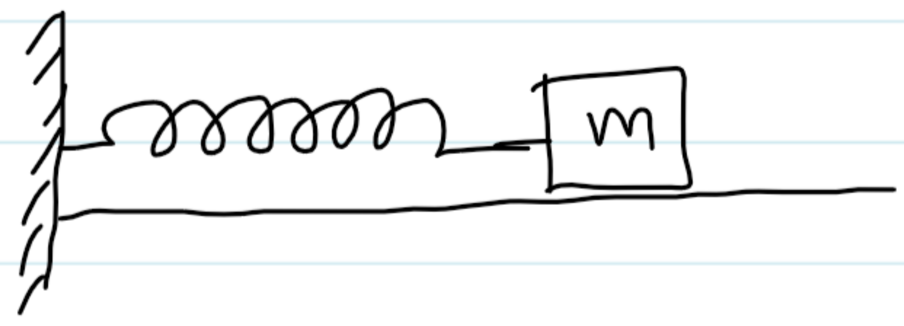
Quantum Simple harmonic oscillator

An example of a simple harmonic oscillator



is a point mass connected to a horizontal spring whose other end fixed and is on a frictionless surface.

The force acting on the point mass is $F = -kx$

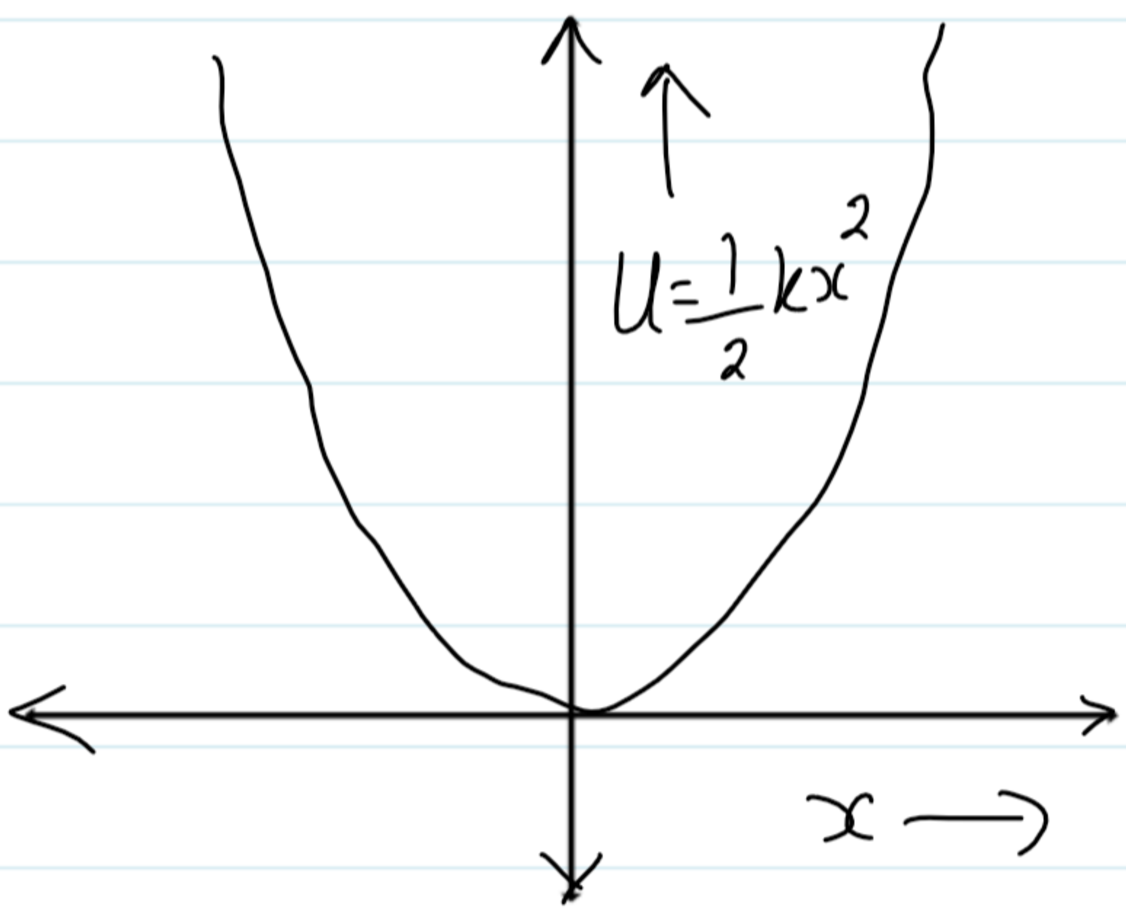
where k is the spring constant and F is the restoring force.

The potential energy $U = -\int F \, dx$

$$U = -\int -kx \, dx$$

$$U = \frac{1}{2} kx^2$$

The total energy of the harmonic oscillator is



$$E = K.E + P.E$$

$$= \frac{1}{2} m v^2 + \frac{1}{2} kx^2$$

$$E = \frac{1}{2} kA^2 = \text{Constant}$$

Where 'A' is the amplitude of oscillation.

As the potential is independent of time, one can solve the time independent Schroedinger's equation to obtain the Eigenfunction.

The one-dimensional time independent Schroedinger's equation is

$$\frac{d^2 \psi_0}{dx^2} + \frac{2m}{\hbar^2}(E - U)\psi_0 = 0$$

Rearranging the above equation we get

$$\frac{\hbar^2}{2m} \frac{d^2 \psi_0}{dx^2} + (E - U)\psi_0 = 0$$

$$\frac{\hbar^2}{2m}\frac{d^2\psi_0}{dx^2} + \left(E - \frac{1}{2}kx^2\right)\psi_0 = 0 \quad -(1) \qquad \left(\because u = \frac{1}{2}kx^2\right)$$

In order to solve the above equation, let us assume the following trial solution $\psi_0 = N e^{-ax^2}$ $-(2)$

where N is the normalization constant and 'a' is a constant.

Differentiating eq (2) wrt 'x' we get

$$\frac{d\psi_0}{dx} = N e^{-ax^2} \times (-2ax)$$

$$\frac{d\psi_0}{dx} = -2aN e^{-ax^2} \times x$$

$$\frac{d^2\psi_0}{dx^2} = -2aN \frac{d}{dx}\left(e^{-ax^2} \times x\right)$$

$$\frac{d^2\psi_0}{dx^2} = -2aN\left[x e^{-ax^2} \times (-2ax) + e^{-ax^2}\right]$$

$$\frac{d^2\psi_0}{dx^2} = -2aN e^{-ax^2}\left[-2ax^2 + 1\right] \quad -(3)$$

Substituting (3) and (2) in (1) we get

$$\frac{\hbar^2}{\cancel{2}m} \times -\cancel{2}a\cancel{N}\cancel{e^{-ax^2}}\left[-2ax^2+1\right] + \left(E - \frac{1}{2}kx^2\right)\cancel{N e^{-ax^2}} = 0$$

$$-\frac{a\hbar^2}{m}\left(-2ax^2+1\right) + E - \frac{1}{2}kx^2 = 0$$

$$E = \frac{a\hbar^2}{m}\left(1 - 2ax^2\right) + \frac{1}{2}kx^2$$

$$E = \frac{a\hbar^2}{m} - \frac{2a^2\hbar^2}{m}x^2 + \frac{1}{2}kx^2 \quad -(4)$$

As we know that the energy E is a constant, the second and third term on the

RHS must vanish.

This implies
$$-\frac{2a^2\hbar^2}{m}x^2 + \frac{1}{2}kx^2 = 0 \quad - (5)$$

$$-\frac{2a^2\hbar^2}{m} + \frac{1}{2}k = 0$$

$$\frac{2a^2\hbar^2}{m} = \frac{1}{2}k$$

$$a^2 = \frac{1}{4}\frac{mk}{\hbar^2} \quad - (6)$$

We know that $\omega_0^2 = \frac{k}{m}$, where ω_0 is the angular frequency of oscillation.

$$\omega_0^2 = \frac{k}{m}$$
$$k = m\omega_0^2 \quad - (7)$$

Substituting (7) in (6) we get

$$a^2 = \frac{1}{4}\frac{m \times m\omega_0^2}{\hbar^2}$$

$$a = \frac{1}{2}\frac{m\omega_0}{\hbar} \quad - (8)$$

Considering eq (5), eq (4) reduces to

$$E = \frac{a\hbar^2}{m} \quad - (9)$$

Substituting eq (9) in eq (8) we get

$$E = \frac{1}{2}\frac{m\omega_0}{\hbar} \times \frac{\hbar^2}{m}$$

$$\boxed{E = \frac{1}{2}\hbar\omega}$$

$E = \frac{1}{2}\hbar\omega$ is the lowest energy level of a quantum harmonic oscillator.

The next energy levels are given as

$$E = \left(n + \frac{1}{2}\right)\hbar\omega \quad \text{where } n = 0, 1, 2, 3, \ldots$$

$$E_0 = \frac{1}{2}\hbar\omega, \quad E_1 = \frac{3}{2}\hbar\omega, \quad E_2 = \frac{5}{2}\hbar\omega, \ldots$$