Assignment – Parameter Estimation

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(1.) Let $(X_1, X_2, \ldots)$ be a random sample of size n taken from a Normal pop. with parameters, mean $= \theta_1$ & variance $= \theta_2$. Find the max. likelihood estimates of these two parameters.

Soln. $X_i \sim N(\theta_1, \theta_2)$ $\qquad X_i \sim N(\mu, \sigma^2)$

The likelihood func. of given sample $(x_1, x_2, \ldots)$ can be written as, pmf,

$$p(x_i, \mu, \sigma^2)\ f(x_i) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-1/2\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

$$p(x_i, \theta_1, \theta_2)\ f(x_i) = \frac{1}{\sqrt{2\pi}\sqrt{\theta_2}} e^{-1/2\left(\frac{x_i-\theta_1}{\sqrt{\theta_2}}\right)^2}$$

The likelihood func is,

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sqrt{\theta_2}} e^{-1/2\left(\frac{x_i-\theta_1}{(\sqrt{\theta_2})}\right)^2}$$

$$\ln L(\theta_1, \theta_2) = \sum_{i=1}^{n} \left[\ln\left(\frac{1}{\sqrt{2\pi}\sqrt{\theta_2}} e^{-1/2\frac{(x-\theta_1)^2}{\theta_2}}\right)\right]$$

$$\ln L(\theta_1, \theta_2) = \sum \left[\ln\frac{1}{\sqrt{\theta_2}} + \ln\frac{1}{\sqrt{2\pi}} + \ln e^{-1/2\frac{(x_i-\theta_1)^2}{\theta_2}}\right]$$

$$= \sum \left[-\ln\sqrt{\theta_2} - \ln\sqrt{2\pi} - \frac{1}{2\theta_2}(x_i - \theta_1)^2 \right.$$

$$= -n\ln\sqrt{\theta_2} - n\ln\sqrt{2\pi} - \frac{1}{2\theta_2}\sum(x_i - \theta_1)^2$$

Taking its partial derivative, (for mean (θ_1))

$$\frac{\partial}{\partial\theta_1}\ln L(\theta_1, \theta_2) = +\frac{1}{\cancel{2}\theta_2}\sum\cancel{2}(x_i - \theta_1)(0+1)$$

$-n\ln\sqrt{\theta_2} = 0$

$-n\ln\sqrt{2\pi} = 0$

For max,

$$\frac{\partial}{\partial\theta_1}\ln L(\theta_1, \theta_2) = 0$$

(equate it to zero)

$$\frac{\sum(x_i - \theta_1)}{\theta_2} = 0$$

$$\sum(x_i - \theta_1) = 0$$

$$\sum x_i - \sum\theta_1 = 0$$

$$\sum x_i - n\theta_1 = 0$$

$$\theta_1 = \frac{\sum x_i}{n}$$

$$\boxed{\hat{\theta}_1 = \bar{x}}$$

$(\mu = \bar{x})$ mean

To ensure it's max., take second partial derivative,

$$\frac{\partial^2}{\partial \theta_1^2} = \frac{\Sigma(-1)}{\theta_2}$$

$$= \frac{-n}{\theta_2} < 0$$

$- n \ln \sqrt{\theta_2} - n \ln \sqrt{2\pi} - \frac{1}{2\theta_2} \Sigma (x_i - \theta_1)^2$

Now taking partial derivative, (for variance (θ_2))

$$\frac{\partial}{\partial \sqrt{\theta_2}} \ln L(\theta_1, \theta_2) = -\frac{n}{\sqrt{\theta_2}} - 0 + \frac{\cancel{2}}{\cancel{2}\theta_2^2} \Sigma (x_i - \theta_1)^2$$

for max, $\frac{\partial}{\partial \sqrt{\theta_2}} \ln L(\theta_1, \theta_2) = 0$ (equate to 0)

$$-\frac{n}{\sqrt{\theta_2}} + \frac{1}{\theta_2^2} \Sigma (x_i - \theta_1)^2 = 0$$

$$+\frac{n}{\sqrt{\theta_2}} = +\frac{1}{\theta_2^2} \Sigma (x_i - \theta_1)^2$$

$$\boxed{\hat{\theta_2} = \frac{\Sigma (x_i - \theta_1)^2}{n}} = \sigma^2 \text{ (variance)}$$

To ensure it's max., take second partial derivative,

$$\frac{\partial^2}{\partial \sqrt{\theta_2}^2} \ln L(\theta_1, \theta_2) = \frac{n}{\theta_2} - \frac{3}{\theta_2^3} \Sigma (x_i - \theta_1)^2$$

$$\left(\theta_2 = \frac{\Sigma (x_i - \theta_1)^2}{n}\right) = \frac{n}{\theta_2} - \frac{3}{\theta_2^{\cancel{3}}} \cdot n \cancel{\theta_2}$$

$$= -\frac{2n}{\theta_2} < 0$$

(2.) Let $x_1, x_2 ---- x_n$ be a random sample from $B(m, \theta)$ distribution, where $\theta \in \Theta = (0,1)$ is 'unknown' and 'm' is a known positive integer. Compute value of θ using the M.L.E.

Soln. M.L.E. for the binomial distribution, pmf,

$$P(x_i | m, \theta) = \binom{m}{x_i} \theta^{x_i} (1-\theta)^{m-x_i},$$

$$x_i \in \{0, 1, 2, ---, m\}$$

The Likelihood function is,

$$L(\theta) = \prod_{i=1}^{n} p(x_i | m, \theta)$$

$$= \prod_{i=1}^{n} \binom{m}{x_i} \theta^{x_i} (1-\theta)^{m-x_i}$$

$$= \prod_{i=1}^{n} \binom{m}{x_i} \prod_{i=1}^{n} \theta^{x_i} \prod_{i=1}^{n} (1-\theta)^{m-x_i}$$

$$= \prod_{i=1}^{n} \binom{m}{x_i} \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{\sum_{i=1}^{n} (m-x_i)}$$

$$= \prod_{i=1}^{n} \binom{m}{x_i} \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{nm - \sum_{i=1}^{n} x_i}$$

The log likelihood is given by,

$$l(\theta) = \log L(\theta)$$

$$= \log\left[\prod_{i=1}^{m} \binom{m}{x_i} \theta^{\sum_{i=1}^{m} x_i} (1-\theta)^{mm-\sum_{i=1}^{m} x_i}\right]$$

$$= \log\left[\prod_{i=1}^{m} \binom{m}{x_i}\right] + \log\left[\theta^{\sum_{i=1}^{m} x_i}\right]$$

$$+ \log\left[(1-\theta)^{mm-\sum_{i=1}^{m} x_i}\right]$$

$$= \log\left[\prod_{i=1}^{m} \binom{m}{x_i}\right] + \log(\theta)\left(\sum_{i=1}^{m} x_i\right) +$$

$$\log(1-\theta)\left(mm - \sum_{i=1}^{m} x_i\right)$$

Differentiating w.r.t θ gives,

$$\frac{\partial l(\theta)}{\partial \theta} = \frac{1}{\theta}\sum_{i=1}^{m} x_i - \frac{1}{1-\theta}\left(mm - \sum_{i=1}^{m} x_i\right)$$

$$\frac{\partial^2 l(\theta)}{\partial \theta^2} = -\frac{1}{\theta^2}\sum_{i=1}^{m} x_i - \frac{1}{(1-\theta)^2}\left(mm - \sum_{i=1}^{m} x_i\right) \quad \text{--- (1)}$$

To find the MLE of θ, $\hat{\theta}$, we solve:
(for max.)

$$\left.\frac{\partial l(\theta)}{\partial \theta}\right|_{\theta=\hat{\theta}} = 0$$

$$\Rightarrow \frac{1}{\hat{\theta}}\sum_{i=1}^{m} x_i - \frac{1}{1-\hat{\theta}}\left(mm - \sum_{i=1}^{m} x_i\right) = 0$$

$$\Rightarrow \frac{1}{1-\hat{\theta}}\left(nm - \sum_{i=1}^{m} x_i\right) = \frac{1}{\hat{\theta}} \sum_{i=1}^{m} x_i$$

$$\Rightarrow \frac{nm - \sum_{i=1}^{m} x_i}{\sum_{i=1}^{m} x_i} = \frac{1-\hat{\theta}}{\hat{\theta}}$$

$$\Rightarrow \frac{nm}{\sum_{i=1}^{m} x_i} - 1 = \frac{1}{\hat{\theta}} - 1$$

$$\Rightarrow \hat{\theta} = \frac{1}{nm} \sum_{i=1}^{m} x_i = \frac{\bar{x}_n}{m}$$

To prove it is maximized,

$$\theta = \hat{\theta} = \frac{\bar{x}_n}{m},$$

we have to show,

$$\left.\frac{\partial^2 l(\theta)}{\partial \theta^2}\right|_{\theta=\hat{\theta}} < 0$$

using eqn. (1),

$$\frac{\partial^2 l(\theta)}{\partial \theta^2} = -\left[\frac{1}{\theta^2} \sum_{i=1}^{m} x_i - \frac{1}{(1-\theta)^2}\left(nm - \sum_{i=1}^{m} x_i\right)\right] < 0$$

$$\forall \theta \in (0,1)$$

So, $\boxed{\text{M.L.E of } \theta \text{ is } \hat{\theta} = \frac{\bar{X}_n}{m}}$

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