3DMHD code

1. The code

The hydrodynamic part of this code has been used extensively to study the descent of cold and dense plumes in a stratified layer (Rast 1998). The magnetic part was added by him between January and February 1998.

1.1. Equations

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\frac{\partial}{\partial x_j}\left(\rho u_j\right) , && (1)\\
\frac{\partial \rho u_i}{\partial t} &= -\frac{\partial}{\partial x_j}\left(\rho u_i u_j\right) - \frac{\partial p}{\partial x_i} + \rho g \delta_{i3} \\
&+ \mu\left[\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{3}\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right)\right] \\
&- 2\rho\varepsilon_{ijk}\Omega_j u_k + \frac{1}{8\pi}\left(2B_j\frac{\partial B_i}{\partial x_j} - \frac{\partial B_j B_j}{\partial x_i}\right) , && (2)\\
\frac{\partial B_i}{\partial t} &= \frac{\partial}{\partial x_j}\left(u_i B_j - B_i u_j\right) + \eta\frac{\partial^2 B_i}{\partial x_j \partial x_j} , && (3)\\
\frac{\partial T}{\partial t} &= -u_i\frac{\partial T}{\partial x_i} - \frac{\mathcal{R}}{c_v\bar{\mu}}T\frac{\partial u_i}{\partial x_i} + \frac{k}{\rho c_v}\frac{\partial^2 T}{\partial x_i \partial x_i} \\
&+ \frac{\eta}{4\pi\rho c_v}\varepsilon_{ijk}\varepsilon_{ilm}\frac{\partial B_k}{\partial x_j}\frac{\partial B_m}{\partial x_l} \\
&+ \frac{\mu}{\rho c_v}\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j} - \frac{2}{3}\left(\frac{\partial u_i}{\partial x_i}\right)^2\right] , && (4)\\
p &= \frac{\mathcal{R}}{\bar{\mu}}\rho T . && (5)
\end{aligned}
$$

In here, z is the depth, $\Omega_i = (\Omega_x, 0, \Omega_z)$ is the angular velocity of the reference frame and the other constants have the following meanings:

$$
\begin{aligned}
\mathcal{R} &= \text{universal gas constant} = 8.31 \times 10^7 \\
\bar{\mu} &= \text{mean molecular weight} \\
&= 1/2 \text{ for fully ionized H} \\
\mu &= \text{viscosity (dynamic)} \\
g &= \text{gravity} \\
\eta &= \text{magnetic diffusivity} \\
k &= \text{thermal conductivity} \\
c_v &= \text{specific heat at constant volume}
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
[\mathcal{R}] &= \text{erg K}^{-1}\text{ g}^{-1} \\
[\bar{\mu}] &= 1 \\
[\mu] &= \text{g cm}^{-1}\text{ sec}^{-1} \\
[g] &= \text{cm sec}^{-2} \\
[\eta] &= \text{cm}^2\text{ sec}^{-1} \\
[k] &= \text{erg cm}^{-1}\text{ sec}^{-1}\text{ K}^{-1} \\
[c_v] &= \text{erg K}^{-1}
\end{aligned}
\tag{7}
$$

1.2. Non-dimensionalization of the equations

The equations are non-dimensionalized using the following constants:

$$
\begin{aligned}
T_0 &= \text{temperature at the top} \\
\rho_0 &= \text{density at the top} \\
d &= \text{initial radius of the magnetic tube} \\
B &= \text{initial maximum magnetic intensity} \\
\Omega &= \text{reference angular velocity}
\end{aligned}
\tag{8}
$$

from which we obtain immediately:

$$
\begin{aligned}
p_0 &= \text{gas pressure at the top} = \mathcal{R}\rho_0 T_0/\bar{\mu} \\
V_0 &= \text{sound speed at const. temp. at the top} \\
&= \sqrt{p_0/\rho_0} = \sqrt{\mathcal{R}\,T_0/\bar{\mu}}
\end{aligned}
\tag{9}
$$

This gives us:

$$
\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_j}\left(\rho u_j\right) \,, \tag{10}
$$

$$
\begin{aligned}
\frac{\partial \rho u_i}{\partial t} &= -\frac{\partial}{\partial x_j}(\rho u_i u_j) - \frac{\partial p}{\partial x_i} + \rho K_0 \delta_{i3} \\
&+ \frac{1}{R_{e0}}\left[\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{3}\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right)\right] \\
&- \frac{\rho}{R_0}\varepsilon_{ijk}\Omega_j u_k + \frac{1}{\beta_0}\left(2B_j\frac{\partial B_i}{\partial x_j} - \frac{\partial B_j B_j}{\partial x_i}\right), \qquad (11)\\
\frac{\partial B_i}{\partial t} &= \frac{\partial}{\partial x_j}(u_i B_j - B_i u_j) + \frac{1}{R_{m0}}\frac{\partial^2 B_i}{\partial x_j \partial x_j}, \qquad (12)\\
\frac{\partial T}{\partial t} &= -u_i\frac{\partial T}{\partial x_i} - \frac{T}{c_v^*}\frac{\partial u_i}{\partial x_i} + \frac{1}{P_r R_{e0}}\frac{1}{\rho c_v^*}\frac{\partial^2 T}{\partial x_i \partial x_i} \\
&+ \frac{2}{\beta_0 R_{m0}}\frac{1}{\rho c_v^*}\varepsilon_{ijk}\varepsilon_{ilm}\frac{\partial B_k}{\partial x_j}\frac{\partial B_m}{\partial x_l} \\
&+ \frac{1}{R_{e0}}\frac{1}{\rho c_v^*}\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j} - \frac{2}{3}\left(\frac{\partial u_i}{\partial x_i}\right)^2\right], \qquad (13)\\
p &= \rho T, \qquad (14)
\end{aligned}
$$

with the non-dimensional constants defined as follows:

$$
\begin{aligned}
K_0 &= dg\bar{\mu} \,/\, (\mathcal{R}T_0) = d/H_0 \\
\beta_0 &= (8\pi\, \rho_0\, \mathcal{R}T_0/\bar{\mu}) \,/\, B^2 \\
R_{e0} &= \rho_0\, d\, \sqrt{\mathcal{R}T_0/\bar{\mu}} \,/\, \mu \\
R_{m0} &= d\sqrt{\mathcal{R}T_0/\bar{\mu}} \,/\, \eta \\
P_r &= \mu\mathcal{R} \,/\, (k\bar{\mu}) \\
R_0 &= \sqrt{\mathcal{R}T_0/\bar{\mu}} \,/\, (2\Omega d) \\
c_v^* &= c_v\bar{\mu} \,/\, \mathcal{R} \ (= 3/2\text{: fully ionized H})
\end{aligned}
\qquad (15)
$$

1.3. Grid stretching

The actual calculation is not performed in the physical (x, y, z) space. Instead, the equations are transformed into a new basis $(X(x), Y(y), Z(z))$, where the $X_i(x_i)$ are defined by:

$$
X_i(x_i) = \frac{\operatorname{atan} a_i - \operatorname{atan}\left(\dfrac{b_i - a_i}{x_{imax}}x_i + a_i\right)}{\operatorname{atan} a_i - \operatorname{atan} b_i}, \qquad (16)
$$

$$
\begin{aligned}
x_i(X_i) = \{&\tan\left[(\operatorname{atan} b_i - \operatorname{atan} a_i)X_i + \operatorname{atan} a_i\right] - a_i\} \\
&\cdot \frac{x_{imax}}{b_i - a_i}, \qquad (17)
\end{aligned}
$$

where a_i, b_i and x_{imax} are constants. Typical values for the a_i's and b_i's are: $a_i = -10^{-9}$, $b_i = 10^{-9}$ to obtain a regularly spaced grid; $a_i = -10^{-9}$, $b_i = 1$, to get a grid which becomes finer towards the top. To concentrate the points in the center of the box in the horizontal direction we would have $a_1 = -1$ and $b_1 = 1$.

Thus, the equations which we actually discretize are equations (10–14) with $\partial/\partial x_i$, $\partial^2/\partial x_i^2$ and $\partial^2/\partial x_i \partial x_j$ replaced by:

$$\frac{\partial}{\partial x_i} = c_i(X_i)\,\frac{\partial}{\partial X_i}\,, \tag{18}$$

$$\frac{\partial^2}{\partial x_i^2} = c_i(X_i)\,c_i{}'(X_i)\,\frac{\partial}{\partial X_i} + c_i^2(X_i)\frac{\partial^2}{\partial X_i^2}\,, \tag{19}$$

$$\frac{\partial^2}{\partial x_i \partial x_j} = c_i(X_i)\,c_j(X_j)\frac{\partial^2}{\partial X_i \partial X_j}\,,\ (i \neq j)\,, \tag{20}$$

respectively, where $c_i(X_i)$ is:

$$c_i(X_i) = \left(\frac{dx_i}{dX_i}\right)^{-1} = \frac{dX_i}{dx_i} \tag{21}$$

$$= \frac{b_i - a_i}{x_{imax}}\;\frac{\cos^2\left[\left(\operatorname{atan} b_i - \operatorname{atan} a_i\right)\,X_i + \operatorname{atan} a_i\right]}{\operatorname{atan} b_i - \operatorname{atan} a_i}\,,$$

$$= \frac{b_i - a_i}{x_{imax}}\;\frac{1}{\operatorname{atan} b_i - \operatorname{atan} a_i}\;\frac{1}{1 + \left(\frac{b_i - a_i}{x_{imax}} x_i\left(X_i\right) + a_i\right)^2}\,,$$

and $c_i'(X_i)$ is its total derivative which gives:

$$c_i(X_i) c_i'(X_i) = \frac{d^2 X_i}{dx_i^2} = -c_i^3\frac{d^2 x_i}{dX_i^2} \tag{22}$$

$$= -2\left(\frac{b_i - a_i}{x_{imax}}\right)^2\left(a_i + \frac{b_i - a_i}{x_{imax}} x_i(X_i)\right)$$

$$\frac{1}{\operatorname{atan} b_i - \operatorname{atan} a_i}\,\frac{1}{\left[1 + \left(a_i + \frac{b_i - a_i}{x_{imax}} x_i(X_i)\right)^2\right]^2}$$

The vectors $c_i(X_i)$ and $c_i'(X_i)$ are calculated once at the beginning of the calculation only. The advantage of using the transformation $X_i(x_i)$ is that this relation is analytic and therefore no error is done by stretching the grid. Moreover, the grid in which the code works, (X_i), is spaced regularly so that the coding of the finite differences is simplified. Note that the actual truncation error of the stretched grid scheme is actually reduced by using a numerical evaluation of the grid Jacobian (Rast 2001).

1.3.1. Different Grid stretching definitions

The grid streching functions $X_i(x_i)$, dX_i/dx_i and d^2X_i/dx_i^2 are defined in the routine MKGRID which is called from XJACOBI, YJACOBI and ZJACOBI (these JACOBI routines actually create the grid but they do that by calling MKGRID). Currently, there are two different kind of grid streching which can be selected. The selection between the two is done by setting NGRID=1 or NGRID=2 in the parameter file (3dmhdparam.f).

NGRID=1: the original grid stretching is used, i.e. the one defined in (16) through (22).

NGRID=2: a function containing two ArcTan is used for the stretching. It produces a dip in grid spacing:

$$
\begin{aligned}
c_i(X_i) &= \left(\frac{dx_i}{dX_i}\right)^{-1} \\
&= k\left[1+\frac{h}{\pi}\left(\operatorname{atan}\left(c\left(x-a-b\right)\right)+\operatorname{atan}\left(c\left(a-x-b\right)\right)\right)\right]
\end{aligned}
\tag{23}
$$

with:

$$
\begin{aligned}
h &= \frac{\pi\left(1-d\right)}{d\ \operatorname{atan}\left(\left(a-b\right)c\right)+2\operatorname{atan}\left(bc\right)-d\ \operatorname{atan}\left(\left(a+b\right)c\right)} \\
k &= 2\pi c x_{imax}\left\{2\pi c+2ch\left(\operatorname{atan}\left(\left(a-b\right)c\right)\left(a-b\right)\right.\right. \\
&\quad -\operatorname{atan}\left(\left(a+b\right)c\right)\left(a+b\right)+\operatorname{atan}((a-b-1)c) \\
&\quad +\operatorname{atan}((a+b-1)c)(a+b-1) \\
&\quad \left.+\operatorname{atan}((1-a+b)c)(a-b)\right) \\
&\quad -h\left[\log(1+(a-b)^2c^2)+\log(1+(a+b)^2c^2)\right. \\
&\quad \left.\left.-\ \log(1+(a+b-1)^2c^2)+\log(1+(1-a+b)^2c^2)\right]\right\}^{-1}
\end{aligned}
$$

where the parameters a, b and d determine the position, the half width and the depth of the dip, respectively. c controls the sharpness of the walls of the dip.

To add a new grid (e.g. NGRID=3) it is very easy: just edit MKGRID.

1.4. Coding

1.4.1. Spatial discretization

For the spatial discretization central finite differences of second order are used. The right hand-sides of equations (10-14) are fully expanded before the finite differences are calculated.

1.4.2. Time integration

Until mid April 1998 the time integration has been achieved using a fully explicit second-order Adams-Bashworth integration scheme. This scheme however is unconditionally unstable and forces us to use a very small safety factor (SFF = 0.125) in the CFL condition.

A new integration scheme has been implemented. It is a 3rd order Runge-Kutta. Representing with U the variables (the ones which appears on the left hand-side of (10–14)) and with N the fluxes(rhs.), the scheme looks like this:

$$U' = U^n + \Delta t\,\gamma_1\,N^n \tag{24}$$

$$U'' = U' + \Delta t\,\gamma_2\,N' + \Delta t\,\xi_1\,N^n \tag{25}$$

$$U^{n+1} = U'' + \Delta t\,\gamma_3\,N'' + \Delta t\,\xi_2\,N' \tag{26}$$

with:

$$\gamma_1 = \frac{8}{15}\,, \quad \gamma_2 = \frac{5}{12}\,, \quad \gamma_3 = \frac{3}{4}\,,$$

$$\xi_1 = \frac{-17}{60}\,, \quad \xi_2 = \frac{-5}{12}\,. \tag{27}$$

Notice that $\gamma_1 + \gamma_2 + \gamma_3 + \xi_1 + \xi_2 = 1$ so that at the end of these 3 steps we have advanced one Δt. The time step is obtained as follows:

$$\Delta t = \mathrm{SFF}\ \min(\Delta t_{adv}, \Delta t_{temp}, \Delta t_{mag}, \Delta t_{visc}) \tag{28}$$

where:

$$\begin{aligned}
\Delta t_{adv} &= \frac{\Delta l}{v_{max}} \\
\Delta t_{temp} &= \Delta l^2\ R_{e0}\,P_r\,c_v^*\,\rho_{min} \\
\Delta t_{mag} &= \Delta l^2\ R_{m0} \\
\Delta t_{visc} &= \Delta l^2\ R_{e0}\,\rho_{min} \\
v_{max} &= \sqrt{\max\left(\sqrt{u^2} + \sqrt{\gamma T + \frac{2B^2}{\beta_0 \rho}}\right)^2}
\end{aligned}$$

and $\Delta l = \min(\Delta x, \Delta y, \Delta z)$

2. Simulation of a rising flux tube

We want to simulate the buoyant rise of a magnetic flux tube through a quiescent stratified layer.

2.1. Input parameters

In this section we list/define the full set of parameters necessary to define completely the problem.

2.1.1. *The grid*

The following 12 parameters are necessary:

$$n_x \; , \quad x_{max} \; , \quad a_x \; , \quad b_x \; , \tag{29}$$

$$n_y \; , \quad y_{max} \; , \quad a_y \; , \quad b_y \; , \tag{30}$$

$$n_z \; , \quad z_{max} \; , \quad a_z \; , \quad b_z \; . \tag{31}$$

The n_i's represent the number of points (without the ghostpoints) in the i-direction.

2.1.2. *The background stratification*

For the background atmosphere we use a polytrope:

$$T = (1 + \theta z) \; , \quad \text{and} \quad \rho = T^m \; . \tag{32}$$

The temperature gradient θ and the polytropic index m are related the one with the other by the equation of hydrostatic equilibrium which implies that

$$K_0 = (m + 1)\theta \; . \tag{33}$$

Note that K_0 is a scaled measure of gravity.

The initial atmosphere can also be specified as two polytropes of different stability. This is done by taking the polytropic index to be a function of depth:

$$m(z) = m_t + \frac{(m_b - m_t)}{2} \left(1 + \tanh\left(\frac{z - z_p}{\sigma} \right) \right) \tag{34}$$

where m_t and m_b are the polytropic indices at the top and bottom of the layer, z_p is the transition depth, and σ measures the width of the transition. In the code m_b, z_p, and σ are specified as parameters and m_t is determined from θ and K_0. The temperature gradient as a function of depth is then

$$\Theta(z) = \frac{K_0}{m(z) + 1} \tag{35}$$

with the temperature given by

$$T(z) = 1 + F(z) \tag{36}$$

where $F(z)$ is determined by integrating the equation

$$\frac{\partial F}{\partial z} = \Theta(z) \ . \tag{37}$$

The hydrostatic pressure is given by integrating

$$\frac{\partial \ln P}{\partial z} = \frac{K_0}{T(z)} \tag{38}$$

simultaneously with Equations (36) and (37) for a given $m(z)$. In the code Equation (37) is actually written as

$$\frac{\partial F}{\partial z} = \frac{\theta}{\kappa(z)} \tag{39}$$

where

$$\kappa(z) = \frac{m(z)+1}{m_t+1} \quad . \tag{40}$$

2.1.3. Alternative characterization

Instead of using m, θ and the initial depth of the tube, h, to characterize the atmosphere that the tube has above itself at the beginning of the calculation, we will use the superadiabatic index,

$$\delta = \nabla - \nabla_{ad} \ , \quad \text{where} \quad \nabla = \frac{d \ln T}{d \ln p} \ , \tag{41}$$

the number of pressure scale heights between the top of the box and the initial height of the tube, $n = \ln p_h$ (here $p_h = p(h)$), and the initial dimension of the tube as compared to the local pressure scale height, $K = d/H$. m, θ and h are then easily obtained from

$$m = \frac{1-\gamma\,\delta}{\gamma - (1-\gamma\,\delta)} \tag{42}$$

$$h = \frac{m+1}{K}\,\frac{e^{n/(m+1)}-1}{e^{n/(m+1)}} \tag{43}$$

$$\theta = \frac{K/(m+1)}{1-h\,K/(m+1)} \tag{44}$$

2.1.4. The initial magnetic tube

Centered at depth $z = h$ we set an axisymmetric flux tube defined by:

$$B_y = \frac{e^{-r^2} - c}{1 - c} , \tag{45}$$

$$B_\phi = B_y \, c_{mt} \, \frac{a \, r^3}{a \, r^3 + 1} , \tag{46}$$

with $r \leq r_{max}$ and $c = \exp(-r_{max}^2)$. The gas pressure excess, $\Delta p = p_i - p_h$, in the interior is then calculated in order to satisfy:

$$-\nabla \left(\Delta p + \frac{B^2}{\beta_0} \right) + \frac{2}{\beta_0} \left(\vec{B} \cdot \nabla \right) \vec{B} = 0 \, . \tag{47}$$

An important structural feature of the initial tube is its degree of twist which can be parameterized with Ψ_{mt} defined as the pitch angle measured where B_ϕ takes its maximum. Ψ_{mt} is uniquely related to c_{mt} by a transcendental equation.

2.1.5. The equations

The parameters, (15), which appear in the equations integrated by the code are not the best suited to characterize this problem. It is convenient to use the following parameters:

$$\begin{aligned}
K &= d g \bar{\mu} \, / \, (\mathcal{R} T_h) = d / H_h \\
\beta &= (8\pi \, \rho_h \, \mathcal{R} T_h / \bar{\mu}) \, / \, B^2 \\
R_e &= \rho_h \, d \, V_{\rm rise} \, / \, \mu \\
R_m &= d \, V_{\rm rise} \, / \, \eta \\
P_r &= \mu \, \mathcal{R} \, / \, (k \bar{\mu}) \\
R &= V_{\rm rise} \, / \, (2 \Omega d) \\
c_v^* &= c_v \, \bar{\mu} \, / \, \mathcal{R} \; (= 3/2\text{: fully ionized H})
\end{aligned} \tag{48}$$

where the subscript h means that T_h, H_h, ρ_h are calculated at the depth $z = h$ corresponding to the initial position of the center of the tube. Notice that here [and only here, i.e. eqns (48) and (49)] T_h, H_h, ρ_h and $V_{\rm rise}$ have *physical dimensions.* $V_{\rm rise}$ is the characteristic speed of rise and is given by (up to a factor unity):

$$V_{\rm rise} \cong \left(\frac{d}{H_h} \, \frac{\Delta\rho}{\rho_h} \, \frac{\pi}{C_D} \right)^{1/2} \; \left(\frac{\mathcal{R} T_h}{\bar{\mu}} \right)^{1/2} \tag{49}$$

Transforming T_h, H_h, ρ_h and $V_{\rm rise}$ back into *non*-dimensional variables the parameters used by the code, (15), are easily obtained from (48):

$$\begin{array}{lcl} K_0 & = & K \;\; T_h \\ \beta_0 & = & \beta \;\; (\rho_h T_h)^{-1} \\ R_{e0} & = & R_e \;\; (\rho_h V_{\rm rise})^{-1} \\ R_{m0} & = & R_m \;\; (V_{\rm rise})^{-1} \\ R_0 & = & R \;\; (V_{\rm rise})^{-1} \\ \text{with:} & & V_{\rm rise} = \left(\frac{K}{\beta}\right)^{1/2} \left(\frac{\pi}{\gamma C_D}\right)^{1/2} T_h^{1/2} \end{array} \tag{50}$$

To obtain the non-dimensional speed of rise it has been assumed that $\Delta\rho/\rho \cong 1/(\gamma\beta)$ (this is valid up to order $1/\beta$ for tubes initially in isentropic equilibrium with the external plasma).

2.1.6. Summary of the input parameters

$$
\begin{array}{cccl}
\delta & m & = & \dfrac{1-\gamma\,\delta}{\gamma-(1-\gamma\,\delta)} \\
n & h & = & \dfrac{m+1}{K}\,\dfrac{e^{n/(m+1)}-1}{e^{n/(m+1)}} \\
K & \theta & = & \dfrac{K/(m+1)}{1-h\,K/(m+1)} \\
 & T_h & = & (1+\theta\,h) \\
 & \rho_h & = & T_h^m \\
 & K_0 & = & K\,T_h \\
 & & & \\
\beta & \beta_0 & = & \beta\,(\rho_h\,T_h)^{-1} \\
\Psi_{mt} & c_{mt} & & \text{have to solve:} \\
a & a & & \text{(typical value is 0.9)} \\
r_{max} & r_{max} & & \text{(typical value is 2.5)} \\
C_D & C_D & & \text{(typical value is 1.0)} \\
 & V_{\rm rise} & = & \left(\dfrac{K}{\beta}\right)^{1/2}\;\left(\dfrac{\pi}{\gamma\,C_D}\right)^{1/2}\;T_h^{1/2} \\
\Longrightarrow & & & \\
R_e & R_{e0} & = & R_e\;\;(\rho V_{\rm rise})^{-1} \\
R_m & R_{m0} & = & R_m\;\;(V_{\rm rise})^{-1} \\
R & R_0 & = & R\;\;(V_{\rm rise})^{-1} \\
P_r & P_r & & \\
c_v^* & c_v^* & & \\
 & & & \\
n_i & n_i & & (i=1,2,3) \\
a_i & a_i & & (i=1,2,3) \\
b_i & b_i & & (i=1,2,3) \\
 & z_{max} & = & h+2.0\,r_{max}\text{ (or more)} \\
 & x_{max} & = & 2\,z_{max} \\
 & y_{max} & = & x_{max}/(n_x-1)
\end{array}
$$

2.2. Physical restrictions

2.2.1. Radius versus the pressure scale height: K

When the radius of the tube is much smaller than the external pressure scale height the magnetic tube feels the external medium as a plasma at almost constant pressure. In this case, although the motion of the tube is due to the buoyancy, the expansion of the magnetic rope remains very small and its deformation is essentially due to the resistance of the external medium to the advance of the tube. This problem has been studied in (Emonet & Moreno-Insertis 1998). Here we want to study the effect of the stratification, or in other words, what happens when the external pressure scale height becomes of the same order or even smaller than the radius of the tube. Thus we want $K \cong 10^{-1}$ at the beginning and several pressure scale heights in the box so that at the end of the rise the radius of the tube be bigger than H.

2.2.2. Plasma β

The plasma beta should be the highest possible. This condition however is likely to be very limited by the number of time steps that we can calculate as $V_{\rm rise} \sim \beta^{-1/2}$.

2.2.3. Diffusion times versus rise time

The time, τ_{mag}, necessary to destroy magnetic field variations over a length-scale d (the radius of the tube) and the corresponding time for the momentum and the temperature are:

$$\tau_{mag}/\tau_{rise} = R_m \frac{1}{h} \gg 1 \ , \tag{51}$$

$$\tau_{visc}/\tau_{rise} = R_e \frac{1}{h} \gg 1 \ , \tag{52}$$

$$\tau_{temp}/\tau_{rise} = R_e \, P_r \, c_v^* \frac{1}{h} \gg 1 \ . \tag{53}$$

We want that the rise time be much smaller than the diffusive time, i.e. we want these values to be much bigger than one. But obviously we are limited by the stability of the code (see below).

Another way to put an under limit to the diffusivities is by requiring that the thickness of the magnetic, viscous and thermal boundary layers be much smaller than the radius of the tube which is equal to 1:

$$\delta_{mag} = {R_m}^{-1/2} \ll 1 \tag{54}$$
$$\delta_{visc} = {R_e}^{-1/2} \ll 1 \tag{55}$$
$$\delta_{temp} = (R_e\, P_r\, c_v^*)^{-1/2} \ll 1 \tag{56}$$

Again, the Reynolds numbers we can afford will put an under limit to the δ's.

2.3. Numerical restrictions

2.3.1. Resolve the stratification

We use central finite differences of second order for the spatial derivatives. For the equation of hydrostatic equilibrium this gives:

$$c_z(Z_i)\,\frac{p_{i+1}-p_{i-1}}{2\,\Delta Z} + K_0\rho_i = \underbrace{c_z(Z_i)\left(\frac{\partial p}{\partial Z}\right)_i + K_0\rho_i}_{=0}$$
$$+ \; c_z(Z_i)\left(\frac{\partial^3 p}{\partial Z^3}\right)_i \frac{\Delta Z^3}{6} \; + \; \ldots \tag{57}$$

The discretization error made in the hydrostatic equilibrium is therefore of order:

$$\begin{aligned} &\left|\left(\frac{\partial^3 p}{\partial Z^3}\right)_i \Big/ \left(\frac{\partial p}{\partial Z}\right)_i\right| \frac{\Delta Z^3}{6} \\ &\quad \cong \left|\left(\frac{\partial^3 p}{\partial z^3}\right)_i \Big/ \left(\frac{\partial p}{\partial z}\right)_i\right| \frac{\Delta z_i^3}{6} \\ &\quad = \frac{(m^2-m)\theta^2}{(1+\theta\, z_i)^2}\frac{\Delta z_i^2}{3}\,. \end{aligned} \tag{58}$$

This error should not be much much bigger than the roundoff error.

2.3.2. "grid scaled" Reynolds numbers

In 1D it can be shown that for the 2nd-order central finite differences to be a stable scheme to solve the advection-diffusion equation,

$$\frac{\partial u}{\partial t} = -V\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right)$$

one must require that: $V\ \Delta x/K < 2$. Applying a similar criteria for each one of our diffusive terms we obtain:

$$\frac{\Delta z_i\, V_{\rm rise}}{\eta} < 2 \tag{59}$$

$$\frac{\Delta z_i\, V_{\rm rise}\,\rho}{\mu} < 2 \tag{60}$$

$$\frac{\Delta z_i\, V_{\rm rise}\,\rho}{\mu} c_v^* P_r < 2 \tag{61}$$

which implies:

$$R_m\, \Delta z_i < 2 \tag{62}$$

$$R_e \left(\frac{1+\theta\, z_i}{1+\theta\, h}\right)^m \Delta z_i < 2 \tag{63}$$

$$R_e\, P_r\, c_v^* \left(\frac{1+\theta\, z_i}{1+\theta\, h}\right)^m \Delta z_i < 2 \tag{64}$$

2.3.3. Rise time

Because β is much bigger than 1 the speed of rise is much smaller than the sound speed. Also because we want to simulate high Reynolds number regimes the diffusion time are much longer than the rise time. Altogether our time step is generally dominated by the sound speed which takes its maximum value at the bottom of the box. If the stratification is very strong the viscous and thermal time scale can become dominant at the bottom of the box because of the scaling of the Reynolds number with the density. However, in all the cases we have considered we were able to push the limits in (62)–(64) up to 4 in the lower part of the box without having problem: this must be due to the fact that the speed of rise is much smaller than $V_{\rm rise}$ in this part of the box.

In any case, $n_t = \tau_{rise}/\Delta t$ should not be bigger than 100'000 or 200'000.

3. Simulations

Case 2:

δ	m	0	1.5
n	h	5	21.61
K	θ	0.1	0.295
	K_0		0.739
β	β_0	100	0.674
Ψ_{mt}	c_{mt}	16.5	0.7
a	a	0.9	0.9
r_{max}	r_{max}	2.5	2.5
C_D	C_D	1.0	1.0
	$V_{\rm rise}$		0.118
R_e	R_{e0}	150	83.28
R_m	R_{m0}	150	1271
R	R_0	0	0
P_r	P_r	1	1
n_x	n_z	1024	1024
a_x	b_x	-4	4
a_z	b_z	$-1e^{-9}$	1.2
x_{max}	z_{max}	55.73	27.87

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