2}{12} \right]$$

From eq (1)

$$I = M \left(\frac{R^2}{4} + \frac{L^2}{12} \right)$$

M.I of a diatomic molecule



consider a diatomic molecule made up of two masses m_1 & m_2 which are separated by a distance (r) Let r_1, r_2 be the distance from the centre c

$$M.I = m_1 r_1^2 + m_2 r_2^2 \quad --(1)$$

$$r = r_1 + r_2 \quad --(2)$$

consider $m_1 r_1 = m_2 r_2$

$$r_1 = \frac{m_2 r_2}{m_1} \quad --(3)$$

From eq(2) $r_2 = r - r_1 \quad --(4)$

sub eq (4) in (3)

$$r_1 = \frac{m_2 (r - r_1)}{m_1} \Rightarrow m_1 r_1 = m_2 r - m_2 r_1$$

$$m_1 r_1 + m_2 r_1 = m_2 r$$

$$r_1 (m_1 + m_2) = m_2 r$$

$$r_1 = \frac{m_2 r}{m_1 + m_2} \quad --(5)$$

Similary: $r_2 = \dfrac{m_1 r}{m_1 + m_2} \quad --(6)$

sub (5) & (6) in eq (1)

$$M.I = m_1 \left[\frac{m_2^2 r^2}{(m_1+m_2)^2}\right] + m_2 \left[\frac{m_1^2 r^2}{(m_1+m_2)^2}\right]$$

$$M.I = \frac{m_1 m_2^2 r^2 + m_2 m_1^2 r^2}{(m_1+m_2)^2}$$

$$= \frac{m_1 m_2 r^2 \, \cancel{(m_2+m_1)}}{(m_1+m_2)^{\cancel{2}}}$$

$$= \frac{m_1 m_2}{m_1+m_2} r^2$$

Let $\frac{m_1 m_2}{m_1+m_2} = \mu$ where μ is the reduced mass of molecule

$$I = \mu r^2$$

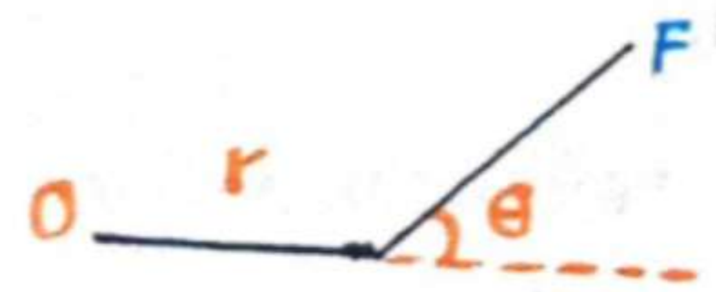
M.I of diatomic molecule about an axis passing through the centre & perpendicular to the bond length is the product of mass of molecule and square of the bond length.

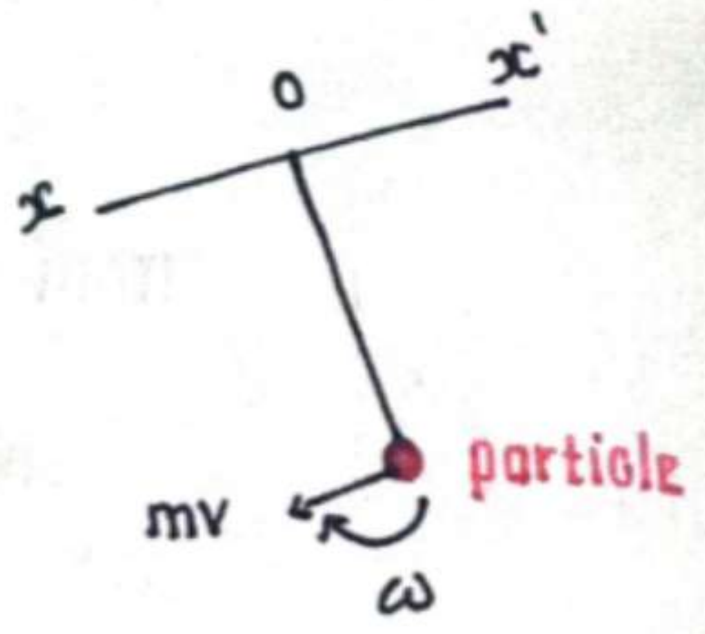
Torque (τ)

Rotational K.E contd in next page

The moment of the applied force is called torque. It is represented by the symbol 'τ'. The unit of torque is Nm

$$\tau = F \times r \sin\theta$$





Angular momentum :-

L = linear momentum x Distance between axis of rotation

$L = I\omega$

$\tau = \frac{dL}{dt}$

Angular momentum $= mv \times r$

$= mvr$

$= mr\omega(r)$

$L = mr^2\omega$

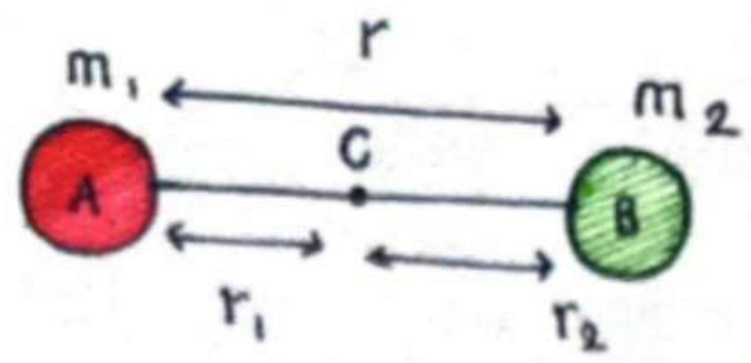
$L = I\omega$

$K.E = \frac{1}{2}mv^2$

$= \frac{1}{2}mr^2\omega^2$

$= \frac{1}{2}I\omega^2$

Rotational energy state of diatomic molecule



Let us consider a diatomic molecule ab with masses m_1 & m_2 joined by a rigid bond of length (r) $r = r_1 + r_2$ the molecule rotates about the point o which is centre of gravity (kinetic energy) of rotation of diatomic molecule $= \frac{1}{2} I\omega^2$ -- (1)

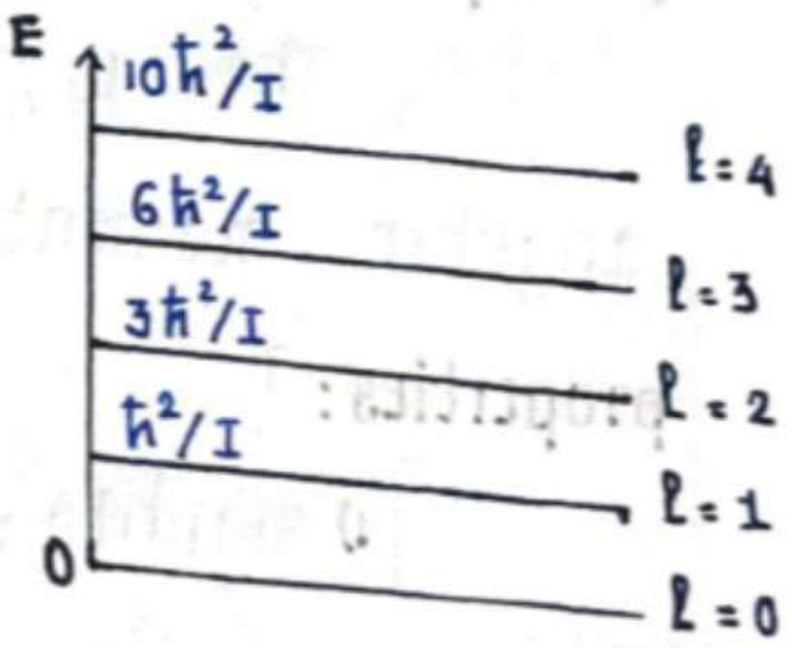
Mul & Divide by I $\Rightarrow$ $K.E(rot) = \frac{1}{2}\frac{I^2\omega^2}{I}$

$$= \frac{1}{2}\frac{L^2}{I} \quad -- (2)$$

where $L = I\omega$ is angular moment of rigid body

Atomic level the rotation leads to quantisation of angular momentum with values given by

$$L = \sqrt{l(l+1)}\,\hbar \quad -- (3)$$



Where $L = 0, 1, 2 \ldots$

$$\Delta E_{(rot)} = E_l - E_{lI}$$

sub eq (3) in (2)

$$K.E_{(rot)} = \frac{1}{2I} l(l+1)\hbar^2$$

$$= \frac{l(l+1)}{2I}\hbar^2 \quad -- (4)$$

The energy of emitted photon are absorbed photon for a transition between rotational energy states with angular momentum quantum number

$$\Delta E = E_l - E_{l-1}$$

$$\Rightarrow E_l = \frac{l(l+1)\hbar^2}{2I}$$

$$\Rightarrow E_{l-1} = \frac{(l-1)(l-1+1)}{2I}\hbar^2$$

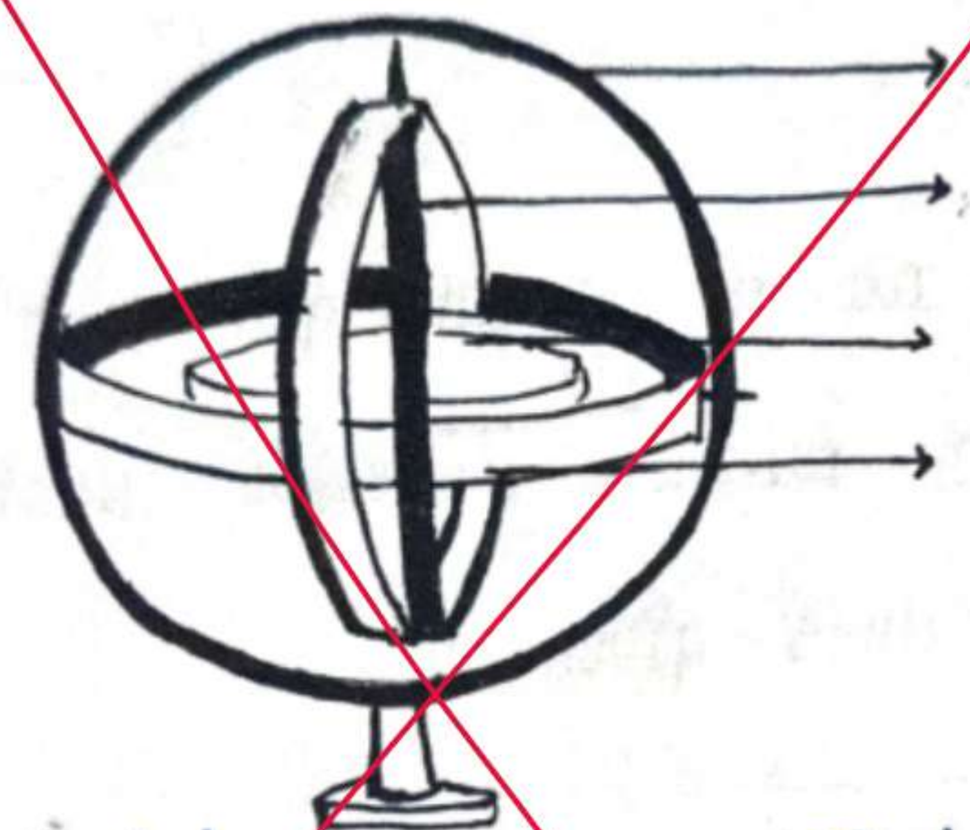
$$\Delta E = \frac{l(l+1)}{2I}\hbar^2 - \frac{(l-1)(l-1+1)\hbar^2}{2I}$$

$$\Delta E = \frac{\hbar^2 l}{I}$$

It is understood that the torque on energy increases directly with the energy levels are not equally spaced.

Gyroscope

A gyroscope is a device consisting of a wheel on disc that spins rapidly about an axis that is free to change direction



principle:

The principle of a gyroscope is based on conservation of angular momentum.

properties:

1) Rigidity : The spin axies remain in a fixed direction in space of no force is applied to it.

2) preccession : The spin axis will turn at right angle to the direction of applied force.

construction:

* The rotor is fixed on its supporting rings known as gimbals
* The rotor is isolated from the external torque with the help of gimbals
* The rotor has high stability as it maintain high speed rotation axis at the central rotation
* The rotor has 3 degrees of rotational freedom

Problems:

1) A circular disc whose radius is 0.5m and 25kg mass is rotating on its axis with 120 rev/minute calculate M.I of the disc and rotational kinetic energy

$\frac{1}{2}mv^2$

M.I of the disc = $\frac{MR^2}{2}$

$\frac{1}{2}mr^2\omega$

K.E rotational = $\frac{1}{2}I\omega^2$

$I\omega$

$\omega = 2\pi F$

$F = \frac{120}{60}$ revolution/sec $\Rightarrow$ 2Hz

M.I $= \frac{25(0.5)^2}{2} \Rightarrow 3.125$ kgm^2

$\omega = 2\pi(2) = 12.56$ radian/sec

K.E $= \frac{1}{2} I(12.56)^2 \Rightarrow 246.5$ joules

2) A flywheel of mass 20kg and radius 100 cm in acted on by torque 20 Nm Determine the angular acceleration

$\tau = I\alpha$

$I = Mr^2$

$= 20 \times 1 \times 1$

$\alpha = \frac{\tau}{I} \Rightarrow \frac{20}{20}$

$= 20$

$\alpha = 1$ rad/s^2

3) The angular position of a particle along a circle of radius 0.5m is given by function of time in seconds

$\theta = t^2 - 0.2t$

Find the Linear velocity of the particle at t=0 seconds

solution:

$V = r\omega$

$\omega = \frac{d\theta}{dt}$

$= 0.5\ (0.2)$

$\frac{d\theta}{dt} = 2t - 0.2$

$= 0.1$ m/s

$= 2(0) - 0.2$

$= -0.2$

4] The moon revolves around the earth in 2.4×10^6 sec. in a circular orbit of radius 3.9×10^5 Km. Determine the acceleration of the moon towards the earth

$T = 2.4 \times 10^6 S$

$r = 3.9 \times 10^5 \text{ km} \Rightarrow 3.9 \times 10^8 \text{ m}$

Angular velocity $\omega = \frac{2\pi}{T} = \frac{2\pi}{2.4 \times 10^6}$

$= 2.62 \times 10^{-6}$ radian /sec

Angular acceleration $a = r\omega^2$

$a = (3.9 \times 10^8)(2.4 \times 10^6)^2$

$a = 2.67716 \times 10^{-3}$

$a = 2.68 \times 10^{-3}$

5] The mass and radius of a solid circular disc are 500 kg and 1m calculate M.I about its axis

$I = \frac{1}{2} MR^2$

$= \frac{1}{2}(500)(1)$

$I = 250$ kg