UNIT : 4 INNER PRODUCT SPACE

Defn: Inner product

Let V be a vector space over the field F. Define a function $< \; >: V \times V \to F$. Then $< \; >$ is called inner product if it satisfies the following :

(i) $\overline{\langle u, v \rangle} = \langle v, u \rangle$ [conjugate]

(ii) $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ iff $u = 0$

(iii) $\langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle \quad \forall u, v \in V$ & $\alpha, \beta \in F$

And a vector space together with an inner product is known as "INNER PRODUCT SPACE"

Note: ⊛ If $F = R$, then the inner product space V is called "real inner product space" or "Euclidean space".

⊛ If $F = C$, then the inner product space V is called "Complex inner product space" or "Unitary space"

⊛ Standard inner product over R is defined as

$$\langle u, v \rangle = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

⊛ Standard inner product over C is defined as

$$\langle u, v \rangle = a_1 \bar{b}_1 + a_2 \bar{b}_2 + \cdots + a_n \bar{b}_n$$

2M. Problems:

(1) ① Prove that in an inner product space $V(F)$,

$$\langle u, \alpha v + \beta w \rangle = \bar{\alpha} \langle u, v \rangle + \bar{\beta} \langle u, w \rangle$$

Proof: LHS : $\langle u, \alpha v + \beta w \rangle = \overline{\langle \alpha v + \beta w, u \rangle}$

$\overline{\langle \alpha (v, u) \rangle} + \overline{\langle \beta (w, u) \rangle}$

$\bar{\alpha} \langle u, v \rangle + \bar{\beta} \langle u, w \rangle$

LHS $= \bar{\alpha} \langle u, v \rangle + \bar{\beta_1} \langle u, w \rangle$

$= \text{RHS}.$

Hence the proof

(2). $F^n(C)$

Practice Problems

1. Let V be a set of all continuous real functions defined on $[0,1]$. The inner product on V be defined by $\langle f,g\rangle = \int_0^1 f(t)\,g(t)\,dt$. Prove that $V(R)$ is an inner product space.

2. Prove that $R^2(R)$ is an inner product space with the inner product defined for $u=(a_1,a_2)$, $v=(b_1,b_2)$ by $\langle u,v\rangle = a_1b_1 - a_2b_1 - a_1b_1 + 2a_2b_2$

2(i)

3. P.T $V_n(R)$ is a real inner product space defined by $\langle x,y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$

Defn: Norm or length of a vector

Let V be a IPS and $v \in V$. The norm of v denoted by $\|v\| = \sqrt{\langle v,v\rangle}$.

The vector v is called a unit vector if $\|v\| = 1$

Problem

1. Let $u = (2, 1+i, i)$, $v = (2-i, 2, 1+2i)$ be vectors in C^3 over C. Compute using standard inner product $\langle u,v\rangle$, $\|u\|$, $\|v\|$, $\|u+v\|$

3(i)

$\longrightarrow$

$u = (2, 1+i, i)$ with a_1, a_2, a_3 $\quad v = (2-i, 2, 1+2i)$ with b_1, b_2, b_3

$$\langle u,v\rangle = a_1\overline{b_1} + a_2\overline{b_2} + a_3\overline{b_3}$$

$$= 2\,\overline{(2-i)} + 1+i\,(\overline{2}) + i\,(\overline{1+2i})$$

$$= 2(2+i) + 1+i(2) + i(1-2i)$$

$$= 4+2i + 2+2i + i + 2 = 8+5i$$

$$\|u\| = \sqrt{\langle u,u\rangle} = \sqrt{a_1\bar{a}_1 + a_2\bar{a}_2 + a_3\bar{a}_3}$$
$$= \sqrt{2(2)+(1+i)(1-i)+i(-i)}$$
$$= \sqrt{4+2+1}$$
$$= \sqrt{7}$$

$$\|v\| = \sqrt{\langle v,v\rangle} = \sqrt{b_1\bar{b}_1 + b_2\bar{b}_2 + b_3\bar{b}_3}$$
$$= \sqrt{(2-i)(2+i)+2(2)+(1+2i)(1-2i)}$$
$$= \sqrt{5+4+5}$$
$$= \sqrt{14}$$

$$u+v = (2+2-i,\ 1+i+2,\ i+1+2i)$$
$$= (4-i,\ 3+i,\ 1+3i)$$

$$\|u+v\| = \sqrt{\langle u+v, u+v\rangle} = \sqrt{(4-i)(4+i)+(3+i)(3-i)+(1+3i)(1-3i)}$$
$$= \sqrt{17+10+10}$$
$$= \sqrt{37}$$

Practice problem

① Let $V = P(R)$, the vector space of polynomials over R with inner product defined by $\langle f,g\rangle = \int_0^1 f(t)\,g(t)\,dt$

Q(i) where $f(t) = t+2$, $g(t) = t^2-2t+3$. find $\|f\|$, $\|g\|$, $\langle f,g\rangle$

Theorem:

Let V be an IPS over F. Then $\forall\ u, v \in V$ &

1M (i) $\|u+v\|^2 + \|u-v\|^2 = 2[\|u\|^2 + \|v\|^2]$. [Parallelogram law]

10.2M (ii) $\|\alpha u\| = |\alpha| \|u\|$.

(iii) $|\langle u,v\rangle| \leq \|u\| \|v\|$ [schwarz inequality]

11.3M (iv) $\|u+v\| \leq \|u\| + \|v\|$ [Triangle inequality]

(v) $\big| \|x\| - \|y\| \big| \leq \|x-y\|$

(vi) $\|x+y\|^2 - \|x-y\|^2 = 4\langle x,y\rangle$

<u>proof:</u>

(i) To proove $\|u+v\|^2 + \|u-v\|^2 = 2[\|u\|^2 + \|v\|^2]$

LHS: $\|u+v\|^2 + \|u-v\|^2 = \langle u+v, u+v\rangle + \langle u-v, u-v\rangle$

$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle + \langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle$

$= 2\langle u,u\rangle + 2\langle v,v\rangle$

$= 2\|u\|^2 + 2\|v\|^2$

$= 2[\|u\|^2 + \|v\|^2]$

$=$ RHS.

(ii) To prove $\|\alpha u\| = |\alpha| \|u\|$

$\|\alpha u\|^2 = \langle \alpha u, \alpha u\rangle$

$= \alpha\bar{\alpha} \langle u,u\rangle$

$= |\alpha|^2 \|u\|^2$

$\therefore \|\alpha u\| = |\alpha| \|u\|$

(iii) To prove $|\langle u,v\rangle| \leq \|u\| \|v\|$.

If $u=0$ or $v=0$ then $\langle u,v\rangle = 0$, $\|u\|=0$, $\|v\|=0$

Then the result is trival.

If $u \neq 0$ and $v \neq 0$

consider $w = v - \frac{\langle v, u \rangle}{\|u\|^2} u$

$\langle w, w \rangle \geq 0$

$\left\langle v - \frac{\langle v, u \rangle}{\|u\|^2} u , v - \frac{\langle v, u \rangle}{\|u\|^2} u \right\rangle \geq 0$

$\langle v, v \rangle - \frac{\langle v, u \rangle \overline{\langle v, u \rangle}}{\|u\|^2} - \frac{\langle v, u \rangle \langle u, v \rangle}{\|u\|^2} + \frac{\langle v, u \rangle \overline{\langle v, u \rangle} \cancel{\langle u, u \rangle}}{\|u\|^2 \cancel{\|u\|^2}} \geq 0$

$\|v\|^2 - \frac{\langle v, u \rangle \overline{\langle v, u \rangle}}{\|u\|^2} - \frac{\langle v, u \rangle \langle u, v \rangle}{\|u\|^2} + \frac{\langle v, u \rangle \overline{\langle v, u \rangle}}{\|u\|^2} \geq 0$

$\|v\|^2 - \frac{\langle v, u \rangle \langle u, v \rangle}{\|u\|^2} \geq 0$

$\|v\|^2 \|u\|^2 - \overline{\langle u, v \rangle} \langle u, v \rangle \geq 0$

$\|u\|^2 \|v\|^2 \geq |\langle u, v \rangle|^2$

$\therefore \ |\langle u, v \rangle| \leq \|u\| \, \|v\|$

(iv) To prove $\|u + v\| \leq \|u\| + \|v\|$

5 (ii) $\|u+v\|^2 = \langle u+v, u+v \rangle$

$= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle$

$= \|u\|^2 + \langle u, v \rangle + \overline{\langle u, v \rangle} + \|v\|^2$

$= \|u\|^2 + 2 \operatorname{Re} \langle u, v \rangle + \|v\|^2$ $\left[\because \frac{z + \bar{z}}{2} = \operatorname{Re}(z)\right]$

$\leq \|u\|^2 + 2 |\langle u, v \rangle| + \|v\|^2$ $[\because \operatorname{Re}(z) \leq |z|]$

$\leq \|u\|^2 + 2\|u\| \, \|v\| + \|v\|^2$ [By schwarz]

$= [\|u\| + \|v\|]^2$

$\therefore \|u+v\| \leq \|u\| + \|v\|$

(v) To prove $\big|\|x\| - \|y\|\big| \leq \|x-y\|$

$\|x\| = \|x-y+y\|$

$\leq \|x-y\| + \|y\|$ by triangular inequality

$\|x\| - \|y\| \leq \|x-y\|$ —— ①

$\|y\| = \|y-x+x\|$

$\leq \|y-x\| + \|x\|$.

$\|y\| - \|x\| \leq \|x-y\|$ —— ② or $[-(\|x\| - \|y\|) \leq \|x-y\|]$

from ① & ②

$\therefore \big|\|x\| - \|y\|\big| \leq \|x-y\|$

2) (vi) (ii). To prove : $\|x+y\|^2 - \|x-y\|^2 = 4\langle x, y\rangle$

LHS : $\|x+y\|^2 - \|x-y\|^2 = \langle x+y, x+y\rangle - \langle x-y, x-y\rangle$

$= \cancel{\langle x, x\rangle} + \langle x, y\rangle + \langle y, x\rangle + \cancel{\langle y, y\rangle}$

$- \cancel{\langle x, x\rangle} + \langle x, y\rangle + \langle y, x\rangle - \cancel{\langle y, y\rangle}$

$= 2\langle x, y\rangle + 2\langle y, x\rangle$

$= 4\langle x, y\rangle$ $[\because \langle x, y\rangle = \langle y, x\rangle$ for Real IPS $]$

$=$ RHS

Practice problem

① Verify Schwarz and triangular inequality, in $C([0,1])$

let $f(t) = t$, $g(t) = e^t$

Defn: Orthogonal vectors:

Let V be an IPS over F. Let $u, v \in V$. The vector u is ~~[illegible]~~ said to be orthogonal to v if $\langle u, v\rangle = 0$ or $\langle v, u\rangle = 0$

Note (i) $\langle u, v\rangle = 0$ then $\langle u, \alpha v\rangle = 0$
(ii) The zero vector is orthogonal to every vector $v \in V$

Problem :

① In an IPS $V(F)$ if u & v are orthogonal then P.T $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ [Pythogoros theorem]

Proof

LHS: $\|u+v\|^2 = \langle u+v, u+v\rangle$
$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$
$= \|u\|^2 + \|v\|^2 \quad [\because \langle u,v\rangle = \langle v,u\rangle = 0]$

Defn : orthonormal set

A set of vectors $S = \{v_1, v_2, \dots v_n\}$ in an IPS V is called an orthonormal set if (i) $\langle v_i, v_j\rangle = 0$ if $i \neq j$
(ii) $\langle v_i, v_i\rangle = 1$

Note: (i) orthonormal set is an orthogonal set consisting of unit vectors.
(ii) for R^2, $\{i, j\}$ is an orthonormal set
(iii) for R^3, $\{i, j, k\}$ is an orthonormal set

Defn : orthonormal basis

A subset S of an IPS V over F is called orthonormal basis for V if it is an ordered basis that is orthonormal.

For instance, $S = \{i, j, k\}$ is an orthonormal basis for R^3
$S = \{i, j\}$ is an orthonormal basis for R^2
$S = \left\{\left(\frac{1}{\sqrt5}, \frac{2}{\sqrt5}\right), \left(\frac{2}{\sqrt5}, \frac{-1}{\sqrt5}\right)\right\}$ is an orthonormal basis in R^2.

* Find an orthogonal basis of IPS $R^3(R)$ with SIP $B = \{(1,1,0), (1,-1,1)\}$

Note:

Frobenius Inner product is defined as $\langle A, B\rangle = tr(B^*A)$

Where $B^* = [\overline{b_{ij}}]^T$ & $tr[B]$ = sum of principal diagonal of B.

Example: If $A = \begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix}$, $B = \begin{pmatrix} 1+i & 0 \\ 0 & -i \end{pmatrix} \in M_{2\times 2}(\mathbb{C})$

using Frobenius Inner product, compute $\langle A, B\rangle$, $\|A\|$, $\|B\|$.

$$\langle A, B\rangle = tr(B^*A)$$

$$= tr\left[\begin{pmatrix} 1-i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix}\right]$$

$(2+i)(1-i)$

$2-2i+i-i^2$

$3-i$

$$= tr\begin{pmatrix} 1-i & 2+i-2i+1 \\ 3i & -1 \end{pmatrix}$$

$$\langle A, B\rangle = -i$$

$$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{tr(A^*A)}$$

(13)

$$\|A\| = \sqrt{tr\left[\begin{pmatrix} 1 & 3 \\ 2-i & -i \end{pmatrix}\begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix}\right]}$$

$$= \sqrt{tr\begin{pmatrix} 1+9 & 2+i+3i \\ 2-i-3i & 4+1+1 \end{pmatrix}}$$

$$= \sqrt{16}$$

$$= 4$$

$$\|B\| = \sqrt{\langle B, B\rangle} = \sqrt{tr(B^*B)}$$

$$= \sqrt{tr\begin{pmatrix} 1-i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 1+i & 0 \\ 0 & -i \end{pmatrix}}$$

$$= \sqrt{tr\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}} = \sqrt{3}$$

) using GSO process. Also find the fourier coefficients of
) relative to orthonormal basis.

Given $B = \{v_1, v_2, v_3\}$
$v_1 = (1,1,0)$ $v_2 = (1,-1,1)$
$v_3 = (-1,1,2)$

Step (i) $u_1 = v_1 = (1,1,0)$

Step (ii)
$$u_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\|u_1\|^2} u_1$$

$\langle v_2, u_1\rangle = \langle (1,-1,1),(1,1,0)\rangle = 1-1 = 0$

$\|u_1\|^2 = \langle u_1, u_1\rangle = \langle (1,1,0),(1,1,0)\rangle = 1+1 = 2$

$$u_2 = (1,-1,1) - \frac{0}{2}(1,1,0) = (1,-1,1)$$

Step (iii)
$$u_3 = v_3 - \frac{\langle v_3, u_1\rangle u_1}{\|u_1\|^2} - \frac{\langle v_3, u_2\rangle u_2}{\|u_2\|^2}$$

$\|u_1\|^2 = 2$

$\|u_2\|^2 = \langle u_2, u_2\rangle = \langle (1,-1,1),(1,-1,1)\rangle = 1+1+1 = 3$

$\langle v_3, u_1\rangle = \langle (-1,1,2),(1,1,0)\rangle = -1+1 = 0$

$\langle v_3, u_2\rangle = \langle (-1,1,2),(1,-1,1)\rangle = -1-1+2 = 0$

$$u_3 = (-1,1,2) - 0 - 0 = (-1,1,2)$$

$\therefore$ The orthogonal basis is $\{u_1, u_2, u_3\}$

$u_1 = (1,1,0)$ $u_2 = (1,-1,1)$ $u_3 = (-1,1,2)$

Now the orthonormal basis is $\{w_1, w_2, w_3\}$

$$w_1 = \frac{u_1}{\|u_1\|} \qquad w_2 = \frac{u_2}{\|u_2\|} \qquad w_3 = \frac{u_3}{\|u_3\|}$$

$$= \frac{(1,1,0)}{\sqrt{2}} \qquad = \frac{(1,-1,1)}{\sqrt{3}} \qquad = \frac{(-1,1,2)}{\sqrt{1+1+4}} = \frac{(-1,1,2)}{\sqrt{6}}$$

The fourier coefficients are $\langle v, w_1\rangle$, $\langle v, w_2\rangle$, $\langle v, w_3\rangle$

Given $v = (2,1,3)$ & $v = \langle v, w_1\rangle w_1 + \langle v, w_2\rangle w_2 + \langle v, w_3\rangle w_3$

$$\langle v, w_1\rangle = \langle (2,1,3), \frac{(1,1,0)}{\sqrt{2}}\rangle = \frac{1}{\sqrt{2}}[2+1] = \frac{3}{\sqrt{2}}$$

$$\langle v, w_2\rangle = \langle (2,1,3), \frac{(1,-1,1)}{\sqrt{3}}\rangle = \frac{1}{\sqrt{3}}[2-1+3] = \frac{4}{\sqrt{3}}$$

① ... that $R^2(R)$ is an inner product space defined for $u=(a_1,a_2)$ and $v=(b_1,b_2)$ by $\langle u,v\rangle = a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2$

Soln: Given $u=(a_1,a_2)$ & $v=(b_1,b_2)$
and the function is defined by
$\langle u,v\rangle = a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2 \quad a_1,a_2,b_1,b_2 \in R.$

To Prove $R^2(R)$ is an inner product space.

(1) $\langle u,v\rangle = \overline{a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2}$

$= a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2$

$= b_1a_1 - b_1a_2 - b_2a_1 + 2b_2a_2 \quad a_i, b_i \in R.$

$= \langle v,u\rangle. \quad$ [if a real $\bar{a}=a$]

(2) $\langle u,u\rangle = a_1a_1 - a_2a_1 - a_1a_2 + 2a_2^2$

$= a_1^2 - 2a_1a_2 + a_2^2 + a_2^2$

$= (a_1-a_2)^2 + a_2^2 \geq 0$

$\langle u,u\rangle = 0$ iff $(a_1-a_2)^2 + a_2^2 = 0.$

iff $a_1 - a_2 = 0 \quad a_2 = 0.$

$\Leftrightarrow \quad a_1 = 0, a_2 = 0.$

$\Leftrightarrow \quad u = 0$

(3) Let $w = (c_1,c_2)$, $\alpha, \beta \in R$

$\langle \alpha u + \beta v, w\rangle = \langle \alpha(a_1,a_2) + \beta(b_1,b_2), (c_1,c_2)\rangle$

$= \langle(\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2), (c_1,c_2)\rangle$

$= (\alpha a_1 + \beta b_1)c_1 - (\alpha a_2 + \beta b_2)c_1 - (\alpha a_1 + \beta b_1)c_2 + 2(\alpha a_1 + \beta b_1)c_2$

$= \alpha a_1c_1 + \beta b_1c_1 - \alpha a_2c_1 - \beta b_2c_1 - \alpha a_1c_2 - \beta b_1c_2 + 2\alpha a_1c_2 + 2\beta b_1c_2$

Let V be the set of all continuous real functions defined on the closed interval $[0,1]$. The inner Product on V be defined by $\langle f, g\rangle = \int_0^1 f(t)\,g(t)\,dt$. Prove that $V(R)$ is a inner Product space.

Soln: we Verify the axioms of inner product on V. $\forall f, g, h \in V$ and $\alpha, \beta \in R$.

(1) $$\langle f, g\rangle = \int_0^1 f(t)\,g(t)\,dt$$

$$= \int_0^1 g(t)\,f(t)\,dt$$

$$= \langle g, f\rangle$$

(2) $$\langle f, f\rangle = \int_0^1 f(t)\,f(t)\,dt$$

$$= \int_0^1 (f(t))^2\,dt \geq 0$$

And $\langle f, f\rangle = 0$ iff $f(t) = 0 \;\forall\, t \in [0,1]$

$\therefore f = 0$

(3) $$\langle \alpha f + \beta g, h\rangle = \int_0^1 (\alpha f(t) + \beta g(t))\,h(t)\,dt$$

$$= \alpha \int_0^1 f(t)\,h(t)\,dt + \beta \int_0^1 g(t)\,h(t)\,dt$$

$$= \alpha\langle f, h\rangle + \beta\langle g, h\rangle$$

From (1), (2), (3) Hence the function $\langle\;\rangle$ is an inner Product over R.

$\therefore$ V is an inner Product space over R.

$= \alpha(a_1c_1 - a_2c_1 - a_1c_2 + 2a_1c_2) + \beta(b_1c_1 - b_2c_1 - b_1c_2 + 2b_1c_2)$.

$= \alpha < u, w > + \beta < v, w >$.

Hence (1), (2) (3) are satisfied ; Hence the fn $<>$ is an inner product on $R^2(R)$

$\therefore R^2(R)$ is an inner product space

$$= \alpha a_1c_1 + \beta b_1c_1 - \alpha a_2c_1 - \beta b_2c_1$$
$$- \alpha a_1c_2 - \beta b_1c_2 + 2\alpha a_2c_2 + 2\beta b_2c_2$$

$$= \alpha(a_1c_1 - a_2c_1 - a_1c_2 + 2a_1c_2) + \beta(b_1c_1 - b_2c_1 - b_1c_2 + 2b_2c_2)$$

$$= \alpha\langle u, v\rangle + \beta\langle v, w\rangle$$

$\Rightarrow$ $\langle\ \rangle$ is an IP on $R^2(R)$

$\Rightarrow$ $R^2(R)$ is an IPS

Adjoint of a matrix is $A^* = (\bar{A})^T$

Adjoint operator

Let $V(F)$ be a finite dimensional inner product space and T be a linear operator on V. Then the adjoint of T on V denoted by T^* is defined as

$$\langle T(u), v\rangle = \langle u, T^*(v)\rangle \quad \forall\, u, v \in V$$

Theorem: If $V(F)$ is an inner product space and S, T are any linear operators on V.

Then
1. $(T^*)^* = T$
2. $(S+T)^* = S^* + T^*$
3. $(\alpha T)^* = \bar{\alpha}\, T^* \quad \forall\, \alpha \in F$
4. $(ST)^* = T^* S^*$

Proof.

(i) for any $u, v \in V$ consider $\langle u, (T^*)^*(v)\rangle = \langle T^*(u), v\rangle$

$= \overline{\langle v, T^*(u)\rangle}$

$= \overline{\langle T(v), u\rangle}$

$= \langle u, T(v)\rangle \quad \forall\, u \in V$

$\Rightarrow (T^*)^*(v) = T(v) \quad \forall\, v \in V$

$\Rightarrow (T^*)^* = T$

(ii) $\langle u, (S+T)^*(v)\rangle = \langle (S+T)(u), v\rangle$

$= \langle S(u) + T(u), v\rangle$

$= \langle S(u), v\rangle + \langle T(u), v\rangle$

$= \langle u, S^*(v)\rangle + \langle u, T^*(v)\rangle$

$= \langle u, S^*(v) + T^*(v)\rangle$

$= \langle u, (S^* + T^*)(v)\rangle \quad \forall\, u \in V$

$\Rightarrow (S+T)^*(v) = (S^* + T^*)(v) \quad \forall\, v \in V$

$\Rightarrow (S+T)^* = S^* + T^*$

(iii) $\langle u, (\alpha T)^*(v)\rangle = \langle (\alpha T)(u), v\rangle$

$= \langle \alpha T(u), v\rangle$

$= \alpha \langle T(u), v\rangle$

$= \alpha \langle u, T^*(v)\rangle$

$= \langle u, \bar{\alpha}\, T^*(v)\rangle \quad \forall\, u \in V$

$\Rightarrow (\alpha T)^*(v) = \bar{\alpha}\, T^*(v) \quad \forall\, v \in V$

$\Rightarrow (\alpha T)^* = \bar{\alpha}\, T^*$

(ii)

$$\langle u, (ST)^*(v)\rangle = \langle (ST)(u), v\rangle$$
$$= \langle S(T(u)), v\rangle$$
$$= \langle T(u), S^*(v)\rangle$$
$$= \langle u, T^*(S^*(v))\rangle$$
$$= \langle u, (T^*S^*)(v)\rangle \quad \forall\, u \in$$

$$\Rightarrow (ST)^*(v) = (T^*S^*)(v) \quad \forall\, v \in V$$
$$\Rightarrow (ST)^* = T^*S^*$$

(*) A linear operator T on $R^2(R)$ is defined by $T(x,y) = (2x+y, x-3y)$ with S.I.P. Find $T^*(x,y)$ and $T^*(3,5)$

$\longrightarrow$ Given $T(x,y) = (2x+y, x-3y)$

Let $B = \{(1,0), (0,1)\}$

$T(1,0) = (2,1)$ coefficient matrix $= \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$

$T(0,1) = (1,-3)$

Matrix $[T]_\beta = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$ $\boxed{* \to T}$

Matrix $[T^*]_\beta \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}^* = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$

Matrix $T^*(x,y) = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 2x+y \\ x-3y \end{bmatrix}$

$T^*(x,y) = (2x+y,\ x-3y)$

$T^*(3,5) = (6+5,\ 3-15) = (11,-12)$

(*) Let $V = P(R)$, the vector space of polynomials over R with inner product defined by

$\langle f, g\rangle = \int_0^1 f(t)\,g(t)\,dt$, $f(t) = t+2$ $g(t) = t^2-2t-3$

Find $\|f\|$, $\|g\|$, $\|f+g\|$, $\langle f,g\rangle$

(i) $\langle f, g\rangle = \int_0^1 f(t)\,g(t)\,dt$

$= \int_0^1 (t+2)(t^2-2t-3)\,dt$

$= \int_0^1 (t^3 - \cancel{2t^2} - 3t + \cancel{2t^2} - 4t - 6)\,dt$

$= \int_0^1 t^3 - 7t - 6)\,dt$

$= \left(\frac{t^4}{4} - \frac{7t^2}{2} - 6t\right)_0^1 = \frac{1}{4} - \frac{7}{2} - 6 = \frac{1-14-24}{4}$

$= -\frac{37}{4}$

$\|f\| = \sqrt{\langle f, f\rangle}$

$\langle f, f\rangle = \int_0^1 f(t)\,f(t)\,dt$

$= \int_0^1 (t+2)^2\,dt = \left.\frac{(t+2)^3}{3}\right]_0^1 = \frac{3^3}{3} - \frac{2^3}{3}$

$= 19/3$

$\|f\| = \sqrt{19/3}$

$$\|g\| = \sqrt{\langle g, g\rangle} \quad = \int_0^1 g(t)\, g(t)\, dt$$

$$= \int_0^1 (t^2-2t-3)^2\, dt$$

$$= \int_0^1 (t^4 + 4t^2 + 9 - 4t^3 + 12t - 6t^2)\, dt$$

$$= \int_0^1 (t^4 - 4t^3 - 2t^2 + 12t + 9)\, dt$$

$$= \left(\frac{t^5}{5} - \frac{4t^4}{4} - \frac{2t^3}{3} + \frac{12t^2}{2} + 9t\right)_0^1$$

$$= \frac{1}{5} - 1 - \frac{2}{3} + 6 + 9$$

$$= \frac{1}{5} - \frac{2}{3} + 14 \quad = \frac{3-10+210}{15} \quad = \frac{203}{15}$$

$$\|g\| = \sqrt{203/15}$$

$$\|f+g\| = \sqrt{\langle f+g, f+g\rangle} \qquad f+g = t^2 - t - 1$$

$$\langle f+g, f+g\rangle = \int_0^1 (t^2-t-1)(t^2-t-1)\, dt$$

$$= \int_0^1 (t^4 - t^3 - \cancel{t^2} - t^3 + \cancel{t^2} + t - t^2 + t + 1)\, dt$$

$$= \int_0^1 (t^4 - 2t^3 - t^2 + 2t + 1)\, dt$$

$$= \left[\frac{t^5}{5} - \frac{2t^4}{4} - \frac{t^3}{3} + \frac{2t^2}{2} + t\right]_0^1$$

$$= \frac{1}{5} - \frac{1}{2} - \frac{1}{3} + 1 + 1$$

$$= \frac{6-15-10+60}{30} = \frac{41}{30}$$

$$\|f+g\| = \sqrt{41/30}.$$

* In an inner product space $R^3(R)$ with the SIP $B = \{(1,0,1), (1,0,-1), (0,3,4)\}$ is a basis. By Gram schmidt orthogonalization process, find the orthogonal basis and hence find an orthonormal basis.

$B = \{v_1, v_2, v_3\}$ is the basis of R^3 $\quad v_1 = (1,0,1) \quad v_2 = (1,0,-1)$

$v_3 = (0,3,4)$.

Step (i) $\quad u_1 = v_1 = (1,0,1)$.

Step (ii) $\quad u_2 = v_2 - \frac{\langle v_2, u_1\rangle u_1}{\|u_1\|^2}$.

$\langle v_2, u_1\rangle = \langle (1,0,-1), (1,0,1)\rangle = 1-1 = 0$

$\|u_1\|^2 = \langle u_1, u_1\rangle = \langle (1,0,1)(1,0,1)\rangle = 1+1 = 2.$

$\therefore \quad u_2 = (1,0,-1) - 0 = (1,0,-1)$

Step (iii) $\quad u_3 = v_3 - \frac{\langle v_3, u_1\rangle u_1}{\|u_1\|^2} - \frac{\langle v_3, u_2\rangle u_2}{\|u_2\|^2}$

$\langle v_3, u_1\rangle = \langle (0,3,4), (1,0,1)\rangle = 4$

$\langle v_3, u_2\rangle = \langle (0,3,4), (1,0,-1)\rangle = -4$

$\|u_1\|^2 = 2 \qquad \|u_2\|^2 = 1+1 = 2$

$\therefore$ $u_3 = (0,3,4) - \frac{(4,0,4)}{2} - \frac{-4(1,0,-1)}{+2}$

$= (0,3,4) - (2,0,2) + (2,0,-2)$

$= (0,3,0)$

$\therefore$ The orthogonal basis is $\{u_1, u_2, u_3\}$

$= \{(1,0,1), (1,0,-1), (0,3,0)\}$

Let The orthonormal basis is $\{w_1, w_2, w_3\}$

where $w_1 = \frac{u_1}{\|u_1\|}$ $\quad w_2 = \frac{u_2}{\|u_2\|}$ $\quad w_3 = \frac{u_3}{\|u_3\|}$

$w_1 = \frac{(1,0,1)}{\sqrt{2}}$ $\quad w_2 = \frac{(1,0,-1)}{\sqrt{2}}$ $\quad w_3 = \frac{(0,3,0)}{\sqrt{9}}$

$\therefore \{w_1, w_2, w_3\} = \left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), (0, 1, 0)\right\}$

0) Verify that the set $\{v_1, v_2, v_3\}$ where $v_1 = (0,1,-1)$ $\quad v_2 = (1+i, 1, 1)$ $\quad v_3 = (1-i, 1, 1)$ In C^3 is basis over C. Construct an orthogonal basis by GSM. Hence find the orthonormal basis with SIP

Given $v_1 = (0,1,-1)$ $v_2 = (1+i,1,1)$ $v_3 = (1-i,1,1)$

we check it is a Basis of $C^3(C)$

we first check L.I or not

$$A = \begin{vmatrix} 0 & 1 & -1 \\ 1+i & 1 & 1 \\ 1-i & 1 & 1 \end{vmatrix}$$

$$= -1(1+i-1+i) - 1(1+i-1+i)$$

$$= -2i - 2i$$

$$= -4i \neq 0$$

$\Rightarrow$ Vectors are L.I

$\dim C^3(C) = 3 \Rightarrow \{v_1, v_2, v_3\}$ is a basis of $C^3(C)$

Step(i) $u_1 = v_1 = (0,1,-1)$

Step(ii) $u_2 = v_2 - \frac{\langle v_2, u_1 \rangle u_1}{\|u_1\|^2}$

$\langle v_2, u_1 \rangle = \langle (1+i,1,1),(0,1,-1) \rangle = 1-1 = 0$

$\therefore u_2 = (1+i,1,1)$

Step (iii) $u_3 = v_3 - \frac{\langle v_3, u_1 \rangle u_1}{\|u_1\|^2} - \frac{\langle v_3, u_2 \rangle u_2}{\|u_2\|^2}$

$\langle v_3, u_1 \rangle = \langle (1-i,1,1)(0,1,-1) \rangle = 0$

$(1-i)(1-i)+1+1$

$\langle v_3, u_2 \rangle = \langle (1-i,1,1)(1+i,1,1) \rangle = 2-2i$

$\|u_1\|^2 = 2$ $\|u_2\| = 1+1+2i+1+1 = 2+2(i) = 4$

$$u_3 = (1-i,1,1) - \frac{(2-2i)(1+i,1,1)}{4}$$

$$u_3 = (1-i, 1, 1) - \frac{(1-i)(1+i, 1, 1)}{2}$$

$$= (1-i, 1, 1) - \left(1, \frac{1-i}{2}, \frac{1-i}{2}\right)$$

$$u_3 = \left(-i, \frac{1+i}{2}, \frac{1+i}{2}\right)$$

The orthogonal basis is $\left\{(0, 1, -1), (1+i, 1, 1), \left(-i, \frac{1+i}{2}, \frac{1+i}{2}\right)\right\}$

The orthonormal basis be $\{w_1, w_2, w_3\}$

$$w_1 = \frac{u_1}{\|u_1\|} \qquad w_2 = \frac{u_2}{\|u_2\|} \qquad w_3 = \frac{u_3}{\|u_3\|}$$

$$= \frac{(0, 1, -1)}{\sqrt{2}} \qquad w_2 = \frac{(1+i, 1, 1)}{\sqrt{4}} \qquad w_3 = \frac{-i, \frac{1+i}{2}, \frac{1+i}{2}}{\sqrt{2}}$$

$$\|u_3\|^2 = \langle u_3, u_3 \rangle = \left\langle \left(-i, \frac{1+i}{2}, \frac{1+i}{2}, i, \frac{1-i}{2}, \frac{1-i}{2}\right)\right\rangle$$

$$= (1 + 1/2 + 1/2)$$

$$= 2.$$

$$\therefore \{w_1, w_2, w_3\} = \left\{\left(0, 1/\sqrt{2}, -1/\sqrt{2}\right), \left(\frac{1+i}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(-i/\sqrt{2}, \frac{1+i}{2\sqrt{2}}, \frac{1+i}{2\sqrt{2}}\right)\right\}$$