Differential Geometry - 3

Contents:

i) 2^{nd} Bianchi identity - Differential form.

ii) Curvature Scalar and Einstien Tensor - "$G_{\mu\nu}$"

iii) Properties and Characteristics of Einstien Tensor.

iv) Mathematical formulation & intuition behind

"Einstein's field Equations"

Takeaways from Differential Geometry - 2

↳ Curvature in geometry can be detected using "Riemann Tensor".

↳ The curvature in Manifold can be expressed as "Geodesic deviations" and how volume elements change along deviating geodesics.

↳ "Ricci Curvature Tensor" ($R_{\mu\nu}$) is used to keep track of volume elements moving along geodesics of the Manifold.

↳ volume element derivative : → Component of "Ricci Curvature Tensor"

$$\boxed{\nabla_{\vec{v}} \nabla_{\vec{v}} V = \frac{d^2 V}{d\lambda^2} = -\underline{R_{xy}}\, v^x v^y \left(V\right) + \dot{\xi}_j^{x_j} \dot{\xi}_k^{x_k} \left(\prod_{\substack{i=1 \\ (i \neq j,k)}}^{n} \xi_i^{x_i} \right) \epsilon_{\mu_1 \ldots \mu_n} \times \sqrt{\det g}}$$

2^{nd} Bianchi Identity:

Some old notations:

(a) $\nabla_{\vec{e_i}}\, g = \nabla_{e_i}\left(g_{mn}\, \epsilon^m \otimes \epsilon^n\right)$

$$= \left(\partial_i(g_{mn}) - g_{mx}\Gamma^x_{in} - g_{xn}\Gamma^x_{im}\right)\epsilon^m \otimes \epsilon^n$$

$$\boxed{\nabla_{e_i}\, g = \left(g_{mn;i}\right)\epsilon^m \otimes \epsilon^n}$$

↳ Co-variant derivative of the "Metric tensor"

(b) Covariant derivative of Riemann Curvature tensor:

Use

Now,

$$\nabla_{\vec{e_i}}(R) = \nabla_{\vec{e_i}}\left(R^{d}{}_{cab}\, \vec{e_d} \otimes \epsilon^{c} \otimes \epsilon^{a} \otimes \epsilon^{b}\right)$$

↓ Formal equation for Riemann tensor's covariant derivative.

$$\boxed{R^{d}{}_{cab;i} = \left(\partial_i (R^{d}{}_{cab}) + (R^{x}{}_{cab})\Gamma_{ix}{}^{d} - (R^{d}{}_{xab})\Gamma_{ic}{}^{x} - R^{d}{}_{cxb}\Gamma_{ia}{}^{x} - R^{d}{}_{cax}\Gamma_{ib}{}^{x}\right)}$$

Given, $$\boxed{R^{d}{}_{cab} = \partial_a(\Gamma_{bc}{}^{d}) - \partial_b(\Gamma_{ac}{}^{d}) - \Gamma_{bc}{}^{k}\Gamma_{ak}{}^{d} - \Gamma_{ac}{}^{j}\Gamma_{bj}{}^{d}}$$

↳ Riemann tensor components.

"Rieman Tensor's Covariant derivative Components"

Note: Solving for "$R^{d}{}_{cab;i}$" by substituting "$R^{d}{}_{cab}$" into the above

equations is very difficult.

Assumption:

#Over a tiny region in space at point 'p'

↳ $g_{ij} = \delta_{ij} \rightarrow$ Metric tensor $= [I]$.

↳ $\Gamma^k_{ij} = 0 \rightarrow$ Connection co-effs $= 0$

This means the space is assumed to be "FLAT" over a tiny region.

↳ "Riemann Normal Coordinates at point p"

(or)

"Local Inertial Frame"

↳ So, the computation for the value of Riemann tensor covariant derivative will be reduced as follows:

$$R^{d}_{\ cab;i} = \partial_i \left(\partial_a (\Gamma^{d}_{bc}) - \partial_b (\Gamma_{ac}^{\ \ d}) + \Gamma^{k}_{bc} \Gamma^{d}_{ak} - \Gamma_{ac}^{\ \ j} \Gamma^{d}_{bj} \right)$$

$$R^{d}_{\ cab;i} = \partial_i \partial_a (\Gamma^{d}_{bc}) - \partial_i \partial_b (\Gamma^{d}_{ac}) + \partial_i (\Gamma^{k}_{bc}) \Gamma^{d}_{ak} + \Gamma^{k}_{bc} \partial_i (\Gamma^{d}_{ak}$$

$$- \partial_i(\Gamma^d_{ac})\Gamma^a_{bj} - \Gamma^j_{ac}\,\partial_i(\Gamma^d_{bj})$$

↳ So, now, any $\boxed{\Gamma_{ij}^{\;k} = 0}$, for the equation can be reduced to as follows.

$$\boxed{R^d_{\;cab;i} = \partial_i\partial_a(\Gamma_{bc}^{\;d}) - \partial_i\partial_b(\Gamma_{ac}^{\;d})}$$

↳ <u>component of covariant derivative</u> "
<u>of Riemann tensor $R^d_{\;cab}$ at Riemann</u>
<u>normal co-ordinates</u>!

<u>Now:</u> Solve for $R^d_{\;cab;i} + R^d_{\;cia;b} + R_{cbi;a}$

$$\hookrightarrow R^{d}{}_{cab;i} = \partial_i \partial_a \left(\Gamma^{d}_{bc} \right) - \partial_i \partial_b \left(\Gamma^{d}_{ac} \right)$$

$$R^{d}{}_{cia;b} = \partial_b \partial_i \left(\Gamma^{d}_{ac} \right) - \partial_b \partial_a \left(\Gamma^{d}_{ic} \right)$$

$$R^{d}{}_{cbi;a} = \partial_a \partial_b \left(\Gamma^{d}_{ic} \right) - \partial_a \partial_i \left(\Gamma^{d}_{bc} \right)$$

Adding the up, the result reduces to '0'.

"Also works for all Co-ordinates but proof sucks ☹"

$\therefore$ At Riemann Normal Coordinates:

$$\boxed{R^{d}{}_{cab;i} + R^{d}{}_{cia;b} + R^{d}{}_{cbi;a} = 0}$$

$\hookrightarrow$ # "2nd Bianchi Identity."

Contracted Bianchi Identity.

we know that, from 2^{nd} Bianchi identity.

$$\boxed{R_{abmn;l} + R_{ablm;n} + R_{abnl;m} = 0}$$

Now take inverse metric tensor with two seperate indices on both sides.

so, $g^{ba} g^{am} \left(R_{abmn;l} + R_{ablm;n} + R_{abnl;m} \right) = 0.$

apply metric tensor "g^{am}" on all elements

Note: There is a covariant derivative on R_{abml} elements and we are applying inverse metric. How do we do that??

solution: Simply contract the indices of Riemann tensor components using inverse metric. So, $[g^{am}][R_{abmn;l}] = R^{m}{}_{bmn;l}$

$= \underline{\underline{R_{bn;l}}}$

Component of Ricci tensor covariant derivative.

How?? $\Rightarrow$ From the properties of Metric tensor, (Metric compatibility).

$\boxed{\nabla_{\vec{e}_j}\, g = 0}$ $\rightarrow$ Dot product remains constant.

Also, $\nabla_{\vec{e}_i}\big(g(R)\big) = \big(\nabla_{\vec{e}_i}(g)R\big) + g\big(\nabla_{\vec{e}_i}R\big)$

$\downarrow 0$

$$\boxed{\therefore \nabla_{\vec{e_i}}\left(g(R)\right) = g\left(\nabla_{\vec{e_i}} R\right)}$$

So, Now the equation becomes

$$\Rightarrow g^{bn}\left(g^{am} R_{abmn;l} + g^{am} R_{ablm;n} + g^{am} R_{abnl;m}\right) = 0$$

"symmetry"

$$\Rightarrow g^{bn}\left(R^{m}{}_{bmn;l} + g^{am}\left(-R_{abml;n}\right) + g^{am} R_{abnl;m}\right) = 0$$

$$\Rightarrow g^{bn}\left(R^{m}{}_{bmn;l} - R^{m}{}_{bml;n} - g^{am} R_{banl;m}\right) = 0$$

these two elements have contraction "
on "2nd index", So they can be "covariant derivative"

components of "Ricci Curvature tensor"

$$\Rightarrow g^{bn}\left(R_{bnjl} + R_{blin} - g^{am} R_{banl;m}\right) = 0$$

Now applying this metric tensor into the inner elements.

$$\Rightarrow \left(R^{n}_{\ njl} - R^{n}_{\ lin} - g^{am} g^{bn} R_{banl;m}\right) = 0.$$

$$\Rightarrow \left(R^{n}_{\ nil} - R^{n}_{\ lin} - g^{am} R^{n}_{\ anl;m}\right) = 0$$

$$\Rightarrow R^{n}_{\ nil} - R^{n}_{\ lin} - g^{am} R_{al;m} = 0$$

now, contract this with a index.

$$\Rightarrow R^n_{n i l} - R^n_{l i n} - \underline{R^m_{l i m}} = 0 \rightarrow$$

here this term is relabeled with n index since it's a dummy index.

$$\hookrightarrow R_{il} - R^n_{l i n} - R^n_{l i n} = 0$$

$$R_{il} - 2R^n_{lin} = 0 \quad \text{or} \quad \boxed{\frac{1}{2}R_{il} - 2R^n_{lin} = 0}$$

Expanding this equation using "kronecker delta"

$$\Rightarrow \boxed{\frac{1}{2}\delta^n_l R_{i\underline{n}} - R^n_{li\underline{n}} = 0} \rightarrow \text{here } \left\{ \delta^i_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \right\}$$

Now we achieved the equation with respect to one "Common Covariant derivative" with direction/index "n".

Now taking inverse metric "g^{ml}" on both sides.

$$\frac{1}{2}\underbrace{g^{ml}\delta^n_l R_{;n}}_{\text{Contracted by index "}l\text{"}} - \underbrace{g^{ml} R^n_{l;m}}_{\text{Contracted by index "}l\text{"}} = 0$$

So the equation becomes

$$\boxed{\frac{1}{2} g^{mn} R_{;n} - R^{nm}{}_{;n} = 0}$$ ← Contracted Bianchi Identity.

Here, we know the fact that covariant derivative of Ricci curvature tensor is $\neq 0$ i.e $\boxed{R^{mn}{}_{;n} \neq 0}$

But by adding a small correction of "$-\frac{1}{2} g^{mn} R$"; it's co-variant derivative is "zero".

Question: Why would someone bother about co-variant derivative of Ricci curvature tensor "R_{mn}"??

Solution:

of solution:

↳ the "Ricci Tensor (R_{mn})" is actually supposed to be equal to a scale value of the "Stress Energy tensor" (T_{mn}) in 4d space time.

↳ But the co-variant derivative of T_{mn} (i.e) $\boxed{T_{mn;n} = 0}$ because the stress imposed in space time manifold due to a mass doesn't further vary in any direction.

↳ Actually the equation is supposed to be formulated like this.

incorrect formulation

$$\boxed{R_{\mu\nu} = \cancel{\phi} T_{\mu\nu}}$$

but on taking co-variant derivative in the direction 'ν')

for

$R_{\mu\nu;\nu} = \phi\, T_{\mu\nu;\nu}$

$\Rightarrow \boxed{R_{\mu\nu;\nu} = 0}$ $\rightarrow$ But the derivative of Ricci tensor will never be equal to "zero"

covariant

False

∴ Einstien had to come up with some "left side Curvature term" which had a "zero covariant derivative" on it's second index so as to satisfy the condition $\boxed{T_{mn;n} = 0}$.

Now the equation can be written as

$\left(R^{mn} + g^{mn} R \right)$

$$\left(R - \frac{1}{2} g \; R\right)_{;n} = 0$$

or

"Einstien Tensor" $\boxed{G^{mn}{}_{;n} = 0}$ where $\boxed{G^{mn} = \left(R^{mn} - \frac{1}{2} g^{mn} R\right)}$

In abstract form

$$\boxed{\nabla_{\vec{e_n}} \left(R^{mn}\left(\vec{e_m} \otimes \vec{e_n}\right) - \frac{1}{2} g^{mn}\left(\vec{e_m} \otimes \vec{e_n}\right) R\right) = 0}$$

$\therefore$ We Finally Arrive at "Einstien tensor" formulation.

$$\left(R^{mn} - \frac{1}{2} g^{mn} R\right)_{;n} = 0.$$

$$G^{mn}{}_{;n} = 0$$

But since it is a covariant derivative component and the covariant derivative is zero, we can add additional "metric tensor" scaled terms to the equation.

so, $$\left(R^{mn} - \frac{1}{2} g^{mn} R\right)_{;n} = 0$$

Also,

$$\left(R^{mn} - \frac{1}{2} g^{mn} R + K g^{mn} + L g^{mn} + \cdots\right)_{;n} = 0$$

$\therefore$ It can be rewritten with another term with a

... it can be re-written with another term with constant ("say Λ") to its sum.

$$\therefore \left(R^{mn} - \frac{1}{2} g^{mn} R\right)_{;n} = 0$$

(or)

$$\boxed{\left(R^{mn} - \frac{1}{2} g^{mn} R + \Lambda g^{mn}\right)_{;n} = 0}$$

↳ "<u>Einstien tensor</u>" ⇒ General representation.

<u>Extending this Concept to</u>

<u>General Relativity</u>.

Basic Concept:

↳ The mass which moves through a "4 dimensional" - Space time imposes some "stress" on the 4d manifold causing it's "Geodesics to curve". This stress is mathematically given by the "Stress Energy Momentum Tensor" ($T_{\mu\nu}$) in 4d

↳ In this "Local" curved 4 dimensional manifold, "Lengths and Clocks change as a result of Geodesic Deviation".

and "Einstein Tensor" ($G_{\mu\nu}$) keeps track of these changes.

only Einstien Tensor ($G_{\mu\nu}$) keeps ... of these ...

↳ Einstien thought that "Stress ($T_{\mu\nu}$) imposed by a massive object causes curvature in space time and other smaller objects moving in this curvature Experience gravity"
(not a force).

↳ General Theory of Relativity in a small sentence.

Einstien formulated the equation as

$$G^{\mu\nu} \propto T^{\mu\nu}$$

$$\Rightarrow G^{\mu\nu} = \Phi T^{\mu\nu}$$

(some constant).

$$\Rightarrow R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} = \Phi T^{\mu\nu}$$

writing it in covariant form

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = \Phi T_{\mu\nu}}$$

$$\Updownarrow$$

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \Phi T_{\mu\nu} + \Lambda g_{\mu\nu}}$$

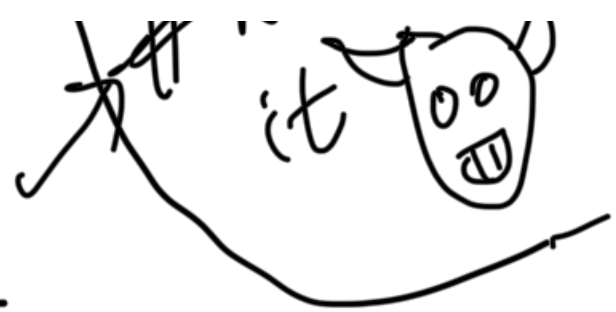
$T^{\mu\nu}{}_{;\nu}\ G^{\mu\nu}{}_{;\nu} = 0$

$\Phi = 8\pi G$

$$\Downarrow$$

If Nailed

Einstien's Field Equation



Summary:

2^{nd} Bianchi Identity:

$$\boxed{R^{d}{}_{cab;i} + R^{d}{}_{cia;b} + R^{d}{}_{cbi;a} = 0}$$

$;$ -> Components of Covariant derivative

Contracted version of 2^{nd} Bianchi identity.

Contracted ...

$$g^{am} g^{bn} \left(R_{abmn;l} + R_{ablm;n} + R_{abnl;m} \right) = 0$$

$$\Rightarrow \boxed{\nabla_{\vec{e}_n} \left(R^{mn} \vec{e}_m \otimes \vec{e}_n + \frac{1}{2} g^{mn} \vec{e}_m \otimes \vec{e}_n (R) \right) = 0}$$

$$(R^{mn} - \frac{1}{2} g^{mn} R)_{;n} = 0$$

ABSTRACT EINSTIEN TENSOR

Einstien tensor → $\boxed{G^{mn}{}_{;n} = 0}$ ⟷ $\nabla_{\vec{e}_n} \left(G^{mn} \vec{e}_m \otimes \vec{e}_n \right) = 0$

$$\boxed{G^{mn} = R^{mn} - \frac{1}{2} g^{mn} R + \Lambda g^{mn}}$$

$T^{\mu\nu}$ = stress tensor

$T^{\mu\nu}{}_{;n} = G^{\mu\nu}{}_{;n} = 0$

Einstien's Field Equations

Finally Nailed it !!

$\Phi = \frac{8\pi G}{c^4}$

So, $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \Phi T_{\mu\nu} - \Lambda g_{\mu\nu}$