23MA301- LINEAR ALGEBRA

Unit wise Notes

UNIT I MATRICES AND SYSTEM OF LINEAR EQUATIONS

Matrices - Row echelon form - Rank - System of linear equations - Consistency - Gauss elimination method - Gauss Jordon method- Gauss Seidal Method.

PART – A

1. **Determine the values of x, y and z for which** $\begin{bmatrix} x & -2 \\ 1 & y \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -z \\ 13 & x+y \end{bmatrix}$

Solution

$$\begin{bmatrix} x & -2 \\ 1 & y \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -z \\ 13 & x+y \end{bmatrix}$$

$$\begin{bmatrix} 4x-6 & -x-4 \\ 4+3y & -1+2y \end{bmatrix} = \begin{bmatrix} 2 & -z \\ 13 & x+y \end{bmatrix}$$

$4x-6=2 \Rightarrow x=2$

$-x-4=-z \Rightarrow z=6$

$4+3y=13 \Rightarrow y=3$

2. **Define row echelon form of a matrix**

Solution

A matrix is in **row echelon form** (ref) when it satisfies the following conditions.

- The first non-zero element in each row, called the **leading entry**, is 1.
- Each leading entry is in a column to the right of the leading entry in the previous row.
- Rows with all zero elements, if any, are below rows having a non-zero element.

Each of the matrices shown below are examples of matrices in row echelon form.

A = $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$, B = $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, C = $\begin{bmatrix} 2 & 1 & 0 & 3 & 4 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ are examples of echelon matrices

while E = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ and F = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are examples of matrices that are not echelon matrices.

3. **State the Elementary row operations.**

Statement

E_1: Interchange the i^{th} and the j^{th} rows. $R_i \leftrightarrow R_j$

E_2: Replace the i^{th} row by a non-zero scalar multiple of itself. $R_i \to \mathbf{k} R_i$, where $\mathbf{k \neq 0}$.

E_3: Replace the i^{th} row by k times the j^{th} row plus the i^{th} row. $R_i \to \mathbf{k}\ R_j + R_i$.

E_1, E_2 and E_3 are called ***Elementary row operations***.

4. **Reduce A =** $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 7 & 8 \\ 5 & 2 & 3 \end{bmatrix}$ **to an echelon form**

	Solution $A=\begin{bmatrix}1&3&4\\2&7&8\\5&2&3\end{bmatrix}$ $\sim\begin{bmatrix}1&3&4\\0&1&0\\0&13&17\end{bmatrix}\begin{matrix}R_2\to R_2-2R_1\\R_3\to 5R_1-R_3\end{matrix}$ $\sim\begin{bmatrix}1&3&4\\0&1&0\\0&0&17\end{bmatrix}R_3\to R_3-13R_2$ Therefore, $\begin{bmatrix}1&3&4\\0&1&0\\0&0&17\end{bmatrix}$ is an echelon form of matrix A.
5.	**Transform A =** $\begin{pmatrix}1&2&-1\\0&1&6\\0&-5&4\end{pmatrix}$ **to row-echelon form** **(April/May 2023)**
	Solution $A=\begin{pmatrix}1&2&-1\\0&1&6\\0&-5&4\end{pmatrix}$ $\sim\begin{pmatrix}1&2&-1\\0&1&6\\0&0&34\end{pmatrix}R_3\to R_3+5R_2$ $\sim\begin{pmatrix}1&2&-1\\0&1&6\\0&0&1\end{pmatrix}R_3\to\frac{R_3}{34}$
6.	**Find the rank of** $\begin{pmatrix}2&2&1\\4&4&2\\6&6&3\end{pmatrix}$
	Solution $A=\begin{pmatrix}2&2&1\\4&4&2\\6&6&3\end{pmatrix}$ $\sim\begin{pmatrix}2&2&1\\0&0&0\\0&0&0\end{pmatrix}\begin{matrix}R_2\to R_2-2R_1\\R_3\to R_3-3R_1\end{matrix}\sim\begin{pmatrix}1&1&0.5\\0&0&0\\0&0&0\end{pmatrix}R_1\to\frac{R_1}{2}$ R(A) = 1
7.	**Do the equations x–3y - 8z = 0, 3x + y = 0 and 2x + 5y + 6z = 0 have a non trivial solution? Why?** **(Apr/May 2022)**
	Solution

	$$\|A\| = \begin{vmatrix} 1 & -3 & -8 \\ 3 & 1 & 0 \\ 2 & 5 & 6 \end{vmatrix} = 1(6) + 3(18) - 8(15-2) = -44 \neq 0$$ A is non singular, therefore there is no nontrivial solution
8.	**If the equations x + 2y + z = 0, 5x + y – z = 0 and x + 5y + λz = 0 have a non trivial solution, find the value of λ.**
	Solution If A is singular, then the system has nontrivial solution $$\|A\| = 0 \Rightarrow \begin{vmatrix} 1 & 2 & 1 \\ 5 & 1 & -1 \\ 1 & 5 & \lambda \end{vmatrix} = 0$$ $$\Rightarrow (\lambda+5) - 2(5\lambda+1) + (25-1) = 0 \Rightarrow -9\lambda + 27 = 0 \Rightarrow \lambda = 3$$
9.	**State the necessary and sufficient condition for the consistency of a system of linear equations.**
	Statement: The system of equations AX = B is consistent, if and only if the coefficient matrix A and the augmented matrix [A, B] are of the same rank.
10.	**Test the consistency of 2x-3y= 5; -4x + 6y = -10 (April/May 2023)**
	Solution $$[A:B] = \begin{pmatrix} 2 & -3 & : & 5 \\ -4 & 6 & : & -10 \end{pmatrix} R_2 \to R_2 + 2R_1$$ $$\sim \begin{pmatrix} 2 & -3 & : & 5 \\ 0 & 0 & : & 0 \end{pmatrix}$$ If R[A : B] = R(A) = 1 < 2 = the number of variables, then system is consistent with infinite number solutions.
11.	**If the augmented matrix of a system of equations is equivalent to** $\begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & \lambda-8 & \mu-11 \end{pmatrix}$**, find the values of λ and μ for which the system has no solution**
	Solution If R[A : B]≠R(A), then system is inconsistent with no solution. $\Rightarrow \lambda - 8 = 0$ and $\mu - 11 \neq 0 \Rightarrow \lambda = 8$ and $\mu \neq 11$
12.	**If the augmented matrix of a system of equations is equivalent to** $\begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 5 & 4 & 2 \\ 0 & 0 & \lambda-2 & \mu-3 \end{pmatrix}$**, find the values of λ and μ for which the system has many solutions (Nov/Dec 2022)**
	Solution If R[A : B] = R(A) = 2< 3 = the number of variables, then system is consistent with many solutions. $\Rightarrow \lambda - 2 = 0$ and $\mu - 3 = 0 \Rightarrow \lambda = 2$ and $\mu = 3$
13.	**Given the linear system $2x_1 - 6\alpha x_2 = 3$; $3\alpha x_1 - x_2 = 3/2$, Find value(s) of α for which the system has no solutions.**

	Solution $[A:B]=\begin{pmatrix} 2 & -6\alpha & 3 \\ 3\alpha & -1 & 3/2 \end{pmatrix}$ $\sim\begin{pmatrix} 1 & -3\alpha & 3/2 \\ 3\alpha & -1 & 3/2 \end{pmatrix} R_1 \to \frac{R_1}{2} \sim \begin{pmatrix} 1 & -3\alpha & 3/2 \\ 0 & 9\alpha^2-1 & 3(1-3\alpha)/2 \end{pmatrix} R_2 \to R_2 - 3\alpha R_1$ If R[A : B] = 2, R(A) = 1, R[A : B] ≠ R(A) then system is inconsistent with no solution. $\Rightarrow 9\alpha^2 - 1 = 0$ and $(3/2)(1-3\alpha) \neq 0 \Rightarrow \alpha = -(1/3)$
14.	**Given the linear system $2x_1 - 6\alpha x_2 = 3$; $3\alpha x_1 - x_2 = 3/2$, Find value(s) of α for which the system has an infinite number of solutions.**
	Solution $[A:B]=\begin{pmatrix} 2 & -6\alpha & 3 \\ 3\alpha & -1 & 3/2 \end{pmatrix}$ $\sim\begin{pmatrix} 1 & -3\alpha & 3/2 \\ 3\alpha & -1 & 3/2 \end{pmatrix} R_1 \to \frac{R_1}{2} \sim \begin{pmatrix} 1 & -3\alpha & 3/2 \\ 0 & 9\alpha^2-1 & 3(1-3\alpha)/2 \end{pmatrix} R_2 \to R_2 - 3\alpha R_1$ If R[A : B] = R(A) = 1 < 2 = the number of variables, then system is consistent with infinite number solutions. $\Rightarrow 9\alpha^2 - 1 = 0$ and $(3/2)(1-3\alpha) = 0 \Rightarrow \alpha = 1/3$
15.	**Given the linear system $2x_1 - 6\alpha x_2 = 3$; $3\alpha x_1 - x_2 = 3/2$, find the value of α, assuming a unique solution exists.**
	Solution $[A:B]=\begin{pmatrix} 2 & -6\alpha & 3 \\ 3\alpha & -1 & 3/2 \end{pmatrix}$ $\sim\begin{pmatrix} 1 & -3\alpha & 3/2 \\ 3\alpha & -1 & 3/2 \end{pmatrix} R_1 \to \frac{R_1}{2} \sim \begin{pmatrix} 1 & -3\alpha & 3/2 \\ 0 & 9\alpha^2-1 & 3(1-3\alpha)/2 \end{pmatrix} R_2 \to R_2 - 3\alpha R_1$ If R[A : B] = R(A) = 2 = the number of variables, then system is consistent with unique solution. $\Rightarrow 9\alpha^2 - 1 \neq 0 \Rightarrow \alpha \neq \pm(1/3)$
16.	**Write the set of simultaneous equation that corresponds to the augmented matrix** $[A:B]=\begin{pmatrix} 1 & 2/3 & 1/3 & -4/3 & : & 1/3 \\ 0 & 1 & -2/5 & 1 & : & -1 \\ 0 & 0 & 0 & 0 & : & 0 \end{pmatrix}$
	Solution $x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3 - \frac{4}{3}x_4 = \frac{1}{3}$ $x_2 - \frac{2}{5}x_3 + x_4 = -1$
17.	**Solve the set of equations associated with the augmented matrix** $\begin{pmatrix} 1 & -2 & 3 & 17 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -4 \end{pmatrix}$
	Solution The equations are $x_1 - 2x_2 + 3x_3 = 17$ $x_2 + 2x_3 = -3$ $x_3 = -4$

	By backward substitution, $x_3 = -4,\ x_2 + 2(-4) = -3 \Rightarrow x_2 = 5$ $x_1 - 2x_2 + 3x_3 = 17 \Rightarrow x_1 - 2(5) + 3(-4) = 17 \Rightarrow x_1 = 39$
18.	**For solving a linear system of equations, compare Gauss Elimination method and Gauss Jordan method.**

S.No.	Gauss-Elimination method	Gauss – Jordan method
1.	Coefficient matrix is transformed into upper triangular matrix	Coefficient matrix is transformed into upper diagonal matrix
2.	Direct method	Direct method
3.	We obtain the solutions by back substitution method	No need of back substitution method

19. **By Gauss elimination method solve $x + y = 2,\ 2x + 3y = 5$**

Solution

The given system equations can be written as AX = B $\Rightarrow \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$$\Rightarrow [A,B] = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} R_2 \to R_2 - 2R_1$$

we have x + y = 2, $\Rightarrow$ y = 1. Hence x = y = 1.

20. **Determine the inverse of** $A = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix}$ **by Jordan method**

Solution

$$[A : I] = \begin{pmatrix} 5 & 3 & : & 1 & 0 \\ 2 & 1 & : & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0.6 & : & 0.2 & 0 \\ 2 & 1 & : & 0 & 1 \end{pmatrix} R_1 \to \tfrac{R_1}{5}$$

$$\sim \begin{pmatrix} 1 & 0.6 & : & 0.2 & 0 \\ 0 & 0.2 & : & -0.4 & 1 \end{pmatrix} R_2 \to R_2 - 2R_1$$

$$\sim \begin{pmatrix} 1 & 0.6 & : & 0.2 & 0 \\ 0 & 1 & : & 2 & -5 \end{pmatrix} R_2 \to R_2 / 0.2$$

$$\sim \begin{pmatrix} 1 & 0 & : & -1 & 3 \\ 0 & 1 & : & 2 & -5 \end{pmatrix} R_1 \to R_1 - 0.6R_2$$

$$A^{-1} = \begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix}$$

PART-B

1 **Find the row reduced echelon form of matrix A and determine its rank, where**

A= $\begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 \\ 14 & 15 & 16 & 17 & 18 \end{bmatrix}$ **(Apr/May 2022, April/May 2023)**

	Solution: $$A=\begin{bmatrix}2&3&4&5&6\\3&4&5&6&7\\4&5&6&7&8\\9&10&11&12&13\\14&15&16&17&18\end{bmatrix}=\begin{bmatrix}2&3&4&5&6\\0&-1&-2&-3&-4\\0&-1&-2&-3&-4\\0&-7&-14&-21&-28\\0&-6&-12&-18&-24\end{bmatrix}\begin{matrix}\\R_2\Rightarrow 2R_2-3R_1\\R_3\Rightarrow R_3-2R_1\\R_4\Rightarrow 2R_4-9R_1\\R_5\Rightarrow R_5-7R_1\end{matrix}$$ $$=\begin{bmatrix}2&3&4&5&6\\0&1&2&3&4\\0&1&2&3&4\\0&1&2&3&4\\0&1&2&3&4\end{bmatrix}\begin{matrix}\\R_2\Rightarrow R_2/-1\\R_3\Rightarrow R_3/-1\\R_4\Rightarrow R_4/-7\\R_5\Rightarrow R_5/-6\end{matrix}$$ $$=\begin{bmatrix}1&1.5&2&2.5&3\\0&1&2&3&4\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\begin{matrix}R_1\Rightarrow R_1/2\\ \\R_3\Rightarrow R_3-R_2\\R_4\Rightarrow R_4-R_2\\R_5\Rightarrow R_5-R_2\end{matrix}$$ Rank is 2
2	**Find the row reduced echelon form of matrix A and determine its rank, where** **A=**$\begin{bmatrix}1&-2&0&0&3\\2&-5&-3&-2&6\\0&5&15&10&0\\2&6&18&8&6\end{bmatrix}$ **(Nov/Dec 2022)**
	Solution Now reduce matrix A to row reduced Echelon form. **A** = $\begin{bmatrix}1&-2&0&0&3\\2&-5&-3&-2&6\\0&5&15&10&0\\2&6&18&8&6\end{bmatrix}$ $$\sim\begin{bmatrix}1&-2&0&0&3\\0&1&3&2&0\\0&5&15&10&0\\0&10&18&8&0\end{bmatrix}\quad\begin{matrix}R_2\to 2R_1-R_2\\R_4\to -2R_1+R_4\end{matrix}$$ $$\sim\begin{bmatrix}1&-2&0&0&3\\0&1&3&2&0\\0&0&0&0&0\\0&0&12&12&0\end{bmatrix}\quad\begin{matrix}R_3\to R_3-5R_2\\R_4\to 10R_2-R_4\end{matrix}$$ $$\sim\begin{bmatrix}1&-2&0&0&3\\0&1&3&2&0\\0&0&12&12&0\\0&0&0&0&0\end{bmatrix}\quad R_3\leftrightarrow R_4$$

	$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to \dfrac{R_3}{12}$ Therefore, the row reduced echelon form of A is $\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and rank (**A**) = 3.
3	**Find the rank of the matrix A=** $\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$
	Solution: **A=** $\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$ $\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{pmatrix} \quad R_1 \Leftrightarrow R_2$ $\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{pmatrix} \quad R_3 \Leftrightarrow R_3 - 3R_1$ $\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix} \quad R_3 \Leftrightarrow R_3 + 5R_2$ It is the row echelon form Therefore rank = 3
4	**Test the consistency of the system** $x_1 - 2x_2 - 3x_3 = 2;\ 3x_1 - 2x_2 = -1;\ -2x_2 - 3x_3 = 2;\ x_2 + 2x_3 = 1$
	Solution : In matrix notation, the system is $AX = B$, where $A = \begin{pmatrix} 1 & -2 & -3 \\ 3 & -2 & 0 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 \\ -1 \\ 2 \\ 1 \end{pmatrix}$. $[A:B] = \begin{pmatrix} 1 & -2 & -3 & : & 2 \\ 3 & -2 & 0 & : & -1 \\ 0 & -2 & -3 & : & 2 \\ 0 & 1 & 2 & : & 1 \end{pmatrix}$ $\sim \begin{pmatrix} 1 & -2 & -3 & : & 2 \\ 0 & 4 & 9 & : & -7 \\ 0 & -2 & -3 & : & 2 \\ 0 & 1 & 2 & : & 1 \end{pmatrix} \mathrm{R}_2 \to \mathrm{R}_2 - 3\mathrm{R}_1$

	$\sim\begin{pmatrix}1 & -2 & -3 & : & 2\\ 0 & 4 & 9 & : & -7\\ 0 & 0 & 3 & : & -3\\ 0 & 0 & -1 & : & 11\end{pmatrix}\begin{matrix}\\ \\ R_3\to 2R_3+R_2\\ R_4\to 4R_4-R_2\end{matrix}$ $\sim\begin{pmatrix}1 & -2 & -3 & : & 2\\ 0 & 4 & 9 & : & -7\\ 0 & 0 & 3 & : & -3\\ 0 & 0 & 0 & : & 30\end{pmatrix}R_4\to 3R_4+R_3$ $\therefore$ R(A : B) = 4; R(A) = 3$\Rightarrow$ R(A : B) $\neq$ R(A). Therefore the system is inconsistent.
5	**Investigate for what values of λ and μ the system of linear equation** $x+2y+z=7, x+y+\lambda z=\mu, x+3y-5z=5$ **has (i) a unique solution (ii) infinite number of solutions (iii) no solution (Apr/May 2022)**
	Solution: $[A,B]=\left[\begin{array}{ccc\|c}1 & 2 & 1 & 7\\ 1 & 1 & \lambda & \mu\\ 1 & 3 & -5 & 5\end{array}\right]\sim\left[\begin{array}{ccc\|c}1 & 2 & 1 & 7\\ 0 & -1 & \lambda-1 & \mu-7\\ 0 & 1 & -6 & -2\end{array}\right]\begin{matrix}\\ R_2\to R_2-R_1\\ R_3\to R_3-R_1\end{matrix}$ $\sim\left[\begin{array}{ccc\|c}1 & 2 & 1 & 7\\ 0 & -1 & \lambda-1 & \mu-7\\ 0 & 0 & \lambda-7 & \mu-9\end{array}\right]R_3\to R_3+R_2$ (i) **a unique solution** If R[A : B] = R(A) = 3 = the number of variables, then system is consistent with unique solution. $\Rightarrow \lambda\neq 7$ and $\mu\neq 9$ (ii) **infinite number of solutions** If R[A : B] = R(A) = 2 < 3 = the number of variables, then system is consistent with many solutions $\Rightarrow\lambda = 7$ and $\mu = 9$ (iii) **no solution** If R[A : B] $\neq$ R(A), then system is inconsistent with no solution.$\Rightarrow\lambda = 7$ and $\mu\neq 9$
6	**Find the value of c for which the system** $x_1+x_2+x_3=2;\ \ 2x_1+x_2+2x_3=5;\ \ 4x_1+3x_2+4x_3=c$ **is (i) consistent (ii) inconsistent**
	Solution : In matrix notation our system is $AX=B$, where $A=\begin{pmatrix}1 & 1 & 1\\ 2 & 1 & 2\\ 4 & 3 & 4\end{pmatrix}$ and $B=\begin{pmatrix}2\\ 5\\ c\end{pmatrix}$. $(A:b)=\begin{pmatrix}1 & 1 & 1 & 2\\ 2 & 1 & 2 & 5\\ 4 & 3 & 4 & c\end{pmatrix}$ Then $\sim\begin{pmatrix}1 & 1 & 1 & 2\\ 0 & -1 & 0 & 1\\ 0 & -1 & 0 & c-8\end{pmatrix}\begin{matrix}R_2\to R_2-2R_1\\ R_3\to R_3-4R_1\end{matrix}$ $\sim\begin{pmatrix}1 & 1 & 1 & 2\\ 0 & 1 & 0 & -1\\ 0 & -1 & 0 & c-8\end{pmatrix}R_2\to -R_2$

	$\sim\begin{pmatrix}1&1&1&2\\0&1&0&-1\\0&0&0&c-9\end{pmatrix}R_3\to R_3+R_2$ If $c\neq 9$ then R(A) = 2 but R[A : B]= 3 and the equations are inconsistent. If $c=9$ then R(A) = R[A : B]= 2 < 3, the system of equations is consistent and the system has infinitely many solutions and it is given by $x_1+x_2+x_3=2$ $x_2=-1.$ Let $x_3=\lambda$, then $x_1=3-\lambda$; $x_2=-1$
7	**Solve the following system of equations, if consistent** $x_1+2x_2-x_3-5x_4=4;\ x_1+3x_2-2x_3-7x_4=5;\ \ 2x_1-x_2+3x_3=3$
	Solution $[A:B]=\begin{pmatrix}1&2&-1&-5&:&4\\1&3&-2&-7&:&5\\2&-1&3&0&:&3\end{pmatrix}$ $\sim\begin{pmatrix}1&2&-1&-5&:&4\\0&1&-1&-2&:&1\\0&-5&5&10&:&-5\end{pmatrix}\begin{matrix}\\R_2\to R_2-R_1\\R_3\to R_3-2R_1\end{matrix}$ $\sim\begin{pmatrix}1&2&-1&-5&:&4\\0&1&-1&-2&:&1\\0&0&0&0&:&0\end{pmatrix}R_3\to R_3+5R_2$ $\therefore$ R[A : B] = 2; R(A) = 2 < 4 = the number of variables. Therefore, the system is consistent with many solutions. From the first two rows of the equivalent matrix $x_1+2x_2-x_3-5x_4=4$ $x_2-x_3-2x_4=1$ As there are only two equations, we can solve for only two variables. $Let\quad x_3=k\ \&\ x_4=k'$ $x_2-x_3-2x_4=1\Rightarrow x_2-k-2k'=1\Rightarrow x_2=1+k+2k'$ $x_1+2x_2-x_3-5x_4=4\Rightarrow x_1+2(1+k+2k')-k-5k'=4\Rightarrow x_1=2-k+k'$
8	**Find the values of k for which the equations** $x+y+z=1;\ x+2y+3z=k\ \ \&\ \ x+5y+9z=k^2$ **have a solution. For these values of k, find the solution also.** **(Nov/Dec 2022)**
	Solution $[A:B]=\begin{pmatrix}1&1&1&:&4\\1&2&3&:&k\\1&5&9&:&k^2\end{pmatrix}$ $\sim\begin{pmatrix}1&1&1&:&4\\0&1&2&:&k-1\\0&4&8&:&k^2-1\end{pmatrix}\begin{matrix}\\R_2\to R_2-R_1\\R_3\to R_3-R_1\end{matrix}$ $\sim\begin{pmatrix}1&1&1&:&4\\0&1&2&:&k-1\\0&0&0&:&k^2-4k+3\end{pmatrix}R_3\to R_3-4R_2$ R(A) = 2. If the system possesses a solution. R[A : B] must also be 2.

Therefore, the last row of the last equivalent matrix must contain only zeros. (i.e) $k^2 - 4k + 3 = 0$
(i.e) k = 1 or 3
For these values of k, R[A : B] = 2; R(A) = 2 < 3 = the number of variables.
Therefore, the system is consistent with many solutions.

Case1: k = 1

From the first two rows of the equivalent matrix $x+y+z=1; \quad y+2z=0$

$$Let \quad z=\lambda$$

$$y+2z=0 \Rightarrow y=-2\lambda$$

$$x+y+z=1 \Rightarrow x-2\lambda+\lambda=1 \Rightarrow x=1+\lambda$$

Case2: k = 3

From the first two rows of the equivalent matrix $x+y+z=1; \quad y+2z=2$

$$Let \quad z=\mu$$

$$y+2z=2 \Rightarrow y=2-2\mu$$

$$x+y+z=1 \Rightarrow x+2-2\mu+\mu=1 \Rightarrow x=-1+\mu$$

9 **Find the value of k such that the following system of equations has (i) a unique solution (ii) many solutions and (iii) no solution kx + y + z = 1; x + ky + z = 1; x + y + kz = 1;**

Solution

$$[A:B]=\begin{pmatrix} k & 1 & 1 & : & 1 \\ 1 & k & 1 & : & 1 \\ 1 & 1 & k & : & 1 \end{pmatrix} \begin{matrix} \\ R_2 \Rightarrow kR_2 - R_1, \quad k \neq 0 \\ R_3 \Rightarrow kR_3 - R_1 \end{matrix}$$

$$\sim \begin{pmatrix} k & 1 & 1 & : & 1 \\ 0 & k^2-1 & k-1 & : & k-1 \\ 0 & k-1 & k^2-1 & : & k-1 \end{pmatrix} \begin{matrix} \\ R_2 \Rightarrow \frac{R_2}{k-1}, \quad k \neq 1 \\ R_3 \Rightarrow \frac{R_3}{k-1} \end{matrix}$$

$$\sim \begin{pmatrix} k & 1 & 1 & : & 1 \\ 0 & k+1 & 1 & : & 1 \\ 0 & 1 & k+1 & : & 1 \end{pmatrix} \begin{matrix} \\ \\ R_3 \Rightarrow (k+1)R_3 - R_2 \end{matrix}$$

$$\sim \begin{pmatrix} k & 1 & 1 & : & 1 \\ 0 & k+1 & 1 & : & 1 \\ 0 & 0 & (k+1)^2-1 & : & k \end{pmatrix} = \begin{pmatrix} k & 1 & 1 & : & 1 \\ 0 & k+1 & 1 & : & 1 \\ 0 & 0 & k^2+2k & : & k \end{pmatrix}$$

(i) a unique solution

If R[A : B] = R(A) = 3 = the number of variables, then system is consistent with unique solution.

$\Rightarrow k^2 + 2k \neq 0$ and $k \neq 0 \Rightarrow k \neq 0, 1, -2$

(ii) many solutions

If R[A : B] = R(A) = 2< 3 = the number of variables, then system is consistent with many solutions.

$\Rightarrow k^2 + 2k = 0$ and $k = 0 \Rightarrow k = 0$

But $k \neq 0$. Clearly if k = 1, then the system of equations reduces to single equation x + y + z = 1

Thus the system is consistent with many solutions.

(iii) no solution

If R[A : B] $\neq$ R(A), then system is inconsistent with no solution.
$\Rightarrow k^2 + 2k = 0$ and $k \neq 0 \Rightarrow k = -2$

10 **Solve by Gauss Jordan method, $2x+y+2z=10,\ x+2y+z=8,\ 3x+y-z=2$**
(Apr/May 2022)

	Solution: $$[A,B]=\left[\begin{array}{ccc\|c}2&1&2&10\\1&2&1&8\\3&1&-1&2\end{array}\right]\sim\left[\begin{array}{ccc\|c}2&1&2&10\\0&3&0&6\\0&-1&-8&-34\end{array}\right]\begin{array}{l}R_2\to 2R_2-R_1\\R_3\to 2R_3-3R_1\end{array}$$ $$\sim\left[\begin{array}{ccc\|c}2&1&2&10\\0&3&0&6\\0&0&-24&-96\end{array}\right]R_3\to 3R_3+R_2\quad\sim\left[\begin{array}{ccc\|c}2&1&2&10\\0&1&0&2\\0&0&1&4\end{array}\right]\begin{array}{l}R_2\to R_2/3\\R_3\to R_3/-24\end{array}$$ $$\sim\left[\begin{array}{ccc\|c}2&1&0&2\\0&1&0&2\\0&0&1&4\end{array}\right]R_1\to R_1-2R_3\quad\sim\left[\begin{array}{ccc\|c}2&0&0&0\\0&1&0&2\\0&0&1&4\end{array}\right]R_1\to R_1-R_2$$ $$\sim\left[\begin{array}{ccc\|c}1&0&0&0\\0&1&0&2\\0&0&1&4\end{array}\right]R_1\to R_1/2$$ $x=0; y=2; z=4$
11	**Solve the system of the following equations using Gauss Jordan method** $\mathbf{2x_1+2x_2-x_3+x_4=4,\ 4x_1+3x_2-x_3+2x_4=6, 8x_1+5x_2-3x_3+4x_4=12}$ **and** $\mathbf{3x_1+3x_2-2x_3+2x_4=6}$ **(Apr/May 2022, April/May 2023)**
	Solution: $$(A,B)=\begin{bmatrix}2&2&-1&1&4\\4&3&-1&2&6\\8&5&-3&4&12\\3&3&-2&2&6\end{bmatrix}$$ $$=\begin{bmatrix}1&1&-0.5&0.5&2\\4&3&-1&2&6\\8&5&-3&4&12\\3&3&-2&2&6\end{bmatrix}R_1\to R_1/2$$ $$=\begin{bmatrix}1&1&-0.5&0.5&2\\0&-1&1&0&-2\\0&-3&1&0&-4\\0&0&-0.5&0.5&0\end{bmatrix}\begin{array}{l}R_2\to R_2-4R_1\\R_3\to R_3-8R_1\\R_4\to R_4-3R_1\end{array}$$ $$=\begin{bmatrix}1&1&-0.5&0.5&2\\0&1&-1&0&2\\0&-3&1&0&-4\\0&0&-0.5&0.5&0\end{bmatrix}R_2\to R_2(-1)$$ $$=\begin{bmatrix}1&0&-0.5&0.5&0\\0&1&-1&0&2\\0&0&-2&0&2\\0&0&-0.5&0.5&0\end{bmatrix}\begin{array}{l}R_2\to R_1-R_2\\R_3\to R_3+3R_2\end{array}$$

	$= \begin{bmatrix} 1 & 0 & 0 & 0.5 & 0.5 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0.5 & -0.5 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 0.5R_3 \\ R_2 \to R_2 + R_3 \\ R_4 \to R_4 + 0.5R_3 \end{matrix}$ $= \begin{bmatrix} 1 & 0 & 0 & 0.5 & 0.5 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 0.5R_4 \\ \\ R_4 \to R_4 / 0.5 \end{matrix}$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ $\therefore x_1 = 1, x_2 = 1, x_3 = -1, x_4 = -1$
12	**Solve the following system of equations by Gauss Jordan method** $27x + 16y - z = 85,$ $x + y + 54z = 110,\ 6x + 15y + 2z = 72$
	$[A, B] = \left[\begin{array}{ccc\|c} 27 & 16 & -1 & 85 \\ 1 & 1 & 54 & 110 \\ 6 & 15 & 2 & 72 \end{array}\right]$ $\sim \left[\begin{array}{ccc\|c} 1 & 1 & 54 & 110 \\ 27 & 16 & -1 & 85 \\ 6 & 15 & 2 & 72 \end{array}\right] \quad R_1 \leftrightarrow R_2$ $\sim \left[\begin{array}{ccc\|c} 1 & 1 & 54 & 110 \\ 0 & -11 & -1459 & -2885 \\ 0 & 9 & -322 & -588 \end{array}\right] \quad R_2 \to R_2 - 27R_1;\ R_3 \to R_3 - 6R_1$ $\sim \left[\begin{array}{ccc\|c} 1 & 1 & 54 & 110 \\ 0 & -11 & -1459 & -2885 \\ 0 & 0 & -16673 & -32433 \end{array}\right] \quad R_3 \to 11R_3 + 9R_2$ $\sim \left[\begin{array}{ccc\|c} 1 & 16679 & 0 & 1678240 \\ 0 & -183403 & 0 & -23775698 \\ 0 & 0 & -16673 & -32433 \end{array}\right] \quad R_1 \to 16673R_1 + 54R_3; R_2 \to 16673R_2 - 1459R_3$ $\sim \left[\begin{array}{ccc\|c} 183403 & 0 & 0 & -887606162 \\ 0 & -183403 & 0 & -23775698 \\ 0 & 0 & -16673 & -32433 \end{array}\right] \quad R_1 \to 183403R_1 + 16679R_2$ By backward subs. x =0.6939 y =4.26306 z =1.9452
13	**Apply Gauss Jordan method to find the solution of the following system:** **x +3 y +3 z = 16, x +4y +3z =18, x +3y +4z = 19.**

	Solution: Given $A=\begin{bmatrix}1&3&3\\1&4&3\\1&3&4\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$, $B=\begin{bmatrix}16\\18\\19\end{bmatrix}$ $(A,B)=\left[\begin{array}{ccc\|c}1&3&3&16\\1&4&3&18\\1&3&4&19\end{array}\right]$ $=\left[\begin{array}{ccc\|c}1&3&3&16\\0&1&0&2\\0&0&1&3\end{array}\right]\begin{array}{l}\\R_2-R_1\\R_3-R_1\end{array}$ $=\left[\begin{array}{ccc\|c}1&0&3&10\\0&1&0&2\\0&0&1&3\end{array}\right]R_1-3R_2$ $=\left[\begin{array}{ccc\|c}1&0&0&1\\0&1&0&2\\0&0&1&3\end{array}\right]R_1-3R_3$ **Therefore x= 1,y= 2,z=3.**
14	**Solve Gauss Jordan method, the equations $2x+y+4z=12$, $8x-3y+2z=20$, $4x+11y-z=33$**
	Solution: $(\mathrm{A},\mathrm{B})=\left[\begin{array}{ccc\|c}2&1&4&12\\8&-3&2&20\\4&11&-1&33\end{array}\right]\Rightarrow\left[\begin{array}{ccc\|c}2&1&4&12\\0&-7&-14&-28\\0&9&-9&9\end{array}\right]\mathrm{R}_2\to\mathrm{R}_2-4\mathrm{R}_1,\ \mathrm{R}_3\to\mathrm{R}_3-2\mathrm{R}_1$ $\Rightarrow\left[\begin{array}{ccc\|c}2&1&4&12\\0&1&2&4\\0&1&-1&1\end{array}\right]\mathrm{R}_2\to\frac{\mathrm{R}_2}{-7},\ \mathrm{R}_3\to\frac{\mathrm{R}_3}{9}$ $\Rightarrow\left[\begin{array}{ccc\|c}2&1&4&12\\0&1&2&4\\0&0&3&3\end{array}\right]\mathrm{R}_3\to\mathrm{R}_2-\mathrm{R}_3$ $\Rightarrow\left[\begin{array}{ccc\|c}6&3&0&24\\0&3&0&6\\0&0&3&3\end{array}\right]\mathrm{R}_1\to3\mathrm{R}_1-4\mathrm{R}_3,\ \mathrm{R}_2\to3\mathrm{R}_2-2\mathrm{R}_3$ $\Rightarrow\left[\begin{array}{ccc\|c}6&0&0&18\\0&3&0&6\\0&0&3&3\end{array}\right]\mathrm{R}_1\to\mathrm{R}_1-\mathrm{R}_2$ $3z=3$; $2y=6$; $6x=18$ Hence the solution is, x = 1, y = 2, z = 3
15	**Solve the following linear system of equations by Gauss elimination method** $2x+3y+z=-1$, $5x+y+z=9$, $3x+2y+4z=11$ **(April/May 2023)**

	Solution: $[A,B]=\left[\begin{array}{ccc\|c} 2 & 3 & 1 & -1 \\ 5 & 1 & 1 & 9 \\ 3 & 2 & 4 & 11 \end{array}\right]$ $\sim\left[\begin{array}{ccc\|c} 2 & 3 & 1 & -1 \\ 0 & -13 & -3 & 23 \\ 0 & -5 & 5 & 25 \end{array}\right] \quad R_2 \to 2R_2 - 5R_1; R_3 \to 2R_3 - 3R_1$ $\sim\left[\begin{array}{ccc\|c} 2 & 3 & 1 & -1 \\ 0 & -13 & -3 & 23 \\ 0 & 0 & 80 & 210 \end{array}\right] \quad R_3 \to 13R_3 - 5R_2$ By Backward Subs. $80z = 210 \Rightarrow z = \frac{210}{80} \Rightarrow z = 2.625$ $-13y - 3z = 23 \Rightarrow y = 2.375$ $2x + 3y + z = -1 \Rightarrow x = -5.375$
16	**Solve the system of equations by Gauss Elimination method** **3x+y–z=3, 2x–8y+z= – 5, x–2y+9z=8.**
	Solution: The given system can be written as matrix form AX=B, where the coefficient matrix $A=\begin{pmatrix} 1 & -2 & 9 \\ 3 & 1 & -1 \\ 2 & -8 & 1 \end{pmatrix}$, $\boldsymbol{X}=\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix}$, $\boldsymbol{B}=\begin{pmatrix} 8 \\ 3 \\ -5 \end{pmatrix}$ The augumented matrix is $[\,\boldsymbol{A} \mid \boldsymbol{I}\,]=\left(\begin{array}{ccc\|c} 1 & -2 & 9 & 8 \\ 3 & 1 & -1 & 3 \\ 2 & -8 & 1 & -5 \end{array}\right)$ by Gauss elimination method, $[\,\boldsymbol{A} \mid \boldsymbol{I}\,]=\left(\begin{array}{ccc\|c} 1 & -2 & 9 & 8 \\ 0 & 7 & -28 & -21 \\ 0 & -4 & -17 & -21 \end{array}\right) \begin{array}{l} \\ \mathbf{R}_2 \to \boldsymbol{R}_2 - 3\boldsymbol{R}_1 \\ \boldsymbol{R}_3 \to \boldsymbol{R}_3 - 2\boldsymbol{R}_1 \end{array}$ $[\,\boldsymbol{A} \mid \boldsymbol{I}\,]=\left(\begin{array}{ccc\|c} 1 & -2 & 9 & 8 \\ 0 & 7 & -28 & -21 \\ 0 & 0 & -231 & -231 \end{array}\right) \boldsymbol{R}_3 \to 4\boldsymbol{R}_2 + 7\boldsymbol{R}_3$ $-231\boldsymbol{z} = -231 \Rightarrow \boldsymbol{z} = 1$ $7\boldsymbol{y} - 28\boldsymbol{z} = -21 \Rightarrow 7\boldsymbol{y} = 7 \Rightarrow \boldsymbol{y} = 1$ $\boldsymbol{x} - 2\boldsymbol{y} + 9\boldsymbol{z} = 8 \Rightarrow \boldsymbol{x} - 2 + 9 = 8 \Rightarrow \boldsymbol{x} = 1$ Hence the solution is $\boldsymbol{x}=1, \boldsymbol{y}=1, \boldsymbol{z}=1$

17 **Solve the system of equations by Gauss Elimination method**

$5x_1+x_2+x_3+x_4 = 4; x_1+7x_2+x_3+x_4 = 12; x_1+x_2+6x_3+x_4 = -5;\ x_1+x_2+x_3+4x_4 = -6$

Solution:

$$(A,B)=\left[\begin{array}{cccc|c}5&1&1&1&4\\1&7&1&1&12\\1&1&6&1&-5\\1&1&1&4&-6\end{array}\right]\sim\left[\begin{array}{cccc|c}5&1&1&1&4\\0&34&4&4&56\\0&4&29&4&-29\\0&4&4&19&-34\end{array}\right]\begin{array}{l}\\R_2\to 5R_2-R_1\\R_3\to 5R_3-R_1\\R_4\to 5R_4-R_1\end{array}$$

$$\sim\left[\begin{array}{cccc|c}5&1&1&1&4\\0&34&4&4&56\\0&0&970&120&-1210\\0&0&120&630&-1380\end{array}\right]\begin{array}{l}\\ \\R_3\to 34R_3-4R_2\\R_4\to 34R_4-4R_2\end{array}$$

$$\sim\left[\begin{array}{cccc|c}5&1&1&1&4\\0&17&2&2&28\\0&0&97&12&-121\\0&0&0&5967&-11934\end{array}\right]R_4\to 970R_4-120R_3$$

Hence $5967x_4=-11934$

$x_4=-2$

$97x_3+12x_4=-121$

$x_3=-1$

$17x_2+2x_3+2x_4=28$

$x_2=2$

$5x_1+x_2+x_3+x_4=4$

$x_1=1$

Hence the solution is $x_1=1; x_2=2; x_3=-1; x_4-2$

18 **Solve the following linear system of equations by Gauss elimination method** $x+2y-5z=-9$**,** $3x-y+2z=5$**,** $2x+3y-z=3$

Solution:

$$[A,B]=\left[\begin{array}{ccc|c}1&2&-5&-9\\3&-1&2&5\\2&3&-1&3\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}1&2&-5&-9\\0&-7&17&32\\0&-1&9&21\end{array}\right]\quad R_2\to R_2-3R_1; R_3\to R_3-2R_1$$

$$\sim\left[\begin{array}{ccc|c}1&2&-5&-9\\0&-7&17&32\\0&0&46&115\end{array}\right]\quad R_3\to 7R_3-R_2$$

By Backward Subs.

	$$46z = 115 \Rightarrow z = \frac{115}{46} \Rightarrow z = 2.5$$ $$-7y + 17z = 32 \Rightarrow y = 1.5$$ $$x + 2y - 5z = -9 \Rightarrow x = 0.5$$
19	**Solve the system of equations by Gauss Elimination method** **(Nov/Dec 2022)** $\boldsymbol{x_1 - 2x_2 - 3x_3 = 2, \;\; 3x_1 - 2x_2 = -1, \;\; -2x_2 - 3x_3 = 2}$
	Solution: $$[A,B] = \left[\begin{array}{ccc\|c} 1 & -2 & -3 & 2 \\ 3 & -2 & 0 & -1 \\ 0 & -2 & -3 & 2 \end{array}\right]$$ $$\sim \left[\begin{array}{ccc\|c} 1 & -2 & -3 & 2 \\ 0 & 4 & 9 & -7 \\ 0 & -2 & -3 & 2 \end{array}\right] \quad R_2 \to R_2 - 3R_1$$ $$\sim \left[\begin{array}{ccc\|c} 1 & -2 & -3 & 2 \\ 0 & 4 & 9 & -7 \\ 0 & 0 & 3 & -3 \end{array}\right] \quad R_3 \to 2R_3 + R_2$$ By Backward Subs. $$\boldsymbol{x_1 - 2x_2 - 3x_3 = 2}$$ $$\boldsymbol{4x_2 + 9x_3 = -7}$$ $$\boldsymbol{3x_3 = -3}$$ $\boldsymbol{\therefore\; x_1 = 0\,;\, x_2 = 0.5\,;\, x_3 = -1}$

UNIT II VECTOR SPACES

Vector spaces, Subspaces, Linear combinations, Linear independence and linear dependence, Bases and dimensions.

PART – A

1.	**Define Vector space.** **(Nov/Dec 2022)**
	Solution: A vector space (or linear space) V over a field F consists of a set on which two operations (called addition and scalar multiplication, respectively) are defined so that for each pair of elements x, y in V there is a unique element x + y in V, and for each element a in F and each element x in V there is a unique element ax in V, such that the following conditions hold. 1. x + y = y + x, $\forall x, y \in V$ (commutativity of addition). 2. (x + y) + z = x + (y + z), $\forall x, y, z \in V$ (associativity of addition). 3. There exists an element in V denoted by 0 such that x + 0 = x, $\forall x \in V$. 4. There exists an element y in V such that x + y = 0, $\forall x \in V$. 5. 1x = x, $\forall x \in V$ 6. (ab)x = a(bx), $\forall\; x \in V \text{ and } a, b \in F$ 7. a(x + y) = ax + ay, $\forall\; x, y \in V \text{ and } a \in F$ 8. (a + b)x = ax + bx, $\forall\; x \in V \text{ and } a, b \in F$
2.	**Let W be the union of the first and third quadrant in the xy-plane i.e.** $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, xy \geq 0 \right\}$ **(a) If u is in W and c is any scalar, then is cu in W? why?** **(b) Find the specific vectors u and v in W such that u + v is not in W.**

	Solution: (a)Let u = $\begin{bmatrix} x \\ y \end{bmatrix} \in W,\ \ xy \geq 0$ Let c be any scalar (+ve or –ve) then $cu = c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix} = c^2 xy \geq 0 \quad \therefore cu \in W$ (b) Let $u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, v = \begin{bmatrix} -1 \\ -5 \end{bmatrix}, u, v \in W$ but $u + v = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \notin W$
3.	**Define Subspace. Give some examples. (Apr/May 2022)**
	Solution: A subspace of a vector space V is a subset H of V that has three properties: 1. The zero vector of V is in H. 2. H is closed under vector addition. i.e.,$u + v \in H,\ \forall u, v \in H$ 3. H is closed under multiplication by scalars. i.e.,$cu \in H, \forall c \in F,\ u \in H$ Examples: 1. The set of all square matrices M_{nxn}(R)is a subspace of set of all matrices M_{mxn}(R). 2. The set of all polynomials of degree at most n with real coefficient P_n(R) is a subspace of P(R).
4.	**If $V = R^3$, then verify whether $W = \{(a_1, a_2, a_3) / 2a_1 - 7a_2 + a_3 = 0\}$ is a subspace or not.**
	Solution: Let $w_1 = (a_1, a_2, a_3), w_2 = (b_1, b_2, b_3); w_1, w_2 \in W\ \ 2a_1 - 7a_2 + a_3 = 0\ \&\ 2b_1 - 7b_2 + b_3 = 0$ $\alpha w_1 + \beta w_2 = \alpha(a_1, a_2, a_3) + \beta(b_1, b_2, b_3)$ $= (\alpha a_1, \alpha a_2, \alpha a_3) + (\beta b_1, \beta b_2, \beta b_3) = (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \alpha a_3 + \beta b_3)$ Consider $2(\alpha a_1 + \beta b_1) - 7(\alpha a_2 + \beta b_2) + (\alpha a_3 + \beta b_3) = \alpha(2a_1 - 7a_2 + a_3) + \beta(2b_1 - 7b_2 + b_3)$ $= \alpha.0 + \beta.0 = 0$ $\Rightarrow \alpha w_1 + \beta w_2 \in W \ \therefore W$ is a subspace of V.
5.	**What are the possible subspaces of R^2?**
	Solution: Subspaces of R^2 are (i) {(0. 0)} is the subspace of R^2 (ii) R^2 is the subspace of R^2 (iii) set of lines through the origin.
6.	**Determine whether the following sets are subspaces of R^3** **(a) R^2 (b) $W = \{(a_1, a_2, a_3) \in R^3 : a_1 = a_3 + 2\}$.**
	Solution: (a) Since the vectors in R^3 have three entries, whereas the vectors in R^2 have only two, The vector space R^2 is not a subspace of R^3, because R^2 is not even a subset of R^3. (b) Let $x = (a_1, a_2, a_3) \in W$, where $a_1 = a_3 + 2$ $y = (b_1, b_2, b_3) \in W$, where $b_1 = b_3 + 2$. $x + y = (a_1, a_2, a_3) + (b_1, b_2, b_3)$ $= (a_1 + b_1, a_2 + b_2, a_3 + b_3)$ Now, $a_1 + b_1 = (a_3 + 2) + (b_3 + 2) = a_3 + b_3 + 4 \neq a_3 + b_3 + 2$ $\Rightarrow x + y \notin W. \quad \because$ [I coordinate of $x+y \neq$ (III coordinate of $x+y$) +2] $\therefore$W is not a subspace of V.

7.	**Define linear combination of vectors.**
	Solution: Let V be a vector space over a field F and S is the subset of V. **Linear combination of vectors:** A vector v $\in$V is a linear combination of vectors of S if and only if there exists a finite number of vectors $u_1, \ldots, u_n \in$ S and $a_1, \ldots, a_n \in$ F such that $v = a_1u_1 + \ldots + a_nu_n$.
8.	**Define span of subset and generator set in vector space.**
	Span of subset S of V: The set consisting of all linear combinations of vectors in S. It is denoted by span(S). **Generator set**: A subset S of a vector space V is said to be generator set of V if span(S) =V.
9.	**Show that the matrices** $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$ **generate $M_{2\times2}$(F). (April/May 2023)**
	Solution: Let $\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$ be any arbitrary matrix in $M_{2\times2}$(F). Let a, b, c and d by any scalars so that $\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} = a\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + b\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + c\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + d\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$ $= \begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ c & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & d\end{pmatrix}$ $= \begin{pmatrix}a & b\\ c & d\end{pmatrix} \Rightarrow a = a_{11}$, $b = a_{12}$, $c = a_{21}$, and $d = a_{22}$. $\therefore$ The given matrices generate M_{2x2} (F).
10.	**For what values of h will y be in the subspace of R^3 spanned by v_1, v_2, v_3** **if** $v_1 = \begin{bmatrix}1\\ -1\\ -2\end{bmatrix}$, $v_2 = \begin{bmatrix}5\\ -4\\ -7\end{bmatrix}$, $v_3 = \begin{bmatrix}-3\\ 1\\ 0\end{bmatrix}$ and $y = \begin{bmatrix}-4\\ 3\\ h\end{bmatrix}$.
	Solution: Let Y is the subspace spanned by v_1, v_2, v_3 If y$\in$Y , then $y = av_1 + bv_2 + cv_3$ $\begin{bmatrix}-4\\ 3\\ h\end{bmatrix} = a\begin{bmatrix}1\\ -1\\ -2\end{bmatrix} + b\begin{bmatrix}5\\ -4\\ -7\end{bmatrix} + c\begin{bmatrix}-3\\ 1\\ 0\end{bmatrix}$ $-4 = a + 5b - 3c \rightarrow (1)$ $3 = -a - 4b + c \rightarrow (2)$ $h = -2a - 7b \rightarrow (3)$ $(1) + (2) \times 3 \Rightarrow 5 = -2a - 7b \rightarrow (4)$ From(3)and(4),we have $h = 5$.
11.	**Define linear dependent set and Independent set of a vector space.**
	Solution: **Linear dependent set:** A subset $\{u_1, u_2, \ldots\ldots, u_n\}$ of a vector space V over F is called linearly dependent set if there exist a finite number of scalars $a_1, a_2, \ldots.., a_n$ in F <u>not all zero</u>, such that $a_1u_1 + a_2u_2 + \ldots\ldots + a_nu_n = 0$. **Linear independent set:** A subset $\{u_1, u_2, \ldots\ldots, u_n\}$ of a vector space V over F is called linearly independent set if there exist a finite number of scalars $a_1, a_2, \ldots.., a_n$ in F <u>all are zero</u>, such that $a_1u_1 + a_2u_2 + \ldots\ldots + a_nu_n = 0$.

12.	**Determine whether the vectors** v_1 **= (1, –2, 3),** v_2 **= (5, 6, –1),** v_3 **= (3, 2, 1) form a linearly dependent or linearly independent set in** R^3**.**
	Solution: $\begin{vmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{vmatrix} = 1[6+2]+5[-2-6]+3[2-18] = 8-40-48 = 0$ Since the determinant value is zero, the vectors form linearly dependent set in R^3.
13.	**Show that the vectors u =(1, 2, 3), v =(2, 5, 7), w =(1, 3, 5) are linearly independent.** **(Nov/Dec 2022)**
	Solution: $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 3 & 7 & 5 \end{vmatrix} = 1[25-21]-2[10-9]+1[14-15] = 4-2-1 = 1 \neq 0$ Since the determinant value is non zero, the vectors are linearly independent.
14.	**Given** v_1 **&** v_2 **in a vector space V and let H = span(**$\{v_1, v_2\}$**). Show that H is a subspace of V.**
	Solution: The zero vector is in H, since $0 = 0v_1 + 0v_2$. To show that H is closed under vector addition, take two arbitrary vectors in H, say, $u = s_1v_1 + s_2v_2$ and $w = t_1v_1 + t_2v_2$ For the vector space V, $u + w = (s_1v_1 + s_2v_2) + (t_1v_1 + t_2v_2)$ $= (s_1 + t_1)v_1 + (s_2 + t_2)v_2$ So u + w is in H. Furthermore, if c is any scalar, $cu = c(s_1v_1 + s_2v_2) = (cs_1)v_1 + (cs_2)v_2$ which shows that cu is in H and H is closed under scalar multiplication. Thus H is a subspace of V.
15.	**Define Basis of a vector space** **(April/May 2023)**
	Solution: A subset S of a vector space is said to be a basis of V if (i) S is a linearly independent set and (ii) S generates V i.e. span(S) =V (i.e) a basis S for a vector space V is a **linearly independent subset** of V that **generates V**
16.	**Write down the standard basis for the vector space** R^n**,** $P_n(R)$ **and** $M_{2x2}(R)$**.**
	(1) For the vector space R^n, $\{e_1, e_2,\dots, e_n\}$ is a standard basis where $e_1 = (1, 0, 0, \dots, 0)$, $e_2 = (0, 1, 0,\dots,0),\dots, e_n = (0, 0,\dots, 0, 1)$; (2) For the vector space of all polynomials of degree $\leq$ n, $P_n(R)$,$\{1,x, x^2,...,x^n\}$ is a standard basis. (3) For the vector space $M_{2x2}(R)$, $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ is a standard basis.
17.	**Define finite dimensional vector space and dimension of vector space with examples.**
	Solution: If the basis of a vector space V consists of finite number of vectors, then the vector space V is called finite dimensional. The unique number of vectors in each basis for V is called the dimension of V and is denoted by dim(V). A vector space that is not finite dimensional is called infinite dimensional. **Example**: 1) Vector space F^n has dimension n. 2) Vector space $P_n(F)$ has dimension n+1. 3) Vector space $M_{mxn}(F)$ has dimension mn.
18.	**Show that dim(R) = 2**
	$C = \{a + ib : a, b \in R\}$ So, any element in C is a linear combination of {1,i}. So C=L({1,i})

	Further $a+ib=0 \Rightarrow a=0,\ b=0$. $\therefore \{1,i\}$ is L.I. Hence {1,i} is a basis of C over R $\therefore \dim C(R)=2$.
19.	**If V =A⊕B, then show that dimV=dim A+ dim B.**
	Proof : We know that, dim (A+B) = dimA+ dimB – dim(A∩B) Since V=A⊕B then V=A+B and A∩B = {0} ⇒ dim(A∩B) = 0. ∴ dim (A+B) = dimA+ dimB. Hence, dimV= dim A+ dim B.
20.	**Find the dimension of W, where $\{(x_1,x_2,x_3)/x_1+x_2+x_3=0\}$.** **(Apr/May 2022)**
	Solution: $x_1+x_2+x_3=0$ $x_3=-x_1-x_2$ $(x_1,x_2,-x_1-x_2)=(x_1,0,-x_1)+(0,x_2,-x_2)=x_1(1,0,-1)+x_2(0,1,-1)$ $\therefore B=\{(1,0,-1),(0,1,-1)\}$ is a basis of W. $\therefore \dim(W)=2$.
PART B	
1	**Prove that F^n, the set of all ordered n-tuples with entries from F is a vector space over F under coordinate wise addition and scalar multiplication.** **(April/May 2023)**
	Solution: Given $F^n=\{(a_1,a_2,...,a_n):\text{each } a_i\in F\}$. **Vector addition defined as coordinate wise addition:** $(a_1,a_2,...,a_n)+(b_1,b_2,...,b_n)=(a_1+b_1,a_2+b_2,...,a_n+b_n)$. **Scalar multiplication defined as coordinate wise scalar multiplication:** $a(a_1,a_2,...,a_n)=(aa_1,aa_2,...,aa_n)$. <u>To check vector addition of two vectors is in </u> F^n : Let $x=(a_1,a_2,...,a_n)\in F^n$, where each $a_i\in F$ and $y=(b_1,b_2,...,b_n)\in F^n$, where each $b_i\in F$. $\therefore x+y=(a_1,a_2,...,a_n)+(b_1,b_2,...,b_n)=(a_1+b_1,a_2+b_2,...,a_n+b_n)\in F^n$ ($\because$ each $a_i+b_i\in F$) <u>To check scalar multiplication of scalar and vector is in</u> F^n : Let $a\in F$ and $x=(a_1,a_2,...,a_n)\in F^n$, where each $a_i\in F$ $\therefore a\,x=(aa_1,aa_2,...,aa_n)\in F^n$ ($\because$ each $aa_i\in F$) (i) $x+y=(a_1,a_2,...,a_n)+(b_1,b_2,...,b_n)=(a_1+b_1,a_2+b_2,...,a_n+b_n)$ $y+x=(b_1,b_2,...,b_n)+(a_1,a_2,...,a_n)=(b_1+a_1,b_2+a_2,...,b_n+a_n)$ $=(a_1+b_1,a_2+b_2,...,a_n+b_n)$ $\therefore x+y=y+x$. (ii) Take $z=(c_1,c_2,...,c_n)\in F^n$, where each $c_i\in F$. $(x+y)+z=((a_1,a_2,...,a_n)+(b_1,b_2,...,b_n))+(c_1,c_2,...,c_n)=(a_1+b_1+c_1,a_2+b_2+c_2,...,a_n+b_n+c_n)$ $x+(y+z)=(a_1,a_2,...,a_n)+((b_1,b_2,...,b_n)+(c_1,c_2,...,c_n))=(a_1+b_1+c_1,a_2+b_2+c_2,...,a_n+b_n+c_n)$ $\therefore (x+y)+z=x+(y+z)$. (iii) $\forall\ x=(a_1,a_2,...,a_n)\in F^n$ there exist a vector $O=(0,0,...,0)\in F^n$ such that $x+O=(a_1,a_2,...,a_n)+(0,0,...,0)=(a_1+0,a_2+0,...,a_n+0)=(a_1,a_2,...,a_n)=x$ Similarly, $O+x=O$. $\therefore x+O=x=O+x$.

	Here, $O=(0,0,...,0)$ is a zero vector of F^n. (iv) For every $x=(a_1,a_2,...,a_n)\in F^n$ there exist a vector $-x=(-a_1,-a_2,...,-a_n)\in F^n$ such that $x+(-x)=(a_1,a_2,...,a_n)+(-a_1,-a_2,...,-a_n)=(a_1-a_1,a_2-a_2,...,a_n-a_n)=(0,0,...,0)=O$ Similarly, $(-x)+x=O$. $\therefore x+(-x)=O=(-x)+x.$ Here, $-x=(-a_1,-a_2,...,-a_n)$ is a additive inverse vector of F^n. (v) $1\in F$ and $x=(a_1,a_2,...,a_n)\in F^n$, where each $a_i\in F$. $1x=(1a_1,1a_2,...,1a_n)=(a_1,a_2,...,a_n)=x$ (vi) Let $a,b\in F$ and $x=(a_1,a_2,...,a_n)\in F^n$, where each $a_i\in F$. $\left(ab\right)x=\left(ab\right)(a_1,a_2,...,a_n)=(aba_1,aba_2,...,aba_n).\ a\left(bx\right)=a(ba_1,ba_2,...,ba_n)=(aba_1,aba_2,...,aba_n).$ $\therefore \left(ab\right)x=a\left(bx\right).$ (vii) Let $a\in F$ and $x=(a_1,a_2,...,a_n)\in F^n$, where each $a_i\in F$ $a(x+y)=a\left((a_1,a_2,...,a_n)+(b_1,b_2,...,b_n)\right)=a(a_1+b_1,a_2+b_2,...,a_n+b_n)$ $=(aa_1+ab_1,aa_2+ab_2,...,aa_n+ab_n)$ $ax+ay)=a(a_1,a_2,...,a_n)+a(b_1,b_2,...,b_n)=(aa_1,aa_2,...,aa_n)+(ab_1,ab_2,...,ab_n)$ $=(aa_1+ab_1,aa_2+ab_2,...,aa_n+ab_n)$ $\therefore a(x+y)=ax+ay$. (viii) Let $a,b\in F$ and $x=(a_1,a_2,...,a_n)\in F^n$, where each $a_i\in F$. $\left(a+b\right)x=\left(a+b\right)(a_1,a_2,...,a_n)=((a+b)a_1,(a+b)a_2,...,(a+b)a_n)$ $=(aa_1+ba_1,aa_2+ba_2,...,aa_n+ba_n).$ $ax+bx=a(a_1,a_2,...,a_n)+b(a_1,a_2,...,a_n)=(aa_1,aa_2,...,aa_n)+(ba_1,ba_2,...,ba_n)$ $=(aa_1+ba_1,aa_2+ba_2,...,aa_n+ba_n).$ $\therefore \left(a+b\right)x=ax+bx.$ Hence F^n is a vector space.
2	**Prove that** $P_n(R)$, **the set of all polynomials of degree at most n with real coefficient is a vector space under usual addition and constant multiplication of polynomial.**
	Solution: Given $P_n(R)=\left\{a_0+a_1t+a_2t^2+\cdots+a_nt^n : \text{polynomial of degree} \le n \text{ and each } a_i\in R\right\}$. **Vector addition defined as addition of polynomial** : $p(t)+q(t)=a_0+a_1t+a_2t^2+\cdots+a_nt^n+b_0+b_1t+b_2t^2+\cdots+b_nt^n$ $=(a_0+b_0)+(a_1+b_1)t+(a_2+b_2)t^2+\cdots+(a_n+b_n)t^n$ **Scalar multiplication defined as constant multiplication of polynomial:** $ap(t)=a\left(a_0+a_1t+a_2t^2+\cdots+a_nt^n\right)=aa_0+aa_1t+aa_2t^2+\cdots+aa_nt^n$ **<u>To check vector addition of two vectors is in</u>** $P_n(R)$ Let $p(t)=a_0+a_1t+a_2t^2+\cdots+a_nt^n$, where each $a_i\in R$ and $q(t)=b_0+b_1t+b_2t^2+\cdots+b_nt^n$, where each $b_i\in R$ then $(p+q)(t)=p(t)+q(t)=(a_0+b_0)+(a_1+b_1)t+(a_2+b_2)t^2+\cdots+(a_n+b_n)t^n\in P_n(R)$ $\because (p+q)(t)$ is a polynomial of degree $\le n$ and each $(a_i+b_i)\in R$ is a polynomial of degree $\le$ n and

	To check scalar multiplication of scalar and vector is in $\mathrm{P_n(R)}: a\mathrm{p(t)} = a\left(a_0 + a_1t + a_2t^2 + \cdots + a_nt^n\right) = aa_0 + aa_1t + aa_2t^2 + \cdots + aa_nt^n \in P_n(R)$ $\because (a\,\mathrm{p})(\mathrm{t})$ is a polynomial of degree $\le n$ and each $aa_i \in R$ (i) $\mathrm{(p+q)(t)} = \left[(a_0+b_0)+(a_1+b_1)t+(a_2+b_2)t^2+\cdots+(a_n+b_n)t^n\right]$ $= \left[(b_0+a_0)+(b_1+a_1)t+(b_2+a_2)t^2+\cdots+(b_n+a_n)t^n\right]$ $= \mathrm{(q+p)(t)}$ (ii) Let $\mathrm{r(t)} = \mathrm{c}_0 + c_1t + c_2t^2 + \cdots + c_nt^n$. $((\mathrm{p+q})+r)(\mathrm{t}) = \mathrm{(p+q)(t)+r(t)}$ $= \left[(a_0+b_0)+(a_1+b_1)t+(a_2+b_2)t^2+\cdots+(a_n+b_n)t^n\right] + (\mathrm{c}_0+c_1t+c_2t^2+\cdots+c_nt^n)$ $= (a_0+b_0+\mathrm{c}_0)+(a_1+b_1+c_1)t+(a_2+b_2+c_2)t^2+\cdots+(a_n+b_n+c_n)t^n$ $= (a_0+(b_0+\mathrm{c}_0))+(a_1+(b_1+c_1))t+(a_2+(b_2+c_2))t^2+\cdots+(a_n+(b_n+c_n))t^n$ $= (\mathrm{a}_0+a_1t+a_2t^2+\cdots+a_nt^n)+\left[(\mathrm{b}_0+c_0)+(\mathrm{b}_1+c_1)t+(\mathrm{b}_2+c_2)t^2+\cdots+(\mathrm{b}_n+c_n)t^n\right]$ $= \mathrm{p(t)+(q+r)(t)}$ $= (\mathrm{p}+(\mathrm{q}+r))(\mathrm{t}) \;\; \forall\, \mathrm{p(t), q(t), r(t)} \in \mathrm{P}_n(R)$ (iii) If all the coefficients are zero, $\mathrm{O(t)} = 0 + 0t + 0t^2 + \cdots + 0t^n$ is a zero polynomial. The zero polynomial is included in $\mathrm{P_n}$ even though its degree is not defined. (iv) $-\mathrm{p(t)} = (-1)\mathrm{p(t)} = (-1)(a_0+a_1t+a_2t^2+\cdots+a_nt^n) = -a_0 - a_1t - a_2t^2 - \cdots - a_nt^n$ $(\mathrm{p}+(-\mathrm{p}))(\mathrm{t}) = (a_0-a_0)+(a_1-a_1)t+(a_2-a_2)t^2+\cdots+(a_n-a_n)t^n = 0+0t+0t^2+\cdots+0t^n = \mathrm{O(t)}$ $-\mathrm{p(t)}$ acts as a additive inverse. (v) 1 p(t) = p(t) $\forall$ p(t) $\in \mathrm{P_n(R)}$ (vi) $(ab)\mathrm{p(t)} = (ab)(a_0+a_1t+a_2t^2+\cdots+a_nt^n)$ $= (aba_0)+(aba_1)t+(aba_2)t^2+\cdots+(aba_n)t^n$ $= (a)(\mathrm{b}\,a_0+ba_1t+ba_2t^2+\cdots+ba_nt^n)$ $= (a)(b\mathrm{p}(t)) \;\; \forall\, \mathrm{p(t)} \in \mathrm{P}_n(R), a,b \in R$ (viii) $(a+b)(\mathrm{p(t)}) = (a+\mathrm{b})(a_0+a_1t+a_2t^2+\cdots+a_nt^n)$ $= (aa_0+aa_1t+aa_2t^2+\cdots+aa_nt^n)+(\mathrm{b}\,a_0+ba_1t+ba_2t^2+\cdots+ba_nt^n)$ $= a\,\mathrm{p(t)} + \mathrm{bp(t)} \;\; \forall\, \mathrm{p(t)} \in \mathrm{P}_n(R), a,b \in R$ Hence, $\mathrm{P_n(R)}$ is the vector space.
3	**Prove that the set of all mxn matrices over F denoted by M_{mxn}(F) is a vector space over F with respect to the operation matrix addition and scalar multiplication of matrix.**
	Solution: Let $\mathrm{A,B,C} \in \mathrm{M(F)}$ and $\alpha, \beta \in \mathrm{F}$. Here $A = (a_{ij})_{mxn}$ $B = (b_{ij})_{mxn}$ $C = (c_{ij})_{mxn}$ (i) Closure: *Let* $A, B \in M(F)$ *Then* $A + B = (a_{ij}) + (b_{ij}) = (d_{ij}) \in M(F)$

(ii) Associative:

$Let\ A, B, C \in M(F)$

$Then\ (A+B)+C = A+(B+C)$

$LHS:$

$$(A+B)+C = \{(a_{ij})+(b_{ij})\}+(c_{ij})$$
$$=(d_{ij})+(c_{ij}) \qquad \because (d_{ij})=(a_{ij})+(b_{ij})$$
$$=(e_{ij}) \in M(F)$$

$RHS:$

$$A+(B+C)=(a_{ij})+\{(b_{ij})+(c_{ij})\}$$
$$=(a_{ij})+(d_{ij}) \qquad \because (d_{ij})=(b_{ij})+(c_{ij})$$
$$=(e_{ij}) \in M(F)$$

(iii) Identity:

Let $A \in$ M(F)

Then A+0=A

LHS: A+0=$(a_{ij})+(0_{ij}) = a_{ij} \in M(F)$

(iv) $Inverse:$

$Let\ A \in M(F)$

$Then\ A - A = 0$

$$A+(-A)=0$$
$$(a_{ij})-(a_{ij})=0_{ij} \ni (-a_{ij}) \in M(F)$$

(v) $Commutative:$

$Let\ A, B \in M(F)$

$Then\ A+B=B+A$

$$(a_{ij})+(b_{ij})=(b_{ij})+(a_{ij})$$

(vi) $\alpha A = A,\ \alpha \in F,\ A \in M(F)$

$$\alpha A = (\alpha(a_{ij})) = (\alpha a_{ij}) = (d_{ij}) \in M(F)$$

(vii) $\alpha(A+B)=\alpha A+\alpha B$

$$LHS: \alpha(A+B)=\alpha((a_{ij})+(b_{ij}))=\alpha(d_{ij})=(e_{ij}) \in M(F)$$
$$RHS: \alpha A+\alpha B=(\alpha(a_{ij}))+(\alpha(b_{ij}))=(\alpha a_{ij})+(\alpha b_{ij})=\alpha(d_{ij})=(e_{ij}) \in M(F)$$

$(viii)$ $(\alpha+\beta)A=\alpha A+\beta A$

$$LHS: (\alpha+\beta)A=(\alpha+\beta)(a_{ij})$$
$$=\alpha(a_{ij})+\beta(a_{ij})=(\alpha a_{ij})+(\beta a_{ij})=(e_{ij}) \in M(F)$$
$$RHS: \alpha A+\beta A=\alpha(a_{ij})+\beta(a_{ij})$$
$$=\alpha(a_{ij})+\beta(a_{ij})=(\alpha a_{ij})+(\beta a_{ij})=(e_{ij}) \in M(F)$$

(ix) $\alpha\beta(A)=\alpha(\beta A)$

$$(\alpha\beta)(A)=\alpha\beta(a_{ij})$$
$$=(\alpha\beta a_{ij})=(e_{ij}) \in M(F)$$

(x) $IA = A$

$$(I_{ij})(a_{ij})=(a_{ij})$$

$\therefore M_{mxn}(F)\ is\ a\ vector\ space\ under\ F.$

4	**Determine whether the set of all 2×2 matrices of the form** $\begin{pmatrix} \mathrm{a} & \mathrm{a+b} \\ \mathrm{a+b} & \mathrm{b} \end{pmatrix}, \mathrm{a,b} \in \mathrm{R}$ **with respect to standard matrix addition and scalar multiplication is a vector space or not? If not list all the axioms that fail to hold.**

Solution:

Let V = Set of all 2×2 matrices of the form $\begin{pmatrix} \mathrm{a} & \mathrm{a+b} \\ \mathrm{a+b} & \mathrm{b} \end{pmatrix}, \mathrm{a,b} \in \mathrm{R}$

(<u>To check vector addition of two vectors is in</u> V

Let $x=\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix} \in V, y=\begin{pmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & b_2 \end{pmatrix} \in V$

$$x+y=\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}+\begin{pmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & b_2 \end{pmatrix}=\begin{pmatrix} a_1+a_2 & a_1+b_1+a_2+b_2 \\ a_1+b_1+a_2+b_2 & b_1+b_2 \end{pmatrix} \in V$$

<u>To check scalar multiplication of scalar and vector is in</u> F^n :

Let $c \in F$ and $x=\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix} \in V$

$$\therefore c\,x=c\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}=\begin{pmatrix} ca_1 & ca_1+cb_1 \\ ca_1+cb_1 & cb_1 \end{pmatrix} \in V$$

(i) Let $x=\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix} \in V, y=\begin{pmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & b_2 \end{pmatrix} \in V$

$$x+y=\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}+\begin{pmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & b_2 \end{pmatrix}$$

$$=\begin{pmatrix} a_1+a_2 & a_1+b_1+a_2+b_2 \\ a_1+b_1+a_2+b_2 & b_1+b_2 \end{pmatrix}$$

$$y+x=\begin{pmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & b_2 \end{pmatrix}+\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}$$

$$=\begin{pmatrix} a_2+a_1 & a_2+b_2+a_1+b_1 \\ a_2+b_2+a_1+b_1 & b_2+b_1 \end{pmatrix}=\begin{pmatrix} a_1+a_2 & a_1+b_1+a_2+b_2 \\ a_1+b_1+a_2+b_2 & b_1+b_2 \end{pmatrix}$$

$\therefore\ x+y=y+x.$

(ii) Take $z=\begin{pmatrix} a_3 & a_3+b_3 \\ a_3+b_3 & b_3 \end{pmatrix} \in V$

$$(x+y)+z=\left[\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}+\begin{pmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & b_2 \end{pmatrix}+\right]\begin{pmatrix} a_3 & a_3+b_3 \\ a_3+b_3 & b_3 \end{pmatrix}$$

$$=\begin{pmatrix} a_1+a_2+a_3 & a_1+b_1+a_2+b_2+a_3+b_3 \\ a_1+b_1+a_2+b_2+a_3+b_3 & b_1+b_2+b_3 \end{pmatrix}$$

$$x+(y+z)=\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}+\left[\begin{pmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & b_2 \end{pmatrix}+\begin{pmatrix} a_3 & a_3+b_3 \\ a_3+b_3 & b_3 \end{pmatrix}\right]$$

$$=\begin{pmatrix} a_1+a_2+a_3 & a_1+b_1+a_2+b_2+a_3+b_3 \\ a_1+b_1+a_2+b_2+a_3+b_3 & b_1+b_2+b_3 \end{pmatrix}$$

$\therefore (x+y)+z=x+(y+z).$

(iii) For all $x=\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}\in V$ there exist a matrix $\mathrm{O}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\in V$ such that

$$x+\mathrm{O}=\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}=x$$

Similarly, $\mathrm{O}+x=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}=\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}=x$

Here, $\mathrm{O}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a zero vector (identity vector) of V.

(iv) For every $x=\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}\in V$ there exist a vector $-x=\begin{pmatrix} -a_1 & -(a_1+b_1) \\ -(a_1+b_1) & -b_1 \end{pmatrix}\in V$ such that $x+(-x)=\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}+\begin{pmatrix} -a_1 & -(a_1+b_1) \\ -(a_1+b_1) & -b_1 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}=\mathrm{O}$

Similarly, $(-x)+x=\mathrm{O}$.

$\therefore x+(-x)=\mathrm{O}=(-x)+x.$

Here, $-x=\begin{pmatrix} -a_1 & -(a_1+b_1) \\ -(a_1+b_1) & -b_1 \end{pmatrix}$ is an additive inverse vector of V .

(v) $1\in F$ and $x=\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}\in V$

$$1x=1\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}=\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}=x$$

(vi) Let $a,b\in F$ and $x=\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}\in V$

$$(ab)x=(ab)\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}=\begin{pmatrix} ab\,a_1 & ab(a_1+b_1) \\ ab(a_1+b_1) & abb_1 \end{pmatrix}=a\begin{pmatrix} ba_1 & b(a_1+b_1) \\ b(a_1+b_1) & bb_1 \end{pmatrix}=a(bx)$$

$\therefore (ab)x=a(bx).$

(vii) Let $a\in F$, $x=\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}\in V$ and $y=\begin{pmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & b_2 \end{pmatrix}\in V$

$$\begin{aligned} a(x+y)&=a\left[\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}+\begin{pmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & b_2 \end{pmatrix}\right] \\ &=\begin{pmatrix} a(a_1+a_2) & a(a_1+b_1+a_2+b_2) \\ a(a_1+b_1+a_2+b_2) & a(b_1+b_2) \end{pmatrix} \\ &=a\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}+a\begin{pmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & b_2 \end{pmatrix} \\ &=ax+ay \end{aligned}$$

$\therefore a(x+y)=ax+ay$.

(viii) Let $a,b\in F$ and $x=\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}$.

$$\left(a+b\right)x=\left(a+b\right)\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}=\begin{pmatrix} \left(a+b\right)a_1 & \left(a+b\right)(a_1+b_1) \\ \left(a+b\right)(a_1+b_1) & \left(a+b\right)b_1 \end{pmatrix}$$

$$=\begin{pmatrix} aa_1+ba_1 & a(a_1+b_1)+b(a_1+b_1) \\ a(a_1+b_1)+b(a_1+b_1 & ab_1+bb_1 \end{pmatrix}$$

$$=a\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}+b\begin{pmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{pmatrix}$$

$\therefore \left(a+b\right)x=ax+bx.$

Hence V is a vector space.

5 **Let V be the set of all positive real numbers. Define the vector addition and scalar multiplication as follows:** $x+y=xy$ and $kx=x^k$. **Determine whether V is a vector space over R with respect to above operations.**

Solution:

Let $V=\{x\in R^+ / x+y=xy$ and $kx=x^k$, where $x,y\in R^+=(0,\infty)\}$

<u>To check vector addition of two vectors is in V</u>

Take $x_1,\ y_1\in V\Rightarrow x_1+y_1=x_1y_1\in V$ ($\because$ product of two positive real numbers is positive).

<u>To check scalar multiplication of scalar and vector is in V</u>

Take $x_1\in V$ and k is any scalar. $k\,x_1=x_1{}^k\in V.$

(i) $x_1+y_1=x_1y_1=y_1x_1=y_1+x_1$

(ii) $\left(x_1+y_1\right)+z_1=(x_1y_1)+z_1$

$$\begin{aligned} &=\left(x_1y_1\right)z_1 \\ &=x_1\left(y_1z_1\right) \\ &=x_1+\left(y_1z_1\right) \\ &=x_1+\left(y_1+z_1\right) \end{aligned}$$

(iii) Let $e\in V, x_1\in V$

$x_1+e=x_1e=x_1 \Rightarrow e=1\in V$

(iv) $(-x_1)+x_1=\left(-x_1\right)x_1$

$$\begin{aligned} &=\left(-x_1\right)(x_1e) \\ &=\left(-x_1x_1\right)e \\ &=e \Rightarrow -x_1=\frac{1}{x_1}\in V \end{aligned}$$

(v) Let $1\in F$ and $x\in V$.

$1x=x^1=x$

(vi) Let $a,b\in F$ and $x\in V$.

$\left(ab\right)x=x^{ab}=(x^b)^a=a(x^b)=a(bx).$

$\therefore \left(ab\right)x=a\left(bx\right).$

(vii) Let $a\in F$ and $x,y\in V$.

$a(x+y)=a(xy)=(xy)^a=(x^a)(y^a)=ax+ay$

$\therefore a\,(x+y)=ax+ay\,.$

	(viii) Let $a,b\in F$ and $x\in V$. $(a+b)x=x^{a+b}=x^a x^b=x^a+x^b=ax+bx.$ $\therefore (a+b)x=ax+bx.$ Hence V is a vector space.
6	**Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1\cup W_2$ is a subspace of V if and only if $W_1\subseteq W_2$ or $W_2\subseteq W_1$. (Apr/May 2022, Nov/Dec 2022)**
	Proof: Assume that $W_1\cup W_2$ is a subspace of V . To prove that $W_1\subseteq W_2$ or $W_2\subseteq W_1$. Assume the contrary that $W_1\not\subseteq W_2$ and $W_2\not\subseteq W_1$. Then there exist elements $x\in W_1$ but $x\notin W_2$ and $y\in W_2$ but $y\notin W_1$. Therefore, x and $y\in W_1\cup W_2$. Since $W_1\cup W_2$ is a subspace of V then $x+y\in W_1\cup W_2$. **Case 1**: Take $x+y\in W_1$. Now $x+y\in W_1$ and $-x\in W_1$ then $-x+(x+y)\in W_1\Rightarrow(-x+x)+y\in W_1$ $y\in W_1$. This is a contradiction. **Case 2**: Take $x+y\in W_2$. Now $x+y\in W_2$ and $-y\in W_2$ then $(x+y)+(-y)\in W_2\Rightarrow x\in W_1$. This is a contradiction. Therefore, $W_1\subseteq W_2$ or $W_2\subseteq W_1$ Conversely, assume that $W_1\subseteq W_2$ or $W_2\subseteq W_1$. To prove that $W_1\cup W_2$ is a subspace of V . Let $W_1\subseteq W_2$. Then $W_1\cup W_2=W_2$, a subspace of V . Let $W_2\subseteq W_1$. Then $W_1\cup W_2=W_1$, a subspace of V .
7	**Prove that intersection of two subspace is again a subspace over a vector space.**
	Proof: Let V be a vector space over a field F . Let W_1,W_2 be two subspace of a vector space V(F). Let $0\in W_1$ & $0\in W_2$ Then $0\in W_1\cap W_2$ $W_1\cap W_2$ is not empty. Let W_1 be a subspace Let $\alpha,\beta\in F,\ u,v\in W_1$ Then $\alpha u+\beta v=W_1------(1)$ Let W_2 be a subspace Let $\alpha,\beta\in F,\ u,v\in W_2$ Then $\alpha u+\beta v=W_2------(2)$ From (1) and (2) $\alpha u+\beta v\in W_1\cap W_2$ $W_1\cap W_2$ is a subspace of V(F).

8 **Prove that the vector $(2,6,8)$ can be expressed as a linear combinations of $(1,2,1)$, $(-2,-4,-2)$, $(0,2,3)$, $(2,0,-3)$, $(-3,8,16)$.** **(April/May 2022, April/May 2023)**

Solution:

Let $u_1=(1,2,1)$; $u_2=(-2,-4,-2)$; $u_3=(0,2,3)$; $u_4=(2,0,-3)$; $u_5=(-3,8,16)$.

Determine the scalars a_1,a_2,a_3,a_4,a_5 such that

$$(2,6,8)=a_1u_1+a_2u_2+a_3u_3+a_4u_4+a_5u_5$$
$$=a_1(1,2,1)+a_2(-2,-4,-2)+a_3(0,2,3)+a_4(2,0,-3)+a_5(-3,8,16)$$
$$=(a_1-2a_2+2a_4-3a_5,2a_1-4a_2+2a_3+8a_5,a_1-2a_2+3a_3-3a_4+16a_5)$$

Therefore,

$$a_1-2a_2+0a_3+2a_4-3a_5=2$$
$$2a_1-4a_2+2a_3+0a_4+8a_5=6$$
$$a_1-2a_2+3a_3-3a_4+16a_5=8$$

Solving the above system by elimination method

$$\left(\begin{array}{ccccc|c} 1 & -2 & 0 & 2 & -3 & 2\\ 2 & -4 & 2 & 0 & 8 & 6\\ 1 & -2 & 3 & -3 & 16 & 8 \end{array}\right)$$

$$\sim\left(\begin{array}{ccccc|c} 1 & -2 & 0 & 2 & -3 & 2\\ 0 & 0 & 2 & -4 & 14 & 2\\ 0 & 0 & 3 & -5 & 19 & 6 \end{array}\right)\begin{array}{l} R_2\leftrightarrow R_2-2R_1\\ R_3\leftrightarrow R_3-R_1 \end{array}$$

$$\sim\left(\begin{array}{ccccc|c} 1 & -2 & 0 & 2 & -3 & 2\\ 0 & 0 & 1 & -2 & 7 & 1\\ 0 & 0 & 0 & 2 & -4 & 6 \end{array}\right)R_3\leftrightarrow 2R_3-3R_2$$

The reduced equations are

$$a_1-2a_2+2a_4-3a_5=2$$
$$a_3-2a_4+7a_5=1$$
$$2a_4-4a_5=6$$

Take $a_2=0$ and $a_5=0$ we get, $a_1=-4$, $a_3=7$, $a_4=3$.

$\therefore (2,6,8)=-4(1,2,1)+7(0,2,3)+3(2,0,-3)$

Thus, $(2,6,8)$ can be expressed as a linear combinations of $(1,2,1)$, $(-2,-4,-2)$, $(0,2,3)$, $(2,0,-3)$, $(-3,8,16)$

Take $a_2=0$ and $a_5=0$ we get

$$a_1+2a_4=2\ ---(1)$$
$$2a_1+2a_3=6\ ---(2)$$
$$a_1+3a_3-3a_4=8\ ---(3)$$

$(3)-(1)\Rightarrow 3a_3-5a_4=6\ ---(4)$

$(2)-2\times(1)\Rightarrow 2a_3-4a_4=2\ ---(5)$

Solving (4) and (5)

2x(4) –3x(5) $\Rightarrow 2a_4=6\Rightarrow a_4=3$

$\therefore a_3=7, a_1=-4$.

9 **Prove that the vector $(2,-5,4)$ can be expressed as a linear combinations of $(1,-3,2)$, $(2,-1,1)$**

Solution:

Let $\alpha_1,\alpha_2\in F$

Let $(2,-5,4)=\alpha_1(1,-3,2)+\alpha_2(2,-1,1)$

$$=(\alpha_1,-3\alpha_1,2\alpha_1)+(2\alpha_2,-\alpha_2,\alpha_2)$$
$$=(\alpha_1+\alpha_2,-3\alpha_1-\alpha_2,2\alpha_1+\alpha_2)$$

	$\alpha_1 + 2\alpha_2 = 2 - - - - - - - (1)$ $-3\alpha_1 - \alpha_2 = -5 - - - - - - (2)$ $2\alpha_1 + \alpha_2 \;\; = 4 - - - - - - - (3)$ $\alpha_1 = 1$ $(1) \Rightarrow 1 + 2\alpha_2 = 2 \Rightarrow \alpha_2 = \frac{1}{2}$
10	**Determine the given set in $P_4(R)$ is linearly dependent or linearly independent for $x^4 - x^3 + 5x^2 - 8x + 6, -x^4 + x^3 - 5x^2 + 5x - 3, x^4 + 3x^2 - 3x + 5$ and $2x^4 + x^3 + 4x^2 + 8x$.** **(Apr/May 2022, Nov/Dec 2022)**
	Solution: Let $v_1 = x^4 - x^3 + 5x^2 - 8x + 6, v_2 = -x^4 + x^3 - 5x^2 + 5x - 3, v_3 = x^4 + 3x^2 - 3x + 5$ $v_4 = 2x^4 + x^3 + 4x^2 + 8x$ Let $a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0$ where a_1, a_2, a_3, a_4 are scalars $\Rightarrow a_1\left(x^4 - x^3 + 5x^2 - 8x + 6\right) + a_2\left(-x^4 + x^3 - 5x^2 + 5x - 3\right)$ $\qquad + a_3\left(x^4 + 3x^2 - 3x + 5\right) + a_4\left(2x^4 + x^3 + 4x^2 + 8x\right) = 0$ $x^4\left(a_1 - a_2 + a_3 + 2a_4\right) + x^3\left(-a_1 + a_2 + a_4\right) + x^2\left(5a_1 - 5a_2 + 3a_3 + 4a_4\right)$ $\qquad + x\left(-8a_1 + 5a_2 - 3a_3 + 8a_4\right) + \left(6a_1 - 3a_2 + 5a_3\right) = 0$ Equating the coefficients $\Rightarrow a_1 - a_2 + a_3 + 2a_4 = 0,$ $-a_1 + a_2 + a_4 = 0,$ $5a_1 - 5a_2 + 3a_3 + 4a_4 = 0,$ $-8a_1 + 5a_2 - 3a_3 + 8a_4 = 0,$ $6a_1 - 3a_2 + 5a_3 = 0$ By solving the above equations $a_1 = 4, a_2 = 3, a_3 = -3, a_4 = 1$. Hence the given set in $P_4(R)$ is linearly dependent.
11	**Verify whether the first polynomial can be expressed as a linear combination of the other two in $P_3(R)$ for the given $x^3 - 8x^2 + 4x, x^3 - 2x^2 + 3x - 1$ and $x^3 - 2x + 3$.**
	Solution: Let $x^3 - 8x^2 + 4x = a\left(x^3 - 2x^2 + 3x - 1\right) + b\left(x^3 - 2x + 3\right)$ $x^3 - 8x^2 + 4x = x^3(a+b) + x^2(-2a) + x(3a - 2b) + (-a + 3b)$ $\Rightarrow 1 = a + b, -8 = -2a, 4 = 3a - 2b, 0 = -a + 3b$ $\Rightarrow a = 4, b = -3$ from equations $a + b = 1$ and $2a = 8$. But $3a - 2b = 4$ & $-a + 3b = 0$ are not satisfied for a=4 and b=-3. Therefore $x^3 - 8x^2 + 4x$ is not a linear combination $x^3 - 2x^2 + 3x - 1$ and $x^3 - 2x + 3$.
12	**In M2×3(R), show that the set $\left\{\begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix}, \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix}, \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix}\right\}$ is linearly dependent**
	Solution: Consider, $a\begin{bmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{bmatrix} + b\begin{bmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{bmatrix} + c\begin{bmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} a & -3a & 2a \\ -4a & 0 & 5a \end{bmatrix} + \begin{bmatrix} -3b & 7b & 4b \\ 6b & -2b & -7b \end{bmatrix} + \begin{bmatrix} -2c & 3c & 11c \\ -c & -3c & 2c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

	Equating $a-3b-2c=0---(1)$ $-3a+7b+3c=0---(2)$ $2a+4b+11c=0---(3)$ $-4a+6b-c=0---(4)$ $-2b-3c=0---(5)$ $5a-7b+2c=0---(6)$ From (5) $c=\frac{-2b}{3}$ Substitute in (1) $a-3b-2\left(\frac{-2b}{3}\right)=0 \Rightarrow a-3b+\left(\frac{4b}{3}\right)=0 \Rightarrow a-\left(\frac{5b}{3}\right)=0 \Rightarrow a=\left(\frac{5b}{3}\right)$ Take b=3 we get, a= 5, c = –2. (they satisfies all the above equations) Hence the given set is linearly dependent.
13	**Show that the polynomials** x^2+3x-2, $2x^2+5x-3$ and $-x^2-4x+4$ **generate P_2(R).**
	Solution: Let $a_2x^2+a_1x+a_0$ be any arbitrary 2nd degree polynomial in P_2(R). Determine the scalars a,b,c such that $a_2x^2+a_1x+a_0=a(x^2+3x-2)+b(2x^2+5x-3)+c(-x^2-4x+4)$ $=(ax^2+3ax-2a)+(2bx^2+5bx-3b)+(-cx^2-4cx+4c)$ $=(a+2b-c)x^2+(3a+5b-4c)x+(-2a-3b+4c)$ On equating we get, $a+2b-c=a_2$ ---(1) $3a+5b-4c=a_1$ ---(2) $-2a-3b+4c=a_0$ ---(3) 2x (1) +(3) gives b + 2c $=2a_2+a_0$ ---(4) 3x (1) – (2) gives b + c $=3a_2-a_1$ ---(5) (4) – (5) gives c $=-a_2+a_1+a_0$ Substitute c in (5) we get, b$+(-a_2+a_1+a_0)=3a_2-a_1$ i.e. b $=4a_2-2a_1-a_0$ Substitute b and c in (1) we get, $a+2(4a_2-2a_1-a_0)-(-a_2+a_1+a_0)=a_2$ i.e. $a=-8a_2+5a_1+3a_0$ $\therefore a_2x^2+a_1x+a_0=(-8a_2+5a_1+3a_0)(x^2+3x-2)+(4a_2-2a_1-a_0)(2x^2+5x-3)$ $+(-a_2+2a_1+a_0)(-x^2-4x+4)$ Hence, the given polynomials generate P_2(R).
14	**Prove that the vectors u_1=(2,-3,1), u_2=(1,4,-2), u_3=(-8,12,-4), u_4=(1,37,-17)and u_5=(-3,-5,8) generate R^3. Find a subset of the set $\{u_1,u_2,u_3,u_4,u_5\}$ that is a basis for R^3. (Nov/Dec 2022)**
	Solution: Let $u_1=(2,-3,1)$, $u_2=(1,4,-2)$, $u_3=(-8,12,-4)$, $u_4=(1,37,-17)$ and $u_5=(-3,-5,8)$ Here $\mathrm{Dim}\,R^3=3$.

We know that

If Dim V = n then any set of more than n is L.D

$\therefore S = \{u_1, u_2, \ldots, u_5\}$

Here we have to find subset of three L.I vectors.

Choose $S_1 = \{u_1, u_2\}$

Then $\alpha_1 u_1 + \alpha_2 u_2 = 0$

$\alpha_1(2,-3,1) + \alpha_2(1,4,-2) = 0$

$(2\alpha_1 + \alpha_2, -3\alpha_1 + 4\alpha_2, \alpha_1 - 2\alpha_2) = (0,0,0)$

Solving we get $\alpha_1 = 0, \alpha_2 = 0$.

$\therefore S_1$ is L.I

Choose u_3 which is not in S

Choose $u_3 \in L(S_1)$

$(-8,12,-4) = \alpha_1 u_1 + \alpha_2 u_2$

$(-8,12,-4) = \alpha_1(2,-3,1) + \alpha_2(1,4,-2)$

$(-8,12,-4) = (2\alpha_1 + \alpha_2, -3\alpha_1 + 4\alpha_2, \alpha_1 - 2\alpha_2)$

$2\alpha_1 + \alpha_2 = -8 -------- (1)$

$-3\alpha_1 + 4\alpha_2 = 12 ------- (2)$

$\alpha_1 - 2\alpha_2 = -4 -------- (3)$

Solving (1), (2) & (3) we get

$\alpha_1 = -4, \alpha_2 = 0$

We know that If S is L.I set then $S \cup \{v\}$ is L.D iff $V \notin L(S)$

$\therefore S_1 \cup \{u_3\}$ is L.D iff $u_3 \in L(S)$

u_3 not belongs to the basis

Choose $u_4 \in L(S_1)$

$(1,37,-17) = \alpha_1 u_1 + \alpha_2 u_2$

$(1,37,-17) = \alpha_1(2,-3,1) + \alpha_2(1,4,-2)$

$(1,37,-17) = (2\alpha_1 + \alpha_2, -3\alpha_1 + 4\alpha_2, \alpha_1 - 2\alpha_2)$

$2\alpha_1 + \alpha_2 = 1 -------- (4)$

$-3\alpha_1 + 4\alpha_2 = 37 ------- (5)$

$\alpha_1 - 2\alpha_2 = -17 -------- (6)$

Solving (4), (5) & (6) we get

$\alpha_1 = -3, \alpha_2 = 7$

$\therefore S_1 \cup \{u_4\}$ is L.D iff $u_4 \in L(S)$

u_4 not belongs to the basis.

Choose $u_5 \in L(S_1)$

$(-3,-3,8) = \alpha_1 u_1 + \alpha_2 u_2$

$(-3,-3,8) = \alpha_1(2,-3,1) + \alpha_2(1,4,-2)$

$(-3,-3,8) = (2\alpha_1 + \alpha_2, -3\alpha_1 + 4\alpha_2, \alpha_1 - 2\alpha_2)$

$2\alpha_1 + \alpha_2 = -3 -------- (7)$

$-3\alpha_1 + 4\alpha_2 = -3 ------- (8)$

$\alpha_1 - 2\alpha_2 = 8 -------- (9)$

Solving (4),(5) & (6) we get

$\alpha_1 = -3, \alpha_2 = 7$

$\therefore S_1 \cup \{u_5\}$ is L.D iff $u_5 \notin L(S)$

u_5 belongs to the basis.

$\therefore S = \{u_1, u_2, \ldots, u_5\}$ form a basis.

15 **Let $S = \{v_1, v_2, v_3\}$ where $v_1 = (1,-3,-2), v_2 = (-3,1,3), v_3 = (-2,-10,-2)$. Verify whether S form a basis or not.**

Solution:

Let $a_1v_1 + a_2v_2 + a_3v_3 = 0$ where a_1, a_2, a_3, a_4 are scalars

$a_1(1,-3,-2) + a_2(-3,1,3) + a_3(-2,-10,-2) = 0$

$(a_1 - 3a_2 - 2a_3, -3a_1 + a_2 - 10a_3, -2a_1 + 3a_2 - 2a_3) = 0$

$\Rightarrow a_1 - 3a_2 - 2a_3 = 0, -3a_1 + a_2 - 10a_3 = 0, -2a_1 + 3a_2 - 2a_3 = 0$

$$\begin{vmatrix} 1 & -3 & -2 \\ -3 & 1 & -10 \\ -2 & 3 & -2 \end{vmatrix} = 0$$

Hence the given vectors are linearly dependent

Therefore, the given set of vectors does not form a basis

The set of vectors is not a basis.

16 **Check whether the sets are basis of R^3 or not** $\{(1,0,-1),(2,5,1),(0,-4,3)\}$

Solution:

$v_1 = (1,0,-1),\ v_2 = (2,5,1),\ v_3 = (0,-4,3)$

Let $S = \{(1,0,-1),(2,5,1),(0,-4,3)\}$

To prove A is basis or not:

(i.e) (i) S is linearly independent

(ii) V=L(S)

$$(i)\quad A = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 5 & 1 \\ 0 & -4 & 3 \end{vmatrix}$$

$$= 1(15+4) - 0 - 1(-8-0) = 27 \neq 0$$

It is linearly independent.

Let $(x,y,z) \in R^3$

Consider $a(1,0,-1) + b(2,5,1) + c(0,-4,3) = (x,y,z)$

$a + 2b = x,\ 5b - 4c = y,\ -a + b + 3c = z$

$$[A/B] \sim \begin{pmatrix} 1 & 2 & 0 & x \\ 0 & 5 & -4 & y \\ -1 & 1 & 3 & z \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & x \\ 0 & 5 & -4 & y \\ 0 & 3 & 3 & z+x \end{pmatrix} R_3 \to R_3 + R_1$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & x \\ 0 & 5 & -4 & y \\ 0 & 0 & 27 & 5z+5x-3y \end{pmatrix} R_3 \to 5R_3 - 3R_2$$

	Using back substitution method, $27c = 5z + 5x - 3y$ $c = \dfrac{5z+5x-3y}{27}$ $5b - 4c = y$ $5b - 4\left(\dfrac{5z+5x-3y}{27}\right) = y$ $5b = y + \dfrac{20z+20x-12y}{27}$ $5b = \dfrac{27y+20z+20x-12y}{27} = \dfrac{15y+20z+20x}{27}$ $b = \dfrac{3y+4z+4x}{27}$ $a + 2b = x$ $a = x - 2\left(\dfrac{3y+4z+4x}{27}\right) = \dfrac{19x-6y-8z}{27}$ $\therefore V = L(S)$
17	**State and prove Completion of basis theorem**
	Statement: Let $\dim V = n$ then any linearly independent subset of V can be extended to form a basis of V. **Proof:** Let $S = \{v_1, v_2, ..., v_n\}$ be a linearly independent set of vectors in V. Given: $\dim V = n$ If $L(S) = V$, then S is a basis of V otherwise $L(S) \neq V$, then there is an element $V_{r+1} \in V$ such that $V_{r+1} \notin L(S)$ We know that $S \cup \{v\}$ is linearly dependent iff $V \in L(S)$ where S be an independent set in V(F) and v be a vector in V(F) that is not in S. $\therefore S \cup \{v_{r+1}\}$ is linearly independent if $V_{r+1} \notin L(S)$ $\therefore S_1 = \{v_1, v_2, ..., v_r, v_{r+1}\}$ is linearly independent If $L(S_1) = V$, then S_1 is basis of V otherwise $L(S_1) \neq V$, then there is element $V_{r+2} \in V$ such that $V_{r+2} \notin L(S_1)$ $\Rightarrow S_2 = \{S_1 \cup V_{r+2}\}$ is linearly independent if $V_{r+2} \notin L(S_1)$. $\therefore S_2$ is linearly independent Proceed until the dimension value. Since $\dim V = n$ $S = (v_1, v_2, ..., v_n)$ spans V It is a basis of V.
18	**Let S be a linearly independent set in V(F) and let v be a vector in V(F) that is not in S. Then $S \cup \{v\}$ is linearly dependent iff $v \in L(S)$.**
	Proof: Let S be a linearly independent set in V(F). Let V be a vector in V(F) that is not in S. Also given $S \cup (V)$ is linearly dependent To prove $V \in L(S)$ If $S \cup (V)$ is linearly dependent then there exists a vector $v_1, v_2, v_3, ...v_n \in S$ *and* $\alpha_1, \alpha_2, \alpha_3, ...\alpha_n \in F$

	Then $\alpha_1 v_1 + \alpha_1 v_2 + ... + \alpha_1 v_n = 0$ not all α_i are zero. Since S is linearly independent one of v_i's is v. (i.e) $v_i = v$ $\alpha_1 v_1 = \alpha_2 v_2 - \alpha_3 v_3 - ...... - \alpha_n v_n$ $v = \alpha_2 \ \alpha_1^{-1} v_2 - \alpha_3 \ \alpha_1^{-1} v_3 - .... - \alpha_n \ \alpha_1^{-1} v_n$ v is a linear combination of element in S. $\therefore v \in L(S)$ Conversely, $Let \ \ v \in L(S)$ To prove: $S \cup (V)$ is linearly dependent then there exists a finite vectors $v_1, v_2, v_3, ...v_n \in S \ and \ \alpha_1, \alpha_2, \alpha_3, ...\alpha_n \in F$ such that $v = \alpha_1 v_1 + \alpha_1 v_2 + ... + \alpha_1 v_n$ v can be expressed as a linear combination of other vectors (i.e) $\alpha_1 v_1 + \alpha_1 v_2 + ... + \alpha_1 v_n - v = 0$ This is the linear combination with not all coefficient as zero. $\therefore v_1, v_2, v_3, ...v_n$ are linearly dependent over R $\therefore S \cup (V)$ is linearly dependent.

UNIT III LINEAR TRANSFORMATION

Linear transformation - Rank space and null space - Rank and nullity - Dimension theorem - Matrix representation of linear transformation - Eigenvalues and eigenvectors of linear transformation.

PART A

1.	**Define Linear transformation on a vector space.**
	Solution: Let V and W be vector spaces over F. We call a function T : V $\rightarrow$ W a linear transformation from V to W if for all x, y $\in$ V and c $\in$ F, we have T(x + y) = T(x) + T(y) and T(cx) = cT(x).
2.	**Show that T: $R^2 \rightarrow R^2$ defined by T(a_1, a_2) = (2a_1 + a_2, a_1) is linear.**
	Solution: Let X, Y $\in R^2$ and c $\in$ R, where X = (b_1, b_2), Y = (d_1, d_2) Since we know that T is linear if and only if T(cX + Y) = cT (X) + T(Y) Now (cX + Y) = (cb_1 + d_1 , cb_2 + d_2) T(cX + Y) = (2(cb_1 + d_1) + cb_2 + d_2 , cb_1 + d_1) Also cT(X) + T(Y) = c(2b_1 +b_2 , b_1) + (2d_1 + d_2 , d_1) = (2cb_1 + cb_2 + 2d_1 + d_2 , cb_1 + d_1) = (2(cb_1 + d_1) + cb_2 + d_2 , cb_1 + d_1) T(cX + Y) = cT(X) + T(Y) Therefore T is linear.
3.	**Verify that $T: R^3 \rightarrow R$ and $T(u) = \lVert u \rVert$ is a linear transformation or not. (Apr/May 2022)**
	Solution: $T(u+v) = \lVert u+v \rVert \le \lVert u \rVert + \lVert v \rVert$, by Triangle inequality $\Rightarrow T(u+v) \le T(u) + T(v) \Rightarrow T(u+v) \ne T(u) + T(v)$ So it is not a linear transformation.
4.	**Determine whether the function T: $R^2 \rightarrow R^2$ defined by T(x, y) = (x^2 , y) is linear? (Nov/Dec 2022)**
	Solution: T((x, y) + (z, w)) =T(x + z, y + w)=($(x + z)^2$, y + w) $\ne$(x^2 , y) + (z^2, w) = T(x, y) + T(z, w) Therefore T is not linear.
5.	**Let T: R$\rightarrow$R be defined byT(x) = 2^x $\forall$ x$\in$R. Is T linear? (April/May 2023)**
	Solution: No. For example, if x = 1, y = 2 then T(x) + T(y) = T(1) + T(2) = 2 + 2^2 =6 But T(x + y) = T(1 + 2) = T(3) = 2^3 = 8