Key notes & takeaways from Differential geometry - I

Christoffel symbol

Transformation law:

$$\boxed{\bar{\Gamma}^{\lambda}_{\rho\sigma} = \Gamma^{\alpha}_{\beta\eta} \frac{\partial x^{\beta}}{\partial \bar{x}^{\rho}} \frac{\partial x^{\eta}}{\partial \bar{x}^{\sigma}} \frac{\partial \bar{x}^{\lambda}}{\partial x^{\alpha}} + \frac{\partial \bar{x}^{\lambda}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial \bar{x}^{\rho} \partial \bar{x}^{\sigma}}}$$

Non tensorial component

Geodesic equation:

Condition: A vector V^{i} transported along itself leaves a geodesic path.

$$\nabla_{\vec{v}} \vec{V} = 0 \Rightarrow v^i \nabla_{\frac{\partial}{\partial i}} (v^j \vec{e}_j)$$

$$\Rightarrow v^i \frac{\partial}{\partial i} (v^j \vec{e}_j) = v^i \left[\frac{\partial v^j}{\partial i} \vec{e}_j + \frac{\partial \vec{e}_j}{\partial i} v^j \right]$$

$$= v^i \left[\frac{\partial v^j}{\partial x^i} \vec{e_j} + \Gamma_{ij}{}^k v^j \vec{e_k} \right]$$

$$= v^i \left[\frac{\partial v^k}{\partial x^i} + \Gamma_{ij}{}^k v^j \right] \vec{e_k}$$

$$= \left(v^i \frac{\partial v^k}{\partial x^i} + v^i v^j \Gamma_{ij}{}^k \right) \vec{e_k} = 0.$$

$\hat{R}$

If a vector $\vec{v}$ is parametrized in a tangent vector space given by "λ"

then Analogy becomes

$$\vec{v} = v^{\mu}\,\vec{e}_{\mu}$$

$$\frac{\partial}{\partial \lambda} = \frac{\partial x^{\mu}}{\partial \lambda}\,\frac{\partial}{\partial x^{\mu}}$$

→ Apply this into equation (2).

<u>The above equation becomes :</u>

$$\left[v^{i}\,\partial_{i} v^{k} + \Gamma_{ij}^{\ \ k}\, v^{i} v^{j} \right] \vec{e}_{k} = 0$$

$$\Updownarrow$$

$$\left[\partial x^{i}\, \partial \left(\partial x^{k} \right) + \Gamma_{ij}^{\ \ k}\, \partial x^{i}\, \partial x^{j} \right] \vec{e}_{k} = 0$$

$$\Rightarrow \left[\frac{\partial}{\partial\lambda}\left(\frac{\partial x^k}{\partial\lambda}\right) + \Gamma_{ij}^{\;\;k}\frac{\partial x^i}{\partial\lambda}\frac{\partial x^j}{\partial\lambda}\right]\vec{e}_j = 0 \Rightarrow \left[\frac{\partial^2 x^k}{\partial\lambda^2} + \Gamma_{ij}^{\;\;k}\frac{\partial x^i}{\partial\lambda}\frac{\partial x^j}{\partial\lambda}\right]\vec{e}_k = 0$$

The Geodesic equation $\Rightarrow$ $$\boxed{\frac{\partial^2 x^k}{\partial\lambda} + \Gamma_{ij}^{\;\;k}\frac{\partial x^i}{\partial\lambda}\frac{\partial x^j}{\partial\lambda} = 0}$$

$\hookrightarrow$ # work on spherical co-ordinate systems.

Curved Space Geometry (Abstract)

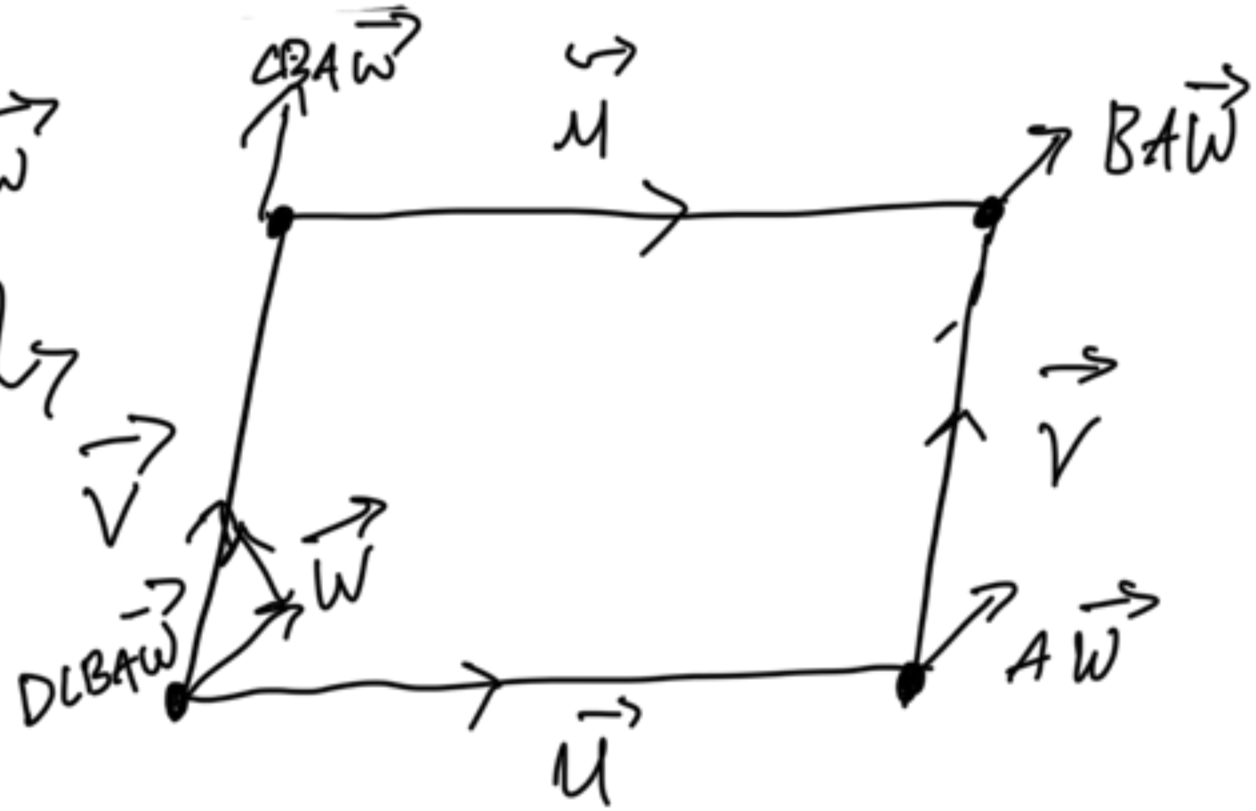
Topics Covered $\rightarrow$

Riemann Curvature tensor.
$R^d_{cab} \rightarrow$ identities and symmetries.
Sectional Curvature and Ricci Curvature (R_{mn}).
Ricci Tensor ...

• Ricci Tensor — components & symmetries.

2^{nd} Bianchi identity & Eienstien's tensor ($G_{\mu\nu}$)

$R(\vec{u},\vec{v})\vec{w}$ ⇝



Assumption
↳ $[\vec{u},\vec{v}]=0$.

$$\boxed{R(\vec{u},\vec{v})\vec{w} = \lim_{r,s\to 0} \frac{DCBA\,\vec{w} - \vec{w}}{rs}}$$

→ 4 transformations.

↳ $\lim_{r,s\to 0} \dfrac{[I]\vec{w} - DCBA\,\vec{w}}{rs}$

rs

$$\underset{r,s\to 0}{\mathrm{lt}} \quad \frac{DCC^{-1}D^{-1}\vec{w} - DCBA\vec{w}}{rs}$$

$$\underset{r,s\to 0}{\mathrm{lt}} \quad \frac{DC\left[\underbrace{C^{-1}D^{-1}\vec{w}} - BA\vec{w}\right]}{rs}$$

$$C^{-1}D^{-1}\vec{w} + C^{-1}\vec{w} - C^{-1}\vec{w} + \vec{w}$$

$$BA\vec{w} + B^{-1}\vec{w} - B^{-1}\vec{w} - \vec{w}$$

$$\Rightarrow \frac{1}{rs}\left[C^{-1}(D^{-1}\vec{w} - \vec{w}) + (C^{-1}\vec{w} - \vec{w})\right]$$

$$-\frac{1}{rs}\left[B(A\vec{w} - \vec{w}) + (B\vec{w} - \vec{w})\right]$$

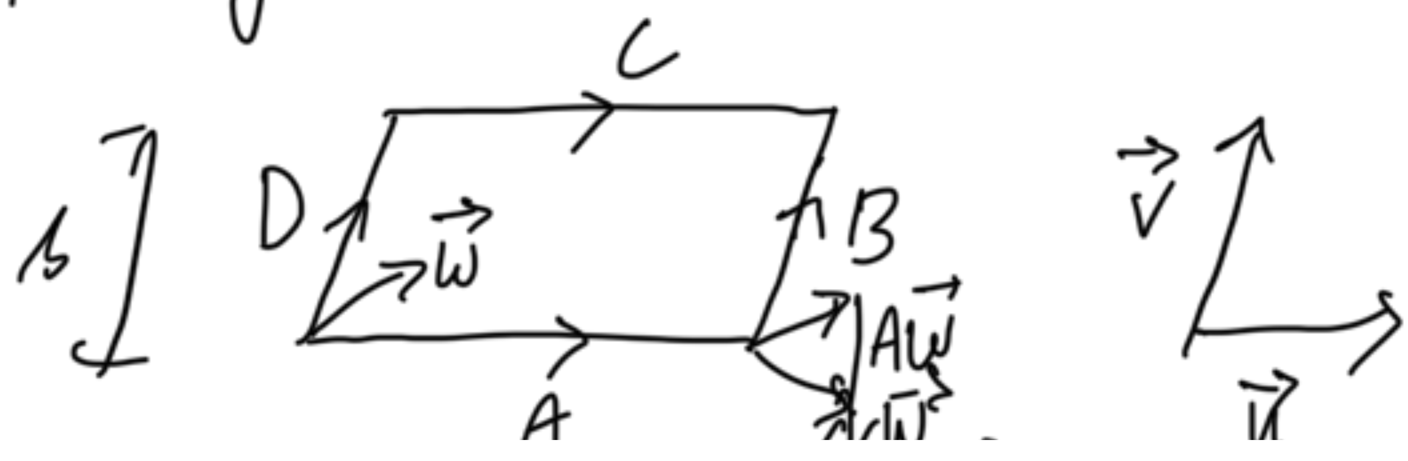
r,s)

$\Downarrow$ distributing denominator accross two terms.

$$\Rightarrow R(\vec{u},\vec{v})\vec{w} = DC \lim_{r,s\to 0}\left[\frac{C^{-1}}{r}\frac{(D^{-1}\vec{w}-\vec{w})}{s} + \frac{1}{s}(C^{-1}\vec{w}-\vec{w})\right]$$

$$-\left[\frac{B}{s}\frac{(A\vec{w}-\vec{w})}{r} + \frac{1}{r}\frac{(B\vec{w}-\vec{w})}{s}\right]$$

Recalling transformation parallelogram



$A\vec{w}-\vec{w}$

r

so, $\dfrac{A\vec{w}-\vec{w}}{r} = \nabla_u \vec{w}$	$\dfrac{B\vec{w}-\vec{w}}{s} = \nabla_v \vec{w}$
$\dfrac{D^{-1}\vec{w}-\vec{w}}{s} = \nabla_v \vec{w}$	$\dfrac{C^{-1}\vec{w}-\vec{w}}{r} = \nabla_u \vec{w}$

$$R(\vec{u},\vec{v})\vec{w} = DC \lim_{r,s\to 0} \frac{C^{-1}\nabla_v \vec{w} - \nabla_u \vec{w}}{r} - \frac{B\nabla_u \vec{w} - \nabla_u \vec{w}}{s}$$

$$R(\vec{u},\vec{v})\vec{w} = \lim_{r,s\to 0} DC\left[\nabla_u \nabla_v \vec{w} - \nabla_v \nabla_u \vec{w}\right]$$

Since $r, s \to 0$

D, C can be considered identity matrices

(For a local point frame).

So, For a closing parallelogram,

$$R(\vec{u},\vec{v})\vec{w} = \nabla_{\vec{u}}\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{v}}\nabla_{\vec{u}}\vec{w}$$

But what if the two vectors $\vec{u}$, & $\vec{v}$ are not closing, i.e their lie bracket $[\vec{u},\vec{v}] \neq 0$,

↳ ## We need to include another term where the "covariant derivative must be taken in the direction where the lie bracket

$[\vec{u}, \vec{v}]$ is non zero."

$$\therefore \quad \boxed{R(\vec{u}, \vec{v})\vec{w} = \nabla_{\vec{u}} \nabla_{\vec{v}} \vec{w} - \nabla_{\vec{v}} \nabla_{\vec{u}} \vec{w} - \nabla_{[\vec{u}, \vec{v}]} \vec{w}}$$

$\hookrightarrow$ Reimann Curvature tensor in an abstract vector space.

Expanding vectors in terms of their basis

$$R(u^i \vec{e_i}, v^j \vec{e_j}) w^k \vec{e_k} = u^i v^j \nabla_i \nabla_j w^k \vec{e_k} - v^j u^i \nabla_j \nabla_i w^k \vec{e_k}$$

$$- u^i v^j \nabla_{[\vec{e_i}, \vec{e_j}]} w^k \vec{e_k} \qquad 0 \; [\vec{e_i}, \vec{e_j}] = 0.$$

By linearity of Riemann Curvature tensor

$$R(\vec{u},\vec{v})\vec{w} = u^i v^j w^k R(\vec{e_i},\vec{e_j})\vec{e_k}$$

Calculating $R(\vec{e_a},\vec{e_b})\vec{e_c} \Rightarrow$

$$= R(\vec{e_a},\vec{e_b})\vec{e_c} = \nabla_a \nabla_b \vec{e_c} - \nabla_b \nabla_a \vec{e_c} - 0$$

$$= \nabla_a\left(\Gamma^i_{bc}\vec{e_i}\right) - \nabla_b\left(\Gamma^j_{ac}\vec{e_j}\right)$$

$\Rightarrow R(\vec{e_a},\vec{e_b})\vec{e_c}$

$$= \partial_a(\Gamma^i_{bc})\vec{e_i} + \Gamma^i_{bc}\,\partial_a\vec{e_i}$$

$$- \partial_b(\Gamma^j_{ac})\vec{e_j} - \Gamma^j_{ac}\,\partial_b\vec{e_j}$$

First we expand partial derivatives of ... vector $\vec{e_i}$, $\vec{e_j}$ ...

first we expand partial derivatives of unit vectors e_i & e_j wrt to a & b to get another set of Christoffel symbols.... Then factor out dummies with first two terms as $\vec{e_d}$ dummy indices.

$$\Rightarrow R(\vec{e_a},\vec{e_b})\vec{e_c} = \partial_a\left(\Gamma_{bc}^{\ i}\right)\vec{e_i} + \Gamma_{bc}^{\ i}\,\Gamma_{ai}^{\ d}\,\vec{e_d}$$

$$- \partial_b\left(\Gamma_{ac}^{\ j}\right)\vec{e_j} - \Gamma_{ac}^{\ j}\,\Gamma_{bj}^{\ f}\,\vec{e_f}$$

$$\Rightarrow R(\vec{e_a},\vec{e_b})\vec{e_c} = \partial_a\left(\Gamma_{bc}^{\ d}\right)\vec{e_d} + \Gamma_{bc}^{\ i}\,\Gamma_{ai}^{\ d}\,\vec{e_d}$$

$$- \partial_b\left(\Gamma_{ac}^{\ d}\right)\vec{e_d} - \Gamma_{ac}^{\ j}\,\Gamma_{bj}^{\ d}\,\vec{e_d}$$

$$\Rightarrow R(\vec{e_a},\vec{e_b})\vec{e_c} = \partial_a\left(\Gamma_{bc}^{\ d}\right) - \partial_b\left(\Gamma_{ac}^{\ d}\right) + \Gamma_{bc}^{\ i}\,\Gamma_{ai}^{\ d} - \Gamma_{ac}^{\ j}\,\Gamma_{bj}^{\ d}$$

It can be written in index notation as as rank 4 tensor.

$$\boxed{R_{cab}^{\ \ \ \ d} = \partial_a\left(\Gamma_{bc}^{\ d}\right) - \partial_b\left(\Gamma_{ac}^{\ d}\right) + \Gamma_{bc}^{\ i}\Gamma_{ai}^{\ d} - \Gamma_{ac}^{\ j}\Gamma_{bj}^{\ d}}$$

↳ Components of Riemann Curvature tensor.

$$\boxed{\therefore R(\vec{u},\vec{v})\vec{w} = u^a v^b w^c \underline{R_{cab}^{\ \ \ \ d}}\, \vec{e}_d}$$

Riemann Curvature Tensor - Symmetries and Identities:

(i) 34 Symmetry.

$$R(\vec{u},\vec{v})\vec{w} = u^a v^b w^c \, R(\vec{e_a},\vec{e_b})\vec{e_c}$$

$$R(\vec{e_a},\vec{e_b})\vec{e_c} = \nabla_a \nabla_b \vec{e_c} - \nabla_b \nabla_a \vec{e_c}$$

similarly

$$R(\vec{v},\vec{u})\vec{w} = v^a u^b w^c \, R(\vec{e_b},\vec{e_a})\vec{e_c}$$

$$R(\vec{e_b},\vec{e_a})\vec{e_c} = \nabla_b \nabla_a \vec{e_c} - \nabla_a \nabla_b \vec{e_c}$$

$$= -\left(R(\vec{e_a},\vec{e_b})\right)$$

$\underline{\text{3-4 Symmetry}}$ #

$$\boxed{\therefore R^d{}_{cab} = -R^d{}_{cba}}$$

(1)

→ 34 Symmetry arises from torsion free property of the basis vectors

(ii) 1st Bianchi Identity:

Considering the expression

$$R(\vec{e_a},\vec{e_b})\vec{e_c} + R(\vec{e_b},\vec{e_c})\vec{e_a} + R(\vec{e_c},\vec{e_a})\vec{e_b}$$

$\Rightarrow$ note, here all terms involving $\nabla_{[\vec{e_i},\vec{e_j}]} = 0$

so,
$$\nabla_a\nabla_b\vec{e_c} - \nabla_b\nabla_a\vec{e_c}$$
$$+ \nabla_b\nabla_c\vec{e_a} - \nabla_c\nabla_b\vec{e_a}$$
$$+ \nabla_c\nabla_a\vec{e_b} - \nabla_a\nabla_c\vec{e_b}$$

$\rightarrow$ By using torsion free property ($[\vec{e_i},\vec{e_j}]=0$) $\forall$ i,j

$\nabla_i\vec{e_j} = \nabla_j\vec{e_i}$

$$\Rightarrow R(\vec{e_a},\vec{e_b})\vec{e_c} + R(\vec{e_b},\vec{e_c})\vec{e_a}$$

$$+ R(\vec{e_c}, \vec{e_a})\vec{e_b} = 0$$

$$u^a v^b w^c \left[R^d{}_{cab} + R^d{}_{abc} + R^d{}_{bca} \right] \vec{e_d} = 0$$

$$\boxed{\therefore R^d{}_{cab} + R^d{}_{bca} + R^d{}_{abc} = 0} \quad \text{—— (2)}$$

↳ First Bianchi identity

from torsion free property.

(iii) 1-2 Symmetry: From metric compatibility property

$$\boxed{\text{Metric Compatability : } \nabla_{\vec{w}}(\vec{u} \cdot \vec{v}) = \vec{v} \cdot (\nabla_{\vec{w}} \vec{u}) + \vec{u} \cdot (\nabla_{\vec{w}} \vec{v})}$$

↳ Co-variant derivative of the dot

product of 2 vectors obey's a sort of chain rule.

Component of Riemann tensor $\rightarrow R^{d}{}_{cab}$

by contracting it with metric tensor

$g_{df} R^{d}{}_{cab} = R_{fcab} \Rightarrow R_{dcab}$ (Since f is a dummy index).

1-2 Symmetry

$$\boxed{R_{dcab} = -R_{cdab}} \quad -(3)$$

(iv) Flip symmetry:

Considering All the symmetries and identities mentioned above.

$$R_{abcd} = -R_{abdc} \quad \text{— (34 Symmetry).}$$

$$-R_{abdc} = -\left(-R_{badc}\right)$$

$$= R_{badc}$$

$$\boxed{\therefore R_{abcd} = R_{badc}} \rightarrow \text{Flip symmetry in Riemann Curvature tensor.}$$

Summarizing Riemann Tensor:

$$R(\vec{u},\vec{v})\vec{w} = \nabla_{\vec{u}}\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{v}}\nabla_{\vec{u}}\vec{w} - \nabla_{[\vec{u},\vec{v}]}\vec{w}$$

$$= u^a v^b w^c R^d_{cab}\,\vec{e_d}$$

$$R^d \ldots = \partial \left(\Gamma^d \ldots\right) - \partial \left(\Gamma^d \ldots\right) \ldots \Gamma^i \Gamma^d \ldots \Gamma^i \Gamma^d \ldots$$

$$\boxed{R_{cab} = \partial_a(\Gamma_{bc}) - \partial_b(\Gamma_{ac}) - \Gamma_{bc}\Gamma_{a} + \Gamma_{ac}\Gamma_{b}}$$

Properties:

1-2	$R_{dabc} = -R_{adbc}$
3-4	$R_{dabc} = -R_{dacb}$
Flip	$R_{abcd} = R_{badc}$
BI	$R^d{}_{cab} + R^d{}_{bca} + R^d{}_{abc} = 0$

Metric compatibility.

→ From both symmetries.

→ Torsion free

↳ Riemann tensor contraction using metric.

Volume elements in curved spaces → # Volume element derivative ✓

Ricci curvature tensor ✓

2^{nd} Bianchi identity.
Einstien Tensor.

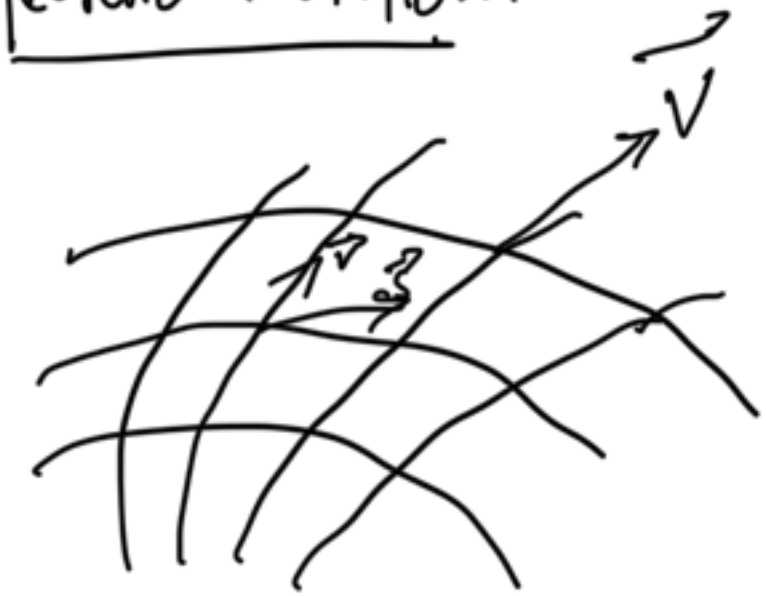
Volumes in space expand and shrink based on convergence and divergence in geodesics.

Volume element derivative is used to calculate the changes in volume of an object.

↳ Skipping the volume element derivative, the final insight came the "Ricci curvature tensor" – a contracted version of the "Reimann curvature tensor".

Mathematically, the only meaningful contraction of the Reimann curvature tensor is this $\rightarrow$ $\boxed{R^{i}_{\ aib}}$

Geodesic deviation:



The basic notation to write a geodesic $\Rightarrow$ $\nabla_{\vec{v}} \vec{v} = 0$, in the direction of tangent vectors $\vec{w}$

Consider a seperation vector $\vec{S}$ whose magnitude is changing based on the geodesic deviation along geodesics of $\vec{V}$.

$\hookrightarrow$ So, we take the Covariant derivative of one direction geodesic with respect to other.

So, $\nabla_{\vec{s}} \nabla_{\vec{v}} \vec{v} = 0$. adding and subtracting

$\nabla_v \nabla_s \vec{v} =$

$$\underbrace{\nabla_s \nabla_v \vec{v} + \nabla_v \nabla_s \vec{v}}_{\Downarrow} - \nabla_v \nabla_s \vec{v} = 0.$$

$$R(\vec{s}, \vec{v}) \vec{v} + \nabla_v \nabla_s \vec{v} = 0$$

since the space time is considered torsion free $\nabla_s \vec{v} \Longleftrightarrow \nabla_{\vec{v}} \vec{s}$

$$\therefore \nabla_{\vec{v}} \nabla_{\vec{v}} \vec{s} = -R(\vec{s}, \vec{v}) \vec{v}$$

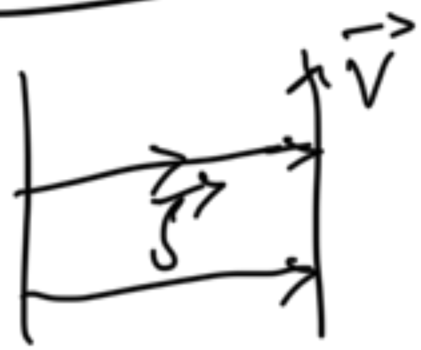
But from symmetry of Riemann tensor.

$$\boxed{\nabla_{\vec{v}} \nabla_{\vec{v}} \vec{s} = R(\vec{v}, \vec{s}) \vec{v}}$$

$$\boxed{\nabla_{\vec V}\nabla_{\vec V}\vec S = -R(\vec S,\vec V)\vec V}$$

The Geodesic deviation can be categorized into 3 types:

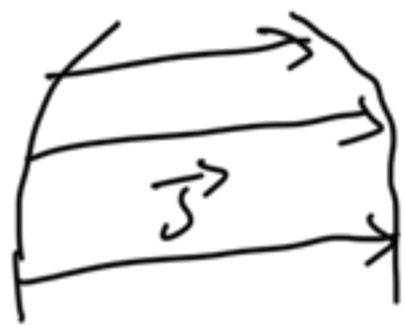
No deviation:



$\nabla_{\vec V}\nabla_{\vec V}\vec S = 0$

↳ because $\vec S$ seperation vector did not change it's magnitude or direction.

"Inward" geodesic deviation:



$\nabla_{\vec V}\vec S < 0$

because $|\vec S|$ is decreasing.

$\therefore \left[\nabla_{\vec V}\nabla_{\vec V}\vec S\right]\cdot\vec S < 0$

"Outward" Geodesic deviation

$\nabla_{\vec V}\vec S > 0$, because on moving separation $\vec S$, it's magnitude actually increases.

$\therefore \left[R(\vec S,\vec V)\vec V\right]\cdot\vec S < 0$

So, $[R(\vec{\xi},\vec{v})\vec{v}]\cdot\vec{\xi} = 0$ | $\therefore [R(\vec{\xi},\vec{v})\vec{v}]\cdot\vec{\xi}\cdot\vec{\xi}\vec{v}$ ↳ greater than '0'. | Less than '0'

<u>Question</u>: Why did we use double derivatives instead of just tracking / finding $\nabla_{\vec{v}}\vec{\xi}$ to observe geodesic deviation??

<u>Answer</u>: <u>Flat Space</u>:

here $\nabla_{\vec{v}}\vec{\xi} \neq 0$

but $\nabla_{\vec{v}}\nabla_{\vec{v}}\vec{\xi} = 0$

The "<u>geodesics are changing</u>"

<u>Curved space</u>:

here $\nabla_v\vec{\xi} \neq 0$

& $\nabla_v\nabla_v\vec{\xi} \neq 0$

the geodesics are not changing constantly, i.e. the $\vec{\xi}$ vector but that change is constant. | constant ... is being accelerated.

Volume elements in differential geometry:

Just like we saw how line elements shrink and expand based on geodesic deviation, same thing happens with volume elements.

↳ The Riemann tensor can't describe it because it only acts on vectors but not volume elements.

Hence, we will use volume element derivative technique and arrive at the fundamental definition of Ricci Curvature Tensor.
to explain volume changes in geodesics.

Volume Elements:

Volume form / Volume tensor: Outputs the volume when we input some vectors of appropriate dimensions.

$\omega(\vec{x}, \vec{y}, \vec{z})$.

Given by $\omega(\vec{a}, \vec{b}) \Rightarrow$ volume of $\vec{a}, \vec{b}$ vectors.

or $\omega(\vec{x}, \vec{y}, \vec{z}) \Rightarrow$ volume of $\vec{x}, \vec{y}, \vec{z}$.

For non-orthonormal basis like the following.

$\vec{u} = u^1 \vec{e_1} + u^2 \vec{e_2}$ $\Rightarrow$ Now, volume is given by.

$\vec{w} = w^1 \vec{e_1} + w^2 \vec{e_2}$ $\qquad \det \begin{bmatrix} u^1 & w^1 \\ u^2 & w^2 \end{bmatrix} = \omega(\vec{u}, \vec{v})$

the value of $\omega(\vec{u}, \vec{w}) = \det \begin{bmatrix} u^1 & w^1 \\ u^2 & w^2 \end{bmatrix}$

can be written in the form of "Levi-Civita connections".

so $\boxed{\omega(\vec{u}, \vec{w}) = \varepsilon_{ij} u^i w^j}$

$$= \varepsilon_{11} u^1 w^1 + \varepsilon_{12} w^2 u^1 + \varepsilon_{21} u^2 w^1 + \varepsilon_{22} w^2 u^2$$

Now in 3-d co-ordinate systems,

$$\boxed{\omega(\vec{u}, \vec{v}, \vec{w}) = \epsilon_{ijk} u^i v^j w^k}$$

Here, $\epsilon_{ijk} = \begin{cases} +1 & ijk \text{ even permutation.} \\ -1 & ijk \text{ odd permutation} \\ 0 & \text{any repeated indices.} \end{cases}$

Levi-Civita Symbols & volume Tensor.

But what if the vector space is "arbitrary"?, i.e "different basis."

??

If $\vec{e}_i$ basis is being transformed from $\vec{\tilde{e}}_i$

then it's expression (let) be.

$$\vec{\tilde{e}}_1 = F_1^1 \vec{e}_1 + F_1^2 \vec{e}_2$$
$$\vec{\tilde{e}}_2 = F_2^1 \vec{e}_1 + F_2^2 \vec{e}_2$$

$\rightarrow$ Then volume will be

$$\det \begin{pmatrix} F_1^1 & F_1^2 \\ F_2^1 & F_2^2 \end{pmatrix} = \det F$$

or in an abstract vector space where basis are just partial derivative operators.

then $\vec{\tilde{e}}_1 = \frac{\partial}{\partial x^1}\frac{\partial x^1}{\partial \tilde{x}^1} + \frac{\partial x^2}{\partial \tilde{x}^1}\frac{\partial}{\partial x}$

then the volume $\omega(\vec{\tilde{e}}_1, \vec{\tilde{e}}_2) = \det \begin{bmatrix} \frac{\partial x^1}{\partial \bar{x}^1} & \frac{\partial x^1}{\partial \bar{x}^2} \\ \frac{\partial x^2}{\partial \bar{x}^1} & \frac{\partial x^2}{\partial \bar{x}^2} \end{bmatrix}$

$= \det[\text{Jacobian}] = \det[J]$

$\therefore \omega(\vec{\tilde{e}}_1, \vec{\tilde{e}}_2) = \det[J]$

From Laws of tensor transformation: (Note: bar '$-$' & tilda '$\sim$' are same).

The lower index metric transforms as follows:

$$\boxed{g_{\bar{i}\bar{j}} = \frac{\partial x^a}{\partial \bar{x}^i} \frac{\partial x^b}{\partial \bar{x}^j} g_{ab}} \Rightarrow \boxed{g_{\bar{i}\bar{j}} = F_i^a F_j^b g_{ab}}$$

Now, applying determinant on both sides.

$$\det \bar{g} = (\det F)(\det F)\det(g) \quad \left(\begin{array}{c}\text{distributive law}\\ \text{of det product}\end{array}\right)$$

Now, if "g" comes from an orthonormal basis

then $\det(g) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$

$$\boxed{\therefore \det \bar{g} = (\det F)^2} \Rightarrow \boxed{\sqrt{\det(\bar{g})} = \det F}$$

But the basic volume of the transformed system is actually "$\det F$"

So $\boxed{\omega(\tilde{\vec{e}}_1, \tilde{\vec{e}}_2) = \det F = \sqrt{\det(\bar{g})}}$ ☆☆☆

Case: Volume created by non-orthonormal basis.

$$\vec{u} = u^1 \vec{\tilde{e}}_1 + u^2 \vec{\tilde{e}}_2$$

$$\vec{v} = v^1 \vec{\tilde{e}}_1 + v^2 \vec{\tilde{e}}_2$$

→ Now, it's volume is given by using volume tensor.

$$\boxed{\omega(\vec{u}, \vec{v}) = \underbrace{\sqrt{\det g}\, \epsilon_{ij}}_{\text{volume form components.}} u^i v^j}$$

General to any metric and manifold.

here $g \to$ metric for $\vec{\tilde{e}}_i$ & $\vec{\tilde{e}}_j$

Tracking Volume changes in Geodesic deviations.

↳ We now have a defined volume element from an abstract & arbitrary vector space defined by the volume form tensor $\boxed{\omega(\vec{u},\vec{v},\vec{w}) = \sqrt{\det g}\; u^i v^j w^k \epsilon_{ijk}}$

↳ Know how it is changing when the geodesics are getting deviated (due to geometry curvature if any). We take it's double co-variant derivative.

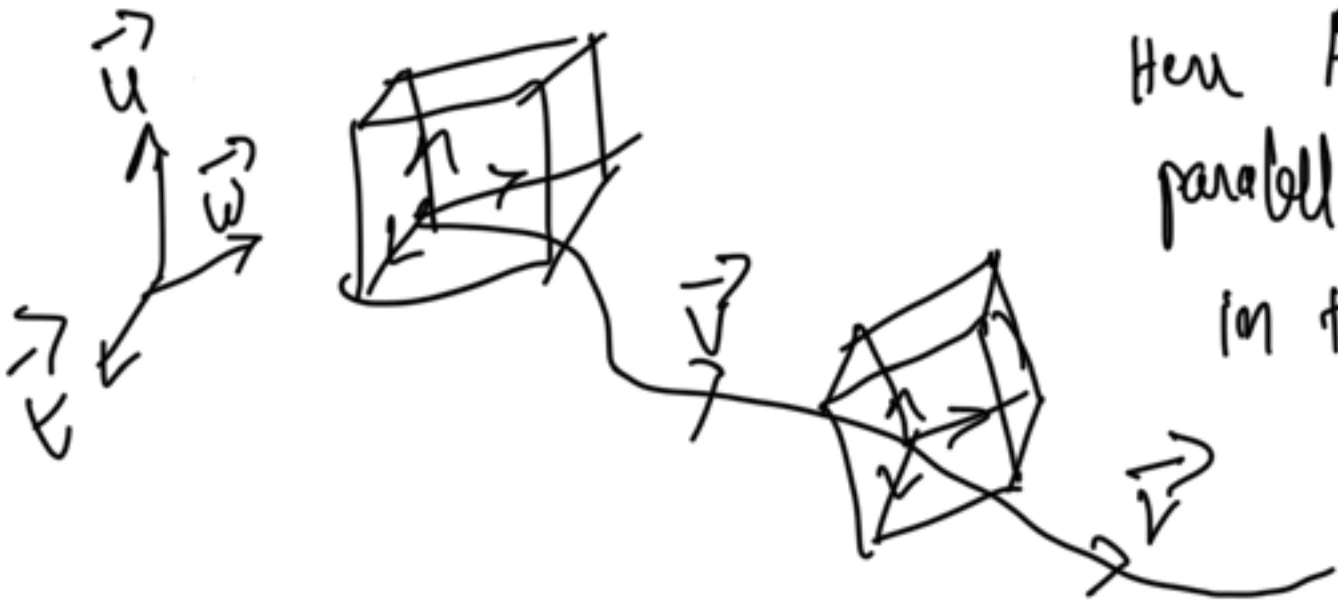
Case: | Problem: # Consider a volume element bounded by 3 vectors in a 3-d space ⇒ "$\omega(\vec{u},\vec{w},\vec{t})$"

Now this volume element is being transported along one of the geodesics in "$\vec{V}$"

↳ That transport is given by it's Co-variant derivative.

But here, we use double Co-variant derivative just like we did with a line in previous analogy. (Operation $\vec{s}$).

So, $\nabla_{\vec{v}} \nabla_{\vec{v}} \omega(\vec{u}, \vec{w}, \vec{t}) \rightarrow (\vec{u}, \vec{w}, \vec{t})$.



Here All the vectors are being parallel transported simultaneously in the direction of vector $\vec{v}$.

$\boxed{\text{↳ Since the parallel transport is taken through Levi-Civita connection, "lengths" and "Angles" are preserved. \# (metric compatibility).}}$

$\therefore$ Covariant derivative of Volume form $= 0$.

$$\boxed{\nabla_{\vec{v}}\, \omega(\vec{w}, \vec{u}, \vec{t}) = 0}$$

$$\boxed{\nabla_{\vec{v}} V = 0}$$

Proof:

$$\nabla_{\vec{v}}\, \omega(\vec{u}, \vec{w}, \vec{t}) = (\nabla_{\vec{v}}\omega)(\vec{u}, \vec{w}, \vec{t}) + \omega(\cancel{\nabla_{\vec{v}}\vec{u}}, \vec{w}, \vec{t})$$

$$+ \omega(\vec{u}, \cancel{\nabla_{\vec{v}}\vec{w}}, \vec{t}) + \omega(\vec{u}, \vec{w}, \cancel{\nabla_{\vec{v}}\vec{t}})$$

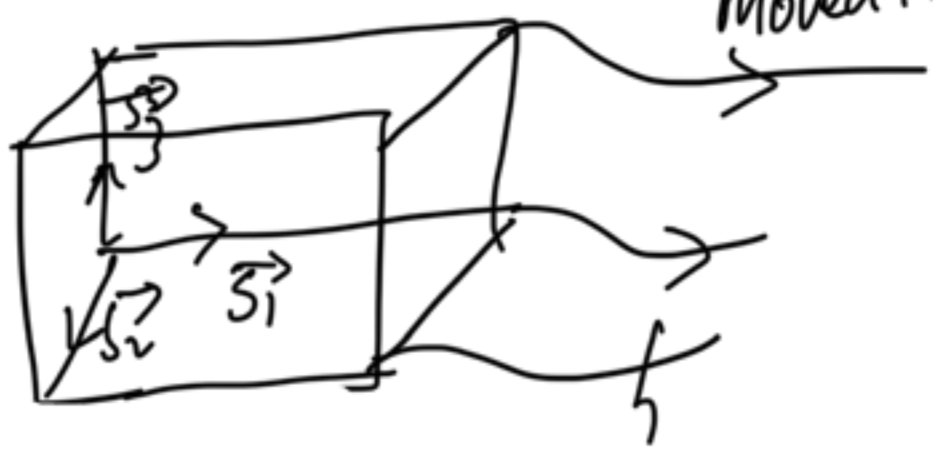
Since inner covariant derivatives are zero because they were parallel transported.

$$\boxed{\therefore \nabla_{\vec{v}}\omega = 0} \Rightarrow \boxed{\frac{\partial}{\partial \lambda}\left(\sqrt{\det g}\, \epsilon_{ijk}\right) = 0}$$

2^{nd} Covariant derivatives of volume spanned by vectors:

... parallelepiped with generating vectors: $\vec{s}_1, \vec{s}_2$ & $\vec{s}_3 \ldots \vec{s}_n$

Consider a parallelogram with separation ... and this n-dimensional parallelogram volume is being moved in a geodesic path in the direction of vector "$\vec{V}$" parametrized by "λ".



we know that Volume $\boxed{V = \sqrt{\det g}\ \epsilon_{ijk} S_1^i S_2^j S_3^k}$

↳ For 3-dimensional space.

↳ For n-dimensional space.

$$V = \sqrt{\det g}\left(\epsilon_{\mu_1 \mu_2 \cdots \mu_n}\right)\left(S_1^{\mu_1} S_2^{\mu_2} \cdots S_i^{\mu_i}\right)$$

$$\boxed{V = \sqrt{\det g}\ \epsilon_{\mu_1 \mu_2 \cdots \mu_n} \left(\prod_{i=1}^{n} S_i^{\mu_i} \right)}$$

↳ Now, that we have arrived with a mathematical expression for a "<u>volume element</u>" with "<u>variable seperation</u>"; we need to take it's double derivative in the direction of $\vec{v}$ which is parametrized by "<u>λ</u>"

$$\boxed{\nabla_{\vec{v}} \nabla_{\vec{v}} V = \frac{d^2 V}{d\lambda^2}}$$

($\because$ Volume is a scalar & $\vec{v}$ is parametrized by λ)

so, Calculating the first derivative

$$\frac{dV}{d\lambda} = \frac{d}{d\lambda}\left[\sqrt{\det g}\ \epsilon_{\mu_1\mu_2\dots\mu_n}\left(\prod_{i=1} S_i^{\mu_i}\right)\right]$$

Here, for a Co-ordinate system, metric determinant & Levi-Civita symbols are constants!

So, $$\frac{dV}{d\lambda} = \sqrt{\det(g)}\ \epsilon_{\mu_1\mu_2\dots\mu_n}\left[\frac{d}{d\lambda}\left(\prod_{i=1}^{n} S_i^{\mu_i}\right)\right]$$

Multivariable Chain rule $\Longleftrightarrow$

$$\boxed{\text{here } \frac{d}{d\lambda}\left(S_1^a S_2^b S_3^c\right) = \dot{S}_1^a S_2^b S_3^c + S_1^a \dot{S}_2^b S_3^c + S_1^a S_2^b \dot{S}_3^c}$$

so, $\frac{dV}{}$ $\dot{S}^{\mu_j}\left(\prod^{n} S_i^{\mu_i}\right)$

$$\boxed{\frac{dV}{d\lambda} = \dot{s}_j^{\mu_j}\left(\prod_{\substack{i=1\\ i\neq j}} s_i^{\mu_i}\right)\sqrt{\det g}\;\epsilon_{\mu_1\mu_2\ldots\mu_n}}$$

Now, the second derivative $\frac{d^2V}{d\lambda^2} = \frac{d}{d\lambda}\left(\frac{dV}{d\lambda}\right)$

$$\Rightarrow \frac{d^2V}{d\lambda^2} = \frac{d}{d\lambda}\left(\dot{s}_j^{\mu_j}\left(\prod_{\substack{i=1\\ i\neq j}}^{n} s_i^{\mu_i}\right)\sqrt{\det g}\;\epsilon_{\mu_1\mu_2\ldots\mu_n}\right)$$

Now, applying the same chain rule will give the following equation.

$$\Rightarrow \frac{dV^2}{d\lambda^2} = \left[\ddot{s}_j^{\mu_j}\left(\prod^{n} s_i^{\mu_i}\right) + \dot{s}_j^{\mu_j}\dot{s}_k^{\mu_k}\left(\prod s_i^{\mu_i}\right)\right](\det g)^{1/2}\,\epsilon_{\mu_1\ldots\mu_n}$$

Here, there are two terms,

Term 1: $\ddot{s}_j^{\mu_j}\left(\prod_{\substack{i=1\\ i\neq j}}^{n} s_i^{\mu_i}\right)$

Term 2: $\dot{s}_j^{\mu_j}\dot{s}_k^{\mu_k}\left(\prod_{\substack{i=1\\ i\neq j,k}} s_i^{\mu_i}\right)$

Term 1: $\ddot{s}_j^{\mu_j}\left(\prod_{\substack{i=1\\ i\neq j}}^{n} s_i^{\mu_i}\right)\sqrt{\det g}\;\epsilon_{\mu_1\mu_2\cdots\mu}$

↳ this second derivative term can be expressed as

This seems ...

a component of Riemann curvature tensor.

$$\ddot{S}_j^{\mu_j} = \nabla_{\vec{V}}\nabla_{\vec{V}}\vec{S} = -R(\vec{S},\vec{V})\vec{V}$$

In its component form, $-R(\vec{S},\vec{V})\vec{V}$ can be written as

$$\boxed{\ddot{S}_j^{\mu_j} = -R^{\mu_j}_{\ xyz}\, S_j^{y} V^{z} V^{x}}$$

So, term 1 now becomes

$$T_1 = \left[-R^{\mu_j}_{\ xyz}\, S_j^{y} V^{z} V^{x} \left(\prod_{\substack{i=0 \\ i\neq j}}^{n} S_i^{\mu_i} \right) \right.$$

Now, in the product, every index is used except μ_j.

$$\times \sqrt{\det g}\ \epsilon_{\mu_1 \mu_2 \dots \mu_n}$$

here the index $\underline{y}$ can now take any value of index from "$\underline{\mu_1 \text{ to } \mu_n}$"

Also, the product is happening such that $S_j^{\mu_j}$ terms donot appear.

In levi-civita symbol, if there are any repeated indices, the term goes to "$\underline{\text{zero}}$".

Given these conditions, the only non zero component of separation vector $\boxed{S_j^{\,y} = S_j^{\,\mu_j}}$

That means $y = \mu_j$ & $T_1 = \left[-R^{\mu_j}_{\ x\mu_j z} V^x V^z S_j^{\mu_j} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n} S_i^{\mu_i} \right) \right] \times \sqrt{\det g}$ $x \in \mu_1, \mu_2 - \mu_n$

Now, this is a contraction of Riemann curvature tensor over the 2nd index.

$$\boxed{-R^{\mu_j}{}_{x\mu_j z} = R_{xz}}$$

"Component of Ricci Curvature tensor"

The term $\delta_j^{\mu_j}$ goes back into the product and $i \neq j$ condition is now revoked.

So, $$\boxed{T_1 = -R_{xz} v^x v^z \left(\prod_{i=1}^{n} \delta_i^{\mu_i} \right) \sqrt{\det g}\, \epsilon_{\mu_1 \mu_2 \ldots \mu_n}}$$

Also, the term $\left(\prod_{i=1}^{n} \delta_i^{\mu_i} \sqrt{\det g}\, \epsilon_{\mu_1 \mu_2 \ldots \mu_n} \right) = \text{Volume } (V)$.

So, $\boxed{T_1 = -R_{xz} V^x V^z \, (V)}$

The Definition of "Ricci Curvature Tensor" $R(\vec{V}, \vec{V})$

and "R_{xz}" is a component of Ricci tensor.

$\therefore$ Ricci Curvature tensor tells us how volumes change as we move along geodesics in space.

The total 2^{nd} derivative of the volume can now be written as

$$\frac{d^2V}{d\lambda^2} = -R_{xz} V^x V^z \, (V) + \dot{S}_j^{\mu_j} \dot{S}_k^{\mu_k} \left(\prod_{i}^{n} S_i^{\mu_i} \right) \sqrt{\det g} \, \epsilon_{\mu_1 \mu_2 \cdots \mu_n}$$

$$\sum_{\substack{i=1 \\ i\neq j,k}}$$

Important note: # Term 1 involving Ricci Curvature tensor keeps track of volumes in "curved geodesics" because the "double derivative" term $\boxed{``\ddot{\delta}_{j}^{ij}" = \nabla_{\vec{v}} \nabla_{\vec{v}} \vec{S}}$ keeps track of accelerating changes of volumes.

go to geodesic deviation & types of curvature.

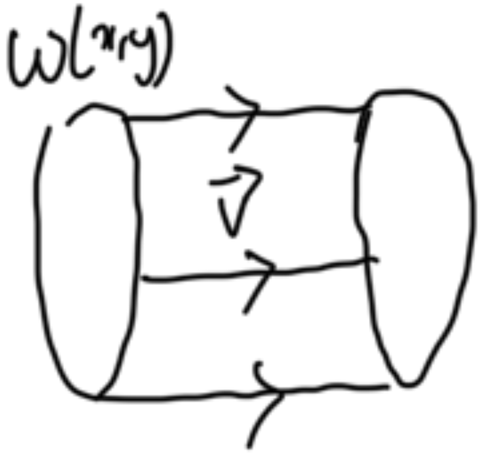
Term 2 involving single derivative keeps track of volume changes involving in "flat spaces / flat geodesics". because single derivatives check with non accelerating changes in volumes over geodesics.

Visual understanding of Ricci Curvature tensor:

Like changes observed while how line elements change their length/seperation while the geodesics deviate due to geometry curvature, the change in their volume can also be visualized

↳ The Ricci curvature can also be classified into 3 types.

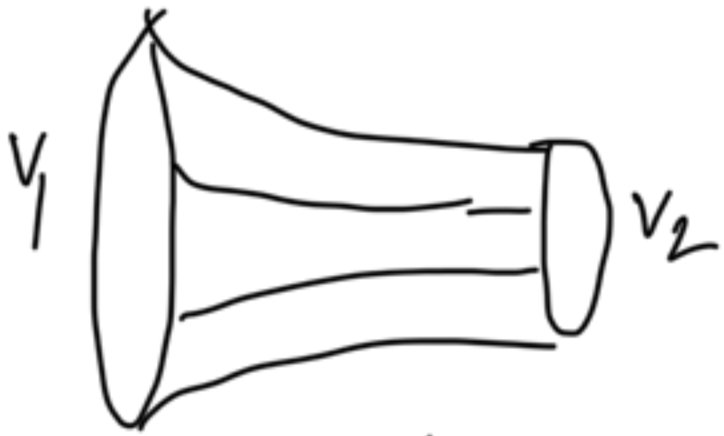
Null Curvature:



No change in volume & no deviation in geodesics.

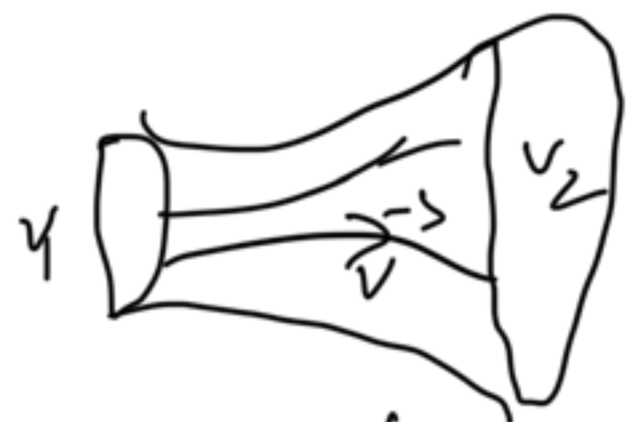
$\boxed{R_{\mu\nu}=0}$

Inward Curvature:



Here the volume element V_1 is reduced in size due to the geodesic deviation (inward)

Outward Curvature:



Here, the volume element seperation is increasing in size due to the outward geodesic

Still no change in volume because some

∴ The Ricci curvature tensor will be

$$Ric(\vec{v},\vec{v}) = R_{ab}v^a v^b$$

$\to 0$

deviation.

∴ Ricci curvature tensor will be.

$$\boxed{Ric(\vec{v},\vec{v}) < 0}$$