population. If this assumption is not valid, then systematic sampling will be less precise than simple random sampling. In conducting systematic sampling, it is also essential that the researcher does not introduce bias into the sample by selecting an inappropriate sampling interval. For instance, when conducting a sample of financial records, or other items that follow a calendar schedule, the researcher would not want to select "7" as the sampling interval because the sample would then be comprised of observations that were all on the same day of the week. Day-of-the-week influences may cause contamination of the sample, giving the researcher biased results.

(*v*) Multi-Stage Sampling: Multi-stage sampling is like cluster sampling, but involves selecting a sample within each chosen cluster, rather than including all units in the cluster. Thus, multi-stage sampling involves selecting a sample in at least two stages. In the first stage, large groups or clusters are selected. These clusters are designed to contain more population units than are required for the final sample.

In the second stage, population units are chosen from selected clusters to derive a final sample. If more than two stages are used, the process of choosing population units within clusters continues until the final sample is achieved.

An example of multi-stage sampling is where, firstly, electoral sub-divisions (clusters) are sampled from a city or state. Secondly, blocks of houses are selected from within the electoral sub-divisions and, thirdly, individual houses are selected from within the selected blocks of houses.

The advantages of multi-stage sampling are convenience, economy and efficiency. Multi-stage sampling does not require a complete list of members in the target population, which greatly reduces sample preparation cost. The list of members is required only for those clusters used in the final stage. The main disadvantage of multi-stage sampling is the same as for cluster sampling: lower accuracy due to higher sampling error.

5.6 PARAMETERS OF STATISTICS

The statistical constants of the population such as mean, the variance etc. are known as the parameters. The statistical concepts of the sample from the members of the sample to estimate the parameters of the population from which the sample has been drawn is known as *statistic*.

Population mean and variance are denoted by μ and σ^2, while those of the samples are given by $\bar{x}$, s^2.

5.7 STANDARD ERROR

The standard deviation of the sampling distribution of a statistic is known as the **standard error (S.E.)**. It plays an important role in the theory of large samples and it forms a basis of the testing of hypothesis. If t is any statistic, for large sample. $z = \dfrac{t - \mathrm{E}(t)}{\mathrm{S.E.}(t)}$ is normally distributed with mean 0 and variance unity.

For large sample, the standard errors of some of the well known statistic are listed below:

n—sample size; σ^2—population variance; s^2—sample variance; p—population proportion; $\mathrm{Q} = 1 - p$; n_1, n_2—are sizes of two independent random samples.

S. No.	Statistic	Standard error
1.	$\bar{x}$	$\sigma/\sqrt{n}$
2.	s	$\sqrt{\sigma^2/2n}$
3.	Difference of two sample means $\bar{x}_1 - \bar{x}_2$	$\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$
4.	Difference of two sample standard deviation $s_1 - s_2$	$\sqrt{\frac{\sigma_1^2}{2n_1} + \frac{\sigma_2^2}{2n_2}}$
5.	Difference of two sample proportions $p_1 - p_2$	$\sqrt{\frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}}$
6.	Observed sample proportion p	$\sqrt{PQ/n}$

5.8 TEST OF SIGNIFICANCE

An important aspect of the sampling theory is to study the test of significance which will enable us to decide, on the basis of the results of the sample, whether

(*i*) *the deviation between the observed sample statistic and the hypothetical parameter value or*

(*ii*) *the deviation between two sample statistics is significant or might be attributed due to chance or the fluctuations of the sampling.*

5.9 TESTING OF STATISTICAL HYPOTHESIS

Step 1. Null hypothesis

For applying the tests of significance, we first set up a hypothesis which is a definite statement about the population parameter called Null Hypothesis. It is denoted by H_0.

Null hypothesis is the hypothesis which is tested for possible rejection under the assumption that it is true. First, we set up H_0 in clear terms.

Step 2. Alternative hypothesis

Any hypothesis which is complementary to the null hypothesis (H_0) is called an alternative hypothesis. It is denoted by H_1.

For example, if we want to test the null hypothesis that the population has a specified mean μ_0 then we have

$$H_0 : \mu = \mu_0$$

then the alternative hypothesis will be

(*i*) $H_1 : \mu \neq \mu_0$ (Two tailed alternative hypothesis)

(*ii*) $H_1 : \mu > \mu_0$ (right tailed alternative hypothesis (or) single tailed)

(*iii*) $H_1 : \mu < \mu_0$ (left tailed alternative hypothesis (or) single tailed)

Hence alternative hypothesis helps to know whether the test is two tailed test or one tailed test. Therefore, we set up H_1 for this decision.

Step 3. Level of significance

The probability of the value of the variate falling in the critical region is known as level of significance. A region corresponding to a statistic t in the sample space S which amounts to rejection of the null hypothesis H_0 is called as **critical region** or region of rejection while which amounts to acceptance of H_0 is called **acceptance region.** The probability α that a random value of the statistic t belongs to the critical region is known as the **level of significance.**

$$P(t \in w/H_0) = \alpha$$

i.e., the level of significance is the size of the type I error (refer art. 5.7) or the maximum producer's risk.

We select the appropriate level of significance in advance depending on the reliability of the estimates.

Step 4. Test statistic (or test criterion): We compute the test statistic z under the null hypothesis. For larger samples corresponding to the statistic t, the variable $z = \dfrac{t - E(t)}{S.E.(t)}$ is normally distributed with mean 0 and variance 1. The value of z given above under the null hypothesis is known as **test statistic.**

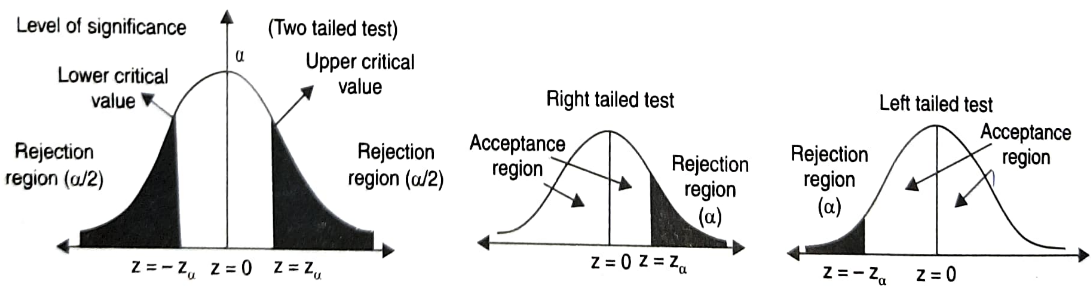
Step 5. Conclusion: We compare the computed value of z with the critical value z_α at level of significance (α). The critical value of z_α of the test statistic at level of significance α for a two tailed test is given by

$$p(|z| > z_\alpha) = \alpha \quad \ldots(1)$$

i.e., z_α is the value of z so that the total area of the critical region on both tails is α. Since the normal curve is symmetrical, from equation (1), we get

$$p(z > z_\alpha) + p(z < -z_\alpha) = \alpha; \; i.e. \; 2p(z > z_\alpha) = \alpha; \; i.e., \; p(z > z_\alpha) = \alpha/2$$

i.e., the area of each tail is $\alpha/2$.



The critical value z_α is that value such that the area to the right of z_α is $\alpha/2$ and the area to the left of $-z_\alpha$ is $\alpha/2$.

In the case of one tailed test,

$p(z > z_\alpha) = \alpha$ if it is right tailed; $p(z < -z_\alpha) = \alpha$ if it is left tailed.

The critical value of z for a single tailed test (right or left) at level of significance α is same as the critical value of z for two tailed test at level of significance 2α.

Using the equation, also using the normal tables, the critical value of z at different levels of significance (α) for both single tailed and two tailed test are calculated and listed below. The equations are

$$p(|z| > z_\alpha) = \alpha; \; p(z > z_\alpha) = \alpha; \; p(z < -z_\alpha) = \alpha$$

	Level of significance		
	1% (0.01)	5% (0.05)	10% (0.1)
Two tailed test	$\lvert z_\alpha \rvert = 2.58$	$\lvert z_\alpha \rvert = 1.96$	$\lvert z_\alpha \rvert = 1.645$
Right tailed test	$z_\alpha = 2.33$	$z_\alpha = 1.645$	$z_\alpha = 1.28$
Left tailed test	$z_\alpha = -2.33$	$z_\alpha = -1.645$	$z_\alpha = -1.28$

If $|z| > z_\alpha$, we reject H_0 and conclude that there is significant difference. If $|z| < z_\alpha$, we accept H_0 and conclude that there is no significant difference.

5.10 ERRORS IN SAMPLING

The main aim of the sampling theory is to draw a valid conclusion about the population parameters on the basis of the sample results. In doing this we may commit the following two types of errors:

Type I Error

When H_0 is true, we may reject it. P(Reject H_0 when it is true) = P(Reject H_0/H_0) = α. α is called the size of the type I error, also referred to as **producer's risk.**

Type II Error: When H_0 is wrong, we may accept it. P(Accept H_0 when it is wrong) = P(Accept H_0/H_1) = β. β is called the size of the type II error, also referred to as **consumer's risk.**

Note. The values of the test statistic which separates the critical region and acceptance region are called the **critical values** or **significant values.** This value is dependent on (*i*) the level of significance used and (*ii*) the alternative hypothesis, whether it is one tailed or two tailed.

5.11 TEST OF SIGNIFICANCE FOR LARGE SAMPLES

If the sample size $n > 30$, the sample is taken as large sample. For such sample we apply distributions, as Binomial, Poisson, which are closely approximated by normal distributions assuming the population as normal.

Under large sample test, the following are the important tests to test the significance:

1. *Testing of significance for single proportion.*
2. *Testing of significance for difference of proportions.*
3. *Testing of significance for single mean.*
4. *Testing of significance for difference of means.*
5. *Testing of significance for difference of standard deviations.*

5.11.1 Testing of Significance for Single Proportion

This test is used to find the significant difference between proportion of the sample and the population. Let X be the number of successes in n independent trials with constant probability P of success for each trial.

$$E(X) = nP;\ V(X) = nPQ;\ Q = 1 - P = \text{Probability of failure.}$$

Let $p = X/n$ called the observed proportion of success.

$$E(p) = E(X/n) = \frac{1}{n}E(X) = \frac{nP}{n} = P$$

$$V(p) = V(X/n) = \frac{1}{n^2}V(X) = \frac{nPQ}{n^2} = PQ/n$$

$$\text{S.E.}(p) = \sqrt{\frac{\text{PQ}}{n}} \;;\; z = \frac{p - \text{E}(p)}{\text{S.E.}(p)} = \frac{p - \text{P}}{\sqrt{\text{PQ}/n}} \sim \text{N}(0, 1)$$

This z is called test statistic which is used to test the significant difference of sample and population proportion.

Note. 1. The probable limits for the observed proportion of successes are $\text{P} \pm 3\sqrt{\text{PQ}/n}$.

2. If p is not known, the probable limits for the proportion in the population are $p \pm z_\alpha \sqrt{pq/n}$, $q = 1 - p$, where sample proportion, p is taken as an estimate of p and z_α is the significant value of z at level of significance α.

ILLUSTRATIVE EXAMPLES

Example 1. *A coin was tossed 400 times and the head turned up 216 times. Test the hypothesis that the coin is unbiased.*

Sol. Null hypothesis:

H_0: The coin is unbiased *i.e.,* $\text{P} = 0.5$

Alternative hypothesis:

H_1: The coin is biased *i.e.,* $\text{P} \neq 0.5$

Hence we use **two tailed test.**

Here, $n = 400$, X = no. of success = 216

$$\therefore \quad p = \text{proportion of success in the sample} = \frac{\text{X}}{n} = \frac{216}{400} = 0.54$$

P = population proportion = 0.5, $\quad \text{Q} = 1 - \text{P} = 0.5$

Test Statistic:

Under H_0, test statistic $z = \dfrac{p - \text{P}}{\sqrt{\text{PQ}/n}}$

$$|z| = \left| \frac{0.54 - 0.5}{\sqrt{\dfrac{0.5 \times 0.5}{400}}} \right| = 1.6$$

Conclusion:

Since $|z| = 1.6 < 1.96$ *i.e.,* $|z| < z_\alpha$ where z_α is the significant value of z at 5% level of significance.

Hence we accept H_0 and conclude that the coin is unbiased.

Example 2. *A machine is producing bolts of which a certain fraction is defective. A random sample of 400 is taken from a large batch and is found to contain 30 defective bolts. Does this indicate that the proportion of defectives is larger than that claimed by the manufacturer where the manufacturer claims that only 5% of his product are defective. Find 95% confidence limits of the proportion of defective bolts in batch.*

Sol. Null hypothesis H_0: The manufacturer claim is accepted *i.e.*, $P = \frac{5}{100} = 0.05$

$$Q = 1 - P = 1 - 0.05 = 0.95$$

Alternative hypothesis: $P > 0.05$

Hence we use **Right tailed test.**

p = observed proportion of sample $= \frac{30}{400} = 0.075$

Test statistic

Under H_0, the test statistic $\quad z = \frac{p - P}{\sqrt{PQ/n}} \quad \therefore \quad z = \frac{0.075 - 0.05}{\sqrt{\frac{0.05 \times 0.95}{400}}} = 2.2941.$

Conclusion: The tabulated value of z at 5% level of significance for right tailed test is $z_\alpha = 1.645$. Since $|z| = 2.2941 > 1.645$, H_0 is rejected at 5% level of significance. *i.e.*, the proportion of defective is larger than the manufacturer claim.

To find 95% confidence limits of the proportion.

It is given by $P \pm z_\alpha \sqrt{PQ/n}$

i.e., $$0.05 \pm 1.96 \sqrt{\frac{0.05 \times 0.95}{400}} = 0.05 \pm 0.02135 = 0.07136, 0.02865$$

Hence 95% confidence limits for the proportion of defective bolts are (0.07136, 0.02865).

TEST YOUR KNOWLEDGE

1. In a hospital 475 female and 525 male babies were born in a week. Do these figures confirm the hypothesis that males and females are born in equal number?
2. In a city a sample of 1000 people were taken and out of them 540 are vegetarian and the rest are non-vegetarian. Can we say that the both habits of eating (vegetarian or non-vegetarian) are equally popular in the city at (*i*) 1% level of significance (*ii*) 5% level of significance?
3. 325 men out of 600 men chosen from a big city were found to be smokers. Does this information support the conclusion that the majority of men in the city are smokers?

Answers

1. H_0 accepted at 5% level
2. H_0 rejected at 5% level, accepted at 1% level
3. H_0 rejected at 5% level.

5.11.2 Testing of Significance for Difference of Proportions

Consider two samples X_1 and X_2 of sizes n_1 and n_2 respectively taken from two different populations. To test the significance of the difference between the sample proportions p_1 and p_2, the test statistic under the null hypothesis H_0, that there is no significant difference between the two sample proportion, is

$$z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}, \quad \text{where} \quad P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} \quad \text{and} \quad Q = 1 - P.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Before an increase in excise duty on tea, 800 people out of a sample of 1000 persons were found to be tea drinkers. After an increase in the duty, 800 persons were known to be tea drinkers in a sample of 1200 people. Do you think that there has been a significant decrease in the consumption of tea after the increase in the excise duty?*

Sol. Here, $n_1 = 800, n_2 = 1200$

$$p_1 = \frac{X_1}{n_1} = \frac{800}{1000} = \frac{4}{5}; p_2 = \frac{X_2}{n_2} = \frac{800}{1200} = \frac{2}{3}$$

$$P = \frac{p_1 n_1 + p_2 n_2}{n_1 + n_2} = \frac{X_1 + X_2}{n_1 + n_2} = \frac{800 + 800}{1000 + 1200} = \frac{8}{11}; Q = \frac{3}{11}$$

Null hypothesis $H_0: p_1 = p_2$ *i.e.*, there is no significant difference in the consumption of tea before and after increase of excise duty.

Alternative hypothesis $H_1: p_1 > p_2$

Hence we use **right tailed test.**

Test statistic:

The test statistic is $$z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{0.8 - 0.6666}{\sqrt{\frac{8}{11} \times \frac{3}{11}\left(\frac{1}{1000} + \frac{1}{1200}\right)}} = 6.842.$$

Conclusion: Since the calculated value of $|z| > 1.645$ and also $|z| > 2.33$, both the significant values of z at 5% and 1% level of significance, hence H_0 is rejected *i.e.*, there is a significant decrease in the consumption of tea due to increase in excise duty.

Example 2. *A machine produced 16 defective articles in a batch of 500. After overhauling it produced 3 defectives in a batch of 100. Has the machine improved?*

Sol. $$p_1 = \frac{16}{500} = 0.032, \quad n_1 = 500$$

$$p_2 = \frac{3}{100} = 0.03, \quad n_2 = 100$$

Null hypothesis:

H_0: The machine has not improved due to overhauling, *i.e.*, $p_1 = p_2$.

Alternative hypothesis:

$H_1: p_1 > p_2$

Hence we use **right tailed test.**

$$\therefore \qquad P = \frac{p_1 n_1 + p_2 n_2}{n_1 + n_2} = \frac{19}{600} \cong 0.032$$

Test Statistic:

Under H_0, the test statistic

$$z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = 0.104$$

Conclusion: The calculated value of $|z| < 1.645$ which is the significant value of z at 5% level of significance, H_0 is accepted *i.e.*, the machine has not improved due to overhauling.

TEST YOUR KNOWLEDGE

1. Random sample of 400 men and 600 women were asked whether they would like to have a school near their residence. 200 men and 325 women were in favour of proposal. Test the hypothesis that the proportion of men and women in favour of the proposal are same at 5% level of significance.
2. In a town A, there were 956 births of which 52.5% were males while in towns A and B combined, this proportion in total of 1406 births was 0.496. Is there any significant difference in the proportion of male births in the two towns ?

Answers

1. H_0 : Accepted 2. H_0 : Rejected.

5.11.3 Testing of Significance for Single Mean

To test whether the difference between sample mean and population mean is significant or not.

Let $X_1, X_2, ..., X_n$ be a random sample of size n from a large population $X_1, X_2, ..., X_N$ of size N with mean μ and variance σ^2.

$\therefore$ the standard error of mean of a random sample of size n from a population with variance σ^2 is $\sigma/\sqrt{n}$.

To test whether the given sample of size n has been drawn from a population with mean μ i.e., to test whether the difference between the sample mean and population mean is significant or not under the null hypothesis that there is no difference between the sample mean and population mean.

The test statistic is $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$, where σ is the standard deviation of the population.

If σ is not known, we use the test statistic $z = \dfrac{\bar{x} - \mu}{s/\sqrt{n}}$, where s is the standard deviation of the sample.

Note. If the level of significance is α and z_α is the critical value $-z_\alpha < |z| = \left|\dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}\right| < z_\alpha$.

The limits of the population mean μ are given by $\bar{x} - z_\alpha \dfrac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_\alpha \dfrac{\sigma}{\sqrt{n}}$.

At 5% level of significance, 95% confidence limits are $\bar{x} - 1.96 \dfrac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.96 \dfrac{\sigma}{\sqrt{n}}$.

At 1% level of significance, 99% confidence limits are $\bar{x} - 2.58 \dfrac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 2.58 \dfrac{\sigma}{\sqrt{n}}$.

These limits are called **confidence limits** or **fiducial limits.**

ILLUSTRATIVE EXAMPLES

Example 1. *A random sample of 900 members has a mean 3.4 cms. Can it be reasonably regarded as a sample from a large population of mean 3.2 cms and S.D. 2.3 cms?*

Sol. Here $n = 900$, $\bar{x} = 3.4$, $\mu = 3.2$, $\sigma = 2.3$

Null hypothesis:

H_0 : Assume that the sample is drawn from a large population with mean 3.2 and S.D. 2.3

Alternative hypothesis:

$H_1 : \mu \neq 3.2$ **(two tailed test)**

Test statistic:

$$\text{Under } H_0 ;\ z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{3.4-3.2}{2.3/\sqrt{900}} = 0.261.$$

Conclusion: As the calculated value of $|z| = 0.261 < 1.96$, the significant value of z at 5% level of significance, H_0 is accepted *i.e.*, the sample is drawn from the population with mean 3.2 and S.D. 2.3.

Example 2. *The mean weight obtained from a random sample of size 100 is 64 gms. The S.D. of the weight distribution of the population is 3 gms. Test the statement that the mean weight of the population is 67 gms at 5% level of significance. Also set up 99% confidence limits of the mean weight of the population.*

Sol. Here $n = 100$, $\mu = 67$, $\bar{x} = 64$, $\sigma = 3$

Null hypothesis:

H_0: There is no significant difference between sample and population mean.

i.e., $\mu = 67$, the sample is drawn from the population with $\mu = 67$.

Alternative hypothesis:

H_1: $\mu \neq 67$ **(Two tailed test).**

Test statistic:

$$\text{Under } H_0,\ z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{64-67}{3/\sqrt{100}} = -10 \quad \therefore \quad |z| = 10.$$

Conclusion: Since the calculated value of $|z| > 1.96$, the significant value of z at 5% level of significance, H_0 is rejected *i.e.*, the sample is not drawn from the population with mean 67.

To find 99% confidence limits. It is given by $\bar{x} \pm 2.58\ \sigma/\sqrt{n} = 64 \pm 2.58(3/\sqrt{100}) =$ 64.774, 63.226.

TEST YOUR KNOWLEDGE

1. A sample of 1000 students from a university was taken and their average weight was found to be 112 pounds with a S.D. of 20 pounds. Could the mean weight of students in the population be 120 pounds?
2. A sample of 400 male students is found to have a mean height of 160 cms. Can it be reasonably regarded as a sample from a large population with mean height 162.5 cms and standard deviation 4.5 cms?

3. A random sample of 200 measurements from a large population gave a mean value of 50 and a S.D. of 9. Determine 95% confidence interval for the mean of population.

Answers

1. H_0 is rejected 2. H_0 accepted 3. 48.8 and 51.2.

5.11.4 Test of Significance for Difference of Means of Two Large Samples

Let $\bar{x}_1$ be the mean of a sample of size n_1 from a population with mean μ_1, and variance σ_1^2. Let $\bar{x}_2$ be the mean of an independent sample of size n_2 from another population with mean μ_2 and variance σ_2^2. The test statistic is given by $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$.

Under the null hypothesis that the samples are drawn from the same population where $\sigma_1 = \sigma_2 = \sigma$ *i.e.*, $\mu_1 = \mu_2$ the test statistic is given by $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$.

Note 1. If σ_1, σ_2 are not known and $\sigma_1 \neq \sigma_2$ the test statistic in this case is $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2 + s_2^2}{n_1 + n_2}}}$.

Note 2. If σ is not known and $\sigma_1 = \sigma_2$. We use $\sigma^2 = \dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}$ to calculate σ.

ILLUSTRATIVE EXAMPLES

Example 1. *The average income of persons was ₹ 210 with a S.D. of ₹ 10 in sample of 100 people of a city. For another sample of 150 persons, the average income was ₹ 220 with S.D. of ₹ 12. The S.D. of incomes of the people of the city was ₹ 11. Test whether there is any significant difference between the average incomes of the localities.*

Sol. Here $n_1 = 100$, $n_2 = 150$, $\bar{x}_1 = 210$, $\bar{x}_2 = 220$, $s_1 = 10$, $s_2 = 12$.

Null hypothesis: The difference is not significant. *i.e.*, there is no difference between the incomes of the localities. H_0: $\bar{x}_1 = \bar{x}_2$

Alternative hypothesis

$H_1 : \bar{x}_1 \neq \bar{x}_2$ **(two tailed test)**

Test statistic:

Under H_0, $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{210 - 220}{\sqrt{\dfrac{10^2}{100} + \dfrac{12^2}{150}}} = -7.1428 \quad \therefore \quad |z| = 7.1428.$

Conclusion: As the calculated value of $|z| > 1.96$, the significant value of z at 5% level of significance, H_0 is rejected *i.e.*, there is significant difference between the average incomes of the localities.

Example 2. *Intelligence tests were given to two groups of boys and girls.*

	Mean	*S.D.*	*Size*
Girls	*75*	*8*	*60*
Boys	*73*	*10*	*100*

Examine if the difference between mean scores is significant.

Sol. Null hypothesis H_0: There is no significant difference between mean scores *i.e.*, $\bar{x}_1 = \bar{x}_2$.

Alternative hypothesis H_1: $\bar{x}_1 \neq \bar{x}_2$ **(two tailed test)**

Test statistic: Under the null hypothesis, $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{75 - 73}{\sqrt{\dfrac{8^2}{60} + \dfrac{10^2}{100}}} = 1.3912.$

Conclusion: As the calculated value of $| z | < 1.96$, the significant value of z at 5% level of significance, H_0 is accepted *i.e.*, there is no significant difference between mean scores.

TEST YOUR KNOWLEDGE

1. Two random samples of sizes 1000 and 2000 farms gave an average yield of 2000 kg and 2050 kg respectively. The variance of wheat farms in the country may be taken as 100 kg. Examine whether the two samples differ significantly in yield.
2. The means of two large samples of 1000 and 2000 members are 168.75 cms and 170 cms respectively. Can the samples be regarded as drawn from the same population of standard deviation 6.25 cms?
3. In a survey of buying habits, 400 women shoppers are chosen at random in supermarket A. Their average weekly food expenditure is ₹ 250 with a S.D. of ₹ 40. For 500 women shoppers chosen at supermarket B, the average weekly food expenditure is ₹ 220 with a S.D. of ₹ 45. Test at 1% level of significance whether the average food expenditures of the two groups are equal.

Answers

1. Highly significant **2.** Not significant **3.** Highly significant.

5.11.5 Test of Significance for the Difference of Standard Deviations

If s_1 and s_2 are the standard deviations of two independent samples, then under the null hypothesis $H_0 : \sigma_1 = \sigma_2$, *i.e.*, the sample standard deviations don't differ significantly, the statistic

$$z = \frac{s_1 - s_2}{\sqrt{\dfrac{\sigma_1^2}{2n_1} + \dfrac{\sigma_2^2}{2n_2}}},$$ where σ_1 and σ_2 are population standard deviations.

When population standard deviations are not known, then $z = \dfrac{s_1 - s_2}{\sqrt{\dfrac{s_1^2}{2n_1} + \dfrac{s_2^2}{2n_2}}}$.

Example. *Random samples drawn from two countries gave the following data relating to the heights of adult males:*

	Country A	*Country B*
Mean height (in inches)	*67.42*	*67.25*
Standard deviation	*2.58*	*2.50*
Number in samples	*1000*	*1200*

(*i*) *Is the difference between the means significant?*
(*ii*) *Is the difference between the standard deviations significant?*

Sol. Given: $n_1 = 1000$, $n_2 = 1200$, $\bar{x}_1 = 67.42$; $\bar{x}_2 = 67.25$, $s_1 = 2.58$, $s_2 = 2.50$
Since the samples size are large we can take $\sigma_1 = s_1 = 2.58$; $\sigma_2 = s_2 = 2.50$.
(*i*) **Null hypothesis** $H_0 : \mu_1 = \mu_2$ *i.e.*, sample means do not differ significantly.
Alternative hypothesis $H_1 : \mu_1 \neq \mu_2$ **(two tailed test)**

Test statistic:
$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{67.42 - 67.25}{\sqrt{\frac{(2.58)^2}{1000} + \frac{(2.50)^2}{1200}}} = 1.56.$$

Conclusion:
Since $|z| < 1.96$ we accept the null hypothesis at 5% level of significance.
(*ii*) **Null hypothesis:**
H_0: $\sigma_1 = \sigma_2$ *i.e.*, the sample S.D.'s do not differ significantly.
Alternative hypothesis H_1: $\sigma_1 \neq \sigma_2$ **(two tailed test)**
Test statistic:
$$z = \frac{s_1 - s_2}{\sqrt{\frac{\sigma_1^2}{2n_1} + \frac{\sigma_2^2}{2n_2}}} = \frac{s_1 - s_1}{\sqrt{\frac{s_1^2}{2n_1} + \frac{s_2^2}{2n_2}}} = 1.0387.$$
Since $|z| < 1.96$ we accept the null hypothesis at 5% level of significance.

TEST YOUR KNOWLEDGE

1. The mean yield of two sets of plots and their variability are as given. Examine
   (*i*) whether the difference in the mean yield of the two sets of plots is significant.
   (*ii*) whether the difference in the variability in yields is significant.

	Set of 40 plots	*Set of 60 plots*
Mean yield per plot	1258 lb	1243 lb
S.D. per plot	34	28

2. The yield of wheat in a random sample of 1000 farms in a certain area has a S.D. of 192 kg. Another random sample of 1000 farms gives a S.D. of 224 kg. Are the S.D's significantly different?

Answers

1. $z = 2.315$, Difference significant at 5% level ; $z = 1.31$, Difference not significant at 5% level
2. $z = 4.851$. The S.D's are significantly different.

5.12 TEST OF SIGNIFICANCE OF SMALL SAMPLES

When the size of the sample is less than 30, then the sample is called small sample. For such sample it will not be possible for us to assume that the random sampling distribution of a statistic is approximately normal and the values given by the sample data are sufficiently close to the population values and can be used in their place for the calculation of the standard error of the estimate.

5.13 STUDENT'S t-DISTRIBUTION (t-Test)

[G.B.T.U. (MBA) 2011 ; G.B.T.U. (MCA) 2010]

This t-distribution is used when sample size is ≤ 30 and the population standard deviation is unknown.

t-statistic is defined as

$$\boxed{t = \frac{\bar{x} - \mu}{S/\sqrt{n}}} \qquad \text{where,} \qquad S = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n - 1}}$$

$\bar{x}$ is the mean of sample, μ is population mean. S is the standard deviation of population and n is sample size.

If the standard deviation of the sample 's' is given then t-statistic is defined as

$$\boxed{t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}}}$$

Note. The relation between s and S is $ns^2 = (n - 1)S^2$.

5.13.1 The *t*-Table

The t-table given at the end is the probability integral of t-distribution. The t-distribution has different values for each degrees of freedom and when the degrees of freedom are infinitely large, the t-distribution is equivalent to normal distribution and the probabilities shown in the normal distribution tables are applicable.

5.13.2 Applications of *t*-Distribution

[G.B.T.U. (MBA) 2011]

Some of the applications of t-distribution are given below:

1. *To test if the sample mean* ($\bar{x}$) *differs significantly from the hypothetical value* μ *of the population mean.*
2. *To test the significance between two sample means.*
3. *To test the significance of observed partial and multiple correlation coefficients.*

5.13.3 Critical Value of *t*

The critical value or significant value of t at level of significance α, degrees of freedom γ for two tailed test is given by

$$P[|t| > t_\gamma(\alpha)] = \alpha$$
$$P[|t| \leq t_\gamma(\alpha)] = 1 - \alpha$$

The significant value of t at level of significance α, for a single tailed test can be got from those of two tailed test by referring to the values at 2α.

5.14 TEST I: *t*-TEST OF SIGNIFICANCE OF THE MEAN OF A RANDOM SAMPLE

To test whether the mean of a sample drawn from a normal population deviates significantly from a stated value when variance of the population is unknown.

H_0: There is no significant difference between the sample mean $\bar{x}$ and the population mean μ, *i.e.*, we use the statistic

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}} \qquad \text{where} \quad S = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n-1}}$$

with degree of freedom $n - 1$.

At given level of significance α and degrees of freedom $(n - 1)$, we refer to *t*-table t_α (two tailed or one tailed). If calculated t value is such that $|t| < t_\alpha$, the null hypothesis is accepted. If $|t| > t_\alpha$, H_0 is rejected.

5.14.1 Fiducial Limits of Population Mean

If t_α is the value of t at level of significance α at $(n - 1)$ degrees of freedom then,

$$\left|\frac{\bar{x} - \mu}{S/\sqrt{n}}\right| < t_\alpha \text{ for acceptance of } H_0.$$

$$\bar{x} - t_\alpha S/\sqrt{n} < \mu < \bar{x} + t_\alpha S/\sqrt{n}$$

95% confidence limits (level of significance 5%) are $\bar{x} \pm t_{0.05} S/\sqrt{n}$.

99% confidence limits (level of significance 1%) are $\bar{x} \pm t_{0.01} S/\sqrt{n}$.

ILLUSTRATIVE EXAMPLES

Example 1. *A random sample of size 16 has 53 as mean. The sum of squares of the deviation' from mean is 135. Can this sample be regarded as taken from the population having 56 as mean? Obtain 95% and 99% confidence limits of the mean of the population.*

Sol. Null hypothesis, H_0: There is no significant difference between the sample mean and hypothetical population mean *i.e.*, $\mu = 56$.

Alternative hypothesis, H_1: $\mu \neq 56$ **(Two tailed test)**

Test statistic Under H_0, test statistic is $t = \dfrac{\bar{x} - \mu}{S/\sqrt{n}}$

Given : $\bar{x} = 53, \mu = 56, n = 16, \Sigma(x - \bar{x})^2 = 135$

$$S = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n-1}} = \sqrt{\frac{135}{15}} = 3$$

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}} = \frac{53 - 56}{3/\sqrt{16}} = -4$$

$|t| = 4$

d.fv. $= 16 - 1 = 15$.

Conclusion: Since $|t| = 4 > t_{0.05} = 2.13$ *i.e.*, the calculated value of t is more than the tabulated value, the null hypothesis is rejected. Hence, the sample mean has not come from a population having 56 as mean.

95% confidence limits of the population mean

$$= \bar{x} \pm \frac{S}{\sqrt{n}} t_{0.05} = 53 \pm \frac{3}{\sqrt{16}} (2.13) = 51.4025, 54.5975$$

99% confidence limits of the population mean

$$= \bar{x} \pm \frac{S}{\sqrt{n}} t_{0.01}, = 53 \pm \frac{3}{\sqrt{16}} (2.95) = 50.7875, 55.2125.$$

Example 2. *The lifetime of electric bulbs for a random sample of 10 from a large consignment gave the following data:*

Item	1	2	3	4	5	6	7	8	9	10
Life in '000 hrs.	4.2	4.6	3.9	4.1	5.2	3.8	3.9	4.3	4.4	5.6

Can we accept the hypothesis that the average lifetime of bulb is 4000 hrs?

Sol. Null hypothesis H_0 : There is no significant difference in the sample mean and population mean. *i.e.*, $\mu = 4000$ hrs.

Alternative hypothesis: $\mu \neq 4000$ hrs (**Two tailed test**)

Test statistic: Under H_0, the test statistic is $t = \dfrac{\bar{x} - \mu}{S/\sqrt{n}}$

x	4.2	4.6	3.9	4.1	5.2	3.8	3.9	4.3	4.4	5.6
$x - \bar{x}$	– 0.2	0.2	– 0.5	– 0.3	0.8	– 0.6	– 0.5	– 0.1	0	1.2
$(x - \bar{x})^2$	0.04	0.04	0.25	0.09	0.64	0.36	0.25	0.01	0	1.44

$$\bar{x} = \frac{\Sigma x}{n} = \frac{44}{10} = 4.4, \qquad \Sigma(x - \bar{x})^2 = 3.12$$

$$S = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n - 1}} = 0.589$$

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}} = \frac{4.4 - 4}{\left(\dfrac{0.589}{\sqrt{10}}\right)} = 2.123$$

For $\gamma = 9$, $t_{0.05} = 2.26$.

Conclusion: Since the calculated value of t is less than the tabulated value of t at 5% level of significance.

∴ The null hypothesis $\mu = 4000$ hrs is accepted *i.e.*, the average lifetime of bulbs could be 4000 hrs.

Example 3. *A sample of 20 items has mean 42 units and S.D. 5 units. Test the hypothesis that it is a random sample from a normal population with mean 45 units.*

Sol. Null hypothesis H_0: There is no significant difference between the sample mean and the population mean. *i.e.*, $\mu = 45$ units

Alternative hypothesis, H_1: $\mu \neq 45$ (**Two tailed test**)

Given: $n = 20$, $\bar{x} = 42$, $s = 5$; $\gamma = 19$ d.f.

Test statistic: Under H_0, the test statistic is

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} = \frac{42 - 45}{5/\sqrt{19}} = -2.615$$

$\therefore$ $\quad |t| = 2.615$

The tabulated value of t at 5% level for 19 d.f. is $t_{0.05} = 2.09$.

Conclusion: Since the calculated value $|t|$ is greater than the tabulated value of t at 5% level of significance, the null hypothesis H_0 is rejected. *i.e.*, there is significant difference between the sample mean and population mean.

i.e., the sample could not have come from this population.

Example 4. *The 9 items of a sample have the following values:*

45, 47, 50, 52, 48, 47, 49, 53, 51.

Does the mean of these values differ significantly from the assumed mean 47.5?

Sol. Here, $n = 9$, $\mu = 47.5$, $\bar{x} = \frac{\Sigma x}{n} = 49.1$

x	45	47	50	52	48	47	49	53	51
$x - \bar{x}$	– 4.1	– 2.1	0.9	2.9	– 1.1	– 2.1	– 0.1	3.9	1.9
$(x - \bar{x})^2$	16.81	4.41	0.81	8.41	1.21	4.41	0.01	15.21	3.61

$$\Sigma(x - \bar{x})^2 = 54.89,$$

$$S^2 = \frac{\Sigma(x - \bar{x})^2}{n - 1} = 6.86$$

$\therefore$ $\quad S = 2.619$

Null hypothesis:

$H_0: \mu = 47.5$

i.e., there is no significant difference between the sample and population means.

Alternative hypothesis:

$H_1: \mu \neq 47.5$

Hence we apply **two-tailed test.**

Test statistic: Under H_0, the test statistic is

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}} = \frac{49.1 - 47.5}{(2.619/\sqrt{9})} = 1.8327$$

$t_{0.05} = 2.31$ for $\gamma = 8$

Conclusion: Since $|t|_{\text{calculated}} < t_{\text{tabulated}}$ at 5% level of significance, the null hypothesis H_0 is accepted *i.e.*, there is no significant difference between their means.

TEST YOUR KNOWLEDGE

1. Ten individuals are chosen at random from a normal population of students and their marks are found to be 63, 63, 66, 67, 68, 69, 70, 70, 71, 71. In the light of these data, discuss the suggestion that mean mark of the population of students is 66.
2. The following values gives the lengths of 12 samples of Egyptian cotton taken from a consignment: 48, 46, 49, 46, 52, 45, 43, 47, 47, 46, 45, 50. Test if the mean length of the consignment can be taken as 46.
3. A sample of 18 items has a mean 24 units and standard deviation 3 units. Test the hypothesis that it is a random sample from a normal population with mean 27 units.
4. A random sample of 10 boys had the I.Q.'s 70, 120, 110, 101, 88, 83, 95, 98, 107 and 100. Do these data support the assumption of a population mean I.Q. of 160?

5.15 TEST II: t-TEST FOR DIFFERENCE OF MEANS OF TWO SMALL SAMPLES (from a Normal Population)

This test is used to test whether the two samples $x_1, x_2, \ldots\ldots, x_{n_1}, y_1, y_2, \ldots\ldots, y_{n_2}$ of sizes n_1, n_2 have been drawn from two normal populations with mean μ_1 and μ_2 respectively under the assumption that the population variance are equal ($\sigma_1 = \sigma_2 = \sigma$).

H_0: The samples have been drawn from the normal population with means μ_1 and μ_2 *i.e.*, H_0: $\mu_1 \neq \mu_2$.

Let $\bar{x}$, $\bar{y}$ be their means of the two samples.

Under this H_0 the test statistic t is given by $t = \dfrac{(\bar{x} - \bar{y})}{S\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$.

Degree of freedom is $n_1 + n_2 - 2$.

Note 1. If the two sample's standard deviations s_1, s_2 are given then we have $S^2 = \dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$.

Note 2. If s_1, s_2 are not given then $S^2 = \dfrac{\Sigma(x_1 - \bar{x}_1)^2 + \Sigma(x_2 - \bar{x}_2)^2}{n_1 + n_2 - 2}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Two samples of sodium vapour bulbs were tested for length of life and the following results were got:*

	Size	*Sample mean*	*Sample S.D.*
Type I	8	*1234 hrs*	*36 hrs*
Type II	7	*1036 hrs*	*40 hrs*

Is the difference in the means significant to generalise that Type I is superior to Type II regarding length of life?

Sol. Null hypothesis:

$H_0 : \mu_1 = \mu_2$ *i.e.*, two types of bulbs have same lifetime.

Alternative hypothesis:

$H_1 : \mu_1 > \mu_2$ *i.e.*, type I is superior to Type II.

Hence we use **right tailed test.**

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{8(36)^2 + 7(40)^2}{8 + 7 - 2} = 1659.076$$

$\therefore$ $S = 40.7317$

Test statistic: Under H_0, the test statistic t is given by

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{1234 - 1036}{40.7317\sqrt{\frac{1}{8} + \frac{1}{7}}} = 18.1480$$

$t_{0.05}$ at d.f. $\gamma = n_1 + n_2 - 2 = 13$ is 1.77.

Conclusion: Since calculated $|t| > t_{\text{tabulated}}$ at 5% level of significance, H_0 is rejected.

$\therefore$ Type I is definitely superior to Type II.

Example 2. *Samples of sizes 10 and 14 were taken from two normal populations with S.D. 3.5 and 5.2. The sample means were found to be 20.3 and 18.6. Test whether the means of the two populations are the same at 5% level.*

Sol. We have, $\bar{x}_1 = 20.3$, $\bar{x}_2 = 18.6$, $n_1 = 10$, $n_2 = 14$, $s_1 = 3.5$, $s_2 = 5.2$

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = 22.775$$

$\therefore$ $S = 4.772$

Null hypothesis:

$H_0 : \mu_1 = \mu_2$ *i.e.*, the means of the two populations are the same.

Alternative hypothesis:

$H_1 : \mu_1 \neq \mu_2$

Test statistic: Under H_0, the test statistic is

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{20.3 - 18.6}{4.772\sqrt{\frac{1}{10} + \frac{1}{14}}} = 0.8604$$

The tabulated value of t at 5% level of significance for 22 *d.f.* is $t_{0.05} = 2.0739$

Conclusion:

Since $t = 0.8604 < t_{0.05}$, the null hypothesis H_0 is accepted; *i.e.*, there is no significant difference between their means.

Example 3. *The height of 6 randomly chosen sailors in inches are 63, 65, 68, 69, 71 and 72. Those of 9 randomly chosen soldiers are 61, 62, 65, 66, 69, 70, 71, 72 and 73. Test whether the sailors are on the average taller than soldiers.*

Sol. Let X_1 and X_2 be the two samples denoting the heights of sailors and soldiers.

$n_1 = 6$, $n_2 = 9$

Null hypothesis H_0: $\mu_1 = \mu_2$.

i.e., the mean of both the population are the same.

Alternative hypothesis H_1: $\mu_1 > \mu_2$ (one tailed test)

Calculation of two sample means :

X_1	63	65	68	69	71	72
$X_1 - \bar{X}_1$	– 5	– 3	0	1	3	4
$(X_1 - \bar{X}_1)^2$	25	9	0	1	9	16

$$\bar{X}_1 = \frac{\Sigma X_1}{n_1} = 68; \ \Sigma(X_1 - \bar{X}_1)^2 = 60$$

X_2	61	62	65	66	69	70	71	72	73
$X_2 - \bar{X}_2$	– 6.66	– 5.66	– 2.66	1.66	1.34	2.34	3.34	4.34	5.34
$(X_2 - \bar{X}_2)^2$	44.36	32.035	7.0756	2.7556	1.7956	5.4756	11.1556	18.8356	28.5156

$$\bar{X}_2 = \frac{\Sigma X_2}{n_2} = 67.66; \ \Sigma(X_2 - \bar{X}_2)^2 = 152.0002$$

$$S^2 = \frac{1}{n_1 + n_2 - 2}[\Sigma(X_1 - \bar{X}_1)^2 + \Sigma(X_2 - \bar{X}_2)^2] = 16.3077$$

$\therefore$ $S = 4.038$

Test statistic:

Under H_0, $$t = \frac{\bar{X}_1 - \bar{X}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{68 - 67.666}{4.038\sqrt{\frac{1}{6} + \frac{1}{9}}} = 0.1569$$

The value of t at 5% level of significance for 13 d.f. is 1.77. (d.f. $= n_1 + n_2 - 2$)

Conclusion: Since $t_{calculated} < t_{0.05} = 1.77$, the null hypothesis H_0 is accepted.

i.e., there is no significant difference between their average.

i.e., the sailors are not on the average taller than the soldiers.

TEST YOUR KNOWLEDGE

1. The mean life of 10 electric motors was found to be 1450 hrs with S.D. of 423 hrs. A second sample of 17 motors chosen from a different batch showed a mean life of 1280 hrs with a S.D. of 398 hrs. Is there a significant difference between means of the two samples?
2. The marks obtained by a group of 9 regular course students and another group of 11 part time course students in a test are given below:

Regular :	56	62	63	54	60	51	67	69	58		
Part time :	62	70	71	62	60	56	75	64	72	68	66

Examine whether the marks obtained by regular students and part time students differ significantly at 5% and 1% level of significance.

3. A group of 5 patients treated with the medicine A weigh 42, 39, 48, 60 and 41 kgs. A second group of 7 patients from the same hospital treated with medicine B weigh 38, 42, 56, 64, 68, 69 and 62 kg. Do you agree with the claim that medicine B increases the weight significantly? It is given that the value of t at 10% level of significance for 10 degree of freedom is 1.81.

[*G.B.T.U. (B. Pharm.) 2010*]

4. Two independent samples of sizes 7 and 9 have the following values:

Sample A :	10	12	10	13	14	11	10		
Sample B :	10	13	15	12	10	14	11	12	11

Test whether the difference between the mean is significant.

5. The average number of articles produced by two machines per day are 200 and 250 with standard deviation 20 and 25 respectively on the basis of records of 25 days production. Can you regard both the machines equally efficient at 5% level of significance?