# MA3151- MATRICES AND CALCULUS

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## Eigenvalue and Eigenvectors

### Definition

A matrix eigenvalue problem considers the vector equation

$$AX = \lambda X \qquad \ldots (1)$$

Here A is given square matrix , $\lambda$ an unknown scalar, and $x$ an unknown vector. In a matrix eigenvalue problem, the task is to determine $\lambda'$ s and $x's$ that satisfy (1). The solutions to (1) are given following names:
The $\lambda'$s that (1) are called **eigenvalue of A** and the corresponding nonzero $x'$ that also that satisfy (1) are called **eigenvectors of A**

## Applications

## Stretching of an elastic membrane

### Problem 1

1.An elastic membrane in the $x_1, x_2$ plane with boundary circle $x_1^2 + x_2^2 = 1$ is stretched so that a point $P : (x_1, x_2)$ goes over into the point $Q : (y_1, y_2)$ given by $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Ax = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$;
Find the principal directions. What shape does the boundary circle take under this deformation.
Solution:
**For Video Explanation of this problem, Click Here**

We are looking for vectors x such that $y = \lambda x$. Since $y = Ax$, we get, $Ax = \lambda x$,
Therefore we have to find the eigenvalue and the eigenvector for the matrix A. The Characteristic equation of A is $|A - \lambda I| = 0$.

$$
\begin{aligned}
\text{That is,} \quad & \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0 \\
\Rightarrow \quad & (5-\lambda)^2 - 9 = 0 \\
\Rightarrow \quad & (5-\lambda)^2 = 9 \\
\Rightarrow \quad & 5-\lambda = \pm 3 \\
\Rightarrow \quad & 5-\lambda = -3 \text{ or } 5-\lambda = 3 \\
\Rightarrow \quad & \lambda = 8 \text{ or } \lambda = 2
\end{aligned}
$$

therefore the eigenvalues are 2,8.
<u>To find Eigenvector</u>
The eigenvector of A is given by $(A - \lambda I)\, X = 0$

$$
\Rightarrow \quad \begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0
$$

$$
\begin{aligned}
(5-\lambda)\, x_1 + 3x_2 &= 0 \\
3x_1 + (5-\lambda)\, x_2 &= 0
\end{aligned}
$$

<u>Case (ii). $\lambda_1 = 2$</u>

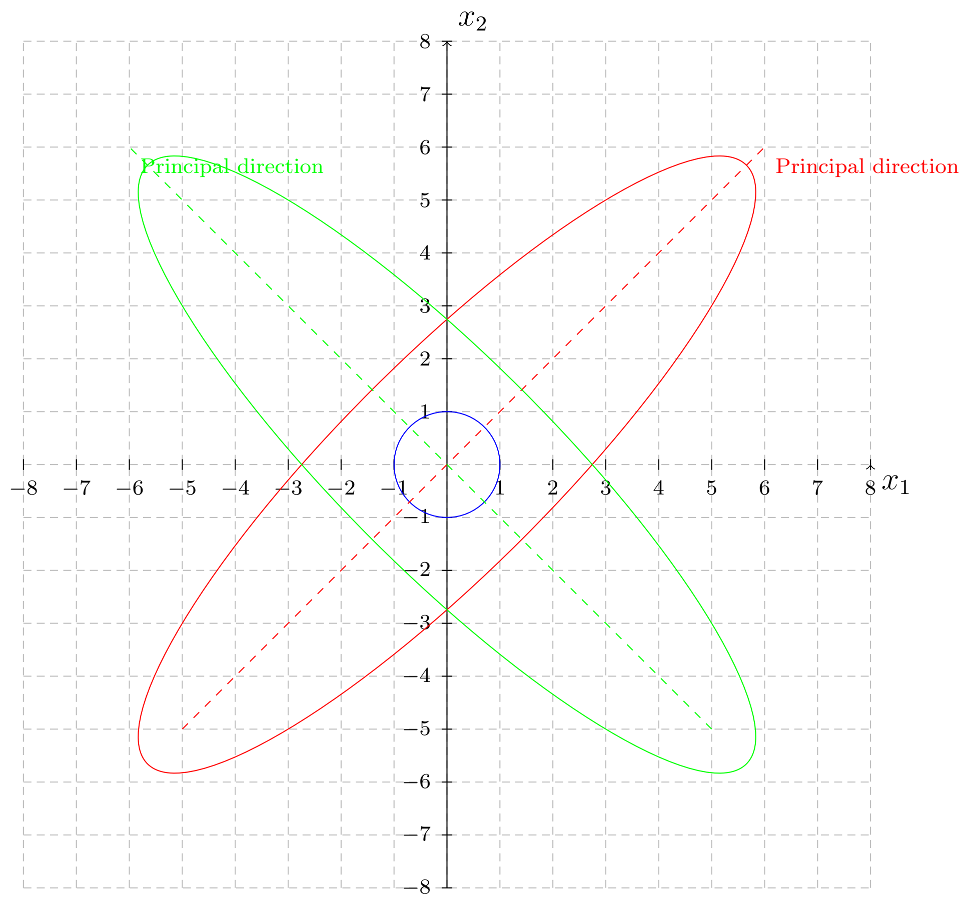
$$
\begin{aligned}
3x_1 + 3x_2 &= 0 \\
3x_1 + 3x_2 &= 0
\end{aligned}
$$

We get $x_1 = -x_2 \quad \Rightarrow \dfrac{x_1}{1} = \dfrac{x_2}{-1}$
The eigen vectors is $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
<u>Case (ii). $\lambda_2 = 8$</u>

$$
\begin{aligned}
-3x_1 + 3x_2 &= 0 \\
3x_1 - 3x_2 &= 0
\end{aligned}
$$

We get $x_1 = x_2 \quad \Rightarrow \dfrac{x_1}{1} = \dfrac{x_2}{1}$
The eigen vectors is $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

When $\lambda_1 = 2$ the eigenvector is $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. This vector make 135 angles with positive $x_1$ direction.
When $\lambda_2 = 8$ the eigenvector is $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This vector make 45 angles with positive $x_1$ direction.
The eigenvalues show that in the principal directions the membrane is stretched by factors 8 and 2 respectively.




## Problem 2

2.Given $A = \begin{bmatrix} 3.0 & 1.5 \\ 1.5 & 3.0 \end{bmatrix}$ in a deformation $y = Ax$, find the principal directions and corresponding factors of extension or contraction.
**For Video Explanation of this problem, Click Here**

We are looking for vectors x such that $y = \lambda x$. Since $y = Ax$, we get, $Ax = \lambda x$,
Therefore we have to find the eigenvalue and the eigenvector for the matrix A. The Characteristic equation of A is $|A - \lambda I| = 0$.

$$\begin{aligned}
\text{That is,} \quad & \begin{vmatrix} 3-\lambda & 1.5 \\ 1.5 & 3-\lambda \end{vmatrix} = 0 \\
\Rightarrow \quad & (3-\lambda)^2 - 2.25 = 0 \\
\Rightarrow \quad & (3-\lambda)^2 = 2.25 \\
\Rightarrow \quad & 3-\lambda = \pm 1.5 \\
\Rightarrow \quad & 3-\lambda = -1.5 \text{ or } 3-\lambda = 1.5 \\
\Rightarrow \quad & \lambda = 4.5 \text{ or } \lambda = 1.5
\end{aligned}$$

therefore the eigenvalues are 1.5,4.5.
<u>To find Eigenvector</u>
The eigenvector of A is given by $(A - \lambda I) X = 0$

$$\Rightarrow \quad \begin{bmatrix} 3-\lambda & 1.5 \\ 1.5 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$
$$(3-\lambda) x_1 + 1.5x_2 = 0$$
$$1.5x_1 + (3-\lambda) x_2 = 0$$

<u>Case (ii). $\lambda_1 = 1.5$</u>

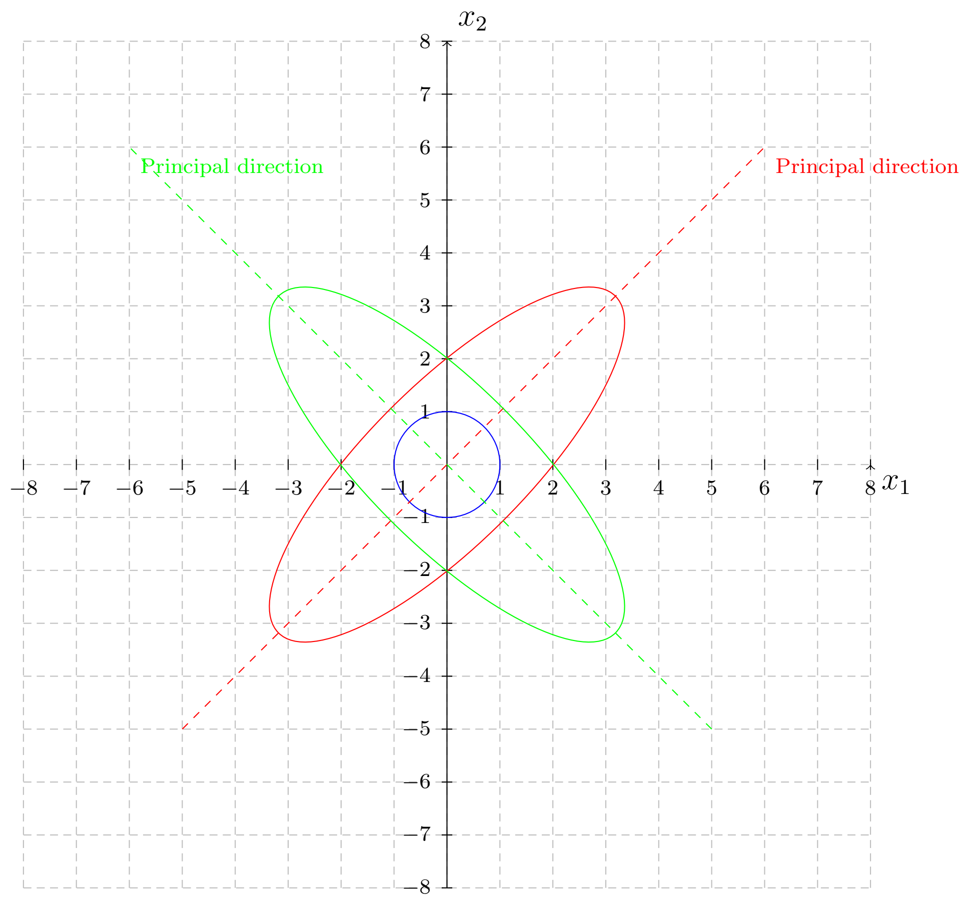
$$1,5x_1 + 1.5x_2 = 0$$
$$1.5x_1 + 1.5x_2 = 0$$

We get $x_1 = -x_2 \quad \Rightarrow \dfrac{x_1}{1} = \dfrac{x_2}{-1}$
The eigenvectors is $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
<u>Case (ii). $\lambda_2 = 4.5$</u>

$$-1.5x_1 + 1.5x_2 = 0$$
$$1.5x_1 - 1.5x_2 = 0$$

We get $x_1 = x_2 \quad \Rightarrow \dfrac{x_1}{1} = \dfrac{x_2}{1}$
The eigen vectors is $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

When $\lambda_1 = 1.5$ the eigenvector is $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. This vector make 135 angles with positive $x_1$ direction.
When $\lambda_2 = 8$ the eigenvector is $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This vector make 45 angles with positive $x_1$ direction.
The eigenvalues show that in the principal directions the membrane is stretched by factors 4.5 and 1.5 respectively.




## Problem 3

3.Given $A = \begin{bmatrix} 2.0 & 0.4 \\ 0.4 & 2.0 \end{bmatrix}$ in a deformation $y = Ax$, find the principal directions and corresponding factors of extension or contraction.
<u>Solution:</u> We are looking for vectors x such that $y = \lambda x$. Since $y = Ax$, we

get, $Ax = \lambda x$,
Therefore we have to find the eigenvalue and the eigenvector for the matrix A. The Characteristic equation of A is $|A - \lambda I| = 0$.

$$
\begin{aligned}
\text{That is,} \quad & \begin{vmatrix} 2-\lambda & 0.4 \\ 0.4 & 2-\lambda \end{vmatrix} = 0 \\
\Rightarrow \quad & (2-\lambda)^2 - 0.16 = 0 \\
\Rightarrow \quad & (2-\lambda)^2 = 0.16 \\
\Rightarrow \quad & 2-\lambda = \pm 0.4 \\
\Rightarrow \quad & 2-\lambda = -0.4 \text{ or } 2-\lambda = 0.4 \\
\Rightarrow \quad & \lambda = 2.4 \text{ or } \lambda = 1.6
\end{aligned}
$$

therefore the eigenvalues are 1.6,2.4.
<u>To find Eigenvector</u>
The eigenvector of A is given by $(A - \lambda I)\, X = 0$

$$
\begin{aligned}
\Rightarrow \quad & \begin{bmatrix} 2-\lambda & 0.4 \\ 0.4 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \\
& (2-\lambda)\, x_1 + 0.4x_2 = 0 \\
& 0.4x_1 + (2-\lambda)\, x_2 = 0
\end{aligned}
$$

<u>Case (ii). $\lambda_1 = 1.6$</u>

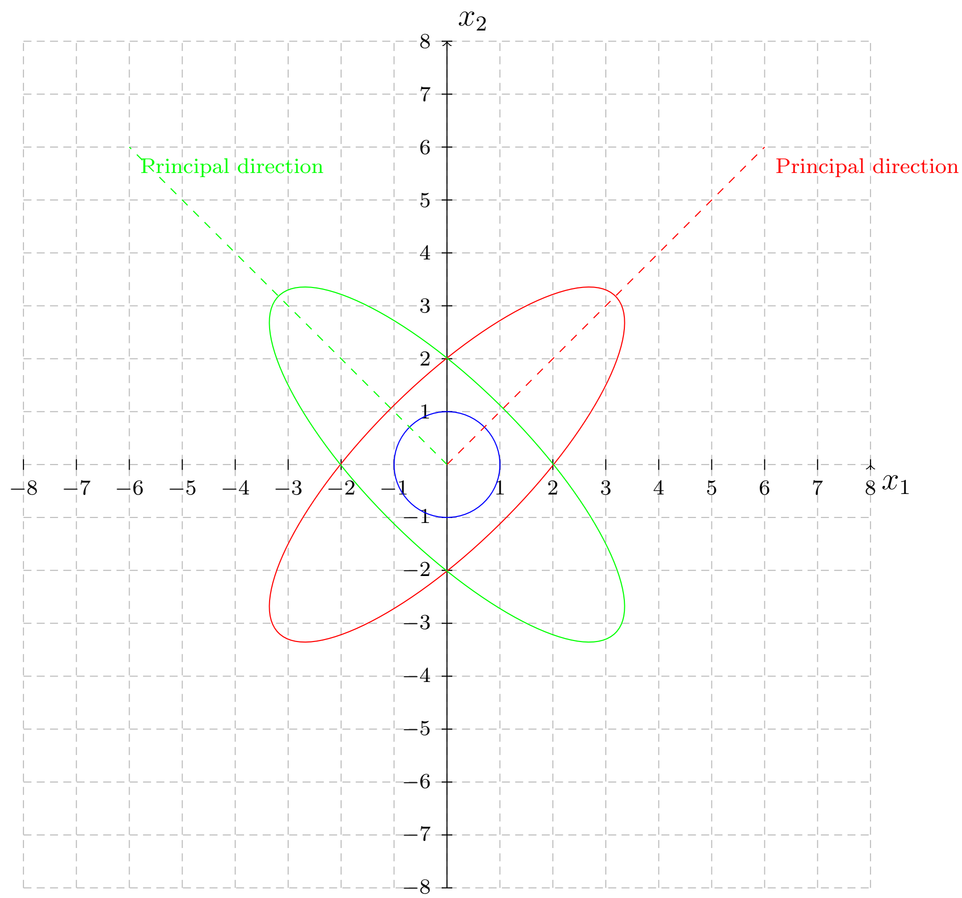
$$
\begin{aligned}
0.4x_1 + 0.4x_2 = 0 \\
0.4x_1 + 0.4x_2 = 0
\end{aligned}
$$

We get $x_1 = -x_2 \quad \Rightarrow \dfrac{x_1}{1} = \dfrac{x_2}{-1}$
The eigenvectors is $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
<u>Case (ii). $\lambda_2 = 4.5$</u>

$$
\begin{aligned}
-1.5x_1 + 1.5x_2 = 0 \\
1.5x_1 - 1.5x_2 = 0
\end{aligned}
$$

We get $x_1 = x_2 \quad \Rightarrow \dfrac{x_1}{1} = \dfrac{x_2}{1}$
The eigen vectors is $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

When $\lambda_1 = 1.5$ the eigenvector is $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. This vector make 135 angles with positive $x_1$ direction.
When $\lambda_2 = 4.5$ the eigenvector is $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This vector make 45 angles with positive $x_1$ direction.
The eigenvalues show that in the principal directions the membrane is stretched by factors 4.5 and 1.5 respectively.




## Problem 4

4.Given $A = \begin{bmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 2 \end{bmatrix}$ in a deformation $y = Ax$, find the principal directions and corresponding factors of extension or contraction.

Solution: We are looking for vectors x such that $y = \lambda x$. Since $y = Ax$, we get, $Ax = \lambda x$,
Therefore we have to find the eigenvalue and the eigenvector for the matrix A. The Characteristic equation of A is $|A - \lambda I| = 0$.

$$
\begin{array}{ll}
\text{That is,} & \begin{vmatrix} 7-\lambda & \sqrt{6} \\ \sqrt{6} & 2-\lambda \end{vmatrix} = 0 \\
\Rightarrow & (7-\lambda)(2-\lambda) - \left(\sqrt{6}\right)^2 = 0 \\
\Rightarrow & 14 - 7\lambda - 2\lambda + \lambda^2 - 6 = 0 \\
\Rightarrow & \lambda^2 - 9\lambda + 8 = 0 \\
\Rightarrow & (\lambda - 1)(\lambda - 8) = 0 \\
\Rightarrow & \lambda = 1 \text{ or } \lambda = 8
\end{array}
$$

therefore the eigenvalues are 1,8.
To find Eigenvector
The eigenvector of A is given by $(A - \lambda I) X = 0$

$$\Rightarrow \begin{bmatrix} 7-\lambda & \sqrt{6} \\ \sqrt{6} & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$(7-\lambda) x_1 + \sqrt{6} x_2 = 0$$
$$\sqrt{6} x_1 + (2-\lambda) x_2 = 0$$

Case (i). $\lambda_1 = 1$

$$6x_1 + \sqrt{6} x_2 = 0$$
$$\sqrt{6} x_1 + x_2 = 0$$

We get $6x_1 = -\sqrt{6} x_2 \quad \Rightarrow \frac{x_1}{\frac{-1}{\sqrt{6}}} = \frac{x_2}{1}$
The eigenvectors is $X_1 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ 1 \end{bmatrix}$
This vector make angle with positive $x_1$ direction is

$$
\begin{aligned}
\tan^{-1}\left(\frac{x_2}{x_1}\right) &= \tan^{-1}\left(\frac{1}{-\frac{1}{\sqrt{6}}}\right) \\
&= -\tan^{-1}\sqrt{6} \\
&= 180 - \tan^{-1}\sqrt{6} = 112.2
\end{aligned}
$$

This vector make 112.2 angles with positive $x_1$ direction.

Case (ii). $\lambda_2 = 8$

$$-x_1 + \sqrt{6}x_2 = 0$$
$$\sqrt{6}x_1 - 6x_2 = 0$$

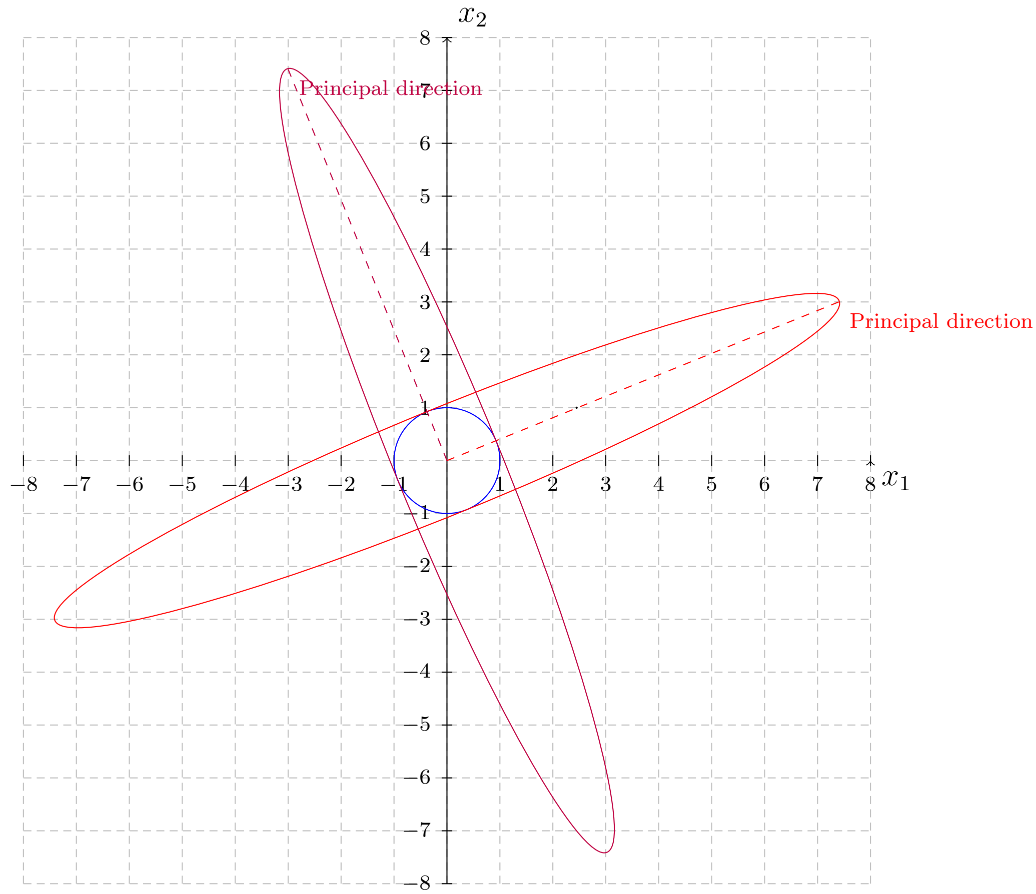
We get $x_1 = \sqrt{6}x_2 \quad \Rightarrow \dfrac{x_1}{1} = \dfrac{x_2}{\frac{1}{\sqrt{6}}}$

The eigenvectors is $X_2 = \begin{bmatrix} 1 \\ \frac{1}{\sqrt{6}} \end{bmatrix}$.

This vector make angle with positive $x_1$ direction is

$$\begin{aligned} \tan^{-1}\left(\frac{x_2}{x_1}\right) &= \tan^{-1}\left(\frac{\frac{1}{\sqrt{6}}}{1}\right) \\ &= \tan^{-1}\left(\frac{1}{\sqrt{6}}\right) \\ &= 180 - \tan^{-1}\sqrt{6} = 22.2 \end{aligned}$$

This vector make 22.2 angles with positive $x_1$ direction.




## Problem 5

5.Given $A = \begin{bmatrix} 5 & 2 \\ 2 & 13 \end{bmatrix}$ in a deformation $y = Ax$, find the principal directions and corresponding factors of extension or contraction.

<u>Solution:</u> We are looking for vectors x such that $y = \lambda x$. Since $y = Ax$, we get, $Ax = \lambda x$,

Therefore we have to find the eigenvalue and the eigenvector for the matrix A. The Characteristic equation of A is $|A - \lambda I| = 0$.

$$\text{That is,} \quad \begin{vmatrix} 5-\lambda & 2 \\ 2 & 13-\lambda \end{vmatrix} = 0$$

$$\begin{aligned}
&\Rightarrow \quad (5-\lambda)(13-\lambda) - (2)^2 = 0 \\
&\Rightarrow \quad 65 - 5\lambda - 13\lambda + \lambda^2 - 4 = 0 \\
&\Rightarrow \quad \lambda^2 - 18\lambda + 61 = 0 \\
&\Rightarrow \quad \lambda = \frac{18 \pm \sqrt{324-244}}{2} \\
&\Rightarrow \quad \lambda = \frac{18 \pm \sqrt{80}}{2} \\
&\Rightarrow \quad \lambda = \frac{18 \pm 4\sqrt{5}}{2} \\
&\Rightarrow \quad \lambda = 9 \pm 2\sqrt{5}
\end{aligned}$$

therefore the eigenvalues are $9 - 2\sqrt{5}, 9 + 2\sqrt{5}$.

<u>To find Eigenvector</u>

The eigenvector of A is given by $(A - \lambda I) X = 0$

$$\Rightarrow \quad \begin{bmatrix} 5-\lambda & 2 \\ 2 & 13-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$(5-\lambda) x_1 + 2x_2 = 0$$

$$2x_1 + (13-\lambda) x_2 = 0$$

<u>Case (i). $\lambda_1 = 9 - 2\sqrt{5}$</u>

$$\left(-4 + 2\sqrt{5}\right) x_1 + 2x_2 = 0$$

$$2x_1 + \left(4 + 2\sqrt{5}\right) x_2 = 0$$

We get $\left(-4 + 2\sqrt{5}\right) x_1 = -2x_2 \quad \Rightarrow \dfrac{x_1}{1} = \dfrac{x_2}{-4.24}$

The eigenvectors is $X_1 = \begin{bmatrix} 1 \\ -4.24 \end{bmatrix}$

This vector make angle with positive $x_1$ direction is

$$\begin{aligned}
\tan^{-1}\left(\frac{x_2}{x_1}\right) &= \tan^{-1}\left(\frac{-4.24}{1}\right) \\
&= -\tan^{-1} 4.24 \\
&= 180 - \tan^{-1} 4.24 = 103.3
\end{aligned}$$

This vector make 166.7 angles with positive $x_1$ direction.

Case (ii). $\lambda_2 = 9 + 2\sqrt{5}$

$$\left(-4 - 2\sqrt{5}\right) x_1 + 2x_2 = 0$$
$$2x_1 + \left(-4 - 2\sqrt{5}\right) x_2 = 0$$

We get $\left(-4 - 2\sqrt{5}\right) x_1 = -2x_2 \quad \Rightarrow \dfrac{x_1}{1} = \dfrac{x_2}{0.24}$

The eigenvectors is $X_2 = \begin{bmatrix} 1 \\ 0.24 \end{bmatrix}$.

This vector make angle with positive $x_1$ direction is

$$\begin{aligned} \tan^{-1}\left(\frac{x_2}{x_1}\right) &= \tan^{-1}\left(\frac{0.24}{1}\right) \\ &= \tan^{-1} 0.24 \\ &= 13.5 \end{aligned}$$

This vector make 13.5 angles with positive $x_1$ direction.