Column Vectors

1 Vectors in $\mathbb{C}^n$

Definition 1.1. Column Vector
A **column vector** is an ordered list of complex numbers, written in a vertical array:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}, \quad \text{where } v_i \in \mathbb{C}.$$

The number in position i of the vector $\mathbf{v}$ is the i^{th} **entry** or **component** of $\mathbf{v}$, written as $[\mathbf{v}]_i$.

Definition 1.2. n-Dimensional Vector
A vector with n entries is called **n-dimensional**.

Definition 1.3. n-Dimensional Space
The set of all n-dimensional vectors is **n-dimensional space**, and is written as $\mathbb{C}^n$.

2 Properties of Vectors in $\mathbb{C}^n$

Definition 2.1. Vector Equality

Two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{C}^n$ are equal, if

$$[\mathbf{u}]_i = [\mathbf{v}]_i, \ i = 1, 2, \ldots, n$$

Definition 2.2. Vector Addition

If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ are vectors in $\mathbb{C}^n$, then the **sum** of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

Definition 2.3. Scalar Multiplication

If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{C}^n$, and $\alpha \in \mathbb{C}$, then $\alpha\mathbf{u} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \\ \vdots \\ \alpha u_n \end{bmatrix}$

The following algebraic properties of vectors can be proven from these 3 definitions.

Theorem 2.1. ***Properties of Column Vectors***
If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are in* $\mathbb{C}^n$, *and* a *and* b *are in* $\mathbb{C}$, *then:*

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$, *where* $-\mathbf{u} = (-1)\mathbf{u}$
5. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
6. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

Definition 2.4. Conjugate of a Column Vector
If $\mathbf{u}$ is in $\mathbb{C}^n$, then the **conjugate** of $\mathbf{u}$ is $\overline{\mathbf{u}}$, where

$$[\overline{\mathbf{u}}]_i = \overline{[\mathbf{u}]_i}, \quad i = 1, \ldots, n$$

Theorem 2.2. ***Conjugate of a Sum of Vectors***
If $\mathbf{u}$ *and* $\mathbf{v}$ *are in* $\mathbb{C}^n$, *then*

$$\overline{\mathbf{u} + \mathbf{v}} = \overline{\mathbf{u}} + \overline{\mathbf{v}}$$

Theorem 2.3. ***Conjugate of a Scalar Product***
If $\mathbf{u} \in \mathbb{C}^n$ *and* $\alpha \in \mathbb{C}$, *then*

$$\overline{\alpha\mathbf{u}} = \overline{\alpha}\ \overline{\mathbf{u}}$$

3 Geometry of Vectors in $\mathbb{R}^2$

Vectors in $\mathbb{R}^2$ can be thought of as points in the plane, and in $\mathbb{R}^3$ as points in 3-dimensional space. For example, the vector

$$\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

can be thought of as the point $(1, -3)$ in the x-y coordinate system. We will refer to vectors as points and points as vectors as it suits our purposes.

Vectors in $\mathbb{R}^2$ can be added geometrically (graphically) using the *parallelogram rule.*

Theorem 3.1. ***Parallelogram Rule***
To add $\mathbf{u}$ *and* $\mathbf{v}$*, draw a parallelogram with vertices at the origin,* (v_1, v_2)*, and* (u_1, u_2)*. The fourth vertex of the parallelogram corresponds to the point* $\mathbf{u} + \mathbf{v}$.

4 Linear Combinations

Definition 4.1. Linear Combination
A **linear combination** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n\}$ is a vector that is formed by multiplying the vectors in the set by scalars and adding them up. We can write a linear combination using sigma notation:

$$\mathbf{u} = \sum_{i=1}^{n} c_i \mathbf{v}_i, \;\; c_i \in \mathbb{C}$$

We would say that $\mathbf{u}$ is a linear combination of the vectors in the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n\}$.

Definition 4.2. Span
The **span** of a set of vectors is the set of all linear combinations of the vectors in the set.

We can write the span of a set in set notation as follows. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n\}$ is a set of vectors in $\mathbb{C}^n$ then

$$\text{The span of } S = \langle S \rangle = \{\mathbf{u} \in \mathbb{C}^n \mid \mathbf{u} = \sum_{i=1}^{n} c_i \mathbf{v}_i, \;\; c_i \in \mathbb{C}\}$$

If W is a set and $\langle S \rangle = W$, then we say S **spans** W.

Example 4.1. Let

$$S = \left\{ \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$$

List 5 different vectors in $\langle S \rangle$, and find the (entire) span of S. How many vectors are in $\langle S \rangle$?

Here are some vectors in S. As you can see, we could come up with lots more.

$$\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \in \langle S \rangle \qquad 2 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} \in \langle S \rangle$$

$$3 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix} \in \langle S \rangle \qquad -2 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} \in \langle S \rangle$$

$$\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \in \langle S \rangle \qquad 3 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} \in \langle S \rangle$$

By definition, $\langle S \rangle$ is the set of all linear combinations of the vectors in S, which we can write using the set notation above.

$$\langle S \rangle = \left\{ c_1 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \mid c_1, c_2, c_3 \in \mathbb{C} \right\}$$

Since there are infinitely many complex numbers to choose from for each of c_1, c_2, and c_3, there are infinitely many vectors in $\langle S \rangle$.

Example 4.2. Let $S = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. Find $\langle S \rangle$.

Since we are not given specific vectors, we can write $\langle S \rangle$ as the set of all linear combinations of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

$$\langle S \rangle = \{c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} \mid c_1, c_2, c_3 \in \mathbb{C}\}$$

Example 4.3. Let

$$S = \left\{ \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$$

Is $\begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$ in $\langle S \rangle$? Does S span $\mathbb{C}^3$?

$\begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$ is in $\langle S \rangle$ if it can be written as a linear combination of the vectors in S.

In other words, if the vector equation

$$c_1 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$$

is consistent.

Using the definitions of vector addition and scalar multiplication, we can write this vector equation as:

$$c_1 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} c_1(-1) \\ c_1(0) \\ c_1(2) \end{bmatrix} + \begin{bmatrix} c_2(3) \\ c_2(1) \\ c_2(-5) \end{bmatrix} + \begin{bmatrix} c_3(1) \\ c_3(1) \\ c_3(4) \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} -c_1 \\ 0c_1 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} 3c_2 \\ 1c_2 \\ -5c_2 \end{bmatrix} + \begin{bmatrix} 1c_3 \\ 1c_3 \\ 4c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$$

Then we can use the definition of vector equality to write a system of equations:

$$\begin{aligned} -c_1 + 3c_2 + 1c_3 &= 3 \\ 0c_1 + 1c_2 + 1c_3 &= 2 \\ 2c_1 - 5c_2 + 4c_3 &= 7 \end{aligned}$$

$$\begin{bmatrix} -1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 2 & -5 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{7}{5} \\ 0 & 1 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{11}{5} \end{bmatrix}$$

Therefore

$$c_1 = -\frac{7}{5} \quad c_2 = -\frac{1}{5} \quad c_3 = \frac{11}{5}$$

and

$$-\frac{7}{5} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{5} \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix} + \frac{11}{5} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$$

Because this vector equation is consistent, we can conclude that $\begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$ is in $\langle S \rangle$.

Now consider the question of whether S spans $\mathbb{C}^3$. This is harder to answer because this is asking if every single vector in $\mathbb{C}^3$ is in $\langle S \rangle$, and there are too many to check them one-by-one like we just did.

As is often the case, we can answer this general question with a theorem.

Theorem 4.1. ***Spanning Sets***
Let A be an $m \times n$ matrix, let S be the set containing the column vectors of A, and let B be the reduced row-echelon form of A. The following statements are logically equivalent:

1. *The system $\mathcal{LS}(A, \mathbf{b})$ is consistent for all $\mathbf{b}$ in $\mathbb{C}^m$.*
2. *Every $\mathbf{b}$ in $\mathbb{C}^m$ can be written as a linear combination of the vectors in S.*
3. *$\langle S \rangle = \mathbb{C}^m$.*
4. *B has a pivot in every row.*

Proof. Consider the system of linear equations $\mathcal{LS}(A, \mathbf{b})$, where A is $m \times n$. If B has a pivot in every row, then there cannot be a pivot in the last column of the augmented matrix, by the definition of reduced row-echelon form. Therefore the system is consistent, and since this result does not depend on $\mathbf{b}$, it is consistent for any (i.e. every) $\mathbf{b} \in \mathbb{C}^m$.

Since $\mathcal{LS}(A, \mathbf{b})$ can be written as a vector equation with the variables as coefficients of the column vectors of A set equal to $\mathbf{b}$, this vector equation is also consistent. That means $\mathbf{b}$ can be written as a linear combination of the vectors in S, which means that $\mathbf{b} \in \langle S \rangle$. Since $\mathbf{b}$ is an arbitrary vector in $\mathbb{C}^m$, every vector in $\mathbb{C}^m$ is in $\langle S \rangle$, and $\langle S \rangle = \mathbb{C}^m$.

Finally, if $\langle S \rangle = \mathbb{C}^m$, then every vector $\mathbf{b}$ in $\mathbb{C}^m$ can be written as a linear combination of the vectors in S, which means there must be a pivot in every row of B, since it is the reduced row-echelon form of the coefficient matrix of the system $\mathcal{LS}(A, \mathbf{b})$. ■

The logical connections in the proof above are worth noting:

$$4 \implies 1 \implies 2 \implies 3 \implies 4$$

That was a lot of work, but it allows us to go back and answer the last question from example 4.3 very easily.

The question was, does S span $\mathbb{C}^3$? Since there was a pivot in every row of the coefficient matrix of the system, by theorem 4.1, the answer is yes, since $\langle S \rangle = \mathbb{C}^3$.

Example 4.4. Let

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & 1 & 1 \\ 2 & -5 & -3 \end{bmatrix}$$

Is $\mathcal{LS}(A, \mathbf{b})$ consistent for all $\mathbf{b} \in \mathbb{C}^3$? Why or why not?

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & 1 & 1 \\ 2 & -5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = B$$

Since B doesn't have a pivot in every row, all the statements in theorem 4.1 are false. So $\mathcal{LS}(A, \mathbf{b})$ is not consistent for all $\mathbf{b}$ in $\mathbb{C}^3$. (It may be consistent for some vectors in $\mathbb{C}^3$ and inconsistent for others.)

5 Linear Independence

Definition 5.1. Relation of Linear Dependence
Given a set of vectors

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

A true equation of the form

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}$$

is a **relation of linear dependence** on S.

Definition 5.2. Trivial Relation of Linear Dependence
Given a set of vectors

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

the **trivial relation of linear dependence** on S is

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n = \mathbf{0}$$

Definition 5.3. Linearly Independent
A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is **linearly independent** if the only relation of linear dependence on S is the trivial one.

Definition 5.4. Linearly Dependent
If a set of vectors is not linearly independent, it is **linearly dependent**.

In other words, a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly dependent if and only if the equation

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}$$

has a non-trivial solution.

Theorem 5.1. ***Linearly Dependent Sets***
An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

Proof. Assume S is linearly dependent. Then there exist scalars $c_1, c_2, \ldots, c_p$, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

and at least one of the scalars is not zero. Call it c_j. So we have

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_j\mathbf{v}_j + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

$$\begin{aligned} c_j\mathbf{v}_j &= -c_1\mathbf{v}_1 - c_2\mathbf{v}_2 - \cdots - c_p\mathbf{v}_p \\ \mathbf{v}_j &= \Big(-\frac{c_1}{c_j}\Big)\mathbf{v}_1 + \Big(-\frac{c_2}{c_j}\Big)\mathbf{v}_2 + \cdots + \Big(-\frac{c_p}{c_j}\Big)\mathbf{v}_p \end{aligned}$$

and $\mathbf{v}_j$ is a linear combination of the other vectors in S.

Conversely, assume that $\mathbf{v}_j$ is a linear combination of the other vectors in S. Then we can write

$$\begin{aligned} \mathbf{v}_j &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{j-1}\mathbf{v}_{j-1} + c_{j+1}\mathbf{v}_{j+1} + \cdots + c_p\mathbf{v}_p \\ 1\mathbf{v}_j &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{j-1}\mathbf{v}_{j-1} + c_{j+1}\mathbf{v}_{j+1} + \cdots + c_p\mathbf{v}_p \\ \mathbf{0} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{j-1}\mathbf{v}_{j-1} - 1\mathbf{v}_j + c_{j+1}\mathbf{v}_{j+1} + \cdots + c_p\mathbf{v}_p \end{aligned}$$

So

$$\sum_{i=1}^{p} c_i\mathbf{v}_i = \mathbf{0}$$

where $c_j = -1$, and S is linearly dependent. ∎

Corollary 5.1.1. *If an indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent, and $\mathbf{v}_1 \neq \mathbf{0}$, then one of the vectors, $\mathbf{v}_j$, where $j > 1$, is a linear combination of the preceding vectors, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.*

Proof. Since S is linearly dependent, there exist scalars $c_1, c_2, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

If $c_p \neq 0$, then

$$\begin{aligned} c_p\mathbf{v}_p &= -c_1\mathbf{v}_1 - c_2\mathbf{v}_2 - \cdots - c_{p-1}\mathbf{v}_{p-1} \\ \mathbf{v}_p &= \Big(-\frac{c_1}{c_p}\Big)\mathbf{v}_1 + \Big(-\frac{c_2}{c_p}\Big)\mathbf{v}_2 + \cdots + \Big(-\frac{c_{p-1}}{c_p}\Big)\mathbf{v}_{p-1} \end{aligned}$$

and $\mathbf{v}_p$ is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p-1}$.

If $c_p = 0$, but $c_{p-1} \neq 0$ then we can write $\mathbf{v}_{p-1}$ as a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p-2}$ as

$$\mathbf{v}_{p-1} = \Big(-\frac{c_1}{c_{p-1}}\Big)\mathbf{v}_1 + \Big(-\frac{c_2}{c_{p-1}}\Big)\mathbf{v}_2 + \cdots + \Big(-\frac{c_{p-2}}{c_{p-1}}\Big)\mathbf{v}_{p-2}$$

If $c_2 = c_3 = \cdots = c_p = 0$, then $c_1\mathbf{v}_1 = \mathbf{0}$ and $c_1 \neq 0$ since at least one scalar is not zero. But if $c_1\mathbf{v}_1 = \mathbf{0}$ and $c_1 \neq 0$, then $\mathbf{v}_1 = \mathbf{0}$, but we are given that $\mathbf{v}_1 \neq \mathbf{0}$, so at least one more scalar must also be non-zero. Call it c_j. Then we can write $\mathbf{v}_j$ as a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$ as

$$\mathbf{v}_j = \Big(-\frac{c_1}{c_j}\Big)\mathbf{v}_1 + \Big(-\frac{c_2}{c_j}\Big)\mathbf{v}_2 + \cdots + \Big(-\frac{c_{j-1}}{c_j}\Big)\mathbf{v}_{j-1}$$

■

Theorem 5.2. ***More Vectors Than Entries***
Any set of p vectors in $\mathbb{C}^n$ is linearly dependent if $p > n$.

Proof. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set of vectors in $\mathbb{C}^n$, where $p > n$. Let

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_p \end{bmatrix}$$

A is an $n \times p$ matrix which has more columns than rows, with at most n pivots. Therefore the system corresponding to the matrix equation $A\mathbf{x} = \mathbf{0}$ has at least one free variable, and infinitely many solutions. In particular, there is a nontrivial solution. Therefore the columns of A form a linearly dependent set. ■

Theorem 5.3. ***Zero Vector Makes Sets Dependent***
Any set of vectors in $\mathbb{C}^n$ that contains the zero vector is linearly dependent.

Proof. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set of vectors in $\mathbb{C}^n$ and assume $\mathbf{v}_j = \mathbf{0}$. Then we can write the non-trivial relation of linear dependence

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 1\mathbf{v}_j + \cdots + 0\mathbf{v}_p = \mathbf{0}$$

and S is linearly dependent. ■

Theorem 5.4. ***Linearly Independent Sets of Column Vectors***
Let A be an $m \times n$ matrix and let B be the reduced row-echelon form of A. The following statements are logically equivalent.

1. *The columns of a matrix A form a linearly independent set.*
2. *The homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has only the trivial solution.*
3. *Every column of B is a pivot column.*
4. *The homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has no free variables.*

Example 5.1. Let

$$S = \left\{ \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$$

Is S linearly independent?

We saw this same set S in example 4.3. We can form a matrix whose columns are the vectors in S and row reduce it:

$$A = \begin{bmatrix} -1 & 3 & 1 \\ 0 & 1 & 1 \\ 2 & -5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B$$

Since B has a pivot in every column, by theorem 5.4, S is linearly independent.