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Unit - 2

Vector Spaces

Definition :

A vector space [V] over a field [F] is a non empty set on which two operation called addition and scalar multiplication satisfying the following axiom for all $u, v, w \in V$ and for all $\alpha, \beta \in F$.

Properties :

1) $u + v \in V$
2) $u + v = v + u$
3) $u + (v + w) = (u + v) + w$
4) There is an element $0 \in V$ such $u + 0 = u$
5) For each $u \in V$, there exists $u \in V$ such that
$$u + (-u) = 0$$
6) $\alpha v \in V$
7) $(\alpha + \beta) u = \alpha u + \beta u$
8) $1 . u = u$
9) $(\alpha\beta) u = \alpha (\beta u)$
10) $2 (u + v) = 2u + 2v$

1. If $F^n = \{(a_1, a_2, \dots a_n), a_i \in F\}$ then PT F^n is a vector space over F with respect to addition and scalar multiplication defined component wise.

Sol:

Given:

$$V = F^n = \{(a_1, a_2, \dots, a_n);\ a_i \in F\}$$

Let $u = (a_1, a_2, \dots, a_n)$

$v = (b_1, b_2, \dots, b_n)$

$w = (c_1, c_2, \dots, c_n)$

Let $\alpha, \beta \in F$.

i) TP: $u + v \in V$

$(a_1, a_2 \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1+b_1, a_2+b_2, \dots a_n+b_n) \in V$

$\therefore$ Hence proved

ii) TP: $u + v = v + u$

LHS:

$u + v = (a_1+b_1, a_2+b_2, \dots, a_n+b_n)$

$= (b_1+a_1, b_2+a_2, \dots, b_n+a_n)$

$= v + u$

$=$ RHS

iii) TP: $u + (v+w) = (u+v) + w$

LHS:

$u + (v+w) = (a_1, a_2, \dots, a_n) + ((b_1+c_1, b_2+c_2, \dots, b_n+c_n))$

$= (a_1+(b_1+c_1), a_2+(b_2+c_2), \dots a_n+(b_n+c_n))$

$= ((a_1+b_1)+c_1, (a_2+b_2)+c_2, \dots (a_n+b_n)+c_n)$

$= (u+v)+w.$

iv) TP: $u + 0 = u$

Let $0 = (0, 0, \dots, 0)$

LHS

$= (a_1, a_2, \dots, a_n) + (0, 0, 0 \dots, 0)$

$= (a_1+0, a_2+0, \dots, a_n+0)$

$= (a_1, a_2, \dots, a_n)$

$= u =$ RHS

v) TP: $u + (-u) = 0$

Let $-u = (-a_1, -a_2, \dots, -a_n)$

LHS:

$u + (-u) = (a_1, a_2, \dots, a_n) + (-a_1, -a_2, \dots, -a_n)$

$= (a_1-a_1, a_2-a_2, \dots, a_n-a_n)$

$= (0, 0, \dots, 0)$

$= 0$

$=$ RHS

vi) TP: $\alpha u \in V$

$\alpha u = \alpha (a_1, a_2, \dots, a_n)$

$= (\alpha a_1, \alpha a_2, \dots \alpha a_n) \in V$

vii) $\alpha (u+v) = \alpha u + \alpha v$

LHS:

$\alpha(u+v) = \alpha((a_1, a_2, \dots a_n) + (b_1, b_2, \dots, b_n))$

$= \alpha (a_1+b_1, a_2+b_2, \dots a_n+b_n)$

$= (\alpha(a_1+b_1), \alpha(a_2+b_2), \ldots \alpha(a_n+b_n))$

$= (\alpha a_1 + \alpha b_1, \alpha a_2 + \alpha b_2, \ldots \alpha a_n + \alpha b_n)$

$= (\alpha a_1, \alpha a_2, \ldots, \alpha a_n) + (\alpha b_1, \alpha b_2, \ldots \alpha b_n)$

$= \alpha U + \alpha V$

viii) $(\alpha+\beta)U = \alpha U + \beta U$

LHS: $(\alpha+\beta)U = (\alpha+\beta)(a_1, a_2, \ldots, a_n)$

$= ((\alpha+\beta)a_1, (\alpha+\beta)a_2, \ldots (\alpha+\beta)a_n)$

$= (\alpha a_1 + \beta a_1, \alpha a_2 + \beta a_2, \ldots \alpha a_n + \beta a_n)$

$= (\alpha a_1, \alpha a_2, \ldots \alpha a_n) + (\beta a_1, \beta a_2, \ldots \beta a_n)$

$= \alpha(a_1, a_2, \ldots, a_n) + \beta(a_1, a_2, \ldots, a_n)$

$= \alpha U + \beta U.$

ix) $\alpha(\beta U) = (\alpha\beta)U$

LHS $\alpha(\beta U) = \alpha(\beta(a_1, a_2, \ldots a_n))$

$= (\alpha\beta)(a_1, a_2, \ldots a_n)$

$= (\alpha\beta)U.$

x) $1.U = U$

Let $1 = (1, 1, \ldots 1)$

LHS

$1.U = (1, 1, \ldots, 1).(a_1, a_2, \ldots a_n)$

$= (1a_1, 1a_2, \ldots, 1a_n)$

$= (a_1, a_2, \ldots, a_n)$

$= U.$

Hence $V = F^n$ satisfying all the conditions of a vector space.

$\therefore F^n$ is a vector space over a field F.

H.W:

1. $27x + 6y - z = 85$

$6x + 15y + 2z = 72$

$x + y + 54z = 110.$

Sol:

$27x + 6y - z = 85$

$27x = 85 - 6y + z$

$x = \frac{1}{27}(85 - 6y + z)$ ——— (1)

$6x + 15y + 2z = 72$

$15y = 72 - 6x - 2z$

$y = \frac{1}{15}(72 - 6x - 2z)$ ——— (2)

$x + y + 54z = 110$

$54z = 110 - x - y$

$z = \frac{1}{54}(110 - x - y)$ ——— (3)

Put in calc:

$A = \frac{1}{27}(85 - 6B + C)$

$$B = \frac{1}{15}(72 - 6A - 2C)$$

$$C = \frac{1}{54}(110 - A - y)$$

Iteration	x	y	z
0	-	0	0
1	3.1481	3.5407	1.91316
2	2.4321	3.5720	1.9258
3	2.4256	3.5729	1.92595
4	2.4254	3.5730	1.92595
5.	2.4254	3.5730	1.92595

$\therefore$ The solutions are $x = 2.4$, $y = 3.5$, $z = 1.9$.

2. $x - y + z = 1$

$-3x + 2y - 3z = -6$

$2x - 5y + 4z = 5$

by Gauss - jordan.

Sol:

$$[A|B] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -3 & 2 & -3 & -6 \\ 2 & -5 & 4 & 5 \end{array}\right]$$

$R_2 \to R_2 + 3R_1$

$R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & -3 & 2 & 3 \end{array}\right]$$

$R_2 \to R_2 \times -1$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & -3 & 2 & 3 \end{array}\right]$$

$R_1 \to R_1 + R_2$

$R_3 \to R_3 + 3R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 12 \end{array}\right]$$

$R_3 \to R_3 \div 2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$R_1 \to R_1 - R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$\therefore$ The solutions are $x = -2$, $y = 3$ $z = 6$.

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2. Prove that the set of all $(m \times n)$ Matrices over F denoted by $M_{m \times n}(F)$ is a vector space over F with respect to Matrix addition and Scalar multiplication.

Sol:

Given :

Let $V = M_{m \times n}(F)$

Let $A = [a_{ij}] \in V$

$B = [b_{ij}] \in V$

$C = [c_{ij}] \in V$

Let $\alpha, \beta \in F$

i) T.P $A + B \in V$

$A + B = [a_{ij}] + [b_{ij}] \in V$

ii) T.P $A + B = B + A$

$A + B = [a_{ij}] + [b_{ij}] = [b_{ij}] + [a_{ij}] = B + A$

iii) T.P $A + (B + C) = (A + B) + C$

$A + (B + C) = [a_{ij}] + ([b_{ij}] + [c_{ij}])$

$= ([a_{ij}] + [b_{ij}]) + [c_{ij}]$

$= (A + B) + C$.

iv) Let $0 = [0_{ij}]$

T.P $A + 0 = A$

$A + 0 = [a_{ij}] + [0_{ij}] = [a_{ij}] = A$

v) Let $-A = [-a_{ij}]$

T.P $A + (-A) = 0$

$[a_{ij}] + [-a_{ij}] = [0_{ij}] = 0$

vi) T.P $\alpha A \in V$

$\alpha A = \alpha [a_{ij}] = [\alpha a_{ij}] \in V$

vii) T.P $\alpha(A + B) = \alpha A + \alpha B$

$\alpha(A + B) = \alpha([a_{ij}] + [b_{ij}])$

$= [\alpha[a_{ij}] + \alpha[b_{ij}]]$

$= [\alpha a_{ij}] + [\alpha b_{ij}]$

$= \alpha A + \alpha B$

viii) T.P $(\alpha + \beta) A = \alpha A + \beta A$

$(\alpha + \beta) A = (\alpha + \beta)[a_{ij}] = [(\alpha + \beta) a_{ij}]$

$= [[\alpha a_{ij}] + [\beta a_{ij}]]$

$= \alpha A + \beta A$

ix) $\alpha(\beta A) = (\alpha\beta)A$

$$\alpha(\beta A) = \alpha[\beta[a_{ij}]]$$
$$= \alpha[\beta[a_{ij}]]$$
$$= [\alpha\beta[a_{ij}]]$$
$$= [\alpha\beta a_{ij}]$$
$$= \alpha\beta(a_{ij})$$
$$= (\alpha\beta)A.$$

x) $1 \cdot A = A$

$$1 = [1_{ij}]$$
$$[1_{ij}]\cdot[a_{ij}] = [a_{ij}] = A$$

Hence $V = M_{m\times n}(F)$ satisfies all the properties of Vector Space. Hence in a vector space under addition and multiplication.

3. Prove that $P_n(R)$, the set of all polynomials of degree atmost n with real co-efficient is a vector space under usual addition and constant multiplication of polynomials.

Sol:

Let ~~V~~ $= P_n(R)$

$U = a_0 + a_1x + a_2x^2 + \dots\dots + a_nx^n, \; a_n \neq 0$

$V = b_0 + b_1x + b_2x^2 + \dots\dots + a_nx^n, \; b_n \neq 0$

$W = c_0 + c_1x + c_2x^2 + \dots\dots + c_nx^n, \; c_n \neq 0$

Let $\alpha, \beta \in F$

i) $U + V \in V$

$$U+V = [a_0 + a_1x + a_2x^2 + \dots + a_nx^n] + [b_0 + b_1x + b_2x^2 + \dots b_nx^n]$$
$$= [(a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \dots (a_n+b_n)x^n]$$
$$\in V$$

ii) $U+V = V+U$

$LHS = U+V$

$$= [a_0 + a_1x + a_2x^2 + \dots + a_nx^n] + [b_0 + b_1x + b_2x^2 + \dots + b_nx^n]$$
$$= [(a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \dots + (a_n+b_n)x^n]$$
$$= [(b_0+a_0) + (b_1+a_1)x + (b_2+a_2)x^2 + \dots + (b_n+a_n)x^n]$$
$$= V+U.$$

iii) $U+(V+W) = (U+V)+W$

$LHS = U+(V+W)$

$$= [a_0 + a_1x + a_2x^2 + \dots + a_nx^n] + [b_0 + b_1x + b_2x^2 + \dots + b_nx^n] + [c_0 + c_1x + c_2x^2 + \dots + c_nx^n]$$
$$= [a_0 + a_1x + a_2x^2 + \dots + a_nx^n + b_0 + b_1x + b_2x^2 + \dots + b_nx^n] + [c_0 + c_1x + c_2x^2 + \dots + c_nx^n]$$
$$= (U+V)+W.$$

iv) $U + 0 = U$

Let $0 = [0 + 0x + 0x^2 + \dots + 0x^n]$

$LHS = U + 0$

$$= [a_0 + a_1x^1 + a_2x^2 + \dots + a_nx^n] + [0 + 0x^1 + 0x^2 + \dots 0x^n]$$
$$= [a_0 + a_1x + a_2x^2 + \dots a_nx^n]$$
$$= U$$
$$= RHS.$$

v) $U + (-U) = 0$

Let $-U = -[a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n]$

LHS $= U + (-U)$

$= [a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n] + (-[a_0 + a_1 x + a_2 x^2 + \dots a_n x^n])$

$= 0$

$=$ RHS.

vi) TP $\alpha U \in V$

$\alpha U = \alpha (a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n)$

$= (\alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \dots + \alpha a_n x^n) \in V$

$\in V$.

vii) $\alpha (U + V) = \alpha U + \alpha V$

LHS

$\alpha (U + V) = \alpha ([a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n] + [b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n])$

$= \alpha (a_0 + b_0, (a_1 + b_1) x, (a_2 + b_2) x^2, \dots (a_n + b_n) x^n)$

$= (\alpha (a_0 + b_0), \alpha ((a_1 + b_1) x), \alpha ((a_2 + b_2) x^2), \dots, \alpha ((a_n + b_n) x^n))$

$= (\alpha a_0 + \alpha b_0, \alpha a_1 x + \alpha b_1 x, \alpha a_2 x^2 + \alpha b_2 x^2, \dots, \alpha a_n x^n + \alpha b_n x^n)$

$= [\alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \dots + \alpha a_n x^n] + [\alpha b_0 + \alpha b_1 x + \alpha b_2 x^2 + \dots + \alpha b_n x^n]$

$= \alpha U + \alpha V$.

viii) $(\alpha + \beta) U = \alpha U + \beta U$

LHS $= (\alpha + \beta) U$

$= (\alpha + \beta)(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n)$

$= ((\alpha + \beta) a_0, (\alpha + \beta) a_1 x, (\alpha + \beta) a_2 x^2, \dots, (\alpha + \beta) a_n x^n)$

$= (\alpha a_0 + \beta a_0, \alpha a_1 x + \beta a_1 x, \alpha a_2 x^2 + \beta a_2 x^2, \dots, \alpha a_n x^n + \beta a_n x^n)$

$= (\alpha a_0, \alpha a_1 x, \alpha a_2 x^2, \dots \alpha a_n x^n) + (\beta a_0, \beta a_1 x, \beta a_2 x^2, \dots \beta a_n x^n)$

$= \alpha (a_0, a_1 x, a_2 x^2, \dots, a_n x^n) + \beta (a_0, a_1 x, a_2 x^2, \dots, a_n x^n)$

$= \alpha U + \beta U$.

ix) $\alpha (\beta U) = (\alpha \beta) U$

LHS $= \alpha (\beta U)$

$= \alpha (\beta (a_0, a_1 x, a_2 x^2, \dots, a_n x^n))$

$= (\alpha \beta)(a_0, a_1 x, a_2 x^2, \dots, a_n x^n)$

$= (\alpha \beta) U$.

x) $1 . U = U$

Let $1 = (1, 1, 1, \dots, 1)$

LHS

$1 . U = (1, 1, 1, \dots, 1) . (a_0, a_1 x, a_2 x^2, \dots, a_n x^n)$

$= (1 a_0, 1 a_1 x, 1 a_2 x^2, \dots, 1 a_n x^n)$

$= (a_0, a_1 x, a_2 x^2, \dots, a_n x^n)$

$= U$.

Example :

Not a Vector space : Let $S = \{(a_1, a_2) ; a_1, a_2 \in R\}$ (ordered pair) Define $(a_1, a_2) + (b_1, b_2) = (a_1+b_1 ; a_2-b_2)$ & $c(a_1, a_2) = (ca_1, ca_2)$

Check whether S is a Vector space or not.

Sol:

i) Let $u = (a_1, a_2)$

$v = (b_1, b_2)$

$u+v = (a_1+b_1, a_2-b_2) \in S$

ii) T.P $u+v = v+u$

$u+v = (a_1+b_1, a_2-b_2)$

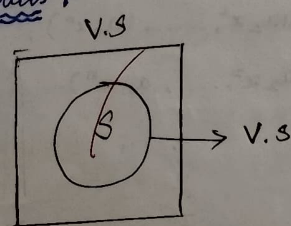
$v+u = (b_1, b_2) + (a_1, a_2)$

$= (b_1+a_1, b_2-a_2)$

$\therefore u+v \neq v+u$

$\therefore S$ is not a vector space.

Subspaces :



i) $0 \in S$

ii) $u+v \in S$

iii) $\alpha u \in W, u \in W$
$\alpha \in F$

(or)

$\alpha u + \beta v \in W$

Definition 1:

A non-empty subset W of a vector space $[V]$ over F is called a subspace of V. If W itself is a vector space over the operation of V.

Definition 2:

A non-empty subset W of V is a subspace of V if and only if

i) $0 \in W$

ii) $u+v \in W$, $u, v \in W$

iii) $\alpha u \in W$, $u \in W$
$\alpha \in F$

Definition 3:

A non empty subset W of V is a subspace of V if and only if $\alpha u + \beta v \in W$, $\alpha, \beta \in F$, $u, v \in W$.

Problem :

1. Let $V = \{(a_1, a_2) ; a_1, a_2 \in R\}$ be a vector space over R. Test whether the following subsets are subspace of V.

i) $W_1 = \{(a, 0) ; a \in R\}$

ii) $W_2 = \{(0, a) ; a \in R\}$

iii) $W_3 = \{(a_1, a_2), 2a_1 + 3a_2 = 0, a_1, a_2 \in R\}$

iv) $W_4 = \{(a_1, a_2) ; 2a_1 + 3a_2 = 2, a_1, a_2 \in R\}$

i) $W_1 = \{(a, 0) ; a \in R\}$

Let $U = (a_1, 0)$

$V = (a_2, 0)$

$\alpha, \beta \in F$

T.P $\alpha U + \beta V \in W_1$

LHS

$\alpha U + \beta V = \alpha (a_1, 0) + \beta (a_2, 0)$

$= (\alpha a_1 + 0) + (\beta a_2, 0)$

$= (\alpha a_1 + \beta a_2, 0) \in W_1$

$\therefore W_1$ is a subspace of V.

ii) $W_2 = \{(0, a) ; a \in R\}$

Let $U = (0_1, a_1)$

$V = (0_2, a_2)$

$\alpha, \beta \in F$

T.P $\alpha U + \beta V \in W_2$

LHS

$\alpha U + \beta V = \alpha (0, a_1) + \beta (0, a_2)$

$= (0, \alpha a_1) + (0, \beta a_2)$

$= (0 + 0, \alpha a_1 + \beta a_2)$

$= (0, \alpha a_1 + \beta a_2) \in W_2$

$\therefore W_2$ is a subspace of V.

iii) $\{(a_1, a_2), 2a_1 + 3a_2 = 0, a_1, a_2 \in R\}$

Let $U = (a_1, a_2) \in W_3 \Rightarrow 2a_1 + 3a_2 = 0$

$V = (b_1, b_2) \in W_3 \Rightarrow 2b_1 + 3b_2 = 0$

Let $\alpha, \beta \in F$

T.P $\alpha U + \beta V \in W_3$

$\alpha U + \beta V = \alpha (a_1, a_2) + \beta (b_1, b_2)$

$= (\alpha a_1, \alpha a_2) + (\beta b_1, \beta b_2)$

$= (\alpha a_1 + \beta b_1, (\alpha a_2 + \beta b_2)$

T.P $2(\alpha a_1 + \beta b_1) + 3(\alpha a_2 + \beta b_2) = 0$

LHS

$= 2\alpha a_1 + 2\beta b_1 + 3\alpha a_2 + 3\beta b_2$

$= \alpha (2a_1 + 3a_2) + \beta (2b_1 + 3b_2)$

$= \alpha (0) + \beta (0)$

$= 0$

$\therefore \alpha U + \beta V \in W_3$

$\therefore W_3$ is a subspace of V.

iv) $W_4 = \{(a_1, a_2) ; 2a_1 + 3a_2 = 2, a_1, a_2 \in R\}$

Let $U = (a_1, a_2) \in W_4 \Rightarrow 2a_1 + 3a_2 = 2$

$V = (b_1, b_2) \in W_4 \Rightarrow 2b_1 + 3b_2 = 2$

Let $\alpha, \beta \in F$

T.P $\alpha U + \beta V \in W_4$

$$\alpha U+\beta V = \alpha(a_1,a_2)+\beta(b_1,b_2)$$
$$=(\alpha a_1,\alpha a_2)+(\beta b_1,\beta b_2)$$
$$=(\alpha a_1+\beta b_1),(\alpha a_2+\beta b_2)$$

T.P $2(\alpha a_1+\beta b_1)+3(\alpha a_2+\beta b_2)=2$

LHS
$$=2\alpha a_1+2\beta b_1+3\alpha a_2+3\beta b_2$$
$$=\alpha(2a_1+3a_2)+\beta(2b_1+3b_2)$$
$$=\alpha(2)+\beta(2)$$
$$=2\alpha+2\beta$$
$$=2(\alpha+\beta)$$
$$\neq RHS$$

$\therefore$ $\alpha U+\beta V$ not belong to W_4

$\therefore$ W_4 is a not subspace of V.

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1. Determine whether the set $W=\{(a_1,a_2,a_3)\in R^3;$ $a_1-3a_2+a_3=3\}$ is a subspace of R^3. three dimension

2. Determine whether the set $W=\{(a,b,c)\in R^3;$ $a^2+b^2+c^2=5\}$ is subspace or not.

3. check whether the following are subspaces of $R^3(R)$

i) $W_1=\{(a_1,a_2,a_3);2a_1-7a_2+a_3=0\}$

ii) $W_2=\{(a,b,c);a-3b+c=3\}$

iii) $W_3=\{(a_1,a_2,a_3);a_1-3a_2$ and $a_3=-a_2\}$

Solutions :

1. Let $U=(a_1,a_2,a_3)\leftarrow W = a_1-3a_2+a_3=3$

$V=(b_1,b_2,b_3)\leftarrow W=b_1-3b_2+b_3=3$

Let $\alpha,\beta\in F$

T.P $\alpha U+\beta V\in W$

$$\alpha U+\beta V=\alpha(a_1,a_2,a_3)+\beta(b_1,b_2,b_3)$$
$$=(\alpha a_1,\alpha a_2,\alpha a_3)+(\beta b_1,\beta b_2,\beta b_3)$$
$$=(\alpha a_1+\beta b_1,\alpha a_2+\beta b_2,\alpha a_3+\beta b_3)$$

To check :

$(\alpha a_1+\beta b_1)-3(\alpha a_2+\beta b_2)+(\alpha a_3+\beta b_3)=3$

LHS
$$=\alpha a_1+\beta b_1-3(\alpha a_2+\beta b_2)+\alpha a_3+\beta b_3$$
$$=\alpha a_1+\beta b_1-3\alpha a_2-3\beta b_2+\alpha a_3+\beta b_3$$
$$=\alpha(a_1-3a_2+a_3)+\beta(b_1-3b_2+b_3)$$
$$=\alpha(3)+\beta(3)$$
$$=3\alpha+3\beta$$
$$=3(\alpha+\beta)$$
$$\neq RHS$$

$\therefore$ $\alpha U+\beta V$ not belong to W

$\therefore$ W is not a subspace of V.

2. Let $U=(a_1,b_1,c_1)\leftarrow W=a_1^2+b_1^2+c_1^2=5$

$V=(a_2,b_2,c_2)\leftarrow W=a_1^2+b_2^2+c_2^2=5$

Let $\alpha,\beta\in F$

T.P $\alpha U+\beta V\in W$

$$\alpha U+\beta V=\alpha(a_1,b_1,c_1)+\beta(a_2,b_2,c_2)$$

$= (\alpha a_1, \alpha b_2, \alpha c_1) + (\beta a_2, \beta b_2, \beta c_2)$

$= (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2)$

To check :

$(\alpha a_1 + \beta a_2)^2 + (\alpha b_1 + \beta b_2)^2 + (\alpha c_1 + \beta c_2)^2 = 5$

LHS

$= \alpha^2 a_1^2 + 2\alpha a_1 \beta a_2 + \beta^2 a_2^2 + \alpha^2 b_1^2 + 2\alpha b_1 \beta b_2 + \beta^2 b_2^2 + \alpha^2 c_1^2 + 2\alpha c_1 \beta c_2 + \beta^2 c_2^2$

$= \alpha^2 a_1^2 + 2\alpha\beta a_1 a_2 + \beta^2 a_2^2 + \alpha^2 b_1^2 + 2\alpha\beta b_1 b_2 + \beta^2 b_2^2 + \alpha^2 c_1^2 + 2\alpha\beta c_1 c_2 + \beta^2 c_2^2$

$= \alpha^2 (a_1^2 + b_1^2 + c_1^2) + \beta^2 (a_2^2 + b_2^2 + c_2^2) + 2\alpha\beta (a_1 a_2 + b_1 b_2 + c_1 c_2)$

$= \alpha^2 (5) + \beta^2 (5) + 2\alpha\beta (a_1 a_2 + b_1 b_2 + c_1 c_2)$

$= 5\alpha^2 + 5\beta^2 + 2\alpha\beta (a_1 a_2 + b_1 b_2 + c_1 c_2)$

$\neq$ RHS

$\therefore$ $\alpha U + \beta V$ not belongs to W

$\therefore$ W is not a subspace of V.

3\. i) $W_1 = \{(a_1, a_2, a_3) ; 2a_1 - 7a_2 + a_3 = 0\}$

Let $U = (a_1, a_2, a_3) \in W_1 = 2a_1 - 7a_2 + a_3 = 0$

$V = (b_1, b_2, b_3) \in W_1 = 2b_1 - 7b_2 + b_3 = 0$

Let $\alpha, \beta \in F$

T.P $\alpha U + \beta V \in W_1$

$\alpha U + \beta V = \alpha (a_1, a_2, a_3) + \beta (b_1, b_2, b_3)$

$= (\alpha a_1, \alpha a_2, \alpha a_3) + (\beta b_1, \beta b_2, \beta b_3)$

$= (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \alpha a_3 + \beta b_3)$

To check :

$2(\alpha a_1 + \beta b_1) - 7(\alpha a_2 + \beta b_2) + (\alpha a_3 + \beta b_3) = 0$

LHS

$= 2(\alpha a_1 + \beta b_1) - 7(\alpha a_2 + \beta b_2) + (\alpha a_3 + \beta b_3)$

$= 2\alpha a_1 + 2\beta b_1 - 7\alpha a_2 - 7\beta b_2 + \alpha a_3 + \beta b_3$

$= \alpha (2a_1 - 7a_2 + a_3) + \beta (2b_1 - 7b_2 + b_3)$

$= \alpha (0) + \beta (0)$

$= 0$

$\therefore$ $\alpha U + \beta V \in W_1$

$\therefore$ W_1 is a subspace of V.

ii) $W_2 = \{(a, b, c), a - 3b + c = 3\}$

Let $U = (a_1, b_1, c_1) \in W_2 = \{a_1 - 3b_1 + c_1 = 3$

$V = (a_2, b_2, c_2) \in W_2 = a_2 - 3b_2 + c_2 = 3\}$

Let $\alpha, \beta \in F$

T.P $\alpha U + \beta V \in W_2$

$\alpha U + \beta V = \alpha (a_1, b_1, c_1) + \beta (a_2, b_2, c_2)$

$= (\alpha a_1, \alpha b_1, \alpha c_1) + (\beta a_2, \beta b_2, \beta c_2)$

$= (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2)$

To check :

$(\alpha a_1 + \beta a_2) - 3(\alpha b_1 + \beta b_2) + (\alpha c_1 + \beta c_2) = 3$

LHS

$= \alpha a_1 + \beta a_2 - 3\alpha b_1 - 3\beta b_2 + \alpha c_1 + \beta c_2$

$= \alpha (a_1 - 3b_1 + c_1) + \beta (a_2 - 3b_2 + c_2)$

$= \alpha (3) + \beta (3)$

$= 3(\alpha + \beta)$

$\neq$ RHS

$\therefore \alpha U + \beta V$ not belong to W_2

$\therefore W_2$ is not a subspace of V.

iii) $W_3 = \{(a_1, a_2, a_3) ; a_1 = 3a_2 \text{ and } a_3 = -a_2\}$

Let $U = (a_1, a_2, a_3) \Rightarrow a_1 = 3a_2, \ a_3 = -a_2$

$V = (b_1, b_2, b_3) \Rightarrow b_1 = 3b_2, \ b_3 = -b_2$

Let $\alpha, \beta \in F$

T.P $\alpha U + \beta V \in W$

$\alpha U + \beta V = \alpha(a_1, a_2, a_3) + \beta(b_1, b_2, b_3)$

$= (\alpha a_1, \alpha a_2, \alpha a_3) + (\beta b_1, \beta b_2, \beta b_3)$

$= (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \alpha a_3 + \beta b_3)$

To prove $a_1 = 3a_2 \qquad a_3 = -a_2$

$b_1 = 3b_2 \qquad b_3 = -b_2$

$\alpha a_1 + \beta b_1 = \alpha(3a_2) + \beta(3b_2)$

$= \alpha 3 a_2 + \beta 3 b_2$

$\alpha a_1 + \beta b_1 = 3(\alpha a_2 + \beta b_2)$

$a_1 = 3a_2$

$\alpha a_3 + \beta b_3 = \alpha(-a_2) + \beta(-b_2)$

$= -\alpha a_2 - \beta b_2$

$\alpha a_3 + \beta b_3 = -(\alpha a_2 + \beta b_2)$

$a_3 = -a_2$

$\therefore W_3$ is a subspace of V.

13:02:25

Theorems on subspaces:

1. Prove that the intersection of 2 subspaces of a vectorspace V is a subspace of V.

Given:

Let W_1, W_2 be subspace of V

T.P $0 \in W_1 \cap W_2$ is a subspace.

First T.P $0 \in W_1 \cap W_2$

W_1 is a subspace $\Rightarrow 0 \in W_1$

W_2 is a subspace $\Rightarrow 0 \in W_2$

$\therefore 0 \in W_1 \cap W_2$

$\therefore$ It is not empty.

Let $U, V \in W_1 \cap W_2$

T.P $\alpha U + \beta V \in W_1 \cap W_2$

$U, V \in W_1$ and $U, V \in W_2$

$\Rightarrow$ Let $\alpha U + \beta V \in W_1$ & $\alpha U + \beta V \in W_2$

Intersection of W_1 & W_2

$\Rightarrow \alpha U + \beta V \in W_1 \cap W_2$

$\therefore$ The subspace $W_1 \cap W_2$ is a subspace in V.

2. Let W_1 & W_2 be subspaces of $V(F)$ then $W_1 \cup W_2$ is a subspace of V iff $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

Proof : [Necessary part]

Given :

Let W_1 & W_2 be subspaces of V

Let $W_1 \cup W_2$ is a subspaces of V

T.P $W_1 \subseteq W_2$ (or) $W_2 \subseteq W_1$

Assume that $W_1 \nsubseteq W_2$ (or) $W_2 \nsubseteq W_1$

$W_1 \nsubseteq W_2 \Rightarrow$ There is an element $u \in W_1$ but $u \notin W_2$ —— (1)

$W_2 \nsubseteq W_1 \Rightarrow$ There is an element $v \in W_2$ but $v \notin W_1$ —— (2)

From $u \in W_1$, $v \in W_2$

$u \in W_1 \cup W_2$, $v \in W_1 \cup W_2$

$u + v \in W_1 \cup W_2$

$\Rightarrow u+v \in W_1$ (or) $u+v \in W_2$ —— (3)

if $u+v \in W_1$ and $u \in W_1$ then

$u+v-u \in W_1$

$\Rightarrow v \in W_1$ (which is a contradication to (2))

if $u+v \in W_2$ and $v \in W_2$ then

$u+v-v \in W_2$

$\Rightarrow u \in W_2$ (which is contraction to (1))

$\therefore$ Hence our assumption is wrong

$\therefore W_1 \cup W_2$ is a subspace then $W_1 \subseteq W_2$ (or) $W_2 \subseteq W_1$

Sufficient part :

Given :

$W_1 \subseteq W_2$ (or) $W_2 \subseteq W_1$

T.P $W_1 \cup W_2$ is a subspace of V

if $W_1 \subseteq W_2$, then $W_1 \cup W_2 = W_2$, which is a subspace of V.

if $W_2 \subseteq W_1$, then $W_1 \cup W_2 = W_1$, which is a subspace of V.

Thus $W_1 \cup W_2$ is a subspace of V

$\therefore$ Hence proved.

Linear Combination :

Let V be a vector space over F and Let $v_1, v_2, \dots, v_n$ be the vectors then an expression of the form $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$. where $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ is called the linear combination of the vectors. $v_1, v_2, \dots, v_n$ over F.

Linear Span :

Let S be a non-empty subset of vector space V then the set of all linear combinations of elements of S is called the linear span of S.

It is denoted by $L(S)$ or $Span(S)$.

Problem:

1. In R^3 over R, test whether $(1,-2,5)$ is a linear combination of $(1,1,1)$ $(1,2,3)$ $(2,-1,1)$

$(1,-2,5) = \alpha(1,1,1) + \beta(1,2,3) + \gamma(2,-1,1)$

$(1,-2,5) = (\alpha,\alpha,\alpha) + (\beta, 2\beta, 3\beta) + (2\gamma, -\gamma, \gamma)$

$(1,-2,5) = (\alpha+\beta+2\gamma, \alpha+2\beta-\gamma, \alpha-3\beta+\gamma)$

$\alpha + \beta + 2\gamma = 1$

$\alpha + 2\beta - \gamma = -2$

$\alpha - 3\beta + \gamma = 5$

By solving the above eqn, we get

$\alpha = -6, \beta = 3, \gamma = 2$.

H.W:

check if $(2,-5,3)$ can be expressed as a linear combination of $(1,-3,2), (2,-4,-1)$ $(1,-5,7)$.

$(2,-5,3) = \alpha(1,-3,2) + \beta(2,-4,-1) + \gamma(1,-5,7)$

$= (\alpha, -3\alpha, 2\alpha) + (2\beta, -4\beta, -\beta) + (\gamma, -5\gamma, 7\gamma)$

$= (\alpha+2\beta+\gamma, -3\alpha-4\beta-5\gamma, 2\alpha-\beta+7\gamma)$

$\alpha + 2\beta + \gamma = 2$

$-3\alpha - 4\beta - 5\gamma = -5$

$2\alpha - \beta + 7\gamma = 3$

~~By~~

~~α~~

~~check whether $2x^3-2x^2+12x-6$ is a linear combination of x^3-2x^2-5x-3 and $3x^3-5x^2-4x-9$.~~

Another method:

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ -3 & -4 & -5 \\ 2 & -1 & 7 \end{bmatrix}$$

$$[A|B] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ -3 & -4 & -5 & -5 \\ 2 & -1 & 7 & 3 \end{array}\right]$$

$R_2 \to R_2 + 3R_1$

$R_3 \to R_3 - 2R_1$

$$[A|B] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 3 & -2 & 1 \\ 0 & -5 & 5 & -3 \end{array}\right]$$

$R_3 \to 2R_3 + 5R_2$

$$[A|B] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

Rank of $A = 2$

Rank of $B = 3$

$\rho(A) \neq \rho(B)$

$\therefore$ It is inconsistent

Check whether $2x^3 - 2x^2 + 12x - 6$ is a linear combination of $1x^3 - 2x^2 - 5x - 3$ and $3x^3 - 5x^2 - 4x - 9$.

$$2x^3 - 2x^2 + 12x - 6 = \alpha(1x^3 - 2x^2 - 5x - 3) + \beta(3x^3 - 5x^2 - 4x - 9)$$

$$= (\alpha x^3 - 2\alpha x^2 - 5\alpha x - 3\alpha) + (3\beta x^3 - 5\beta x^2 - 4\beta x - 9\beta)$$

$$= (\alpha x^3 + 3\beta x^3, -2\alpha x^2 - 5\beta x^2, -5\alpha x - 4\beta x, -3\alpha - 9\beta)$$

$$\alpha + 3\beta = 2$$

$$-2\alpha - 5\beta = -2$$

$$-5\alpha - 4\beta = 12$$

$$-3\alpha - 9\beta = -6$$

By solving above any two equation, we get

$$\alpha = -4$$

$$\beta = 2.$$

Linear Dependence and Linear

Linear Dependent :

Let V be a vector space over F a finite number vectors $(v_1, v_2, \ldots, v_n)$ are called linearly dependent if there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all zero such that $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$

Linear Independent :

Let V be a vector space over F a finite number vectors $(v_1, v_2, \ldots, v_n)$ are called linearly independent if there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ all zero such that $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$.

Example :

Show that the set of vectors $S = \{(1,0,0), (0,1,0), (1,1,0)\}$ in R^3 is linearly dependent over R.

$$v_1 = (1, 0, 0)$$

$$v_2 = (0, 1, 0)$$

$$v_3 = (1, 1, 0)$$

Let $\alpha_1 (1,0,0) + \alpha_2 (0,1,0) + \alpha_3 (1,1,0) = 0$

$(\alpha_1, 0, 0) + (0, \alpha_2, 0) + (\alpha_3, \alpha_3, 0) = 0$

$\alpha_1 + \alpha_3 = 0 \Rightarrow \alpha_1 = -\alpha_3$

$\alpha_2 + \alpha_3 = 0 \Rightarrow \alpha_2 = -\alpha_3$

where $\alpha_3 = 1$

$\Rightarrow \alpha_1, \alpha_2 = -1$.

So, not all α_i's are zero.

$\therefore v_1, v_2, v_3$ are linearly dependent.

2. Show that the set of vectors $S = \{(1,2,3), (2,3,1)\}$ in $\mathbb{R}^3$ is linearly independent over $\mathbb{R}$.

$v_1 = (1, 2, 3)$

$v_2 = (2, 3, 1)$

Let $\alpha_1, \alpha_2 \in F$

Let $\alpha_1 (1, 2, 3) + \alpha_2 (2, 3, 1) = 0$

$(\alpha_1, 2\alpha_1, 3\alpha_1) + (2\alpha_2, 3\alpha_2, \alpha_2) = 0$

$\alpha_1 + 2\alpha_2 = 0$

$2\alpha_1 + 3\alpha_2 = 0$

$3\alpha_1 + \alpha_2 = 0$

By Solving, we get
$\alpha_1 = 0, \alpha_2 = 0$

So, all α_i's are zero

$\therefore v_1, v_2$ are linearly independent

05:02:35

1. Determine that the set $S = \left\{ \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -4 & 8 \\ 6 & 0 & -2 \end{bmatrix} \right\}$ in $M_{2\times 3}(\mathbb{R})$ is linearly dependent.

Let $A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -1 \end{bmatrix}$

$B = \begin{bmatrix} 2 & -4 & 8 \\ 6 & 0 & -2 \end{bmatrix}$

It is observed that $B = 2A$

$\therefore$ They are dependent.

2. Verify whether the $S = \left\{ \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix}, \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix}, \begin{pmatrix} -2 & 3 & 1 \\ 1 & -3 & 2 \end{pmatrix} \right\}$ in $M_{2\times 3}(\mathbb{R})$ is linearly dependent.

Sol:

Let $\alpha, \beta, \gamma \in \mathbb{R}$

$v_1 = \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix}$ $v_2 = \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix}$ $v_3 = \begin{pmatrix} -2 & 3 & 1 \\ 1 & -3 & 2 \end{pmatrix}$

Let $\alpha v_1 + \beta v_2 + \gamma v_3 = 0$

$\alpha \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} + \beta \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix} + \gamma \begin{pmatrix} -2 & 3 & 1 \\ 1 & -3 & 2 \end{pmatrix} = 0$

$$\alpha - 3\beta - 2\gamma = 0$$
$$-3\alpha + 7\beta + 3\gamma = 0$$
$$2\alpha + 4\beta + \gamma = 0$$
$$-4\alpha + 6\beta + \gamma = 0$$
$$-2\beta - 3\gamma = 0$$
$$5\alpha - 7\beta + 2\gamma = 0$$

$$\begin{array}{ccc} \alpha & -\beta & \gamma \\ \begin{vmatrix} -3 & -2 \\ 7 & 3 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ -3 & 3 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ -3 & 7 \end{vmatrix} \end{array}$$

$$\frac{\alpha}{-9+14} = \frac{-\beta}{3-6} = \frac{\gamma}{7-9}$$

$$\frac{\alpha}{5} = \frac{-\beta}{-3} = \frac{\gamma}{-2}$$

$\alpha = 5,\ \beta = 3,\ \gamma = -2$

Since all scalars is not zero, S is linearly dependent.

3. Determine x such that the vectors $(1, -1, x-1)$ $(2, x, -4)$ $(0, x+2, -8)$ in R^3 are linearly dependent.

NOTE :
Determinant is 0 if it is linearly dependent.

$$\begin{vmatrix} 1 & -1 & x-1 \\ 2 & x & -4 \\ 0 & x+2 & -8 \end{vmatrix} = 0$$

$$1(-8x - (-4)(x+2)) + 1(2(-8) - 0(-4)) + (x-1)(2(x+2) - x(0)) = 0$$

$$(-8x + 4x + 8 - 16 + [x - 1(2x+4)]) = 0$$

$$-8x + 4x + 8 - 16 + 2x^2 + 4x - 2x - 4 = 0$$

$$2x^2 - 2x - 12 = 0$$

$$x = 3, -2$$

4. Prove that vectors $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ in $R^3(R)$ are linearly dependent.

$$\alpha v_1 + \beta v_2 + \gamma v_3 = 0$$

$$\alpha \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + \gamma \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\alpha + 4\beta + 2\gamma = 0$$
$$2\alpha + 5\beta + \gamma = 0$$
$$3\alpha + 6\beta = 0$$

$$\frac{\alpha}{\begin{vmatrix} 4 & 2 \\ 5 & 1 \end{vmatrix}} = \frac{-\beta}{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}} = \frac{\gamma}{\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix}}$$

$$\frac{\alpha}{4-10} = \frac{-\beta}{1-4} = \frac{\gamma}{5-8}$$

$$\frac{\alpha}{-6} = \frac{-\beta}{-3} = \frac{\gamma}{-5}$$

$\alpha = -6,\ \beta = 3,\ \gamma = -5$

Since all scalars are not zero. It is a linearly dependent.

Bases and Dimensions

A subset B of a vector space V is called a bases of V if B is

1) Linearly independent

$L(S) = V$ Linear combinations.

The Number of in the Bases is called dimension of V.

1) $R^3 \to \{(1,0,0),(0,1,0),(0,0,1)\}$

$\dim(R^3) = 3$

2) $R^2 \to \{(1,0),(0,1)\}$

$\dim(R^2) = 2$

3) $M_{2\times 2}(R) = \left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$

$\dim(M_{2\times 2}(R)) = 4$

4) $M_{3\times 3}(R) = \left\{\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}, \ldots \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right\}$

$\dim(M_{3\times 3}(R)) = 9$

5) $\dim(M_{n\times n}(R)) = n^2$

6) $P_2(R) \to \{1, x, x^2\}$

$\dim P_2(R) = 3$

7) $P_3(R) \to \{1, x, x^2, x^3\}$

$\dim P_3(R) = 4$

8) $\dim(P_n(R)) = n+1$.

1. Check whether the set $S = \{v_1, v_2, v_3\}$ where $v_1 = (2,1,0)$, $v_2 = (-3,-3,1)$ and $v_3 = (2,1,-1)$ is a basis in the vector space

T.P i) S is linear independent

ii) $L(S) = V$

$$\begin{vmatrix}2&1&0\\-3&-3&1\\2&1&-1\end{vmatrix} = 2\begin{vmatrix}-3&1\\1&-1\end{vmatrix} - 1\begin{vmatrix}-3&1\\2&-1\end{vmatrix}$$

$= 2(3-1) - (3-2)$

$= 2(2) - 1$

$= 4 - 1$

$= 3 \neq 0$

S is linearly independent

Also By if $\dim V = n$, then any n linearly independent vectors in V forms a basis of V.

Here $\dim(R^3) = 3$

No. of linearly independent vector $= 3$

$\therefore S = \{v_1, v_2, v_3\}$ forms a basis of V.

determine whether $\{1+2x+x^2, 3+x^2, x+x^2\}$ is a basis for $P_2(\mathbb{R})$

Sol: $\{1, x, x^2\} \to$ standard basis for $P_2(\mathbb{R})$

Let $v_1 = 1 + 2x + x^2$

$v_2 = 3 + x^2$

$v_3 = x + x^2$

Let $\alpha_1, \alpha_2, \alpha_3 \in F$

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

TP. $\alpha_1 = \alpha_2 = \alpha_3 = 0$

$\alpha_1(1+2x+x^2) + \alpha_2(3+x^2) + \alpha_3(x+x^2) = 0$

$x^2[\alpha_1 + \alpha_2 + \alpha_3] + x[2\alpha_1 + \alpha_3] + [\alpha_1 + 3\alpha_2] = 0$

$\Rightarrow$ since $\{1, x, x^2\}$ is a basis of $\mathbb{R}$

$\alpha_1 + \alpha_2 + \alpha_3 = 0$

$2\alpha_1 + \alpha_3 = 0$

$\alpha_1 + 3\alpha_2 = 0$

Solving this we get $\alpha_1 = \alpha_2 = \alpha_3 = 0$

$\Rightarrow v_1, v_2, v_3$ dimension linearly independent.

By if dim $V = n$, then any linearly independent vector forms a basis of V. Here dim $P_2(\mathbb{R}) = 3$

No. of linearly independent $= 3$

$\therefore \{v_1\, v_2\, v_3\}$ forms the basis

3. State that the matrices $\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ generates $M_{2\times 2}(\mathbb{R})$

Sol:

Let $S = \left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$

To prove $L(S) = M_{2\times 2}(\mathbb{R})$

we know that $L(S) \subseteq M_{2\times 2}(\mathbb{R})$ —— ①

To prove $M_{2\times 2}(\mathbb{R}) \subseteq L(S)$

Let $\begin{pmatrix}a&b\\c&d\end{pmatrix} \in M_{2\times 2}(\mathbb{R})$

$\begin{pmatrix}a&b\\c&d\end{pmatrix} = a_{11}\begin{pmatrix}1&0\\0&0\end{pmatrix} + a_{12}\begin{pmatrix}0&1\\0&0\end{pmatrix} + a_{21}\begin{pmatrix}0&0\\1&0\end{pmatrix} + a_{22}\begin{pmatrix}0&0\\0&1\end{pmatrix}$

$= \begin{pmatrix}a_{11}&0\\0&0\end{pmatrix} + \begin{pmatrix}0&a_{12}\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\a_{21}&0\end{pmatrix} + \begin{pmatrix}0&0\\0&a_{22}\end{pmatrix}$

$\begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}$

$a = a_{11}$

$b = a_{12}$

$c = a_{21}$

$d = a_{22}$

Any Matrix can be generated using S

$\therefore M_{2\times 2}(R) \subseteq L(S)$ —— ②

From ① & ②

$L(S) = M_{2\times 2}(R)$

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The vectors $v_1 = (2,-3,1)$, $v_2 = (1,4,-2)$, $v_3 = (-8,12,-4)$, $v_4 = (1,37,-17)$, $v_5 = (-3,-3,8)$ generate R^3 over R. Find the subset which is a basis of R^3.

Sol:

$\text{Dim}(R^3) = 3$

Basis has only 3 elements

consider v_1, v_2

Let $\alpha_1, \alpha_2 \in F$

$\alpha_1 v_1 + \alpha_2 v_2 = 0$

$\alpha_1(2,-3,1) + \alpha_2(1,4,-2) = 0$

$2\alpha_1 \; -3\alpha_1 \; 1\alpha_1 + \alpha_2 + 4\alpha_2 - 2\alpha_2 = 0$

$2\alpha_1 + \alpha_2 = 0$

$-3\alpha_1 + 4\alpha_2 = 0$

$\alpha_1 + \alpha_2 = 0$

on solving this

$\alpha_1 = \alpha_2 = 0$

v_1, v_2 are linearly independent

$L(B) = \{ 2\alpha_1 + \alpha_2, -3\alpha_1 + 4\alpha_2, \alpha - 2\alpha_2 \quad \alpha_1, \alpha_2 \in F \}$

Case ii $v_3 \in L(B)$

$(-8, 12, -4) = 2\alpha_1 + \alpha_2, -3\alpha_1 + 4\alpha_2, \alpha - 2\alpha_2$

$2\alpha_1 + \alpha_2 = 8$

$-3\alpha_1 + 4\alpha_2 = 12$

$\alpha_1 - 2\alpha_2 = -4$

$\alpha_1 = -4, \alpha_2 = 0$

$12 = -3(-4) + 0 = 12$

$+12 = 12$

$\therefore$ It is satisfied

$v_3 \notin L(B)$

Case iii $v_4 \in L(B)$

$(1, 37, -17) = 2\alpha_1 + \alpha_2, -3\alpha_1 + 4\alpha_2, \alpha_1 - 2\alpha_2$

$2\alpha_1 + \alpha_2 = 1$

$-3\alpha_1 + 4\alpha_2 = 37$

$\alpha_1 - 2\alpha_2 = -17$

$\alpha_1 = -3 \quad \alpha_2 = 7$

$9 + 28 = 37$

$37 = 37$

$\therefore$ It is satisfied $v_4 \notin L(B)$

Case iii) $v_5 \in L(B)$

$(-3,-3,8) = 2\alpha_1 + \alpha_2 ,\ -3\alpha_1 + 4\alpha_2 ,\ \alpha - 2a_2$

$$2\alpha_1 + \alpha_2 = -3$$

$$-3\alpha_1 + 4\alpha_2 = -3$$

$$\alpha_1 - 2\alpha_2 = 8$$

$$\alpha_1 = 0.4 \qquad \alpha_2 = -3.8$$

$$-3 = -3(0.4) + 4(-3.8) = -3$$

$$= -1.2 + (-15.2) = -3$$

$$-16.4 = -3$$

$\therefore$ It is not satisfied

$$v_5 \in L(B)$$

$B = \{ v_1\ v_2\ v_5 \}$ is a basis of R^3.

5. Determine whether the polynomial $x^2 + 3x - 2$, $2x^2 + 5x - 3$, $-x^2 - 4x + 4$ generates the vector space of polynomial.

Sol: