12 Business Mathematics

Tamil Nadu State Board Syllabus

(Solutions of all the text book problems are provided with video explanations)

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \; n \neq -1$$

Manjusha

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EDITION 1

Hello to all students and teachers. I have uploaded an e-book (pdf) with all concepts, definitions, formulae, diagrams and exercises related to the topics for 12th standard business Mathematics via my YouTube channel Simha’s Classes and also the solutions as a pdf. On clicking any question will take you to the appropriate video and pdf solution .This video is designed in such a way that rural students can easily understand and read on their own without a teacher . Following this course, videos for subsequent chapters, exercises and PDF solutions are being prepared which are soon planned to be uploaded in our channel. It can also be used by students and teachers. I kindly ask students who have used it and found beneficial to subscribe to the SIMHASCLASS YouTube channel and press the bell button nearby. More exercises, solutions are yet to come....Watch out the channel for "Learning Mathematics as a fun"

அனைத்து மாணவர்களுக்கும் ஆசிரியர்களுக்கும் வணக்கம் .நான் SIMHAS CLASS எனும் யூடியூப் சேனல் மூலமாக பன்னிரண்டாம் வகுப்பு வணிக கணிதம் (பிசினஸ் மேத்தமேடிக்ஸ்) பாடத்திற்கான அனைத்து கருத்துருக்கள் வரையறை சூத்திரங்கள் வரைபடங்கள் மற்றும் அது சார்ந்த பயிற்சி கணக்குகளுக்கான தீர்வினை காணொளி மூலமாகவும் தொடர்ந்து அதனுடைய பின்னூட்டமாக பிடிஎஃப் வடிவில் அதற்கான தீர்வையும் வழங்கியுள்ளேன்.

வினா என்னை நீங்கள் தொடு திரையில் தொட்டால் அதற்கான காணொளி மற்றும் பிடிஎஃப் தீர்வுக்கும் உங்களை அழைத்துச் செல்லும் .கிராமப்புற மாணவ மாணவிகள் எளிமையாக ஆசிரியரே இல்லாமல் புரிந்துகொண்டு அவர்களே தானாக படிக்கும் வகையில் இந்த காணொளி வடிவமைக்கப்பட்டுள்ளது .ஆசிரியர் பெருமக்களுக்கும் ஒரு வழிகாட்டியாக இருக்கும் என நம்புகிறேன். இந்த பாடப்பகுதியை தொடர்ந்து அடுத்தடுத்த படங்களுக்கான காணொளிகள் தொடர்ந்து அடுத்தடுத்த பாடப் பகுதிகளுக்கான காணொளிகளும் பிடிஎஃப் வடிவிலான தீர்வுகளும் தயார் செய்யப்பட்டு தயார் நிலையில் உள்ளது. அதையும் மாணவர்கள் மற்றும் ஆசிரியர்கள் பயன்படுத்திக்கொள்ளலாம். பயன்படுத்திக் கொண்ட மாணவர்கள் SIMHASCLASS யூடியூப் சேனலை சப்ஸ்கிரைப் செய்து அருகிலுள்ள பெல் பட்டனை அழுத்துமாறு பணிவோடு கேட்டுக்கொள்கிறேன்.

காணொளியும் தீர்வுகளும் மாணவச் செல்வங்கள் எளிமையாக புரிந்து கொள்ளும்படி விரிவாக தெளிவாக விளக்கங்களும் அதற்கான வரைபடங்களும் தெளிவாக வழங்கப்பட்டுள்ளது என்பது குறிப்பிடத்தக்கது

Contents

2 Integral Calculus – I

Introduction

Calculus divides naturally into two parts, namely (i) differential calculus and (ii) integral calculus. Differential calculus deals with the derivatives of a function whereas, integral calculus deals with the anti derivative of the derived function that is, finding a function when its rate of change / marginal function is known. So integration is the technique to find the original function from the derived function, the function obtained is called the indefinite integral. Definite integral is the evaluation of the indefinite integral between the specified limits, and is equal to the area bounded by the graph of the function (curve) between the specified limits and the axis. The area under the curve is approximately equal to the area obtained by summing the area of the number of inscribed rectangles and the approximation becomes exact in the limit that the number of rectangles approaches infinity. Therefore both differential and integral calculus are based on the theory of limits. The word 'integrate' literally means that 'to find the sum'. So, we believe that the name "Integral Calculus" has its origin from this process of summation. Calculus is the mathematical tool used to test theories about the origins of the universe, the development of tornadoes and hurricanes. It is also used to find the surplus of consumer and producer, identifying the probability density function of a continuous random variable, obtain an original function from its marginal function and etc., in business applications. In this chapter, we will study about the concept of integral and some types of method of indefinite and definite integrals.

Some Basic Formulas of integrals

1. $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a\,(n+1)} + C,\ n \neq -1$
2. $\int \frac{dx}{(ax+b)} = \frac{1}{a}\log|ax+b| + C$
3. $\int x^n dx = \frac{x^{n+1}}{(n+1)} + C,\ n \neq -1$
4. $\int \frac{dx}{x} = \log|x| + C$
5. $\int \sin ax dx = -\frac{1}{a}\cos ax + C$
6. $\int \sin x dx = -\cos x + C$
7. $\int \cos ax dx = \frac{1}{a}\sin ax + C$
8. $\int \cos x dx = \sin x + C$
9. $\int \sec^2 ax dx = \frac{1}{a}\tan ax + C$
10. $\int \sec^2 x dx = \tan x + C$
11. $\int cosec^2 ax dx = -\frac{1}{a}\cot ax + C$
12. $\int cosec^2 x dx = -\cot x + C$
13. $\int e^{ax} dx = \frac{e^{ax}}{a} + C$
14. $\int e^x dx = e^x + C$
15. $\int a^x dx = \frac{1}{\log a}a^x + C, a > 0$ and $a \neq 1$
16. $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C, n \neq -1$

17. $\boxed{\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C}$

18. $\boxed{\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C}$

19. $\boxed{\int e^x \left[f(x) + f'(x)\right] \, dx = e^x f(x) + C}$

20. $\boxed{\int e^{ax} \left[af(x) + f'(x)\right] \, dx = e^{ax} f(x) + C}$

21. $\boxed{\int \frac{1}{a^2 - x^2} \, dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + C}$

22. $\boxed{\int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C}$

23. $\boxed{\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \log\left|x + \sqrt{x^2 - a^2}\right| + C}$

24. $\boxed{\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \log\left|x + \sqrt{x^2 + a^2}\right| + C}$

25. $\boxed{\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left|x + \sqrt{x^2 - a^2}\right| + C}$

26. $\boxed{\int \sqrt{x^2 + a^2} \, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log\left|x + \sqrt{x^2 + a^2}\right| + C}$

Exercise 2.1

Problem 1

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1.Integrate the following with respect to x. $\sqrt{3x+5}$

Solution: $\boxed{\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)},\ n \neq -1}$

$$\begin{aligned}\int (3x+5)^{\frac{1}{2}}\,dx &= \frac{(3x+5)^{\frac{1}{2}+1}}{3\left(\frac{1}{2}+1\right)}+C\\ &= \frac{(3x+5)^{\frac{3}{2}}}{3\left(\frac{3}{2}\right)}+C\\ &= \frac{2(3x+5)^{\frac{3}{2}}}{9}+C\end{aligned}$$

Problem 2

Simhas Class (video Explanation)

2. Integrate with respect to x. $\left(9x^2-\frac{4}{x^2}\right)^2$

Solution: $\boxed{(A-B)^2 = A^2-2AB+B^2}$

$$\begin{aligned}\int \left(9x^2-\frac{4}{x^2}\right)^2 dx &= \int \left(81x^4-2\left(9\cancel{x^2}\right)\frac{4}{\cancel{x^2}}+\frac{16}{x^4}\right) dx\\ &= \int \left(81x^4-72+16x^{-4}\right) dx\\ &= 84\int x^4\,dx-72\int dx+16\int x^{-4}\,dx\\ &= 81\frac{x^{4+1}}{4+1}-72x+16\frac{x^{-4+1}}{-4+1}+C\\ &= 81\frac{x^5}{5}-72x+16\frac{x^{-3}}{-3}+C\\ &= \frac{81}{5}x^5-72x-\frac{16}{3x^3}+C\end{aligned}$$

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Problem 3

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3. Integrate with respect to x. $(3+x)\,(2-5x)$

Solution:

$$(i)\int x^n\,dx=\frac{x^{n+1}}{n+1}+C,\ n\neq-1 \qquad (ii)\int dx=x+C$$

$$\begin{aligned}
\int (3+x)\,(2-5x)\,dx &= \int \left(6-15x+2x-5x^2\right)\,dx\\
&= \int \left(6-13x-5x^2\right)\,dx\\
&= \int 6\,dx-\int 13x\,dx-\int 5x^2\,dx\\
&= 6x-13\frac{x^{1+1}}{1+1}-5\frac{x^{2+1}}{2+1}+C\\
&= 6x-\frac{13x^2}{2}-\frac{5x^3}{3}+C
\end{aligned}$$

Problem 4

For Video Explanation of this problem Click Here

4. Integrate with respect to x. $\sqrt{x}\,(x^3-2x+3)$

Solution: $$\int x^n\,dx=\frac{x^{n+1}}{n+1}+C,\ n\neq-1$$

$$\begin{aligned}
\int \sqrt{x}\left(x^3-2x+3\right)dx &= \int x^{\frac{1}{2}}\left(x^3-2x+3\right)dx\\
&= \int \left(x^{3+\frac{1}{2}}-2x^{1+\frac{1}{2}}+x^{\frac{1}{2}}\right)dx\\
&= \int x^{\frac{7}{2}}\,dx-\int 2x^{\frac{3}{2}}\,dx+\int 3x^{\frac{1}{2}}dx\\
&= \frac{x^{\frac{7}{2}+1}}{\left(\frac{7}{2}+1\right)}-2\frac{x^{\frac{3}{2}+1}}{\left(\frac{3}{2}+1\right)}+3\frac{x^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)}+C\\
&= \frac{x^{\frac{9}{2}}}{\left(\frac{9}{2}\right)}-2\frac{x^{\frac{5}{2}}}{\left(\frac{5}{2}\right)}+3\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}+C\\
&= \frac{2}{9}x^{\frac{9}{2}}-\frac{4}{5}x^{\frac{5}{2}}+2x^{\frac{3}{2}}+C
\end{aligned}$$

Problem 5

For Video Explanation of this problem Click Here

5. Integrate with respect to x. $\dfrac{8x+13}{\sqrt{4x+7}}$

Solution: $\boxed{\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C, n \neq -1, \int K dx = Kx + C}$

$$\begin{aligned}
\int \frac{8x+13}{\sqrt{4x+7}}dx &= \int \frac{2\left(4x+\frac{13}{2}\right)}{\sqrt{4x+7}}\,dx \\
&= \int \frac{2\left(4x+7-7+\frac{13}{2}\right)}{\sqrt{4x+7}}\,dx \\
&= \int \frac{2(4x+7)}{\sqrt{4x+7}}\,dx + \int \frac{2\left(-7+\frac{13}{2}\right)}{\sqrt{4x+7}}\,dx \\
&= \int \frac{2(4x+7)}{(4x+7)^{\frac{1}{2}}}\,dx + \int \frac{2\left(\frac{-14+13}{2}\right)}{(4x+7)^{\frac{1}{2}}}\,dx \\
&= \int 2(4x+7)^{1-\frac{1}{2}}\,dx + \int \cancel{2}\left(\frac{-1}{\cancel{2}}\right)(4x+7)^{-\frac{1}{2}}\,dx \\
&= \int 2(4x+7)^{\frac{1}{2}}\,dx - \int (4x+7)^{-\frac{1}{2}}\,dx \\
&= \frac{2(4x+7)^{\frac{1}{2}+1}}{4\left(\frac{1}{2}+1\right)} - \frac{(4x+7)^{-\frac{1}{2}+1}}{4\left(-\frac{1}{2}+1\right)} + C \\
&= \frac{2(4x+7)^{\frac{3}{2}}}{4\left(\frac{3}{2}\right)} - \frac{(4x+7)^{\frac{1}{2}}}{4\left(\frac{1}{2}\right)} + C \\
&= \frac{\cancel{2}(4x+7)^{\frac{3}{2}}}{\cancel{4}\cancel{2}\left(\frac{3}{\cancel{2}}\right)} - \frac{(4x+7)^{\frac{1}{2}}}{\cancel{4}2\left(\frac{1}{\cancel{2}}\right)} + C \\
&= \frac{(4x+7)^{\frac{3}{2}}}{3} - \frac{(4x+7)^{\frac{1}{2}}}{2} + C
\end{aligned}$$

Problem 6

For Video Explanation of this problem Click Here

6. Integrate with respect to x. $\dfrac{1}{\sqrt{x+1}+\sqrt{x-1}}$

Solution:

$$\begin{aligned}
\int \frac{1}{\sqrt{x+1}+\sqrt{x-1}}\,dx &= \int \frac{1}{\sqrt{x+1}+\sqrt{x-1}}.\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}\,dx \\
&= \int \frac{\sqrt{x+1}-\sqrt{x-1}}{\left(\sqrt{x+1}\right)^2-\left(\sqrt{x-1}\right)^2}\,dx \\
&= \int \frac{\sqrt{x+1}-\sqrt{x-1}}{x+1-(x-1)}\,dx \\
&= \int \frac{1}{\sqrt{x+1}+\sqrt{x-1}}.\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}\,dx \\
&= \int \frac{\sqrt{x+1}-\sqrt{x-1}}{\left(\sqrt{x+1}\right)^2-\left(\sqrt{x-1}\right)^2}\,dx \\
&= \int \frac{\sqrt{x+1}-\sqrt{x-1}}{\cancel{x}+1-(\cancel{x}-1)}\,dx \\
&= \int \frac{\sqrt{x+1}-\sqrt{x-1}}{2}\,dx \\
&= \int \frac{\sqrt{x+1}-\sqrt{x-1}}{2}\,dx \\
&= \frac{1}{2}\int (x+1)^{\frac{1}{2}}\,dx - \frac{1}{2}\int (x-1)^{\frac{1}{2}}\,dx \\
&= \frac{1}{2}\frac{(x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{1}{2}\frac{(x-1)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C \\
&= \frac{1}{\cancel{2}}\frac{(x+1)^{\frac{3}{2}}}{\frac{3}{\cancel{2}}} - \frac{1}{\cancel{2}}\frac{(x-1)^{\frac{3}{2}}}{\frac{3}{\cancel{2}}} + C \\
&= \frac{(x+1)^{\frac{3}{2}}}{3} - \frac{(x-1)^{\frac{3}{2}}}{3} + C
\end{aligned}$$

Problem 7

For Video Explanation of this problem Click Here

7. If $f'(x) = x + b, f(1) = 5$ and $f(2) = 13$, then find $f(x)$

Solution: Given $f'(x) = x + b$

Integrating, both sides with respect to x, we get

$$\begin{aligned}\int f'(x)\ dx &= \int (x+b)\ dx \\ \int d(f(x)) &= \int (x+b)\ dx \\ f(x) &= \int x\ dx + \int b\ dx \\ &= \frac{x^{1+1}}{1+1} + bx + C \\ &= \frac{x^2}{2} + bx + C\end{aligned}$$

Given $f(1) = 5 \ \Rightarrow \frac{1^2}{2} + b + C = 5 \ \Rightarrow b + C = 5 - \frac{1}{2} = \frac{9}{2} \ldots (1)$

$f(2) = 13 \Rightarrow \frac{2^2}{2} + 2b + C = 13 \Rightarrow 2b + C = 11 \ldots (2)$

$$\begin{aligned}b + C &= \frac{9}{2} \ldots (1) \\ 2b + C &= 11 \ldots (2)\end{aligned}$$

$(1) - (2), -b = \frac{9}{2} - 11 = \frac{9-22}{2} = \frac{-13}{2}$

$\Rightarrow \ b = \frac{13}{2}$

Substitute $b = \frac{13}{2}$ in (1), we get $C = \frac{9}{2} - \frac{13}{2} = \frac{-4}{2} = -2$

$$f(x) = \frac{x^2}{2} + \frac{13}{2}x - 2$$

Problem 8

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8. If $f'(x) = 8x^3 - 2x$ and $f(2) = 8$, then find $f(x)$

<u>Solution:</u> Given $f'(x) = 8x^3 - 2x$

Integrating, both sides with respect to x, we get

$$\begin{aligned}
\int f'(x)\ dx &= \int \left(8x^3 - 2x\right) dx \\
\int d\left(f(x)\right) &= \int \left(8x^3 - 2x\right) dx \\
f(x) &= 8\int x^3\,dx - 2\int x\,dx \\
&= 8\frac{x^{3+1}}{3+1} - 2\frac{x^{1+1}}{1+1} + C \\
&= 8\frac{x^{3+1}}{3+1} - 2\frac{x^{1+1}}{1+1} + C \\
&= \cancel{8}^{2}\frac{x^4}{\cancel{4}} - \cancel{2}\frac{x^2}{\cancel{2}} + C \\
f(x) &= 2x^4 - x^2 + C \\
f(2) &= 8 \text{ Given} \\
2\left(2^4\right) - 2^2 + C &= 8 \\
C &= 8 - 32 + 4 \\
C &= -20
\end{aligned}$$

$$f(x) = 2x^4 - x^2 - 20$$

Exercise 2.2

Problem 1

For Video Explanation of this problem Click Here

1. Integrate with respect to x. $\left(\sqrt{2x} - \frac{1}{\sqrt{2x}}\right)^2$

Solution: $\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + C\ n \neq -1, \int dx = x + C \int \frac{1}{x}\,dx = \log|x| + C}$

$$\begin{aligned}
\int \left(\sqrt{2x} - \frac{1}{\sqrt{2x}}\right)^2 dx &= \int \left(\left(\sqrt{2x}\right)^2 - 2\left(\cancel{\sqrt{2x}}\right)\left(\frac{1}{\cancel{\sqrt{2x}}}\right) + \left(\frac{1}{\sqrt{2x}}\right)^2\right) dx \\
&= \int \left(2x - 2 + \frac{1}{2x}\right) dx \\
&= 2\int x\,dx - 2\int dx + \frac{1}{2}\int \frac{1}{x}\,dx \\
&= 2\left(\frac{x^{1+1}}{1+1}\right) - 2x + \frac{1}{2}\log|x| + C \\
&= \cancel{2}\left(\frac{x^2}{\cancel{2}}\right) - 2x + \frac{1}{2}\log|x| + C \\
&= x^2 - 2x + \frac{1}{2}\log|x| + C
\end{aligned}$$

Problem 2

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2. Integrate with respect to x. $\frac{x^4 - x^2 + 2}{x-1}$

Solution: $\boxed{\int x^n dx = \frac{x^n + 1}{n+1} + C, \int \frac{1}{ax+b} dx = \frac{1}{a}\log|ax+b| + C}$

$$\begin{aligned}
\int \left(\frac{x^4 - x^2 + 2}{x-1}\right) dx &= \int \left(\frac{x^4 - x^2}{x-1}\right) dx + \int \frac{2}{x-1}\,dx \\
&= \int \left(\frac{x^2\left(x^2-1\right)}{x-1}\right) dx + \int \frac{2}{x-1}\,dx \\
&= \int \left(\frac{x^2\cancel{(x-1)}(x+1)}{\cancel{x-1}}\right) dx + \int \frac{2}{x-1}\,dx \\
&= \int \left(x^3 + x^2\right) dx + \int \frac{2}{x-1}\,dx \\
&= \int \left(x^3 + x^2\right) dx + \int \frac{2}{x-1}\,dx \\
&= \frac{x^4}{4} + \frac{x^3}{3} + 2\log|x-1| + C
\end{aligned}$$

Problem 3

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3. Integrate with respect to x. $\dfrac{x^3}{x+2}$

<u>Solution:</u>

$$
\begin{array}{r|llll}
 & x^2 & -2x & +4 & \\
\hline
x+2 & \cancel{x^3} & & & \\
 & \cancel{x^3} & +2x^2 & & \\
 & - & - & & \\
\hline
 & & \cancel{-2x^2} & & \\
 & & \cancel{-2x^2} & -4x & \\
 & & - & - & \\
\hline
 & & & \cancel{4x} & \\
 & & & \cancel{4x} & +8 \\
 & & & - & - \\
\hline
 & & & & -8
\end{array}
$$

Quotient = $x^2 - 2x + 4$ and Remainder = -8.

Therefore $\dfrac{x^3}{x+2} = x^2 - 2x + 4 + \dfrac{-8}{x+2}$

$$
\begin{aligned}
\int \frac{x^3}{x+2}\,dx &= \int \left(x^2 - 2x + 4 - \frac{8}{x+2}\right) dx \\
\int \frac{x^3}{x+2}\,dx &= \int x^2 dx - 2\int x\,dx + 4\int dx - \int \frac{8}{x+2}\,dx \\
&= \frac{x^{2+1}}{2+1} - 2\frac{x^{1+1}}{1+1} + 4x - 8\log|x+2| + C \\
&= \frac{x^3}{3} - \cancel{2}\left(\frac{x^2}{\cancel{2}}\right) + 4x - 8\log|x+2| + C \\
&= \frac{x^3}{3} - x^2 + 4x - 8\log|x+2| + C
\end{aligned}
$$

Problem 4

4. Integrate with respect to x. $\dfrac{x^3 + 3x^2 - 7x + 11}{x+5}$

For Video Explanation of this problem Click Here

<u>Solution:</u>

$$
\begin{array}{r|llll}
 & x^2 & -2x & +3 & \\ \hline
x+5 & \cancel{x^3} & +3x^2 & -7x & +11 \\
 & \cancel{x^3} & +5x^2 & & \\
 & - & - & & \\ \hline
 & & \cancel{-2x^2} & -7x & \\
 & & \cancel{-2x^2} & -10x & \\
 & & +- & + & \\ \hline
 & & & \cancel{3x} & +11 \\
 & & & \cancel{3x} & +15 \\
 & & & - & - \\ \hline
 & & & & -4
\end{array}
$$

Quotient = $x^2 - 2x + 3$ and Remainder = -4.

Therefore $\dfrac{x^3 + 3x^2 - 7x + 11}{x+5} = x^2 - 2x + 3 + \dfrac{-4}{x+5}$

$$
\begin{aligned}
\int \frac{x^3 + 3x^2 - 7x + 11}{x+5}\,dx &= \int \left(x^2 - 2x + 3 + \frac{-4}{x+5}\right)dx \\
\int \frac{x^3}{x+2}\,dx &= \int x^2 dx - 2\int x\,dx + 3\int dx - \int \frac{4}{x+5}\,dx \\
&= \frac{x^{2+1}}{2+1} - 2\frac{x^{1+1}}{1+1} + 3x - 4\log|x+5| + C \\
&= \frac{x^3}{3} - \cancel{2}\left(\frac{x^2}{\cancel{2}}\right) + 3x - 4\log|x+5| + C \\
&= \frac{x^3}{3} - x^2 + 3x - 4\log|x+5| + C
\end{aligned}
$$

Problem 5

5. Integrate with respect to x. $\dfrac{3x+2}{(x-2)(x-3)}$

For Video Explanation of this problem Click Here

<u>Solution:</u> Given $\dfrac{3x+2}{(x-2)(x-3)}$

$$\frac{3x+2}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$3x+2 = \frac{A\cancel{(x-2)}(x-3)}{\cancel{(x-2)}} + \frac{B(x-2)\cancel{(x-3)}}{\cancel{x-3}}$$

$$3x+2 = A(x-3) + B(x-2)$$

If $x = 2,\ 3(2) + 2 = A(2-3) \ \Rightarrow\ -A = 8 \ \Rightarrow A = -8$
If $x = 3,\ 3(3) + 2 = B(3-2) \ \Rightarrow\ B = 11$

$$\frac{3x+2}{(x-2)(x-3)} = \frac{-8}{x-2} + \frac{11}{x-3}$$

$$\int \frac{3x+2}{(x-2)(x-3)}\,dx = -8\int \frac{dx}{x-2} + 11\int \frac{dx}{x-3}$$

we Know that $\boxed{\int \frac{dx}{ax+b} = \frac{1}{a}\log|ax+b| + C}$

$$\int \frac{3x+2}{(x-2)(x-3)}\,dx = -8\log|x-2| + 11\log|x-3| + C$$

Problem 6

6. Integrate with respect to x. $\dfrac{4x^2+2x+6}{(x+1)^2(x-3)}$

For Video Explanation of this problem Click Here

<u>Solution:</u> Given $\dfrac{4x^2+2x+6}{(x+1)^2(x-3)}$

$$\frac{4x^2+2x+6}{(x+1)^2(x-3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x-3)}$$

$$4x^2+2x+6 = \frac{A(x+1)^{\cancel{2}}(x-3)}{\cancel{(x+1)}} + \frac{B\cancel{(x+1)^2}(x-3)}{\cancel{(x+1)^2}} + \frac{C(x+1)^2\cancel{(x-3)}}{\cancel{x-3}}$$

$$4x^2+2x+6 = A(x+1)(x-3) + B(x-3) + C(x+1)^2$$

If $x = -1,\ 4(-1)^2 + 2(-1) + 6 = B(-1-3) \ \Rightarrow\ -4B = 8 \ \Rightarrow B = -2$
If $x = 3,\ 4(3)^2 + 2(3) + 6 = C(3+1)^2 \ \Rightarrow\ 16C = 48 \ \Rightarrow C = 3$
Coefficient of x^2 : $4 = A + C \ \Rightarrow A = 4 - C \ \Rightarrow A = 4 - (3) = 1$

$$\frac{4x^2+2x+6}{(x+1)^2(x-3)} = \frac{1}{x+1} + \frac{-2}{(x+1)^2} + \frac{3}{(x-3)}$$

$$\int \frac{4x^2+2x+6}{(x+1)^2(x-3)}\,dx = \int \frac{dx}{x+1} - 2\int \frac{dx}{(x+2)^2} + 3\int \frac{dx}{x-3}$$

$$\int \frac{4x^2+2x+6}{(x+1)^2(x-3)}\,dx = \int \frac{dx}{x+1} - 2\int (x+2)^{-2}\,dx + 3\int \frac{dx}{x-3}$$

we Know that

$$\boxed{(i) \int \frac{dx}{ax+b} = \frac{1}{a}\log|ax+b| + C}$$

$$\boxed{(ii) \int (ax+b)^n \ dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, \ n \neq -1}$$

$$\begin{aligned}\int \frac{4x^2+2x+6}{(x+1)^2(x-3)}\,dx &= \log|x+1| - 2\frac{(x+2)^{-2+1}}{1(-2+1)} + 3\log|x-3| + C \\ &= \log|x+1| + \frac{2}{(x+2)} + 3\log|x-3| + C\end{aligned}$$

Problem 7

7. Integrate with respect to x. $\dfrac{3x^2-2x+5}{(x-1)(x^2+5)}$

For Video Explanation of this problem Click Here

Solution: Given $\dfrac{3x^2-2x+5}{(x-1)(x^2+5)}$

$$\begin{aligned}\frac{3x^2-2x+5}{(x-1)(x^2+5)} &= \frac{A}{x-1} + \frac{Bx+C}{x^2+5} \\ 3x^2-2x+5 &= \frac{A\cancel{(x-1)}(x^2+5)}{\cancel{(x-1)}} + \frac{(Bx+C)(x-1)\cancel{(x^2+5)}}{\cancel{x^2+5}} \\ 3x^2-2x+5 &= A\left(x^2+5\right) + (Bx+C)(x-1)\end{aligned}$$

If $x=1,\ 3(1)^2-2(1)+5 = A(1^2+5) \ \Rightarrow\ 6A=6 \ \Rightarrow\ A=1$
If $x=0,\ 3(0)^2-2(0)+5 = A(0+5)+C(0-1) \ \Rightarrow\ -C = 5-5A = 5-5=0$
Coefficient of x^2 : $3 = A+B \ \Rightarrow\ B = 3-A = 3-1 = 2$

$$\frac{3x^2-2x+5}{(x-1)(x^2+5)} = \frac{1}{x-1} + \frac{2x}{x^2+5}$$

$$\begin{aligned}\int \frac{3x^2-2x+5}{(x-1)(x^2+5)}\,dx &= \int \frac{dx}{x-1} + \int \frac{2xdx}{x^2+5} \\ \int \frac{3x^2-2x+5}{(x-1)(x^2+5)}\,dx &= \int \frac{dx}{x-1} + \text{put } x^2+5=u, \text{ then } 2xdx = du \\ &= \int \frac{dx}{x-1} + \int \frac{du}{u}\end{aligned}$$

we Know that $\boxed{\int \frac{dx}{ax+b} = \frac{1}{a}\log|ax+b| + C}$

$$\begin{aligned}\int \frac{3x^2-2x+5}{(x-1)\left(x^2+5\right)}\,dx &= \log|x-1| + \log|u| + C\\ &= \log|x-1| + \log\left|x^2+5\right| + C\\ &= \log\left|(x-1)\left(x^2+5\right)\right| + C\\ &= \log\left|x^3-x^2+5x-5\right| + C\end{aligned}$$

Problem 8

8. If $f'(x) = \frac{1}{x}$ and $f(1) = \frac{\pi}{4}$, then find $f(x)$
For Video Explanation of this problem Click Here

Solution: Given $f'(x) = \frac{1}{x}$
Integrating, both sides with respect to x, we get

$$\begin{aligned}\int f'(x)\;dx &= \int \frac{1}{x}\,dx\\ \int d(f(x)) &= \int \frac{1}{x}\,dx\\ f(x) &= \log|x| + C\\ f(1) &= \frac{\pi}{4}\ \text{Given}\\ \log|1| + C &= \frac{\pi}{4}\\ C &= \frac{\pi}{4} - 0\\ C &= \frac{\pi}{4}\end{aligned}$$

$$f(x) = \log|x| + \frac{\pi}{4}$$

Exercise 2.3

Problem 1

1. Integrate the following with respect to x. $e^{x\log a} + e^{a\log a} - e^{n\log x}$
For Video Explanation of this problem Click Here

Solution: Given $e^{x\log a} + e^{a\log a} - e^{n\log x} = e^{\log a^x} + e^{\log a^a} - e^{\log x^n} = a^x + a^a - x^n$

$$\begin{aligned}\int \left(e^{x\log a} + e^{a\log a} - e^{n\log x}\right)\,dx &= \int \left(a^x + a^a - x^n\right)\,dx \\ &= \int a^x\,dx + \int a^a\,dx - \int x^n\,dx\end{aligned}$$

We know that $\boxed{\int a^x dx = \frac{1}{\log a}a^x + C, a > 0 \text{ and } a \neq 1}$ $\boxed{\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1}$

$$\int \left(e^{x\log a} + e^{a\log a} - e^{n\log x}\right)\,dx = \frac{1}{\log a}a^x + a^a x - \frac{x^{n+1}}{n+1} + C$$

Problem 2

2. Integrate the following with respect to x. $\frac{a^x - e^{x\log b}}{e^{x\log a}b^x}$

For Video Explanation of this problem Click Here

Solution: Given $\frac{a^x - e^{x\log b}}{e^{x\log a}b^x} = \frac{a^x - e^{\log b^x}}{e^{\log a^x}b^x} = \frac{a^x - b^x}{a^x b^x} = \frac{\cancel{a^x}}{\cancel{a^x}b^x} - \frac{\cancel{b^x}}{a^x\cancel{b^x}} = b^{-x} - a^{-x}$

$$\begin{aligned}\int \left(\frac{a^x - e^{x\log b}}{e^{x\log a}b^x}\right)\,dx &= \int \left(b^{-x} - a^{-x}\right)\,dx \\ &= \int b^{-x}\,dx - \int a^{-x}\,dx\end{aligned}$$

We know that $\boxed{\int a^x dx = \frac{1}{\log a}a^x + C, a > 0 \text{ and } a \neq 1}$

$$\begin{aligned}\int \left(\frac{a^x - e^{x\log b}}{e^{x\log a}b^x}\right)\,dx &= -\frac{1}{\log b}b^{-x} + \frac{1}{\log a}a^{-x} + C \\ &= \frac{1}{a^x\log a} - \frac{1}{b^x\log b} + C\end{aligned}$$

Problem 3

3. Integrate the following with respect to x. $(e^x + 1)^2\, e^x$

For Video Explanation of this problem Click Here

Solution: Given $\left(e^x+1\right)^2 e^x = e^x\left(e^{2x}+2e^x+1\right) = e^{3x}+2e^{2x}+e^x$

$$\begin{aligned}\int \left(e^x+1\right)^2 e^x\, dx &= \int \left(e^{3x}+2e^{2x}+e^x\right) dx \\ &= \int e^{3x}\, dx + 2\int e^{2x}\, dx + \int e^x\, dx\end{aligned}$$

We know that $\boxed{\int e^{ax} dx = \frac{e^{ax}}{a}+C}$

$$\begin{aligned}\int \left(e^x+1\right)^2 e^x\, dx &= \frac{e^{3x}}{3} + \cancel{2}\frac{e^{2x}}{\cancel{2}} + e^x + C \\ &= \frac{e^{3x}}{3} + e^{2x} + e^x + C\end{aligned}$$

Problem 4

4. Integrate the following with respect to x. $\frac{e^{3x}-e^{-3x}}{e^x}$

For Video Explanation of this problem Click Here

Solution: Given $\frac{e^{3x}-e^{-3x}}{e^x} = \frac{e^{3x}}{e^x} - \frac{e^{-3x}}{e^x} = e^{(3x-x)} - e^{(-3x-x)} = e^{2x} - e^{-4x}$

$$\begin{aligned}\int \left(\frac{e^{3x}-e^{-3x}}{e^x}\right) dx &= \int \left(e^{2x}-e^{-4x}\right) dx \\ &= \int e^{2x}\, dx - \int e^{-4x}\, dx\end{aligned}$$

We know that $\boxed{\int e^{ax} dx = \frac{e^{ax}}{a}+C}$

$$\begin{aligned}\int \left(\frac{e^{3x}-e^{-3x}}{e^x}\right) dx &= \frac{e^{2x}}{2} - \frac{e^{-4x}}{-4} + C \\ &= \frac{e^{2x}}{2} + \frac{e^{-4x}}{4} + C\end{aligned}$$

Problem 5

5. Integrate the following with respect to x. $\frac{e^{3x}+e^{5x}}{e^x+e^{-x}}$

For Video Explanation of this problem Click Here

Solution: Given $\dfrac{e^{3x}+e^{5x}}{e^x+e^{-x}} = \dfrac{e^{4x}\left(\dfrac{1}{e^x}+e^x\right)}{e^x+e^{-x}} = \dfrac{e^{4x}\cancel{(e^x+e^{-x})}}{\cancel{e^x+e^{-x}}} = e^{4x}$

$$\int \frac{e^{3x}+e^{5x}}{e^x+e^{-x}}\,dx = \int e^{4x}\,dx$$

We know that $\boxed{\int e^{ax}dx = \dfrac{e^{ax}}{a}+C}$

$$\int \frac{e^{3x}+e^{5x}}{e^x+e^{-x}}\,dx = \frac{e^{4x}}{4}+C$$

Problem 6

6. Integrate the following with respect to x. $\left(1-\dfrac{1}{x^2}\right)e^{\left(x+\frac{1}{x}\right)}$

For Video Explanation of this problem Click Here

Solution: Given $\left(1-\dfrac{1}{x^2}\right)e^{\left(x+\frac{1}{x}\right)}$

$\int \left[1-\dfrac{1}{x^2}\right]e^{\left(x+\frac{1}{x}\right)}\,dx$

Let $x+\dfrac{1}{x}=t \Rightarrow \left(1-\dfrac{1}{x^2}\right)dx = dt$

$\int \left(1-\dfrac{1}{x^2}\right)e^{\left(x+\frac{1}{x}\right)}\,dx = \int e^t\,dt$

We know that $\boxed{\int e^x dx = e^x + C}$

$$\begin{aligned}\int \left(1-\frac{1}{x^2}\right)e^{\left(x+\frac{1}{x}\right)}\,dx &= e^t + C\\ &= e^{\left(x+\frac{1}{x}\right)}+C\end{aligned}$$

Problem 7

7. Integrate the following with respect to x. $\dfrac{1}{x\,(\log x)^2}$

For Video Explanation of this problem Click Here

Solution: Given $\dfrac{1}{x\,(\log x)^2}$

$$\int \frac{1}{x(\log x)^2}\, dx$$

Let $\log x = u \Rightarrow \frac{1}{x}dx = du$

$$\int \frac{1}{x(\log x)^2}\, dx = \int \frac{1}{u^2}\, du = \int u^{-2}\, du$$

We know that $\boxed{\int x^n dx = \frac{x^{n+1}}{n+1} + C \ n \neq -1}$

$$\begin{aligned}\int \frac{1}{x(\log x)^2}\, dx &= \frac{u^{-2+1}}{-2+1} + C \\ &= -u^{-1} + C \\ &= -\frac{1}{u} + C \\ &= -\frac{1}{\log x} + C\end{aligned}$$

Problem 8

8. If $f'(x) = e^x$ and $f(0) = 2$, then find $f(x)$

For Video Explanation of this problem Click Here

Solution: Given $f'(x) = e^x$

Integrating, both sides with respect to x, we get

$$\begin{aligned}\int f'(x)\, dx &= \int e^x\, dx \\ \int d(f(x)) &= e^x + C \\ f(x) &= e^x + C \\ f(0) &= 2 \text{ Given} \\ e^0 + C &= 2 \\ C &= 2 - 1 = 1\end{aligned}$$

$$f(x) = e^x + 1$$

Exercise 2.4

Problem 1

1. Integrate the following with respect to x.
$2\cos x - 3\sin x + 4\sec^2 x - 5cosec^2 x$
For Video Explanation of this problem Click Here
Solution:
$$\int \left(2\cos x - 3\sin x + 4\sec^2 x - 5cosec^2 x\right)\, dx$$
We know that,
$\boxed{\int \cos x dx = \sin x + C}$ $\boxed{\int \sin x dx = -\cos x + C}$
$\boxed{\int \sec^2 x dx = \tan x + C}$ $\boxed{\int cosec^2 x dx = -\cot x + C}$
$$= 2\int \cos x\, dx - 3\int \sin x\, dx + 4\int \sec^2 x\, dx - 5\int cosec^2 x\, dx$$
$$= 2\sin x + 3\cos x + 4\tan x + 5\cot x + C$$

Problem 2

2. Integrate the following with respect to x. $\sin^3 x$
For Video Explanation of this problem Click Here
Solution:
We know that, $\boxed{\sin 3x = 3\sin x - 4\sin^3 x}$
$$\begin{aligned}\int \sin^3 x\, dx &= \int \left(\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right)\, dx \\ &= \frac{3}{4}\int \sin x\, dx - \frac{1}{4}\int \sin 3x\, dx\end{aligned}$$
We know that $\boxed{\int \sin x dx = -\cos x + C}$ $\boxed{\int \sin ax dx = -\frac{1}{a}\cos ax + C}$
$$\begin{aligned}\int \sin^3 x\, dx &= -\frac{3}{4}\cos x - \frac{1}{4}\left(-\frac{1}{3}\cos 3x\right) + C \\ &= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C\end{aligned}$$

Problem 3

3. Integrate the following with respect to x. $\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$

For Video Explanation of this problem Click Here

Solution: Given $\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$

We know that, $\boxed{\cos 2x = 1 - 2\sin^2 x}$

$$\frac{\cos 2x + 2\sin^2 x}{\cos^2 x} = \frac{1 - \cancel{2\sin^2 x} + \cancel{2\sin^2 x}}{cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}\, dx = \int \sec^2 x\, dx$$

We know that $\boxed{\int \sec^2 x dx = \tan x + C}$

$$\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}\, dx = \tan x + C$$

Problem 4

4. Integrate the following with respect to x. $\frac{1}{\sin^2 x \cos^2 x}$ $\left[\text{Hint } \sin^2 x + \cos^2 x = 1\right]$

For Video Explanation of this problem Click Here

Solution: Given

$$\frac{1}{\sin^2 x \cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} = \frac{\cancel{\sin^2 x}}{\cancel{\sin^2 x} \cos^2 x} + \frac{\cancel{\cos^2 x}}{\sin^2 x \cancel{\cos^2 x}} = \sec^2 x + cosec^2 x$$

We know that $\boxed{\int \sec^2 x dx = \tan x + C}$ $\boxed{\int cosec^2 x dx = -\cot x + C}$

$$\begin{aligned}\int \frac{1}{\sin^2 x \cos^2 x}\, dx &= \int \sec^2 x\, dx + \int cosec^2 x\, dx \\ &= \tan x - \cot x + C\end{aligned}$$

Problem 5

5. Integrate the following with respect to x. $\sqrt{1 - \sin 2x}$

Solution: Given $\sqrt{1 - \sin 2x}$

For Video Explanation of this problem Click Here

We know that $\boxed{\sin^2 x + \cos^2 x = 1}$ $\boxed{\sin 2x = 2\sin x \cos x}$

$$\begin{aligned}
\sqrt{1-\sin 2x} &= \sqrt{\sin^2 x + \cos^2 x - 2\sin x \cos x} \\
&= \sqrt{(\sin x - \cos x)^2} \\
&= \sin x - \cos x \\
\int \sqrt{1-\sin 2x}\, dx &= \int (\sin x - \cos x)\ dx \\
&= \int \sin x\ dx - \int \cos x\ dx
\end{aligned}$$

We know that $\boxed{\int \sin x dx = -\cos x + C}$ $\boxed{\int \cos x dx = \sin x + C}$

$$\begin{aligned}
\int \sqrt{1-\sin 2x}\, dx &= -\cos x - \sin x + C \\
&= -(\cos x + \sin x) + C + C
\end{aligned}$$

Exercise 2.5

Problem 1

1. Integrate the following with respect to x. xe^{-x}

For Video Explanation of this problem Click Here

<u>Solution:</u> Given xe^{-x}

$$\begin{aligned}
u &= x \\
u' &= 1 \\
u'' &= 0
\end{aligned}$$

$$\begin{aligned}
v_1 &= \int e^{-x}\, dx = -e^{-x} \\
v_2 &= \int -e^{-x}\, dx = e^{-x}
\end{aligned}$$

We know that $\boxed{\int udv = uv_1 - u'v_2 + u''v_3 - \cdots}$

$$\begin{aligned}
\int xe^{-x} dx &= x\left(-e^{-x}\right) - 1\left(e^{-x}\right) + C \\
&= -xe^{-x} - e^{-x} + C \\
&= -e^{-x}(x+1) + C
\end{aligned}$$

Problem 2

2. Integrate the following with respect to x. x^3e^{3x}

For Video Explanation of this problem Click Here

Solution: Given x^3e^{3x}

$$\begin{aligned} u &= x^3 \\ u' &= 3x^2 \\ u'' &= 6x \\ u''' &= 6 \\ u'''' &= 0 \end{aligned}$$

$$\begin{aligned} v_1 &= \int e^{3x}\,dx = \frac{e^{3x}}{3} \\ v_2 &= \int \frac{e^{3x}}{3}\,dx = \frac{e^{3x}}{3^2} = \frac{e^{3x}}{9} \\ v_3 &= \int \frac{e^{3x}}{9}\,dx = \frac{1}{9}\frac{e^{3x}}{3} = \frac{e^{3x}}{27} \\ v_4 &= \int \frac{e^{3x}}{27}\,dx = \frac{1}{27}\frac{e^{3x}}{3} = \frac{e^{3x}}{81} \end{aligned}$$

We know that $\boxed{\int udv = uv_1 - u'v_2 + u''v_3 - \cdots}$

$$\begin{aligned} \int x^3e^{3x}dx &= x^3\left(\frac{e^{3x}}{3}\right) - 3x^2\left(\frac{e^{3x}}{9}\right) + 6x\left(\frac{e^{3x}}{27}\right) - 6\left(\frac{e^{3x}}{81}\right) + C \\ &= e^{3x}\left(\frac{x^3}{3} - \frac{x^2}{3} + \frac{2x}{9} - \frac{2}{27}\right) + C \end{aligned}$$

Problem 3

3. Integrate the following with respect to x. $\log x$

For Video Explanation of this problem Click Here

Solution: Given $\int \log x\, dx$

$$\begin{aligned} u &= \log x \\ du &= \frac{1}{x}dx \end{aligned}$$

$$\begin{aligned} dv &= dx \\ v &= x \end{aligned}$$

We know that $\boxed{\int udv = uv - \int v\, du}$

$$\begin{aligned}\int \log x dx &= x\log x - \int \cancel{x}\left(\frac{1}{\cancel{x}}\right)\,dx\\ &= x\log x - \int\,dx + C\\ &= x\log x - x + C\\ &= x\,(\log x - 1) + C\end{aligned}$$

Problem 4

4. Integrate the following with respect to x. $x\log x$

For Video Explanation of this problem Click Here

Solution: Given $\int x\log x\,dx$

$$\begin{aligned} u &= \log x\\ du &= \frac{1}{x}dx\end{aligned}$$

$$\begin{aligned}\int dv &= \int x dx\\ v &= \frac{x^2}{2}\end{aligned}$$

We know that $\boxed{\int udv = uv - \int v\,du}$

$$\begin{aligned}\int \log x d\left(\frac{x^2}{2}\right) &= \left(\frac{x^2}{2}\right)\log x - \int\left(\frac{x^{\cancel{2}}}{2}\right)\left(\frac{1}{\cancel{x}}\right)\,dx\\ &= \frac{x^2\log x}{2} - \frac{1}{2}\int x\,dx + C\\ &= \frac{x^2\log x}{2} - \frac{1}{2}\left(\frac{x^2}{2}\right) + C\\ &= \frac{x^2}{2}\left(\log x - \frac{1}{2}\right) + C\end{aligned}$$

Problem 5

5. Integrate the following with respect to x. $x^n\log x$

For Video Explanation of this problem Click Here

Solution: Given $\int x^n\log x\,dx$

$$\begin{aligned} u &= \log x \\ du &= \frac{1}{x}dx \end{aligned} \qquad \Bigg| \qquad \begin{aligned} \int dv &= \int x^n dx \\ v &= \frac{x^{n+1}}{n+1} \end{aligned}$$

We know that $\boxed{\int udv = uv - \int v\,du}$

$$\begin{aligned} \int \log x d\left(\frac{x^{n+1}}{n+1}\right) &= \left(\frac{x^{n+1}}{n+1}\right)\log x - \int \left(\frac{x^{\cancel{n+1}n}}{n+1}\right)\left(\frac{1}{\cancel{x}}\right)\,dx \\ &= \frac{x^{n+1}\log x}{n+1} - \frac{1}{n+1}\int x^n\,dx + C \\ &= \frac{x^{n+1}\log x}{n+1} - \frac{1}{n+1}\left(\frac{x^n+1}{n+1}\right) + C \\ &= \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right) + C \end{aligned}$$

Problem 6

6. Integrate the following with respect to x. $x^5e^{x^2}$

For Video Explanation of this problem Click Here

<u>Solution:</u> Given $\int x^5e^{x^2}\,dx\ldots(1)$

Let $x^2 = t \;\Rightarrow 2xdx = dt \;\Rightarrow xdx = \frac{1}{2}$

Therefore, (1) $\Rightarrow \int x^5e^{x^2}\,dx = \int x^4e^{x^2}\,dt = \frac{1}{2}\int t^2e^t\,dt$

$$\begin{aligned} u &= t^2 \\ u' &= 2t \\ u'' &= 2 \\ u''' &= 0 \end{aligned} \qquad \Bigg| \qquad \begin{aligned} v_1 &= \int e^t\,dt = e^t \\ v_2 &= \int e^t\,dt = e^t \\ v_3 &= \int e^t\,dt = e^t \end{aligned}$$

We know that $\boxed{\int udv = uv_1 - u'v_2 + u''v_3 - \cdots}$

$$\int t^2 e^t \, dt = \frac{1}{2}\left[t^2 e^t - 2te^t + 2e^t\right] + C$$
$$= \frac{e^t}{2}\left(t^2 - 2t + 2\right) + C$$
$$= \frac{e^{x^2}}{2}\left(x^4 - 2x^2 + 2\right) + C$$

Exercise 2.6

Problem 1

1. Integrate with respect to x. $\dfrac{2x+5}{x^2+5x-7}$

For Video Explanation of this problem Click Here

<u>Solution:</u> Given $\displaystyle\int \frac{2x+5}{x^2+5x-7}\,dx$

Let $x^2+5x+7 = t \;\Rightarrow (2x+5)\,dx = dt$

Therefore, (1) $\Rightarrow \displaystyle\int \frac{2x+5}{x^2+5x-7}\,dx = \int \frac{dt}{t}$

We know that $\boxed{\displaystyle\int \frac{dx}{x} = \log|x| + C}$

$$\int \frac{dt}{t} = \log|t| + C$$
$$= \log\left|x^2+5x-7\right| + C$$

Problem 2

2. Integrate with respect to x. $\dfrac{e^{3\log x}}{x^4+1}$

For Video Explanation of this problem Click Here

<u>Solution:</u> Given $\dfrac{e^{3\log x}}{x^4+1} = \dfrac{e^{\log x^3}}{x^4+1} = \dfrac{x^3}{x^4+1}$

$\displaystyle\int \frac{x^3}{x^4+1}\,dx \ldots (1)$

Let $x^4+1 = t \;\Rightarrow 4x^3dx = dt \;\Rightarrow x^3dx = \dfrac{1}{4}dt$

Therefore, (1) $\Rightarrow \int \frac{x^3}{x^4+1}\,dx = \frac{1}{4}\int \frac{dt}{t}$

We know that $\boxed{\int \frac{dx}{x} = \log|x| + C}$

$$\begin{aligned}\frac{1}{4}\int \frac{dt}{t} &= \frac{1}{4}\log|t| + C\\ &= \frac{1}{4}\log\left|x^4+1\right| + C\end{aligned}$$

Problem 3

3. Integrate with respect to x. $\frac{e^{2x}}{e^{2x}-2}$

For Video Explanation of this problem Click Here

Solution: Given $\int \frac{e^{2x}}{e^{2x}-2}\,dx \ldots (1)$

Let $e^{2x} - 2 = t \;\Rightarrow 2e^{2x}dx = dt \;\Rightarrow e^{2x}dx = \frac{1}{2}dt$

Therefore, (1) $\Rightarrow \int \frac{e^{2x}}{e^{2x}-2}\,dx = \frac{1}{2}\int \frac{dt}{t}$

We know that $\boxed{\int \frac{dx}{x} = \log|x| + C}$

$$\begin{aligned}\frac{1}{2}\int \frac{dt}{t} &= \frac{1}{2}\log|t| + C\\ &= \frac{1}{2}\log\left|e^{2x}-2\right| + C\end{aligned}$$

Problem 4

4. Integrate with respect to x. $\frac{(\log x)^3}{x}$

For Video Explanation of this problem Click Here

Solution: Given $\int \frac{(\log x)^3}{x}\,dx \ldots (1)$

Let $\log x = t \;\Rightarrow \frac{1}{x}dx = dt$

Therefore, (1) $\Rightarrow \displaystyle\int \frac{(\log x)^3}{x}\, dx = \int t^3\, dt$

We know that $\boxed{\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + C \; n \neq -1}$

$$\begin{aligned} \int t^3\, dt &= \frac{t^{3+1}}{3+1} + C \\ &= \frac{t^4}{4} + C \\ &= \frac{(\log x)^4}{4} + C \end{aligned}$$

Problem 5

5. Integrate with respect to x. $\dfrac{6x+7}{\sqrt{3x^2+7x-1}}$

For Video Explanation of this problem Click Here

Solution: Given $\displaystyle\int \frac{6x+7}{\sqrt{3x^2+7x-1}}\, dx \ldots (1)$

Let $3x^2 + 7x - 1 = t \;\Rightarrow (6x+7)\, dx = dt$

Therefore, (1) $\Rightarrow \displaystyle\int \frac{6x+7}{\sqrt{3x^2+7x-1}}\, dx = \int \frac{dt}{\sqrt{t}}$

We know that $\boxed{\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + C \; n \neq -1}$

$$\begin{aligned} \int \frac{dt}{\sqrt{t}} &= \int t^{-\frac{1}{2}}\, dt \\ &= \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C \\ &= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C \\ &= 2\sqrt{t} + C \\ &= 2\sqrt{3x^2+7x-1} + C \end{aligned}$$

Problem 6

6. Integrate with respect to x. $(4x+2)\sqrt{x^2+x+1}$
For Video Explanation of this problem Click Here
Solution: Given $\int (4x+2)\sqrt{x^2+x+1}\,dx \ldots (1)$
Let $x^2+x+1=t \implies (2x+1)\,dx = dt$
Therefore, (1) $\Rightarrow \int (4x+2)\sqrt{x^2+x+1}\,dx = 2\int (2x+1)\sqrt{x^2+x+1}\,dx$

$= 2\int \sqrt{t}\,dt$

We know that $\boxed{\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1}$

$$\begin{aligned}
2\int \sqrt{t}\,dt &= 2\int t^{\frac{1}{2}}\,dt \\
&= 2\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C \\
&= 2\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C \\
&= \frac{4\left(x^2+x+1\right)^{\frac{3}{2}}}{3} + C
\end{aligned}$$

Problem 7

7. Integrate with respect to x. $x^8\left(1+x^9\right)^5$
For Video Explanation of this problem Click Here
Solution: Given $\int x^8\left(1+x^9\right)^5\,dx \ldots (1)$
Let $1+x^9=t \implies 9x^8dx = dt \implies x^8dx = \frac{1}{9}dt$
Therefore, (1) $\Rightarrow \int x^8\left(1+x^9\right)^5\,dx = \frac{1}{9}\int t^5\,dt$

We know that $\boxed{\int x^n dx = \frac{x^{n+1}}{n+1} + C, \; n \neq -1}$

$$\begin{aligned} \frac{1}{9}\int t^5 \, dt &= \frac{1}{9}\frac{t^{5+1}}{5+1} + C \\ &= \frac{1}{9}\frac{t^6}{6} + C \\ &= \frac{1}{54}\left(1 + x^9\right)^6 + C \end{aligned}$$

Problem 8

8. Integrate with respect to x. $\frac{x^{e-1} + e^{x-1}}{x^e + e^x}$

For Video Explanation of this problem Click Here

Solution: Given $\int \frac{x^{e-1} + e^{x-1}}{x^e + e^x} \, dx \ldots (1)$

Let $x^e + e^x = t \Rightarrow (ex^{e-1} + e^x)\, dx = dt \Rightarrow e\left(x^{e-1} + e^{x-1}\right) dx = dt \Rightarrow \left(x^{e-1} + e^{x-1}\right) dx = \frac{1}{e} dt$

Therefore, (1) $\Rightarrow \int \frac{x^{e-1} + e^{x-1}}{x^e + e^x} \, dx = \frac{1}{e}\int \frac{dt}{t}$

We know that $\boxed{\int \frac{dx}{x} = \log|x| + C}$

$$\begin{aligned} \int \frac{x^{e-1} + e^{x-1}}{x^e + e^x} \, dx &= \frac{1}{e}\int \frac{dt}{t} \\ &= \frac{1}{e}\log|t| + C \\ &= \frac{1}{e}\log\left|x^e + e^x\right| + C \end{aligned}$$

Problem 9

9. Integrate with respect to x. $\frac{1}{x(\log x)}$

For Video Explanation of this problem Click Here

Solution: Given $\int \frac{1}{x(\log x)} \, dx \ldots (1)$

Let $\log x = t \Rightarrow \frac{1}{x} dx = dt$

Therefore, (1) $\Rightarrow \int \frac{1}{x(\log x)}\,dx = \int \frac{dt}{t}$

We know that $\boxed{\int \frac{dx}{x} = \log|x| + C}$

$$\begin{aligned}\int \frac{1}{x(\log x)}\,dx &= \int \frac{dt}{t}\\ &= \log|t| + C\\ &= \log|\log x| + C\end{aligned}$$

Problem 10

10. Integrate with respect to x. $\frac{x}{2x^4 - 3x^2 - 2}$

For Video Explanation of this problem Click Here

<u>Solution:</u> Given $\int \frac{x}{2x^4 - 3x^2 - 2}\,dx \ldots (1)$

Let $x^2 = t \Rightarrow 2xdx = dt \Rightarrow xdx = \frac{1}{2}dt$

Therefore, (1) $\Rightarrow \int \frac{x}{2x^4 - 3x^2 - 2}\,dx = \frac{1}{2}\int \frac{dt}{2t^2 - 3t - 2} \ldots (2)$

consider

$$\begin{aligned}2t^2 - 3t - 2 &= 2t^2 - 4t + t - 2\\ &= 2t(t-2) + 1(t-2)\\ &= (2t+1)(t-2)\end{aligned}$$

Therefore, $\frac{1}{2t^2 - 3t - 2} = \frac{1}{(2t+1)(t-2)}$

$$\begin{aligned}\frac{1}{(2t+1)(t-2)} &= \frac{A}{(2t+1)} + \frac{B}{(t-2)}\\ 1 &= \frac{A\cancel{(2t+1)}(t-2)}{\cancel{(2t+1)}} + \frac{B(2t+1)\cancel{(t-2)}}{\cancel{(t-2)}}\\ 1 &= A(t-2) + B(2t+1)\end{aligned}$$

If $t = -\frac{1}{2},\ 1 = A\left(-\frac{1}{2} - 2\right) \Rightarrow -\frac{5A}{2} = 1 \Rightarrow A = -\frac{2}{5}$

If $t = 2,\ 1 = B(4+1) \Rightarrow 5B = 1 \Rightarrow B = \frac{1}{5}$

Therefore, $\dfrac{1}{2t^2-3t-2} = \dfrac{-\frac{2}{5}}{(2t+1)} + \dfrac{\frac{1}{5}}{(t-2)}$

(2) $\Rightarrow$

$$\begin{aligned}
\int \frac{x}{2x^4-3x^2-2}\,dx &= \frac{1}{2}\int \frac{dt}{2t^2-3t-2} \\
&= \frac{1}{2}\int \frac{-\frac{2}{5}}{(2t+1)}\,dt + \frac{1}{2}\int \frac{\frac{1}{5}}{(t-2)}\,dt \\
&= -\frac{1}{5}\int \frac{1}{(2t+1)}\,dt + \frac{1}{10}\int \frac{1}{(t-2)}\,dt
\end{aligned}$$

We know that $\boxed{\int \dfrac{dx}{ax+b} = \dfrac{1}{a}\log|ax+b| + C}$

$$\begin{aligned}
\int \frac{x}{2x^4-3x^2-2}\,dx &= -\frac{1}{5}\int \frac{1}{(2t+1)}\,dt + \frac{1}{10}\int \frac{1}{(t-2)}\,dt \\
&= -\frac{1}{5}\left[\frac{1}{2}\log|2t+1|\right] + \frac{1}{10}\log|t-2| + C \\
&= -\frac{1}{10}\log|2t+1| + \frac{1}{10}\log|t-2| + C \\
&= \frac{1}{10}\left[\log|t-2| - \log|2t+1|\right] + C \\
&= \frac{1}{10}\log\left|\frac{t-2}{2t+1}\right| + C \\
&= \frac{1}{10}\log\left|\frac{x^2-2}{2x^2+1}\right| + C
\end{aligned}$$

Problem 11

11. Integrate with respect to x. $e^x(1+x)\log(xe^x)$

For Video Explanation of this problem Click Here

<u>Solution:</u> Given $\int e^x(1+x)\log\left(xe^x\right)\,dx \ldots (1)$

Let $xe^x = t \Rightarrow (xe^x + e^x)\,dx = dt \Rightarrow e^x(1+x)\,dx = dt$

Therefore, (1) $\Rightarrow \int e^x(1+x)\log\left(xe^x\right)\,dx = \int \log t\,dt$

$$\begin{aligned} u &= \log t \\ du &= \frac{1}{t}dt \end{aligned} \qquad \begin{aligned} dv &= dt \\ v &= t \end{aligned}$$

We know that $\boxed{\int udv = uv - \int v\, du}$

$$\begin{aligned} \int e^x (1+x) \log \left(xe^x\right) dx &= \int \log t dt \\ &= t \log t - \int \cancel{t} \left(\frac{1}{\cancel{t}}\right) dt \\ &= t \log t - \int dt + C \\ &= t \log t - t + C \\ &= t (\log t - 1) + C \\ &= xe^x \left(\log \left(xe^x\right) - 1\right) + C \end{aligned}$$

Problem 12

12. Integrate with respect to x. $\frac{1}{x\left(x^2+1\right)}$

For Video Explanation of this problem Click Here

<u>Solution:</u> Given $\int \frac{1}{x\left(x^2+1\right)} dx \ldots (1)$

$$\begin{aligned} \frac{1}{x\left(x^2+1\right)} &= \frac{A}{x} + \frac{Bx+C}{\left(x^2+1\right)} \\ 1 &= \frac{A\cancel{x}\left(x^2+1\right)}{\cancel{x}} + \frac{(Bx+C)\, x\cancel{\left(x^2+1\right)}}{\cancel{\left(x^2+1\right)}} \\ 1 &= A\left(x^2+1\right) + (Bx+C)\, x \end{aligned}$$

If $x = 0,\ 1 = A(0+1)\ \Rightarrow A = 1$
Coefficient of x^2 : $0 = A + B\ \Rightarrow B = 0 - A = -1\ \Rightarrow\ B = -1$
If $x = 1,\ 1 = A(1+1) + (B+C)\, 1\ \Rightarrow c = 1 - 2A - B = 1 - 2 + 1\ \Rightarrow C = 0$
Therefore, $\frac{1}{x\left(x^2+1\right)} = \frac{1}{x} + \frac{-x}{\left(x^2+1\right)}$

(1) $\Rightarrow$

$$\begin{aligned}\int \frac{1}{x\left(x^2+1\right)}\,dx &= \int \frac{1}{x}\,dx + \int \frac{-x}{\left(x^2+1\right)}\,dx \\ &= \int \frac{dx}{x} + \text{put } x^2+1=t, \text{ then } 2xdx = dt\; xdx\frac{1}{2}dt \\ &= \int \frac{dx}{x} - \frac{1}{2}\int \frac{dt}{t}\end{aligned}$$

We know that $\boxed{\int \frac{dx}{ax+b} = \frac{1}{a}\log|ax+b| + C}$

$$\begin{aligned}\int \frac{1}{x\left(x^2+1\right)}\,dx &= \log|x| - \frac{1}{2}\log|t| + C \\ &= \log|x| - \frac{1}{2}\log\left|x^2+1\right| + C\end{aligned}$$

Problem 13

13. Integrate with respect to x. $e^x\left[\frac{1}{x^2} - \frac{2}{x^3}\right]$

For Video Explanation of this problem Click Here

<u>Solution:</u> Given $\int e^x\left[\frac{1}{x^2} - \frac{2}{x^3}\right]\,dx\ldots(1)$

Let $f(x) = \frac{1}{x^2} = x^{-2}$

then $f'(x) = -2x^{-2-1} = -2x^{-3} = -\frac{2}{x^3}$

we Know that $\boxed{\int e^x\left[f(x) + f'(x)\right]\,dx = e^x f(x) + C}$

$$\begin{aligned}\int e^x\left[\frac{1}{x^2} - \frac{2}{x^3}\right]\,dx &= e^x\left[\frac{1}{x^2}\right] + C \\ &= \frac{e^x}{x^2} + C\end{aligned}$$

Problem 14

14. Integrate with respect to x. $e^x\left[\frac{x-1}{(x+1)^3}\right]$

For Video Explanation of this problem Click Here

Solution: Given

$$\begin{aligned} e^x\left[\frac{x-1}{(x+1)^3}\right] &= e^x\left(\frac{x+1-1-1}{(x+1)^3}\right) \\ &= e^x\left(\frac{\cancel{x+1}}{(x+1)^{\cancel{3}2}}-\frac{2}{(x+1)^3}\right) \\ &= e^x\left(\frac{1}{(x+1)^2}-\frac{2}{(x+1)^3}\right) \end{aligned}$$

Let $f(x)=\frac{1}{(x+1)^2}=(1+x)^{-2}$

then $f'(x)=-2\,(1+x)^{-2-1}=-\frac{2}{(x+1)^3}$

we Know that $\boxed{\int e^x\left[f(x)+f'(x)\right]\,dx=e^x f(x)+C}$

$$\begin{aligned} \int e^x\left[\frac{x-1}{(x+1)^3}\right]\,dx &= e^x\left[\frac{1}{(x+1)^2}\right]+C \\ &= \frac{e^x}{(x+1)^2}+C \end{aligned}$$

Problem 15

15. Integrate with respect to x. $e^{3x}\left[\frac{3x-1}{9x^2}\right]$

For Video Explanation of this problem Click Here

Solution: Given

$$\begin{aligned} e^{3x}\left[\frac{3x-1}{9x^2}\right] &= e^{3x}\left(\frac{3x}{9x^2}-\frac{1}{9x^2}\right) \\ &= e^{3x}\left(\frac{\cancel{3x}}{\cancel{9x^{\cancel{2}}}3x}-\frac{1}{9x^2}\right) \\ &= \frac{1}{9}e^{3x}\left(\frac{3}{x}-\frac{1}{x^2}\right) \end{aligned}$$

Let $f(x)=\frac{1}{x}=(x)^{-1}$

then $f'(x)=-(x)^{-1-1}=-\frac{1}{(x)^2}$

we Know that $\boxed{\int e^{ax}\left[af(x)+f'(x)\right]\, dx = e^{ax}f(x)+C}$

$$\begin{aligned}
\int e^{3x}\left[\frac{3x-1}{9x^2}\right] dx &= \int \frac{1}{9}e^{3x}\left(\frac{3}{x}-\frac{1}{x^2}\right) dx \\
&= \frac{1}{9}\int e^{3x}\left(3\frac{1}{x}-\frac{1}{x^2}\right) dx \\
&= \frac{1}{9}e^{3x}\left(\frac{1}{x}\right)+C \\
&= \frac{e^{3x}}{9x}+C
\end{aligned}$$

Exercise 2.7

Problem 1

1. Integrate with respect to x : $\frac{1}{9-16x^2}$

For Video Explanation of this problem Click Here

<u>Solution:</u> Given

$$\begin{aligned}
\int \frac{1}{9-16x^2}\, dx &= \int \frac{1}{16\left(\frac{9}{16}-x^2\right)}\, dx \\
&= \frac{1}{16}\int \frac{1}{\left(\frac{3}{4}\right)^2-x^2}\, dx
\end{aligned}$$

We know that $\boxed{\int \frac{1}{a^2-x^2}\, dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C}$

$$\begin{aligned}
\int \frac{1}{9-16x^2}\, dx &= \frac{1}{16}\int \frac{1}{\left(\frac{3}{4}\right)^2-x^2}\, dx \\
&= \frac{1}{16}\left[\frac{1}{\cancel{2}\left(\frac{3}{\cancel{4}2}\right)}\log\left|\frac{\frac{3}{4}+x}{\frac{3}{4}-x}\right|\right]+C \\
&= \frac{1}{\cancel{16}8}\left[\frac{\cancel{2}}{3}\log\left|\frac{\frac{3+4x}{\cancel{4}}}{\frac{3-4x}{\cancel{4}}}\right|\right]+C \\
&= \frac{1}{24}\log\left|\frac{3+4x}{3-4x}\right|+C
\end{aligned}$$

Problem 2

2. Integrate with respect to x : $\dfrac{1}{9-8x-x^2}$

For Video Explanation of this problem Click Here

Solution: Given $\displaystyle\int \frac{1}{9-8x-x^2}\,dx$

Consider

$$\begin{aligned}
9-8x-x^2 &= -\left(x^2+8x-9\right)\\
&= -\left(x^2+8x+16-16-9\right)\\
&= -\left(x^2+8x+16-25\right)\\
&= -(x+4)^2+5^2\\
&= 5^2-(x+4)^2
\end{aligned}$$

We know that $\boxed{\displaystyle\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C}$

$$\begin{aligned}
\int \frac{1}{9-8x-x^2}\,dx &= \int \frac{1}{5^2-(x+4)^2}\,dx\\
&= \frac{1}{2\,(5)}\log\left|\frac{5+(x+4)}{5-(x+4)}\right| + C\\
&= \frac{1}{10}\log\left|\frac{9+x}{1-x}\right| + C
\end{aligned}$$

Problem 3

3. Integrate with respect to x : $\dfrac{1}{2x^2-9}$

For Video Explanation of this problem Click Here

Solution: Given

$$\begin{aligned}
\int \frac{1}{2x^2-9}\,dx &= \int \frac{1}{2\,(x^2-\frac{9}{2})}\,dx\\
&= \frac{1}{2}\int \frac{1}{x^2-\left(\frac{3}{\sqrt{2}}\right)^2}\,dx
\end{aligned}$$

We know that $\boxed{\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C}$

$$\begin{aligned}
\int \frac{1}{2x^2 - 9}\, dx &= \frac{1}{2}\int \frac{1}{x^2 - \left(\frac{3}{\sqrt{2}}\right)^2}\, dx \\
&= \frac{1}{2}\left[\frac{1}{\cancel{2}\sqrt{2}\left(\frac{3}{\cancel{\sqrt{2}}}\right)} \log\left|\frac{x - \frac{3}{\sqrt{2}}}{x + \frac{3}{\sqrt{2}}}\right|\right] + C \\
&= \frac{1}{6\sqrt{2}} \log\left|\frac{\frac{\sqrt{2}x-3}{\cancel{\sqrt{2}}}}{\frac{\sqrt{2}x+3}{\cancel{\sqrt{2}}}}\right| + C \\
&= \frac{1}{6\sqrt{2}} \log\left|\frac{\sqrt{2}x - 3}{\sqrt{2}x + 3}\right| + C
\end{aligned}$$

Problem 4

4. Integrate with respect to x : $\frac{1}{x^2 - x - 2}$

For Video Explanation of this problem Click Here

<u>Solution:</u> Given $\int \frac{1}{x^2 - x - 2}\, dx$

Consider

$$\begin{aligned}
x^2 - x - 2 &= x^2 - x + \frac{1}{4} - \frac{1}{4} - 2 \\
&= \left(x - \frac{1}{2}\right)^2 - \frac{9}{4} \\
&= \left(x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2
\end{aligned}$$

We know that $\boxed{\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C}$

$$\begin{aligned}
\int \frac{1}{x^2 - x - 2}\, dx &= \int \frac{1}{\left(x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\, dx \\
&= \frac{1}{\cancel{2}\left(\frac{3}{\cancel{2}}\right)} \log\left|\frac{\left(x - \frac{1}{2}\right) - \frac{3}{2}}{\left(x - \frac{1}{2}\right) + \frac{3}{2}}\right| + C \\
&= \frac{1}{3} \log\left|\frac{x-2}{x+1}\right| + C
\end{aligned}$$

Problem 5

5. Integrate with respect to x : $\dfrac{1}{x^2 + 3x + 2}$

For Video Explanation of this problem Click Here

<u>Solution:</u> Given $\int \dfrac{1}{x^2 + 3x + 2}\, dx$

Consider

$$\begin{aligned}
x^2 + 3x + 2 &= x^2 + 3x + \frac{9}{4} - \frac{9}{4} + 2 \\
&= \left(x + \frac{3}{2}\right)^2 - \frac{1}{4} \\
&= \left(x + \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2
\end{aligned}$$

We know that $\boxed{\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C}$

$$\begin{aligned}
\int \frac{1}{x^2 + 3x + 2}\, dx &= \int \frac{1}{\left(x + \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\, dx \\
&= \frac{1}{\cancel{2}\left(\frac{1}{\cancel{2}}\right)} \log\left|\frac{\left(x + \frac{3}{2}\right) - \frac{1}{2}}{\left(x + \frac{3}{2}\right) + \frac{1}{2}}\right| + C \\
&= \log\left|\frac{x+1}{x+2}\right| + C
\end{aligned}$$

Problem 6

6. Integrate with respect to x : $\dfrac{1}{2x^2+6x-8}$

For Video Explanation of this problem Click Here

Solution: Given $\displaystyle\int \frac{1}{2x^2+6x-8}\,dx$

Consider

$$\begin{aligned}
2x^2+6x-8 &= 2\left(x^2+3x-4\right)\\
&= 2\left(x^2+3x+\frac{9}{4}-\frac{9}{4}-4\right)\\
&= 2\left(\left(x+\frac{3}{2}\right)^2-\frac{25}{4}\right)\\
&= 2\left(\left(x+\frac{3}{2}\right)^2-\left(\frac{5}{2}\right)^2\right)
\end{aligned}$$

We know that $\boxed{\displaystyle\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C}$

$$\begin{aligned}
\int \frac{1}{2x^2+6x-8}\,dx &= \frac{1}{2}\int \frac{1}{\left(x+\frac{3}{2}\right)^2-\left(\frac{5}{2}\right)^2}\,dx\\
&= \frac{1}{2}\left(\frac{1}{\cancel{2}\left(\frac{5}{\cancel{2}}\right)}\log\left|\frac{\left(x+\frac{3}{2}\right)-\frac{5}{2}}{\left(x+\frac{3}{2}\right)+\frac{5}{2}}\right|\right)+C\\
&= \frac{1}{10}\log\left|\frac{x-1}{x+4}\right|+C
\end{aligned}$$

Problem 7

7. Integrate with respect to x : $\dfrac{e^x}{e^{2x}-9}$

For Video Explanation of this problem Click Here

Solution: Given $\displaystyle\int \frac{e^x}{e^{2x}-9}\,dx = \int \frac{e^x}{(e^x)^2-3^2}\,dx \ldots (1)$

Let $e^x = t \;\Rightarrow e^x dx = dt$

Then (1) $\Rightarrow \displaystyle\int \frac{e^x}{(e^x)^2-3^2}\,dx = \int \frac{dt}{t^2-3^2}$

We know that $\boxed{\displaystyle\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C}$

$$\begin{aligned}\int \frac{e^x}{(e^x)^2 - 3^2}\, dx &= \int \frac{dt}{t^2 - 3^2} \\ &= \frac{1}{2\,(3)} \log\left|\frac{t-3}{t+3}\right| + C \\ &= \frac{1}{6} \log\left|\frac{e^x - 3}{e^x + 3}\right| + C\end{aligned}$$

Problem 8

8. Integrate with respect to x : $\frac{1}{\sqrt{9x^2 - 7}}$

For Video Explanation of this problem Click Here

Solution: Given

$$\begin{aligned}\int \frac{1}{\sqrt{9x^2 - 7}}\, dx &= \int \frac{1}{3\left(\sqrt{x^2 - \frac{7}{9}}\right)}\, dx \\ &= \frac{1}{3}\int \frac{1}{\sqrt{x^2 - \left(\frac{\sqrt{7}}{3}\right)^2}}\, dx\end{aligned}$$

We know that $\boxed{\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x+\sqrt{x^2-a^2}\right| + C}$

$$\begin{aligned}
\int \frac{1}{\sqrt{9x^2-7}}\,dx &= \frac{1}{3}\int \frac{1}{\sqrt{x^2-\left(\frac{\sqrt{7}}{3}\right)^2}}\,dx \\
&= \frac{1}{3}\log\left|x+\sqrt{x^2-\left(\frac{\sqrt{7}}{3}\right)^2}\right| + C \\
&= \frac{1}{3}\log\left|x+\sqrt{x^2-\frac{7}{9}}\right| + C \\
&= \frac{1}{3}\log\left|x+\sqrt{\frac{9x^2-7}{9}}\right| + C \\
&= \frac{1}{3}\log\left|\frac{3x+\sqrt{9x^2-7}}{3}\right| + C \\
&= \frac{1}{3}\log\left|3x+\sqrt{9x^2-7}\right| - \frac{1}{3}\log 3 + C \\
&= \frac{1}{3}\log\left|3x+\sqrt{9x^2-7}\right| + C_1 \text{ where } C_1 = -\frac{1}{3}\log 3 + C
\end{aligned}$$

Problem 9

9. Integrate with respect to x : $\frac{1}{\sqrt{x^2+6x+13}}$

For Video Explanation of this problem Click Here

Solution: Given $\int \frac{1}{\sqrt{x^2+6x+13}}\,dx$

Consider

$$\begin{aligned}
x^2+6x+13 &= x^2+6x+9-9+13 \\
&= (x+3)^2+4 \\
&= (x+3)^2+2^2
\end{aligned}$$

We know that $\boxed{\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x+\sqrt{x^2+a^2}\right| + C}$

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+6x+13}}\,dx &= \int \frac{1}{\sqrt{(x+3)^2+2^2}}\,dx \\ &= \log\left|x+3+\sqrt{(x+3)^2+2^2}\right| + C \\ &= \log\left|x+3+\sqrt{x^2+6x+13}\right| + C\end{aligned}$$

Problem 10

10. Integrate with respect to x : $\frac{1}{\sqrt{x^2-3x+2}}$

For Video Explanation of this problem Click Here

Solution: Given $\int \frac{1}{\sqrt{x^2-3x+2}}\,dx$

Consider

$$\begin{aligned}x^2-3x+2 &= x^2-3x+\frac{9}{4}-\frac{9}{4}+2 \\ &= \left(x-\frac{3}{2}\right)^2-\frac{1}{4} \\ &= \left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2\end{aligned}$$

We know that $\boxed{\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x+\sqrt{x^2-a^2}\right| + C}$

$$\begin{aligned}\int \frac{1}{\sqrt{x^2-3x+2}}\,dx &= \int \frac{1}{\sqrt{(x-\frac{3}{2})^2-(\frac{1}{2})^2}}\,dx \\ &= \log\left|\left(x-\frac{3}{2}\right)+\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right| + C \\ &= \log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right| + C\end{aligned}$$

Problem 11

11. Integrate with respect to x : $\frac{x^3}{\sqrt{x^8-1}\,dx}$

For Video Explanation of this problem Click Here

Solution: Given $\int \frac{x^3}{\sqrt{x^8-1}}\,dx = \int \frac{x^3}{\sqrt{\left(x^4\right)^2-1}}\,dx \ldots(1)$

Let $x^4 = t \;\Rightarrow 4x^3dx = dt \;\Rightarrow x^3dx = \frac{1}{4}dt$

Therefore (1) $\Rightarrow \int \frac{x^3}{\sqrt{x^8-1}}\,dx = \frac{1}{4}\int \frac{dt}{\sqrt{t^2-1}}$

We know that $\boxed{\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x+\sqrt{x^2-a^2}\right| + C}$

$$\begin{aligned}
\int \frac{x^3}{\sqrt{x^8-1}}\,dx &= \frac{1}{4}\int \frac{dt}{\sqrt{t^2-1}} \\
&= \frac{1}{4}\log\left|t+\sqrt{t^2-1}\right| + C \\
&= \frac{1}{4}\log\left|x^4+\sqrt{x^8-1}\right| + C
\end{aligned}$$

Problem 12

12. Integrate with respect to x : $\sqrt{1+x+x^2}$

Solution: Given $\int \sqrt{1+x+x^2}\,dx \ldots(1)$

For Video Explanation of this problem Click Here

consider

$$\begin{aligned}
1+x+x^2 &= x^2+x+\frac{1}{4}-\frac{1}{4}+1 \\
&= \left(x+\frac{1}{2}\right)^2+\frac{3}{4}
\end{aligned}$$

We know that $\boxed{\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C}$

$$\begin{aligned}
\int \sqrt{1+x+x^2}\,dx &= \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx \\
&= \frac{x+\frac{1}{2}}{2}\sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\frac{3}{4}}{2}\log\left|\left(x+\frac{1}{2}\right) + \sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\right| + C \\
&= \frac{2x+1}{4}\sqrt{1+x+x^2} + \frac{3}{8}\log\left|\left(x+\frac{1}{2}\right) + \sqrt{1+x+x^2}\right| + C
\end{aligned}$$

Problem 13

13. Integrate with respect to x : $\sqrt{x^2-2}$

For Video Explanation of this problem Click Here

Solution: Given $\int \sqrt{x^2 - \left(\sqrt{2}\right)^2}\;dx \ldots (1)$

We know that $\boxed{\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C}$

$$\begin{aligned}
\int \sqrt{x^2-2}\,dx &= \int \sqrt{x^2 - \left(\sqrt{2}\right)^2}\;dx \\
&= \frac{x}{2}\sqrt{x^2-2} - \frac{2}{2}\log\left|x + \sqrt{x^2 - \left(\sqrt{2}\right)^2}\right| + C \\
&= \frac{x}{2}\sqrt{x^2-2} - \log\left|x+\sqrt{x^2-2}\right| + C
\end{aligned}$$

Problem 14

14. Integrate with respect to x : $\sqrt{4x^2-5}$

For Video Explanation of this problem Click Here

Solution: Given $\int \sqrt{4x^2-5}\,dx = \int 2\sqrt{x^2 - \frac{5}{4}}\,dx \ldots (1)$

We know that $\boxed{\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C}$

$$\begin{aligned}
\int \sqrt{4x^2 - 5}\,dx &= 2\int \sqrt{x^2 - \left(\frac{\sqrt{5}}{2}\right)^2}\,dx \\
&= 2\left[\frac{x}{2}\sqrt{x^2 - \frac{5}{4}} - \frac{\frac{5}{4}}{2}\log\left|x + \sqrt{x^2 - \frac{5}{4}}\right|\right] + C \\
&= \frac{x}{2}\sqrt{4x^2 - 5} - \frac{5}{4}\log\left|x + \sqrt{x^2 - \frac{5}{4}}\right| + C \\
&= \frac{x}{2}\sqrt{4x^2 - 5} - \frac{5}{4}\log\left|2x + \sqrt{4x^2 - 5}\right| + C
\end{aligned}$$

Problem 15

15. Integrate with respect to x : $\sqrt{2x^2 + 4x + 1}$

For Video Explanation of this problem Click Here

Solution: Given $I = \int \sqrt{2x^2 + 4x + 1}\,dx \ldots (1)$

consider

$$\begin{aligned}
2x^2 + 4x + 1 &= 2\left(x^2 + 2x + 1 - 1 + \frac{1}{2}\right) \\
&= 2\left((x+1)^2 - \frac{1}{2}\right)
\end{aligned}$$

We know that

$$\boxed{\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C}$$

$$\int \sqrt{2x^2+4x+1}\,dx$$

$$= \sqrt{2}\int \sqrt{(x+1)^2-\left(\frac{1}{\sqrt{2}}\right)^2}\,dx$$

$$= \sqrt{2}\left[\frac{(x+1)}{2}\sqrt{(x+1)^2-\left(\frac{1}{\sqrt{2}}\right)^2}-\frac{\frac{1}{2}}{2}\log\left|(x+1)+\sqrt{(x+1)^2-\left(\frac{1}{\sqrt{2}}\right)^2}\right|\right]+C$$

$$= \frac{(x+1)}{\sqrt{2}}\sqrt{(x+1)^2-\frac{1}{2}}-\frac{\sqrt{2}}{4}\log\left|(x+1)+\sqrt{x^2+2x+1-\frac{1}{2}}\right|+C$$

$$= \frac{(x+1)}{2}\sqrt{2\left(x^2+2x+1\right)-1}-\frac{\sqrt{2}}{4}\log\left|\frac{\sqrt{2}\,(x+1)+\sqrt{2x^2+4x+1}}{\sqrt{2}}\right|+C$$

$$= \frac{(x+1)}{2}\sqrt{2x^2+4x+1}-\frac{\sqrt{2}}{4}\log\left|\sqrt{2}\,(x+1)+\sqrt{2x^2+4x+1}\right|+C'$$

where $C' = -\frac{\sqrt{2}}{4}\log\sqrt{2}+C$

Problem 16

16. Integrate with respect to x : $\frac{1}{x+\sqrt{x^2-1}}$

For Video Explanation of this problem Click Here

Solution: Given $\int \frac{1}{x+\sqrt{x^2-1}}\,dx$

$$\int \frac{1}{x+\sqrt{x^2-1}}\,dx = \int \frac{1}{x+\sqrt{x^2-1}}.\frac{x-\sqrt{x^2-1}}{x-\sqrt{x^2-1}}\,dx$$

$$= \int \frac{x-\sqrt{x^2-1}}{x^2-\left(\sqrt{x^2-1}\right)^2}\,dx$$

$$= \int \frac{x-\sqrt{x^2-1}}{x^2-\left(x^2-1\right)}\,dx$$

$$= \int \left(x-\sqrt{x^2-1}\right)\,dx$$

$$= \int x\,dx-\int \sqrt{x^2-1}\,dx$$

We know that $\boxed{\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C}$

$$\int \frac{1}{x+\sqrt{x^2-1}}\,dx = \frac{x^2}{2} - \left[\frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\log\left|x+\sqrt{x^2-1}\right|\right] + C$$

Exercise 2.8

Problem 1

I. Using second fundamental theorem, evaluate the following: 1. $\int_0^1 e^{2x}\,dx$

For Video Explanation of this problem Click Here

Solution: second fundamental theorem of Integral Calculus:

If $f(x)$ be a continuous function on $[a,b]$, if $F(x)$ is anti derivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\begin{aligned}\int_0^1 e^{2x}\,dx &= \left[\frac{e^{2x}}{2}\right]_0^1 \\ &= \frac{1}{2}\left[e^{2(1)} - e^{2(0)}\right] \\ &= \frac{1}{2}\left[e^2 - 1\right]\end{aligned}$$

Problem 2

I. Using second fundamental theorem, evaluate the following: 2. $\int_0^{\frac{1}{4}} \sqrt{1-4x}\,dx$

For Video Explanation of this problem Click Here

Solution: second fundamental theorem of Integral Calculus:

If $f(x)$ be a continuous function on $[a,b]$, if $F(x)$ is anti derivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

We know that $\boxed{\int (ax+b)^n \ dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, \ n \neq -1}$

$$\begin{aligned}
\int_0^{\frac{1}{4}} \sqrt{1-4x}\,dx &= \int_0^{\frac{1}{4}} (1-4x)^{\frac{1}{2}} \ dx \\
&= \left[\frac{(1-4x)^{\frac{1}{2}+1}}{-4\left(\frac{1}{2}+1\right)}\right]_0^{\frac{1}{4}} \\
&= \left[\frac{(1-4x)^{\frac{3}{2}}}{-\cancel{4}2\left(\frac{3}{\cancel{2}}\right)}\right]_0^{\frac{1}{4}} \\
&= -\frac{1}{6}\left[\left(1-\cancel{4}\left(\frac{1}{\cancel{4}}\right)\right)^{\frac{3}{2}} - (1-4(0))^{\frac{3}{2}}\right] \\
&= -\frac{1}{6}\left[0-(1)^{\frac{3}{2}}\right] = \frac{1}{6}
\end{aligned}$$

Problem 3

I. Using second fundamental theorem, evaluate the following: 3. $\int_1^2 \frac{xdx}{x^2+1}$

For Video Explanation of this problem Click Here

Solution: Second fundamental theorem of Integral Calculus:
If $f(x)$ be a continuous function on $[a,b]$, if $F(x)$ is anti derivative of $f(x)$, then

$$\int_a^b f(x) \ dx = F(b) - F(a)$$

Put $x^2+1 = t$, then $2xdx = dt \ \Rightarrow xdx = \frac{1}{2}$

When $x = 1, t = x^2+1 = 1+1 = 2$

When $x = 2, t = x^2+1 = 4+1 = 5$

We know that $\boxed{\int \frac{dx}{x} = \log|x| + C}$

$$\begin{aligned}\int_1^2 \frac{x dx}{x^2+1} &= \frac{1}{2}\int_2^5 \frac{dt}{t} \\ &= \frac{1}{2}\left[\log|t|\right]_2^5 \\ &= \frac{1}{2}\left[\log 5 - \log 2\right] \\ &= \frac{1}{2}\log\left[\frac{5}{2}\right]\end{aligned}$$

Problem 4

I. Using second fundamental theorem, evaluate the following: 4. $\int_0^3 \frac{e^x dx}{1+e^x}$

For Video Explanation of this problem Click Here

<u>Solution:</u> Second fundamental theorem of Integral Calculus:
If $f(x)$ be a continuous function on $[a,b]$, if $F(x)$ is anti derivative of $f(x)$, then

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

Put $1+e^x = t$, then $e^x dx = dt$
When $x = 0, t = 1+e^x = 1+1 = 2$
When $x = 3, t = 1+e^x = 1+e^3$
We know that $\boxed{\int \frac{dx}{x} = \log|x| + C}$

$$\begin{aligned}\int_0^3 \frac{e^x dx}{1+e^x} &= \int_2^{1+e^3} \frac{dt}{t} \\ &= \left[\log|t|\right]_2^{1+e^3} \\ &= \left[\log\left(1+e^3\right) - \log 2\right] \\ &= \log\left[\frac{1+e^3}{2}\right]\end{aligned}$$

Problem 5

I. Using second fundamental theorem, evaluate the following: 5. $\int_0^1 xe^{x^2}\, dx$

For Video Explanation of this problem Click Here

Solution: Second fundamental theorem of Integral Calculus:
If $f(x)$ be a continuous function on $[a,b]$, if $F(x)$ is anti derivative of $f(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Put $x^2 = t$, then $2xdx = dt \Rightarrow xdx = \frac{1}{2}dt$

When $x = 0, t = x^2 = 0$

When $x = 1, t = x^2 = 1^2 = 1$

We know that $\boxed{\int e^x\, dx = e^x + C}$

$$\begin{aligned}\int_0^1 xe^{x^2}\, dx &= \frac{1}{2}\int_0^1 e^t\, dt \\ &= \frac{1}{2}\left[e^t\right]_0^1 \\ &= \frac{1}{2}\left[e^1 - e^0\right] \\ &= \frac{1}{2}\left[e - 1\right]\end{aligned}$$

Problem 6

I. Using second fundamental theorem, evaluate the following: 6. $\int_1^e \frac{dx}{x(1+\log x)^3}$

For Video Explanation of this problem Click Here

Solution: Second fundamental theorem of Integral Calculus:
If $f(x)$ be a continuous function on $[a,b]$, if $F(x)$ is anti derivative of $f(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Put $1 + \log x = t$, then $\frac{1}{x} dx = dt$
When $x = 1, t = 1 + \log x = 1 + \log 1 = 1 + 0 = 1$
When $x = e, t = 1 + \log e = 1 + 1 = 2$

We know that $\boxed{\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, n \neq -1}$

$$
\begin{aligned}
\int_1^e \frac{dx}{x(1+\log x)^3} &= \int_1^2 \frac{dt}{t^3} \\
&= \int_1^2 t^{-3} \, dt \\
&= \left[\frac{t^{-3+1}}{-3+1}\right]_1^2 \\
&= \left[\frac{t^{-2}}{-2}\right]_1^2 \\
&= -\frac{1}{2}\left[\frac{1}{t^2}\right]_1^2 \\
&= -\frac{1}{2}\left[\frac{1}{4} - \frac{1}{1}\right] = \frac{3}{8}
\end{aligned}
$$

Problem 7

I. Using second fundamental theorem, evaluate the following: 7. $\int_{-1}^{1} \frac{2x+3}{x^2+3x+7} \, dx$

For Video Explanation of this problem Click Here

Solution: Second fundamental theorem of Integral Calculus:
If $f(x)$ be a continuous function on $[a, b]$, if $F(x)$ is anti derivative of $f(x)$, then

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

Put $x^2 + 3x + 7 = t$, then $(2x + 3) \, dx = dt$
When $x = -1, t = x^2 + 3x + 7 = (-1)^2 + 3(-1) + 7 = 1 - 3 + 7 = 5$
When $x = 1, t = x^2 + 3x + 7 = (1)^2 + 3(1) + 7 = 1 + 3 + 7 = 11$

We know that $\boxed{\int \frac{dx}{x} = \log|x| + C}$

$$\int_{-1}^{1} \frac{2x+3}{x^2+3x+7}\,dx = \int_{5}^{11} \frac{dt}{t}$$
$$= [\log|t|]_5^{11}$$
$$= [\log 11 - \log 5]$$
$$= \log\left[\frac{11}{5}\right]$$

Problem 8

I. Using second fundamental theorem, evaluate the following: 8. $\int_0^{\frac{\pi}{2}} \sqrt{1+\cos x}\,dx$

For Video Explanation of this problem Click Here

Solution: Second fundamental theorem of Integral Calculus:
If $f(x)$ be a continuous function on $[a,b]$, if $F(x)$ is anti derivative of $f(x)$, then

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

We know that $\boxed{cos2\theta = 2\cos^2\theta - 1}$

$$\int_0^{\frac{\pi}{2}} \sqrt{1+\cos x}\ dx = \int_0^{\frac{\pi}{2}} \sqrt{2\cos^2\frac{x}{2}}\ dx$$
$$= \sqrt{2}\int_0^{\frac{\pi}{2}} \cos\frac{x}{2}\ dx$$
$$= \sqrt{2}\left[\frac{\sin\frac{x}{2}}{\frac{1}{2}}\right]_0^{\frac{\pi}{2}}$$
$$= 2\sqrt{2}\left[\sin\frac{\pi}{4} - \sin 0\right]$$
$$= 2\sqrt{2}\left[\frac{1}{\sqrt{2}} - 0\right] = 2$$

Problem 9

I. Using second fundamental theorem, evaluate the following: 9. $\int_1^2 \frac{x-1}{x^2}\, dx$

For Video Explanation of this problem Click Here

Solution: Second fundamental theorem of Integral Calculus:
If $f(x)$ be a continuous function on $[a, b]$, if $F(x)$ is anti derivative of $f(x)$, then

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

$$\begin{aligned}\int_1^2 \frac{x-1}{x^2}\, dx &= \int_1^2 \frac{\cancel{x}}{x^{\cancel{2}}}\, dx - \int_1^2 \frac{1}{x^2}\, dx \\ &= \int_1^2 \frac{1}{x}\, dx - \int_1^2 x^{-2}\, dx\end{aligned}$$

We know that $\boxed{\int \frac{dx}{x} = \log|x| + C}$, $\boxed{\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, n \neq -1}$

$$\begin{aligned}\int_1^2 \frac{x-1}{x^2}\, dx &= \int_1^2 \frac{1}{x}\, dx - \int_1^2 x^{-2}\, dx \\ &= [\log|x|]_1^2 - \left[\frac{x^{-2+1}}{-2+1}\right]_1^2 \\ &= \log 2 - \log 1 - \left[\frac{x^{-1}}{-1}\right]_1^2 \\ &= \log 2 - 0 + \left[\frac{1}{x}\right]_1^2 \\ &= \log 2 + \frac{1}{2} - 1 \\ &= \log 2 - \frac{1}{2} = \frac{1}{2}[2\log 2 - 1]\end{aligned}$$

Problem 1

II. Evaluate the following: 1. $\int_1^4 f(x)\ dx$ where $f(x) = \begin{cases} 4x+3, & 1 \le x \le 2 \\ 3x+5, & 2 \le x \le 4 \end{cases}$

For Video Explanation of this problem Click Here

Solution:

$$\begin{aligned}
\int_1^4 f(x)\ dx &= \int_1^2 (4x+3)\ dx + \int_2^4 (3x+5)\ dx \\
&= 4\int_1^2 x\ dx + 3\int_1^2 dx + 3\int_2^4 x\ dx + 5\int_2^4 dx \\
&= 4\left[\frac{x^2}{2}\right]_1^2 + 3\,[x]_1^2 + 3\left[\frac{x^2}{2}\right]_2^4 + 5\,[x]_2^4 \\
&= 2\,[4-1] + 3\,[2-1] + \frac{3}{2}\,[16-4] + 5\,[4-2] \\
&= 2\,[3] + 3\,[1] + \frac{3}{2}\,[12] + 5\,[2] \\
&= 6 + 3 + 18 + 10 = 37
\end{aligned}$$

Problem 2

2. $\int_0^2 f(x)\ dx$ where $f(x) = \begin{cases} 3-2x-x^2, & 0 < x \le 1 \\ x^2+2x-3, & 1 < x \le 2 \end{cases}$

For Video Explanation of this problem Click Here

Solution:

$$\int_0^2 f(x)\ dx = \int_0^1 \left(3-2x-x^2\right)\ dx + \int_1^2 \left(x^2+2x-3\right)\ dx$$

$$= 3\int_0^1 dx - 2\int_0^1 x\ dx - \int_0^1 x^2\ dx + \int_1^2 x^2\ dx + 2\int_1^2 x\ dx - 3\int_1^2 dx$$

$$= 3\,[x]_0^1 - 2\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3}\right]_1^2 + 2\left[\frac{x^2}{2}\right]_1^2 - 3\,[x]_1^2$$

$$= 3\,[1-0] - [1-0] - \frac{1}{3}[1-0] + \frac{1}{3}[8-1] + [4-1] - 3\,[2-1]$$

$$= 3 - 1 - \frac{1}{3} + \frac{7}{3} + 3 - 3 = 4$$

Problem 3

3. $\int_{-1}^{1} f(x)\ dx$ where $f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

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Solution:

$$\int_{-1}^{1} f(x)\ dx = \int_{-1}^{0} -x\ dx + \int_0^1 x\ dx$$

$$= \left[-\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^1$$

$$= -\frac{1}{2}[0-1] + \frac{1}{2}[1-0]$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1$$

Problem 4

4. $f(x) = \begin{cases} cx, & 0 < x < 1 \\ 0, & otherwise \end{cases}$. Find 'c' if $\int_0^1 f(x)\ dx = 2$

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Solution:

$$\begin{aligned}
\int_0^1 f(x)\ dx &= 2 \\
\int_0^1 cx\, dx &= 2 \\
c\left[\frac{x^2}{2}\right]_0^1 &= 2 \\
\frac{c}{2}[1-0] &= 2 \\
\frac{c}{2} &= 2 \\
c &= 4
\end{aligned}$$

Exercise 2.9

Evaluate the following using properties of definite integrals:

Problem 1

1. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^3 \cos^3 x\ dx$

For Video Explanation of this problem Click Here

Solution: Let

$$\begin{aligned}
f(x) &= x^3 \cos^3 x \\
f(-x) &= (-x)^3 (\cos(-x))^3 \\
&= -x^3 (\cos x)^3 \text{ Since } \cos -x = \cos x \\
&= -\left(x^3 \cos^3 x\right) \\
&= -f(x)
\end{aligned}$$

$f(x)$ is an odd function, We know that $\boxed{\text{If f(x) is odd, } \int_{-a}^{a} f(x)\,dx = 0}$

Therefore, $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^3 \cos^3 x\,dx = 0$

Problem 2

2. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\,d\theta$

For Video Explanation of this problem Click Here

Solution: Let

$$\begin{aligned} f(\theta) &= \sin^2\theta \\ f(-\theta) &= (\sin(-\theta))^2 \\ &= (-\sin\theta)^2 \quad \text{Since } \sin(-x) = -\sin x \\ &= \sin^2\theta \\ &= f(\theta) \end{aligned}$$

$f(x)$ is an even function,

We know that

$\boxed{\text{If f(x) is even, } \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx}$

Therefore,

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\, d\theta &= 2\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2}\, d\theta \\
&= \int_0^{\frac{\pi}{2}} d\theta - \int_0^{\frac{\pi}{2}} \cos 2\theta d\theta \\
&= [\theta]_0^{\frac{\pi}{2}} - \left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{2} - 0 - \frac{1}{2}[\sin\pi - \sin 0] \\
&= \frac{\pi}{2}
\end{aligned}$$

Problem 3

3. $\displaystyle\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right) dx$

For Video Explanation of this problem Click Here

Solution: Let

$$\begin{aligned}
f(x) &= \log\left(\frac{2-x}{2+x}\right) \\
&= \log(2-x) - \log(2+x) \\
f(-x) &= \log(2+x) - \log(2-x) \\
&= -(\log(2-x) - \log(2+x)) \\
&= -\log\left(\frac{2-x}{2+x}\right) \\
&= -f(x)
\end{aligned}$$

$f(x)$ is an odd function,

We know that $\boxed{\text{If f(x) is odd, } \int_{-a}^{a} f(x)\, dx = 0}$

Therefore, $\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right) dx = 0$

Problem 4

4. $\int_{0}^{\frac{\pi}{2}} \frac{\sin^7 x}{\sin^7 x + \cos^7 x}\, dx$

For Video Explanation of this problem Click Here

Solution: Let $I = \int_{0}^{\frac{\pi}{2}} \frac{\sin^7 x}{\sin^7 x + \cos^7 x}\, dx \ldots (1)$

We know that $\boxed{\int_{0}^{a} f(x)\, dx = \int_{0}^{a} f(a-x)\, dx}$

Therefore,

$$\begin{aligned}
\int_{0}^{\frac{\pi}{2}} \frac{\sin^7 x}{\sin^7 x + \cos^7 x}\, dx &= \int_{0}^{\frac{\pi}{2}} \frac{\sin^7\left(\frac{\pi}{2}-x\right)}{\sin^7\left(\frac{\pi}{2}-x\right) + \cos^7\left(\frac{\pi}{2}-x\right)}\, dx \\
&= \int_{0}^{\frac{\pi}{2}} \frac{\cos^7 x}{\cos^7 x + \sin^7 x}\, dx \\
&= \int_{0}^{\frac{\pi}{2}} \frac{\cos^7 x}{\sin^7 x + \cos^7 x}\, dx \ldots (2)
\end{aligned}$$

(1)+(2)

$\Rightarrow 2I = \int_{0}^{\frac{\pi}{2}} \frac{\sin^7 x + \cos^7 x}{\sin^7 x + \cos^7 x}\, dx = \int_{0}^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

$\Rightarrow I = \frac{\pi}{4}.$

Problem 5

5. $\int_0^1 \log\left(\frac{1}{x}-1\right) dx$

For Video Explanation of this problem Click Here

Solution: Let $I = \int_0^1 \log\left(\frac{1}{x}-1\right) dx = \int_0^1 \log\left(\frac{1-x}{x}\right) dx \ldots (1)$

We know that $\boxed{\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx}$

Therefore,

$$\begin{aligned} I &= \int_0^1 \log\left(\frac{1-x}{x}\right) dx \\ &= \int_0^1 \log\left(\frac{1-(1-x)}{1-x}\right) dx \\ &= \int_0^1 \log\left(\frac{x}{1-x}\right) dx \ \ldots (2) \end{aligned}$$

(1)+(2) $\Rightarrow$

$$\begin{aligned} 2I &= \int_0^1 \left(\log\left(\frac{1-x}{x}\right) + \log\left(\frac{x}{1-x}\right)\right) dx \\ &= \int_0^1 \log\left(\frac{1-x}{x}\right)\left(\frac{x}{1-x}\right) dx \\ &= \int_0^1 \log 1\, dx \\ 2I &= 0 \\ I &= = 0 \end{aligned}$$

Problem 6

6. $\int_0^1 \frac{x}{(1-x)^{\frac{3}{4}}}\, dx$

For Video Explanation of this problem Click Here

Solution: Let $I = \int_0^1 \frac{x}{(1-x)^{\frac{3}{4}}}\, dx \ldots (1)$

We know that $\boxed{\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx}$

Therefore,

$$
\begin{aligned}
I &= \int_0^1 \frac{x}{(1-x)^{\frac{3}{4}}}\, dx \\
&= \int_0^1 \frac{1-x}{(1-(1-x))^{\frac{3}{4}}}\, dx \\
&= \int_0^1 \frac{1-x}{(x)^{\frac{3}{4}}}\, dx \\
&= \int_0^1 \frac{1}{x^{\frac{3}{4}}}\, dx - \int_0^1 \frac{x}{x^{\frac{3}{4}}}\, dx \\
&= \int_0^1 x^{-\frac{3}{4}}\, dx - \int_0^1 x^{\frac{1}{4}}\, dx \\
&= \left[\frac{x^{-\frac{3}{4}+1}}{\left(-\frac{3}{4}+1\right)}\right]_0^1 - \left[\frac{x^{\frac{1}{4}+1}}{\left(\frac{1}{4}+1\right)}\right]_0^1 \\
&= \left[\frac{x^{\frac{1}{4}}}{\left(\frac{1}{4}\right)}\right]_0^1 - \left[\frac{x^{\frac{5}{4}}}{\left(\frac{5}{4}\right)}\right]_0^1 \\
&= 4\left[1^{\frac{1}{4}} - 0\right] - \frac{4}{5}\left[1^{\frac{5}{4}} - 0\right] \\
&= 4\,[1] - \frac{4}{5}\,[1] = 4 - \frac{4}{5} = \frac{16}{5}
\end{aligned}
$$

Exercise 2.10

1. Evaluate the following:

Problem (i)

For Video Explanation of this problem Click Here

$\Gamma(4)$

Solution: We know that $\boxed{\Gamma(n+1) = n!, n \text{ is a positive integer}}$

$\Gamma(4) = \Gamma(3+1) = 3! = 6$

Problem (ii)

For Video Explanation of this problem Click Here

$\Gamma\left(\frac{9}{2}\right)$

For Video Explanation of this problem Click Here

Solution: We know that $\boxed{\Gamma(n+1) = n\Gamma(n), n > 0}$

$$\begin{aligned}
\Gamma\left(\frac{9}{2}\right) &= \frac{7}{2}\Gamma\left(\frac{7}{2}\right) \\
&= \left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right) \\
&= \left(\frac{35}{4}\right)\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right) \\
&= \left(\frac{105}{8}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) \\
&= \frac{105}{16}\sqrt{\pi}
\end{aligned}$$

Problem (iii)

$$\int_0^\infty e^{-mx}x^6\,dx$$

For Video Explanation of this problem Click Here

Solution: We know that $\boxed{\int_0^\infty e^{-ax}x^n\,dx = \frac{n!}{a^{n+1}}}$

$$\int_0^\infty e^{-mx}x^6\,dx = \frac{6!}{m^{6+1}} = \frac{6!}{m^7}$$

Problem (iv)

$$\int_0^\infty e^{-4x}x^4\,dx$$

For Video Explanation of this problem Click Here

Solution: We know that $\boxed{\int_0^\infty e^{-ax}x^n\,dx = \frac{n!}{a^{n+1}}}$

$$\int_0^\infty e^{-4x}x^4\,dx = \frac{4!}{4^{4+1}} = \frac{24}{4^5} = \frac{3}{128}$$

Problem (v)

$$\int_0^\infty e^{-\frac{x}{2}}x^5\,dx$$

For Video Explanation of this problem Click Here

Solution: We know that $\boxed{\int_0^\infty e^{-ax}x^n\,dx = \frac{n!}{a^{n+1}}}$

$$\int_0^\infty e^{-\frac{x}{2}}x^5\,dx = \frac{5!}{\left(\frac{1}{2}\right)^{5+1}} = \frac{5!}{\left(\frac{1}{2}\right)^6} = \left(2^6\right)5!$$

Problem 2.

If $f(x) = \begin{cases} x^2 e^{-2x}, & x \geq 0 \\ 0, & otherwise \end{cases}$, then evaluate $\int_0^{\infty} f(x)\ dx$

For Video Explanation of this problem Click Here

Solution: We know that $\boxed{\int_0^{\infty} e^{-ax} x^n\ dx = \frac{n!}{a^{n+1}}}$

$$\begin{aligned} \int_0^{\infty} f(x)\ dx &= \int_0^{\infty} e^{-2x} x^2\ dx \\ &= \frac{2!}{2^{2+1}} = \frac{2}{2^3} = \frac{2}{8} = \frac{1}{4} \end{aligned}$$

Exercise 2.11

Evaluate the following integrals as the limit of the sum:

Problem 1

$$\int_0^1 (x+4)\ dx$$

For Video Explanation of this problem Click Here

Solution:

If $f(x)$ be a continuous real valued function in $[a, b]$, which is divided into n equal parts of width h, then

$$\boxed{\int_a^b f(x)\ dx = \lim_{\substack{n\to\infty \\ h\to 0}} \sum_{r=1}^{n} hf(a+rh)\text{, where } h = \frac{b-a}{n}}$$

Here $a = 0, b = 1, h = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$ and $f(x) = x + 4$

Now $f(a+rh) = f\left(0 + \frac{r}{n}\right) = f\left(\frac{r}{n}\right) = \frac{r}{n} + 4$

$$\begin{aligned}\int_0^1 (x+4)\ dx &= \lim_{\substack{n\to\infty\\h\to 0}} \sum_{r=1}^{n} hf(a+rh)\\ &= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{1}{n}\right)\left(\frac{r}{n}+4\right)\\ &= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{r}{n^2}+4\frac{1}{n}\right)\\ &= \lim_{n\to\infty} \frac{1}{n^2}\sum_{r=1}^{n} r + \lim_{n\to\infty}\frac{4}{n}\sum_{r=1}^{n} 1 \ldots (1)\end{aligned}$$

We know that

$\sum_{r=1}^{n} r = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$	$\sum_{r=1}^{n} 1 = 1+1+1+\cdots$+1(n times)=n

Therefore (1) $\Rightarrow$

$$\begin{aligned}\int_0^1 (x+4)\ dx &= \lim_{n\to\infty} \frac{1}{n^2}\sum_{r=1}^{n} r + \lim_{n\to\infty}\frac{4}{n}\sum_{r=1}^{n} 1\\ &= \lim_{n\to\infty} \frac{1}{n^2}\left(\frac{n(n+1)}{2}\right) + \lim_{n\to\infty}\frac{4}{n}(n)\\ &= \lim_{n\to\infty} \frac{1}{n^2}\left(\frac{n^2\left(1+\frac{1}{n}\right)}{2}\right)+4\\ &= \frac{(1+0)}{2}+4 = \frac{9}{2}\end{aligned}$$

Problem 2

$$\int_1^3 x\,dx$$

For Video Explanation of this problem Click Here

Solution:

If $f(x)$ be a continuous real valued function in $[a,b]$, which is divided into n equal parts of width h, then

$$\boxed{\int_a^b f(x)\ dx = \lim_{\substack{n\to\infty\\h\to 0}} \sum_{r=1}^{n} hf(a+rh),\ \text{where}\ \ h=\frac{b-a}{n}}$$

Here $a = 1, b = 3, h = \dfrac{b-a}{n} = \dfrac{3-1}{n} = \dfrac{2}{n}$ and $f(x) = x$

Now $f(a+rh) = f\left(1+\frac{2r}{n}\right) = 1 + \dfrac{2r}{n} = \dfrac{2r}{n} + 1$

$$\begin{aligned}
\int_1^3 x\,dx &= \lim_{\substack{n\to\infty\\ h\to 0}} \sum_{r=1}^{n} hf(a+rh)\\
&= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{2}{n}\right)\left(\frac{2r}{n}+1\right)\\
&= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{4r}{n^2}+\frac{2}{n}\right)\\
&= \lim_{n\to\infty} \frac{4}{n^2}\sum_{r=1}^{n} r + \lim_{n\to\infty} \frac{2}{n}\sum_{r=1}^{n} 1 \ldots (1)
\end{aligned}$$

We know that

$\sum_{r=1}^{n} r = 1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$	$\sum_{r=1}^{n} 1 = 1+1+1+\cdots$+1(n times)=n

Therefore (1) $\Rightarrow$

$$\begin{aligned}
\int_1^3 x\,dx &= \lim_{n\to\infty} \frac{4}{n^2}\sum_{r=1}^{n} r + \lim_{n\to\infty} \frac{2}{n}\sum_{r=1}^{n} 1\\
&= \lim_{n\to\infty} \frac{4}{n^2}\left(\frac{n(n+1)}{2}\right) + \lim_{n\to\infty} \frac{2}{n}(n)\\
&= \lim_{n\to\infty} \frac{4}{n^2}\left(\frac{n^2\left(1+\frac{1}{n}\right)}{2}\right) + 2\\
&= 4\left(\frac{(1+0)}{2}\right) + 2 = 4
\end{aligned}$$

Problem 3

$$\int_1^3 (2x+3)\ dx$$

Solution:

For Video Explanation of this problem Click Here

If $f(x)$ be a continuous real valued function in $[a,b]$, which is divided into n equal parts of width h, then

$$\boxed{\int_a^b f(x)\ dx = \lim_{\substack{n\to\infty \\ h\to 0}} \sum_{r=1}^{n} hf(a+rh),\ \text{where}\ \ h = \frac{b-a}{n}}$$

Here $a = 1, b = 3, h = \dfrac{b-a}{n} = \dfrac{3-1}{n} = \dfrac{2}{n}$ and $f(x) = 2x+3$

Now $f(a+rh) = f\left(1+\frac{2r}{n}\right) = 2\left(1+\dfrac{2r}{n}\right)+3 = 2+\dfrac{4r}{n}+3 = \dfrac{4r}{n}+5$

$$\begin{aligned}
\int_1^3 x\ dx &= \lim_{\substack{n\to\infty \\ h\to 0}} \sum_{r=1}^{n} hf(a+rh) \\
&= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{2}{n}\right)\left(\frac{4r}{n}+5\right) \\
&= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{8r}{n^2}+\frac{10}{n}\right) \\
&= \lim_{n\to\infty} \frac{8}{n^2} \sum_{r=1}^{n} r + \lim_{n\to\infty} \frac{10}{n} \sum_{r=1}^{n} 1 \ldots (1)
\end{aligned}$$

We know that

$\boxed{\sum_{r=1}^{n} r = 1+2+3+\cdots+n = \dfrac{n(n+1)}{2}}$ $\boxed{\sum_{r=1}^{n} 1 = 1+1+1+\cdots+1\text{(n times)=n}}$

Therefore (1) $\Rightarrow$

$$\begin{aligned}
\int_1^3 x\ dx &= \lim_{n\to\infty} \frac{8}{n^2} \sum_{r=1}^{n} r + \lim_{n\to\infty} \frac{10}{n} \sum_{r=1}^{n} 1 \\
&= \lim_{n\to\infty} \frac{8}{n^2} \left(\frac{n(n+1)}{2}\right) + \lim_{n\to\infty} \frac{10}{n}(n) \\
&= \lim_{n\to\infty} \frac{8}{n^2} \left(\frac{n^2\left(1+\frac{1}{n}\right)}{2}\right) + 10 \\
&= 8\left(\frac{(1+0)}{2}\right) + 10 = 14
\end{aligned}$$

Problem 4

$$\int_0^1 x^2\,dx$$

For Video Explanation of this problem Click Here

Solution:

If $f(x)$ be a continuous real valued function in $[a,b]$, which is divided into n equal parts of width h, then

$$\boxed{\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty\\h\to 0}} \sum_{r=1}^{n} hf(a+rh), \text{ where } h = \frac{b-a}{n}}$$

Here $a=0, b=1, h=\frac{b-a}{n}=\frac{1-0}{n}=\frac{1}{n}$ and $f(x)=x^2$

Now $f(a+rh) = f\left(0+\frac{r}{n}\right) = \left(\frac{r}{n}\right)^2 = \frac{r^2}{n^2}$

$$\begin{aligned}\int_0^1 x^2\,dx &= \lim_{\substack{n\to\infty\\h\to 0}} \sum_{r=1}^{n} hf(a+rh)\\ &= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{1}{n}\right)\left(\frac{r^2}{n^2}\right)\\ &= \lim_{n\to\infty} \frac{1}{n^3}\sum_{r=1}^{n} r^2 \ldots(1)\end{aligned}$$

We know that $\boxed{\sum_{r=1}^{n} r^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}}$

Therefore (1) $\Rightarrow$

$$\begin{aligned}\int_0^1 x^2\,dx &= \lim_{n\to\infty} \frac{1}{n^3}\sum_{r=1}^{n} r^2\\ &= \lim_{n\to\infty} \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\\ &= \lim_{n\to\infty} \frac{1}{n^3}\left(\frac{n^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}\right)\\ &= \left(\frac{(1+0)(2+0)}{6}\right) = \frac{2}{6} = \frac{1}{3}\end{aligned}$$