Lorrentz Transformation

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Task: To find an acceptable solution to the problem of "Relative Motion at higher velocities" based on improvising the Galilean Transformation Laws.

Generaling the Problem to a Scenario

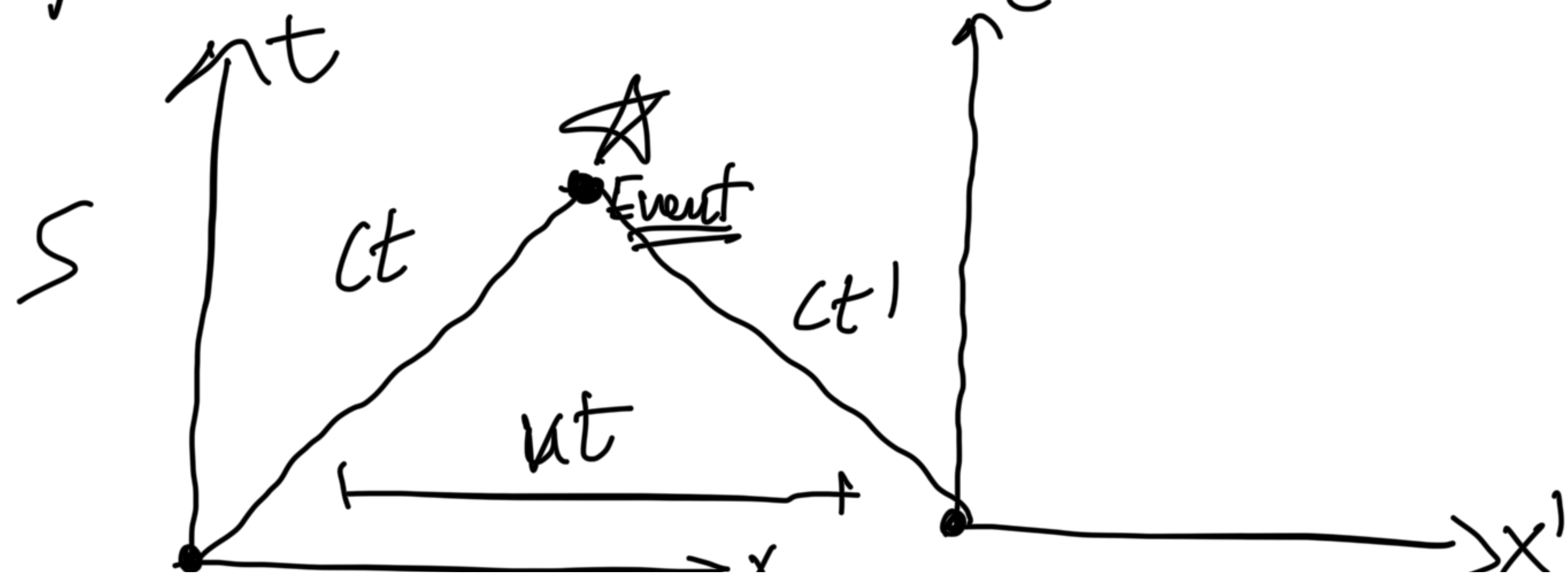
↳ There is an event of a "Light Pulse" flashed at an instant in space and time.

↳ This Event is measured by two different observers traveling relative to each other at a constant relative velocity (V).

↳ Let the two frames of reference be S and S'.

↳ Let their relative motion be only restricted to x-axis direction. or y-axis direction.

Diagramatic Representations.



↳ We need to find the measuring co-ordinates of the event in one frame by using other frame as a reference.

So,
$$x' = f(x, t)$$
$$t' = f(x, t)$$

But this cannot be equal to a general Galilean transformation.

So, lets write it as a linear transformation.

So $\begin{bmatrix} x' \\ t' \end{bmatrix} = \begin{bmatrix} a & b \\ d & f \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}$; we need to solve for the coefficient a,b,d,f.

So, $x' = ax + bt$

$t' = dx + ft$.

Initial Condition

Event $(x, y, z), t$

S: $x = vt$, $y = 0$, $z = 0$, t.

S': $x' = 0$, $y = 0$, $z = 0$, t'

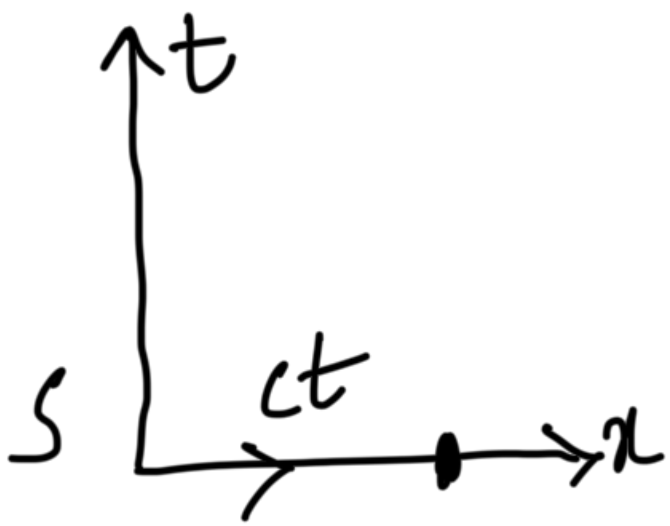
$x' = ax + bt$.

$\Rightarrow \quad 0 = avt + bt \Rightarrow (av+b)t = 0$

$av + b = 0 \Rightarrow \boxed{b = -av}$

Situation - I

↳ From origin, a light ray has travelled for time t in "S" frame.



So, $x = ct$; $y = 0$, $z = 0$; t

↳ The same situation is observed from "S'" reference and the equations are as follows.

$x' = a(x - vt) \Rightarrow x' = a(ct - vt) = at(c - v)$.

↳ But frome "S'" frame of reference, ($x' = ct'$) because light travels the same distance ct' in t'!

so, $$ct' = at(c-v)$$

also, $$t' = dx + ft$$

$$= dct + ft \Rightarrow (dc+f)t = t'$$

$$\Rightarrow c(dc+f)\cancel{t} = a\cancel{t}(c-v) \Rightarrow$$

$$\Rightarrow \boxed{dc^2 + fc = a(c-v)} \quad \text{--- (1)}$$

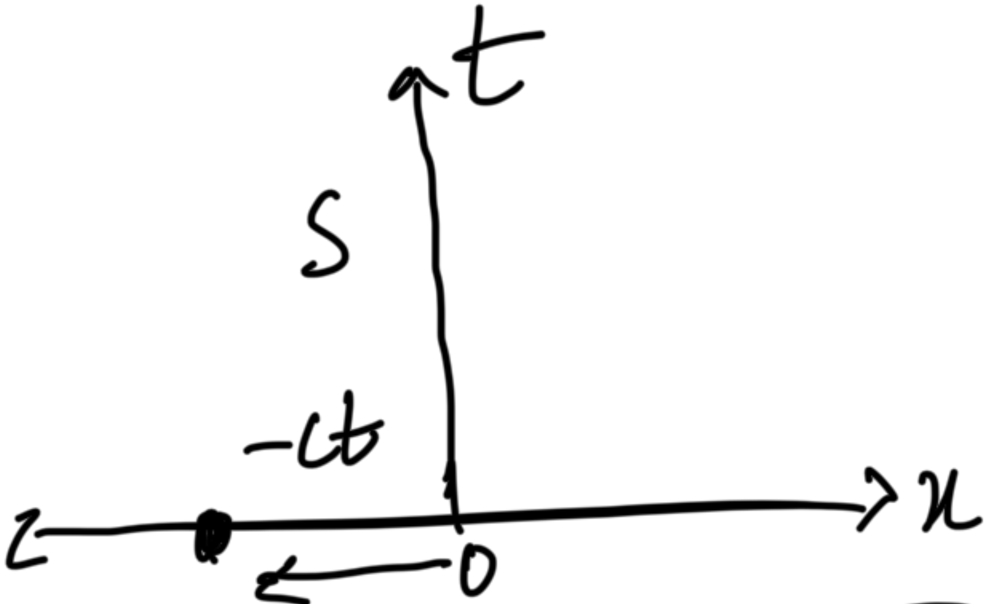
<u>Situation-2</u>

$S: x = -ct,\ y = 0,\ z = 0,\ t$

↳ $x' = -ct',\ y' = 0,\ z' = 0,\ t'$

↳ Now, the light beam is travelling from the Origin of the 'S' frame to it's left (i.e −ve x-axis) direction.

So,



So the equations will be

$$x' = a(x - vt)$$

$$x' = -(c+v)at$$

Also, $x' = -ct' \implies \cancel{-}ct' = \cancel{-}(c+v)at$

$$ct' = (c+v)at$$

Also, $t' = dx + ft$

$$= -dct + ft$$

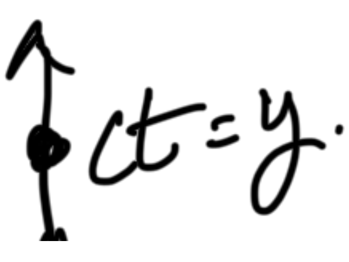
$$= t(f - dc)$$

$$c\cancel{t}(f-dc) = a\cancel{t}(c+v)$$

$$\boxed{-dc^2+fc = a(c+v)} \longrightarrow (2)$$

Situation-3

The light ray now is passed from the origin of the frame "S" for time "t" in Y axis direction.



S: $x=0$, $y=ct$, $z=0$, t

$S': x'=0,\ y'=y,\ z'=0,\ t'$

Say $x' = a(x - vt)$ $\quad$ $t' = dx + ft$

$x' = a(0 - vt)$ $\quad$ $t' = 0 + ft \Rightarrow \underline{t' = ft}$

$\underline{x' = -avt}$

Calulating the distance of the event from S' frame

$$S^2 = x'^2 + y'^2 + z'^2 = a^2v^2t^2 + c^2t^2$$

Also S = distance of the light pulse $= ct'$

$$c^2t'^2 = a^2v^2t^2 + c^2t^2$$

Also, $t' = ft$

So, $c^2 f^2 t^2 = a^2 v^2 t^2 + c^2 t^2$

$$\boxed{c^2 f^2 = av^2 + c^2} \text{ — } (3)$$

<u>Summarizing Our Equations:</u>

$$\cancel{d}c^2 + fc = a(c-v) \text{ — } (1)$$

$$-\cancel{d}c^2 + fc = a(c+v) \text{ — } (2)$$

(1)+(2) $$0 + 2fc = 2ac \Rightarrow \boxed{a = f}$$

(4) $\Longleftarrow$ $\boxed{\therefore a = f}$ $\Longrightarrow$ $dc^2 + fc = ac - av$

$$dc^2 + \cancel{ac} = \cancel{ac} - av$$

$$\boxed{d = -\frac{av}{c^2}} \quad \text{— (5)}$$

$$\boxed{c^2 f^2 = a^2 v^2 + c^2} \quad \text{— (3)}$$

$$c^2 a^2 = a^2 v^2 + c^2$$

$$\Rightarrow a^2 (c^2 - v^2) = c^2$$

$$\Rightarrow a^2 = \frac{c^2}{c^2 - v^2}$$ $\Rightarrow$ dividing and multiplying by c^2. on RHS.

$$\Rightarrow a^2 = \frac{1}{1 - \frac{v^2}{c^2}} \Rightarrow a = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\boxed{\therefore a = f = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}}$$

and $d = \dfrac{-av}{c^2}$

$$\Rightarrow d = \frac{-v}{c^2} \times \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow \boxed{d = \frac{-v}{\sqrt{c^2-v^2}}}$$

$$\therefore x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t' = \left(t - \frac{vx}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

Let $\gamma = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

So, $x' = \gamma(x - vt)$

$t' = \gamma\left(t - \frac{vx}{\ldots}\right)$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

Since Lorentz Transformation is a linear transformation we can write it as follows.

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma v \\ -\frac{\gamma v}{c^2} & \gamma \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}$$

Concepts of Invariants in Special Relativity.

↳ In special relativity, we use lorentz transformations to find out the position and time between reference frames.

↳ The spatial and temporal Co-ordinates changes with respect to relative velocity.

But there will be some Geometric objects and Measurements that remain constant even after applying a lorrentz transform.

These are Called "Invariants". Examples of invariants would be:

↳ Angles b/w two vectors.

↳ Their Dot products.

↳ The locus of equidisant points (accross at x&t).