Geometry of Space and Time 

Takeaways from Lorrentz Transformations:

↳ Lorrentz transformation is a Linear Transformation.

↳ It has special Geometric objects called Invariants.

Topics:

The $\boxed{c^2 \Delta t^2 - \Delta x^2 = c^2 \Delta t'^2 - \Delta x'^2}$ Invariant.

How is relativity a theory of Geometry??

How is the concept of Time extendend to 4D??

Time dilation and length contraction.

The $\boxed{c^2\Delta t^2 - \Delta x^2 = s^2}$ Invariant.

↳ Consider two reference frames (Inertial reference) and let them be called S and S'.

$\Rightarrow$ The position co-ordinates change based on the frame obeying Lorentz Transform.

↳ They can be mathematically written as

$$\underset{\underline{\underline{\text{"}S'\text{"}}}}{\begin{bmatrix} x' \\ t' \end{bmatrix}} = \gamma \underset{\text{Lorentz Transform}}{\begin{bmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{bmatrix}} \underset{\underline{\underline{\text{"}S\text{"}}}}{\begin{bmatrix} x \\ t \end{bmatrix}}$$

$$x' = \gamma(x - vt)$$

$$t' = \gamma\left(-\frac{vx}{c^2} + t\right)$$

$\Rightarrow$ These can be written into respective displacement vectors

↳ Now lets solve for the squared difference between spatial and temporal displacement.

$$\Rightarrow \quad c^2 \Delta t'^2 - \Delta x'^2$$, substitute the Lorentz equations in the adjacent equation.

$$c^2\left[\gamma\left(\frac{-vx}{c^2}+t\right)\right]^2 - \left[\gamma\left(x-vt\right)\right]^2$$

$$\Rightarrow c^2\gamma^2\left[\frac{-v^2x^2}{c^4}+t^2-\frac{2vxt}{c^2}\right]-\gamma^2\left[x^2+v^2t^2-2xvt\right]$$

$$\Rightarrow \gamma^2\left[-\frac{v^2x^2}{c^2}+c^2t^2-\cancel{2vxt}-x^2-v^2t^2+\cancel{2xvt}\right]$$

$$\Rightarrow \gamma^2\left[t^2(c^2-v^2) - x^2\left(1-\frac{v^2}{c^2}\right)\right]$$

$$\Rightarrow \boxed{\gamma^2 = \frac{1}{1-\frac{v^2}{c^2}} \Rightarrow \frac{c^2}{c^2-v^2}}$$

$$\Rightarrow \frac{c^2}{\cancel{c^2-v^2}} \times t^2\cancel{(c^2-v^2)} - x^2 \times \frac{1}{\cancel{\left(1-\frac{v^2}{c^2}\right)}} \times \cancel{\left(1-\frac{v^2}{c^2}\right)}$$

$$\Rightarrow \boxed{c^2t^2 - x^2}$$

$$\therefore \boxed{c^2\Delta t'^2 - \Delta x'^2 = c^2\Delta t^2 - \Delta x^2}$$

This holds true for any reference frames. This quantity is called "Space-Time interval"

Extending this to the fundamentals of Geometry.

The term or the equation which we called "Invariant" describes more than what it appears.

In mathematical terms, any locus with an equation $\Delta x^2 - \Delta y^2 = s^2$ represents a "hyperbola".

So, no matter in what reference we observe a relative motion in, this $c^2\Delta t^2 - \Delta x^2 = s^2$ remains constant.

The reality we choose to believe is a 3 dimensional reality with ticking clocks. (A dynamic reality in 3 space).

But this geometric object $-\Delta x^2 + c^2 \Delta t^2 = s^2$ extends this approach into "another higher dimension"

where the geometry can now be called

$$\boxed{\text{" Static in 4 Space "}}$$

where all the events accross space and time can be plotted on a stationary set of manifolds.

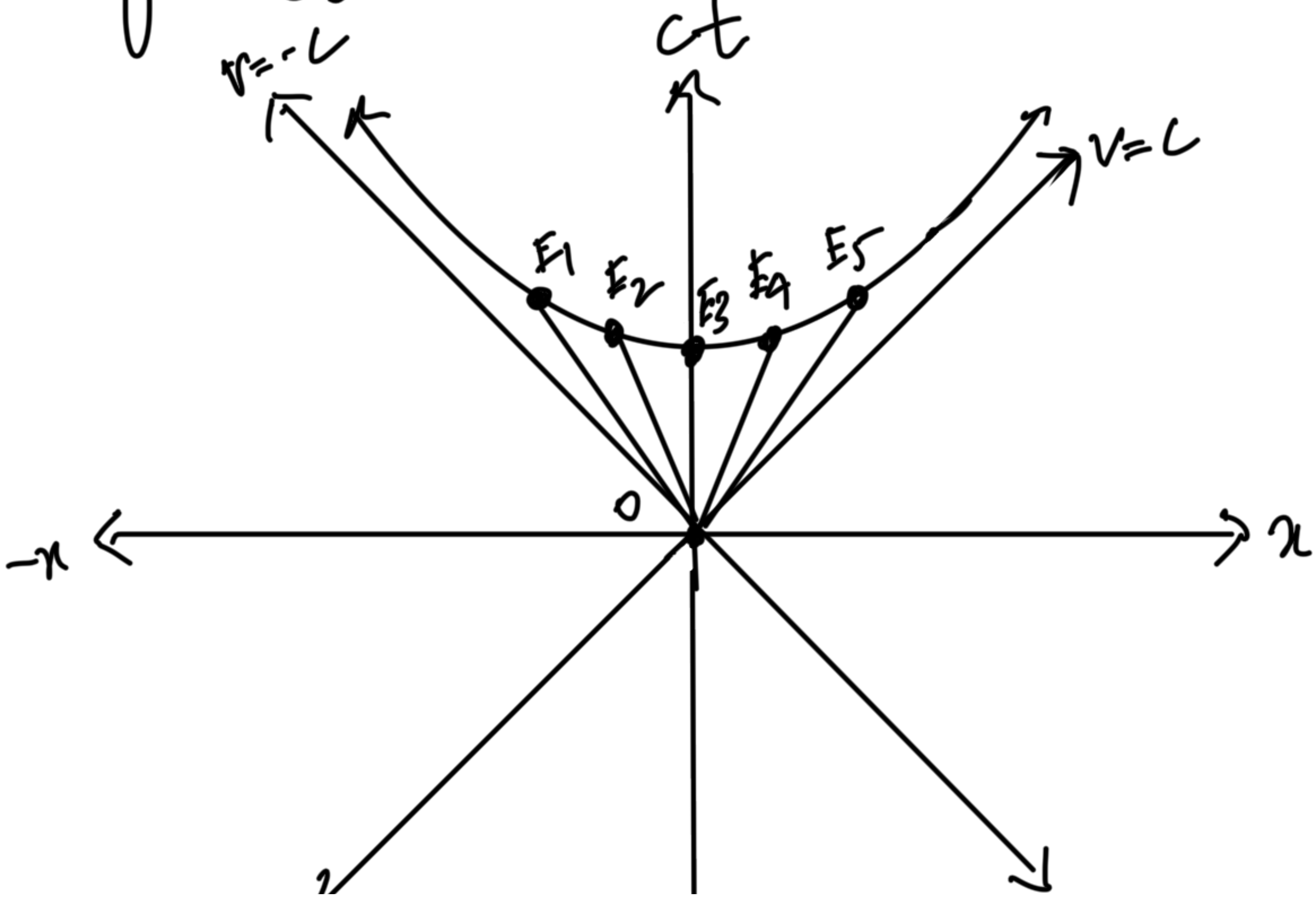
Nature of the Static Geometry [Space-Time]

From this equation of Invariant $-\Delta x^2 + c^2 \Delta t^2 = s^2$; it's clear that all set of events in the "4d-spacetime" lie on the locus of a hyperbola.

So, the static geometry of our reality is a type of geometry that can be called "Non-Euclidian" Geometry.

Non Euclidian spaces entirely change the existing paradigms of straight lines and shortest paths.

Spacetime diagrams help us understand and track changes b/w frames of references.



In this spacetime diagrams, the points on that 4dimensional space (drawn as 1 space+ 1 time for easy understanding) are called "**Events**" which represent the time and position on the diagram.

In Geometry, the space is defined based on the "locus its Equidistant points" lie on.

Note: In normal space, the equidistant points lie on a circle (in 2d). So, that space is euclidian.

The sum of angle on a triangles in 180°. (on that surface).

But in this 4d spacetime, the equidistance points no matter viewed from different frames lie on on a Hyperbola.

It's impossible to create a Non Euclidean subspace inside an almost Euclidian space. [i.e 3d earth].

Bizzareness inside a Hyperbolic space:

If a chart from the hyperbolic space is cut out

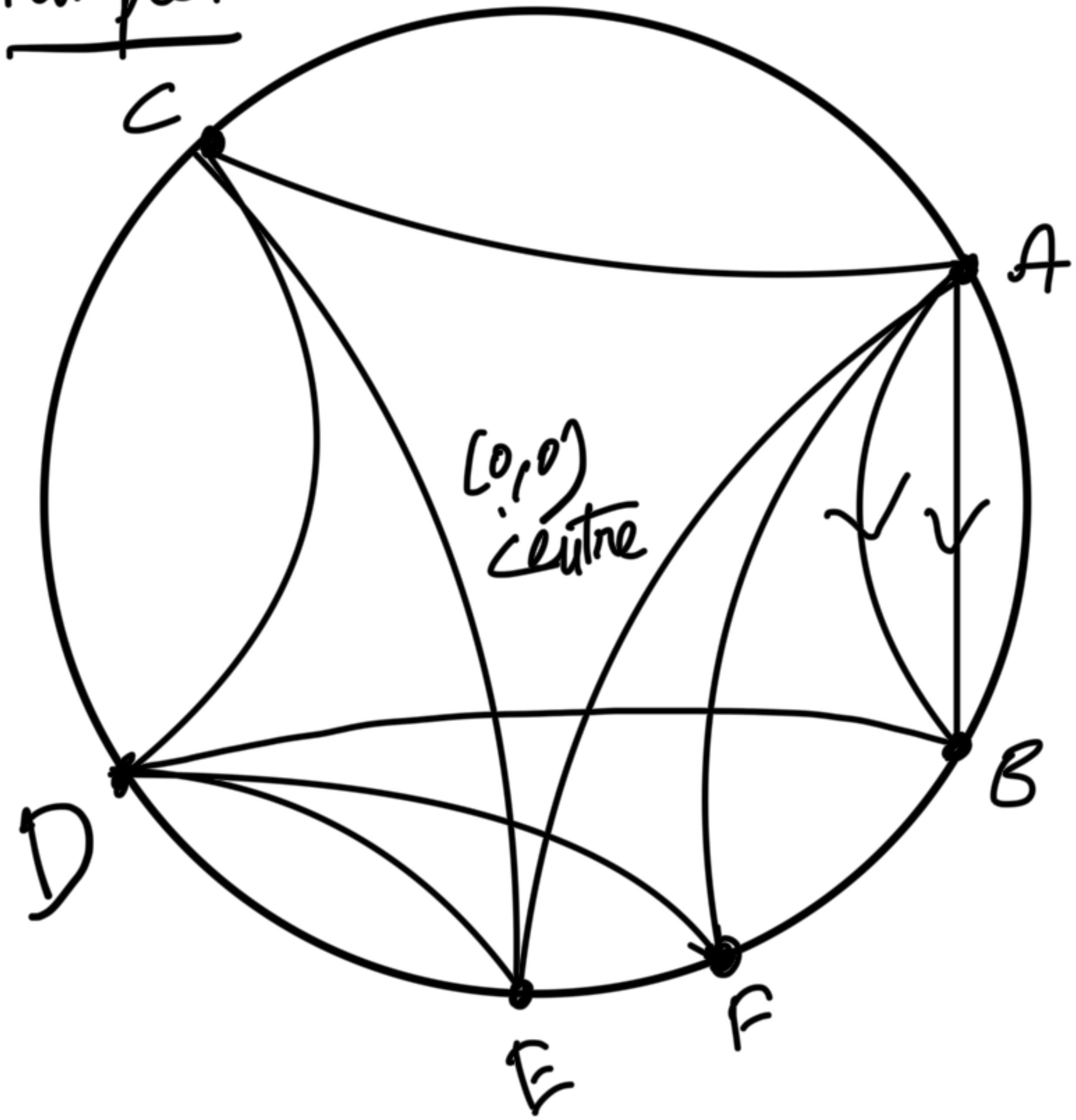
If a sheet from the hyperbolic space is cut out and we draw a triangle using those points, the sum of Angles inside the triangle

ie $\angle A + \angle B + \angle C < 180^\circ$

Also, any projection from a hyperbolic space, be it Orthographic, stereographic etc, result in different Area and volume element changes varying from the centre of the space. (Klein, Lobachevski Models).

Fo

Example:



According to normal human perception the shortest path between points is the line segment $\overrightarrow{AB}$

But due to the mathematics of hyperbolic geometry

where $x = \cosh\alpha$;
$y = \sinh\alpha$

the shortest path between AB is the hyperbolic geodesic where the magnitude varies Exponentially high.

Also, the rectangles with same area will be larger towards centre and smaller away from the centre.
& [Also, lengths]

The Ricci tensor acting on a volume element of a hyperbolic manifold will show Negative

curvature.

$$R^{\rho}{}_{\mu\rho\nu} = \boxed{R_{\mu\nu} < 0} \Longrightarrow \text{Negative Ricci Tensor.}$$

$R^{\alpha}{}_{\beta\gamma\delta}$ = Riemann Curvature

$$R(\vec{u},\vec{v})\vec{w} = \nabla_u \nabla_v \vec{w} - \nabla_v \nabla_u \vec{w} - \nabla_{[u,v]}\vec{w}$$