23MA101-MATRICES AND CALCULUS
QUESTION BANK
UNIT – 3
FUNCTIONS OF SEVERAL VARIABLES

PART A

1.If u = $\frac{y}{z} + \frac{z}{x}$ then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

Solution

Given u = $\frac{y}{z} + \frac{z}{x}$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$ = $x(-\frac{z}{x^2}) + y(\frac{1}{z}) + z(-\frac{y}{z^2} + \frac{1}{x})$

$= -\frac{z}{x} + \frac{y}{z} - \frac{y}{z} + \frac{z}{x} = 0$

2.If u=f(x-y,y-z,z-x) then find $u_x+u_y+u_z$

Solution

Let r = x-y, s = y-z, t = z-x

$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}(-1) = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}$

Similarly , $u_y = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$ and $u_z = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}$

$\therefore u_x+u_y+u_z = 0$

3.If $z= x^2+y^2$ and $x=at^2$, y=2at ,find $\frac{dz}{dt}$

Solution

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$ = (2x)(2at)+(2y)(2a)

$= 2(at^2)(2at)+2(2at)(2a)$

$= 4a^2t^3+8a^2t$

4. If u= $\frac{x}{y}$ and x=e^t, y=logt, find $\frac{du}{dt}$

Solution

$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = (\frac{1}{y})(e^t) + (-\frac{x}{y^2})(\frac{1}{t}) = (\frac{1}{logt})(e^t) + (-\frac{e^t}{(logt)^2})(\frac{1}{t})$

$= \frac{e^t}{logt}(1-\frac{1}{t(logt)})$

5. Find $\frac{du}{dt}$, if u = x^3+y^3 where x = a cost , y = b sint

Solution

$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$ = $(3x^2)$(-a sint) + $(3y^2)$(b cost)

$= 3(a^2 \cos^2 t)(-a \sin t) + 3(b^2 \sin^2 t)(b \cos t) = -3a^3\cos^2 t \sin t + 3b^3 \sin^2 t \cos t$

$= 3\cos t \sin t(-a^3 \cos t + b^3 \sin t)$

6. Find $\frac{dy}{dx}$, if $x^y+y^x = c$

Solution

Let $f(x,y) = x^y+y^x - c$

$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{yx^{y-1}+y^x \log y}{x^y \log x + xy^{x-1}}$

7. Find $\frac{dy}{dx}$, if $x^3+y^3 = 3axy$

Solution

Let $f(x,y) = x^3+y^3 - 3axy$

$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{3x^2-3ay}{3y^2-3ax} = -\frac{x^2-ay}{y^2-ax}$

8. If u = (x-y)(y-z)(z-x) , Show that $u_x+u_y+u_z = 0$

Solution

$u_x = (1)(y-z)(z-x) + (x-y)(0)(z-x) + (x-y)(y-z)(-1) = (y-z)(z-x) + (x-y)(y-z)(-1)$ …….(1)

$u_y = (-1)(y-z)(z-x) + (x-y)(1)(z-x) + (x-y)(y-z)(0) = (x-y)(z-x) + (y-z)(z-x)(-1)$ …….(2)

$u_z = (0)(y-z)(z-x) + (x-y)(-1)(z-x) + (x-y)(y-z)(1) = (x-y)(y-z)+ (x-y)(z-x)(-1)$ …….(3)

(1)+(2)+(3) we get , $u_x+u_y+u_z = 0$

9.If $w = xy + \frac{e^y}{y^2+1}$, then find $\frac{\partial^2 w}{\partial x \partial y}$

Solution

$\frac{\partial w}{\partial x} = y$ and $\frac{\partial w}{\partial y} = x + \frac{(y^2+1)e^y - e^y(2y)}{(y^2+1)^2} = x + \frac{e^y(y^2-2y+1)}{(y^2+1)^2}$

$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial x}\left(x + \frac{e^y(y^2-2y+1)}{(y^2+1)^2}\right) = 1$

10. Prove that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, if $f = x^3+y^3+ z^3 + 3xyz$

Solution

$\frac{\partial f}{\partial x} = 3x^2 + 3yz$; $\frac{\partial f}{\partial y} = 3y^2 + 3xz$; $\frac{\partial f}{\partial z} = 3z^2 + 3xy$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3y^2 + 3xz) = 3z$ ------(1)

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(3x^2 + 3yz) = 3z$ ------(2)

$\therefore$ from (1) & (2) , $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

11.State any two properties of Jacobian

Solution

(i)If u and v are functions of r and s & r and s are functions of x and y, then $\frac{\partial(u,v)}{\partial(r,s)} \frac{\partial(r,s)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,y)}$

(ii)If u and v are functions of x and y then $\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = 1$

12. If x = uv, y = $\frac{u}{v}$ then find $\frac{\partial(x,y)}{\partial(u,v)}$

Solution

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = (v)\left(\frac{-u}{v^2}\right) - (u)\left(\frac{1}{v}\right)$$

$$= -\frac{u}{v} - \frac{u}{v} = -\frac{2u}{v}$$

13. Find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$, if x+y = u and y = uv

Solution

u= x + y & y= uv

u=x + uv ∴ x = u – uv & y = uv

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u(1-v) - (-u)(v) = u - uv + uv = u$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}} = \frac{1}{u}$$

14. Find the Taylor's series expansion of x^y near the point (1,1) upto first degree.

Solution

here (a,b) = (1,1)

$f(x,y) = f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)] + \dots$

$f(x,y) = x^y$ $f(1,1) = 1$

$f_x(x,y) = y\, x^{y-1}$ $f_x(1,1) = 1$

$f_y(x,y) = x^y \log x$ $f_y(x,y) = 0$

$\therefore f(x,y) = 1 + [(x-1)(1) + (y-1)(0)] + \dots = x$

15.State the conditions for maxima and minima of *f*(x,y)

Solution

If $f_x(a,b) = 0$, $f_y(a,b) = 0$ and $f_{xx}(a,b) = r$, $f_{xy}(a,b) = s$, $f_{yy}(a,b) = t$ then

(i)f(x,y) attains its maximum value at (a,b) if $rt-s^2 > 0$ and $r < 0$

(ii)f(x,y) attains its minimum value at (a,b) if $rt-s^2 > 0$ and $r > 0$

PART B

1. If z = f(x,y), where $x = e^x \cos y$, $y = e^x \sin y$, then prove that $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = (u^2 + v^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$

2.If z = f(x,y), where $x = u^2-v^2$, y = 2uv , then prove that $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4(u^2 + v^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$

3.If $u = \sin^{-1}\left(\frac{x^3 - y^3}{x+y}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2 \tan u$

4.Let u = x+y+z , v = xy + yz + zx and w = $x^2 + y^2 + z^2$. Are u,v and w functionally dependent?

If so, find the relationship.

5.Find the Jacobian of u,v,w with respect to x,y,z if u = $\frac{yz}{x}$, v = $\frac{xz}{y}$, w = $\frac{xy}{z}$

6.Expand the function f(x,y) = $x^3 + y^3 + xy^2$ in powers of (x-1) and (y-2) as a Taylor's series.

7. Expand the function f(x,y) = $e^x \log(1+y)$ in powers of x and y as a Taylor's series upto third degree terms.

8.Find the maximum and minimum values of $x^3 + y^3 – 12x - 3y +20$

9. Examine $x^4 + y^4 –2x^2 + 4xy -2y^2$ for its extreme values.

10.A rectangular box open at the top is to have a volume of 32 cc. Find the dimensions of the box that requires the least material for its construction.

11.Find the shortest distance and longest distance from the point (1,2,-1) to the sphere $x^2 + y^2 + z^2 = 24$, using Lagrange's method of constrained maxima and minima.

12.The temperature T at any point (x,y,z) in space is T = $400xyz^2$. Find the highest temperature on the surfaceof the unit sphere $x^2 + y^2 + z^2 = 1$

UNIT 4

INTEGRAL CALCULUS

PART A

1.Find $\int e^{x\log 2}\, e^x\ dx$

Solution

$\int e^{x\log 2}\, e^x\ dx = \int e^{\log 2^x}\, e^x\ dx = \int 2^x e^x\ dx = \int (2e)^x\ dx = \frac{(2e)^x}{\log 2e} + c$

2.Find the derivative of G(x) = $\int_x^1 \cos\sqrt{t}\ dt$

Solution

Fundamental theorem of calculus

"Suppose f is continuous on [a,b] and if g(x) = $\int_a^x f(t)dt$, then $g'(x) = f(x)$ "

Given G(x) = $\int_x^1 \cos\sqrt{t}\ dt$, here f(t) = $\cos\sqrt{t}$

By using Fundamental theorem of calculus

G'(x) = f(x) = $\cos\sqrt{x}$

3.Find $\int \frac{\cos x}{\sqrt{\sin x}}\, dx$ by substitution method

Solution

Put u = sin x $\Longrightarrow$ du = cos x dx

I = $\int \frac{du}{\sqrt{u}}$ = $2\sqrt{u}$ +c = $2\sqrt{\sin x}$ +c

4.Evaluate $\int \frac{dx}{\sqrt{x^2 + 3x + 1}}$

Solution

$\int \frac{dx}{\sqrt{x^2+3x+1}} = \int \frac{dx}{\sqrt{(x+\frac{3}{2})^2 - \frac{5}{4}}} = \int \frac{dx}{\sqrt{(x+\frac{3}{2})^2 - (\frac{\sqrt{5}}{2})^2}} = \cosh^{-1}(\frac{x+\frac{3}{2}}{\frac{\sqrt{5}}{2}})$ $\left[∵ \int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + c \right]$

5. Evaluate $\int x \sin x\, dx$ using integration by parts

Solution

Integration by Parts $\int u dv = uv - \int v du$

Put u = x and dv = sin x dx

du = dx, $v = \int dv = \int \sin x\, dx$ = - cos x

$\int x \sin x\, dx$ = (x)(- cos x) - $\int (-\cos x)\, dx$ = - x cos x + sin x + c

6.Evaluate $\int \tan^3 x\, dx$

Solution

$\int \tan^3 x\, dx = \int \tan x \,.\tan^2 x\, dx = \int \tan x\,.(\sec^2 x - 1)dx = \int \tan x\,.\sec^2 x\, dx - \int \tan x\, dx$

Put u = tan x ⟹ du = $\sec^2$x dx in the first integral

∴ I = $\int u\, du - \int \tan x\, dx$

$= \frac{u^2}{2}$ – log(sec x) + c = $\frac{\tan^2 x}{2}$ – log(sec x) + c

7.Evaluate $\int_1^2 \left(-3x^{\frac{1}{2}} + \frac{1}{x^2}\right) dx$

Solution

$\int_1^2 \left(-3x^{\frac{1}{2}} + \frac{1}{x^2}\right) dx = \left[\frac{-3x^{\frac{3}{2}}}{3/2} - \frac{1}{x}\right]_1^2 = \left[-2x^{\frac{3}{2}} - \frac{1}{x}\right]_1^2 = \left[\left(-2\left(2^{3/2}\right) - \frac{1}{2}\right) - \left(-2\left(1^{3/2}\right) - 1\right)\right]$

$= \left[-2(2\sqrt{2}) - \frac{1}{2} + 3\right]$

$= -4\sqrt{2} + \frac{5}{2}$

8. Evaluate $\int_0^{\frac{\pi}{2}} (\cos^8 x\, dx)\, dx$

Solution

$$\int_0^{\frac{\pi}{2}} (\cos^n x\, dx) = \begin{cases} \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots\frac{1}{2}\frac{\pi}{2}, & if\ n\ is\ even \\ \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots\frac{2}{3}\mathbf{1}, & if\ n\ is\ odd \end{cases}$$

Here n = 8 even

$\int_0^{\frac{\pi}{2}} (\cos^8 x\, dx) = \frac{8-1}{8}\,\frac{8-3}{8-2}\frac{8-5}{8-4}\frac{8-7}{8-6}\,\frac{\pi}{2} = \frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2} = \frac{105}{768}\pi$

9. Evaluate $\int_0^{\frac{\pi}{2}} (\sin^6 x \cos^5 x\, dx)$

Solution

$\int_0^{\frac{\pi}{2}} (\sin^m x \cos^n x\, dx) = \frac{n-1}{m+n}\,\frac{n-3}{m+n-2}\frac{n-5}{m+n-4}\cdots\frac{1}{m+1}$, where m is an even and n is an odd integer

Here m = 6 , n = 5

$\int_0^{\frac{\pi}{2}}(\sin^6 x\cos^5 x\,dx) = \frac{4}{11}\,\frac{2}{9}\,\frac{1}{7} = \frac{8}{693}$

10.Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \mathbf{x^3\sin^2x\,dx}$

Solution

$f(x) = x^3\sin^2 x$

$f(-x) = (-x)^3[\sin(-x)]^2 = -x^3\sin^2 x = -f(x) \implies f(x)$ is an odd function.

$\therefore \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \mathbf{x^3\sin^2x\,dx} = 0$ ∵By property $\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & ,\ \text{if } f(x) = \text{odd function} \\ 2\int_0^a f(x)\,dx & ,\ \text{if } f(x) = \text{even function} \end{cases}$

11.If f is continuous and $\int_0^4 f(x)dx = 10$, then find $\int_0^2 f(2x)dx$

Solution

Let $2x = t \implies 2dx = dt \implies dx = dt/2$

x	0	2
t	0	4

$\therefore \int_0^2 f(2x)dx = \int_0^4 f(t)\frac{dt}{2} = \frac{1}{2}\int_0^4 f(t)dt$

$= \frac{\mathbf{10}}{\mathbf{2}} = 5$

12.Given that $\int_0^{10} f(x)dx = 17$ and $\int_0^8 f(x)dx = 12$, then find $\int_8^{10} f(x)dx$

Solution

$\int_0^{10} f(x)dx = \int_0^8 f(x)dx + \int_8^{10} f(x)dx$

$17 = 12 + \int_8^{10} f(x)dx$

$\therefore \int_8^{10} f(x)dx = 17 - 12 = 5$

13. What is wrong with the equation $\int_{-1}^{2}\frac{4}{x^3}dx = \left[\frac{-2}{x^2}\right]_{-1}^{2} = \frac{3}{2}$?

Solution

The given function $\frac{4}{x^3}$ is not continuous at x = 0.

Hence the integral $\int_{-1}^{2}\frac{4}{x^3}dx$ does not exist.

14. Determine whether the integral $\int_1^{\infty}\frac{\log(x)}{x}\,dx$ is convergent or divergent.

Solution

$\int_1^{\infty}\frac{\log(x)}{x}dx = \lim_{b\to\infty}\int_1^b\frac{\log(x)}{x}dx = \lim_{b\to\infty}\int_1^b u\,du$ put $u = \log x \implies du = \frac{1}{x}dx$

$= \lim_{b\to\infty}\left[\frac{u^2}{2}\right]_1^b = \lim_{b\to\infty}\left[\frac{(\log x)^2}{2}\right]_1^b$

$= \lim_{b\to\infty}\left[\frac{(\log b)^2}{2} - \mathbf{0}\right] = \frac{(\log\infty)^2}{2} = \infty$

The limit does not exist. Hence the given integral is divergent.

15. Evaluate $\int_0^\infty \frac{dx}{a^2 + x^2}$, a > 0, if it exists?

Solution

$\int_0^\infty \frac{dx}{a^2 + x^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{a^2 + x^2} = \lim_{b\to\infty} \left[\frac{1}{a} \tan^{-1}\frac{x}{a}\right]_0^b$

$= \lim_{b\to\infty} \frac{1}{a}\left[\tan^{-1}\frac{b}{a} - \tan^{-1}0\right]$

$= \frac{1}{a}\left[\tan^{-1}\infty - 0\right] = \frac{1}{a}\frac{\pi}{2} = \frac{\pi}{2a}$

$\therefore \int_0^\infty \frac{dx}{a^2 + x^2}$ is converges to $\frac{\pi}{2a}$

PART B

1.Using integration by parts, evaluate $\int \frac{(\log x)^2}{x^2}$ dx

2.Evaluate $\int x \sin x \, dx$, by using integration by parts

3. Evaluate,$\int e^{-ax} \cos bx \, dx$

4. Establish a reduction formula for $I_n = \int \cos^n x \, dx$. Hence find $\int_0^{\pi/2} \cos^n x \, dx$.

5. Evaluate $\int \frac{3x+1}{(x-1)^2(x+3)} \, dx$

6. Evaluate$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} \, dx$

7. Evaluate $\int \frac{2x+3}{x^2 + x + 1} \, dx$

8. Evaluate $\int \frac{dx}{\sqrt{3x - x^2 - 2}}$

9. Evaluate (i)$\int \frac{\sqrt{9 - x^2}}{x^2} \, dx$ (ii) $\int \frac{dx}{(1+x^2)^2}$

10.For what value of p, is $\int_1^\infty \frac{1}{x^p} \, dx$ convergent?

11. *Prove that* $\int_0^{\pi/2} \log(\sin x) \, dx = -\frac{\pi}{2} \log 2$

12. Evaluate $\int_0^{\pi/4} \log(1 + \tan x) \, dx$