Schwarzchild Solution For Einstien's Equation.

- What is Schwarzchild Metric?? ✓
- why is it used?? ✓
- What are the conditions under which it is derived?? ✓
- Derivation. ✓
- Analysis of the Schwarzchild radius Parameter. ✓

What is Schwarzchild Metric??

It describes the gravitational field outside a

↳ It describes the gravitational field outside a spherical mass on the assumption that

↳ Electric charge $= 0$

↳ Angular Momentum $= 0$ (non rotating mass)

↳ Cosmological constant $= \Lambda = 0$. (no universal Expansion)

↳ It's used in approximately describing slowly rotating astronomical objects like <u>Earth</u> around the <u>Sun</u> etc

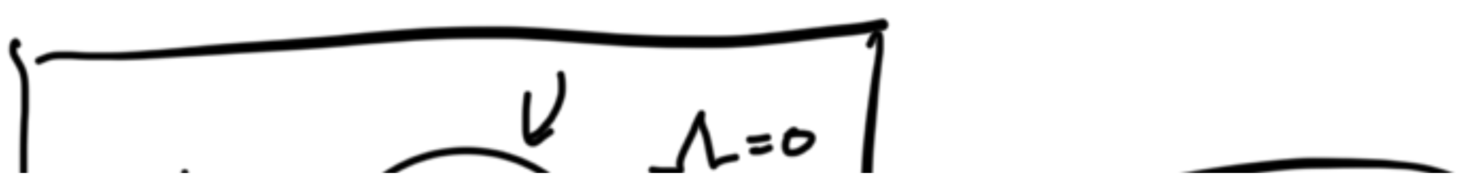
↳ The trajectories in space and time can be calculated from the geodesic equation after solving for the metric at given conditions.

Initial Conditions to establish the metric tensor:

- → The mass is spherical.
- → the mass is chargeless.
- → the mass is non-rotatating.

↳ So, The space arround the Mass has zero stress acting on it. $\boxed{T_{\mu\nu}=0}$

↳ the spacetime is assumed to be non expanding. $\boxed{\Lambda=0}$



$$\boxed{R^{\sigma}_{\ \alpha\beta\gamma} \neq 0}$$

solving for the Ricci tensor components

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \underbrace{\Lambda}_{0} g_{\mu\nu} = \frac{8\pi G}{c^4} \underbrace{T_{\mu\nu}}_{0}$$

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0}$$

contracting the equation using inverse metric

$$R_{\mu\nu} g^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} g_{\mu\nu} = 0$$

$$R^{\sigma}_{\ \sigma} - \frac{1}{2} R \delta^{\sigma}_{\ \sigma} = 0.$$

$$R - \frac{1}{2} R(4) = 0$$

$$\boxed{R = 0}$$

Now,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0.$$

$$R_0 \quad \frac{2}{2} \quad \frac{\sigma}{4}.$$
$$\downarrow$$
$$R$$

$$R_{\mu\nu} - \frac{1}{2}(0)\, g_{\mu\nu} = 0$$

$$\boxed{R_{\mu\nu} = 0}$$

The values of all "$R_{\mu\nu}$" equals to "zero". That means the space is "Ricci Flat". There are no immediate volume changes

Note: If $R_{\mu\nu} = 0$ for all μ, ν, that doesn't mean the Riemann curvature tensor is zero.

so, even if $\boxed{R_{\mu\nu} = 0; \; R^{\sigma}_{\alpha\beta\gamma} \neq 0}$

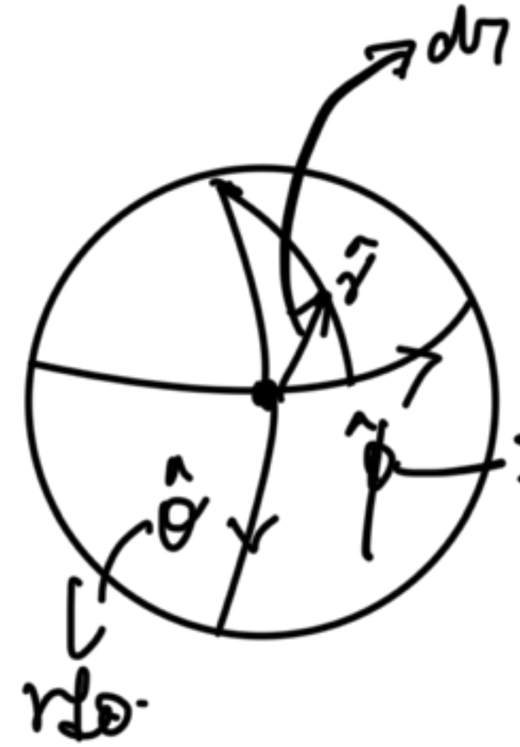
Calculate Cur... Minkowski Metric:

Spherical Co-ordinates and Minkowski Metric.

↳ In 3-d space, a vector can be represented in Spherical co-ordinates.

$\hat{r} = \frac{\partial}{\partial r}$; $\hat{\phi} = \frac{\partial}{\partial \phi}$

$\hat{\theta} = \frac{\partial}{\partial \theta}$



→ $r\sin\theta\, d\phi$

$$\vec{V} = dr\,\hat{r} + r d\theta\,\hat{\theta} + r\sin\theta\,\hat{\phi}$$

so, Metric is defined from the distance dot product.

$$\boxed{\vec{V}\cdot\vec{V} = g_{ij}\, dx^i\, dx^j}$$

$$\boxed{g_{ij} = \frac{\partial}{\partial x^i}\cdot\frac{\partial}{\partial x^j}}$$

↳ inner product of unit vector.

So, for $i \neq j$, since the unit vectors are orthogonal to each other. So, $g_{ij} = 0$,

for $i = j$, we have 3 components. g_{rr}, $g_{\theta\theta}$, and $g_{\phi\phi}$

$$g_{rr} = dr\,\hat{r} \cdot dr\,\hat{r} = (dr)^2 (1), \quad \boxed{g_{rr} = 1}$$

$$g_{\theta\theta} = r\,d\theta\,\hat{\theta} \cdot r\,d\theta\,\hat{\theta} = r^2 (d\theta)^2, \quad \boxed{g_{\theta\theta} = r^2}$$

$$g_{\phi\phi} = r\sin\theta\,\hat{\phi} \cdot r\sin\theta\,\hat{\phi} = r^2 \sin^2\theta (d\phi)^2, \quad \boxed{g_{\phi\phi} = r^2 \sin^2\theta}$$

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

↳ In spherical co-ordinate system.

$R_{\alpha\beta\gamma} \neq 0 \to$ # curved 3d space.

Now, the Four-dimensional version includes the time parameter.

$$\vec{X} = \text{Position 4-vector} = icdt\,\vec{e_t} + dr\,\vec{e_r} + r\,d\theta\,\vec{e_\theta} + r\sin\theta\,d\phi\,\vec{e_\phi}$$

or $$\boxed{\vec{X} = c\,dt\,\vec{e_t} + i\left(dr\,\vec{e_r} + r\,d\theta\,\vec{e_\theta} + r\sin\theta\,d\phi\,\vec{e_\phi}\right)}$$

Metric in Flat space time can be written as

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix} \Rightarrow \text{Minkowski Flat-Space Metric } (\eta_{\mu\nu})$$

To this Metric, $\boxed{R^{\sigma}_{\alpha\beta\gamma} = 0}$ – The curvature is zero. (at least time components?)

Derivation of the Schwarzchild Metric:

↳ Let the Metric be denoted by $g_{\mu\nu}$

↳ The metric is designed in such a way that it only changes / depends on the radial parameter ('r').

↳ The angular momentum of the Mass is considered to be zero, so, there won't be any apparent changes in the position 4-vector and hence in the metric.

↳ Condition we will obtain the metric is that the spacetime is static.

Static spacetime:

Metric doesn't change with time

i.e $\boxed{\partial_t g_{\mu\nu} = 0}$

interchanging signs on the time parameter doesn't make a difference.

$t \to -t$ because metric will be

$$(\cancel{-}c\,dt\,\vec{e}_t)\cdot(\cancel{-}c\,dt\,\vec{e}_t)$$

$\& \eta_{00}$

↳ Here, $g_{r\theta}$ is determined closer to the centre of mass because at $r \approx \infty$, the value of $g_{r\theta} \approx \eta_{r\theta}$ i.e the space is assumed to be flat at greater distance.

<u>Components as variables with functions of 'r'</u>.

↳ First determine the spatial components of the Schwarzchild metric.

↳ Since the spatial basis vectors are orthogonal, $\boxed{g_{ij} = 0}$ if $\boxed{i \neq j}$.

↳ $g_{\theta\theta}$ & $g_{\phi\phi}$ remain the same since the angular momentum of the mass is zero in both 'θ' & 'ϕ' components.

↳ The g_{tt} and g_{rr} components of the Schwarzschild metric vary only with the distance (radial) parameter. So, they can be arbitrarily written as functions of 'r'.

↳ let $g_{tt} = A(r)$ and $g_{rr} = -B(r)$, $A(r)$ and $B(r)$ be arbitrary functions varying with radius (r).

Now the metric looks like.

$$g_{\mu\nu} = \begin{bmatrix} A(r) & 0 & 0 & 0 \\ 0 & -B(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix}$$

Now, we find the values for $A(r)$ and $B(r)$ by following a few steps.

✓ Step-1: Solve for all Christoffel symbols (Connection Co-efficients).

✓ Step-2: Find the Ricci Curvature ($R_{\mu\nu}$) components from Step-I values

✓ Step-3: Substitute the values of $R_{\mu\nu}$ from Step 2 in the equation obtained from beginning.

$$\boxed{R_{\mu\nu} = 0}$$

✓ Step-4: Obtain the differential equations for individual variables ($A(r)$ & $B(r)$) seperately solve them.

✓ Step-5: From general solutions, establish a relationship b/w $A(r)$ and $B(r)$. Use boundary conditions.

✓ Final Step: Force the Schwarzchild metric into the "Weak Field limit" conditions and apply Newtonian gravity equations to find constants in $A(r)$ & $B(r)$

Step 1: Solving for Christoffel symbols

Step-1: Solving for Christoffel symbols

(as functions of $A(r), B(r), r, \theta$ & ϕ)

The generic connection co-efficients can be solved through this relation formula by pluggin in the metric.

$$\boxed{\Gamma^{\sigma}_{\mu\nu} = \frac{g^{\sigma\alpha}}{2}\left(\partial_{\nu} g_{\alpha\mu} + \partial_{\mu} g_{\alpha\nu} - \partial_{\alpha} g_{\mu\nu}\right)}$$

In this scenario, the christoffel symbols in 4d have '13' independent components ($^4P_3 = 24$ – zero components).

Note: Due to static condition veiled in space time, the metric tensor derivatives taken with respect the metric ... with respect to time are zero. $\boxed{\partial_t g_{\mu\nu} = 0}$

$\Gamma_{01}^{0} = \Gamma_{10}^{0} = \frac{1}{2A}\left(\partial_r(A)\right)$	
$\Gamma_{00}^{1} = \frac{1}{2B}\partial_r A$;	$\Gamma_{11}^{1} = \frac{1}{2B}\partial_r(B)$
$\Gamma_{33}^{1} = -\frac{r\sin^2\theta}{B}$	; $\Gamma_{33}^{2} = -\sin\theta\cos\theta$
$\Gamma_{12}^{2} = \Gamma_{21}^{2} = \frac{1}{r}$	; $\Gamma_{13}^{3} = \Gamma_{31}^{3} = \frac{1}{r}$
$\Gamma_{23}^{3} = \Gamma_{32}^{3} = \cot\theta$	

Step-2: Calculate the Ricci Curvature ($R_{\mu\nu}$) Components.

Note: From the initial conditions we considered

- No charge.
- Non rotating ($\partial_t(\theta,\phi)=0$)
- No immediate volume changes. ($T_{\mu\nu}=0$)
- No cosmological expansion ($\Lambda=0$)

Zero Ricci Curvature

$$\boxed{R_{\mu\nu}=0}$$

$\Rightarrow$ This will be used to obtain relations with $R_{\mu\nu}$ which form from the basic notations of Riemann Curvature.

Now, it's important to note that the Riemann Curvature

Also, [illegible]
arround this 4d spacetime is `non zero'.

i.e $\boxed{R^{\sigma}{}_{\alpha\beta\gamma} \neq 0}$

Early takeaways from differential geometry, Ricci Curvature tensor components can be obtained by contracting the Riemann Curvature tensor components on their second indices.

So, Solving for Riemann Tensor components from the values of metric and Connection co-efficients as functions of A(r), B(r), r, θ and ϕ.

Generic formulation of Riemann Curvature components when the values of Connection Co-efficients are available.

$$\boxed{R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\rho}_{\mu\alpha} - \Gamma^{\beta}_{\mu\sigma}\Gamma^{\rho}_{\nu\beta}}$$

Contracting $R^{\rho}{}_{\sigma\mu\nu} \Rightarrow R^{\rho}{}_{\sigma\rho\nu}$; we obtain the Ricci Curvature tensor.

so, $$\boxed{R_{\sigma\nu} = R^{\rho}{}_{\sigma\rho\nu} = \partial_\rho \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\rho\sigma} + \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\rho}_{\rho\alpha} - \Gamma^{\beta}_{\rho\sigma}\Gamma^{\rho}_{\nu\beta}}$$

Calculating the diagonal elements of the Ricci Tensor

Matrix -

$$R_{00} = \frac{\partial_r \partial_r A}{2B} - \frac{\partial_r A \, \partial_r B}{4B^2} + \frac{\partial_r A}{rB} - \frac{(\partial_r A)^2}{4AB}$$

$$= 0$$

Simplifying the differential notations.....

$$\boxed{\begin{array}{ll} \partial_r A \rightarrow A' & \partial_r \partial_r A \rightarrow A'' \\ \partial_r B \rightarrow B' & \end{array}}$$

now, $R_{00} = \frac{A''}{2B} - \frac{A'B'}{4B^2} + \frac{A'}{rB} - \frac{(A')^2}{4AB} = 0$

Multiplying both sides with "$4AB^2r$"

$$R_{00} = \underline{2rABA''} - rAA'B' + 4ABA' - rB(A')^2 = 0$$

Similarly,

$$R_{11} = -\underline{2rABA''} + rB(A')^2 + rAA'B' + 4A^2B' = 0$$

And

$$R_{22} = -2AB + 2AB^2 - A'Br + rAB' = 0.$$

Step-3 and Step-4

But from the initial conditions of the Schwarzchild metric

$R_{\mu\nu}\ \forall\ \mu\ \&\ \nu = 0$ so, $\boxed{R_{\alpha\beta} + R_{\gamma\delta} = 0}$

so $R_{11} + R_{00} = 0.$

Adding them $R_{11} + R_{00} = \boxed{4A^2B' + 4ABA' = 0}$

$$AB' + BA' = 0$$

$$\Rightarrow \quad A\partial_r B + B\partial_r A = 0$$

$$\Rightarrow \quad A\frac{\partial B(r)}{\partial r} + B\frac{\partial A(r)}{\partial r} = 0.$$

on using chain rule in reverse.

$$\boxed{\partial_r (AB) = 0}$$

solution to this partial differential equation is

$\boxed{AB = \text{Constant}}$ For all values of "r"

↳ Now considering the value of Schwarzchild Metric tensor matrix at an extreme boundary.

At Farther distance from the centre of Mass ($r \to \infty$), the Schwarzchild metric is almost identical to Minkowski Metric.

Mathematically, at $r \to \infty$; $g_{\mu\nu} = \eta_{\mu\nu}$

$$g_{\mu\nu} = \begin{bmatrix} A(r) & 0 & 0 & 0 \\ \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ \end{bmatrix}$$

$$\begin{bmatrix} 0 & -B(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix} \quad \& \quad \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \eta_{\mu\sigma}$$

So, we have the solution

$$A(r)\, B(r) = 0 \;; \text{ for all values of } 'r',$$

So, $r = \infty$; $A(\infty)\, B(\infty) = \text{Constant}.$

$$\boxed{(1)(1) = \text{Constant}}$$

Substituting this value in $\boxed{AB = \text{Constant}}$

$AB = 1 \quad \therefore \boxed{A(r) = \frac{1}{B(r)}}$

$\boxed{B(r)}$

Step-5

So, we have new relations

$\hookrightarrow$ $\boxed{B(r) = \frac{1}{A(r)}}$; taking derivative with respect to 'r' on both sides.

$$B'(r) = \frac{-1}{A^2(r)} \times \frac{\partial A(r)}{\partial r}$$

$\boxed{B' = -\frac{A'}{A^2}}$; Now use these relation's R_{22} equation so as to obtain an equation for single variable (here $A(r)$).

$$R_{22} = -2AB + 2AB' - rA'B + rAB' = 0$$

$$R_{22} = -2A(A)^{-1} + 2A\left(\frac{1}{A^2}\right) - \frac{rA'}{A} + rA\left(\frac{-A'}{A^2}\right) = 0$$

$$\Rightarrow \quad -2 + 2A^{-1} - rA'A^{-1} - rA'A^{-1} = 0$$

$$\Rightarrow \quad -2 + 2A^{-1} - 2rA'A^{-1} = 0$$

Multiplying the equation with "$\frac{A}{2}$" on both sides.

$$-A + 1 - rA' = 0$$

$$\Rightarrow \boxed{rA' = 1 - A(r)} \rightarrow$$ Final differential equation for getting

the value of $A(r)$.

This equation can be re-arranged as

$$\boxed{r\frac{\partial A(r)}{\partial x^r} - \frac{A(r)}{r} = \frac{1}{g}}$$ $\Rightarrow$ obtained as as a solvable form of differential equation.

The solution to the above equation

$$\boxed{A(r) = \left(1 - \frac{k}{r}\right)}$$ $\Rightarrow$ k being some constant.

Also, $B(r) = \frac{1}{A(r)}$; So, $\boxed{B(r) = \left(1 - \frac{k}{r}\right)^{-1}}$

$A(r)$

Final Step: Forcing the Schwarzchild Metric into "Weak field Limit" to get value of "k":

Main Conditions & Notations of "Weak Field Limit" $\Rightarrow$

$t = \tau$	$T_{00}^{\,i} = \frac{\partial \phi}{\partial x^i} \times \frac{1}{c^2}$
$U^0 = c$	$g_{\mu 0} = h_{\mu 0} + \eta_{\mu 0}$
$U^i = 0$	$g = -\nabla \phi$

↳ So, at weak field limit, at a local region, the einstien's field equations will be reduced to Newtonian gravity.

↳ The metric is assumed to be flat with a small added

Component. (Not yet forced to Newtonian)

so, $\boxed{g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}}$ at weak field limit.

$\boxed{\text{given } \|h_{\mu\nu}\| \ll 1}$

Now, we obtain an expression for h_{00} component so as to find a value of $A(r)$ and hence $B(r)$ in the schwarzchild metric.

↳ From the geodesic equation, it's clear that $\boxed{\Gamma^{\nu}_{00} = \frac{1}{c^2}\nabla\phi}$

'c' being the velocity of light and 'ϕ'

being the scalar gravitational potential function.

To find "h_{00}" component, we use "Γ_{00}^{i}" component of the connection co-efficients and obtain h_{00} as a function of it.

so, Generally, $\boxed{\Gamma^{\sigma}_{\mu\nu} = \frac{g^{\sigma\alpha}}{2}\left(\partial_{\nu} g_{\alpha\mu} + \partial_{\mu} g_{\alpha\nu} - \partial_{\alpha} g_{\mu\nu}\right)}$

so, $\Gamma_{00}^{i} = \frac{g^{i\alpha}}{2}\left(\partial_0 g_{\alpha 0} + \partial_0 g_{\alpha 0} - \partial_{\alpha} g_{00}\right)$

From static spacetime condition

$$\boxed{\partial_t g_{\mu\nu} = 0}$$

So $\quad T^{i}_{00} = -\frac{g^{i\alpha}}{2}\left(\partial_\alpha g_{00}\right)$

$$\Rightarrow T^{i}_{00} = -\frac{g^{i\alpha}}{2}\left(\underbrace{\partial_\alpha \eta_{00}}_{0} + \partial_\alpha h_{00}\right)$$

$$\boxed{T^{i}_{00} = -\frac{g^{i\alpha}}{2} \times \left(\partial_\alpha h_{00}\right)}$$

$\Downarrow$

But if $g_{\mu\nu}$ is forced to weak field limit,

at lower velocities, $\boxed{g_{\mu\nu} \approx \eta_{\mu\nu} \text{ \& } g^{\mu\nu} \approx \eta^{\mu\nu}.}$

So $$\Gamma^i_{00} = -\frac{\eta^{i\alpha}}{2}(\partial_\alpha h_{00})$$

here if $i \neq \alpha$, $\eta^{i\alpha} = 0$ (off diagonal)

so, only diagonal elements are non zero.

i.e η^{ii} are non zero.

So, $$\Gamma^i_{00} = -\frac{\eta^{ii}}{2}(\partial_i h_{00}) \Rightarrow \Gamma^i_{00} = -1 \times (-1)(\partial_i h_{00})$$

$$\boxed{\therefore \; \Gamma^i_{00} = \frac{1}{2}\partial_i h_{00}}$$ Also, $$\boxed{\Gamma^i_{00} = \frac{1}{c^2}\partial_i \phi}$$

comparing both equations -

$$\frac{1}{2}\partial_i h_{00} = \frac{1}{c^2}\partial_i \phi$$ (it's a summation not a single value).

Taking the "contravariant derivative" on both sides
(or)
(Inverse Gradient operator).

So, $$\partial^i\left(\frac{1}{2}\partial_i h_{00}\right) = \partial^i\left(\frac{1}{c^2}\partial_i \phi\right)$$ or Covariant Integral??

$$\Rightarrow \frac{1}{2}\partial^i(\partial_i h_{00}) = \frac{1}{c^2}\partial^i(\partial_i \phi)$$

$$\Rightarrow \frac{h_{00}}{2} = \frac{\phi}{c^2} \Rightarrow \boxed{h_{00} = \frac{2\phi}{c^2}}$$

Since the conditions are for weak field limit, the gravitation potential scalar "ϕ" is also Newtonian.

$$\therefore \boxed{\phi = -\frac{GM}{r}}$$

"M" being the mass of the object causing curvature in spacetime at a distance "r"

so $h_{00} = \frac{2\phi}{c^2} = -\frac{2GM}{rc^2}$; $h_{00} = \frac{-2GM}{rc^2}$

But our initial schwarzchild metric's g_{00} component is

$$g_{00} = \eta_{00} + h_{00}$$

$$\boxed{g_{00} = 1 - \frac{2GM}{rc^2}} \; ; \text{ Also} \rightarrow \boxed{g_{00} = A(r) = \left(1 - \frac{k}{r}\right)}$$

Comparing both equations.

$$1 - \frac{2GM}{rc^2} = 1 - \frac{k}{r}$$

$$\Rightarrow \boxed{k = \frac{2GM}{c^2}}$$

Main Take Aways:-

$$g_{\mu\nu} = \begin{bmatrix} \left(1 - \frac{r_s}{r}\right) & 0 & 0 & 0 \\ & \left(1 - \frac{r_s}{r}\right)^{-1} & 0 & 0 \end{bmatrix} \Rightarrow$$

Schwarzchild Metric

$$\begin{bmatrix} \cdots & & & \\ 0 & \left(1-\frac{\cdot}{r}\right) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$ Metric solution.

↳ The expression $k = \frac{2GM}{rc^2}$ is also written as

$$\boxed{r_s = \frac{2GM}{rc^2}}$$ and called "<u>Schwarzschild Radius</u>". of a Non Rotating Mass.

$1.482 \times 10^{-27} \times$ Mass meters.

If $r \geq r_s$; then the field equations are valid as the time metric doesn't change sign.

But if $r < r_s$; then field equation lose their meaning as Space and time behave opposite to what "they actually do." # R.I.P A.Einstien.