Tensors & Differential Geometry:

Vectors and Co-vectors.
Linear maps.
Matrix notations.
Tensors and Tensor products.

Vectors & Co-vectors:

Vectors -> points in space with specific direction.

written with a column matrix notation.
have co-variant basis & contravariant components.

Example:

$$\vec{V} = \begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix} \rightarrow \text{vector in 3d space and some basis} \rightarrow \vec{V} = a^1\vec{e_1} + a^2\vec{e_2} + a^3\vec{e_3}$$

or $\vec{V} = a^i\vec{e_i}$ (Einstien Summation convention)

here a^i is component of vector in $\vec{e_i}$ direction.

Basis vectors: Given by $\vec{e_i}$. In Cartesian system, $\perp$ to eachother (orthonormal). Example $\Rightarrow \vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Transformations: Vectors can be expressed in any different basis but they retain properties like lengths and angles.

let initial basis be $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

let that basis now transform into other basis.

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Now let's say vector $\vec{V} = a^1 \vec{e}_1 + a^2 \vec{e}_2$

after transformation $\vec{V'} = b^1 \vec{u}_1 + b^2 u_2$

It's transformation can be written as.

$$\vec{V'} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a^1 \\ a^2 \end{bmatrix} = \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}$$ (right to transfrom matrix)

This is a "linear transformation" because the transformed basis can be written as a linear function of old basis

↳ The "determinant" of the transformation matrix $\det[T]$ gives us the area of each block in that co-ordinate basis.

Co-vectors:

Just like we defined geometric objects using Vectors, we can define them using Co-vectors.

So, $\begin{bmatrix} a^1 \\ a^2 \end{bmatrix} \rightarrow$ column vector $\qquad [b_1\ b_2] \rightarrow$ 'row vector' (or) Co-vector.

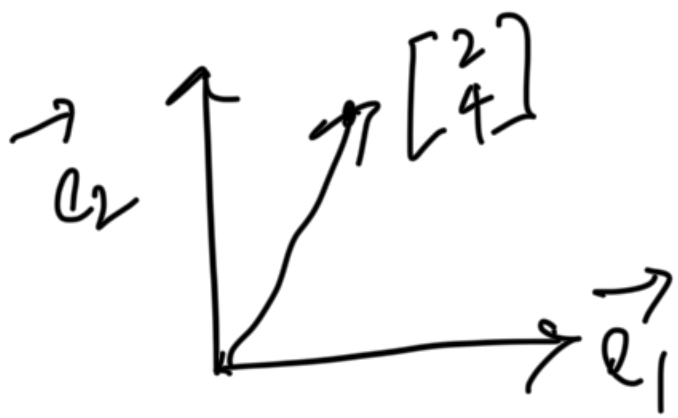
How to define Co-vectors:

They are essentially functions that map Vectors to a scalar value.

Example: take a co-vector $[2\ 4]$ and apply it to a vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$. Since we defined it as a function call it $\alpha(\vec{v})$.

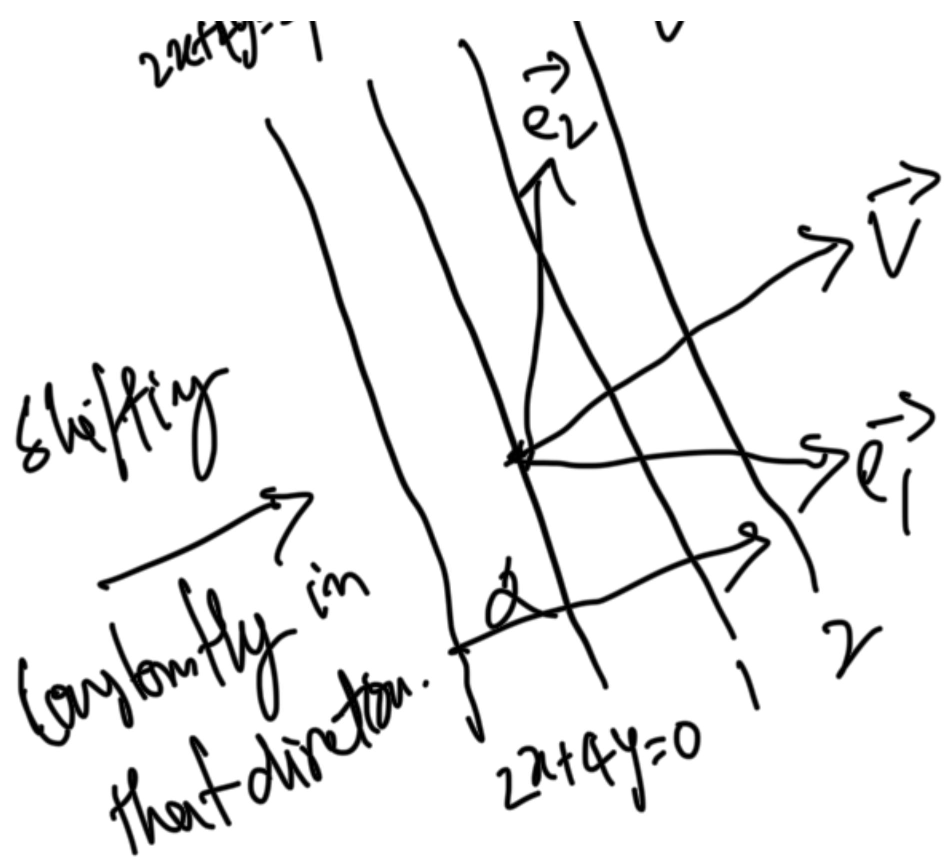
$\therefore \alpha(\vec{v}) = [2\ 4]\begin{bmatrix} x \\ y \end{bmatrix} = 2x + 4y.$

So, 2x+4y is a scalar function which can be represented in a "vector space".



→ vector representation.

For all values of 2x+4y



$2x+4y=0 \Rightarrow y = -x/2$

$2x+4y=1 \Rightarrow y = \frac{1-2x}{4}$

Here $\alpha(\vec{V})$ is just the grid of straight lines which are shifting constantly in one direction.

$\therefore$ "α" can be represented as a direction oriented geometric object just like vectors.

Note: i) Vectors and co-vectors lie in different spaces.

(i.e) Covectors won't lie in vector space and vice versa.

ii) Covectors have their own basis.

(iii) they are linear and scalable.

Co-vector Basis:

↳ In the previous example, we called [2 4] as a co-vector.

↳ A vector like $\begin{bmatrix} a^1 \\ a^2 \end{bmatrix} = a^1\vec{e_1} + a^2\vec{e_2}$ i.e it's represented with a set of basis vectors. $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

↳ How do we define a basis for a co-vector??

Fact: (i) Covector basis is not similar to vector basis because they lie in "different spaces".

(ii) Like co-vectors, co-vector basis also convert vectors (basis) into scalars.

Let's define a new set of basis for co-vectors just like we did for vectors.

Vector basis = $\{\vec{e_1}, \vec{e_2}\}$.

Co-vector basis → functions that map vector basis to a scalar.

So let co-vector basis = $\{\epsilon^1, \epsilon^2\}$.

They exist such that: (For "orthonormal basis").

$$\epsilon^1(\vec{e_1}) = 1 \qquad \epsilon^2(\vec{e_1}) = 0$$

$$\epsilon^1(\vec{e}_2) = 0 \qquad \epsilon^2(\vec{e}_2) = 1$$

or $\boxed{\epsilon^i(\vec{e}_j) = \underline{\underline{\delta^i_j}}}$ → "<u>Kronecker delta</u>"

$$\delta^i_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Also, when covector basis "$\underline{\underline{\epsilon^i}}$" acts on a vector, it spits out the respective component "$\underline{\underline{a^i}}$"

<u>Ex:</u> $$\epsilon^1(a^1\vec{e}_1 + a^2\vec{e}_2) = a^1\epsilon^1(\vec{e}_1) + a^2\epsilon^1(\vec{e}_2)$$
$$= a^1(1) + 0 = a^1$$

$$\boxed{\therefore\ \epsilon^i(\vec{v}) = a^i}$$

Now consider a covector $\underline{\underline{\alpha}}$ which maps vector $\underline{V}$ into a scalar.

$$\alpha(\vec{V}) = \alpha(a^1\vec{e_1} + a^2\vec{e_2})$$

$$= a^1\alpha(\vec{e_1}) + a^2\alpha(\vec{e_2})$$

let's say α maps $\vec{e_i}$ to some value α_i

so, $\boxed{\alpha(\vec{e_i}) = \alpha_i}$

Also, $a^i = \epsilon^i(\vec{V})$

so, $\alpha(\vec{V}) = \epsilon^1(\vec{V})\alpha_1 + \epsilon^2(\vec{V})\alpha_2$

$$\Rightarrow \alpha(\vec{V}) = (\alpha_1\epsilon^1 + \alpha_2\epsilon^2)(\vec{V})$$

On comparing, we can write

'$\underset{=}{\alpha}$' as a linear combination

of it's basis covectors

$$\boxed{\therefore \alpha = \alpha_1 \epsilon^1 + \alpha_2 \epsilon^2}$$

Now the vector $\vec{v}$ which is a "geometric object" in "vector space" (V) can be represented as "one form / co-vector" in "dual-space" (V^*)

Note: The components and basis might change but their geometric meaning and properties remain the same.

Covector Transformations:

II How vectors transform:

$$\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \Rightarrow$$ Vector Transformation using a Linear map.

Here $F_{11}, F_{21} \rightarrow \vec{u_1} = F_{11}\vec{e_1} + F_{21}\vec{e_2}$ $\longrightarrow$ New basis

$F_{12}\ F_{22} \rightarrow \vec{u_2} = F_{12}\vec{e_1} + F_{22}\vec{e_2}$ & vector

$$v' = x' u_1' + y' u_2'$$

here $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x' \\ y' \end{bmatrix}$ being initial and transformed vectors by a forward linear matrix $\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$.

→ Now consider a co-vector pair $[\alpha_1 \ \alpha_2]$ and $[\alpha_1' \ \alpha_2']$ before after transformation.

→ The initial basis co-vectors and transformed basis co-vectors are defined as follows.

$$\tilde{\epsilon} = \alpha_1 \epsilon^1 + \alpha_2 \epsilon^2$$

$$\tilde{\epsilon}_1 = Q_{11}\epsilon^1 + Q_{22}\epsilon^2$$

$$\tilde{\epsilon}_2 = Q_{21}\epsilon^1 + Q_{22}\epsilon^2$$

Generalizing this to higher dimension

$$\boxed{\tilde{\epsilon}^i = Q_{ij}\,\epsilon^j}$$

apply a vector basis on this co-vector basis.

$$\tilde{\epsilon}^i(\tilde{\vec{e}}_k) = Q_{ij}\,\epsilon^j(\tilde{\vec{e}}_k)$$

From the new vector basis

from the rule of vector transformation.

$$\vec{\tilde{e}}_k = \sum F_{lk} \vec{e}_l$$

$$\delta_{ik} = \sum_{j=1}^{n} Q_{ij} t^j \left(\sum_{l=1}^{n} F_{lk} \vec{e}_l \right)$$

$$\delta_{ik} = \sum_{j=1}^{n} \sum_{l=1}^{n} Q_{ij} F_{lk} \, t^j(\vec{e}_l)$$

$$\delta_{ik} = \sum^{n} \sum^{n} Q_{ij} F_{lk} \delta_{jl}$$

$j=1 \quad k=1$

$\downarrow$

Blowing this summation.

$$S_{ik} = \sum_{j=1}^{n} Q_{ij} F_{jk} \Rightarrow \boxed{Q_{ij} = B_{ij}}$$

$$\boxed{G_i = \sum_{i=0}^{n} B_{ij} t^j}$$