Gravitational Time dilation & Length Contraction due to a stationary Black Hole

(net Angular momentum = 0).

Schwarzschild metric ($\partial_t, \partial_\phi, \partial_\theta = 0$)

$$g_{\mu\nu} = \text{diagonal}\left[\left(1-\frac{r_s}{r}\right)dt^2 - \left(1-\frac{r_s}{r}\right)^{-1}dr^2\right]$$

$$r_s = \frac{2GM}{c^2}$$

Calculating the proper time.

τ = proper time which changes with distance from the mass because the time metric is not constant.

So, to find proper time of the object at a distance 'r' from the mass, we must integrate the tangent vectors along the path parameter (λ).

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu.$$

Assuming $\boxed{d\theta, d\phi = 0}$

$$= g_{tt}\,[d(ct)]^2 + g_{rr}\, dr^2.$$

$$= c^2 g_{tt}\, dt^2 + g_{rr}\, dr^2.$$

$= c^2 g_{tt} dt^2 + g_{rr} dr^2$

So, if the object is fixed at 'r', then $dr = 0$.

And ds^2 will be the change in proper time of the object since the integrated path parameter moves along tangent vectors and τ changes on acceleration (geodesic deviation)

so, $\boxed{ds^2 = c^2 d\tau^2}$

& dr^2.

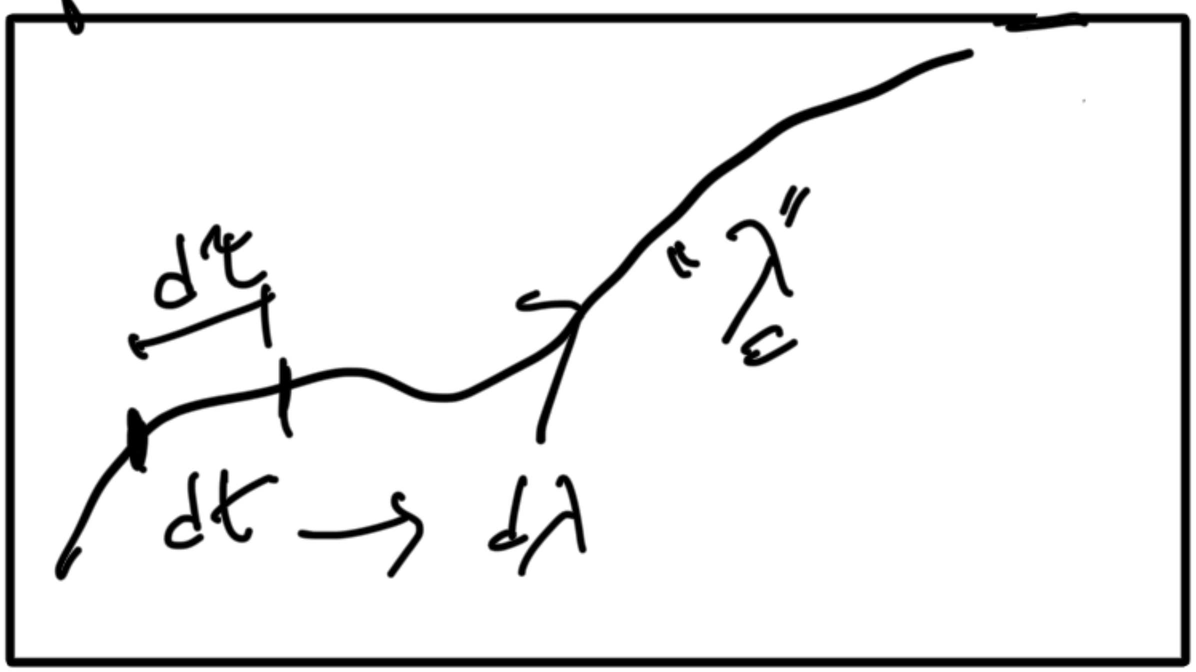
$\cancel{c^2} d\tau^2 = \cancel{c^2} g_{tt} dt^2$

so, $d\tau^2 = g_{tt} dt^2$

$$d\tau = \sqrt{g_{tt}}\, dt.$$

so, $$\boxed{d\tau = \sqrt{1 - \frac{r_s}{r}}\, dt}$$

Now this is the change in proper time of a small piece of the path λ.



To find the whole time dilation, integrate on both sides.

$$\int^{\lambda} d\tau = \int^{\lambda} \sqrt{1-\frac{r_s}{r}}\, dt$$

$$r_s = \frac{2GM}{c^2}$$

$$\boxed{\tau = \sqrt{1-\frac{r_s}{r}}\; t} \Rightarrow$$ Gravitational time dilation equation.

At constant "R".

Comparing gravitational time dilation to kinematic time dilation.

Kinematic Time dilation:

Gravitational time dilation:

$$\tau = \sqrt{1 - \frac{v^2}{c^2}}\, t$$

$$\tau = \sqrt{1 - \frac{2GM}{rc^2}}\, t$$

Comparing both equations;

when acceleration / staying position R:-

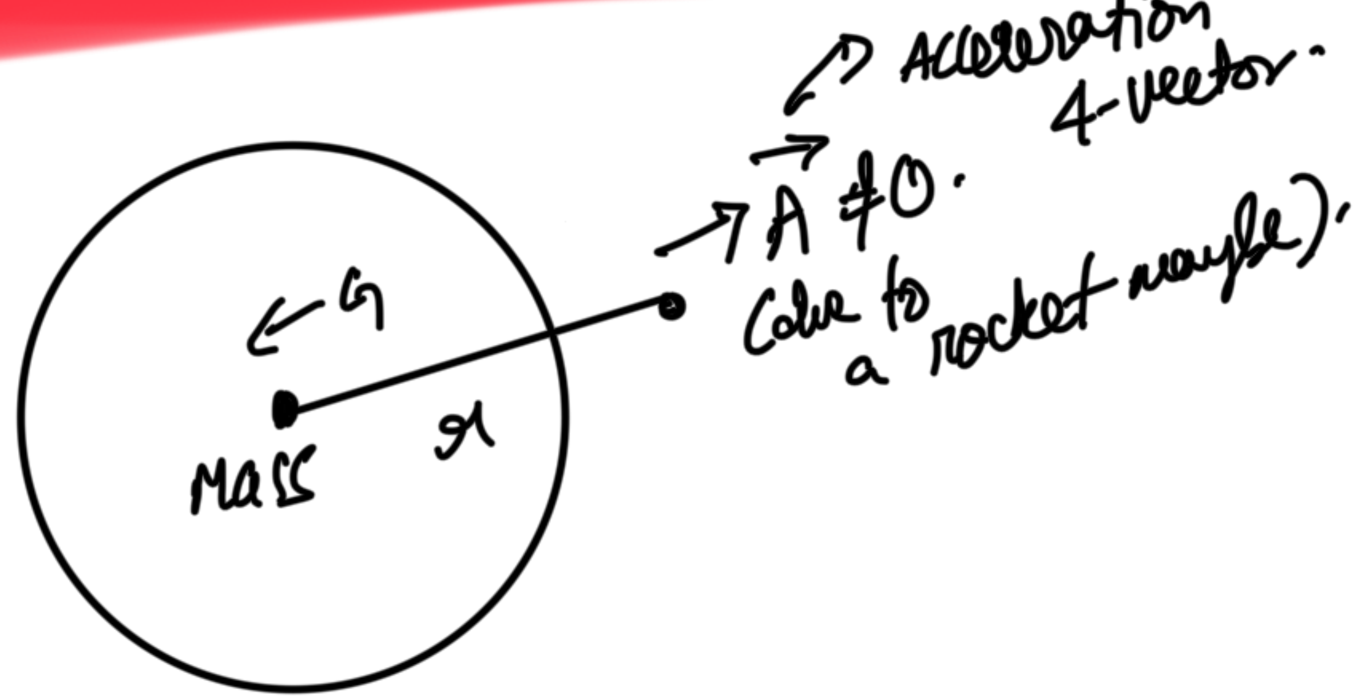
$$v^2 = \frac{2GM}{r} \Rightarrow \boxed{v = \sqrt{\frac{2GM}{r}}}$$

The notation for escape velocity.

\# This is a special case where "r" is constant.

But in reality, when a body is in space, it's naturally free falling towards the mass.

So, there is apparently another force (Acceleration 4 vector) opposite to the gravity to hold the object still.



Length contraction due to Schwarzschild metric.

↳ Earlier, time dilation is calculated a given point at a distance 'r' (point being stationary due to opposite force).

so, revisiting our standard formula

$$ds^2 = g_{tt}\, c^2 dt^2 + g_{rr}\, dr^2$$

(assuming $d\theta, d\phi = 0$).

Now, length contraction also occours and it can be measured in certain point in time.

so, $\boxed{dt = 0}$ & ds = proper length of path → $= dL_0$.

$$ds^2 = g_{rr}\, dr^2.$$

or

$$dL_0^2 = g_{rr}\, dr^2 \Rightarrow dL_0 = \sqrt{g_{rr}}\, dr.$$

Solving the integral over the path λ.

$$\int dL_0 = \int \sqrt{g_{rr}}\, dr.$$

[note: this is only valid for $r > r_s$]

From Schwarzschild metric:

$$g_{rr} = \left(1 - \frac{r_s}{r}\right)^{-1} = \left(\frac{r - r_s}{r}\right)^{-1} = \left(\frac{r}{r - r_s}\right)$$

so,

$$\boxed{L_0 = \int^{\lambda} \sqrt{\frac{r}{r - r_s}}\, dr}$$

$\lambda \Rightarrow r_1$ to r_2 (two point in space)

To solve this integral, we use substitution.

$r = r_s \sec^2\theta$.

so, $dr = 2r_s \sec^2\theta \tan\theta \, d\theta$.

so, $L_0 = \int \sqrt{\dfrac{\cancel{r_s}\sec^2\theta}{\cancel{r_s}(\sec^2\theta - 1)}} \times 2r_s \sec^2\theta \tan\theta \, d\theta$.

$\Rightarrow L_0 = 2r_s \int \dfrac{\sec\theta}{\cancel{\tan\theta}} \times \sec^2\theta \cancel{\tan\theta} \, d\theta$. $\Rightarrow L_0 = 2r_s \int \sec^3\theta \, d\theta$.

let $I = \int \sec^3\theta \cdot d\theta$. , so, $L_0 = 2r_s \times I$.

so, $I = \int \underset{v'}{\sec^2\theta} \, \underset{u}{\sec\theta} \, d\theta$. $\Rightarrow$ applying $\int u v' = uv - \int u'v$.

$v' = \sec^2\theta \, d\theta$. $u' = \dfrac{d}{d\theta}\sec\theta = \sec\theta \tan\theta$.

$v = \tan\theta$.

so, $I = \sec\theta\tan\theta - \int \sec\theta\tan\theta \times \tan\theta \, d\theta$.

$I = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta \, d\theta \Rightarrow I = \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta \, d\theta$.

$$I = \sec\theta\tan\theta + \int \sec\theta \, d\theta - \underbrace{\int \sec^3\theta \, d\theta}_{I}.$$

$$I = \sec\theta\tan\theta + \int \sec\theta \, d\theta - I$$

$$2I = \sec\theta\tan\theta + \int \sec\theta \, d\theta.$$

$$2I = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|$$

$$I = \frac{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|}{2}$$

But $r = r_s \sec^2\theta \Rightarrow \sqrt{\frac{r}{r_s}} = \sec\theta$.

$\sqrt{r_s}$

Also, $\tan^2\theta = \sec^2\theta - 1$

$$\tan^2\theta = \frac{r}{r_s} - 1 \Rightarrow \tan\theta = \sqrt{\frac{r - r_s}{r_s}}$$

So, $$I = \frac{\sqrt{\frac{r}{r_s}} \times \sqrt{\frac{r-r_s}{r_s}} + \ln\left|\sqrt{\frac{r}{r_s}} + \sqrt{\frac{r-r_s}{r_s}}\right|}{2}.$$

$$I = \frac{\frac{1}{r_s}\sqrt{r(r-r_s)} + \ln\left|\frac{1}{\sqrt{r_s}}\left[\sqrt{r} + \sqrt{r-r_s}\right]\right|}{2}$$

From First substitution, $L_0 = 2r_s \times I$.

so, $$L_0 = \cancel{2} r_s \times \left[\frac{\frac{1}{r_s}\sqrt{r(r-r_s)} + \ln\left|\frac{1}{\sqrt{r_s}}\left(\sqrt{r} + \sqrt{r-r_s}\right)\right|}{\cancel{2}} \right]$$

$$L_0 = \sqrt{r}\,\sqrt{r-r_s} + r_s \ln\left|\frac{\sqrt{r}+\sqrt{r-r_s}}{\sqrt{r_s}}\right|$$

$$L_0 = \left[\sqrt{r}\,\sqrt{r-r_s} + r_s\ln\left(\sqrt{r}+\sqrt{r-r_s}\right) - r_s\ln\left(\sqrt{r_s}\right)\right]_{r_1}^{r_2}$$

so, Finally..... the gravitational length Contraction at a given point in time is. . .

$$\boxed{L_0 = \left[\sqrt{r}\,\sqrt{r-r_s} + r_s\cdot\ln\left|\sqrt{r}+\sqrt{r-r_s}\right| - \frac{r_s}{2}\ln|r_s|\right]_{r_1}^{r_2}}$$