Machine Learning
有向图模型
Bayes nets

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第6章：有向图模型

(Bayes nets)

第6章：有向图模型

- Introduction
- Examples
- Inference
- Learning
- Conditional independence properties of DGMs

概述

问题的提出

- 我们需要观察多个相关变量，例如
- 文档中的单词
- 生物芯片中的基因
- 如何方便表示相关变量的联合分布$p(x|\theta)$？
- 给定一组变量，如何在合理的计算时间内使用这个分布来推断另一组变量？
- 如何学习这个分布的参数？

概率计算的链式法则

- 链式法则公式：

$$p(x_{1:V}) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2, x_1)\cdots p(x_V \mid x_{1:V-1})$$

- 当t 变大时，$p(x_t|\mathbf{x}_{1:t-1})$ 的表示将变得越来越复杂
- 例如, 设所有变量都有 K 个状态.
  - $p(x1)$ 可表示为一个含 $O(K)$ 个参数的表,
  - $p(x2|x1)$可表示为一个含 $O(K^2)$个参数的表
  - $p(x3|x1, x2)$可表示为一个含 $O(K3)$个参数的3*d*表.
  - 以此类推，模型将包含 $O(K^V)$ 个参数.

随机矩阵（*stochastic matrix*）

- 随机矩阵是一个条件概率的2维表:
  - $p(x2 = j|x1 = i) = T_{ij}$;
  - 满足约束：$\Sigma_j T_{ij} = 1$ for all rows i, $0 \le T_{ij} \le 1$ for all entries
- 例如

x	**1**	**…**	**K**
p	0.1	…	0.15

x_2/x_1	**1**	**…**	**K**
1	0.1	…	0.15
¦	¦	…	¦
K	0.2	…	0.3

条件概率表（CPTs）

- 对于多维变量，其联合发布参数就变得很多.
  - *因此，需要用多个表格来表达这些参数.*
- *这些表格称为条件概率表*

x1	x2	x3=	1	…	K
K					
¦					
K					

x1	x2	x3=	1	…	K
¦					
¦					
¦					

x1	x2	x3=	1	…	K
1	1		0.1	…	0.14
¦	¦		¦		¦
1	K		0.13	…	0.16

条件独立性 (CI)

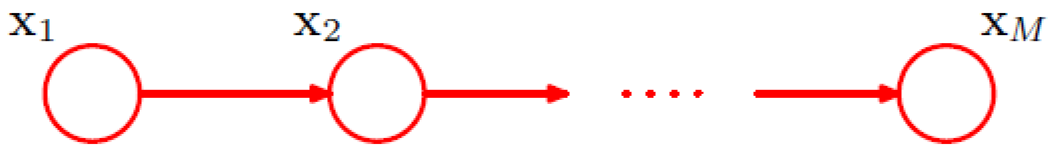
- 当存在：$p(X, Y \mid Z) = p(X|Z)p(Y|Z)$
  - 称 X 、Y 在给定条件Z下，是条件独立的
  - *记为:* $X \perp Y \mid Z,$
  - 即有，$X \perp Y \mid Z \Leftarrow\Rightarrow p(X, Y \mid Z) = p(X|Z)p(Y|Z)$

Markov 链

- 一阶 ***Markov* 假设*:***
  - $x_{t+1} \perp x_{1:t-1} | x_t$,
  - 这表示“*给定现在，未来独立于过去*”
- 基于这个假设,联合分布可表示成:

$$p(x_{1:V}) = p(x_1) \prod_{1 \le t \le V} p(x_t \mid x_{t-1})$$

- 这称为一阶 **Markov 链**



图模型 (GM)

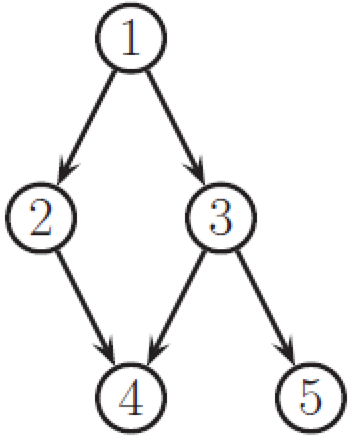
- 定义*1d*序列上的分布时，一阶马尔可夫模型是有用的
- *定义二维图像、三维视频或更高维变量的分布时，使用图模型更方便*
- 图模型用图来表示联合分布
  - *节点：随机变量，边：条件依赖*
- 图模型类型
  - 有向图模型
  - 无向图模型
  - 有向与无向结合的模型

有向图模型(DGM)

- 如果是有向无环图模型（DAG），被称为贝叶斯网络
  - 这些模型也被称为信念网络。
- 有向图模型举例：

$$p(x_{1:5}) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_1, \cancel{x_2})p(x_4 \mid \cancel{x_1}, x_2, x_3)p(x_5 \mid \cancel{x_1}, \cancel{x_2}, x_3, \cancel{x_4})$$

$$p(x_{1:5}) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_1)p(x_4 \mid x_2, x_3)p(x_5 \mid x_3)$$



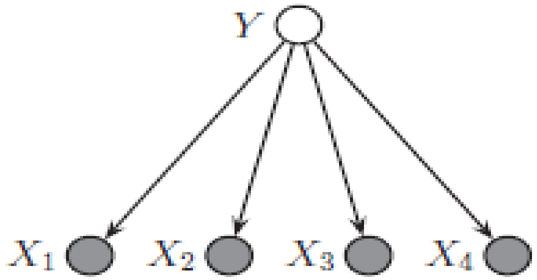
- 一般， $p(x_{1:V} \mid G) = \prod_{t=1}^{V} p(x_t \mid x_{pa(t)})$
- 如果每个节点有$O(F)$个父节点、K个状态
  - 模型的参数就有 $O(VK^F)$个

几个例子

朴素贝叶斯分类器

- 朴素贝叶斯分类器假定
  - 给定类标签后，特征是条件独立的
- 联合分布的表达式与网络图：

$$p(y,x)=p(y)\prod_{j=1}^{D}p(x_j\mid y)$$

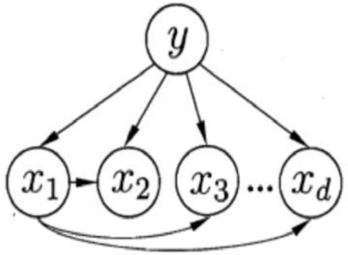


半朴素贝叶斯分类器

- 朴素贝叶斯假设往往很难成立，因此，希望对属性条件独立性假设放松一点
  - 产生了“半朴素贝叶斯分类器”(semi-naive Bayes classifiers),基本思路：
    - 适当考虑一部分属性间比较强的相互依赖信息
    - 采用“独依赖估计”(One-Dependent Estimator, ODE)：假设每个属性除类别外，最多只依赖一个其他属性

$$p(C|x) \propto p(C)\prod_{i=1}^{d} p(x_i|C, pa_i)$$

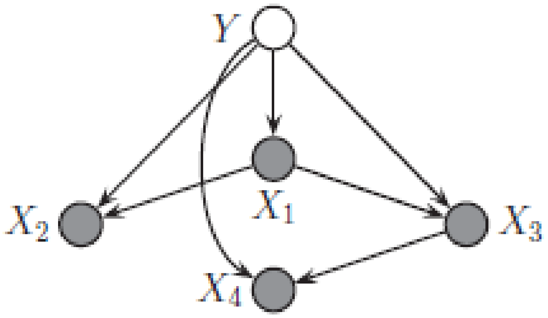
  - 例如，超父结构（super-parent）：假设所有属性都依赖于同一个属性
    - SPODE (Super-Parent ODE)模型：



SPODE

树增强（tree-augmented）的朴素贝叶斯分类器

- 如果特征间不完全满足朴素贝叶斯分类器假设
- 但可通过特征间的互相关性计算，将网络图化成一颗树
- 这个模型称为树增强朴素贝叶斯分类器
  - 这实际上是一个半朴素贝叶斯分类器



Markov 模型

❖ 一阶和二阶Markov链是有向无环图:

(a) (b)

$$p(x_{1:T}) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2)\ldots = p(x_1)\prod_{t=2}^{T} p(x_t \mid x_{t-1})$$

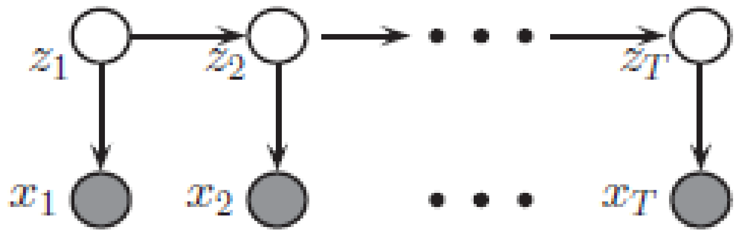
$$p(x_{1:T}) = p(x_1, x_2)p(x_3 \mid x_1, x_2)p(x_4 \mid x_2, x_3)\ldots = p(x_1, x_2)\prod_{t=3}^{T} p(x_t \mid x_{t-1}, x_{t-2})$$

❖ 类似地可创建高阶马尔可夫模型

❖ 对于高阶模型，参数的数量将激增。

隐马尔科夫模型(HMM)

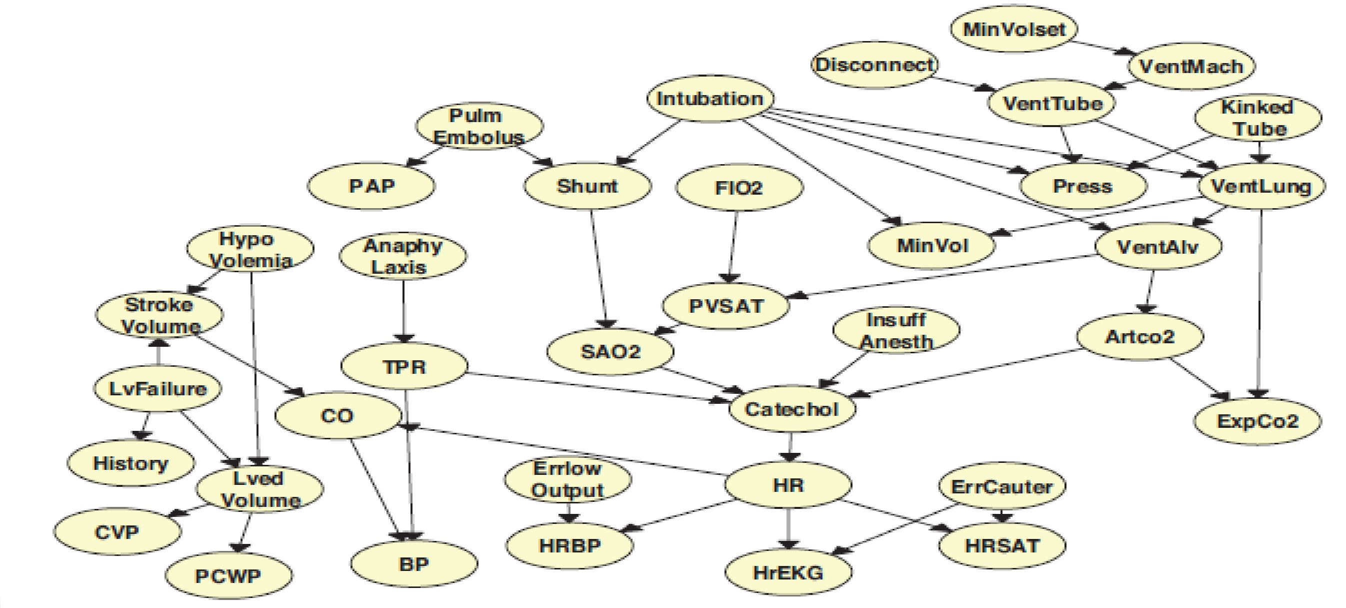
- 由一阶马尔科夫链定义的隐过程
  - 其观察过程是带噪声的



- z_t 为t时刻的隐变量
- x_t 为t时刻的观察变量
- $p(z_t|z_{t-1})$：**转移模型，** $p(x_t|z_t)$：观测模型

医疗诊断

- 如果我们希望对重症监护室（ICU）中的病人的疾病进行建模
  - 需要测量患者的多种变量，包括呼吸频率、血压等
  - 需要描述这些变量之间的关系
- Beinlich et al. 1989提出一种“报警网络”
  - 它表达了变量间的依赖关系
  - 该模型有37个变量和504个参数。
- 这样的系统，又称为概率专家系统

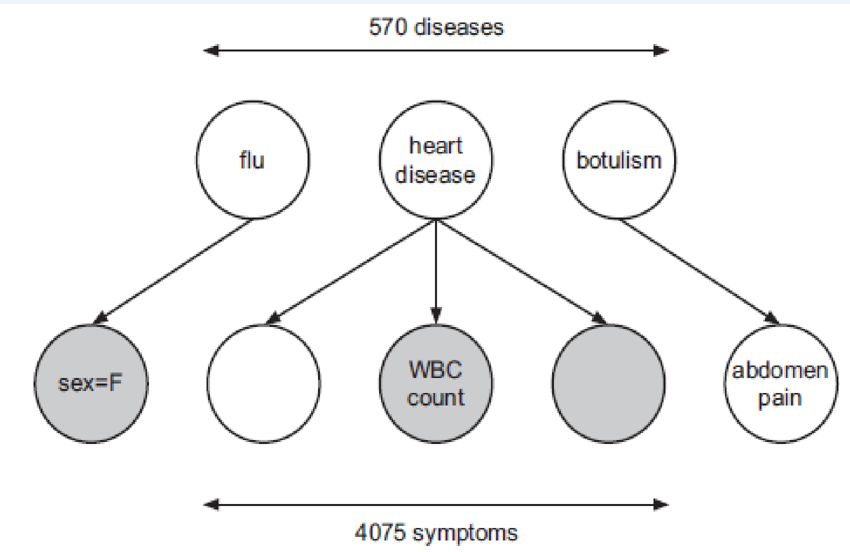


另一种医疗诊断网络

- Shwe et al. 1991提出，称为快速医学参考（QMR网络）
- 这是设计的为传染病进行建模
- QMR模型是二分图结构，上部表示疾病，下部表示症状或发现
  - 所有根节点都服从伯努利分布，代表发生该疾病的先验概率
  - 许多叶节点的父节点数量非常多
    - ⑩ 叶节点（症状）用条件概率表(CPT)表示，需要太多参数
- 一种替代方案：用逻辑回归表示条件概率分布

$$p\left(v_t = 1 \middle| h_{pa(t)}\right) = sigm\left(\omega^T h_{pa(t)}\right)$$

  - 用有向图表示，称为sigmoid belief net (Neal 1992).



$$p(v,h) = \prod_s p(h_s) \prod_t p(v_t \mid h_{pa(t)})$$

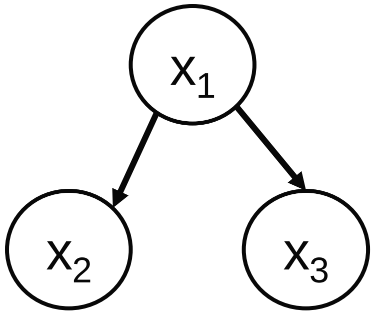
h_s : 隐藏节点 (*疾病diseases*)

v_t : 观察节点 (*症状symptoms*)

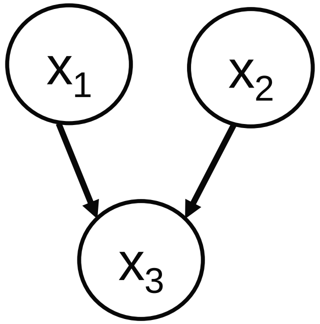
贝叶斯网络结构

贝叶斯网络结构

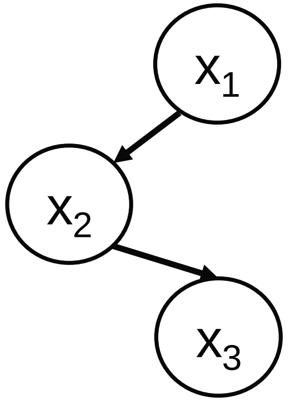
- 贝叶斯网有效地表达了属性间的条件独立性。
- 贝叶斯网络假设每个属性与它的非后裔属性独立
- 三个变量的典型依赖关系



同父结构



V型结构



顺序结构

推理（*Inference*）

推理的定义

- 概率推理系统的基本任务
  - 给定一组证据，计算一组查询变量的后验概率.
- 一组完整的变量包括：**X**={X}∪**E** ∪**Y**.
  - X 表示查询变量;
  - **E 表示一组证据变量** $E_1, . . . , E_m$,
  - **Y : 即不是证据，也不是查询变量** $Y_1, . . . , Y_l$ (隐变量).
- 一个典型的推理任务：计算 **P**(X | **e**).
  - **e**： 特定观测到的证据;
- 推理方法分为精确推理和近似推理两类

基于枚举的推理

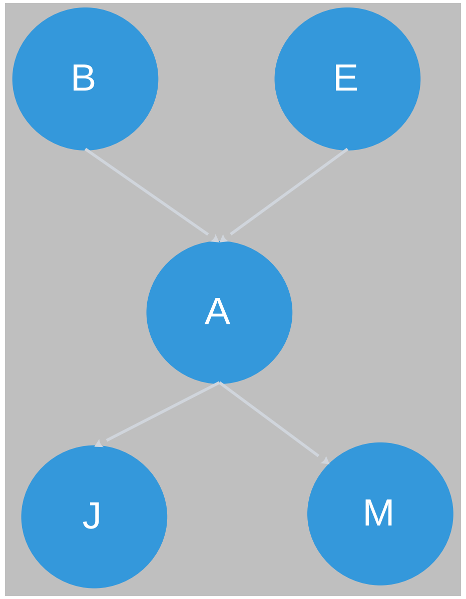
- 一个推理 **P**(X | **e**) 可以用下式进行计算:

$$p(X \mid \mathrm{e}) = \alpha p(X, \mathrm{e}) = \alpha \sum_{y} p(X, \mathrm{e}, \mathrm{y})$$

- 贝叶斯网络给出了完整的联合概率分布.
  - 推理可以通过贝叶斯网络进行
    - ⑩计算网络上条件概率乘积的和.

分析一个例子: *Alarm*

- 场景：当地震发生或者有入室盗窃，报警器会响，若报警器响了， John 和 Mary 会打电话
- 推理 ：P(Burglary | JohnCalls =true,MaryCalls =true).
- 设计网络结构，其中:
  - *B: Burglary*
  - *E: Earthquake*
  - *J：John calls*
  - *M：Mary calls*
  - *A：Alarm*
  - 这个问题里，隐变量是Earthquake和 Alarm



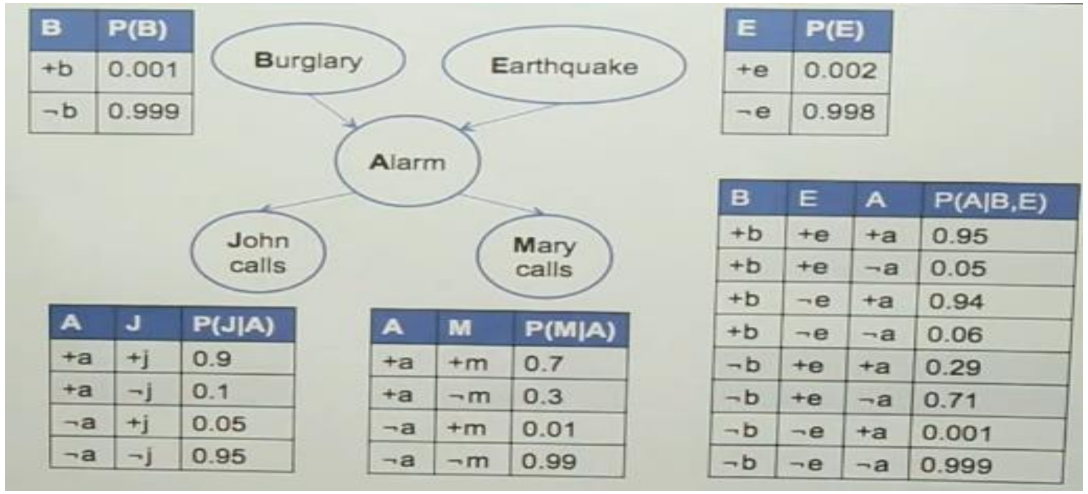
基于枚举直接推理

- 根据公式，增加隐变量，有：

$$p(B \mid j, m) = \alpha p(B, j, m) = \alpha \sum_{e} \sum_{a} p(B, j, m, e, a)$$

- 根据贝叶斯网络，可得到：

$$p(b \mid j, m) = \alpha \sum_{e} \sum_{a} p(b) p(e) p(a \mid b, e) p(j \mid a) p(m \mid a)$$



B	P(B)
+b	0.001
¬b	0.999

E	P(E)
+e	0.002
¬e	0.998

B	E	A	P(A|B,E)
+b	+e	+a	0.95
+b	+e	¬a	0.05
+b	¬e	+a	0.94
+b	¬e	¬a	0.06
¬b	+e	+a	0.29
¬b	+e	¬a	0.71
¬b	¬e	+a	0.001
¬b	¬e	¬a	0.999

A	J	P(J|A)
+a	+j	0.9
+a	¬j	0.1
¬a	+j	0.05
¬a	¬j	0.95

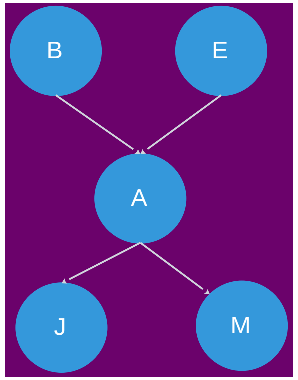
A	M	P(M|A)
+a	+m	0.7
+a	¬m	0.3
¬a	+m	0.01
¬a	¬m	0.99

直接枚举推理的复杂度

- 贝叶斯网络算法:

$$p(b \mid j, m) = \alpha \sum_{e} \sum_{a} p(b)p(e)p(a \mid b, e)p(j \mid a)p(m \mid a)$$

  - 仅仅n 个布尔变量的网络，复杂度都是 $(n2^n)$

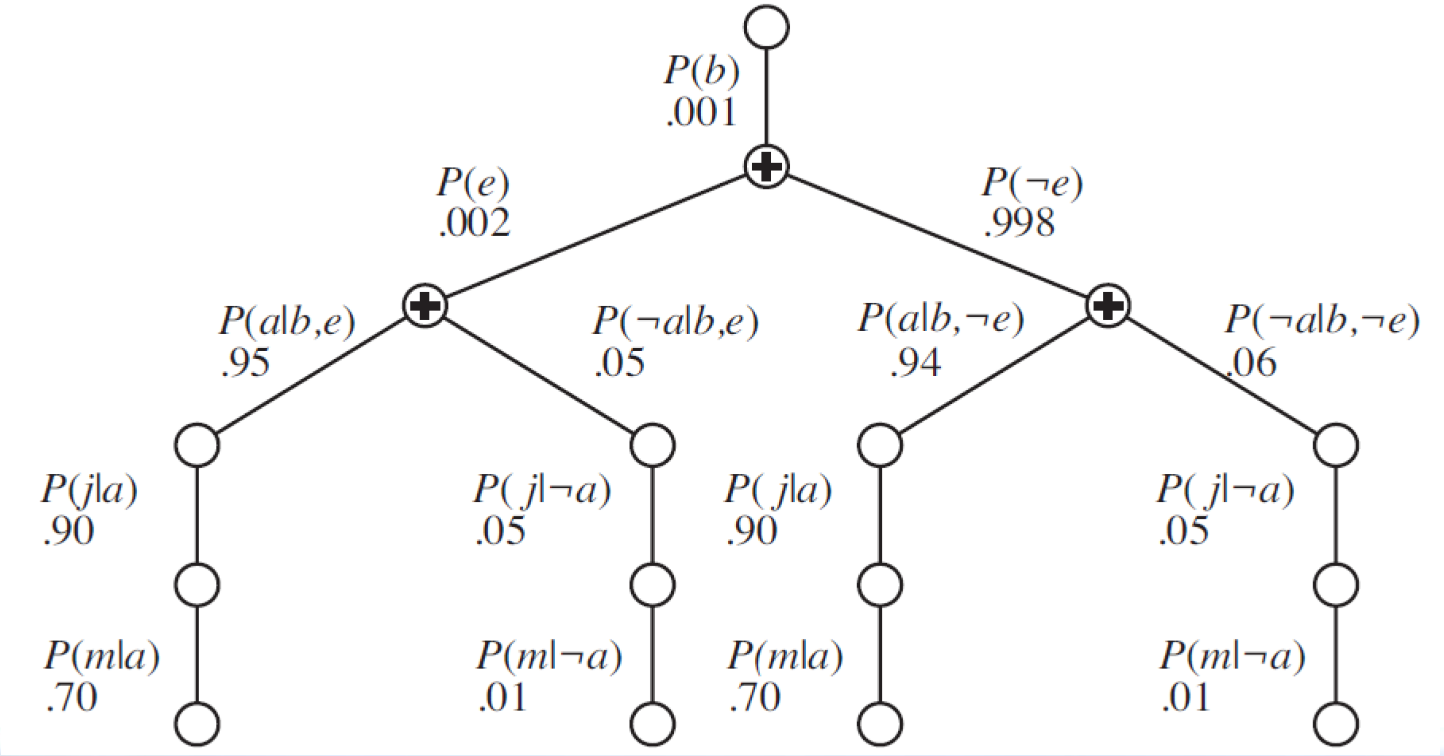


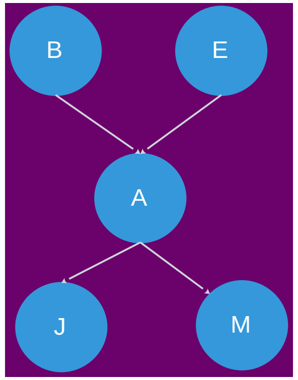
- 这种推理方法过于复杂

加速方法：做点简单的改进

- 一种简单的改进方法 :将网络表示成一个树结构

$$p(b \mid j, m) = \alpha p(b) \sum_e p(e) \sum_a p(a \mid b, e) p(j \mid a) p(m \mid a)$$





加速方法：变量消除

- 变量消除算法：消除枚举算法中重复变量计算，改进算法
- 想法很简单：计算一次后保存结果，后面直接使用。
  - 变量消除通过从右到左计算表达式来实现存储中间结果
  - 对每个变量的求和仅针对表达式中依赖于该变量的部分。
- 例如

$$P(d)=\sum_{a,b,c}P(a,b,c,d)$$

$$=\sum_{a,b,c}P(a)P(b|a)P(c|b)P(d|c)=\sum_{b,c}P(c|b)P(d|c)\underbrace{\sum_{a}P(a)P(b|a)}_{\phi_a(b)}=\sum_{c}P(d|c)\underbrace{\sum_{b}P(c|b)\phi_a(b)}_{\phi_b(c)}=\sum_{c}P(d|c)\phi_b(c)$$

- 存在问题：不容易找到消解顺序

因子分解

- 如果要计算下列公式:

$$p(b \mid j,m) = \alpha \underbrace{p(B)}_{f_1(B)} \sum_e \underbrace{p(e)}_{f_2(E)} \sum_a \underbrace{p(a \mid b,e)}_{f_3(A,B,E)} \underbrace{p(j \mid a)}_{f_4(A)} \underbrace{p(m \mid a)}_{f_5(A)}$$

  - f_i :表示因子，以矩阵形式存储表达式中某部分的信息

- 例如:
  - f_4(A) 和f_5(A) 对应于P(j | a) 和 P(m| a)，都依赖于 A

$$f_4(A) = \begin{pmatrix} p(j \mid a) \\ p(j \mid \neg a) \end{pmatrix} = \begin{pmatrix} 0.90 \\ 0.05 \end{pmatrix} \qquad f_5(A) = \begin{pmatrix} p(m \mid a) \\ p(m \mid \neg a) \end{pmatrix} = \begin{pmatrix} 0.70 \\ 0.01 \end{pmatrix}$$

  - f_3(A,B,E)： 2×2×2 矩阵,(张量)

因子的逐点乘积

- 将表达式:

$$p(b \mid j,m) = \alpha \underbrace{p(B)}_{f_1(B)} \sum_e \underbrace{p(e)}_{f_2(E)} \sum_a \underbrace{p(a \mid b,e)}_{f_3(A,B,E)} \underbrace{p(j \mid a)}_{f_4(A)} \underbrace{p(m \mid a)}_{f_5(A)}$$

- 转换成 ：

$$p(B \mid j,m) = \alpha f_1(B) \times \sum_e f_2(E) \times \sum_a f_3(A,B,E) \times f_4(A) \times f_5(A)$$

  - 这里“×”不是普通的矩阵乘法，而是逐点相乘

逐点相乘的定义

- 因子逐点相乘: $f_1 \times f_2 = f$
  - 假设两个因子有共同变量$Y_1, . . . , Y_k$.
    - ✓ $f_1(X_1 ...X_j, Y_1 ...Y_k)\times f_2(Y_1 ...Y_k, Z_1 ...Z_l)=f(X_1 ...X_j, Y_1 ...Y_k, Z_1 ...Z_l)$
    - ✓ 如果变量都是二值的, 则 f_1 、f_2 分别有 2^{j+k} 和2^{k+l} 个项
    - ✓ 逐点乘积有 2^{j+k+l} 个项.
- 看个两个因子逐点乘积的例子

逐点乘积举例

- 给定因子$f_1(A,B)$ 和 $f_2(B,C)$
  - 逐点乘积 $f_1(A,B)\times f_2(B,C)=f_3(A,B,C)$ 有 $2^{1+1+1}=8$ 项, 如下表

A	B	$\mathbf{f}_1(A,B)$	B	C	$\mathbf{f}_2(B,C)$	A	B	C	$\mathbf{f}_3(A,B,C)$
T	T	0.3	T	T	0.2	T	T	T	$0.3\times0.2=0.06$
T	F	0.7	T	F	0.8	T	T	F	$0.3\times0.8=0.24$
F	T	0.9	F	T	0.6	T	F	T	$0.7\times0.6=0.42$
F	F	0.1	F	F	0.4	T	F	F	$0.7\times0.4=0.28$
						F	T	T	$0.9\times0.2=0.18$
						F	T	F	$0.9\times0.8=0.72$
						F	F	T	$0.1\times0.6=0.06$
						F	F	F	$0.1\times0.4=0.04$

变量消除的步骤(1)

- 对于表达式:

$$p(B \mid j, m) = \alpha f_1(B) \times \sum_e f_2(E) \times \sum_a f_3(A, B, E) \times f_4(A) \times f_5(A)$$

- 步骤1:

f_3, f_4, f_5 乘积里对A求和，得到 2×2 的新因子 $f_6(B,E)$

$$\begin{aligned} f_6(B, E) &= \sum_a f_3(A, B, E) \times f_4(A) \times f_5(A) \\ &= f_3(\mathrm{a}, B, E) \times f_4(\mathrm{a}) \times f_5(\mathrm{a}) + f_3(\neg\mathrm{a}, B, E) \times f_4(\neg\mathrm{a}) \times f_5(\neg\mathrm{a}) \end{aligned}$$

表达式变成为:

$$p(B \mid j, m) = \alpha f_1(B) \times \sum_e f_2(E) \times f_6(B, E)$$

变量消除的步骤(2)

- 对于表达式：

$$p(B \mid j, m) = \alpha f_1(B) \times \sum_e f_2(E) \times f_6(B, E)$$

- 步骤2：

f_2, f_6,乘积里对E求和，得到 2×2的新因子 f_7(B)

$$f_7(B) = \sum_e f_2(E) \times f_6(B, E)$$
$$= f_2(\mathrm{e}) \times f_6(B, \mathrm{e}) + f_2(\neg\mathrm{e}) \times f_6(B, \neg\mathrm{e})$$

表达式变成为：

$$p(B \mid j, m) = \alpha f_1(B) \times f_7(B)$$

学习

学习

- 推理是指计算 $p(\mathbf{x}_h|\mathbf{x}_v, \boldsymbol{\theta})$,
  - $\boldsymbol{\theta}$ 是模型的参数，假定已知.
- 学习通常是指
  - *给定数据后，计算* ***MAP 来估计参数***

$$\hat{\theta} = \arg\max_{\theta} \sum_{i=1}^{N} \log p(x_{i,v} \mid \theta) + \log p(\theta)$$

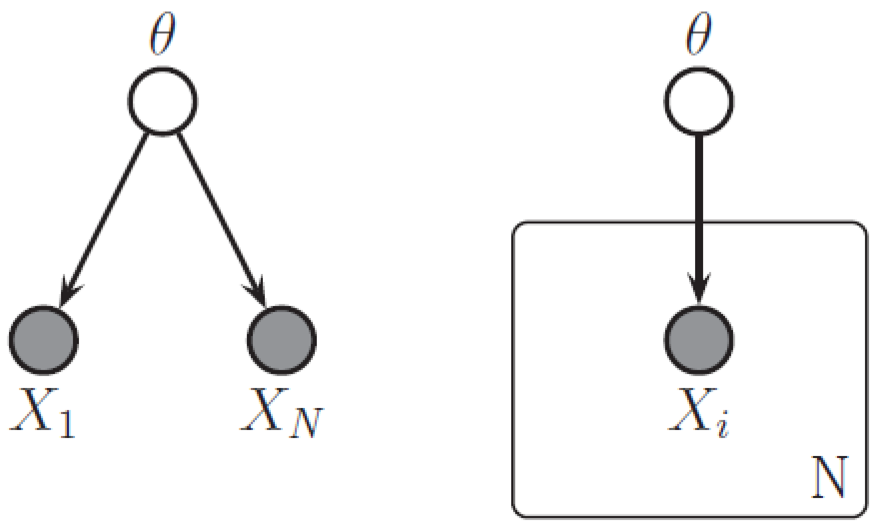
- 如果是均匀先验，$p(\boldsymbol{\theta}) \propto 1$, 后验估计就退化为最大似然估计

贝叶斯观点

- 将参数也看成是未知变量进行推理。
- 推理和学习之间没有区别：
  - 将参数作为节点添加到图中，条件为D推断所有节点的值。
- 隐变量和参数之间的主要区别
  - 隐变量的数量随着训练数据的数量而增长，
  - 参数的数量通常是固定的

盘标记

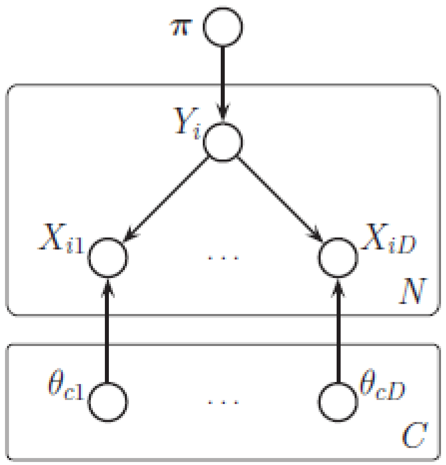
- 当从数据中推断参数时，通常假设数据是独立同分布的（iid）
- 将这个假设表示为
  - 给定θ，数据点x_i是条件独立的
  - 这个假定可用盘标记



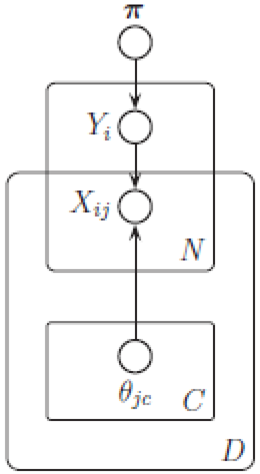
$$p(\theta, D) = p(\theta)\prod_{i=1}^{N} p(x_i \mid \theta)$$

嵌套盘标记

- 一个稍微复杂一点的例子：朴素贝叶斯分类器
  - （a） 表示已将D个特征“展开”
  - （b） 用嵌套板标记相同的模型



(a)



(b)

$$p(\mathbf{x} \mid y=c, \boldsymbol{\theta}) = \prod_{j=1}^{D} p(x_j \mid y=c, \theta_{jc})$$

完全数据（Complete data）

- 如果在每种情况下所有的变量都被完全观察到
  - 没有丢失数据，也没有隐藏变量
- 我们称这数据是完全的

从完全数据中学习

- 对于一个完全数据的有向图模型(DGM), 其似然函数：

$$p(D\,|\,\boldsymbol{\theta}) = \prod_{i=1}^{N} p(x_i\,|\,\theta) = \prod_{i=1}^{N}\prod_{t=1}^{V} p(x_{it}\,|\,x_{i,pa(t)},\theta_t) = \prod_{t=1}^{V} p(D_t\,|\,\theta_t)$$

这里D_t 是与节点t 相关联的数据以及它们的父节点,

  - 这就是根据图的似然分解.

- 假设先验为：

$$p(\boldsymbol{\theta}) = \prod_{t=1}^{V} p(\boldsymbol{\theta}_i)$$

- 后验也因子分解为：

$$p(\boldsymbol{\theta}) \propto p(D\mid\boldsymbol{\theta})p(\boldsymbol{\theta}) = \prod_{t=1}^{V} p(D_t\mid\boldsymbol{\theta}_t)p(\boldsymbol{\theta}_t)$$

Conditional independence properties of DGMs

(directed graphical model)

the heart of graphical model

- At the heart of any graphical model
  - a set of conditional indepence (CI) assumptions:
- $x_A \perp_G x_B | x_C$: A is independent of B given C in the graph G
  - I-map (independence map) : the set of all such CI statements
  - I(G) : the set of all such CI statements encoded by the graph
  - I(p) is the set of all CI *statements that hold for distribution p.*
  - *G is an I-map for p,* iff $I(G) \subseteq I(p)$
    - ✓ There are many I-map for p,
  - *I(G) = I(P), Graph G represent the distribution P*
    - ✓ The *I(G) is also called* P-map (Perfect-map)。

d-separation

- Conditional independence of two variables in graph G
  - determined by using a very important conception called **d-separation**
  - **d-separation : Directed Separation**
- *We say path **P** is **d-separated by a set of*** nodes *E*
  - E contains the evidence
  - iff at least one of the following conditions hold:

1. P contains a chain, $s \to m \to t$ *or* $s \leftarrow m \leftarrow t$, $(m \in E)$

2. P contains a tent or fork, $s \swarrow m \searrow t$, *where* $m \in E$

3. P contains a **v-structure,** $\boldsymbol{s \searrow m \swarrow t}$**,** *$m$ is not in **E** and nor is any* descendant of *m.*

d-separated two sets of nodes

- a *set of nodes A is d-separated from a different set of nodes B*
  - *given a* third observed set *E*
  - *iff each path from every node a* $\in$ *A to every node b* $\in$ *B is* d-separated by *E.*

Why to need d-separate?

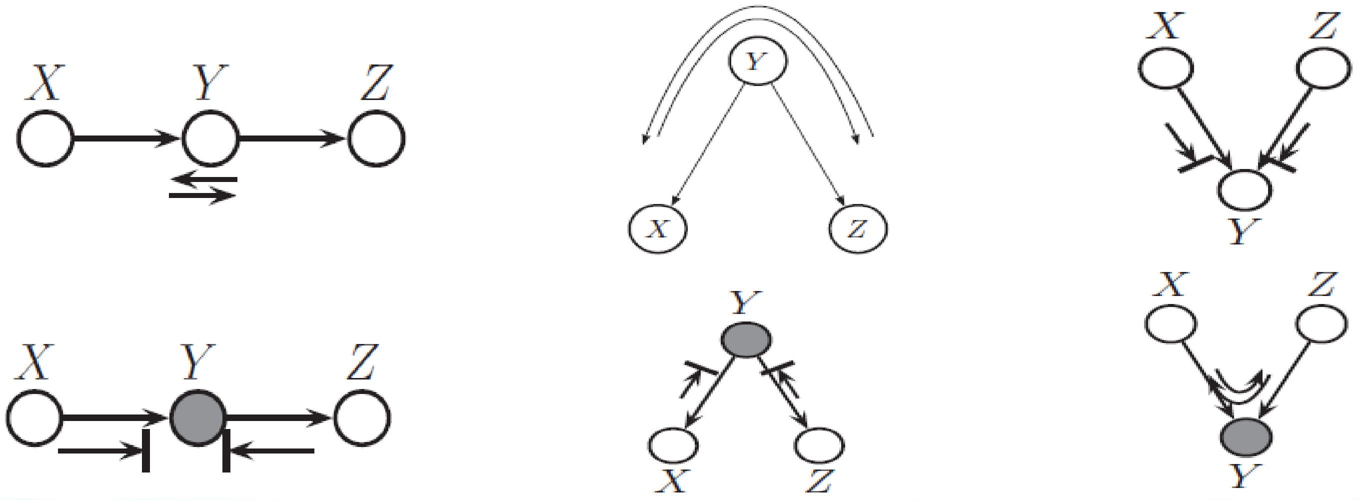
- *if the observation variable (evidence node) set E is known*
  - whether node set A and node set B are conditionally independent with respect to E?
    - ✓ *we can follow all path from A to B. If these path all are d-separated by E*
    - ✓ *it shows that A and B are conditionally independent with respect to E*

global Markov properties

- Conditional independence properties of a DGM
  - $\mathbf{x}_A \perp_G \mathbf{x}_B \mid \mathbf{x}_E \iff$ *A is d-separated from B given E.*

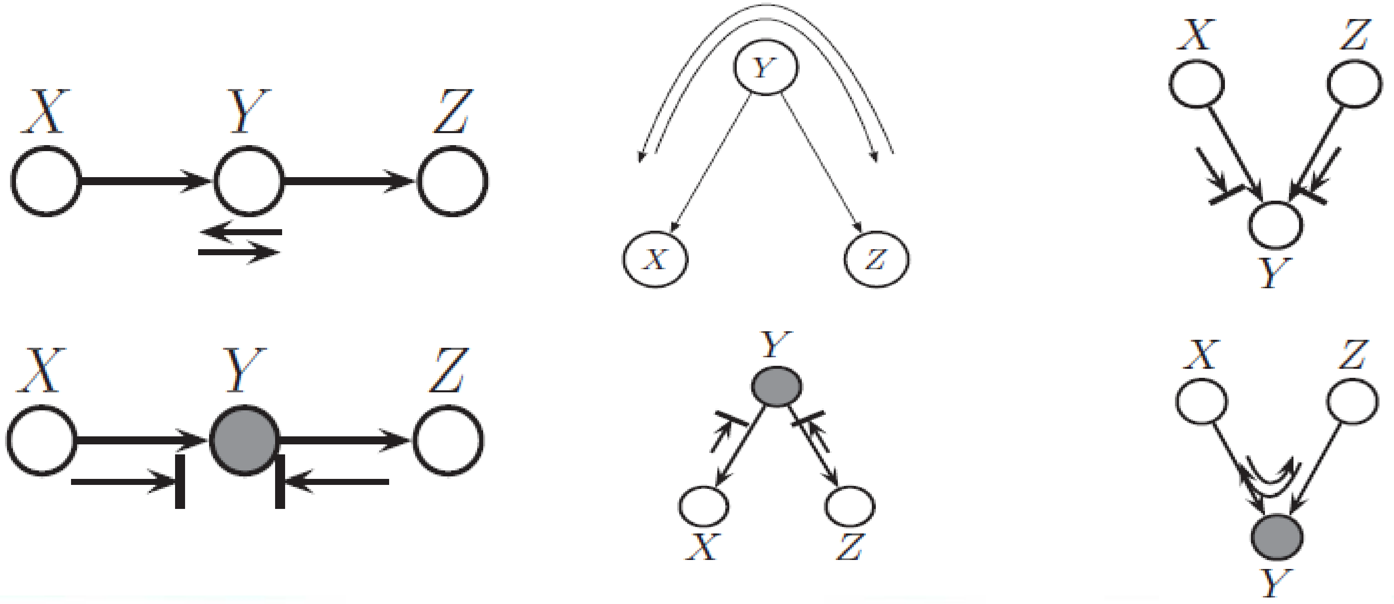
Bayes ball algorithm

- a simple way to see if *A is d-separated from B* given *E*
- *The idea of the algorithm:*
  - *“shade” all nodes in E means* they are observed.
  - place “balls” at each node in *A, let them “bounce around”* according to some rules, ask if any of the balls reach any of the nodes in *B.*



Rules of Bayes ball

- The three main rules
  - *a ball can pass through a chain, but not if it is shaded in the middle.*
  - *a ball can pass through a fork, but not if it is shaded in the middle.*
  - *a ball cannot pass through a v-structure, unless it is shaded in the middle*

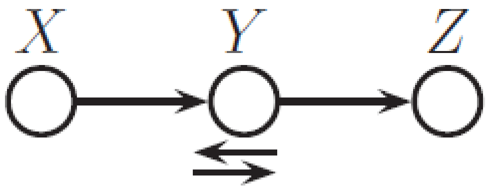


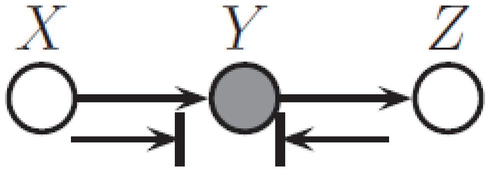
justify Rule 1 of Bayes ball

- consider a chain structure $X \rightarrow Y \rightarrow Z$

$$p(x, y, z) = p(x)p(y \mid x)p(z \mid y)$$

$$p(x, z \mid y) = \frac{p(x, z \mid y)p(y)}{p(y)} = \frac{p(x, y, z)}{p(y)} = \frac{p(x)p(y \mid x)p(z \mid y)}{p(y)} = p(x \mid y)p(z \mid y)$$



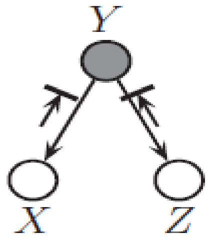


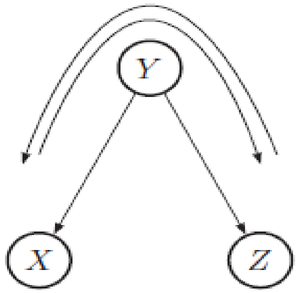
justify Rule 2 of Bayes ball

- consider a tent or fork, $X \swarrow Y \searrow Z$

$$p(x, y, z) = p(y)p(x \mid y)p(z \mid y)$$

$$p(x, z \mid y) = \frac{p(x, z \mid y)p(y)}{p(y)} = \frac{p(x, y, z)}{p(y)} = \frac{p(y)p(x \mid y)p(z \mid y)}{p(y)} = p(x \mid y)p(z \mid y)$$





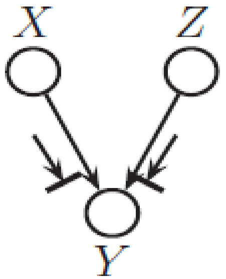
justify Rule 3 of Bayes ball

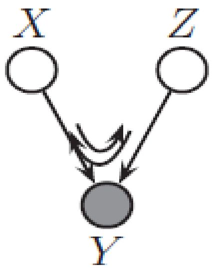
- consider a v-structure, X↘Y↙ Z

$$p(x, y, z) = p(x)p(z)p(y \mid x, z)$$

$$p(x, z \mid y) = \frac{p(x, z \mid y)p(y)}{p(y)} = \frac{p(x, y, z)}{p(y)} = \frac{p(x)p(z)p(y \mid x, z)}{p(y)}$$

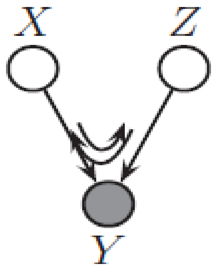
- So, $X \not\perp Z \mid Y$, *but*, in the unconditional distribution, X ⊥ Z

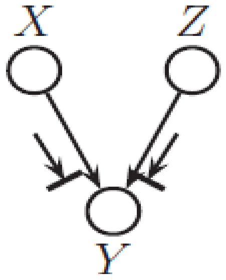




explaining away

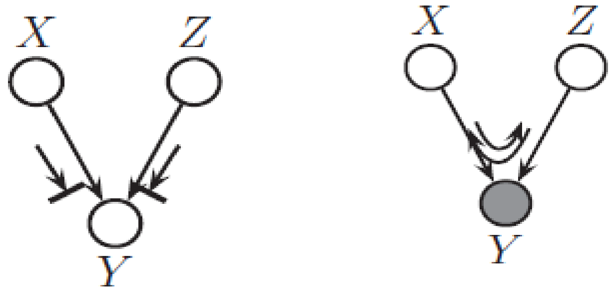
- in conditioning on a common child at the bottom of a v-structure
  - we see that
    - ⓾*x* and *z* are marginally independent.
    - ⓾ but it makes its parents become dependent.
- This important effect is called
  - **explaining away**
  - **inter-causal reasoning**
  - **Berkson's paradox**





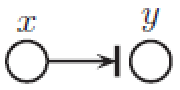
Example of explaining away

- suppose we toss two coins (0 ,1), the coins are independent
  - we observe the “sum” of their values.
  - but once we observe their sum, they become coupled, e.g.
    - if the sum is 1, and the first coin is 0,
    - then we know the second coin is 1

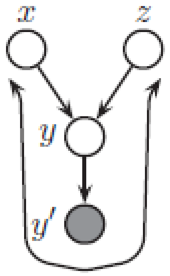


Boundary conditions of Bayes ball

- For a v-structure
  - we need the “boundary conditions of Bayes Ball
  - To explain
    - ❿ besides y, the descendant of Y play the same role as y

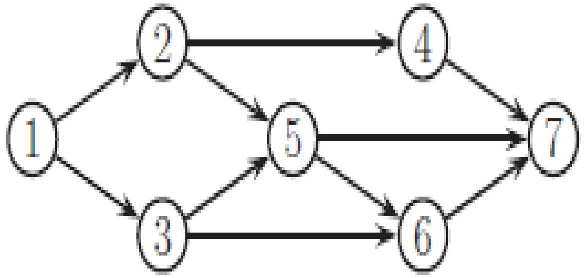


- Let us analyse it
  - The Suppose Y’ is a noise-free copy of Y .
  - if we observe Y’ , we effectively observe Y as well,
  - so the parents X and Z have to compete to explain this.
  - So if we send a ball down X → Y → Y’ , it should “bounce back” up along Y’ → Y → Z.
  - if Y and all its children are hidden, the ball does not bounce back.



directed local Markov property

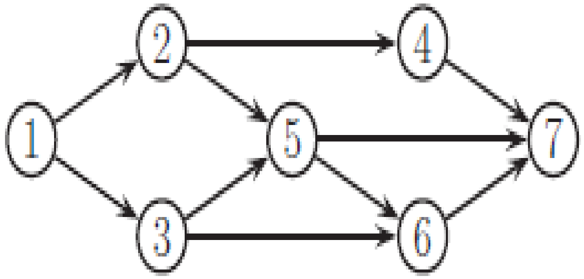
- From the d-separation criterion, one can conclude
  - $t \perp \mathrm{nd}(t) \setminus \mathrm{pa}(t) | \mathrm{pa}(t)$
  - nd(t): **non-descendants** of a node t
    - ✓ all the nodes except for its descendants,
    - ✓ $\mathrm{nd}(t) = V \setminus \{t \cup \mathrm{desc}(t)\}$.



- For example
  - Suppose t =3, then nd(3) = {1, 2, 4}, and pa(3) = 1,
  - So we have $3 \perp 2, 4 | 1$.

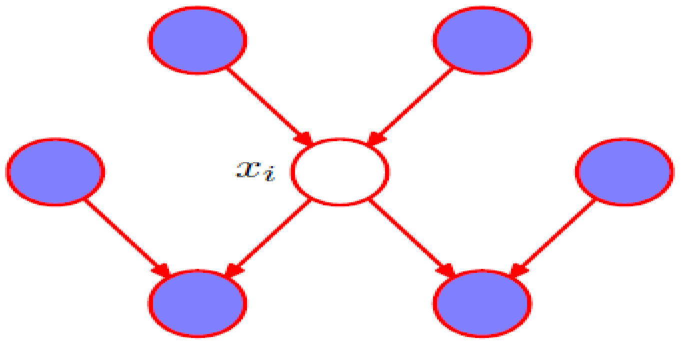
ordered Markov property

- A special case of directed local Markov property
  - $t \perp \text{pred}(t) \setminus \text{pa}(t) \mid \text{pa}(t)$
    - ✓ since $\text{pred}(t) \subseteq \text{nd}(t)$.
    - ✓ pred(t): **predecessors** of a node t
  - we only look at predecessors of a node
    - ✓ according to some topological ordering.
  - For example
    - ✓ suppose topological ordering 1, 2, . . . , 7.
      - pred(3) = {1, 2} and pa(3) = 1
      - So, we have $3 \perp 2|1$



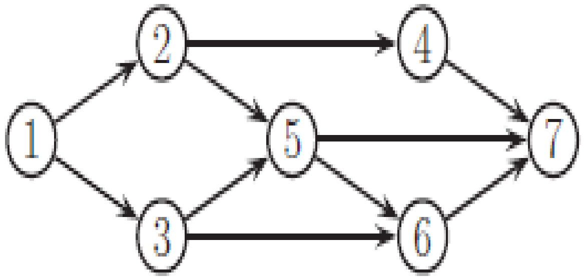
Markov blanket

- Markov blanket is the set of nodes
  - it renders t conditionally independent of all the other nodes in the graph
  - It is called **Markov blanket** of t : mb(t)
- Markov blanket is equal to
  - the parents + the children + the co-parents
  - mb(t)= ch(t) ∪ pa(t) ∪ copa(t)
- the co-parents:
  - these are also parents of its children



Example of Markov blanket

- Given a graph, we have
  - mb(5) = *{6, 7}* $\cup$ *{2, 3}* $\cup$ *{4}* = *{2, 3, 4, 6, 7}*



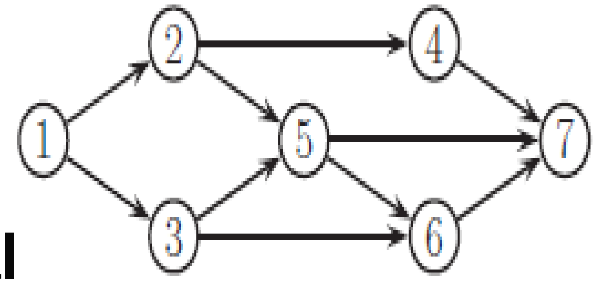
full conditional

- why the co-parents are in the Markov blanket ?
  - Let us see:

$$P(x_t \mid x_{-t}) = \frac{P(x_t, x_{-t})}{P(x_{-t})} = \frac{P(x_t \mid x_{pa(t)})P(x_{ch(t)} \mid x_{pa(ch(t))})P(x_B \mid x_{pa(B)})}{\int P(x_t \mid x_{pa(t)})P(x_{ch(t)} \mid x_{pa(ch(t))})P(x_B \mid x_{pa(B)})dx_t}$$

$$P(x_t \mid x_{-t}) \propto P(x_t \mid x_{pa(t)}) \prod_{s \in ch(t)} P(x_s \mid x_{pa(s)})$$

Where x_{-t} : all the terms that do not involve x_t

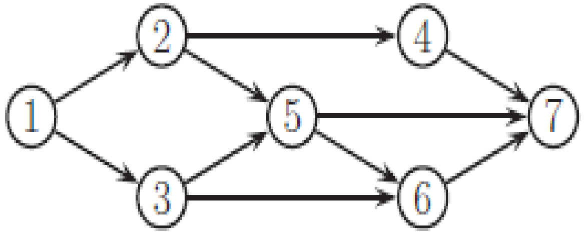


- The expression is called t's **full conditional**
- For example

$p(x_5|\mathbf{x}_{-5}) \propto p(x_5|x_2, x_3)p(x_6|x_3, x_5)p(x_7|x_4, x_5, x_6)$

Example of condition independence

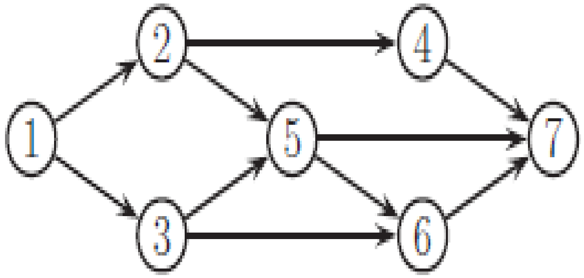
- we see that $x2 \perp x6 | x5$, *since*
  - *the* $2 \rightarrow 5 \rightarrow 6$ *path is blocked* by *x5 (which is observed),*
  - *the* $2 \rightarrow 4 \rightarrow 7 \rightarrow 6$ *path is blocked by x7 (which is hidden),*
  - the $2 \rightarrow 1 \rightarrow 3 \rightarrow 6$ *path is blocked by x1 (which is hidden).*
- *we also see that* $x2 \not\perp x6 | x5, x7$, *since*
  - *the* $2 \rightarrow 4 \rightarrow 7 \rightarrow 6$ *path is no longer blocked by x7 (which is* observed).



Example of full conditional

- Given the graph, we have

$$P(x_5 \mid x_{-5}) \propto P(x_5 \mid x_2, x_3) P(x_6 \mid x_3, x_5) P(x_7 \mid x_4, x_5, x_6)$$

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Thank You !