Unit-III

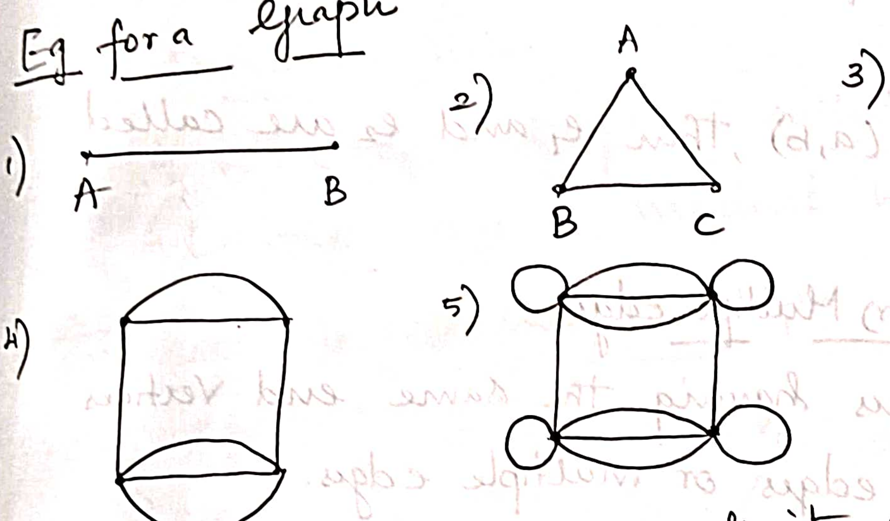
Graph Theory

Definition :- Graph

A Graph $G = (V, E)$ consists of a non-empty set V of elements called Vertices and a set E of elements called edges.

End points :- Each edge is associated with one or more Vertices, called its end points.

Eg for a graph



Note :- 1) The set V may be finite or infinite.

2) If V is finite then the graph G is a finite graph.

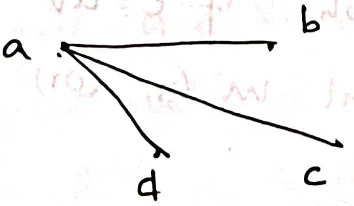
3) If V is infinite then the graph G is an infinite graph.

4) $|V|$ — No. of Vertices of V.

5) Edge set E may be empty.

Ex Draw a graph $G = (V, E)$ where $V = \{a, b, c, d\}$

$$E = \{e_1 = ab, e_2 = ac, e_3 = ad\}$$



Def:- (p, q) graph:-

A graph $G = (V, E)$ with p vertices and q edges is called a (p, q) graph.

Def:- Trivial graph:-

A graph with only one vertex and no edges is called a trivial graph. ($(1, 0)$ graph)

Def:- loop:-

If the end vertices of an edge coincide, then the edge is called a loop. (ie) $e = (a, a)$ is a loop.

Def:- parallels:-

If $e_1 = (a, b)$ $e_2 = (a, b)$, then e_1 and e_2 are called parallels.

Def:- parallel Edges (or) Multiple edges

Two or more edges having the same end vertices are called parallel edges or multiple edges.

Def:- Simple graph:-

A graph $G = (V, E)$ without loops and without parallel edges is called a simple graph.

Def:- pseudo graph:-

Graphs having loops and multiple edges is called a pseudograph.

Def:- Adjacent:-

Let $G = (V, E)$ be a graph. If $e = uv$ is an edge of G then u and v are adjacent in G. (or) e is incident with u and v.

Def:- Isolated Vertex :-

A vertex which is not adjacent to every other vertex is called an isolated vertex.

Def:- Null graph :-

~~A vertex which is not adjacent to every other~~

A graph having only isolated vertices (and no edges) is called a null graph.

Def:- Adjacent edges :-

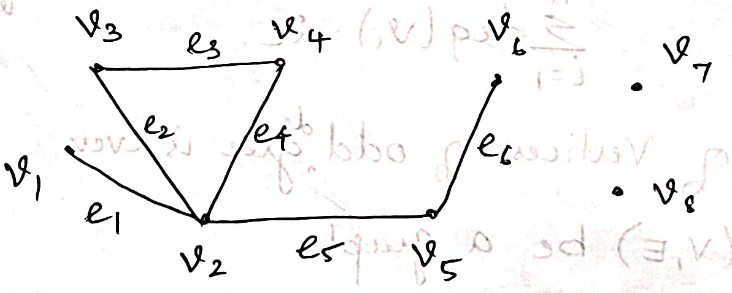
If two or more edges are incident with the same vertex, then they are said to be adjacent edges.

Def:- Pendant Vertex

A vertex of a graph is called a pendant vertex if only one edge is incident with it.

Ex:-

Consider the following graph



$v_1, v_6 \rightarrow$ pendant vertices

$v_7, v_8 \rightarrow$ isolated vertices

$v_1, v_2; v_2, v_3; v_2, v_4; v_3, v_4; v_2, v_5; v_5 v_6 \rightarrow$ adjacent vertices.

e_1, e_2, e_4, e_5 are adjacent edges as they are all incident with v_2.

Def:- Degree of a vertex

The degree of a vertex in a graph G is the number of edges incident with it. A loop of a vertex contribute degree 2 to a vertex.

Note: 1) If v is an isolated vertex $\deg(v) = 0$

2) If v is a pendant vertex $\deg(v) = 1$

(QB) Hand shaking Theorem

Thm: Let $G = (V, E)$ be a graph and let e denote the number of edges and let $v_1, v_2, \dots v_n$ be n vertices. Then

$$\sum_{i=1}^{n} \deg(v_i) = 2e.$$

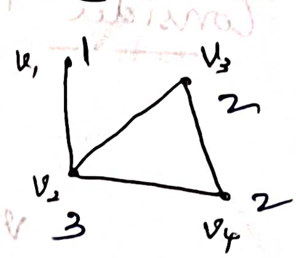
Given: Let $G = (V, E)$ be a graph. Let e denote the no of edges. Let $v_1, v_2, \dots v_n$ be n vertices.

Proof: Every non-loop edge is incident with two vertices and so contributes 2 to the degree. Every loop edge contributes 2 to the degree.

$\therefore$ Every edge (loop or not) contributes 2 to the sum of degrees of the vertices. So all the e edges contribute $2e$ degrees.

$\therefore$ sum of the degrees of vertices $= 2e$

$$\sum_{i=1}^{n} \deg(v_i) = 2e.$$



Thm (QB) The number of vertices of odd degree is even.

Proof: Let $G = (V, E)$ be a graph

$8 = 2(4)$

Let V_1 and V_2 be the set of vertices of even degree and the set of vertices of odd degree respectively in G.

$$\text{Then } \sum_{i=1}^{n} \deg(v_i) = 2e$$

$$\sum_{v \in V_1} \deg(v_i) + \sum_{v \in V_2} d(v) = 2e$$

$$2k + \sum_{v \in V_2} d(v) = 2e \quad (\because \text{Sum of even degree is even})$$

$$\sum_{v \in V_2} d(v) = 2e - 2k = 2(e-k), \text{ even}$$

Since each $d(v)$ is odd, each term is odd and their sum is even and hence the number of terms must be even.

Hence there are even number of vertices of odd degree

Some special simple graphs

Def:-

Complete Graph:-

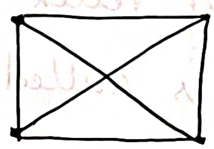
A simple graph is called a Complete graph if there is exactly one edge between every pair of vertices.

A Complete graph on n vertices is denoted by K_n.

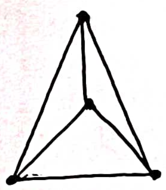
K_1

K_2

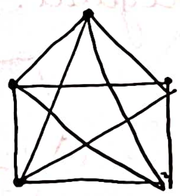
K_3



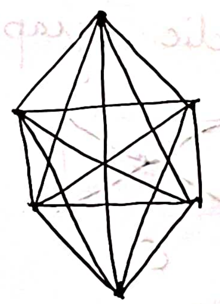
K_4



K_4



K_5



K_6

Note:- 1) No. of edges of $K_n = nC_2 = \frac{n(n-1)}{2}$

Degree of each vertex is $n-1$

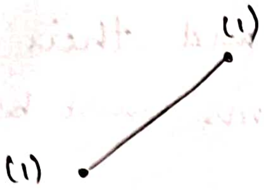
Ex In K_6, No. of edges of $K_6 = 6C_2 = \frac{6\times5}{2\times1} = 15$

Degree of each vertex $= n-1 = 6-1 = 5$

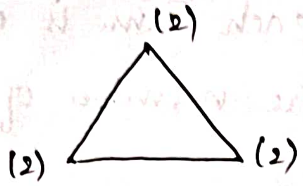
2) No. of edges in K_n = No. of handshakes.

Def:- Regular graphs

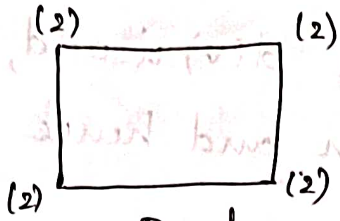
A simple graph is called regular if every vertex of the graph has the same degree. The degree of each vertex is called the degree of the graph



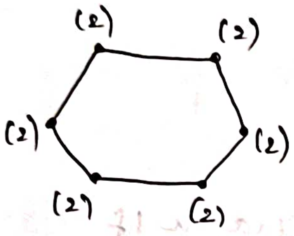
1-Regular



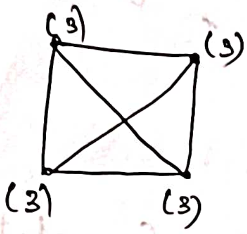
2-Regular



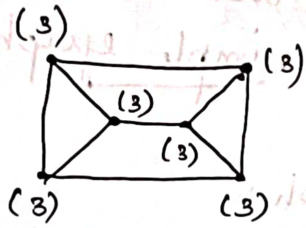
2-Regular



2-Regular



3-Regular



3-Regular.

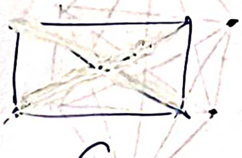
Def:- Cycles (or) Cyclic graph or circuit

If every vertex of a simple graph G is of degree 2, then G is called a cycle or Cyclic graph or Circuit

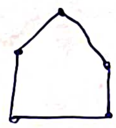
A cycle with n vertices is denoted by $C_n, n \geq 3$.

Thus a cyclic graph is 2-regular.

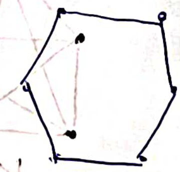
C_3



C_4



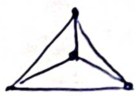
C_5



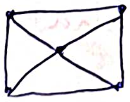
Def:- Wheel graph W_n $(n \geq 3)$

W_n is obtained from C_n by adding a vertex v inside C_n and connecting v to every vertex of C_n by new edges.

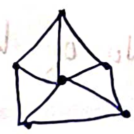
The wheels W_3, W_4, W_5 and W_6 are shown below.



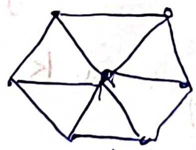
W_3



W_4



W_5



W_6

Note:- 1) All complete graphs are regular.

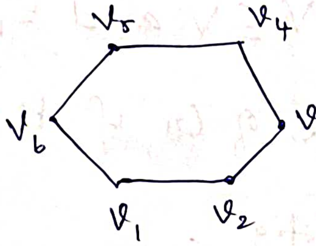
2) All regular graphs need not be complete

3) Wheels are not regular graph & not a complete graph

Bipartite graph (Bi-graph)

A simple graph G is called bipartite graph if its vertex set V can be partitioned into two disjoint non-empty sets V_1 and V_2 such that every edge in G connects a vertex in V_1 and a vertex in V_2.

Ex.1 C_6 is Bipartite : $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

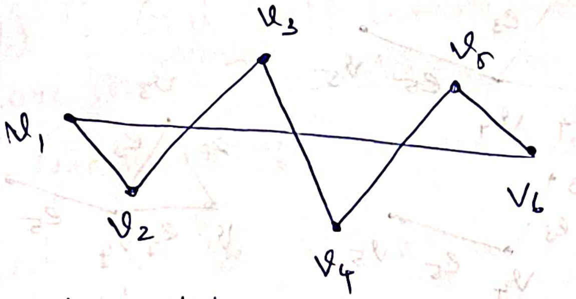


We can partition it as

$V_1 = \{v_1, v_3, v_5\}$

$V_2 = \{v_2, v_4, v_6\}$

Then the bipartite graph is



So C_6 is bipartite.

Ex-2 Is K_3 bipartite? No.

Complete Bipartite graph:

Let $G = (V, E)$ be a bipartite graph with bipartition (V_1, V_2). If there is an edge of G connecting every vertex in V_1 and in V_2 then G is called a complete bipartite graph.

If $|V_1| = m$, $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m,n}$, where m and n are the number of vertices in V_1 and in V_2 respectively.

Note In a complete bigraph $K_{m,n}$

1) The number of vertices is $m+n$
2) The number of edges is $m \times n$.
3) $K_{m,n}$ is not regular if $m \neq n$.

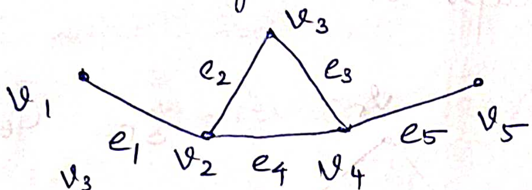
~~Prob Draw the complete graph K_5 with Vertices A, B, C, D, E. Draw all complete subgraphs of K_5 with 4 Vertices.~~ Ex Draw a complete bipartite graph of $K_{3,3}$, $K_{2,3}$

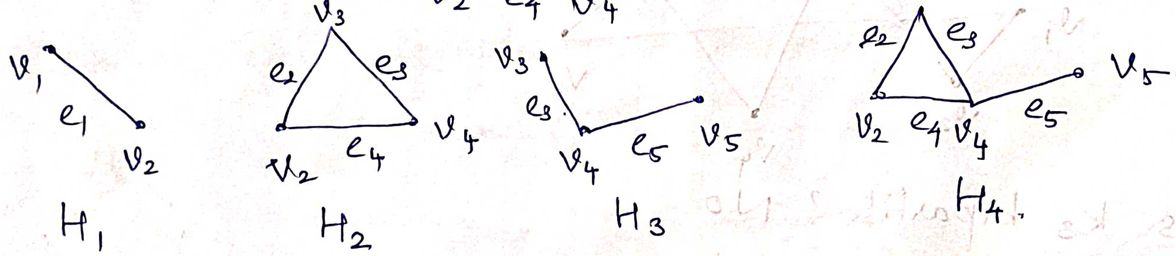
Def :- Subgraph :

Let $G = (V, E)$ be a graph. A graph $H = (V_1, E_1)$ is called a subgraph of G if $V_1 \subseteq V$ and $E_1 \subset E$.

H is a proper subgraph of G if $H \neq G$.

Ex Let G be the graph in fig



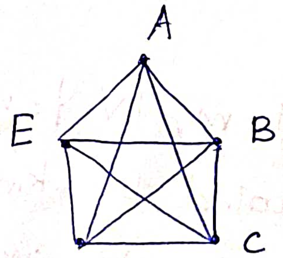


QB

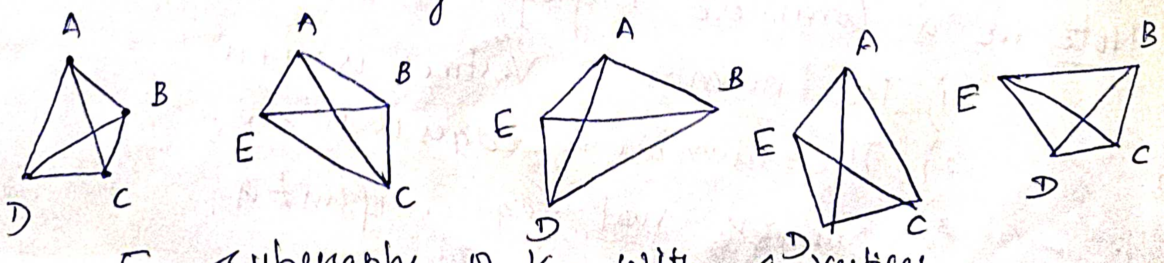
$H_1, H_2, H_3, H_4 \rightarrow$ Subgraphs of G.

Prob :- Draw the complete graph K_5 with Vertices A, B, C, D, E. Draw all complete subgraphs of K_5 with 4 Vertices.

Solution Complete graph K_5 $\longrightarrow$



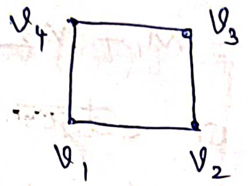
Omit a vertex, The edges with it will also be omitted.



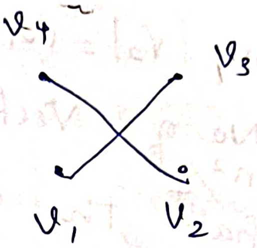
5 subgraphs of K_5 with 4 Vertices.

Let G be a simple graph, then the graph $\overline{G}$ is called the complement graph of G if the vertex set of $\overline{G}$ is same as that of G and such that two vertices of $\overline{G}$ are adjacent iff those vertices are non-adjacent in G.

Eg.



G



$\overline{G}$

$|V(G)| = |V(\overline{G})|$

$|E(G)| = |E(\overline{G})|$

Note :- 1) $G \cup \overline{G} = K_n$.

QB 2) If G is self complementary, then ~~$|V_1| = |V_2|$~~

Thm Prove that the maximum number of edges in a simple graph with n vertices is $nC_2 = \frac{n(n-1)}{2}$.

Proof :- Let G be a simple graph

By hand shaking theorem,

$$\sum_{i=1}^{n} d(v_i) = 2e$$

where e is the number of edges with n vertices in the graph G.

(ie) $d(v_1) + d(v_2) + \cdots + d(v_n) = 2e \longrightarrow (1)$.

Since we know that the maximum degree of each vertex in the graph G can be $n-1$

$(1) \Rightarrow (n-1) + (n-1) + \cdots + n \text{ times} = 2e$

$$n(n-1) = 2e$$

$$\Rightarrow e = \frac{n(n-1)}{2}.$$

Hence the maximum number of edges with n vertices in a simple graph is $\frac{n(n-1)}{2}$.

Hence Proved

QB

Thm:- prove that the number of edges in a bipartite graph with n vertices atmost $\frac{n^2}{4}$.

Soln Given:- Let G be a bipartite graph with n vertices

Let V_1 and V_2 be the bipartition of G

Let $|V_1| = m_1$, $|V_2| = m_2$

No. of vertices $n = m_1 + m_2$

T.P:- $e \leq n^2/4$

We know that the number of edges of a bipartite graph is maximum when it is complete bipartite graph $K_{m,n}$.

So maximum no. of edges of $G \leq mn$

$$\Rightarrow e \leq mn \longrightarrow (2)$$

W.K.T $\quad \frac{m+n}{2} \geq \sqrt{mn} \quad$ (A.M $\geq$ G.M)

squaring on both sides,

$$\left(\frac{m+n}{2}\right)^2 \geq \left(\sqrt{mn}\right)^2$$

$$\frac{(m+n)^2}{4} \geq mn \geq e \longrightarrow (3)$$

(from (2))

$$\Rightarrow e \leq \frac{(m+n)^2}{4} = \frac{n^2}{4}$$

$$\Rightarrow e \leq \frac{n^2}{4}.$$

Hence the number of edges in a bipartite graph with n vertices is atmost $\frac{n^2}{4}$

(Q8)

Thm Prove that any self complementary graph has $4n$ or $4n+1$ vertices. $n \equiv 0$ or $1 \pmod 4$

Proof: Let $G = (V, E)$ be a self complementary graph with $\not{p}^{n}$ vertices.

T.P $\not{p}^{n} = 4p$ or $4p+1$.

Since G is self complementary,

$$|V(G)| = |V(\bar{G})| \text{ \& } |E(G)| = |E(\bar{G})| \longrightarrow (1)$$

W.K.T $G \cup \bar{G} = K_p$, K_p – Complete graph with p vertices.

$$\therefore |E(K_p)| = \not{p}^{n}C_2 = \frac{p(p-1)}{2} \quad \left(\frac{n(n-1)}{2}\right)$$

$$\Rightarrow |E(G)| + |E(\bar{G})| = \frac{p(p-1)}{2} \quad \left(\frac{n(n-1)}{2}\right)$$

$$|E(G)| + |E(G)| = \frac{p(p-1)}{2} \quad (\text{by } (1))$$

$$2\,|E(G)| = \frac{p(p-1)}{2} \quad \left(\frac{n(n-1)}{2}\right).$$

$$|E(G)| = \frac{p(p-1)}{4} \quad \frac{n(n-1)}{4}$$

Since one of $\not{p}^{n}$ or $\not{p}^{n}-1$ must be odd.

4 must divide one of $\not{p}^{n}$ or $\not{p}^{n}-1$.

$\therefore \not{p}^{n}$ or $\not{p}^{n}-1$ is a multiple of 4

$\Rightarrow \not{p}^{n} = 4p$ or $\not{p}^{n}-1 = 4p \Rightarrow \not{p} = 4p+1$

Hence $\not{p}^{n} = 4p$ (or) $4p+1$

(ie) $n \equiv 0 \pmod 4$

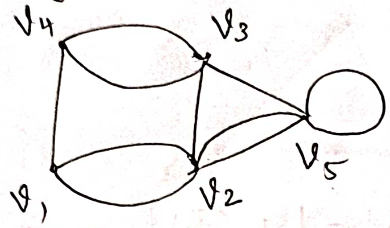
(or) $n \equiv 1 \pmod 4$

Hence proved.

Degree sequence of a graph

The degree sequence of a graph G is the sequence of degrees of the vertices of G in a non-increasing order.

Ex



$d(v_1) = 3$

$d(v_2) = 5$

$d(v_3) = 4$

$d(v_4) = 3$

$d(v_5) = 5$

$\therefore$ degree sequence $= 5, 5, 4, 3, 3$.

Matrix representation of Graphs

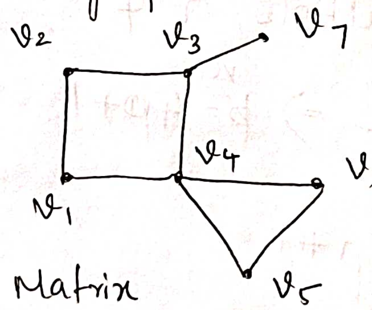
Adjacency Matrix

Let $G = (V, E)$ be a simple graph with n vertices. Let the vertices of G be denoted as $v_1, v_2, v_3, \ldots v_n$. The adjacency matrix of G is the $n \times n$ matrix (a_{ij})

where $a_{ij} = \begin{cases} 1, & \text{if } v_i \,\&\, v_j \text{ are adjacent.} \\ 0, & \text{otherwise.} \end{cases}$

and it is denoted by A (or) A_G (or) $A(G)$.

Ex Consider the graph

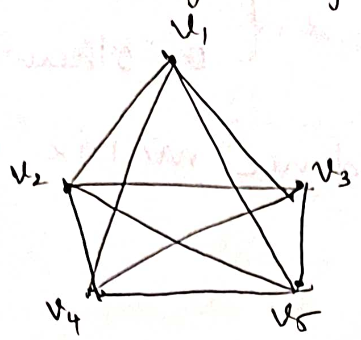


$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array}
\begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \end{array} \\
\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Note

1) Adjacency Matrix is a binary matrix
2) Symmetric matrix with all diagonal elements 0.
3) Row total or Col. total is the degree of the corresponding vertex.
4) A is called a bit Matrix or Boolean matrix.

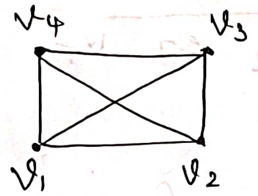
Ex Find the adjacency matrix of K_5



$$\begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 1 & 1 & 1 & 1 \\ v_2 & 1 & 0 & 1 & 1 & 1 \\ v_3 & 1 & 1 & 0 & 1 & 1 \\ v_4 & 1 & 1 & 1 & 0 & 1 \\ v_5 & 1 & 1 & 1 & 1 & 0 \end{array}$$

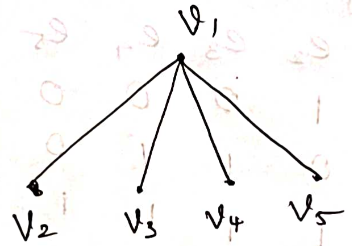
Ex Represent each of the following graphs with an adjacency matrix K_4, $K_{1,4}$, C_4, W_4

Solution: 1) K_4



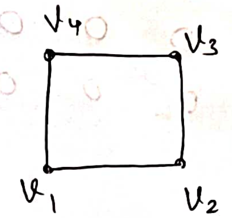
$$A = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 1 & 1 \\ v_2 & 1 & 0 & 1 & 1 \\ v_3 & 1 & 1 & 0 & 1 \\ v_4 & 1 & 1 & 1 & 0 \end{array}$$

2) $K_{1,4}$



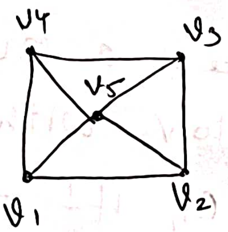
$$A = \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 1 & 1 & 1 & 1 \\ v_2 & 1 & 0 & 0 & 0 & 0 \\ v_3 & 1 & 0 & 0 & 0 & 0 \\ v_4 & 1 & 0 & 0 & 0 & 0 \\ v_5 & 1 & 0 & 0 & 0 & 0 \end{array}$$

3) C_4



$$A = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 0 & 1 \\ v_2 & 1 & 0 & 1 & 0 \\ v_3 & 0 & 1 & 0 & 1 \\ v_4 & 1 & 0 & 1 & 0 \end{array}$$

4) W_4



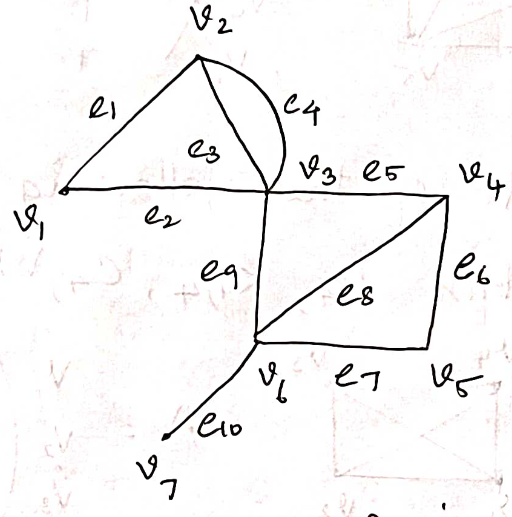
$$A = \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 1 & 0 & 1 & 1 \\ v_2 & 1 & 0 & 1 & 0 & 1 \\ v_3 & 0 & 1 & 0 & 1 & 1 \\ v_4 & 1 & 0 & 1 & 0 & 1 \\ v_5 & 1 & 1 & 1 & 1 & 0 \end{array}$$

Incidence Matrix

Def: Let $G = (V, E)$ be a (n, m) graph. Let $v_1, v_2 \dots v_n$ be the vertices and $e_1, e_2 \dots e_m$ be the edges of G. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix.

$M = (m_{ij})$ where $m_{ij} = \begin{cases} 1, & \text{when the edge } e_j \text{ is incident with } v_i \\ 0, & \text{otherwise} \end{cases}$

<u>Ex</u> Write down the incidence matrix of the graph G given below



The incidence matrix of G is

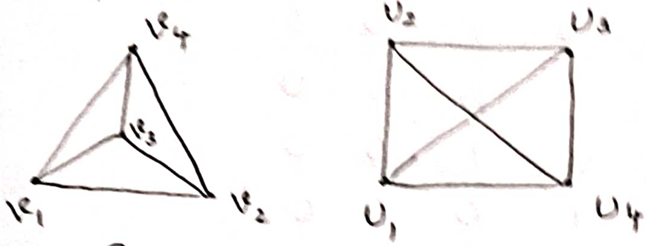
$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{cccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

<u>Def</u>: Isomorphism of Graphs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. G_1 and G_2 are isomorphic if there is a one to one and onto function or map f from V_1 to V_2 with the property that a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2 $\forall\ a, b \in V_1$.

The function f is called isomorphism.

(Q8) Examine whether the following pair of graphs are isomorphic. If not isomorphic, give reasons.



G_1 G_2

Sln Let $V_1 = \{v_1, v_2, v_3, v_4\}$ $V_2 = \{u_1, u_2, u_3, u_4\}$

$|V_1| = 4, \; |V_2| = 4$

$|E_1| = 6, \; |E_2| = 6$

The degree sequence of G_1 is $3,3,3,3$.

The degree sequence of G_2 is $3,3,3,3$.

Define $f : G_1 \to G_2$ by $f(v_i) = u_i$

The mapping preserves adjacency & it is one-one & onto.

Hence $G_1 \approx G_2$

(or)

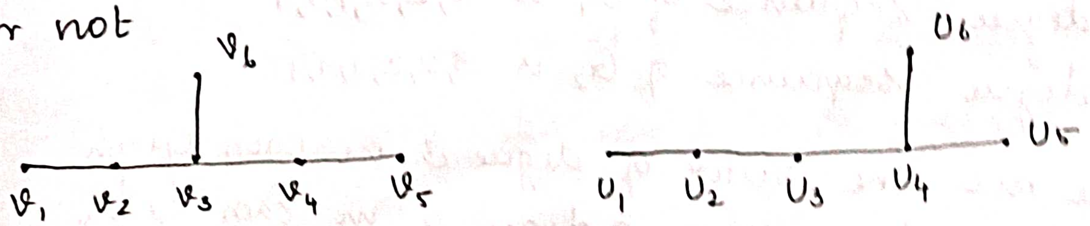
$$A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array} \quad A(G_2) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

$A(G_1)$ & $A(G_2)$ are similar matrices.

(ie) $A(G_1) \sim A(G_2)$

$\therefore$ G_1 and G_2 are isomorphic

ii) Examine whether the following graphs are isomorphic or not



G_1 G_2

$$A(G_1) = \begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & & & & & & \\ v_2 & & & & & & \\ v_3 & & & & & & \\ v_4 & & & & & & \\ v_5 & & & & & & \\ v_6 & & & & & & \end{array}$$

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$A(G_2) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Soln:- Given:- Let G_1 and G_2 be the graph with 6 Vertices.

Let $V_1 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

$V_2 = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be the set of Vertices in G_1 and G_2 respectively.

$|V_1| = 6$, $|V_2| = 6$

$|E_1| = 5$, $|E_2| = 5$

The degree sequence of G_1 is 3,2,2,1,1,1

The degree sequence of G_2 is 3,2,2,1,1,1

There are one Vertex of degree 3 in each Graph.

There are two Vertices of degree 2 in each graph

There are three Vertices of degree 1 in each graph

But Consider the Vertex v_3, in G_1 the degree of the adjacent vertexes are 2, 2, 1.

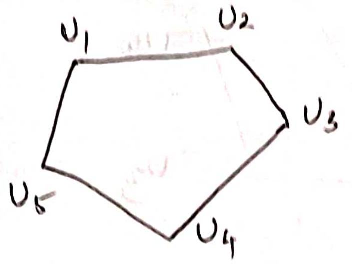
Consider the Vertex U_4 in G_2

choose a Vertex v_3 in G_1, It has degree 2 and the three adjacent Vertices of v_3 having the degree sequence (1, 2, 2). But in G_2, U_4 only the third degree vertex and the degree sequence is (1, 1, 2)

$\therefore$ v_3 is not mapped to any of the Vertex in G_1,

$\Rightarrow$ G & H are not isomorphic.

Ex When do we say that two graphs are isomorphic? Examine whether the following graphs are isomorphic or not.



G

H

Solution

Let G & H be the graph with 5 Vertices.

Let $V_1 = \{U_1, U_2, U_3, U_4, U_5\}$
$V_2 = \{v_1, v_2, v_3, v_4, v_5\}$ be the set of Vertices in G & H respectively.

$|V_1| = 5, |V_2| = 5$

$|E_1| = 5, |E_2| = 5$.

The degree sequence of G is 2, 2, 2, 2, 2

The degree sequence of H is 2, 2, 2, 2, 2.

There are 5 Vertices of degree 2 in G & H.

(Consider, a vertex U_1 in G_1. It has degree 2 and the adjacent vertices of U_1 having degree sequence $(2,2)$) X

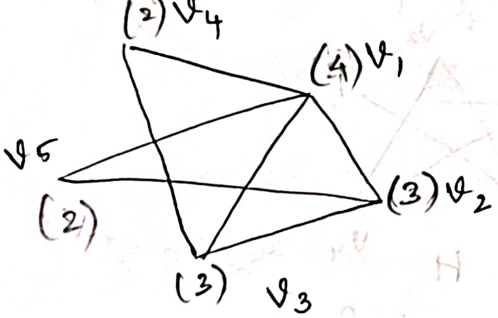
Define $f: G \to H$ by $f(U_i) = V_i$

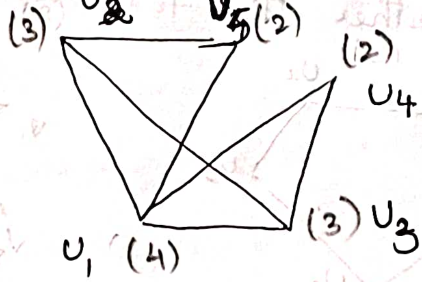
The mapping preserves adjacency and it is 1-1 & onto

$\therefore$ f is isomorphism b/w G & H

Hence $G \simeq H$.

Ex Examine wheather the following pair of graphs are isomorphic. If isomorphic, label the vertices of the two graphs to show that their adjacency matrices are same.





$\{4,3,3,2,2\}$ $\{4,3,3,2,2\}$

$$\begin{array}{c|ccccc} & V_1 & V_2 & V_3 & V_4 & V_5 \\ \hline V_1 & 0 & 1 & 1 & 1 & 1 \\ V_2 & 1 & 0 & 1 & 0 & 1 \\ V_3 & 1 & 1 & 0 & 1 & 0 \\ V_4 & 1 & 0 & 1 & 0 & 0 \\ V_5 & 1 & 1 & 0 & 0 & 0 \end{array} \qquad \begin{array}{c|ccccc} & U_1 & U_2 & U_3 & U_4 & U_5 \\ \hline U_1 & 0 & 1 & 1 & 1 & 1 \\ U_2 & 1 & 0 & 1 & 0 & 1 \\ U_3 & 1 & 0 & 0 & 1 & 0 \\ U_4 & 1 & 0 & 1 & 0 & 0 \\ U_5 & 1 & 1 & 0 & 0 & 0 \end{array}$$

Let $V_1 = \{V_1, V_2, V_3, V_4, V_5\}$

$V_2 = \{U_1, U_2, U_3, U_4, U_5\}$ be the vertices of G_1 & G_2 respectively.

$|V_1| = 5$, $|V_2| = 5$

$|E_1| = 7$, $|E_2| = 7$.

The degree sequence of G_1 is 4, 3, 3, 2, 2.

The degree sequence of G_2 is 4, 3, 3, 2, 2.

There are 1 vertices of degree 4.

There are 2 vertices of degree 3

There are 2 vertices of degree 2.

Define $f : G_1 \rightarrow G_2$ by $f(u_i) = v_i$

The mapping preserves adjacency & it is 1-1 & onto.

f is isomorphism b/w G_1 & G_2

Hence $G_1 \simeq G_2$.

<u>Thm</u> P.T the simple graph with n vertices and k components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges.

<u>proof</u>: Let $n_1, n_2, n_3 \ldots n_k$ be the number of vertices in each k components of the graph G.

Then $n_1 + n_2 + \ldots + n_k = n = |V(G)|$

$$\sum_{i=1}^{k} n_i = n$$

Now, $$\sum_{i=1}^{k} (n_i - 1) = (n_1 - 1) + (n_2 - 1) + \ldots + (n_k - 1)$$

$$= \sum_{i=1}^{n} n_i - (1+1+1+\ldots k\text{ times})$$

$$= \sum_{i=1}^{n} n_i - k$$

$$\sum_{i=1}^{k} (n_i - 1) = n - k$$

$[(n_1-1)+(n_2-1)+\ldots(n_k-1)]^2 = (n-k)^2$

Squaring on both sides

$(n_1-1)^2+(n_2-1)^2+\ldots+(n_k-1)^2 \leq (n-$

$$\left(\sum_{i=1}^{k}(n_i - 1)\right)^2 = (n-k)^2$$

$$[(n_1-1)^2 + (n_2-1)^2 + \ldots + (n_k-1)]^2 = (n-k)^2$$

$$(n_1-1)^2 + (n_2-1)^2 + \ldots + (n_k-1)^2 \leq (n-k)^2 = n^2 + k^2 - 2nk$$

$$n_1^2 - 2n_1 + 1 + n_2^2 - 2n_2 + 1 + \ldots + n_k^2 - 2n_k + 1 \leq n^2 + k^2 - 2nk$$

$$\sum n_i^2 + k - 2n \leq n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} n_i^2 \le n^2+k^2-2nk-k+2n$$

$$\le n^2+k^2-k-2n(k-1)$$

$$\le n^2+k(k-1)-2n(k-1)$$

$$\le n^2+(k-1)(k-2n) \longrightarrow (1)$$

Since G is simple, the maximum number of edges of G in k components is $\frac{n_i(n_i-1)}{2}$.

$$\text{Maximum no. of edges of } G = \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$$

$$= \sum_{i=1}^{k} \left(\frac{n_i^2-n_i}{2}\right)$$

$$= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n$$

$$\le \frac{1}{2}\left[n^2+(k-1)(k-2n)\right]-\frac{1}{2}n$$

$$\le \frac{1}{2}\left[n^2+\cancel{k^2}-2nk-\cancel{k}+2n-n\right]$$

$$\le \frac{1}{2}\left[n^2-2nk+k^2+n-k\right]$$

$$\le \frac{1}{2}\left[(n-k)^2+(n-k)\right]$$

$$= \frac{(n-k)(n-k+1)}{2}$$

∴ Maximum number of edges with k components

$$= \frac{(n-k)(n-k+1)}{2}$$

Euler and Hamilton paths

Euler ~~Circuit~~ :-

path :- A path of length n from the vertex v_0 to vertex v_n is a sequence of the form

$$v_0, e_1, v_1, e_2, \cdots v_{n-1}\, e_n, v_n \text{ where } e_i = v_{i-1}v_i \quad i = 1, 2, \ldots n$$

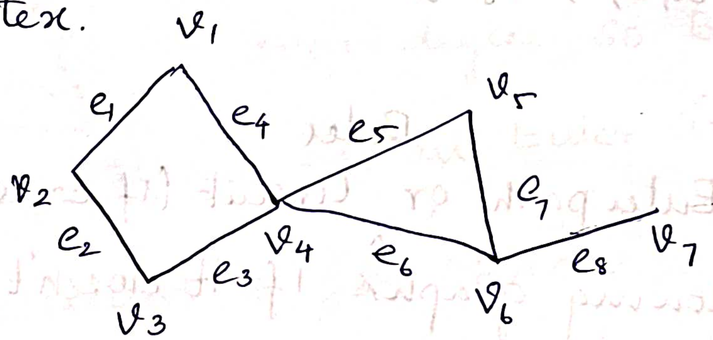
The vertices v_0 and v_n are called the end points of the paths, v_0 is the initial point and v_n is the terminal point of the path.

Simple path :- A path from v_0 to v_n which does not contain repeated vertices is called a simple ~~graph~~ path.

Trivial path :- A path of length 0 (ie) it contains only one vertex is called a trivial path.

Circuit or Cycle :- A non-trivial path is called a circuit or cycle if it starts and ends with the same vertex.

Eg



A path from v_1 to v_5 is $P_1 \Rightarrow v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_5 v_5$

A path from v_2 to v_7 is $P_2 \Rightarrow v_2 e_2 v_3 e_3 v_4\, e_6 v_6 e_8 v_7$

A cycle is $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_1$

Connected :- A graph is connected if there is a path between every pair of distinct vertices of the graph. Otherwise 'H is' disconnected.

Def:- Euler circuit (or cycle)

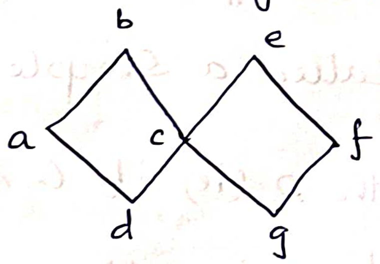
An Euler circuit in a graph G is a simple Circuit containig every edge of G.

An Euler path in a graph G is a simple path Containing every edge of G.

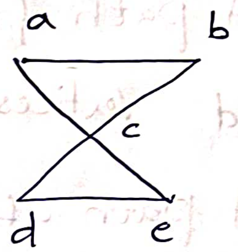
Def:- Euler graph

A Connected graph with an Euler Circuit is called an Euler graph

Ex:-



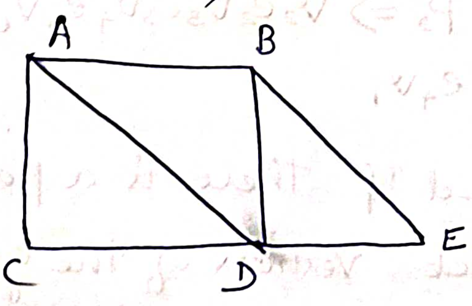
G_1



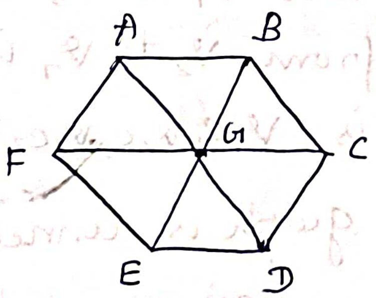
G_2

G_1 is Eulerian, because it Contains an Euler circuit ~~abcd~~ a, b, c, e, f, g, c, d, a, it contains each edge only once.

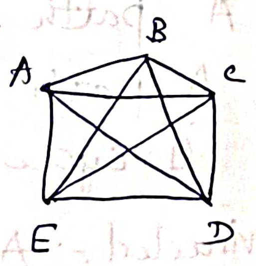
(QB) Find an Euler path or Euler Circuit (If exists in each of the following graphs. If it doesn't exits explain why?)



G_1

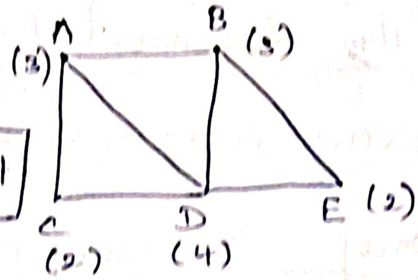


G_2



G_3

1) [In G_1, There are two Vertices A and B of odd degree and other Vertices are of even degree 2 and 4]



$\therefore$ G_1 has Euler path but no Euler Circuit.

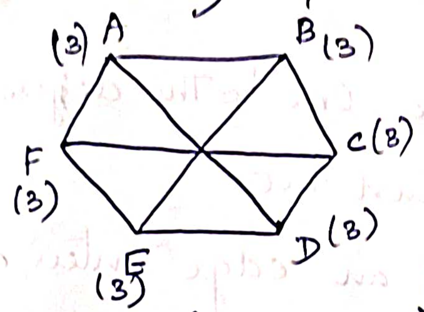
(or)

[Also By "A connected Graph is Eulerian iff every Vertex of G is of Even degree" $\longrightarrow$ (*)

Here there are two Vertices A & B of odd degree

$\Rightarrow$ G_1 is not Eulerian.]

2)

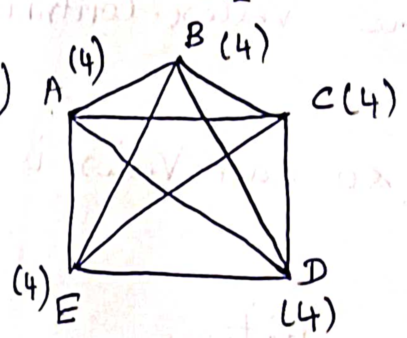


G_2

Since all the 6 Vertices are of ~~even~~ odd degree, so it does not Contains Euler Circuit

$\Rightarrow$ G_2 is not Eulerian.

3)



Since all the Vertices are of even degree so By the Thm (*) G has an Euler Circuit.

The circuit is ABCDEACEBDA.

(All the edges are traced exactly once)

$\therefore$ G is Eulerian.

(QB) Thm A Connected graph G is Eulerian If and only if every vertex of G is of even degree.

Proof:- Necessary part

Given:- Let G be a connected graph.

Let G be Eulerian

T.P Every Vertex of G is of even degree.

Since G is Eulerian, G Contains an Euler circuit, say $v_0, e_1, v_1, e_2, v_2, e_3, \ldots v_{n-1}, e_n, v_0$

Both the edges e_1 and e_n Contribute one to the degree of v_0 and so degree of v_0 is atleast two.

In tracing this circuit we find an edge enters a Vertex and another edge leaves the Vertex Contributing 2 to the degree of the Vertex.

This is true for all Vertices and so each Vertex is of degree 2, an even integer.

Hence Every vertex of G is of even integer.

Sufficient part

Given:- Every Vertex of G is of even degree.

Let G be a connected graph.

T.P G is Eulerian.

Let v be an arbitrary Vertex in G.

Begining with v form a Circuit C : $v, v_1, v_2, \ldots v_{n-1}, v$

This is possible because every Vertex of even degree.

We can leave a vertex ($\neq v$) along an edge not used to enter it. This tracing clearly stops only at a vertex v because v is also of even degree and we started from v.

Thus we get a circuit or cycle C.

If C includes all the edges of G, then C is an Euler circuit and so G is Eulerian.

If C does not includes all the edges of G, consider the subgraph H of G obtained by deleting all the edges of C from G and vertices not incident with the remaining edges.

Note that all the vertices of H have even degree.

Since G is connected, H and C must have a common vertex u.

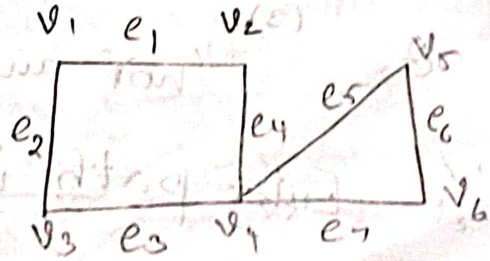
Beginning with u construct a circuit C_1 for H.

Now combine C_1 and C_2 to form a larger circuit C_0.

If it contains all the edges of G, then G is Eulerian.

Otherwise, continue the process until we get an Eulerian circuit.

Hence G is Eulerian.



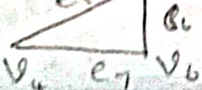
G

$C = v_1 e_1 v_2 e_4 v_4 e_4 v_3 e_2 v_1$

Common vertex $= v_4$

$C_2 = v_4 e_5 v_5 e_6 v_6 e_7 v_4$

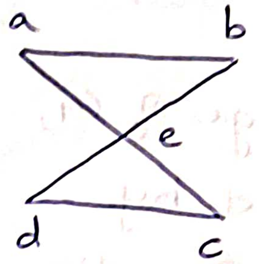
H



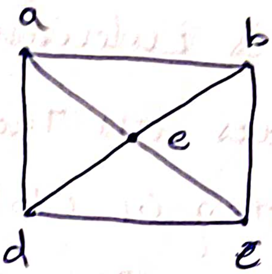
$v_1 e_1 v_2 e_4 v_4 e_5 v_5 e_6 v_6 e_7 v_4 e_3 v_3 e_2 v_1$

Note: A connected graph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

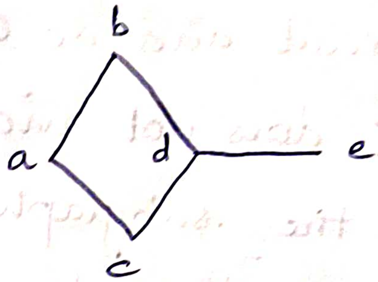
QB pblm Find an Euler path or an Euler circuit, if it exists in each of the following three graphs. If it doesn't exist explain why?



G_1

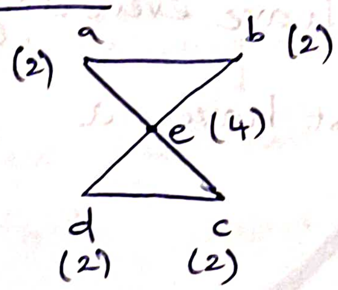


G_2



G_3.

solution



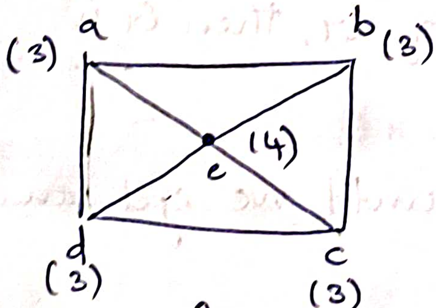
G_1

G_1 contains all the vertices of even degree. So Euler path and Euler circuit exists.

~~Euler path abecde~~

Euler Circuit abecdea

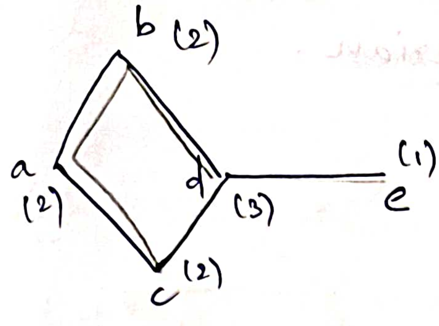
So G_1 is Eulerian.



G_2

There are 4 vertices of odd degree and 1 vertex are of even degree so G_2 has ~~an~~ neither Euler path ~~but~~ nor an Euler circuit.

~~Euler path is abcdaeb~~



G_3

There are exactly 2 vertices of odd degree ∴

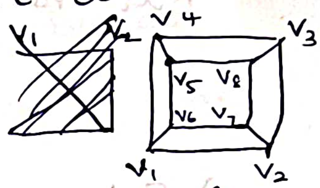
∴ G_3 has an Euler path but not Euler circuit.

Euler path is e d c a b d

G is not Eulerian.

<u>Hamilton path</u> :- A simple path in connected graph G is called a Hamilton path if it contains every vertex of G exactly once. Eg.

→ P: $V_1 V_2 V_3 V_4 V_5 V_6 V_7 V_8$



<u>Hamilton Circuit (or) cycle</u>

A simple circuit of a connected graph G is called a Hamilton circuit or Hamilton cycle if it contains every vertex of G exactly once.

C :- $V_1 V_2 V_3 V_4 V_5 V_8 V_7 V_6 V_1$

<u>Hamilton graph</u>

A connected graph that contains Hamilton circuit is called Hamilton graph (or) Hamiltonian graph.

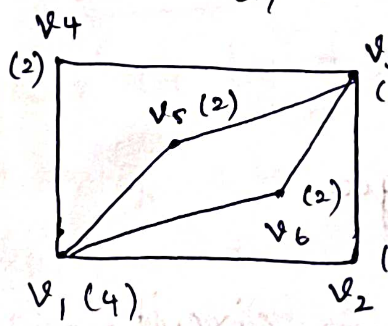
<u>QB</u> Illustrate with an example for graphs which are

a) Eulerian but not Hamiltonian

b) Hamiltonian but not Eulerian

c) Eulerian and Hamiltonian

d) Neither Eulerian nor Hamiltonian

a) 

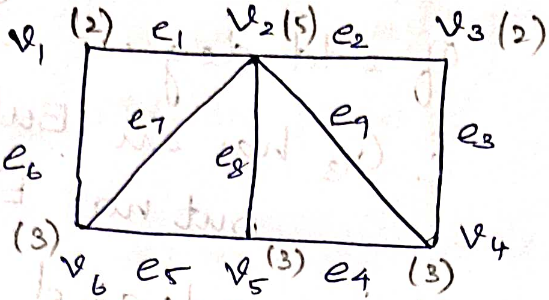
Every vertex is of even degree, G has an Euler circuit.

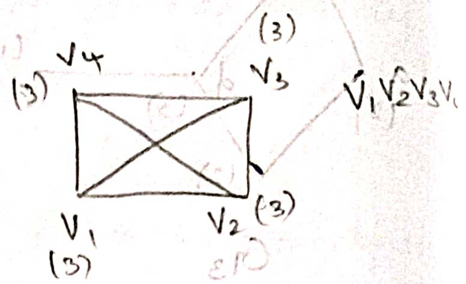
∴ G is Eulerian.

But it is not Hamiltonian, because every circuit containing every vertex contains a vertex twice.

Eg. $V_1 V_2 V_3 V_4 V_1 V_6 V_3 V_4 V_1$

b) Hamiltonian but not Eulerian.





The graph has 4 vertices of odd degree

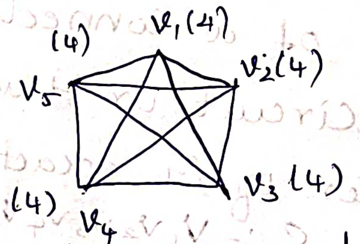
⇒ It is not Eulerian

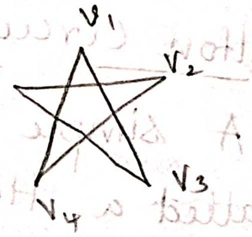
But the graph contain Hamiltonian circuit

$v_1 v_2 v_3 v_4 v_5 v_6 v_1$

⇒ It is Hamiltonian.

c) Euler and Hamiltonian.





The graph has even degree for all vertices

∴ It is Eulerian

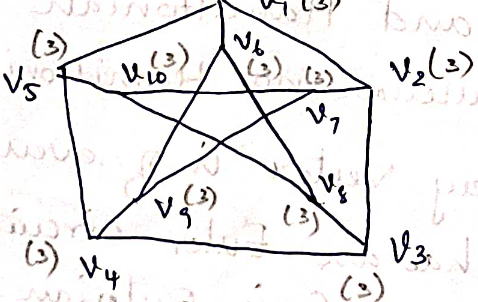
Eulerian circuit is $v_1 v_2 v_3 v_4 v_5 v_2 v_4 v_1 v_3 v_5 v_1$

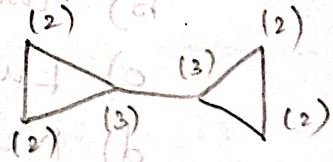
Also the graph has Hamiltonian circuit

$v_1 v_2 v_3 v_4 v_5 v_1$

∴ It is Hamiltonian

d) Neither Hamiltonian nor Eulerian





Hamitonian Circuit

All the vertices are of odd degree

∴ It is not Eulerian.

Also the graph has no Hamiltonian circuit

∴ It is not Hamiltonian