## Relativistic Momentum:

**Event**: A mass is being split into two parts. It's veiwed from '2' reference frames $S$ & $S'$ where $S'$ is moving $\rightarrow$ relative to $S$ at velocity '$\underline{v}$'.

**Initial Assumption**: Mass in two reference frames are identical.

| Frame S | Frame S' |
|---|---|
| $\bullet\ M, v = 0$ | $\leftarrow\bullet\ M'v$ |
| $\xleftarrow{\ \ }\bullet\ \ \dfrac{M}{2}v$     $\bullet\xrightarrow{\ \ }\ \ \dfrac{M}{2}v$ | $\xleftarrow{\ \ }\bullet\ \ \dfrac{M'}{2} \times \dfrac{2v}{1 + \frac{v^2}{c^2}}$     $\bullet\ \dfrac{M'}{2}, v = 0$ |
| Initial momentum $= 0$ | Initial momentum $= -Mv$ |
| Final momentum $= \dfrac{M}{2}v - \dfrac{M}{2}v$ | Final momentum |

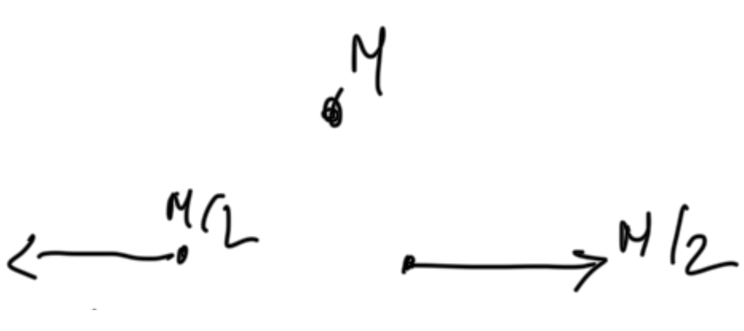
| | |
|---|---|
| $= 0$ <br> ∴ Momentum is conserved in frame S | Final momentum <br> $= 0 + -\left(\frac{M'}{\cancel{2}} \times \frac{\cancel{2}v}{1+\frac{v^2}{c^2}}\right)$ <br> ∴ Initial momentum ≠ Final momentum. |

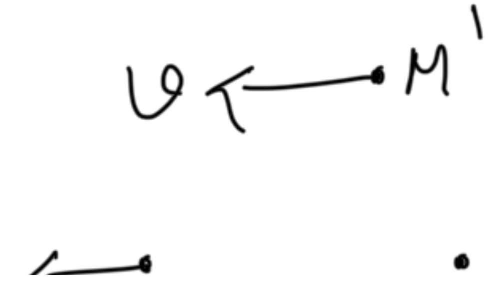
So, the considered assumption is incorrect and mass is not the same for moving objects (inside a reference frame)

**Note:** But mass will Always be conserved in a reference frame.

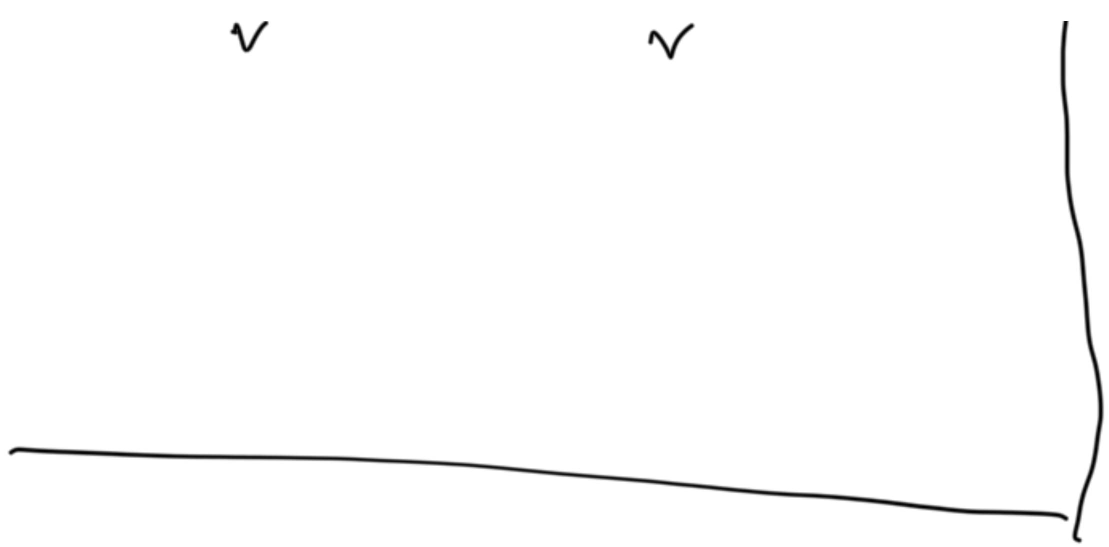
Sketching the same scenario.

| | |
|---|---|
| <u>Frame S</u> | <u>Frame S'</u> |




Here, '$m_0$' is the rest mass



$m_0$

$m, V = \frac{2v}{1+\frac{v^2}{c^2}}$ & $m'$ is the relativistic mass

here $M' = m_0 + m$

According to conservation of momentum,

$$M'\cancel{V} = \frac{2m\cancel{V}}{1+\frac{v^2}{c^2}}$$

$$M' = \frac{2m}{1+\frac{v^2}{c^2}}$$

So,

$$M' = m_0 + m$$

$$\frac{2m}{1+\frac{v^2}{c^2}} = m_0 + m$$

$$m\left(\frac{2}{1+\frac{v^2}{c^2}} - 1\right) = m_0$$

$$\Rightarrow m\left(\frac{2-1-\frac{v^2}{c^2}}{1+\frac{v^2}{c^2}}\right) = m_0 \Rightarrow m\left(\frac{1-\frac{v^2}{c^2}}{1+v^2}\right) = m_0$$

Now writing $v$ as a function $V$ (relativistic velocity of moving mass)

Consider the term

$$1-\frac{v^2}{c^2} \Rightarrow 1+\frac{\left(\frac{2V}{1+\frac{V^2}{c^2}}\right)^2}{c^2}$$

$$\Rightarrow 1-\frac{\cancel{4}V^2}{\left(1+\frac{V^2}{c^2}\right)^2 c^2} \Rightarrow \frac{\left(1+\frac{V^2}{c^2}\right)^2-\frac{4V^2}{c^2}}{\left(1+\frac{V^2}{c^2}\right)^2}$$

$$\Rightarrow \left[\frac{\left(1+\frac{V^2}{c^2}\right)^2-\left(\frac{4V^2}{c^2}\right)}{\left(1+\frac{V^2}{c^2}\right)^2}\right] \Rightarrow \left(1-\frac{v^2}{c^2}\right)=\left(\frac{1-\frac{V^2}{c^2}}{1+\frac{V^2}{c^2}}\right)^2$$

$$\Rightarrow \sqrt{1-\frac{v^2}{c^2}} = \frac{1-\frac{v^2}{c^2}}{1+\frac{v^2}{c^2}}$$

$$\therefore m_0 = m\left(1-\frac{v^2}{c^2} \Big/ 1+\frac{v^2}{c^2}\right) \Rightarrow \boxed{m_0 = m_{rel}\sqrt{1-\frac{v^2}{c^2}}.}$$

So,

Momentum in $S'$ is given by

$$\vec{p} = m_{rel}\vec{V} \Rightarrow \boxed{\vec{p} = \frac{m_0\vec{V}}{\sqrt{1-\frac{v^2}{c^2}}}}$$

## Relativistic Kinetic Energy:

In a given reference frame

A small change in kinetic energy is caused due to a small force leading in some displacement

so, we have

$$\vec{p} = \frac{m_0 \vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

but $\vec{F}$ = force on an object = $\frac{d\vec{p}}{dt} = \frac{m_0 \, d\left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right)}{dt}$

Also, work done = Change in KE

10)

$$\vec{F}\cdot\vec{ds} = \vec{dk}$$

$$m_0 \frac{d\left[\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right]}{dt} \times dx = dk$$

$$\Rightarrow m_0 v \, d\left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right) = dk \Rightarrow \frac{d\left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right)}{dv} = \frac{\sqrt{1-\frac{v^2}{c^2}} - \frac{v}{\cancel{2}}\times\frac{\cancel{2}v}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-1/2}}{1-\frac{v^2}{c^2}}$$

$$\Rightarrow \frac{\sqrt{1-\frac{v^2}{c^2}} + \frac{v^2}{c^2}\times\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}}{1-\frac{v^2}{c^2}} = \frac{d\left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right)}{dv}$$

$$\Rightarrow \quad \frac{1-\frac{v^2}{c^2}+\frac{v^2}{c^2}}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}\,dv = d\left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

$$m_0 v\left(\frac{1-\frac{v^2}{c^2}+\frac{v^2}{c^2}}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}\,dv\right) = dK$$

$$\Rightarrow \quad m_0 v\left(\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}\right)dv = dK$$

$$\Rightarrow \quad dK = \frac{m_0\, v\,dv}{\ldots}$$

now

take $1-\frac{v^2}{\ldots} = x^2 \rightarrow x = \sqrt{1-\frac{v^2}{\ldots}}$

$$\left(1-\frac{v^2}{c^2}\right)^{3/2}$$

$$\frac{-\cancel{2}v\,dv}{c^2} = \cancel{2}x\,dx$$

$$\Rightarrow v\,dv = -c^2 x\,dx$$

<u>Integrating on both sides</u>

$$\text{So, } \int_0^k dk = \int_0^v \frac{m_0 v\,dv}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}$$

Also, $v$ from $0 \to v$

$x$ from $1 \to \left(1-\frac{v^2}{c^2}\right)^{1/2}$

$$k = \int_1^{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \frac{m_0 \times -c^2 x\,dx}{x^3}$$

$$k = -m_0c^2 \int_1^x x^{(1-3)}\,dx \quad \Rightarrow k = -m_0c^2 \int_1^x x^{-2}\,dx$$

$$= -m_0c^2 \left(\frac{x^{-1}}{-1}\right)_1^x$$

$$\Rightarrow k = m_0c^2\left[\frac{1}{x} - 1\right] \quad \text{but } x = \sqrt{1-\frac{v^2}{c^2}}$$

So, $K = m_0c^2\left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right] \Rightarrow K = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0c^2$

$$\Rightarrow \boxed{\underbrace{K}_{\text{Kinetic energy}} + \underbrace{m_0c^2}_{\text{Rest mass Energy}} = \underbrace{\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}}_{\text{Total energy}}}$$

Let Total Energy = E

So, $E = m_0c^2 + KE$

Comparing E with adjacent equation

$E = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \boxed{E = \gamma m_0c^2}$

At non relativistic speeds, $v \lll c$, so

$\left(1-\frac{v^2}{c^2}\right)^{-1/2} = \left(1+\frac{v^2}{2c^2}\right)$, So, $K = m_0c^2\left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right]$

$\Rightarrow K = m_0c^2\left[1+\frac{v^2}{2c^2} - 1\right] \Rightarrow \boxed{K = \frac{1}{2}m_0v^2}$

Question:

It speed of causality is $\frac{1}{c}$,

then why $dt \to 0$ in calculus,

shouldn't it be $\boxed{dt \to \frac{1}{c}}$.

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<u>Vectors</u> <u>and</u> <u>Tensors</u>: <u>Relativity</u>.

<u>Contravariant</u> <u>Vectors</u>:

Note: " — " - based co-ordinates.

else - normal co-ordinates.

Component transformation

Law:

$$\boxed{A^{\bar{\mu}} = \Lambda^{\bar{\mu}}_{\ \alpha} A^{\alpha}}$$

$\Lambda^{\bar{\mu}}_{\ \alpha}$ being the Lorentz transformation

General vector notation:

$$\vec{V} = A^{\alpha} \vec{e}_{\alpha}$$

"Contravariant" components and "Co-variant" basis

For transformation of Basis

$$\vec{e}_{\alpha} = \Lambda^{\bar{\mu}}_{\ \alpha} \vec{e}_{\bar{\mu}}$$

Taking inverse Lorentz transformation on both sides

$$\vec{e}_\alpha \left(\Lambda^{\bar{\mu}}{}_{\alpha}\right)^{-1} = I\,\vec{e}_{\bar{\mu}}$$

↳ since it's a summation, switch to new variable.

$$\boxed{\vec{e}_{\bar{\mu}} = \Lambda^{\beta}{}_{\bar{\mu}}\,\vec{e}_{\beta}}$$ → Basis transformation.

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Invariant quantities:

for flat space time geometry: distance b/w two events is

$$\Delta s^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2.$$

Important ⇒ $c \to 1$ (for convention).

For dot product b/w two vectors.

$$\vec{A}\cdot\vec{B} = (A^\alpha \vec{e}_\alpha)\cdot(B^\beta \vec{e}_\beta).$$

$$= A^\alpha B^\beta (e_\alpha \cdot e_\beta).$$

$$\boxed{\vec{A}\cdot\vec{B} = \eta_{\alpha\beta} A^\alpha B^\beta}$$

For "flat" non euclidian space time

$$\eta_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To measure distance put $\vec{B} = \vec{A}$.

so, $$\vec{A}^2 = \boxed{\vec{A}\cdot\vec{A} = \eta_{\alpha\beta} A^\alpha A^\beta.}$$

or more easily.

$$\boxed{\Delta s^2 = \Delta x^\alpha \Delta x^\beta \eta_{\alpha\beta}}$$

Here $\eta_{\alpha\beta}$ is called the metric tensor and it's a rank 2 tensor which is invariant "for" a coordinate system

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$ for 4d spherical system

## Velocity - 4 Vector Analogy.

$\vec{v} = d\vec{x}$ $\quad$ $d\tau \to$ being change in "Proper time"

$u = \frac{dx}{d\tau}$ ... proper time

(time measured by the person on the trajectory).

Also,

$\vec{x} = (x^0, x^1, x^2, x^3)$.

& $dx^0 = c\,dt$.

and $\frac{c\,dt}{d\tau} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma$ (from time dilation)

Now, finding $\vec{u}\cdot\vec{u}$ (assuming the space co-ordinates are Cartesian (orthogonal)).

So, $\vec{u}\cdot\vec{u} = x^0 \cdot x^0 + x^1 x^1 + x^2 x^2 + x^3 x^3$

$= -\gamma^2 c^2 + |u^2| \gamma^2$

for a body at rest.

$|u^2| = 0$ & $\gamma^2 = 1$

so, $\vec{U} \cdot \vec{U} = -\gamma^2 c^2$.

<u>Same Analogy for 4-momentum:</u>

$$4\text{ momentum} = \vec{P} = m\vec{U}.$$

so, $\vec{P} \cdot \vec{P} = m^2 \; \vec{U} \cdot \vec{U}$.

$$= m^2 \left(-\gamma^2 c^2 + \gamma^2 |u|^2\right)$$

$$\vec{P}\cdot\vec{P} = |P|^2 = \frac{-m^2c^2}{\left(1-\frac{v^2}{c^2}\right)} + \frac{|\vec{p}|^2}{\left(1-\frac{v^2}{c^2}\right)}$$

For a body at rest, $v=0$.

$$|P|^2 = -m^2c^2 + \underbrace{|\vec{p}|^2}$$ → spatial moment

multiply both sides with $c^2$

$$\underbrace{|P|^2c^2}_{E_{rel}} = -m^2c^4 + \underbrace{|\vec{p}|^2c^2}_{E_{rest}}.$$

$$\Rightarrow \quad |\vec{p}|^2c^2 - |P|^2c^2 = m^2c^4$$

$$\boxed{E^2 - p^2c^2 = m^2c^4}$$ → Invariant quantity.